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THE POLYLOGARITHM AS A LIOUVILLE FRACTIONAL INTEGRAL AND A RELATED FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. We study the polylogarithm $\mathrm{Li}_s(z)$ from the perspective of fractional calculus. For $\mathrm{Re}\, s > 0$ and $\mathrm{Re}\, w < 0$, we express $\mathrm{Li}_s(e^w)$ as the left Liouville fractional integral of order s of the function $e^w/(1-e^w)$. This gives the classical Bose–Einstein integral representation.

We then consider a linear fractional differential equation with forcing term $\mathrm{Li}_s(e^w)$. Using the eigenfunction property of exponentials under the Liouville derivative, we obtain an explicit solution in the form of a series. This series defines a parameter-dependent function $R_{\alpha,s}(z;\lambda)$, for which we establish several properties, including a reduction to a classical polylogarithm, an expansion in polylogarithms, differential relations, and its meromorphic dependence on λ.

1. Introduction

The polylogarithm is a classical special function [5, 4]: it contains the geometric series, the dilogarithm and, at $z = 1$, the Riemann zeta function as special cases. Fractional calculus, which extends differentiation and integration to non-integer orders, is treated in the monographs [6, 10, 9, 8].

This paper is inspired by the work of Fernandez [1], who expressed the Lerch zeta function as a fractional derivative of an elementary function, and by the fractional differential relations subsequently obtained for it in [2]. The polylogarithm is a specialization of the Lerch transcendent, since $\mathrm{Li}_s(z) = z\Phi(z, s, 1)$; in the notation of [1, 2], this corresponds to setting the shift parameter equal to 1. Their fractional representation therefore has a polylogarithmic specialization; however, after the natural change of variables, its integration direction differs from the horizontal left Liouville integral used here. Lin and Srivastava [3] studied related families of Hurwitz–Lerch zeta functions and associated fractional-derivative and integral representations. These works place the present study in an established context. A further motivation is to introduce and examine the parameter-dependent function $R_{\alpha,s}(z;\lambda)$, which, to our knowledge, has not been isolated in this form and

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has useful connections with polylogarithms and fractional differential equations.

Our first result records a direct left Liouville integral representation for $\mathrm{Li}_s(e^w)$ in the variable w (Section 2). The resulting Bose–Einstein integral is classical; see [4, Eq. 25.12.11]. We include it as a consequence of the fractional-integral identity, rather than as a new integral formula. We then take $\mathrm{Li}_s(e^w)$ as the forcing term in a linear fractional differential equation and obtain an explicit series solution (Section 3). Finally, we study the parameter-dependent function $R_{\alpha,s}(z;\lambda)$ defined by that series (Section 4). In particular, we establish its reduction to a polylogarithm, its expansion in polylogarithms for $|\lambda| < 1$, its Euler-operator relation, and its meromorphic structure in λ.

Preliminaries. For $s, z \in \mathbb{C}$ with $|z| < 1$ the polylogarithm is the Dirichlet-type series

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \tag{1}$$

where $n^s = e^{s \log n}$ with $\log n \in \mathbb{R}$ [4, §25.12(ii)]; it converges absolutely for $|z| < 1$ by comparison with a geometric series.

For a real lower limit c and real $w > c$, the Riemann–Liouville fractional integral of order s is

$${}_cD_w^{-s} f(w) = \frac{1}{\Gamma(s)} \int_c^w (w-u)^{s-1} f(u)\, du, \qquad \operatorname{Re} s > 0. \tag{2}$$

In this paper we use the Liouville case $c = -\infty$. For complex w, its integral is defined along the horizontal ray extending to the left from w. Substituting $u = w - r$ with $0 \le r < \infty$ it becomes

$${}_{-\infty}D_w^{-s} f(w) = \frac{1}{\Gamma(s)} \int_0^{\infty} r^{s-1} f(w-r)\, dr, \qquad \operatorname{Re} s > 0. \tag{3}$$

For complex w the integral in (3) is taken along the horizontal ray extending to the left from w, with $r \in (0,\infty)$ and $\arg r = 0$; see [1].

Definition 1 (Riemann–Liouville fractional derivative [6, 1])**.** Let $\operatorname{Re}\alpha \ge 0$ and let $m = \lfloor \operatorname{Re}\alpha \rfloor + 1$ be the smallest integer strictly greater than $\operatorname{Re}\alpha$. The Riemann–Liouville fractional derivative of order α is

$$\begin{aligned} {}_cD_w^{\alpha} f(w) &= \frac{d^m}{dw^m}\, {}_cD_w^{-(m-\alpha)} f(w) \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dw^m} \int_c^w (w-u)^{m-\alpha-1} f(u)\, du. \end{aligned} \tag{4}$$

For $\operatorname{Re}\alpha < 0$ we set ${}_cD_w^{\alpha} f := {}_cD_w^{-\alpha} f$, which is the Riemann–Liouville fractional integral of order $-\alpha$.

When $c = -\infty$ the operator is the *Liouville fractional derivative*. If α is a positive integer, then $m = \alpha + 1$ and (4) reduces to the ordinary derivative d^{α}/dw^{α}. For the theory of these operators see [6, 10].

We shall use the Gamma integral in the form

$$\int_0^{\infty} r^{\sigma-1} e^{-nr}\, dr = \frac{\Gamma(\sigma)}{n^{\sigma}}, \qquad \operatorname{Re}\sigma > 0,\ n > 0, \tag{5}$$

which follows from Euler's integral for Γ [4, §5.2(i)] by the substitution $r \mapsto nr$.

Finally, the key property that connects the polylogarithm with the Liouville derivative is that exponentials are eigenfunctions of it.

Lemma 1 (Eigenfunction property)**.** *Let $s \in \mathbb{C}$ and let $k > 0$ be real. Then*

$${}_{-\infty}D_t^{s}\big(e^{kt}\big) = k^{s} e^{kt}, \qquad k^{s} = e^{s \log k},\ \log k \in \mathbb{R}.$$

This is [1, Lemma 1.2]; see also [2, Lemma 2.1]. We use it with $k = n \in \mathbb{N}$ and with $-s$ in place of s:

$$n^{-s} e^{nw} = {}_{-\infty}D_w^{-s}\big(e^{nw}\big), \qquad s \in \mathbb{C},\ n \in \mathbb{N}. \tag{6}$$

2. Fractional integral representation of the polylogarithm

In this section we establish the representation of the polylogarithm in terms of the Liouville fractional integral, and show that it yields the classical Bose–Einstein integral representation of the polylogarithm [4, §25.12(ii)].

Theorem 1 (Liouville representation of the polylogarithm)**.** *Let $\operatorname{Re} s > 0$ and let $w \in \mathbb{C}$ satisfy $\operatorname{Re} w < 0$. Then*

$$\operatorname{Li}_s(e^{w}) = {}_{-\infty}D_w^{-s}\left(\frac{e^{w}}{1 - e^{w}}\right).$$

The idea is to write $\operatorname{Li}_s(e^w)$ as a series of exponentials that matches the form of Lemma 1, and then to sum the geometric series inside the integral, after justifying the interchange of the infinite sum and the integration.

Proof. Let $z^n = e^{n \log z}$ and put $w = \log z$, so that $|z| = e^{\operatorname{Re} w} < 1$. From definition (1) we obtain

$$\operatorname{Li}_s(z) = \sum_{n=1}^{\infty} n^{-s} e^{n \log z}, \qquad \text{or} \qquad \operatorname{Li}_s(e^{w}) = \sum_{n=1}^{\infty} n^{-s} e^{nw}. \tag{7}$$

Applying Lemma 1, in the form (6), gives $n^{-s} e^{nw} = {}_{-\infty}D_w^{-s}(e^{nw})$, so we can write (7) as

$$\mathrm{Li}_s(e^w) = \sum_{n=1}^{\infty} {}_{-\infty}D_w^{-s}\big(e^{nw}\big). \tag{8}$$

By the definition (2) of the Liouville fractional integral, (8) becomes

$$\mathrm{Li}_s(e^w) = \sum_{n=1}^{\infty} \frac{1}{\Gamma(s)} \int_{-\infty}^{w} (w-u)^{s-1} e^{nu}\, du,$$

or equivalently, by (3),

$$\mathrm{Li}_s(e^w) = \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-1} e^{nw} e^{-nt}\, dt.$$

Since $\left|t^{s-1}e^{nw}e^{-nt}\right| = t^{\operatorname{Re} s-1}e^{n(\operatorname{Re} w - t)}$, the Gamma integral (5) gives

$$\sum_{n=1}^{\infty} \int_0^{\infty} t^{\operatorname{Re} s-1} e^{n(\operatorname{Re} w - t)}\, dt = \Gamma(\operatorname{Re} s) \sum_{n=1}^{\infty} n^{-\operatorname{Re} s} e^{n \operatorname{Re} w} < \infty.$$

This fact is direct since $\operatorname{Re} w < 0$. Now we can apply the Fubini–Tonelli theorem [11, Thm. 8.8] (in its series form, [11, Thm. 1.38]) to interchange the sum and the integral:

$$\mathrm{Li}_s(e^w) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \Big(\sum_{n=1}^{\infty} e^{n(w-t)} \Big) dt.$$

For $t \geq 0$ we have $|e^{w-t}| < 1$, so the geometric series sums to $e^{w-t}/(1 - e^{w-t})$, resulting in

$$\mathrm{Li}_s(e^w) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\, \frac{e^{w-t}}{1-e^{w-t}}\, dt = {}_{-\infty}D_w^{-s}\left(\frac{e^w}{1-e^w} \right). \qquad \square$$

The theorem gives at once a famous integral representation of the polylogarithm.

Corollary 1 (Bose–Einstein integral representation)**.** *For* $\operatorname{Re} s > 0$ *and* $|z| < 1$*,*

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - z}\, dt.$$

Proof. For $z = 0$ both sides vanish. Otherwise choose w with $z = e^w$, where $\operatorname{Re} w = \log|z| < 0$, and apply Theorem 1:

$$\mathrm{Li}_s(z) = {}_{-\infty}D_w^{-s}\left(\frac{e^w}{1-e^w} \right).$$

Writing down the definition (3) of the Liouville integral gives

$$\mathrm{Li}_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{w-t}}{1-e^{w-t}}\, dt.$$

After substituting $z = e^w$ and simplifying, using $\dfrac{e^{w-t}}{1-e^{w-t}} = \dfrac{z}{e^t - z}$ (multiply numerator and denominator by e^t; the denominator does not vanish because $e^t \geq 1 > |z|$), we have the desired result:

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - z}\, dt. \qquad \square$$

3. Applications to fractional differential equations

Lemma 1 implies that the exponentials e^{nw} are eigenfunctions of the Liouville derivative, and Theorem 1 says that the polylogarithm is built from exactly these exponentials. Together they suggest applications to fractional differential equations. More precisely, we use $\mathrm{Li}_s(e^w)$ as the forcing term of the equation

$$\big({}_{-\infty}D_w^\alpha - \lambda\big) y(w) = \mathrm{Li}_s(e^w), \qquad \alpha > 0,\ s, \lambda \in \mathbb{C},\ \lambda \neq n^\alpha \quad (n \in \mathbb{N}). \tag{9}$$

Here ${}_{-\infty}D_w^\alpha$ is the left-sided Liouville fractional derivative of Definition 1 with $c = -\infty$. We work on the half-plane $\operatorname{Re} w < 0$ and with functions given by series in e^{nw}; on this class the derivative and the series operations below are well defined, as the computation shows.

Solution.

We seek a solution of (9) by the method of undetermined coefficients [9, 10]. Since the forcing term is a power series in e^w, we take an exponential-series ansatz and determine its coefficients by substitution and comparison:

$$y(w) = \sum_{n=1}^{\infty} a_n e^{nw}. \tag{10}$$

To substitute (10) into (9), we first express its fractional derivative using the defining Liouville integral. Let $m = \lfloor \alpha \rfloor + 1$, so that $0 < m - \alpha \leq 1$. Definition 1 and (3) give

$${}_{-\infty}D_w^\alpha y(w) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dw^m} \int_0^\infty t^{m-\alpha-1} \Big(\sum_{n=1}^{\infty} a_n e^{n(w-t)} \Big) dt. \tag{11}$$

Interchange of sum and integral.

We want to write

$$\int_0^\infty t^{m-\alpha-1} \sum_{n=1}^{\infty} a_n e^{n(w-t)}\, dt = \sum_{n=1}^{\infty} a_n e^{nw} \int_0^\infty t^{m-\alpha-1} e^{-nt}\, dt. \tag{12}$$

To justify the interchange we need a careful consideration of the domains. Fix a compact subset K of the half-plane $\operatorname{Re} w < 0$. There exists a constant $c > 0$ such that

$$\operatorname{Re} w \le -c \qquad \text{for every } w \in K.$$

So $|e^{n(w-t)}| = e^{n(\operatorname{Re} w - t)} \le e^{-nc} e^{-nt}$, and therefore

$$\sum_{n=1}^{\infty} \int_0^{\infty} \left| a_n t^{m-\alpha-1} e^{n(w-t)} \right| dt \le \sum_{n=1}^{\infty} |a_n| e^{-nc} \int_0^{\infty} t^{m-\alpha-1} e^{-nt}\, dt.$$

By the Gamma integral (5) with $\sigma = m - \alpha > 0$,

$$\int_0^{\infty} t^{m-\alpha-1} e^{-nt}\, dt = \frac{\Gamma(m-\alpha)}{n^{m-\alpha}}, \tag{13}$$

and from this we obtain

$$\sum_{n=1}^{\infty} \int_0^{\infty} \left| a_n t^{m-\alpha-1} e^{n(w-t)} \right| dt \le \Gamma(m-\alpha) \sum_{n=1}^{\infty} \frac{|a_n| e^{-nc}}{n^{m-\alpha}}.$$

Now let us assume that $|a_n| \le C n^p$ for some constants $C > 0$ and $p \in \mathbb{R}$ (we check at the end that the coefficients found below satisfy this). Then

$$\sum_{n=1}^{\infty} \int_0^{\infty} \left| a_n t^{m-\alpha-1} e^{n(w-t)} \right| dt \le \Gamma(m-\alpha)\, C \sum_{n=1}^{\infty} n^{p-m+\alpha} e^{-nc} < \infty.$$

Since this is finite, the Fubini–Tonelli theorem [11, Thm. 8.8] (applied with $f_n(t) = a_n t^{m-\alpha-1} e^{n(w-t)}$) allows us to interchange the summation and the integration, which proves (12). By (13) we now have

$$\int_0^{\infty} t^{m-\alpha-1} \sum_{n=1}^{\infty} a_n e^{n(w-t)}\, dt = \Gamma(m-\alpha) \sum_{n=1}^{\infty} \frac{a_n e^{nw}}{n^{m-\alpha}}. \tag{14}$$

Differentiation.

Next we take the m-th derivative of (14). Let us define

$$F_N(w) = \sum_{n=1}^{N} \frac{a_n e^{nw}}{n^{m-\alpha}}.$$

Since the F_N are entire functions, we can differentiate them:

$$F_N^{(j)}(w) = \sum_{n=1}^{N} \frac{a_n n^j e^{nw}}{n^{m-\alpha}}, \qquad j = 0, 1, 2, \ldots, m.$$

We show that these derivatives converge uniformly on every compact subset K of $\operatorname{Re} w < 0$. Recall that $\operatorname{Re} w \le -c$ on K. Then

$$\Big|\frac{a_n n^j e^{nw}}{n^{m-\alpha}}\Big| \le \frac{|a_n| n^j e^{-nc}}{n^{m-\alpha}} \le C n^{p+j-m+\alpha} e^{-nc},$$

and since $\sum_{n=1}^{\infty} C n^{p+j-m+\alpha} e^{-nc}$ converges, it follows from the Weierstrass M-test [12, Thm. 7.10] that the series $F_N^{(j)}$ converge uniformly on K for every $j \le m$. By Weierstrass's theorem on locally uniform limits of holomorphic functions [11, Thm. 10.28], the limit $F = \lim_{N\to\infty} F_N$ is holomorphic on $\operatorname{Re} w < 0$ and $F^{(m)} = \lim_{N\to\infty} F_N^{(m)}$. Therefore, for $j = m$ we have

$$F_N^{(m)}(w) = \sum_{n=1}^{N} \frac{a_n n^m e^{nw}}{n^{m-\alpha}} = \sum_{n=1}^{N} a_n n^\alpha e^{nw} \xrightarrow[N\to\infty]{} \sum_{n=1}^{\infty} a_n n^\alpha e^{nw}.$$

Combining this with (11) and (14), we find

$$ {}_{-\infty}D_w^\alpha y(w) = \sum_{n=1}^{\infty} a_n n^\alpha e^{nw}. \tag{15}$$

Substitution and comparison of coefficients.

The left-hand side of (9) is therefore

$$\big({}_{-\infty}D_w^\alpha - \lambda\big) y(w) = \sum_{n=1}^{\infty} a_n n^\alpha e^{nw} - \lambda \sum_{n=1}^{\infty} a_n e^{nw}.$$

Both series converge absolutely, so we may combine them term by term, and we end up with

$$\big({}_{-\infty}D_w^\alpha - \lambda\big) y(w) = \sum_{n=1}^{\infty} a_n \left(n^\alpha - \lambda\right) e^{nw}.$$

Setting the left-hand side equal to the polylogarithm, by (9) and (1), we obtain

$$\sum_{n=1}^{\infty} a_n \left(n^\alpha - \lambda\right) e^{nw} = \sum_{n=1}^{\infty} \frac{e^{nw}}{n^s}.$$

This holds for all w with $\operatorname{Re} w < 0$, in particular for real $w < 0$, that is, with $x = e^w$ ranging over $(0, 1)$. Both sides are power series in x that converge on $(-1, 1)$ and agree on $(0, 1)$, a set with limit points in $(-1, 1)$. By the uniqueness theorem for power series (the uniqueness theorem for power series [12, Thm. 8.5]) we may compare the coefficients of e^{nw} to find a_n:

$$\frac{1}{n^s} = a_n \left(n^\alpha - \lambda\right).$$

Therefore, since $\lambda \neq n^\alpha$,

$$a_n = \frac{1}{n^s (n^\alpha - \lambda)}.$$

Finally, these coefficients satisfy the growth condition that we assumed. Indeed, $|n^\alpha - \lambda|/n^\alpha = |1 - \lambda n^{-\alpha}|$ tends to 1 as $n \to \infty$ and is never 0 (because $\lambda \neq n^\alpha$), so it is bounded below by some $c_0 > 0$ for all n. Hence $|a_n| \leq c_0^{-1} n^{-\operatorname{Re} s - \alpha}$, which is of the form Cn^p. Thus all the steps above are justified, and a solution of the fractional differential equation (9) on $\operatorname{Re} w < 0$ is

$$y(w) = \sum_{n=1}^{\infty} \frac{e^{nw}}{n^s(n^\alpha - \lambda)}. \tag{16}$$

Remark. Within the class (10) with polynomially bounded coefficients, the coefficients a_n are forced, so (16) is the only solution of this form. The equation can have other solutions: if $\lambda = \mu^\alpha$ for some real $\mu > 0$, then $e^{\mu w}$ solves the homogeneous equation by Lemma 1 and may be added to (16).

The solution (16) introduces an interesting series, whose properties we study in the next section.

4. Properties of the function $R_{\alpha,s}(z;\lambda)$

Definition 2. For $\alpha > 0$, $s, \lambda \in \mathbb{C}$ with $\lambda \neq n^\alpha$ for all $n \in \mathbb{N}$, and $|z| < 1$, define

$$R_{\alpha,s}(z;\lambda) = \sum_{n=1}^{\infty} \frac{z^n}{n^s(n^\alpha - \lambda)}.$$

Thus the solution (16) is $y(w) = R_{\alpha,s}(e^w;\lambda)$ for $\operatorname{Re} w < 0$.

Convergence.

Let $L \subset \mathbb{C} \setminus \{n^\alpha : n \in \mathbb{N}\}$ be compact and put $M = \max_{\lambda \in L} |\lambda|$. If $n^\alpha \geq 2M$, then for every $\lambda \in L$

$$|n^\alpha - \lambda| \geq n^\alpha - M \geq \tfrac{1}{2} n^\alpha.$$

Since $\alpha > 0$, only finitely many n satisfy $n^\alpha < 2M$; for each of them, $|n^\alpha - \lambda|/n^\alpha$ is continuous and positive on the compact set L, hence bounded below by a positive constant. Therefore there is $c_L > 0$ with

$$|n^\alpha - \lambda| \geq c_L\, n^\alpha \qquad \text{for all } n \in \mathbb{N},\ \lambda \in L. \tag{17}$$

The same argument gives (17) for all $n \neq N$ if L is allowed to contain N^α but no other point n^α. For $|z| \leq r < 1$ and $\lambda \in L$ we get

$$\Big|\frac{z^n}{n^s(n^\alpha - \lambda)}\Big| \leq \frac{r^n\, n^{-\operatorname{Re} s - \alpha}}{c_L},$$

and the right-hand side is summable. By the Weierstrass M-test [12, Thm. 7.10] the series converges absolutely and uniformly for $|z| \leq r$, $\lambda \in L$. Since every term is holomorphic in z and in λ, Weierstrass's theorem [11,

Thm. 10.28] shows that $R_{\alpha,s}(z;\lambda)$ is jointly holomorphic on $\{(z,\lambda) : |z| < 1,\ \lambda \notin \{n^\alpha : n \in \mathbb{N}\}\}$. In particular, it is holomorphic in either variable when the other is fixed away from the excluded values.

The following proposition collects several properties of $R_{\alpha,s}$. In particular, the case $\lambda = 0$ reduces to a classical polylogarithm of shifted order.

Proposition 1. *Let $\alpha > 0$ and $s \in \mathbb{C}$.*

(1) (Reduction to Li.) *For $\lambda = 0$,*

$$R_{\alpha,s}(z;0) = \mathrm{Li}_{s+\alpha}(z).$$

(2) (Expansion in polylogarithms.) *For $|\lambda| < 1$,*

$$R_{\alpha,s}(z;\lambda) = \sum_{k=0}^{\infty} \lambda^k \,\mathrm{Li}_{s+\alpha(k+1)}(z).$$

(3) (Euler operator.) *Let $\theta = z\,\dfrac{d}{dz}$. Then*

$$\theta R_{\alpha,s}(z;\lambda) = R_{\alpha,s-1}(z;\lambda).$$

(4) (Derivatives in λ.) *For $m = 0, 1, 2, \ldots$,*

$$\frac{\partial^m}{\partial \lambda^m} R_{\alpha,s}(z;\lambda) = m! \sum_{n=1}^{\infty} \frac{z^n}{n^s (n^\alpha - \lambda)^{m+1}}.$$

(5) (Poles and residues.) *For fixed z with $0 < |z| < 1$, the function $\lambda \mapsto R_{\alpha,s}(z;\lambda)$ is meromorphic on $\mathbb{C}$, with simple poles exactly at $\lambda = N^\alpha$, $N \in \mathbb{N}$, and*

$$\mathrm{Res}_{\lambda = N^\alpha} R_{\alpha,s}(z;\lambda) = -\frac{z^N}{N^s}.$$

(6) (Coefficientwise fractional Euler operator.) *Let $\theta = z\,d/dz$. For a holomorphic function $f(z) = \sum_{n=1}^{\infty} c_n z^n$ on the unit disk, define*

$$\theta^\alpha f(z) = \sum_{n=1}^{\infty} n^\alpha c_n z^n.$$

Then

$$(\theta^\alpha - \lambda) R_{\alpha,s}(z;\lambda) = \mathrm{Li}_s(z).$$

Proof of (1). Setting $\lambda = 0$ (allowed, since $0 \neq n^\alpha$),

$$R_{\alpha,s}(z;0) = \sum_{n=1}^{\infty} \frac{z^n}{n^s n^\alpha}.$$

Trivially this is $\mathrm{Li}_{s+\alpha}(z)$, because $n^s n^\alpha = n^{s+\alpha}$.

□

Proof of (2). Since $|\lambda| < 1 \le n^\alpha$ we have $|\lambda n^{-\alpha}| < 1$, so

$$\frac{1}{n^\alpha - \lambda} = \frac{1}{n^\alpha} \cdot \frac{1}{1 - \lambda n^{-\alpha}}.$$

Expanding the geometric series gives

$$\frac{1}{n^\alpha - \lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{n^{\alpha(k+1)}}.$$

Substituting this into the definition yields

$$R_{\alpha,s}(z;\lambda) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \sum_{k=0}^{\infty} \frac{\lambda^k}{n^{\alpha(k+1)}} = \sum_{k=0}^{\infty} \lambda^k \sum_{n=1}^{\infty} \frac{z^n}{n^{s+\alpha(k+1)}}.$$

The rearrangement of the two series is justified by the conditions $|\lambda| < 1$ and $|z| < 1$. To see this, let $a_{nk} = \lambda^k z^n n^{-s-\alpha(k+1)}$. Since $n^{-\alpha k} \le 1$ we have $|a_{nk}| \le |z|^n n^{-\operatorname{Re} s - \alpha} |\lambda|^k$, hence

$$\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} |a_{nk}| \le \Big(\sum_{n=1}^{\infty} \frac{|z|^n}{n^{\operatorname{Re} s + \alpha}} \Big) \Big(\sum_{k=0}^{\infty} |\lambda|^k \Big) < \infty,$$

and the theorem on rearrangement of absolutely convergent double series [12, Thm. 8.3] applies. The inner series is the polylogarithm $\operatorname{Li}_{s+\alpha(k+1)}(z)$, making

$$R_{\alpha,s}(z;\lambda) = \sum_{k=0}^{\infty} \lambda^k \operatorname{Li}_{s+\alpha(k+1)}(z). \qquad \square$$

Proof of (3). The partial sums of the series defining $R_{\alpha,s}(z;\lambda)$ are holomorphic in z and converge uniformly on compact subsets of $|z| < 1$ (see the convergence discussion above). By Weierstrass's theorem on locally uniform limits of holomorphic functions [11, Thm. 10.28] the series may be differentiated term by term:

$$\theta R_{\alpha,s}(z;\lambda) = z \frac{d}{dz} \Big(\sum_{n=1}^{\infty} \frac{z^n}{n^s (n^\alpha - \lambda)} \Big) = \sum_{n=1}^{\infty} \frac{n\, z^n}{n^s (n^\alpha - \lambda)}.$$

Since $n/n^s = 1/n^{s-1}$, this equals

$$\sum_{n=1}^{\infty} \frac{z^n}{n^{s-1} (n^\alpha - \lambda)} = R_{\alpha,s-1}(z;\lambda). \qquad \square$$

Proof of (4). We differentiate term by term. For the n-th term

$$f_n(\lambda) = \frac{z^n}{n^s (n^\alpha - \lambda)}$$

the first, second and third derivatives with respect to λ are

$$f_n'(\lambda) = \frac{z^n}{n^s(n^\alpha-\lambda)^2}, \qquad f_n''(\lambda) = \frac{2z^n}{n^s(n^\alpha-\lambda)^3}, \qquad f_n'''(\lambda) = \frac{3!\,z^n}{n^s(n^\alpha-\lambda)^4},$$

and in general (by induction on m, since $\frac{d}{d\lambda}(n^\alpha-\lambda)^{-j} = j\,(n^\alpha-\lambda)^{-j-1}$)

$$f_n^{(m)}(\lambda) = \frac{m!\,z^n}{n^s(n^\alpha-\lambda)^{m+1}}.$$

To justify termwise differentiation, let $L \subset \mathbb{C}\setminus\{n^\alpha : n\in\mathbb{N}\}$ be compact. By (17), there is a constant $c_L>0$ such that

$$|n^\alpha-\lambda| \geq c_L n^\alpha, \qquad \lambda\in L,\ n\in\mathbb{N}.$$

Consequently,

$$|f_n^{(m)}(\lambda)| \leq m!\,|z|^n n^{-\operatorname{Re} s-\alpha(m+1)} c_L^{-(m+1)}, \qquad \lambda\in L.$$

The right-hand side is summable because $|z|<1$. Thus, for every $m\geq 0$, the series of m-th derivatives converges uniformly on compact subsets of the parameter domain. The theorem on termwise differentiation of locally uniformly convergent series of holomorphic functions therefore gives

$$\frac{\partial^m}{\partial\lambda^m} R_{\alpha,s}(z;\lambda) = m!\sum_{n=1}^{\infty}\frac{z^n}{n^s(n^\alpha-\lambda)^{m+1}}. \qquad \square$$

Proof of (5). By the convergence discussion, $R_{\alpha,s}(z;\cdot)$ is holomorphic on $\mathbb{C}\setminus\{n^\alpha : n\in\mathbb{N}\}$. The points n^α are distinct and isolated, since $\alpha>0$. It remains to study the behaviour near each of them. For a simple pole at N^α the residue is given by (see [13, Ch. V])

$$\operatorname{Res}_{\lambda=N^\alpha} R_{\alpha,s}(z;\lambda) = \lim_{\lambda\to N^\alpha}(\lambda-N^\alpha)\,R_{\alpha,s}(z;\lambda).$$

We can write $R_{\alpha,s}(z;\lambda)$ by separating the N-th term of the series:

$$R_{\alpha,s}(z;\lambda) = \sum_{\substack{n=1\\ n\neq N}}^{\infty}\frac{z^n}{n^s(n^\alpha-\lambda)} + \frac{z^N}{N^s(N^\alpha-\lambda)} = H_N(\lambda) + \frac{z^N}{N^s(N^\alpha-\lambda)},$$

where

$$H_N(\lambda) = \sum_{\substack{n=1\\ n\neq N}}^{\infty}\frac{z^n}{n^s(n^\alpha-\lambda)}.$$

Choose $r>0$ such that the closed disc $\overline{D} = \{|\lambda-N^\alpha|\leq r\}$ contains no other point n^α. The bound (17) for $n\neq N$ (with $L=\overline{D}$) and the Weierstrass M-test show that the series for H_N converges uniformly on $\overline{D}$, so H_N is holomorphic in the open disc by Weierstrass's theorem on locally uniform limits of holomorphic functions [11, Thm. 10.28]; in particular H_N is bounded near N^α. Substituting this form in the residue formula,

$$\operatorname{Res}_{\lambda=N^\alpha} R_{\alpha,s}(z;\lambda) = \lim_{\lambda\to N^\alpha}(\lambda - N^\alpha)\Big[H_N(\lambda) + \frac{z^N}{N^s(N^\alpha-\lambda)}\Big]$$

$$= \lim_{\lambda\to N^\alpha}(\lambda - N^\alpha)H_N(\lambda) + \frac{z^N}{N^s}\lim_{\lambda\to N^\alpha}\frac{\lambda - N^\alpha}{N^\alpha-\lambda}.$$

We take each limit separately. The first is

$$\lim_{\lambda\to N^\alpha}(\lambda - N^\alpha)H_N(\lambda) = 0,$$

since H_N is bounded near N^α, and the second is

$$\lim_{\lambda\to N^\alpha}\frac{\lambda - N^\alpha}{N^\alpha-\lambda} = -1.$$

Therefore

$$\operatorname{Res}_{\lambda=N^\alpha} R_{\alpha,s}(z;\lambda) = -\frac{z^N}{N^s}.$$

Since $z \neq 0$ this residue is non-zero, and H_N is holomorphic at N^α, so $R_{\alpha,s}(z;\cdot)$ has a *simple* pole at N^α. As N was arbitrary, $R_{\alpha,s}(z;\cdot)$ is meromorphic on $\mathbb{C}$ with simple poles exactly at the points N^α.

□

Proof of (6). The defining series of $R_{\alpha,s}$ converges locally uniformly in the unit disk. Multiplication of its n-th coefficient by n^α preserves the radius of convergence, since it introduces only polynomial growth in n. Hence the series defining $\theta^\alpha R_{\alpha,s}$ converges locally uniformly for $|z| < 1$. By the definition of $R_{\alpha,s}$,

$$(\theta^\alpha - \lambda)R_{\alpha,s}(z;\lambda) = \sum_{n=1}^{\infty}\frac{n^\alpha z^n}{n^s(n^\alpha-\lambda)} - \lambda\sum_{n=1}^{\infty}\frac{z^n}{n^s(n^\alpha-\lambda)}$$

$$= \sum_{n=1}^{\infty}\frac{z^n}{n^s} = \operatorname{Li}_s(z).$$

This proves the identity. □

5. Conclusion

We have studied the polylogarithm from the point of view of fractional calculus. The first result expresses $\operatorname{Li}_s(e^w)$ as a left Liouville fractional integral for $\operatorname{Re} s > 0$ and $\operatorname{Re} w < 0$. The classical Bose–Einstein integral representation follows by writing this fractional integral explicitly.

We have also considered the fractional differential equation with forcing term $\operatorname{Li}_s(e^w)$. The exponential eigenfunction property of the Liouville derivative leads to the series solution

$$y(w) = \sum_{n=1}^{\infty}\frac{e^{nw}}{n^s(n^\alpha-\lambda)},$$

under the stated condition $\lambda \neq n^\alpha$. The solution is unique within the class of exponential series with polynomially bounded coefficients; no uniqueness claim is made here for a larger function space.

The series defines $R_{\alpha,s}(z;\lambda)$. Its properties include the reduction $R_{\alpha,s}(z;0) = \mathrm{Li}_{s+\alpha}(z)$, an expansion in polylogarithms for $|\lambda| < 1$, the relation with the Euler operator, explicit formulas for derivatives with respect to λ, and a meromorphic continuation in λ with simple poles at $\lambda = N^\alpha$ and residues $-z^N/N^s$ when $0 < |z| < 1$. The fractional Euler-operator identity in Proposition 1 gives another way to express the equation satisfied by this function.

Several questions remain open. One may study the analytic continuation of $R_{\alpha,s}(z;\lambda)$ beyond the disk $|z| < 1$ and its behaviour as the parameter s varies. It would also be natural to consider an analogue in which n is replaced by $n + x$, in connection with the Lerch transcendent and the work in [1]. Other fractional operators, such as the Weyl or Caputo derivatives, may lead to related equations; see [9, 10]. Finally, uniqueness of solutions of (9) in suitable function spaces remains a separate question, as noted in Section 3.

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