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COLLISION PRIMES OF $J_0(23)$: FORCED CONGRUENCES, EXCEPTIONAL LOCAL FACTORS, AND A PRE-REGISTERED TEST OF THE MURTY–VAN HIRTUM CONSTANT

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Abstract. Let f be the newform of weight 2 and level 23, with coefficient field $\mathbb{Q}(\sqrt{5})$. Call a prime $p \neq 2, 23$ a *collision prime* if $a_p(f) \in \mathbb{Z}$, that is, if the two Galois-conjugate Hecke eigenvalues coincide. We prove that at a collision prime $a_p(f) \equiv p + 1 \pmod{11}$, and that $a_p(f) \bmod 2$ is determined by the class of p in the Hilbert class field of $\mathbb{Q}(\sqrt{-23})$. Hence $a_p(f) \bmod 22$ is a function of p alone. We also record a mod-4 lemma and state a mod-4 conjecture, supported by 1.19×10^6 cases without exception.

On the quantitative side, we replace the empirically fitted exceptional local factors in Van Hirtum's heuristic for $\#\{p \leq x : a_p(f) \in \mathbb{Q}\}$ by factors computed from explicit, hypothesised models of the Galois image: an Eisenstein model at 11 and an icosahedral model at 3. The factor at 2 remains empirical. A model fixed in advance using the generic factor at 3 was refuted ($z = +8.4$). The corrected model was frozen before computing the range $(2{\cdot}10^7, 6{\cdot}10^7]$, and there predicted 914.9 collisions against 909 observed. The asymptotic constant obtained is $c \approx 4.864$. The $\approx 10\%$ gap between Van Hirtum's fitted (5.490) and predicted (4.982) constants for this form is numerically consistent with the known finite-range effect of replacing $\sum_{p \leq x} p^{-1/2}$ by $2\sqrt{x}/\log x$. All data (3,562,113 primes), code and frozen pre-registrations are public.

1. Introduction

Let $f = \sum a_n q^n \in S_2(\Gamma_0(23))$ be a newform. Its Hecke field is $K = \mathbb{Q}(\sqrt{5})$; write $\phi = (1 + \sqrt{5})/2$ and $\mathcal{O} = \mathbb{Z}[\phi]$. The Galois conjugate f^σ is the other newform, and $J_0(23)$ is a simple abelian surface with real multiplication by $\mathcal{O}$ over $\mathbb{Q}$ [4, 7]. Write

$$a_p(f) = u_p + v_p \phi, \qquad u_p, v_p \in \mathbb{Z}.$$

Definition 1.1. A prime $p \neq 2, 23$ is a *collision prime* if $a_p(f) = a_p(f^\sigma)$, i.e. $v_p = 0$, i.e. $a_p(f) \in \mathbb{Z}$.

The first collision primes are $13, 19, 29, 43, 109, 223, 229, 271, 463, 673, \ldots$. Since f has no inner twist, Ribet's theorems [11] imply that collision primes have density zero. A conjecture of V. K. Murty [9], extending Lang and Trotter [6], in the form stated by Van Hirtum [15, Conj. 1.1], predicts

$$\#\{p \leq x : \ a_p(f) \in \mathbb{Q}\} \ \sim \ c_f \, \frac{\sqrt{x}}{\log x}, \qquad c_f = \frac{16\sqrt{D}\, \widehat{F}_f}{3\pi^2}, \tag{1}$$

where $D = 5$ and $\widehat{F}_f = \lim_m F_m$ is a limit of local divisibility densities. Van Hirtum computed the unramified large-image factors and tested level 23 up to 10^8; for the ramified and exceptional primes he used empirical frequencies [15, §7]. Amri [1, App. A] has recently proved that $\widehat{F}_f$

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exists and is positive in general, and that it factors as a *joint* exceptional factor times the unramified factors. No explicit exceptional factors for a specific form are computed there.

This note does three things for level 23.

(1) Section 3: collision primes satisfy forced congruences modulo 11 and 2 (Theorem 3.2), together with a mod-4 lemma and a mod-4 conjecture.
(2) Section 4: the exceptional factors at 3, 11 and 5 are derived by exact counting from explicit models of the Galois image (stated as hypotheses), instead of being fitted.
(3) Section 5: the resulting constant is tested against 3,562,113 primes, using predictions frozen (by SHA-256 hash) before the data were computed. This includes an honest record of a failed first model.

We claim no new theorem of depth. Theorem 3.2 is a short consequence of Mazur's work and of the structure of $J_0(23)[2]$; we have not found it stated in the literature. The infinitude of collision primes remains open.

Data and code. Everything is available at `https://github.com/Dojo-1/J0-23-collision-primes`: code, ten regression tests, the values (u_p, v_p) for all primes up to $6 \cdot 10^7$, and both pre-registrations with their hashes. This work grew out of round R14 of the Bolyai Research Atlas [2].

2. Computing the coefficients

Let $\theta_1 = \sum_{x,y} q^{x^2+xy+6y^2}$ and $\theta_2 = \sum_{x,y} q^{2x^2+xy+3y^2}$ be the theta series of the two reduced forms of discriminant -23 (up to $GL_2(\mathbb{Z})$). They have weight 1 and character $\left(\frac{-23}{\cdot}\right)$. Put

$$B = \tfrac{1}{2}(\theta_1 - \theta_2) = \eta(z)\eta(23z) = q - q^2 - q^3 + q^6 + q^8 - q^{13} - \cdots .$$

Proposition 2.1. *In the normalisation $a_p(f) = u_p + v_p\phi$ with $\phi^2 = \phi + 1$,*

$$\sum_n v_n q^n = \eta(z)^2\eta(23z)^2, \qquad \sum_n u_n q^n = \theta_2\,\eta(z)\eta(23z).$$

Proof. The three products $\theta_1^2, \theta_1\theta_2, \theta_2^2$ lie in $M_2(\Gamma_0(23))$, which has dimension 3. Comparing q-expansions, both identities are linear relations in this space. By Sturm's bound [14] it suffices to check coefficients up to $n = 2 \cdot 24/12 = 4$. We checked them up to $n = 10^4$ against PARI/GP's `mfcoefs` [10]. □

Proposition 2.1 reduces the computation of (u_p, v_p) for $p \le N$ to two FFT convolutions of length $O(N)$. For $N = 6 \cdot 10^7$ the largest rounding residual at a prime was below $4 \cdot 10^{-11}$. As an independent check, the resulting Euler factors agree with point counts on the genus-2 model $y^2 + (x^3 + x + 1)y = -x^5$ of $X_0(23)$ (PARI's `hyperellcharpoly`) at 468 primes below 12,000, with no mismatch.

3. Forced congruences

We use the following standard inputs:

(I1) (Mazur [8]) For prime level N, the Eisenstein ideal $I \subset \mathbb{T}$ is generated by $T_\ell - 1 - \ell$ ($\ell \neq N$) and $1 + w_N$, and $\mathbb{T}/I \cong \mathbb{Z}/n\mathbb{Z}$ with $n = \mathrm{num}\big((N-1)/12\big)$. For $N = 23$, $n = 11$.
(I2) $\mathrm{End}(J_0(23)) = \mathcal{O}$ [4, 7]. Since T_2 satisfies $x^2 + x - 1 = 0$, we get $\mathcal{O} = \mathbb{Z}[T_2] \subseteq \mathbb{T} \subseteq \mathcal{O}$, so $\mathbb{T} = \mathcal{O}$.
(I3) (Eichler–Shimura [13]) For $\ell \neq p$, the Tate module $T_\ell J_0(23)$ is free of rank 2 over $\mathcal{O} \otimes \mathbb{Z}_\ell$, and Frob_p acts on it with trace T_p and determinant p.

Let H be the Hilbert class field of $\mathbb{Q}(\sqrt{-23})$, the splitting field of x^3-x-1, with $\mathrm{Gal}(H/\mathbb{Q}) \cong S_3$. A prime $p \neq 23$ splits completely in H iff $p = x^2+xy+6y^2$. It has Frobenius of order 3 iff $p = 2x^2+xy+3y^2$, and Frobenius of order 2 iff $\left(\frac{-23}{p}\right) = -1$.

Lemma 3.1. $\mathbb{Q}(J_0(23)[2]) = H$.

Proof. Completing the square in $y^2+(x^3+x+1)y = -x^5$ gives $(2y+x^3+x+1)^2 = F(x)$ with

$$F(x) = (x^3+x+1)^2 - 4x^5 = (x^3-2x^2-3x-1)(x^3-2x^2+x-1).$$

Both cubic factors are irreducible over $\mathbb{Q}$ (no rational roots) and have discriminant -23. The cubic field of discriminant -23 is unique up to isomorphism, so both factors split in H, and the splitting field of F is H. For a genus-2 curve $y^2 = F(x)$ with $\deg F = 6$, the field generated by the 2-torsion of the Jacobian is the splitting field of F. (Schoof uses another model of the same curve [12].) □

Theorem 3.2. *Let $p \neq 2, 23$ be a collision prime and $a = a_p(f) \in \mathbb{Z}$. Then*

$$a \equiv p+1 \pmod{11}, \qquad a \equiv \begin{cases} 1, & p = 2x^2+xy+3y^2, \\ 0, & \text{otherwise,} \end{cases} \pmod 2.$$

In particular $a \bmod 22$ is determined by p, and $a_p(f) = 0$ implies $p \equiv 10 \pmod{11}$.

Proof. Mod 11. By (I2), $T_p = a \in \mathbb{Z} \subset \mathbb{T}$. By (I1), $a-1-p \in I \cap \mathbb{Z}$, and $I \cap \mathbb{Z} = 11\mathbb{Z}$ because $\mathbb{Z} \to \mathbb{T}/I \cong \mathbb{Z}/11$ is surjective.

Mod 2. By (I3) and Lemma 3.1, $G_{\mathbb{Q}}$ acts on $J[2] \cong \mathbb{F}_4^2$ through $\mathrm{Gal}(H/\mathbb{Q}) \cong S_3$, $\mathbb{F}_4$-linearly ($\mathcal{O}/2 \cong \mathbb{F}_4$, since 2 is inert in K), with determinant the mod-2 cyclotomic character, which is trivial. The image is therefore a subgroup of $\mathrm{SL}_2(\mathbb{F}_4) \cong A_5$ of order dividing 6. In $\mathrm{SL}_2(\mathbb{F}_4)$, elements of order 1 or 2 have trace 0, elements of order 3 have trace 1, and only elements of order 5 have trace in $\mathbb{F}_4 \setminus \mathbb{F}_2$. Hence $T_p \bmod 2\mathcal{O} \in \mathbb{F}_2$, and it equals 1 exactly when Frob_p has order 3 in $\mathrm{Gal}(H/\mathbb{Q})$ (the action is faithful because $H = \mathbb{Q}(J[2])$). At a collision prime $T_p = a$, so $a \bmod 2$ is this trace. □

Corollary 3.3. *For every $p \neq 2, 23$, v_p is even.*

Proof. $T_p \bmod 2\mathcal{O} \in \mathbb{F}_2$ means ϕ-coordinate $v_p \equiv 0 \pmod 2$. An elementary second proof: by Proposition 2.1, $v_p = \sum_{m=1}^{p-1} b_m b_{p-m}$ where $B = \sum b_n q^n$. For odd p the terms pair off under $m \leftrightarrow p-m$ with $m \neq p-m$. □

Lemma 3.4. *If $p = x^2+xy+6y^2$, then $T_p \equiv 1+p \pmod{4\mathcal{O}}$. In particular $4 \mid v_p$.*

Proof. Frob_p acts trivially on $J[2]$, so on $T_2 J$ it equals $1+2X$ with $X \in M_2(\mathcal{O}_2)$. Then $T_p = 2+2\,\mathrm{tr}\,X$ and $p = \det(1+2X) = 1+2\,\mathrm{tr}\,X+4\det X$. Hence $1+p-T_p = 4\det X$. □

Conjecture 3.5. *If $p = 2x^2+xy+3y^2$, then $v_p \equiv p-1 \pmod 4$.*

Conjecture 3.5 is not a formal consequence of the mod-2 structure. Write $\mathrm{Frob}_p = A+2\phi B$ with $A, B \in M_2(\mathbb{Z}_2)$ in a basis where the mod-2 image is $\mathbb{F}_2$-rational. Then $v_p = 2\,\mathrm{tr}\,B$, while $p \equiv \det A \pmod 4$, and nothing above links $\mathrm{tr}\,B \bmod 2$ to $\det A \bmod 4$. So Conjecture 3.5 is a constraint on the 2-adic image itself. It plausibly follows from Schoof's analysis of the 2-divisible group of $J_0(23)$ [12], which we have not carried out.

For inert p, $v_p \bmod 4$ is equidistributed in our data (0.4998 vs. 0.5002).

Remark 3.6 (Verification). For all 3,562,113 primes $3 \le p \le 6 \cdot 10^7$, $p \neq 23$, the following hold with no exceptions: Theorem 3.2 (2,386 collision primes), Corollary 3.3, the congruence $a_p(f) \equiv 1+p \pmod{(11, \phi-4)}$, Lemma 3.4 (593,491 primes) and Conjecture 3.5 (1,187,172 primes). The only primes up to $6 \cdot 10^7$ with $a_p(f) = 0$ are 43, 109 and 1033, all $\equiv 10 \pmod{11}$.

4. Exceptional local factors from image models

Following [15, 1], for a prime ℓ set

$$F_\ell^{(k)} = \ell^k \cdot \Pr\left[\operatorname{tr} g \in \mathbb{Z}/\ell^k\right],$$

where g is Haar-distributed on the image of $G_{\mathbb{Q}}$ in $\mathrm{GL}_2(\mathcal{O}/\ell^k)$. Then $F_\ell = \lim_k F_\ell^{(k)}$, and empirically $F_\ell^{(k)}$ equals ℓ^k times the frequency of primes with $\ell^k \mid v_p$. Van Hirtum's closed forms [15, Props. 5.5–5.8] hold for unramified ℓ with image $\{g : \det g \in \mathbb{Z}_\ell^\times\}$. Our exact enumeration reproduces them: for example $F_3^{(1)} = 87/80$, $F_3^{(2)} = 263/240$, with limit 1.096875.

4.1. $\ell = 11$**: Eisenstein.** 11 splits in K. At the Eisenstein prime $\mathfrak{P} = (11, \phi - 4)$, Mazur shows (for prime level with n odd) that $J[\mathfrak{P}]$ is the direct sum of the cuspidal group $\mathbb{Z}/11$ and the Shimura subgroup μ_{11} [8]. So $\bar\rho_{\mathfrak{P}} \simeq 1 \oplus \chi_{11}$, and we take the $\mathfrak{P}$-adic image to be the full preimage of the diagonal $\{\mathrm{diag}(1, *)\}$. At the other prime above 11 we assume full image, and the two components share their determinant. Exact counting gives

$$F_{11}^{(1)} = 1.09083, \quad F_{11}^{(2)} = 1.09917, \quad F_{11}^{(3)} = 1.09924.$$

The empirical values are 1.0948 and 1.1083 (standard errors about 0.01). A Borel-type model, which is inconsistent with the split kernel, would give $F_{11}^{(2)} = 1.0916$. For comparison, the generic split factor at 11 is 1.0076.

4.2. $\ell = 3$**: an icosahedral model.** Billerey and Dieulefait state, for this form and the prime above 3, that the projective mod-3 image is A_5, with kernel field the splitting field of $X^5 + 3X^3 + 6X^2 + 9$; they attribute the example to Ribet and to Kiming–Verrill [3, §5.3], [5, Ex. 3.2]. We have not re-derived this identification theoretically (but see Remark 4.1), and nothing below depends on it beyond motivating the model. The data show which coset structure is consistent with the traces. For $p \equiv 1 \pmod 3$, $T_p \bmod 3 \in \mathbb{F}_9$ never equals $\pm\phi^2$; for $p \equiv 2 \pmod 3$, it never equals ± 1. The remaining frequencies are 0.250, 0.175 and 0.100. These coincide to three decimals with the trace distribution of $G_3 = \langle i \rangle \cdot 2.A_5 \subset \mathrm{GL}_2(\mathbb{F}_9)$, where $2.A_5 \cong \mathrm{SL}_2(\mathbb{F}_5)$ is the binary icosahedral group embedded in $\mathrm{SL}_2(\mathbb{F}_9)$, $i = \phi^2$ and $i^2 = -1$. In $2.A_5$, the trace is 0 on 30 of 120 elements, ± 1 on 21 each, and each of the four values $(\pm 1 \pm \sqrt{5})/2$ on 12. Under the *model assumption* that the 3-adic image is the full preimage of G_3, we get

$$F_3^{(1)} = \tfrac{51}{40}, \quad F_3^{(2)} = \tfrac{53}{40}, \qquad F_3 \approx 1.332,$$

compared with the empirical values 1.2750, 1.3258, 1.3328 and 1.3333 for $k = 1, \ldots, 4$. A frequency match does not by itself determine the image; the extrapolated limit is heuristic.

Remark 4.1 (An independent consistency check). The data can be tested against the field reported in [3, §5.3]. Let $g \in \mathrm{GL}_2(\mathbb{F}_9)$ have trace t and determinant $d \in \mathbb{F}_3^\times$, and put $r = t^2/d$. In characteristic 3, an element of order 3 in $\mathrm{PGL}_2(\mathbb{F}_9)$ is unipotent, and an element of order 5 has eigenvalue ratio a primitive fifth root of unity ζ. So the image of g in A_5 has order 2 iff $t = 0$, has order 1 or 3 iff $r = 4 = 1$, and has order 5 iff $r = 2 + \zeta + \zeta^{-1} \notin \mathbb{F}_3$. On the other side, the cycle type of Frob_p on the roots of $X^5 + 3X^3 + 6X^2 + 9$ is 1^5, $(3, 1, 1)$, $(2, 2, 1)$ or (5) according to its order in A_5.

For all 17,981 primes $p \le 2 \cdot 10^5$ with $p \nmid \mathrm{disc} = 3^8 \cdot 23^2$, we compared the factorisation pattern of the quintic mod p with r computed from $T_p \bmod 3$ and $d = p \bmod 3$. There are no mismatches. The four classes occur 296, 6,004, 4,495 and 7,186 times, close to the A_5 proportions $1 : 20 : 15 : 24$ out of 60. This is strong evidence for the reported projective image.

It is not a proof, and it says nothing about the 3-adic image beyond the residual level. The script is `scripts/mod3_quintic_check.py` in the repository.

4.3. $\ell = 5$ **and** $\ell = 2$**.** 5 ramifies in K. With full image, $F_5^{(1)} = 31/30$ and $F_5^{(2)} = 388/375$, so $F_5 \approx 1.0347$; the empirical values are 1.0304 and 1.0259.

At 2, Theorem 3.2 puts the mod-2 image inside $\mathrm{GL}_2(\mathbb{F}_2)$. The full preimage of that image gives $F_2^{(k)} = 2, \frac{7}{3}, \frac{29}{12}, \frac{39}{16} \to \frac{22}{9}$. The data give 2, 2.332, 2.501, 2.542, 2.554. The model therefore fails at $k \geq 3$, as Conjecture 3.5 predicts: the full preimage gives $\Pr[4 \mid v_p] = 7/12$ for both classes p mod 4, whereas the data give 3/4 for $p \equiv 1$ and 5/12 for $p \equiv 3$. We therefore use the empirical value $F_2 = 2.554$ and leave the 2-adic image open.

4.4. **Independence.** The model below multiplies the exceptional factors, which assumes independence across ℓ; Amri's joint factor [1] allows for dependence. The one entanglement visible here is between the 2-adic and 23-adic images, through $\sqrt{-23} \in H \cap \mathbb{Q}(\zeta_{23})$. It changes the product by less than $2 \cdot 10^{-4}$, because the 23-adic factor depends on $\left(\frac{p}{23}\right)$ only at relative order $2/23^3 \approx 1.6 \cdot 10^{-4}$.

5. A pre-registered test

The archimedean factor comes from the Sato–Tate group $\mathrm{SU}(2) \times \mathrm{SU}(2)$: $\Pr[v_p = 0] \approx \frac{8\sqrt{5}}{3\pi^2} \widehat{F}\, p^{-1/2}$ [15, Lemma 4.5]. We predict

$$N(x) = \#\{p \leq x : \ v_p = 0\} \ \approx \ \frac{8\sqrt{5}\,\widehat{F}}{3\pi^2} \sum_{3 \leq p \leq x,\ p \neq 23} p^{-1/2}.$$

Model M1 was frozen before any computation beyond $p \leq 101$ (SHA-256 `f3d823ab...`). It used $F_2 = 22/9$, F_{11} as above, and the *generic* F_3, giving $\widehat{F} = 3.1729$. It was refuted: at $x = 2 \cdot 10^7$ it predicted 1187.9 against 1477 observed ($z = +8.4$). Its pre-registered local tests located the failure at $\ell = 3$ ($z = +147$) and at $\ell = 2$, $k \geq 3$, while $\ell = 5$ and $\ell = 11$ passed. This is how the icosahedral structure above was found.

Model M2 is exploratory, being chosen after M1 failed. It uses the icosahedral $F_3 = 1.332$ and the empirical $F_2 = 2.554$. The latter is estimated from local frequencies over all primes, not from collision counts. This gives $\widehat{F} = 4.026$ and $c_f = 16\sqrt{5}\widehat{F}/(3\pi^2) = 4.864$. M2 was frozen (SHA-256 `c71a78e4...`) before the range $(2 \cdot 10^7, 6 \cdot 10^7]$ was computed.

held-out range	observed	M2	z_{M2}	z_{M1}
$(2 \cdot 10^7, 4 \cdot 10^7]$	504	524.6	-0.90	$+4.45$
$(4 \cdot 10^7, 6 \cdot 10^7]$	405	390.3	$+0.75$	$+5.55$
$(2 \cdot 10^7, 6 \cdot 10^7]$	909	914.9	-0.19	$+7.00$

TABLE 1. Held-out test. $z = (\mathrm{obs} - \mathrm{pred})/\sqrt{\mathrm{pred}}$.

M2 overestimates at small x: the ratio is 0.75 at 10^5, 0.90 at 10^6, and 0.985 to 0.99 from 10^7 on. The $p^{-1/2}$ approximation ignores finite-p effects: the lattice of admissible $(a_p(f), a_p(f^\sigma))$, restricted by Theorem 3.2 to classes mod 22, sampled against the Sato–Tate density near the edges of the Hasse box. We have not modelled this.

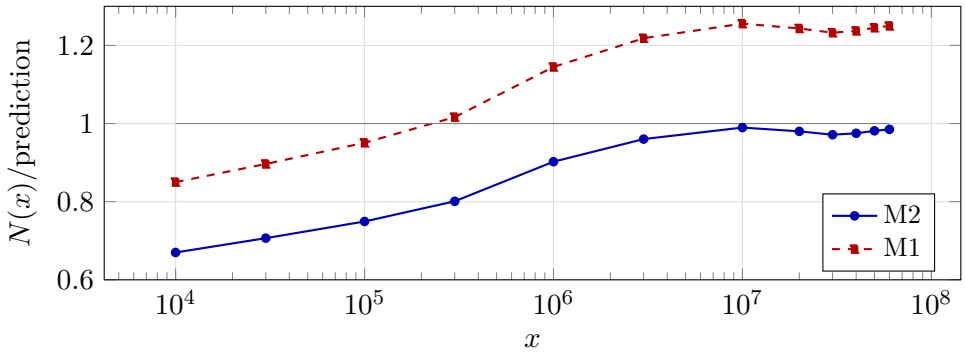

FIGURE 1. Cumulative ratio of observed to predicted collision counts.

5.1. **Comparison with Van Hirtum's fitted constant.** For level 23, Van Hirtum reports a predicted $\hat{c} = 4.982$ (using empirical exceptional factors) and a constant $\tilde{c} = 5.490$ fitted on $x \le 10^8$, a ratio of 1.10 [15, Table 1]. Our $c_f = 4.864$ is within 2.4% of his prediction. The finite-range effect of normalising by $2\sqrt{x}/\log x$ is already recognised in prior work; we only quantify it for this form. The quantity

$$\Big(\sum_{p \le x} p^{-1/2}\Big) \Big/ \big(2\sqrt{x}/\log x\big)$$

equals 1.174, 1.160 and 1.146 at $x = 10^7$, $3 \cdot 10^7$ and 10^8. A least-squares fit of $c\sqrt{x}/\log x$ to M2-expected counts at 50 equally spaced points up to 10^8 returns 5.60, within 2% of 5.490. This is numerically consistent with the gap being a normalisation effect, but it does not rule out omitted local corrections, and we did not reconstruct Van Hirtum's fitting window. Comparisons with (1) at accessible ranges are cleaner with $\sum p^{-1/2}$ than with its leading asymptotic.

6. Open problems

(1) Determine the 2-adic image of $J_0(23)$. Prove Conjecture 3.5, and check whether the full preimage of that image gives $F_2 = 2.554$ exactly.
(2) Independently determine the mod-3 and 3-adic images (rather than inferring them from trace frequencies), check whether the latter is the full preimage of $\langle i \rangle \cdot 2.A_5$, and compute F_3 exactly; the data and $k \le 2$ suggest $F_3 = 4/3$.
(3) Give a finite-p refinement of the archimedean factor that explains the deficit at $x \le 10^6$.
(4) Formalise Theorem 3.2 given (I1)–(I3); its arithmetic steps are finite.

Use of AI. The computations, the search for congruences and local-factor models, the proofs and the text were produced with substantial assistance from an AI system (Claude, Anthropic), working under the author's direction within the Bolyai Research Atlas project. Prompts and transcripts are available on request. Every numerical claim is reproducible from the public repository.

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