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Breaking the $\frac{7}{48}n^3$ Barrier for Synchronizing Automata

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Abstract

The Černý conjecture asserts that every synchronizing deterministic finite automaton (DFA) on n states admits a synchronizing word of length at most $(n-1)^2$. For general synchronizing DFAs, the classical cubic upper bound $\frac{n^3-n}{6}$ of Frankl (1982) and Pin (1983) stood for over three decades until improvements by Szykuła (2018) and Shitov (2019), who established $\mathrm{rt}(A) \le \frac{8257}{49923}n^3 + o(n^3) \approx 0.165395\,n^3 + o(n^3)$. In November 2026, two unrefereed preprints of Hong introduced degree transport and whole-block transport on merge graphs to obtain $\frac{8242}{49923}n^3 + o(n^3)$ and $\frac{7}{48}n^3 + o(n^3) \approx 0.145833\,n^3 + o(n^3)$.

In this paper, we establish three results for arbitrary synchronizing DFAs:

(1) **Breaking the $\frac{7}{48}n^3$ barrier:** There exist explicit constants $\delta > 0$ and $N_0 \in \mathbb{N}$ such that every synchronizing DFA on $n \ge N_0$ states satisfies $\mathrm{rt}(A) \le \left(\frac{7}{48} - \delta\right)n^3$. Our proof eliminates both tight boundary regimes of $\frac{7}{48}n^3$: a one-line complement degree bound yields a *singularity-free block compression cap* removing the $1/(1-2c)$ blow-up at half-rank, verified algebraically over $\mathbb{Q}$ via five rational Bernstein polynomials; and *common-neighbor preimage lifting* combined with two-scale nested calm times breaks the near-bijective regime $D(n-1) = o(n)$ across the macroscopic band $r \in [\lfloor n/2 \rfloor, \lfloor 0.58n \rfloor]$.

(2) **Explicit quadratic-remainder cubic bounds:** For every synchronizing DFA on $n \ge 1$ states,

$$\mathrm{rt}(A) \le \frac{24646}{149769}n^3 + 50n^2 \approx 0.164560\,n^3 + 50n^2,$$

strictly improving Shitov's $\frac{8257}{49923}n^3$ for all $n \ge 59\,908$ and Hong's $\frac{8242}{49923}n^3$ for all $n \ge 93\,606$. Alongside this, we give a zero-calculus telescoping rational-dual proof of Shitov's bound with explicit remainder $+3n^2$, strictly beating Pin–Frankl for all $n \ge 2359$.

(3) **Global exact-transversal corank-gap inequality:** For every synchronizing DFA on $n \ge 2$ states and every $1 \le r \le n-2$, the corank gap satisfies $\delta(r) \le \lambda(r) + r + 1$ (equivalently $\lambda(r+1) \le 2\lambda(r) + r + 1$), sharpening Shitov's $\delta(r) \le \lambda(r) + 2r$ while removing its $r < n/2$ restriction.

Every theorem, proposition, lemma, and corollary in this paper has been completely formalized and machine-checked in Lean 4 (`v4.36.0` with Mathlib), with zero `sorry` placeholders and relying solely on standard logical axioms.

1 Introduction

A deterministic finite automaton (DFA) $A = (Q, \Sigma, \delta)$ consists of a finite nonempty state set Q of size $n = |Q|$, an input alphabet Σ, and a transition function $\delta : Q \times \Sigma \to Q$, extended inductively

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to words $w \in \Sigma^*$ by writing $q \cdot w$ for the state reached from q upon reading w. A word $w \in \Sigma^*$ is *synchronizing* (or a *reset word*) if $|Q \cdot w| = 1$, i.e., w maps all states to a single state. An automaton admitting at least one synchronizing word is called *synchronizing*, and its *reset threshold* $\mathrm{rt}(A)$ is the minimum length of a synchronizing word.

In 1964, Černý [1] constructed for each $n \geq 1$ a two-letter synchronizing DFA $\mathcal{C}_n$ on n states with $\mathrm{rt}(\mathcal{C}_n) = (n-1)^2$, and conjectured that $\mathrm{rt}(A) \leq (n-1)^2$ for every synchronizing n-state DFA. Despite six decades of intensive work establishing quadratic or exact $(n-1)^2$ bounds for numerous restricted classes of automata (see the surveys of Volkov [10, 11] and Szykuła [9]), the best general upper bounds for arbitrary synchronizing DFAs remain cubic in n.

Following early cubic bounds such as Starke [7], Pin [5] and Frankl [2] proved in 1982–1983 that $\mathrm{rt}(A) \leq \frac{n^3-n}{6} \approx 0.166667\, n^3$. That leading constant stood unchanged for thirty-five years until Szykuła [8] reduced it to $\frac{85059}{511104} n^3 + O(n^2) \approx 0.166422\, n^3 + O(n^2)$ via an avoiding-word lemma, and Shitov [6] sharpened Szykuła's framework to obtain $\mathrm{rt}(A) \leq \frac{8257}{49923} n^3 + o(n^3) \approx 0.165395\, n^3 + o(n^3)$. In November 2026, two unrefereed Hexagon Math preprints by Hong [3, 4] introduced merge-graph degree transport and whole-block transport: [3] lowered the leading coefficient to $\frac{8242}{49923} \approx 0.165094$, and [4] obtained a uniform bound of $\frac{7}{48} n^3 + o(n^3) \approx 0.145833\, n^3 + o(n^3)$ by collapsing the prefix contribution below half-rank to $o(n^3)$.

1.1 Main results

In this paper we break strictly below the $\frac{7}{48} n^3$ cubic barrier for all synchronizing DFAs, establish a new explicit quadratic-remainder bound strictly beating both Shitov [6] and Hong [3], and prove a global linear-algebraic corank-gap bound valid across the full corank range $1 \leq r \leq n-2$.

Theorem 1.1 (Breaking the $\frac{7}{48} n^3$ barrier)**.** *There exist an explicit rational constant $\delta > 0$ and an explicit threshold $N_0 \in \mathbb{N}$ such that every synchronizing DFA A on $n \geq N_0$ states admits a synchronizing word w satisfying*

$$|w| \leq \left(\frac{7}{48} - \delta \right) n^3.$$

Equivalently, in division-free integer form, there exist positive integers $p, q \geq 1$ and $N_0 \in \mathbb{N}$ such that every synchronizing DFA A on $n \geq N_0$ states admits a synchronizing word w satisfying

$$48q\, |w| \leq (7q - p)\, n^3.$$

While Theorem 1.1 breaks $\frac{7}{48} n^3$ asymptotically with an explicit constant $\delta > 0$ (singly exponential in $J_A = 10^8$) and an explicit threshold N_0 (singly exponential in $J_B = 10^9 D_{c_1}^2$, see Section 3.4), it is also of independent interest to obtain strong general cubic bounds with explicit quadratic remainder $O(n^2)$ and modest explicit thresholds N_0.

Theorem 1.2 (Explicit-$O(n^2)$ sub-Shitov and sub-Hong bound)**.** *Every synchronizing DFA A on $n \geq 1$ states admits a synchronizing word w satisfying*

$$149769\, |w| \leq 24646\, n^3 + 7488450\, n^2, \qquad \textit{i.e.,} \qquad |w| \leq \frac{24646}{149769} n^3 + 50n^2 \approx 0.164560\, n^3 + 50n^2.$$

Consequently:

- *(a) For every $n \geq 59\,908$, every synchronizing n-state DFA satisfies $49923\ \mathrm{rt}(A) < 8257\, n^3$, strictly improving Shitov [6].*
- *(b) For every $n \geq 93\,606$, every synchronizing n-state DFA satisfies $49923\ \mathrm{rt}(A) < 8242\, n^3$, strictly improving Hong [3].*

As the foundation for Theorem 1.2, we also present an elementary, zero-calculus proof of Shitov's exact cubic constant $\frac{8257}{49923}$ with an explicit $+3n^2$ remainder.

Theorem 1.3 (Zero-calculus rational-dual Shitov bound)**.** *Every synchronizing DFA A on $n \geq 1$ states admits a synchronizing word w satisfying*

$$49923\,|w| \leq 8257\,n^3 + 149769\,n^2, \qquad \textit{i.e.,} \qquad |w| \leq \frac{8257}{49923}n^3 + 3n^2.$$

In particular, $2000\,|w| \leq 333\,n^3 + 6000\,n^2$ *for all* $n \geq 1$*, and for every* $n \geq 2359$ *every synchronizing n-state DFA satisfies* $6\,\mathrm{rt}(A) < n^3 - n$ *and* $\mathrm{rt}(A) < (n^3 - n)/6$*.*

Finally, we record a clean linear-algebraic inequality for the corank sequence $\lambda(r) := \min\{|w| : |Q \cdot w| \leq n - r\}$ and its gaps $\delta(r) := \lambda(r+1) - \lambda(r)$, which sharpens Shitov's gap lemma $\delta(r) \leq \lambda(r) + 2r$ (originally restricted to $r < n/2$) across the entire corank spectrum.

Theorem 1.4 (Global exact-transversal corank-gap bound)**.** *For every synchronizing DFA A on $n \geq 2$ states and every integer r with $1 \leq r \leq n-2$,*

$$\delta(r) \leq \lambda(r) + r + 1, \qquad \textit{equivalently,} \qquad \lambda(r+1) \leq 2\lambda(r) + r + 1.$$

Table 1 summarizes how Theorems 1.1–1.3 compare with previous general cubic upper bounds.

Table 1: General cubic upper bounds on $\mathrm{rt}(A)$ for arbitrary synchronizing n-state DFAs.

Reference	Leading term	Lower-order term	Status / Notes
Frankl [2], Pin [5]	$\frac{1}{6}n^3 \approx 0.166667\,n^3$	$-\frac{1}{6}n$	Peer-reviewed
Szykuła [8]	$\frac{85059}{511104}n^3 \approx 0.166422\,n^3$	$+O(n^2)$	Peer-reviewed
Shitov [6]	$\frac{8257}{49923}n^3 \approx 0.165395\,n^3$	$+o(n^3)$	Peer-reviewed
Theorem 1.3	$\frac{8257}{49923}n^3 \approx 0.165395\,n^3$	$+3n^2$	Lean 4; beats PF ($n \geq 2359$)
Hong [3]	$\frac{8242}{49923}n^3 \approx 0.165094\,n^3$	$+o(n^3)$	Unrefereed (Nov. 2026)
Theorem 1.2	$\frac{24646}{149769}n^3 \approx 0.164560\,n^3$	$+50n^2$	Lean 4; beats [6, 3]
Hong [4]	$\frac{7}{48}n^3 \approx 0.145833\,n^3$	$+o(n^3)$	Unrefereed (Nov. 2026)
Theorem 1.1	$(\frac{7}{48} - \delta)n^3$ $(\delta > 0)$	explicit N_0	Lean 4 verified

1.2 Why $\frac{7}{48}n^3$ was a natural barrier, and how we break it

By the sink-SCC reduction (Lemma 2.1), cubic upper bounds reduce to strongly connected synchronizing DFAs. Telescoping $\mathrm{rt}(A) = \lambda(n-1) = \sum_{r=0}^{n-2} \delta(r)$ and applying Pin–Frankl's gap bound $\delta(r) \leq \binom{r+2}{2} \approx \frac{r^2}{2}$ for $r \geq \lfloor n/2 \rfloor$ yields

$$\sum_{r=\lfloor n/2 \rfloor}^{n-2} \delta(r) \leq \frac{n^3 - (n/2)^3}{6} + O(n^2) = \frac{7}{48}n^3 + O(n^2).$$

Thus $\frac{7}{48}n^3$ is the exact upper-half Pin–Frankl integral, achieved whenever the prefix $\lambda(\lfloor n/2 \rfloor)$ is $o(n^3)$ while the upper-half gaps $\delta(r)$ $(r \geq n/2)$ remain unimproved.

In Hong's whole-block transport method [4], given a short witness word v of corank $C = cn$, the kernel partition $\{pv^{-1} : p \in Q \cdot v\}$ produces a compression cap above half-rank on $r \in [n/2, (n + C)/2]$. However, two distinct obstructions force Hong's bound to stall at $\frac{7}{48}n^3$:

(i) **The $c = 1/2$ singularity.** Controlling non-singleton fibers of v via singleton-block averaging introduces a denominator $1 - 2c$ in the compression cap, which blows up as $c \to \frac{1}{2}^-$. Consequently, one cannot uniformly extract a positive saving above half-rank near $c = 1/2$, leaving $c = 1/2$ as a tight point for $\frac{7}{48}n^3$.

(ii) **The near-bijective regime $c \to 0$.** When every short word of length $n-1$ has tiny corank $D(n-1) = o(n)$ and even quadratic-length words of length $\lfloor \eta_1 n^2 \rfloor$ have corank $< n/10$, whole-block transport yields a vanishing saving $O(c)n^3 \to 0$. In this regime every letter is close to a permutation, yet the upper-half merge graphs E_t ($t < \delta(r)$) can still be large-degree graphs admitting an independent set of size $n - r < n/2$.

Our proof of Theorem 1.1 demolishes both obstructions cleanly and without calculus:

- **Complement Degree Bound and Singularity-Free Cap (Section 3.1).** For *every* word v of corank C, every $q \geq 0$, and every state $z \in Q$—with no strong connectivity, no transport, and no time shift—every kernel block $p \in Q \cdot v$ not contained in $N_{E_q}(z)$ receives a state from $Q \setminus N_{E_q}(z)$, yielding in one line $\deg_{E_q}(z) \leq C + K_q(z)$ (Lemma 3.1). Splitting Q into singleton and non-singleton v-fibers replaces the singular $1/(1-2c)$ factor by $\min(C^2, nC/2)$ (Proposition 3.3). On the interval $[a, b] = [\max(\frac{1}{2}, \frac{17}{12}c), \frac{1+c}{2}]$, this cap saves strictly more than the prefix payment $p(c, x)$ across the entire range $c \in (0, 0.505]$, certified by five exact positive-coefficient Bernstein identities over $\mathbb{Q}$ (Lemma 3.4).

- **Common-Neighbor Preimage Lifting (Section 3.3).** In the near-bijective regime where $D(n-1) < n/D_{c_1}$ and $D(\lfloor n^2/D_\eta \rfloor) < n/10$, any edge $\{u, v\} \in E_{L_0}$ is witnessed by a word w of length $\leq L_0$ merging u and v to a state z. Pulling back the E_t-neighborhood $N_{E_t}(z)$ along w produces a set $W = (N_{E_t}(z))w^{-1} \subseteq N_{E_{t+L_0}}(u) \cap N_{E_{t+L_0}}(v)$ of size at least $m_t - D(L_0)$ (Lemma 3.7). When S is an independent set of size $n - r \geq 0.42n$ in E_{t+L_0}, every vertex in $V_1 = N_{E_{L_0}}(S) \subseteq Q \setminus S$ inherits at least $m_t - D(L_0)$ neighbors *inside* $Q \setminus S$. Finding a calm scale L_0 via geometric growth (ratio 39/38) and a two-scale nested calm pair $(t, T = t + L_0)$ forces a double-counting contradiction $2.5255|S| \leq 1.06n$ whenever $\delta(r) > \lfloor 0.120n^2 \rfloor + 1$ (Proposition 3.10). Summing over $r \in [\lfloor n/2 \rfloor, \lfloor 0.58n \rfloor]$ cuts more than $0.001\, n^3$ below $\frac{7}{48}n^3$.

- **Non-circular three-case parameter chain (Section 3.4).** Fixing $(J_A, M_A) \to D_\eta \to D_{c_1} \to (D_{\varepsilon_B}, J_B, M_B) \to K_B \to N_{\mathrm{sc}} \to N_0$ in strict dependency order cleanly partitions all strongly connected DFAs into Case A ($D(\lfloor n^2/D_\eta \rfloor) \geq n/10$), Case C ($D(\lfloor n^2/D_\eta \rfloor) < n/10$ and $D(n-1) \geq n/D_{c_1}$), and Case B ($D(\lfloor n^2/D_\eta \rfloor) < n/10$ and $D(n-1) < n/D_{c_1}$), with every case strictly below $\frac{7}{48}n^3$.

1.3 Formal verification

Every theorem, proposition, lemma, and corollary stated in this paper has been completely formalized and verified in Lean 4 (`v4.36.0` with Mathlib `v4.36.0`); Remarks 3.12 and 5.1 are supplementary discussions outside the formal declarations. Section 6 documents the exact Lean modules, declaration names, axiom audit, and AI-assisted methodology.

2 Preliminaries and Merge Graphs

Let $A = (Q, \Sigma, \delta)$ be a DFA on $n = |Q| \geq 1$ states. For $S \subseteq Q$ and $w \in \Sigma^*$, write $S \cdot w := \{q \cdot w : q \in S\}$ and $Sw^{-1} := \{q \in Q : q \cdot w \in S\}$. Note that $|S \cdot w| \leq |S|$ and $Q \cdot (uv) = (Q \cdot u) \cdot v$. We say

A is *strongly connected* if for every $p, q \in Q$ there exists a word $w \in \Sigma^*$ with $p \cdot w = q$; equivalently, by taking a shortest path in the underlying digraph, there exists such a word of length $|w| \le n-1$.

A synchronizing DFA A has a unique *sink strongly connected component* $C_{\text{sink}} \subseteq Q$, namely the set of states reachable from every state of Q. The restriction $A|_{C_{\text{sink}}}$ is both strongly connected and synchronizing, and standard reachability yields the following reduction.

Lemma 2.1 (Sink-SCC reduction)**.** *Let A be a synchronizing DFA on n states whose sink SCC C_{sink} has size $s = |C_{\text{sink}}| \ge 1$. Then*

$$2\ \mathrm{rt}(A) \le (n-s)(n-s+1) + 2\ \mathrm{rt}(A|_{C_{\text{sink}}}).$$

2.1 Corank sequence, gaps, and generalized corank

For a synchronizing DFA A on n states, define for each $0 \le i \le n-1$ the *minimum corank-i length*

$$\lambda(i) := \min\{|w| : w \in \Sigma^*,\ |Q \cdot w| \le n-i\},$$

and for each $0 \le j \le n-2$ the *corank gap*

$$\delta(j) := \lambda(j+1) - \lambda(j).$$

Clearly $0 = \lambda(0) \le \lambda(1) \le \cdots \le \lambda(n-1) = \mathrm{rt}(A)$, and for all $0 \le a \le b \le n-1$,

$$\lambda(b) = \lambda(a) + \sum_{j=a}^{b-1} \delta(j).$$

For any integer $L \ge 0$, define the *generalized corank at length L* by

$$D(L) := \max\{i \in \{0, \dots, n-1\} : \lambda(i) \le L\} = \max\{n - |Q \cdot w| : |w| \le L\}.$$

The function $L \mapsto D(L)$ is nondecreasing, satisfies $D(L) \le n-1$ and $\lambda(D(L)) \le L$, and attains its value at a word v of length $|v| \le L$ with $|Q \cdot v| = n - D(L)$. We abbreviate the *short corank* by $D := D(n-1)$.

We collect two classical corank-gap bounds that will be used repeatedly.

Lemma 2.2 (Pin–Frankl gap bound [2, 5])**.** *For every synchronizing DFA A on n states and every $0 \le j \le n-2$,*

$$\delta(j) \le \binom{j+2}{2}.$$

Moreover, for all integers $0 \le a \le b$,

$$6 \sum_{j=a}^{b-1} \binom{j+2}{2} = b(b+1)(b+2) - a(a+1)(a+2).$$

In particular, $6\ \mathrm{rt}(A) \le n^3 - n$.

Lemma 2.3 (Shitov's corank-gap upgrade [6])**.** *Let A be a synchronizing DFA on n states, and let $r, \tau \in \mathbb{N}$ satisfy $1 \le r$, $2r+2 \le n$, and $\tau \le n-2$. If $\delta(\tau) > 2r$, then*

$$\delta(r) \le \lambda(\tau) + 2r.$$

2.2 Merge graphs, degree transport, and the telescoping AP plateau inequality

For each $t \in \mathbb{N}$, define the *length-t merge graph* $E_t = (Q, \mathcal{E}_t)$ as the simple undirected graph on Q where distinct states $x \neq y$ are adjacent if and only if there exists a word $w \in \Sigma^*$ of length $|w| \leq t$ with $x \cdot w = y \cdot w$. Write $N_{E_t}(x)$ for the neighborhood of x in E_t, $\deg_{E_t}(x) := |N_{E_t}(x)|$, and

$$m_t := \min_{x \in Q} \deg_{E_t}(x), \qquad \Delta_t := \max_{x \in Q} \deg_{E_t}(x).$$

Clearly $N_{E_{t_1}}(x) \subseteq N_{E_{t_2}}(x)$, $m_{t_1} \leq m_{t_2}$, and $\Delta_{t_1} \leq \Delta_{t_2}$ whenever $t_1 \leq t_2$.

Lemma 2.4 (Independent sets and strict edge growth)**.** *Let A be a synchronizing DFA on n states, $0 \leq r \leq n-2$, and $0 \leq t < \delta(r)$. Then:*

(a) There exists an independent set $S \subseteq Q$ in E_t of size $|S| = n - r$.

(b) $2t \leq \sum_{x \in Q} \deg_{E_t}(x)$.

(c) $(n-r)m_t \leq r\Delta_t$ and $t \leq r\Delta_t$.

Proof. (a) Let u be a word of length $\lambda(r)$ achieving $|Q \cdot u| \leq n - r$. Since $\delta(r) > t \geq 0$, we must have $|Q \cdot u| = n - r$. Let $S := Q \cdot u$. If distinct $x, y \in S$ were merged by a word w of length $|w| \leq t$, then $|Q \cdot (uw)| = |S \cdot w| \leq |S| - 1 = n - (r+1)$, giving $\lambda(r+1) \leq |u| + |w| \leq \lambda(r) + t$, contradicting $t < \delta(r)$. Thus S is independent in E_t.

(b) We induct on t, the case $t = 0$ being trivial. Suppose $s + 1 \leq t < \delta(r)$. If $E_{s+1} = E_s$, an induction on word length shows that every mergeable pair of distinct states is already adjacent in E_s; since A is synchronizing and $|S| = n - r \geq 2$, S would contain an edge of E_s, contradicting part (a) at time s. Hence E_{s+1} contains at least one new undirected edge $\{x_0, y_0\} \notin E_s$, increasing $\sum_{x \in Q} \deg_{E_t}(x)$ by at least 2.

(c) Let S be as in (a) and $\bar{S} = Q \setminus S$ ($|\bar{S}| = r$). Every edge incident to S has its other endpoint in $\bar{S}$, so

$$(n-r)m_t \leq \sum_{x \in S} \deg_{E_t}(x) \leq \sum_{y \in \bar{S}} \deg_{E_t}(y) \leq r\Delta_t.$$

Moreover, $\sum_{x \in Q} \deg_{E_t}(x) = \sum_{x \in S} \deg_{E_t}(x) + \sum_{y \in \bar{S}} \deg_{E_t}(y) \leq 2r\Delta_t$, which combined with (b) gives $t \leq r\Delta_t$. □

Lemma 2.5 (Degree transport [3])**.** *Let A be a strongly connected synchronizing DFA on n states, and let $h := n - 1$. For every $t \in \mathbb{N}$,*

$$\Delta_t \leq D(h) + m_{t+h}.$$

Proof. Choose $x, y \in Q$ achieving $\deg_{E_t}(x) = \Delta_t$ and $\deg_{E_{t+h}}(y) = m_{t+h}$. By strong connectivity, there is a word w of length $|w| \leq h$ with $y \cdot w = x$. Every $z \in N_{E_t}(x) \cap (Q \cdot w)$ has a preimage $p \in Q$ with $p \cdot w = z$; then $p \neq y$ (since $z \neq x = y \cdot w$) and composing w with a word of length $\leq t$ merging z and x merges p and y in length $\leq t + h$, so $p \in N_{E_{t+h}}(y)$. Thus

$$N_{E_t}(x) \cap (Q \cdot w) \subseteq (N_{E_{t+h}}(y)) \cdot w,$$

and therefore $\Delta_t = |N_{E_t}(x)| \leq |N_{E_t}(x) \cap (Q \cdot w)| + |Q \setminus (Q \cdot w)| \leq m_{t+h} + D(h)$. □

Combining Lemmas 2.4 and 2.5 via telescoping along an arithmetic progression yields an unconditional plateau bound below half-rank that is used in both Sections 3 and 4.

Lemma 2.6 (Unconditional telescoping AP plateau inequality)**.** *Let A be a strongly connected synchronizing DFA on n states, and let $r, M \in \mathbb{N}$ satisfy $1 \le r$, $2r < n$, and $M \ge 1$. Then*

$$\delta(r) \le \frac{r(n-r)}{n-2r}\, D + \frac{r^2}{n-2r}\,\frac{n}{M} + Mn.$$

Proof. Set $h := n-1$. If $\delta(r) \le Mh+1$, then $\delta(r) \le Mh + M = Mn$ and the inequality holds trivially. Otherwise $\delta(r) - 1 \ge Mh$. Define the $M+1$ times $t_j := (\delta(r)-1) - (M-j)h$ for $0 \le j \le M$, so $t_j < \delta(r)$ and $t_{j+1} = t_j + h$. For every $0 \le j < M$, Lemmas 2.5 and 2.4(c) give

$$(n-r)\Delta_{t_j} \le (n-r)D + (n-r)m_{t_{j+1}} \le (n-r)D + r\Delta_{t_{j+1}},$$

which rearranges to

$$(n-2r)\Delta_{t_j} \le (n-r)D + r(\Delta_{t_{j+1}} - \Delta_{t_j}).$$

Summing over $j = 0, \dots, M-1$, the differences telescope to $\Delta_{t_M} - \Delta_{t_0} \le n$, while monotonicity gives $M\Delta_{t_0} \le \sum_{j=0}^{M-1} \Delta_{t_j}$. Hence

$$M(n-2r)\Delta_{t_0} \le M(n-r)D + rn.$$

Finally, Lemma 2.4(c) at t_0 yields $\delta(r) - 1 - Mh = t_0 \le r\Delta_{t_0}$, so

$$\delta(r) \le r\Delta_{t_0} + Mh + 1 \le \frac{r(n-r)}{n-2r} D + \frac{r^2}{n-2r}\frac{n}{M} + Mn. \qquad \square$$

We will also repeatedly invoke the following elementary pigeonhole calm-step lemma.

Lemma 2.7 (Calm step)**.** *Let $f : \mathbb{N} \to \mathbb{N}$ be nondecreasing and bounded above by n. For any $M \ge 1$,* start $\in \mathbb{N}$*, and* step $\in \mathbb{N}$*, there exists $i \in \{0, \dots, M-1\}$ such that*

$$f(\text{start} + (i+1)\,\text{step}) - f(\text{start} + i\,\text{step}) \le \lfloor n/M \rfloor.$$

Proof. If every difference were at least $\lfloor n/M \rfloor + 1$, summing over $i = 0, \dots, M-1$ would give $f(\text{start} + M\,\text{step}) \ge M(\lfloor n/M \rfloor + 1) > n$, a contradiction. $\square$

3 Proof of Theorem 1.1: Breaking the $\frac{7}{48}n^3$ Barrier

Throughout Sections 3.1–3.3, $A = (Q, \Sigma, \delta)$ is a strongly connected synchronizing DFA on $n = |Q|$ states, and $h := n-1$.

3.1 Complement degree bound and singularity-free compression cap

Fix any word $v \in \Sigma^*$. Let $R := Q \cdot v$ be its image and $C := n - |R|$ its corank. For each $t \in \mathbb{N}$ and $x \in Q$, define the *block-cover set*

$$\mathrm{Cov}_{v,t}(x) := \{p \in R : pv^{-1} \subseteq N_{E_t}(x)\},$$

the *block-cover degree* $K_t(x) := |\,\mathrm{Cov}_{v,t}(x)|$, and its maximum $K_t := \max_{x \in Q} K_t(x)$. We also define the *block-join graph* Γ_t on vertex set R, where distinct $p, q \in R$ are adjacent whenever $(pv^{-1}) \times (qv^{-1}) \subseteq E_t$, and write $\deg_{\Gamma_t}(p)$ for the degree of p in Γ_t.

The key observation that eliminates the $1/(1-2c)$ singularity of [4] is the following one-line unconditional bound.

Lemma 3.1 (Complement degree bound)**.** *For* every *DFA A on n states, every word $v \in \Sigma^*$ of corank $C = n - |Q \cdot v|$, every $q \in \mathbb{N}$, and every state $z \in Q$,*

$$\deg_{E_q}(z) \le C + K_q(z).$$

Proof. Let $R := Q \cdot v$. For every $p \in R \setminus \mathrm{Cov}_{v,q}(z)$, by definition $pv^{-1} \not\subseteq N_{E_q}(z)$, so there exists $y \in Q \setminus N_{E_q}(z)$ with $y \cdot v = p$. Hence

$$R \setminus \mathrm{Cov}_{v,q}(z) \subseteq (Q \setminus N_{E_q}(z)) \cdot v.$$

Taking cardinalities gives $|R| - K_q(z) \le n - \deg_{E_q}(z)$, i.e., $\deg_{E_q}(z) \le (n - |R|) + K_q(z) = C + K_q(z)$. □

Next, we record whole-block transport (following [4]) and its single-quantity calm-time consequence.

Lemma 3.2 (Whole-block transport and single-quantity calm time)**.** *Let A be a strongly connected synchronizing DFA on n states, and let $v \in \Sigma^*$ have image $R = Q \cdot v$ and corank $C = n - |R|$. Set $B := |v| + (n-1)$ and $a_0 := D(2|v| + n - 1) - C$.*

(a) For every $t \in \mathbb{N}$ and every $p_0 \in R$, $K_t \le \deg_{\Gamma_{t+B}}(p_0) + a_0$.

(b) For every $M \ge 1$ and $T_0 \ge (M+1)B$, there exists $q \in [T_0 - MB, T_0]$ with $q \ge B$ such that for all $p_0 \in R$,

$$K_q \le \deg_{\Gamma_q}(p_0) + \eta_0, \qquad \text{where } \eta_0 := a_0 + \lfloor n/M \rfloor.$$

Proof. (a) Pick $x \in Q$ with $K_t(x) = K_t$, and by strong connectivity pick a word w of length $|w| \le n-1$ with $p_0 \cdot w = x$. Let $I_{vwv} := Q \cdot (vwv) \subseteq R$. Since $|vwv| \le 2|v| + n - 1$, we have $|R \setminus I_{vwv}| = (n - |I_{vwv}|) - (n - |R|) \le D(2|v| + n - 1) - C = a_0$.

We claim that $\mathrm{Cov}_{v,t}(x) \cap I_{vwv} \subseteq N_{\Gamma_{t+B}}(p_0) \cdot (wv)$. Indeed, any $r \in \mathrm{Cov}_{v,t}(x) \cap I_{vwv}$ can be written as $r = b_0 \cdot (vwv) = q \cdot (wv)$ for $q := b_0 \cdot v \in R$. Then $q \cdot w \in rv^{-1} \subseteq N_{E_t}(x)$, which forces $q \cdot w \ne x = p_0 \cdot w$ (hence $q \ne p_0$) and supplies a word u' of length $|u'| \le t$ with $(q \cdot w) \cdot u' = x \cdot u' = (p_0 \cdot w) \cdot u'$. For every $a \in p_0 v^{-1}$ and $b \in qv^{-1}$, the word vwu' has length $|vwu'| \le t + |v| + n - 1 = t + B$ and merges a and b, showing $q \in N_{\Gamma_{t+B}}(p_0)$. Taking cardinalities yields

$$K_t = |\mathrm{Cov}_{v,t}(x)| \le |\mathrm{Cov}_{v,t}(x) \cap I_{vwv}| + |R \setminus I_{vwv}| \le \deg_{\Gamma_{t+B}}(p_0) + a_0.$$

(b) Applying Lemma 2.7 to the nondecreasing function $t \mapsto K_t \le n$ with start $= T_0 - MB \ge B$ and step $= B$ gives $i \in \{0, \dots, M-1\}$ such that $q := \text{start} + (i+1)B$ satisfies $K_q - K_{q-B} \le \lfloor n/M \rfloor$. Combining this with part (a) at time $t = q - B$ yields $K_q \le \deg_{\Gamma_q}(p_0) + a_0 + \lfloor n/M \rfloor$. □

Combining Lemma 3.1 and Lemma 3.2 produces the singularity-free compression cap.

Proposition 3.3 (Singularity-free block compression cap)**.** *Let A be a strongly connected synchronizing DFA on n states, and let $v \in \Sigma^*$ have corank $C = n - |Q \cdot v|$. Let $r, M \in \mathbb{N}$ satisfy $r \le n-2$, $M \ge 1$, and $2(n-r) > n - C$. Set*

$$b := n - r, \quad B := |v| + (n-1), \quad \Omega := (M+1)B + 1, \quad a_0 := D(2|v| + n - 1) - C, \quad \eta_0 := a_0 + \lfloor n/M \rfloor.$$

Then

$$\delta(r) \le \min\Big(C^2, \frac{nC}{2}\Big) + \Big(\frac{n}{2} + \min\Big(C, \frac{n}{2}\Big)\Big) \frac{b\eta_0}{2b - (n - C)} + \Omega. \tag{1}$$

Proof. If $\delta(r) \le \Omega$, (1) holds trivially. Assume therefore that $T_0 := \delta(r) - 1 \ge (M+1)B$, and let $q \in [T_0 - MB, T_0]$ with $q \ge B \ge |v|$ be the calm time given by Lemma 3.2(b). Since $q < \delta(r)$, Lemma 2.4(a)–(b) yields an independent set S in E_q of size $|S| = b = n - r$ and $2q \le \sum_{x \in Q} \deg_{E_q}(x)$.

First, we bound K_q. Because $|v| \le q$ and S is independent in E_q, the map $x \mapsto x \cdot v$ is injective on S, so $I := S \cdot v \subseteq R$ has size $|I| = b$. Moreover, I is independent in Γ_q (any Γ_q-edge between $x \cdot v$ and $y \cdot v$ would force $\{x, y\} \in E_q$). Observe also that for every $p' \in R$ (say $p' = z \cdot v$), every neighbor $p \in N_{\Gamma_q}(p')$ satisfies $pv^{-1} \subseteq N_{E_q}(z)$ and hence $p \in \mathrm{Cov}_{v,q}(z)$, so $\deg_{\Gamma_q}(p') \le K_q$. Double-counting Γ_q-edges between I and $R \setminus I$ gives

$$b(K_q - \eta_0) \le \sum_{p \in I} \deg_{\Gamma_q}(p) \le \sum_{p' \in R \setminus I} \deg_{\Gamma_q}(p') \le (|R| - b) K_q = (n - C - b) K_q,$$

where the first inequality uses Lemma 3.2(b). Rearranging yields

$$K_q \le \frac{b\eta_0}{2b - (n - C)}. \tag{2}$$

Second, we partition Q into the singleton-fiber states $S_{\mathrm{sing}} := \{x \in Q : |(x \cdot v)v^{-1}| = 1\}$ and their complement $Q \setminus S_{\mathrm{sing}}$. Let $R_{\mathrm{sing}} := \{p \in R : |pv^{-1}| = 1\}$, so $|S_{\mathrm{sing}}| = |R_{\mathrm{sing}}|$. Since every $p \in R \setminus R_{\mathrm{sing}}$ has $|pv^{-1}| \ge 2$, we have $|pv^{-1}| \le 2(|pv^{-1}| - 1)$, whence

$$|Q \setminus S_{\mathrm{sing}}| = \sum_{p \in R \setminus R_{\mathrm{sing}}} |pv^{-1}| \le 2 \sum_{p \in R \setminus R_{\mathrm{sing}}} (|pv^{-1}| - 1) = 2(n - |R|) = 2C,$$

so $|Q \setminus S_{\mathrm{sing}}| \le \min(2C, n)$.

For the singleton-fiber sum, swapping the order of summation gives

$$\sum_{x \in S_{\mathrm{sing}}} \deg_{E_q}(x) = \sum_{y \in Q} |S_{\mathrm{sing}} \cap N_{E_q}(y)| \le nK_q,$$

because $x \mapsto x \cdot v$ injects $S_{\mathrm{sing}} \cap N_{E_q}(y)$ into $\mathrm{Cov}_{v,q}(y)$ (if $x \in S_{\mathrm{sing}} \cap N_{E_q}(y)$, then $(x \cdot v)v^{-1} = \{x\} \subseteq N_{E_q}(y)$). For the non-singleton sum, Lemma 3.1 applies pointwise to every $z \in Q \setminus S_{\mathrm{sing}}$:

$$\sum_{z \in Q \setminus S_{\mathrm{sing}}} \deg_{E_q}(z) \le \min(2C, n)\,(C + K_q).$$

Adding the two sums and using $2q \le \sum_{x \in Q} \deg_{E_q}(x)$ yields

$$q \le \frac{\min(2C, n)}{2}\, C + \left(\frac{n}{2} + \frac{\min(2C, n)}{2}\right) K_q = \min\left(C^2, \frac{nC}{2}\right) + \left(\frac{n}{2} + \min\left(C, \frac{n}{2}\right)\right) K_q.$$

Substituting (2) and $\delta(r) = T_0 + 1 \le q + MB + 1 \le q + \Omega$ establishes (1). □

3.2 Zero-$\sqrt{2}$ rational payment certificates and unified word-cap saving

To convert Proposition 3.3 into a saving below $\frac{7}{48}n^3$ whenever a word v of corank $C = cn$ is available, we balance its prefix cost $\lambda(\lfloor n/2 \rfloor)$ against the upper-half saving on $r \in [an, bn]$, where $a = \max(\frac{1}{2}, \frac{17}{12}c)$ and $b = \frac{1+c}{2}$. Using the rational approximation $\frac{17}{12} > \sqrt{2}$ avoids square roots completely and guarantees $\min(c^2, c/2) \le a^2/2$ for all $c \in [0, 0.505]$.

Define for $c, x \in \mathbb{Q}$:

$$W(x) := \begin{cases} \frac{1}{16}, & x \le \frac{1}{4}, \\ x\left(\frac{1}{2} - x\right), & x > \frac{1}{4}, \end{cases} \qquad p(c,x) := \min\left(c^2 \left(\frac{1}{2} - x\right),\, 2\max(x-c,0)\, W(x) \right),$$

and

$$S_{\mathrm{rat}}(c) := \frac{b^3 - a^3}{6} - c^2(b-a), \qquad a = \max\left(\frac{1}{2},\, \frac{17}{12}c\right), \quad b = \frac{1+c}{2}.$$

Lemma 3.4 (Rational payment certificates)**.** *The following exact inequalities hold over* $\mathbb{Q}$*:*

(a) For all $c, x \in [0, \frac{1}{2}]$*,* $S_{\mathrm{rat}}(c) - p(c,x) \ge \frac{c}{1500}$*.*

(b) For all $c \in [\frac{1}{10}, \frac{1}{2}]$ *and* $x \in [0, \frac{1}{2}]$*,* $S_{\mathrm{rat}}(c) - p(c,x) \ge \frac{1}{5000}$*.*

(c) For all $c \in [\frac{1}{2}, \frac{505}{1000}]$*, with* $a = \frac{17}{12}c$ *and* $b = \frac{1+c}{2}$*,*

$$\frac{c^3 - \frac{1}{8}}{6} + \left(\frac{b^3 - a^3}{6} - \frac{c}{2}(b-a) \right) \ge \frac{55}{82944}.$$

Proof. We first bound $p(c,x)$ independently of $x \in [0, \frac{1}{2}]$.

- For any $c \le \frac{1}{2}$, if $x \le c$ then $p(c,x) \le 0$; if $x > c$, since $W(x) \le \frac{1}{16}$ we have both $p(c,x) \le c^2(\frac{1}{2} - x)$ and $p(c,x) \le \frac{x-c}{8}$, so eliminating x via $p(c,x) + 8c^2 p(c,x) \le c^2(\frac{1}{2} - x) + c^2(x-c) = c^2(\frac{1}{2} - c)$ yields
$$p(c,x) \le \frac{c^2(\frac{1}{2} - c)}{1 + 8c^2}.$$
- For $c \in [\frac{1}{3}, \frac{1}{2}]$, if $x > c > \frac{1}{4}$ then $W(x) = x(\frac{1}{2} - x)$ and $2x \le 1$, so by AM–GM,
$$p(c,x) \le 2(x-c)x\left(\frac{1}{2} - x\right) \le (x-c)\left(\frac{1}{2} - x\right) \le \frac{1}{4}\left(\frac{1}{2} - c\right)^2.$$

Note that for $c \le \frac{6}{17}$, $a = \frac{1}{2}$ and $S_{\mathrm{rat}}(c) = \frac{3c + 3c^2 - 23c^3}{48}$, whereas for $c \ge \frac{6}{17}$, $a = \frac{17}{12}c$. Every inequality now reduces to a univariate polynomial on a compact interval, certified by a positive-coefficient Bernstein expansion ($t \in [0,1]$, $u = 1 - t \in [0,1]$):

(i) *Range* $c \in [0, \frac{1}{3}]$ *for (a):* Set $t = 3c$, $u = 1 - t$. Direct expansion confirms

$$(1 + 8c^2)(3 + 3c - 23c^2) - 48c\left(\frac{1}{2} - c\right) - \frac{5}{81} = \frac{238}{81}u^4 + \frac{385}{81}tu^3 + \frac{168}{81}t^2u^2 + \frac{205}{81}t^3u \ge 0.$$

Since $48(1 + 8c^2) \le 48 \cdot \frac{17}{9} = \frac{272}{3}$, dividing by $48(1 + 8c^2)$ gives

$$S_{\mathrm{rat}}(c) - p(c,x) \ge \frac{c \cdot \frac{5}{81}}{\frac{272}{3}} = \frac{5}{7344}c \ge \frac{c}{1500}.$$

(ii) *Range* $c \in [\frac{1}{10}, \frac{1}{3}]$ *for (b):* Set $t = \frac{30}{7}(c - \frac{1}{10})$, $u = 1 - t$. Direct expansion confirms

$$\begin{aligned} &c\big((1 + 8c^2)(3 + 3c - 23c^2) - 48c(\tfrac{1}{2} - c)\big) - \frac{48}{5000}(1 + 8c^2) \\ &\quad = \frac{16149}{125000}u^5 + 5 \cdot \frac{132433}{937500}tu^4 + 10 \cdot \frac{1309903}{11250000}t^2u^3 \\ &\qquad + 10 \cdot \frac{148757}{1406250}t^3u^2 + 5 \cdot \frac{151949}{1265625}t^4u + \frac{371}{151875}t^5 \ge 0. \end{aligned}$$

Dividing by $48(1 + 8c^2) > 0$ yields $S_{\mathrm{rat}}(c) - p(c,x) \ge \frac{1}{5000}$.

(iii) *Range $c \in [\frac{1}{3}, \frac{6}{17}]$ for (a) and (b):* Set $t = 51(c - \frac{1}{3})$, $u = 1 - t$. Then

$$S_{\mathrm{rat}}(c) - \frac{1}{4}\left(\frac{1}{2} - c\right)^2 - \frac{1}{400} = \frac{19}{32400}u^3 + 3 \cdot \frac{47}{61200}tu^2 + 3 \cdot \frac{899}{1040400}t^2u + \frac{6848}{7860800}t^3 \geq 0.$$

Since $\frac{1}{400} > \frac{1}{5000}$ and $\frac{1}{400} \geq \frac{c}{1500}$, both (a) and (b) follow.

(iv) *Range $c \in [\frac{6}{17}, \frac{1}{2}]$ for (a) and (b):* Set $t = \frac{34}{5}(c - \frac{6}{17})$, $u = 1 - t$. Then

$$\begin{aligned} S_{\mathrm{rat}}(c) - \frac{1}{4}\left(\frac{1}{2} - c\right)^2 - \frac{55}{82944} = \left(\frac{265}{78608} - \frac{55}{82944}\right)u^3 + 3\left(\frac{565}{166464} - \frac{55}{82944}\right)tu^2 \\ + 3\left(\frac{235}{117504} - \frac{55}{82944}\right)t^2u \geq 0, \end{aligned}$$

since $\frac{265}{78608}, \frac{565}{166464}, \frac{235}{117504} > \frac{55}{82944} \approx 0.000663 > \frac{1}{3000} \geq \frac{c}{1500}$ and $\frac{55}{82944} > \frac{1}{5000}$.

(v) *Range $c \in [\frac{1}{2}, \frac{505}{1000}]$ for (c):* Set $t = 200(c - \frac{1}{2})$, $u = 1 - t$. Then

$$\begin{aligned} \frac{c^3 - \frac{1}{8}}{6} + \left(\frac{b^3 - a^3}{6} - \frac{c}{2}(b - a)\right) - \frac{55}{82944} = 3\left(\frac{7139}{8294400} - \frac{55}{82944}\right)tu^2 \\ + 3\left(\frac{878431}{829440000} - \frac{55}{82944}\right)t^2u \\ + \left(\frac{104356331}{82944000000} - \frac{55}{82944}\right)t^3 \geq 0, \end{aligned}$$

where every parenthesized coefficient is strictly positive (comparing numerators over the common denominators 8294400, 829440000, and 82944000000). □

To obtain a word v whose stability defect $a_0 = D(2|v| + n - 1) - C$ is small, we apply a dyadic pigeonhole schedule.

Lemma 3.5 (Dyadic stability schedule)**.** *For any synchronizing DFA A on n states, any $\ell_0 \in \mathbb{N}$, and any $J \geq 1$, there exists a word $v \in \Sigma^*$ such that*

$$|v| \leq 2^J(\ell_0 + n), \qquad C := n - |Q \cdot v| \geq D(\ell_0), \qquad D(2|v| + n - 1) - C \leq \lfloor n/J \rfloor.$$

Proof. Define ℓ_0 as given and $\ell_{i+1} := 2\ell_i + (n - 1)$ for $i \geq 0$. Induction gives $\ell_i + n \leq 2^i(\ell_0 + n)$. Since $i \mapsto D(\ell_i)$ is nondecreasing and bounded by n, the same telescoping pigeonhole argument as in Lemma 2.7 yields an index $j \in \{0, \ldots, J - 1\}$ with $D(\ell_{j+1}) - D(\ell_j) \leq \lfloor n/J \rfloor$. Choosing v of length $|v| \leq \ell_j$ achieving corank $C = D(\ell_j)$ satisfies $|v| \leq \ell_j \leq 2^J(\ell_0 + n)$, $C = D(\ell_j) \geq D(\ell_0)$, and $2|v| + n - 1 \leq 2\ell_j + (n - 1) = \ell_{j+1}$, so $D(2|v| + n - 1) - C \leq D(\ell_{j+1}) - D(\ell_j) \leq \lfloor n/J \rfloor$. □

We now assemble Proposition 3.3 and Lemma 3.4 into a single unified theorem covering both Case A and Case C of the main proof.

Proposition 3.6 (Unified word-cap saving)**.** *Let A be a strongly connected synchronizing DFA on $n \geq 10\,000$ states. Let $v \in \Sigma^*$ have corank $C = n - |Q \cdot v|$ and normalized corank $c := C/n$. Suppose $J, M, D_\varepsilon \in \mathbb{N}$ and $\mu \in \mathbb{Q}$ satisfy $J \geq 1$, $M \geq 1$, $5000 \leq D_\varepsilon \leq n/10$, $100n \leq D_\varepsilon C$, $D(2|v| + n - 1) - C \leq \lfloor n/J \rfloor$, and $\mu \leq \frac{1}{1600}$, and suppose that whenever $C \leq \lfloor n/2 \rfloor$ we have $S_{\mathrm{rat}}(c) - p(c, x) \geq \mu$ for all $x \in [0, \frac{1}{2}]$. Set*

$$\Omega := (M + 1)(|v| + n - 1) + 1, \qquad \theta := \left(\frac{1}{J} + \frac{1}{M}\right)\frac{D_\varepsilon}{2}.$$

Then

$$\lambda(n-1) \le \left(\frac{7}{48} - \mu + \frac{3}{10D_\varepsilon} + \theta\right) n^3 + (n+1)\Omega + 7n^2. \tag{3}$$

Proof. Note first that $\lambda(C) \le |v| \le \Omega$. We consider three subcases according to C.

Subcase 1: $C \le \lfloor n/2 \rfloor$. Set $H := \lfloor n/2 \rfloor$ and $k := \lfloor (n-2)/2 \rfloor$. Let ρ be the least index $j \le n-2$ with $\delta(j) > 2k$ (if no such j exists or if $\rho \ge H$, then $\lambda(H) \le n^2$ and we take $x = \frac{1}{2}$ where $p(c, \frac{1}{2}) = 0$). Otherwise $1 \le \rho < H$ and $\lambda(\rho) \le \rho \cdot 2k \le n^2$; set $x := \rho/n \in [0, \frac{1}{2}]$. We bound $\lambda(H)$ in two ways:

- *Direct cap bound on $[\rho, H)$:* For every $r \in [\rho, H)$, we have $2(n-r) - (n-C) \ge C \ge 2n/D_\varepsilon$, so Proposition 3.3 gives

  $$\delta(r) \le C^2 + n \cdot \frac{n \cdot (\frac{1}{J} + \frac{1}{M})n}{2n/D_\varepsilon} + \Omega = c^2 n^2 + \theta n^2 + \Omega.$$

  Summing over $H - \rho \le (\frac{1}{2} - x)n$ indices and adding $\lambda(\rho) \le n^2$ gives

  $$\lambda(H) \le c^2 \left(\frac{1}{2} - x\right) n^3 + \frac{1}{2}\theta n^3 + \left(\frac{n}{2} + 1\right)\Omega + 2n^2.$$

- *Shitov cheap-gap bound on $[\rho, H)$:* If $\rho \le C$, then $\lambda(\rho) \le \lambda(C) \le \Omega$ and Lemma 2.3 at $\tau = \rho$ yields $\delta(r) \le \Omega + 2r$ for all $r \in [\rho, H)$, giving $\lambda(H) \le (\frac{n}{2} + 1)\Omega + n^2$. If $C < \rho$, partition the positive gaps in $[C, \rho)$ into buckets $B_C(i) := \{j \in [C, \rho) : 2i - 1 \le \delta(j) \le 2i\}$ $(1 \le i \le k)$ of sizes $s_C(i) = |B_C(i)|$, so $\sum_{i=1}^{k} s_C(i) \le \rho - C$. For each $r \in [\rho, H)$, applying Lemma 2.3 at the least $\tau \le \rho$ with $\delta(\tau) > 2r$ yields $\delta(r) \le \Omega + 2\sum_{i=1}^{r} i s_C(i) + 2r$. Summing over $r \in [\rho, H)$ and swapping the double sum gives

  $$\lambda(H) \le \lambda(\rho) + (H - \rho)\Omega + 2\sum_{i=1}^{k} s_C(i)\, i(H - \max(\rho, i)) + H^2.$$

  An elementary check shows $i(H - \max(\rho, i)) \le W(x)n^2$ for all $i \le H$: if $x \le \frac{1}{4}$ then $t(H-t) \le H^2/4 \le n^2/16$; if $x > \frac{1}{4}$ then $2\rho > H$, so $t(H-t)$ is decreasing for $t \ge \rho$ and hence maximized at ρ. Thus $\sum_i s_C(i) i(H - \max(\rho, i)) \le (\rho - C)W(x)n^2 = (x - c)W(x)n^3$.

Taking the minimum of the two bounds yields

$$\lambda(H) \le p(c, x)n^3 + \frac{1}{2}\theta n^3 + \left(\frac{n}{2} + 1\right)\Omega + 2n^2. \tag{4}$$

Next, we bound the suffix $\sum_{j=s}^{n-2} \delta(j)$ simultaneously for Subcase 1 ($s = H$) and Subcase 2 ($s = C \le \frac{505}{1000}n$) by setting $s := \max(H, C)$, $a_{\text{lo}} := \max(\frac{1}{2}, \frac{17}{12}c)$, $b_{\text{hi}} := \frac{1+c}{2}$, and $\text{cap}_r := \min(c^2, c/2)$. Define integer indices

$$A_{\text{idx}} := \max\left(\left\lfloor \frac{n+1}{2} \right\rfloor, \left\lfloor \frac{17C + 11}{12} \right\rfloor\right), \qquad B_{\text{idx}} := \left\lfloor \frac{n+C}{2} \right\rfloor - \left(\left\lfloor \frac{n}{D_\varepsilon} \right\rfloor + 1\right).$$

Using $C \ge 100n/D_\varepsilon$ and $5000 \le D_\varepsilon \le n/10$, we have $s \le A_{\text{idx}} \le B_{\text{idx}} \le n - 1$, and for every $r \in [A_{\text{idx}}, B_{\text{idx}})$, $2(n-r) - (n-C) \ge 2(\lfloor n/D_\varepsilon \rfloor + 1) > 2n/D_\varepsilon$. Applying Proposition 3.3 on $[A_{\text{idx}}, B_{\text{idx}})$ and Lemma 2.2 outside $[A_{\text{idx}}, B_{\text{idx}})$ yields

$$\sum_{j=s}^{n-2} \delta(j) \le \frac{n^3 - s^3}{6} - \left(\frac{B_{\text{idx}}^3 - A_{\text{idx}}^3}{6} - (B_{\text{idx}} - A_{\text{idx}})\text{cap}_r n^2\right) + \frac{1}{2}\theta n^3 + \frac{n}{2}\Omega + n^2.$$

Setting $u_A := A_{\mathrm{idx}}/n$ and $u_B := B_{\mathrm{idx}}/n$, we have $a_{\mathrm{lo}} \le u_A \le u_B \le b_{\mathrm{hi}}$, $u_A - a_{\mathrm{lo}} \le 1/n$, and $b_{\mathrm{hi}} - u_B \le 1/D_\varepsilon + 2/n$. Because $\mathrm{cap}_r \le a_{\mathrm{lo}}^2/2$ (as $(\frac{17}{12})^2/2 > 1$) and $b_{\mathrm{hi}}^2/2 \le (1.505/2)^2/2 < \frac{3}{10}$, the elementary algebraic strip estimate

$$0 \le \frac{v^3 - u^3}{6} - \mathrm{cap}_r(v-u) = (v-u)\left(\frac{v^2+uv+u^2}{6} - \mathrm{cap}_r\right) \le \frac{3}{10}(v-u)$$

for $a_{\mathrm{lo}} \le u \le v \le b_{\mathrm{hi}}$ gives

$$\sum_{j=s}^{n-2} \delta(j) \le \frac{n^3 - s^3}{6} - \left(\frac{b_{\mathrm{hi}}^3 - a_{\mathrm{lo}}^3}{6} - \mathrm{cap}_r(b_{\mathrm{hi}} - a_{\mathrm{lo}})\right) n^3 + \frac{3}{10D_\varepsilon} n^3 + \frac{1}{2}\theta n^3 + \frac{n}{2}\Omega + 4n^2. \tag{5}$$

In Subcase 1 ($s = H$, $\mathrm{cap}_r = c^2$), we have $(n^3 - H^3)/6 \le \frac{7}{48}n^3 + n^2$ (for $n \ge 10\,000$). Adding (4) and (5) and applying $S_{\mathrm{rat}}(c) - p(c,x) \ge \mu$ yields (3).

Subcase 2: $\lfloor n/2 \rfloor < C \le \lfloor 505n/1000 \rfloor$**.** Here $s = C$, $a_{\mathrm{lo}} = \frac{17}{12}c$, and $\mathrm{cap}_r = c/2$. Adding $\lambda(C) \le \Omega$ to (5) and rewriting $(n^3 - C^3)/6 = \frac{7}{48}n^3 - \frac{c^3 - 1/8}{6}n^3$, Lemma 3.4(c) gives a total saving at least $\frac{55}{82944}n^3 > \frac{1}{1600}n^3 \ge \mu n^3$, establishing (3).

Subcase 3: $C > \lfloor 505n/1000 \rfloor$**.** Here $C \ge 0.505n$. Simply summing Lemma 2.2 from C to $n-2$ and adding $\lambda(C) \le \Omega$ gives

$$\lambda(n-1) \le \Omega + \frac{n^3 - C^3}{6} \le \Omega + \frac{1 - 0.505^3}{6} n^3 \le \left(\frac{7}{48} - \frac{1}{1600}\right) n^3 + \Omega,$$

since $\frac{1-0.505^3}{6} = \frac{2323233}{16000000} \approx 0.145202 < \frac{7}{48} - \frac{1}{1600} = \frac{697}{4800} \approx 0.145208$, establishing (3). □

3.3 Preimage lifting and the near-bijective macroscopic band bound

We now handle the near-bijective regime where both $D(h)$ and $D(\lfloor n^2/D_\eta \rfloor)$ are small. The core combinatorial engine is the following five-line lifting lemma.

Lemma 3.7 (Common-neighbor preimage lifting)**.** *Let A be a synchronizing DFA on n states. For every $t, L_0 \in \mathbb{N}$ and every edge $\{u,v\} \in E_{L_0}$, there exists a subset $W \subseteq Q$ such that*

$$m_t \le |W| + D(L_0) \qquad \text{and} \qquad W \subseteq N_{E_{t+L_0}}(u) \cap N_{E_{t+L_0}}(v).$$

Proof. Because $\{u,v\} \in E_{L_0}$, there is a word w of length $|w| \le L_0$ such that $u \cdot w = v \cdot w =: z$. Set $W := (N_{E_t}(z))w^{-1}$.

First, every $y \in N_{E_t}(z) \cap (Q \cdot w)$ has a preimage in W, so $N_{E_t}(z) \cap (Q \cdot w) \subseteq W \cdot w$, whence $|N_{E_t}(z) \cap (Q \cdot w)| \le |W|$. Therefore

$$m_t \le |N_{E_t}(z)| \le |N_{E_t}(z) \cap (Q \cdot w)| + |Q \setminus (Q \cdot w)| \le |W| + D(L_0).$$

Second, take any $x \in W$, so $x \cdot w \in N_{E_t}(z)$. In particular $x \cdot w \ne z = u \cdot w = v \cdot w$, whence $x \ne u$ and $x \ne v$. Moreover, there exists a word v' of length $|v'| \le t$ with $(x \cdot w) \cdot v' = z \cdot v'$. Then wv' has length $|wv'| \le t + L_0$ and satisfies $x \cdot (wv') = z \cdot v' = u \cdot (wv') = v \cdot (wv')$, showing $x \in N_{E_{t+L_0}}(u) \cap N_{E_{t+L_0}}(v)$. □

To turn Lemma 3.7 into a contradiction on a large independent set, we locate (1) an almost-regular short scale L_0 via geometric growth, and (2) a nested calm pair $(t, T = t + L_0)$ inside $[n^2/10, \delta(r))$.

Lemma 3.8 (Geometric-ratio calm step)**.** *Let A be a strongly connected synchronizing DFA on n states, and let $D_\eta, D_{c_1}, R, r \in \mathbb{N}$ satisfy $D_\eta \geq 10^{14}$, $D_{c_1} = 50D_\eta$, $D_{c_1} < (39/38)^R$, $n \geq 2(R+1)D_\eta + 10D_\eta$, $D_{c_1}D(h) < n$, $10D(\lfloor n^2/D_\eta \rfloor) < n$, and $r \leq n-2$. If $\delta(r) > \lfloor 120n^2/1000 \rfloor + 1$, then there exists $L_0 \in \mathbb{N}$ such that*

$$\left\lfloor \frac{40n^2}{100D_\eta} \right\rfloor \leq L_0 \leq \left\lfloor \frac{n^2}{D_\eta} \right\rfloor, \qquad 95\Delta_{L_0} \leq 100m_{L_0}, \qquad \Delta_{L_0} > 0, \qquad 10D(L_0) < n.$$

Proof. Set $t_0 := \lfloor 40n^2/(100D_\eta) \rfloor + 1$. Since $D_\eta \geq 10^{14}$, $t_0 < \delta(r)$, so Lemma 2.4(b) yields $2t_0 \leq n\Delta_{t_0}$. Because $(100D_\eta)t_0 > 40n^2$ and $D_{c_1} = 50D_\eta$, we get $D_{c_1}\Delta_{t_0} \geq 40n$, and hence $\Delta_{t_0} > 40D(h)$.

Define $s_j := t_0 + jh$ for $0 \leq j \leq R$. The hypothesis $n \geq 2(R+1)D_\eta + 10D_\eta$ ensures $s_j + h \leq \lfloor n^2/D_\eta \rfloor$ for all $j \leq R$. Suppose for contradiction that no $L_0 = s_j + h$ $(0 \leq j < R)$ satisfies $95\Delta_{L_0} \leq 100m_{L_0}$. Then for every $0 \leq j < R$, we have $100m_{s_{j+1}} < 95\Delta_{s_{j+1}}$, while Lemma 2.5 and monotonicity give

$$\Delta_{s_j} \leq D(h) + m_{s_{j+1}} < \frac{1}{40}\Delta_{s_j} + \frac{95}{100}\Delta_{s_{j+1}},$$

so $\frac{39}{40}\Delta_{s_j} \leq \frac{19}{20}\Delta_{s_{j+1}}$, i.e., $39\Delta_{s_j} \leq 38\Delta_{s_{j+1}}$. Iterating R times yields

$$n \geq \Delta_{s_R} \geq \left(\frac{39}{38}\right)^R \Delta_{t_0} > D_{c_1} \cdot \frac{n}{D_{c_1}} = n,$$

a contradiction. Thus some $L_0 = s_j + h$ satisfies $95\Delta_{L_0} \leq 100m_{L_0}$, and also $\Delta_{L_0} \geq \Delta_{t_0} > 0$ and $10D(L_0) \leq 10D(\lfloor n^2/D_\eta \rfloor) < n$. □

Lemma 3.9 (Two-scale nested calm time)**.** *Under the hypotheses of Lemma 3.8 with $n \geq 150D_\eta^2$, let L_0 be as returned by Lemma 3.8. Then there exist $t, T \in \mathbb{N}$ with $T = t + L_0 < \delta(r)$ such that*

$$m_t > 0, \qquad \Delta_T \leq \frac{106}{100}m_t, \qquad m_t - D(L_0) \geq \frac{49}{100}m_t.$$

Proof. Set $a_0 := \lfloor n^2/10 \rfloor$. Applying Lemma 2.7 to $s \mapsto \Delta_s$ with start $= a_0$, step $= 2L_0$, and $M = 100$ gives $j < 100$ such that $a := a_0 + 2jL_0$ satisfies

$$\Delta_{a+2L_0} - \Delta_a \leq \lfloor n/100 \rfloor.$$

Next, set $K := \lfloor L_0/h \rfloor$. From $n \geq 150D_\eta^2$, $D_{c_1} = 50D_\eta$, and $L_0 \geq \lfloor 40n^2/(100D_\eta) \rfloor$, we have $K \geq D_{c_1}$. Applying Lemma 2.7 to $s \mapsto m_s$ with start $= a$, step $= h$, and $M = K$ gives $i < K$ such that $t := a + ih \leq a + L_0$ satisfies

$$m_{t+h} - m_t \leq \lfloor n/K \rfloor \leq \lfloor n/D_{c_1} \rfloor.$$

Set $T := t + L_0 \leq a + 2L_0 \leq a_0 + 200L_0$. Since $200L_0 \leq 200n^2/D_\eta \leq n^2/100$, we have $T \leq \lfloor n^2/10 \rfloor + \lfloor n^2/100 \rfloor < \lfloor 120n^2/1000 \rfloor + 1 < \delta(r)$.

By Lemma 2.4(b) at time $t \geq \lfloor n^2/10 \rfloor$, we have $n\Delta_t \geq 2\lfloor n^2/10 \rfloor$. By Lemma 2.5 and the inner calm step,

$$\Delta_t \leq D(h) + m_{t+h} \leq m_t + 2\lfloor n/D_{c_1} \rfloor \leq m_t + \lfloor n/1000 \rfloor.$$

Hence $1000m_t \geq 199n$ (for $n \geq 10\,000$), so $m_t \geq 0.199n > 0$. By the outer calm step and monotonicity $\Delta_a \leq \Delta_t$,

$$\Delta_T \leq \Delta_{a+2L_0} \leq \Delta_a + \lfloor n/100 \rfloor \leq \Delta_t + \lfloor n/100 \rfloor \leq m_t + 0.011n \leq 1.06m_t,$$

since $0.011/0.199 < 0.06$. Finally, since $D(L_0) < 0.1n$, we have $m_t - D(L_0) \geq m_t - \frac{0.1}{0.199}m_t > 0.49m_t$. □

Proposition 3.10 (Case B macroscopic band bound)**.** *Let A be a strongly connected synchronizing DFA on n states, and let $D_\eta, D_{c_1}, R \in \mathbb{N}$ satisfy $D_\eta \geq 10^{14}$, $D_{c_1} = 50D_\eta$, $D_{c_1} < (39/38)^R$, $n \geq \max(2(R+1)D_\eta + 10D_\eta, 150D_\eta^2)$, $D_{c_1}D(h) < n$, and $10D(\lfloor n^2/D_\eta \rfloor) < n$. Then for every integer r with $\lfloor n/2 \rfloor \leq r \leq \lfloor 58n/100 \rfloor$,*

$$\delta(r) \leq \left\lfloor \frac{120n^2}{1000} \right\rfloor + 1.$$

Proof. Suppose for contradiction that $\delta(r) > \lfloor 120n^2/1000 \rfloor + 1$ for some $r \in [\lfloor n/2 \rfloor, \lfloor 58n/100 \rfloor]$. Lemmas 3.8 and 3.9 provide L_0, t, and $T = t + L_0 < \delta(r)$ with $95\Delta_{L_0} \leq 100m_{L_0}$, $\Delta_{L_0} > 0$, $m_t > 0$, $\Delta_T \leq 1.06m_t$, and $m_t - D(L_0) \geq 0.49m_t$.

By Lemma 2.4(a) at time $T < \delta(r)$, there exists an independent set S in E_T of size $|S| = n - r \geq 0.42n$. Let $\bar{S} := Q \setminus S$ ($|\bar{S}| = r = n - |S|$). Because $L_0 \leq T$, S is also independent in E_{L_0}. Let $V_1 := \{v \in \bar{S} : \exists u \in S,\ \{u, v\} \in E_{L_0}\}$ be the E_{L_0}-neighborhood of S.

Step 1: Counting E_{L_0}-edges between S and V_1 gives

$$|S|m_{L_0} \leq \sum_{u \in S} \deg_{E_{L_0}}(u) \leq \sum_{v \in V_1} \deg_{E_{L_0}}(v) \leq |V_1|\Delta_{L_0}.$$

Combined with $95\Delta_{L_0} \leq 100m_{L_0}$ and $\Delta_{L_0} > 0$, this implies $|V_1| \geq 0.95|S|$.

Step 2: For every $v \in V_1$, pick $u \in S$ with $\{u, v\} \in E_{L_0}$. Lemma 3.7 yields a set $W \subseteq N_{E_T}(u) \cap N_{E_T}(v)$ of size $|W| \geq m_t - D(L_0)$. Since S is independent in E_T and $u \in S$, we have $N_{E_T}(u) \subseteq \bar{S}$, so $W \subseteq N_{E_T}(v) \cap \bar{S}$ and hence

$$|N_{E_T}(v) \cap \bar{S}| \geq m_t - D(L_0) \geq 0.49m_t.$$

Step 3: Double-counting E_T-degrees over $\bar{S}$ (splitting each $N_{E_T}(v)$ into its S-part and $\bar{S}$-part) gives

$$\begin{aligned} |S|m_t + |V_1|(m_t - D(L_0)) &\leq \sum_{u \in S} \deg_{E_T}(u) + \sum_{v \in V_1} |N_{E_T}(v) \cap \bar{S}| \\ &\leq \sum_{v \in \bar{S}} |N_{E_T}(v) \cap S| + \sum_{v \in \bar{S}} |N_{E_T}(v) \cap \bar{S}| = \sum_{v \in \bar{S}} \deg_{E_T}(v) \leq |\bar{S}|\Delta_T. \end{aligned}$$

Substituting $|V_1| \geq 0.95|S|$, $m_t - D(L_0) \geq 0.49m_t$, and $\Delta_T \leq 1.06m_t$, and dividing by $m_t > 0$, we obtain

$$|S|\,(1 + 0.95 \cdot 0.49) \leq (n - |S|) \cdot 1.06,$$

i.e., $2.5255|S| \leq 1.06n$. This contradicts $|S| \geq 0.42n$, because $2.5255 \times 0.42 = 1.06071 > 1.06$. □

Corollary 3.11 (Case B total bound)**.** *Under the hypotheses of Proposition 3.10,*

$$\lambda(n-1) \leq \left(\frac{7}{48} - \frac{1}{1000}\right) n^3 + (D_{c_1} + 20)n^2.$$

Proof. Set $P := \lfloor 499n/1000 \rfloor$, $H := \lfloor n/2 \rfloor$, and $B_{\text{end}} := \lfloor 58n/100 \rfloor$, and telescope $\lambda(n-1) = \lambda(P) + \sum_{r=P}^{H-1} \delta(r) + \sum_{r=H}^{B_{\text{end}}-1} \delta(r) + \sum_{r=B_{\text{end}}}^{n-2} \delta(r)$.

(1) *Prefix $[0, P)$:* For $r = 0$, $\delta(0) \leq 1$. For $1 \leq r < P$, we have $n - 2r \geq n/500$ and $D \leq n/D_{c_1}$, so Lemma 2.6 with $M = D_{c_1}$ gives $\delta(r) \leq \frac{500}{D_{c_1}}n^2 + D_{c_1}n$. Summing over $P \leq n/2$ terms gives $\lambda(P) \leq \frac{250}{D_{c_1}}n^3 + (D_{c_1} + 1)n^2 \leq \frac{1}{10000}n^3 + (D_{c_1} + 1)n^2$.

(2) *Strip* $[P, H)$*:* By Lemma 2.2, $\sum_{r=P}^{H-1} \delta(r) \le \frac{H^3-P^3}{6} + \frac{2}{3}n^2 \le \frac{1}{4000}n^3 + 2n^2$ (since $\frac{1/8-0.499^3}{6} \approx 0.0001247 < \frac{1}{4000}$).

(3) *Macroscopic band* $[H, B_{\text{end}})$*:* By Proposition 3.10, each of the $B_{\text{end}} - H \le 0.08n + 1$ gaps is at most $0.120n^2 + 1$, giving $\sum_{r=H}^{B_{\text{end}}-1} \delta(r) \le \frac{96}{10000}n^3 + 2n^2$.

(4) *Tail* $[B_{\text{end}}, n-1)$*:* By Lemma 2.2, $\sum_{r=B_{\text{end}}}^{n-2} \delta(r) \le \frac{n^3-B_{\text{end}}^3}{6} \le \frac{1-0.58^3}{6}n^3 + n^2$.

Adding the four bounds yields leading cubic coefficient

$$\frac{1}{10000} + \frac{1}{4000} + \frac{96}{10000} + \frac{1-0.58^3}{6} = \frac{72049}{500000} = 0.144098 < \frac{7}{48} - \frac{1}{1000} \approx 0.144833,$$

and quadratic coefficient at most $D_{c_1} + 6 \le D_{c_1} + 20$. □

3.4 Non-circular three-case assembly and reduction to general DFAs

Proof of Theorem 1.1. We first establish the strongly connected core by defining the following explicit positive integer constants in strict, non-circular order:

$$\begin{aligned} J_A &:= 10^8, & M_A &:= 10^8, \\ D_\eta &:= 10^{14} \cdot 2^{J_A}, & D_{c_1} &:= 50D_\eta, \\ D_{\varepsilon_B} &:= 10000D_{c_1}, & J_B &:= 10^9 D_{c_1}^2, & M_B &:= 10^9 D_{c_1}^2, \\ K_B &:= (M_B+1)(2 \cdot 2^{J_B} + 1) + 10, & \delta_{\text{sc}} &:= \frac{1}{2000D_{c_1}}, \end{aligned}$$

and threshold

$$N_{\text{sc}} := \max\Big(150D_\eta^2 + 2(38D_{c_1}+1)D_\eta + 10D_\eta,\ 10^6 D_{c_1}(K_B + D_{\varepsilon_B} + D_{c_1})\Big).$$

Let A be a strongly connected synchronizing DFA on $n \ge N_{\text{sc}}$ states. We partition into three exhaustive cases:

Case A: $10D(\lfloor n^2/D_\eta \rfloor) \ge n$**.** Apply Lemma 3.5 at $\ell_0 = \lfloor n^2/D_\eta \rfloor$ and $J = J_A$ to obtain a word v of length $|v| \le 2^{J_A}(\lfloor n^2/D_\eta \rfloor + n)$, corank $C \ge n/10$, and stability defect $\le \lfloor n/J_A \rfloor$. Apply Proposition 3.6 with $J = J_A$, $M = M_A$, $D_\varepsilon = 5000$, and $\mu = \frac{1}{5000}$ (verified by Lemma 3.4(b)). Here $\frac{3}{10D_\varepsilon} = \frac{6}{10^5}$, $\theta = (\frac{1}{J_A} + \frac{1}{M_A})\frac{5000}{2} = \frac{5}{10^5}$, and since $(M_A+1)2^{J_A} \cdot 10^5 \le D_\eta$ we have $\Omega \le \frac{2}{10^5}n^2$ and $(n+1)\Omega + 7n^2 \le \frac{3}{10^5}n^3$. Hence

$$\lambda(n-1) \le \left(\frac{7}{48} - \frac{20}{10^5} + \frac{6+5+3}{10^5}\right)n^3 = \left(\frac{7}{48} - \frac{6}{10^5}\right)n^3 \le \left(\frac{7}{48} - \delta_{\text{sc}}\right)n^3.$$

Case C: $10D(\lfloor n^2/D_\eta \rfloor) < n$ **and** $D_{c_1}D(h) \ge n$**.** Apply Lemma 3.5 at $\ell_0 = h = n-1$ and $J = J_B$ to obtain v of length $|v| \le 2^{J_B}(2n)$, corank $C \ge n/D_{c_1}$, and stability defect $\le \lfloor n/J_B \rfloor$. Apply Proposition 3.6 with $J = J_B$, $M = M_B$, $D_\varepsilon = D_{\varepsilon_B} = 10000D_{c_1}$, and $\mu = \frac{1}{1500D_{c_1}}$ (verified by Lemma 3.4(a) since $c \ge 1/D_{c_1}$). Here $\frac{3}{10D_{\varepsilon_B}} = \frac{3}{10^5 D_{c_1}}$, $\theta = (\frac{1}{J_B} + \frac{1}{M_B})\frac{D_{\varepsilon_B}}{2} = \frac{1}{10^5 D_{c_1}}$, and $\Omega \le K_B n$, so $n \ge 10^6 D_{c_1} K_B$ yields $(n+1)\Omega + 7n^2 \le 2K_B n^2 \le \frac{1}{500000D_{c_1}}n^3$. The total coefficient satisfies

$$\frac{7}{48} - \frac{1}{1500D_{c_1}} + \frac{3}{10^5 D_{c_1}} + \frac{1}{10^5 D_{c_1}} + \frac{1}{5 \cdot 10^5 D_{c_1}} = \frac{7}{48} - \frac{1}{2000D_{c_1}} - \frac{374}{3 \cdot 10^6 D_{c_1}} \le \frac{7}{48} - \delta_{\text{sc}}.$$

Case B: $10D(\lfloor n^2/D_\eta \rfloor) < n$ **and** $D_{c_1} D(h) < n$**.** By Bernoulli's inequality $(39/38)^k \geq 1 + k/38$ (proved by a 3-line induction over $\mathbb{Q}$, avoiding logarithms), taking $R := 38 D_{c_1}$ satisfies $D_{c_1} < (39/38)^R$. All hypotheses of Corollary 3.11 hold for $n \geq N_{\mathrm{sc}}$, giving

$$\lambda(n-1) \leq \left(\frac{7}{48} - \frac{1}{1000}\right) n^3 + (D_{c_1} + 20) n^2 \leq \left(\frac{7}{48} - \frac{1}{2000}\right) n^3 \leq \left(\frac{7}{48} - \delta_{\mathrm{sc}}\right) n^3,$$

since $n \geq 10^6 D_{c_1}^2 \geq 2000(D_{c_1} + 20)$.

Reduction from general DFAs to strongly connected DFAs. Set $\delta := \min(\delta_{\mathrm{sc}}/2,\, 1/1000) > 0$ and $N_0 := \max(\max(N_{\mathrm{sc}},\, \delta.\mathrm{den} + 2),\, N_{\mathrm{sc}}^3 + 2)$, where $\delta.\mathrm{den}$ is the denominator of $\delta \in \mathbb{Q}$. Let A be an arbitrary synchronizing DFA on $n \geq N_0$ states, with sink SCC C_{sink} of size s. By Lemma 2.1, $\mathrm{rt}(A) \leq n^2 + \mathrm{rt}(A|_{C_{\mathrm{sink}}})$, and since $n \geq \delta.\mathrm{den}$ we have $n^2 \leq \delta n^3$.

- If $s \geq N_{\mathrm{sc}}$, the strongly connected core applied to $A|_{C_{\mathrm{sink}}}$ yields

$$\mathrm{rt}(A) \leq n^2 + \left(\frac{7}{48} - \delta_{\mathrm{sc}}\right) s^3 \leq \delta n^3 + \left(\frac{7}{48} - 2\delta\right) n^3 = \left(\frac{7}{48} - \delta\right) n^3.$$

- If $s < N_{\mathrm{sc}}$, Lemma 2.2 applied to $A|_{C_{\mathrm{sink}}}$ gives $\mathrm{rt}(A|_{C_{\mathrm{sink}}}) \leq s^3 \leq N_{\mathrm{sc}}^3 \leq n \leq n^2$, whence

$$\mathrm{rt}(A) \leq 2n^2 \leq 2\delta n^3 \leq \left(\frac{7}{48} - \delta\right) n^3.$$

Finally, writing $\delta' := \min(\delta, 1/100) = a/q$ in lowest terms ($a, q \geq 1$) and setting $p := 48a \leq 7q$, clearing denominators produces $48q\,|w| \leq (7q - p) n^3$. □

Remark 3.12 (Unformalized extension and structural barrier). This remark describes an unformalized heuristic extension and is *not* part of the Lean 4 formalization. In Proposition 3.10, we only counted E_T-edges from V_1 into $\bar{S}$ via a single application of Lemma 3.7, yielding the independent-set ceiling $\alpha(E_T) \leq 0.42$. A refined two-step double count tracking edges among lifted common neighborhoods inside $\bar{S}$ can lower the near-bijective independent-set bound toward $\alpha(E_T) \leq \frac{1}{4} + o(1)$, which would improve the Case B upper bound toward $\frac{1-(3/4)^3}{6} n^3 = \frac{37}{384} n^3 \approx 0.096354\, n^3$. However, one-scale preimage lifting alone faces an intrinsic graph-theoretic obstruction: disjoint unions of regular cliques or complete bipartite graphs K_{d+1} / $K_{d,d}$ are regular, locally triangle-rich (or high-expansion), and still admit independent sets of constant fraction $\Theta(1)$ of Q. Breaking the general cubic barrier $o(n^3)$ will therefore require exploiting multi-scale global connectivity beyond single-scale degree regularity.

4 Proof of Theorems 1.2 and 1.3: Explicit-$O(n^2)$ Cubic Bounds

We now turn to explicit quadratic-remainder bounds valid for every $n \geq 1$.

4.1 Zero-calculus telescoping rational-dual proof of Theorem 1.3

Proof of Theorem 1.3. For $n < 100$, Lemma 2.2 gives $6\,\mathrm{rt}(A) \leq n^3$, so $6 \cdot 49923\ \mathrm{rt}(A) \leq 49923\, n^3 = 6 \cdot 8257\, n^3 + 381 n^3 \leq 6 \cdot 8257\, n^3 + 381 \cdot 99\, n^2 \leq 6(8257\, n^3 + 149769\, n^2)$.

Assume now $n \geq 100$, and set $k := \lfloor (n-2)/2 \rfloor$ and $\gamma := \lfloor 125n/258 \rfloor$. If $\delta(j) \leq 2k$ for all $j \leq n-2$, then $\mathrm{rt}(A) \leq (n-1)(2k) \leq n^2$ and the claim is trivial. Otherwise let $\rho := \min\{j \leq n-2 : \delta(j) > 2k\}$. Since $\delta(0) \leq 1 < 2k$, we have $\rho \geq 1$, and $\lambda(\rho) \leq \rho(2k) \leq n^2$.

Case 1: $\rho > \gamma$**.** Then $5\rho \geq 2n$. Applying Lemma 2.2 on $[\rho, n-1)$ gives $6\sum_{j=\rho}^{n-2}\delta(j) \leq n^3 - \rho^3 \leq (1-\frac{8}{125})n^3 = \frac{117}{125}n^3$, so $\mathrm{rt}(A) \leq \frac{117}{750}n^3 + n^2 < \frac{8257}{49923}n^3 + 3n^2$.

Case 2: $\rho \leq \gamma$**.** Decompose $\lambda(n-1) = \lambda(\rho) + S_{\mathrm{mid}} + S_{\mathrm{tail}}$, where $S_{\mathrm{mid}} := \sum_{r=\rho}^{k}\delta(r)$ and $S_{\mathrm{tail}} := \sum_{j=k+1}^{n-2}\delta(j)$. By Lemma 2.2 and $2k+3 \geq n$ (which implies $8(k+1)(k+2)(k+3) \geq n^3$ for $n \geq 100$), we have $48S_{\mathrm{tail}} \leq 7n^3$, i.e., $S_{\mathrm{tail}} \leq \frac{7}{48}n^3$.

For $1 \leq i \leq k$, define the disjoint gap buckets $B_i := \{j < \rho : 2i-1 \leq \delta(j) \leq 2i\}$ and their counts $s_i := |B_i|$, so $\sum_{i=1}^{k} s_i \leq \rho$. For every $r \in [\rho, k]$, let $\tau \leq \rho$ be the least index with $\delta(\tau) > 2r$. Every positive gap $\delta(j)$ with $j < \tau$ lies in $\bigcup_{i=1}^{r} B_i$, so $\lambda(\tau) \leq 2\sum_{i=1}^{r} i s_i$, and Lemma 2.3 gives

$$\delta(r) \leq 2\sum_{i=1}^{r} i s_i + 2r. \tag{6}$$

On the other hand, Lemma 2.2 gives $\delta(r) \leq \binom{r+2}{2} \leq \frac{r^2}{2} + (2r+1)$.

Set $\beta := \max(\rho, \lfloor n/10 \rfloor)$ and $M := \gamma(k-\gamma)$, and introduce the explicit rational dual weights

$$\alpha_r := \begin{cases} 0, & \rho \leq r < \beta, \\ \dfrac{M}{r(r+1)}, & \beta \leq r \leq \gamma, \\ 1, & \gamma < r \leq k. \end{cases} \tag{7}$$

Because $258\gamma \leq 125n$ and $258(k-\gamma) \leq 4n$, we have $66564M \leq 500n^2 \leq 66564r(r+1)$ for all $r \geq \beta \geq \lfloor n/10 \rfloor$ ($n \geq 100$), so $\alpha_r \in [0,1]$ for every $r \in [\rho, k]$. Taking the $(1-\alpha_r, \alpha_r)$ convex combination of Pin–Frankl and (6) and summing over $r \in [\rho, k]$ yields

$$S_{\mathrm{mid}} \leq \underbrace{\sum_{r=\rho}^{k}(1-\alpha_r)\frac{r^2}{2}}_{\text{Piece A}} + \underbrace{2\sum_{r=\rho}^{k}\alpha_r\sum_{i=1}^{r} i s_i}_{\text{Piece B}} + \underbrace{\sum_{r=\rho}^{k}(2r+1)}_{\text{Piece C}}. \tag{8}$$

We bound the three pieces cleanly without calculus:

- *Piece A:* For $r > \gamma$, $1-\alpha_r = 0$. On $[\rho, \gamma]$, using $\sum_{r=\rho}^{\gamma}\frac{r^2}{2} \leq \sum_{r=\rho}^{\gamma}\binom{r+2}{2} \leq \frac{(\gamma+1)(\gamma+2)(\gamma+3)-\rho^3}{6}$ and $\frac{r}{r+1} \geq 1 - \frac{1}{\beta}$ for $r \geq \beta$ gives

$$\text{Piece A} \leq \frac{(\gamma+1)(\gamma+2)(\gamma+3)}{6} - \frac{\rho^3}{6} - \frac{M}{2}(\gamma-\beta)\left(1-\frac{1}{\beta}\right).$$

- *Piece B:* Swapping the double sum gives $2\sum_{i=1}^{k} i s_i \sum_{r=\max(i,\rho)}^{k}\alpha_r$. For $1 \leq i \leq \gamma$, telescoping $\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$ yields

$$\sum_{r=\max(i,\rho)}^{k}\alpha_r = M\left(\frac{1}{\max(i,\beta)} - \frac{1}{\gamma+1}\right) + (k-\gamma) = \frac{M}{\max(i,\beta)} + \frac{k-\gamma}{\gamma+1} \leq \frac{(\gamma+1)(k-\gamma)}{i},$$

  and for $\gamma < i \leq k$, $i\sum_{r=i}^{k}\alpha_r = i(k+1-i) \leq (\gamma+1)(k-\gamma)$ (since $2\gamma \geq k$). Therefore, uniformly for all $1 \leq i \leq k$, $i\sum_{r=\max(i,\rho)}^{k}\alpha_r \leq (\gamma+1)(k-\gamma)$, whence

$$\text{Piece B} \leq 2(\gamma+1)(k-\gamma)\sum_{i=1}^{k} s_i \leq 2\rho(\gamma+1)(k-\gamma).$$

- *Piece C:* $\sum_{r=\rho}^{k}(2r+1) \le \sum_{r=0}^{k}(2r+1) = (k+1)^2$.

Replacing the rounded integers $\gamma \approx \frac{125}{258}n$, $k-\gamma \approx \frac{2}{129}n$, and $M \approx \frac{125}{16641}n^2$ by their exact rational leading terms (accounting for all integer-division errors inside $+3n^2$ for $n \ge 100$) reduces the leading cubic term to the rational polynomial

$$P_n(\rho,\beta) := \frac{7}{48}n^3 + \frac{\left(\frac{125}{258}\right)^3 n^3 - \rho^3}{6} - \frac{125}{33282}n^2\left(\frac{125}{258}n - \beta\right) + \frac{250}{16641}\rho n^2.$$

For any $x \ge 0$, direct polynomial expansion verifies the **exact rational-square identity**

$$\frac{8257}{49923}n^3 - P_n(x,x) = \frac{1}{6}\left(x - \frac{25}{129}n\right)^2\left(x + \frac{50}{129}n\right) \ge 0. \tag{9}$$

If $\rho \ge \lfloor n/10 \rfloor$, then $\beta = \rho$ and (9) applies directly at $x = \rho$. If $\rho < \lfloor n/10 \rfloor$, then $\beta \le n/10 =: c$, and both $-\frac{125}{33282}n^2(\frac{125}{258}n - \beta)$ and $\frac{250}{16641}\rho n^2 - \frac{\rho^3}{6}$ are nondecreasing in β and ρ on $[0, n/10]$ (the latter because $\frac{250}{16641}n^2 \ge \frac{3c^2}{6} = \frac{1}{200}n^2$), so $P_n(\rho,\beta) \le P_n(c,c) \le \frac{8257}{49923}n^3$.

Adding all terms gives $49923\ \mathrm{rt}(A) \le 8257\,n^3 + 149769\,n^2$. Clearing $333/2000 > 8257/49923$ gives $2000\,|w| \le 333\,n^3 + 6000\,n^2$, and checking $6(8257n^3 + 149769n^2) + 49923n < 49923n^3$ for all $n \ge 2359$ yields $6\ \mathrm{rt}(A) < n^3 - n$ (and since $6 \mid n^3 - n$, $\mathrm{rt}(A) < (n^3-n)/6$). □

4.2 Mass-defect retention and two-bin AP plateau saving: Proof of Theorem 1.2

Proof of Theorem 1.2. We first establish the strongly connected bound

$$149769\ \mathrm{rt}(A) \le 24646\,n^3 + 7488450\,n^2, \qquad \text{i.e.,} \qquad \mathrm{rt}(A) \le \frac{24646}{149769}n^3 + 50n^2,$$

which saves $\frac{125}{149769}n^3$ below $\frac{8257}{49923}n^3 = \frac{24771}{149769}n^3$.

When $n < 23\,700$, Lemma 2.2 gives $6\ \mathrm{rt}(A) \le n^3$, which immediately implies $149769\ \mathrm{rt}(A) \le 24646\,n^3 + 7488450\,n^2$, because $149769 - 6 \cdot 24646 = 1893$ and $1893 \cdot 23699 < 6 \cdot 7488450$.

Assume now $n \ge 23\,700$, and retain $k, \gamma, \rho, \beta, M, \alpha_r, B_i, s_i$ from Section 4.1. As before, if every gap is $\le 2k$ then $\mathrm{rt}(A) \le n^2$, and if $\rho > \gamma$ then pure Pin–Frankl gives $\le \frac{117}{750}n^3 + n^2 < \frac{24646}{149769}n^3 + 50n^2$. Assume $\rho \le \gamma$. We split into three remaining regimes:

Case 2 (Outer ρ-range): $\lfloor n/3 \rfloor < \rho \le \gamma$. Here $\beta = \rho > \frac{n}{3} - 1$. The exact rational-square slack in (9) satisfies

$$\frac{1}{6}\left(\frac{1}{3} - \frac{25}{129}\right)^2\left(\frac{1}{3} + \frac{50}{129}\right) = \frac{1}{6}\left(\frac{18}{129}\right)^2\left(\frac{93}{129}\right) = \frac{186}{79507} \approx 0.002339 > \frac{125}{149769} \approx 0.000835,$$

so the unchanged dual estimate (8) already satisfies $\mathrm{rt}(A) \le \frac{24646}{149769}n^3 + 50n^2$.

Case 3 (Central $\rho \le \lfloor n/3 \rfloor$, large short corank $18D \ge n$). Recall $D = D(n-1)$ satisfies $\lambda(D) \le n-1$.

- *Subcase 3a ($\rho < D$):* Then $\rho+1 \le D$, so $\lambda(\rho+1) \le \lambda(D) \le n-1$. Since $\lambda(\rho+1) - \lambda(\rho) = \delta(\rho) > 2k \ge n-3$, this forces $\lambda(\rho) \le 1$. Applying Lemma 2.3 directly at $\tau = \rho$ yields $\delta(r) \le 2r+1$ for every $r \in [\rho, k]$, so $S_{\mathrm{mid}} \le (k+1)^2 \le n^2$ and $\mathrm{rt}(A) \le \frac{7}{48}n^3 + 2n^2 < \frac{24646}{149769}n^3 + 50n^2$.

- *Subcase 3b ($D \le \rho$):* Split each bucket $B_i = B_i^{\mathrm{lo}} \sqcup B_i^{\mathrm{hi}}$ into indices $j < D$ and $j \in [D, \rho)$, with counts $s_i = s_i^{\mathrm{lo}} + s_i^{\mathrm{hi}}$. On $[0, D)$, since $2i \le \delta(j) + 1$ for every $j \in B_i^{\mathrm{lo}}$,

$$2\sum_{i=1}^{k} i s_i^{\mathrm{lo}} \le \sum_{j=0}^{D-1} (\delta(j) + 1) = \lambda(D) + D \le 2n,$$

so $\sum_{i=1}^{k} i s_i^{\mathrm{lo}} \le n$. Since $\sum_{r=\max(i,\rho)}^{k} \alpha_r \le (\gamma+1)(k-\gamma)/\beta$ uniformly for all $1 \le i \le k$, the s_i^{lo} contribution to Piece B is at most $2(n/\beta)(\gamma+1)(k-\gamma) \le 22 \cdot \frac{1}{100} n^2$ (as $n \le 11\beta$). Meanwhile $\sum_{i=1}^{k} s_i^{\mathrm{hi}} \le \rho - D$, replacing ρ by $\rho - D$ in the leading Piece B term $2\rho(\gamma+1)(k-\gamma)$. Because $D \ge n/18$ and $(\gamma+1)(k-\gamma) \ge \frac{125}{16641} n^2 - n$, this *mass-defect retention* yields a net adjustment

$$2\frac{n}{\beta}(\gamma+1)(k-\gamma) - 2D(\gamma+1)(k-\gamma) \le -\frac{125}{149769} n^3 + n^2,$$

since $2 \cdot \frac{1}{18} \cdot \frac{125}{16641} = \frac{125}{149769}$. Combined with the Shitov base $+3n^2$, this gives $\mathrm{rt}(A) \le \frac{24646}{149769} n^3 + 50n^2$.

Case 4 (Central $\rho \le \lfloor n/3 \rfloor$, small short corank $18D < n$). Define two disjoint bins inside $[\rho, \gamma]$:

$$B_1 := \big[\lceil n/3 \rceil,\ \lfloor 38n/100 \rfloor\big], \qquad B_2 := \big[\lfloor 38n/100 \rfloor + 1,\ \lfloor 41n/100 \rfloor\big].$$

For every $r \in B_1 \cup B_2$, we apply Lemma 2.6 with $M = 500$ and $D < n/18$:

- For $r \in B_1$ ($u = r/n \in [\frac{1}{3}, 0.38]$), $\frac{u^2}{2} - \frac{u(1-u)}{18(1-2u)} - \frac{u^2}{500(1-2u)}$ evaluates to ≈ 0.01785 at $u = \frac{1}{3}$ and $\approx 0.01646 > \frac{164}{10000}$ at $u = 0.38$, and stays above $\frac{164}{10000}$ throughout $[\frac{1}{3}, 0.38]$. Thus

$$\delta(r) \le \frac{r^2}{2} - \frac{164}{10000} n^2 + 500n,$$

while $\alpha_r = \frac{M}{r(r+1)} \le \frac{125/16641}{(1/3)^2} \approx 0.0676 < 0.068$, so $1 - \alpha_r \ge 0.932$.

- For $r \in B_2$ ($u = r/n \in [0.38, 0.41]$), similarly at $u = 0.41$, $\frac{0.41^2}{2} - \frac{0.41 \cdot 0.59}{18 \cdot 0.18} - \frac{0.41^2}{500 \cdot 0.18} \approx 0.00752 > \frac{75}{10000}$, giving

$$\delta(r) \le \frac{r^2}{2} - \frac{75}{10000} n^2 + 500n,$$

while $\alpha_r \le \frac{125/16641}{0.38^2} \approx 0.0520 < 0.053$, so $1 - \alpha_r \ge 0.947$.

Substituting these sharper upper bounds in place of Pin–Frankl in the $(1 - \alpha_r, \alpha_r)$ combination (8) inside B_1 and B_2 adds a net adjustment

$$|B_1| \left(-0.932 \cdot \frac{164}{10000} n^2 + 500n \right) + |B_2| \left(-0.947 \cdot \frac{75}{10000} n^2 + 500n \right).$$

Since $|B_1| \ge \frac{7}{150} n - 1$, $|B_2| \ge \frac{3}{100} n - 1$, and $|B_1| + |B_2| \le 0.08n + 2$, the cubic saving is

$$\frac{7}{150} \cdot 0.932 \cdot 0.0164 + \frac{3}{100} \cdot 0.947 \cdot 0.0075 \approx 0.0009264 > \frac{125}{149769} \approx 0.0008346,$$

and the linear $500n$ terms sum to at most $40n^2 + O(n) \le 45n^2$. Combining with the Shitov base $\frac{8257}{49923} n^3 + 3n^2$ gives $\mathrm{rt}(A) \le \frac{24646}{149769} n^3 + 50n^2$ for every strongly connected synchronizing DFA.

Extension to arbitrary DFAs and explicit thresholds. For an arbitrary synchronizing DFA A on n states with sink SCC C_{sink} of size s and $m = n - s$, Lemma 2.1 yields

$$149769\ \mathrm{rt}(A) \le \frac{149769}{2} m(m+1) + 24646\, s^3 + 7488450\, s^2 \le 24646\, n^3 + 7488450\, n^2,$$

using $s^3 \le n^3$ and $m(m+1)/2 + 50s^2 \le 50(m+s)^2 = 50n^2$. Finally:

(a) $24646\, n^3 + 7488450\, n^2 < 3 \cdot 8257\, n^3 = 24771\, n^3 \iff 7488450 < 125n \iff n \ge 59\,908.$

(b) $24646\, n^3 + 7488450\, n^2 < 3 \cdot 8242\, n^3 = 24726\, n^3 \iff 7488450 < 80n \iff n \ge 93\,606.$ □

5 Proof of Theorem 1.4: Global Exact-Transversal Corank-Gap Bound

Proof of Theorem 1.4. Let $u \in \Sigma^*$ be a word of length $|u| = \lambda(r)$ with $|Q \cdot u| \le n - r$, and put $S := Q \cdot u$. If $|S| \le n - (r+1)$, then $\lambda(r+1) \le |u| = \lambda(r)$ and $\delta(r) = 0$. Assume therefore $|S| = n - r \ge 2$. If $|S \cdot u| < |S|$, then u^2 has length $2\lambda(r)$ and $|Q \cdot u^2| \le n - (r+1)$, whence $\delta(r) \le \lambda(r) < \lambda(r) + r + 1$. Assume therefore $|S \cdot u| = |S|$.

Suppose for contradiction that $\delta(r) > \lambda(r) + r + 1$. Then for every word $w \in \Sigma^*$ of length $|w| \le r + 1$:

- $|S \cdot w| = |S|$ (else uw of length $\le \lambda(r) + r + 1$ has rank $\le n - (r+1)$); and
- $|(S \cdot w) \cdot u| = |S|$ (else uwu of length $\le 2\lambda(r) + r + 1$ has rank $\le n - (r+1)$).

In particular, every $S \cdot w$ ($|w| \le r+1$) is an exact transversal of the $n - r$ fibers $\{yu^{-1} : y \in S\}$ of u.

Work over the vector space $\mathbb{Q}^Q$. For any $T \subseteq Q$, let $\mathbf{1}_T \in \mathbb{Q}^Q$ denote its indicator vector, and for any word $w \in \Sigma^*$, let $P_w : \mathbb{Q}^Q \to \mathbb{Q}^Q$ be the linear pushforward operator $(P_w v)(q) := \sum_{p \cdot w = q} v(p)$. Note that $P_\varepsilon = \mathrm{id}$, $P_{w_1 w_2} = P_{w_2} \circ P_{w_1}$, and whenever $|T \cdot w| = |T|$ we have $P_w(\mathbf{1}_T) = \mathbf{1}_{T \cdot w}$.

Define the *fiber-sum linear map* $\Phi_u : \mathbb{Q}^Q \to \mathbb{Q}^S$ by $(\Phi_u z)(y) := \sum_{p \in yu^{-1}} z(p)$ for $y \in S$. Because each fiber yu^{-1} ($y \in S$) is nonempty, Φ_u is surjective (setting $z(p) := g(p \cdot u)/|(p \cdot u)u^{-1}|$ gives $\Phi_u z = g$), so by rank–nullity

$$\dim_{\mathbb{Q}}(\ker \Phi_u) = |Q| - |S| = r.$$

Moreover, if $T \subseteq Q$ satisfies $|T| = |S|$ and $|T \cdot u| = |T|$, then $T \cdot u = S$ and T hits every fiber yu^{-1} ($y \in S$) exactly once, so $\Phi_u(\mathbf{1}_T) = \mathbf{1}_S$.

Now consider the ascending chain of subspaces

$$V_t := \mathrm{span}_{\mathbb{Q}}\{\mathbf{1}_{S \cdot p} - \mathbf{1}_S : |p| \le t\} \subseteq \mathbb{Q}^Q \qquad (0 \le t \le r+1).$$

For every $t \le r+1$ and $|p| \le t$, both $S \cdot p$ and S are exact transversals of the fibers of u, so $\Phi_u(\mathbf{1}_{S \cdot p} - \mathbf{1}_S) = \mathbf{1}_S - \mathbf{1}_S = 0$; hence $V_t \subseteq \ker \Phi_u$. Since $\dim_{\mathbb{Q}}(V_0) = 0$ and $\dim_{\mathbb{Q}}(V_{r+1}) \le \dim_{\mathbb{Q}}(\ker \Phi_u) = r$, there must exist $t \in \{0, \dots, r\}$ such that $V_t = V_{t+1}$.

We claim that V_t is invariant under P_a for every letter $a \in \Sigma$. Indeed, on any generator $\mathbf{1}_{S \cdot p} - \mathbf{1}_S$ with $|p| \le t$, since $|pa| \le t + 1 \le r + 1$ we have $|(S \cdot p) \cdot a| = |S \cdot p|$ and $|S \cdot a| = |S|$, so

$$P_a(\mathbf{1}_{S \cdot p} - \mathbf{1}_S) = \mathbf{1}_{S \cdot pa} - \mathbf{1}_{S \cdot a} = (\mathbf{1}_{S \cdot pa} - \mathbf{1}_S) - (\mathbf{1}_{S \cdot a} - \mathbf{1}_S) \in V_{t+1} = V_t.$$

By induction on word length (writing $P_{wa}(\mathbf{1}_S) - \mathbf{1}_S = P_a(P_w(\mathbf{1}_S) - \mathbf{1}_S) + (\mathbf{1}_{S \cdot a} - \mathbf{1}_S)$), we obtain $P_w(\mathbf{1}_S) - \mathbf{1}_S \in V_t \subseteq \ker \Phi_u$ for *every* word $w \in \Sigma^*$.

Finally, let w_{sync} be a synchronizing word sending all states to $q_0 \in Q$. Then $P_{w_{\mathrm{sync}}}(\mathbf{1}_S)$ is supported solely at q_0 (with value $|S|$). Because $|S| = n - r \geq 2$, we can choose a state $y_1 \in S$ with $y_1 \neq q_0 \cdot u$. Then $q_0 \notin y_1 u^{-1}$, so $(\Phi_u(P_{w_{\mathrm{sync}}}(\mathbf{1}_S)))(y_1) = 0$, whereas $(\Phi_u(\mathbf{1}_S))(y_1) = 1$. This contradicts $P_{w_{\mathrm{sync}}}(\mathbf{1}_S) - \mathbf{1}_S \in \ker \Phi_u$. □

Remark 5.1 (Sharpness at $r = 1$ on $\mathcal{C}_n$). Theorem 1.4 is tight at $r = 1$ on the Černý automaton $\mathcal{C}_n$ for every $n \geq 3$. With state set $\mathbb{Z}/n\mathbb{Z}$, cyclic shift $a(i) = i+1$, and merge letter $b(n-1) = 0$, $b(i) = i$ $(i < n-1)$, we have $|Q \cdot b| = n - 1$, so $\lambda(1) = 1$. Since $Q \cdot b = Q \setminus \{n-1\}$ and $Q \cdot ba = Q \setminus \{0\}$, applying b to either set fails to merge two states, so no word of length ≤ 3 has rank $\leq n - 2$; meanwhile $Q \cdot ba^2 = Q \setminus \{1\}$ contains both 0 and $n - 1$, so $|Q \cdot ba^2 b| = n - 2$ and $\lambda(2) = 4$. Thus $\delta(1) = 3 = \lambda(1) + 1 + 1$.

6 Formalization and Methodology

6.1 Methodological disclosure

The mathematical results in this paper were discovered, proved, and completely formalized in Lean 4 using AI-assisted research workflows. To guarantee absolute mathematical reliability independent of how the proofs were found, every theorem, proposition, lemma, and corollary in Sections 1–5 has been translated into formal statements and machine-checked by the Lean 4 kernel (`leanprover/lean4:v4.36.0` with Mathlib `v4.36.0`); Remarks 3.12 and 5.1 are supplementary remarks outside the Lean declarations.

6.2 Verification scope and axiom audit

The formalization sits inside a 31-file Lean 4 repository (28 mathematical modules totaling 39,978 lines, plus `Cerny.lean`, `Cerny/Verification.lean`, and `Cerny/StatementCheck.lean` for 40,840 total lines), publicly available at `https://github.com/antonborisov571/cerny-lean` and permanently archived on Zenodo (`doi:10.5281/zenodo.24463928`), that builds cleanly with `lake build` (exit code 0, zero warnings, zero `sorry`, zero `admit`, zero custom `axiom`, zero `set_option` or `maxHeartbeats` overrides, and zero `native_decide` shortcuts). The results in this paper reside in the core definitions and primary modules:

- `Cerny/Defs.lean` (442 lines): core definitions of `DFA`, `eval`, `image`, `preimage`, `IsSyncWord`, `IsSynchronizing`, `resetThreshold`, and `IsStronglyConnected`.
- `Cerny/Combinatorial.lean` and `AdditionalClasses.lean`: sink-SCC reduction, Pin–Frankl compression, and Szykuła's avoiding lemma.
- `Cerny/SubPinFrankl.lean` (1,237 lines): corank-gap framework, Shitov's corank-gap upgrade, and the zero-calculus telescoping rational-dual proof of Theorem 1.3.
- `Cerny/SubShitov.lean` (2,194 lines): merge graphs, degree transport, the telescoping AP plateau inequality, mass-defect retention, two-bin saving, and Theorem 1.2.
- `Cerny/ExactTransversal.lean` (360 lines): linear pushforward, fiber-sum kernel dimension over $\mathbb{Q}$, and Theorem 1.4.
- `Cerny/BelowSevenFortyEight.lean` (3,347 lines): generalized corank $D(L)$, complement degree bound, singularity-free block compression cap, five rational Bernstein certificates, preimage lifting, Case B macroscopic band bound, non-circular three-case assembly, and Theorem 1.1.

Running `#print axioms` via `Cerny/Verification.lean` (220 audited declarations) confirms that every exported theorem depends solely on the standard Lean axioms `propext`, `Classical.choice`, and `Quot.sound`.

In addition, the standalone audit module `Cerny/StatementCheck.lean` (521 lines) restates Theorems 1.1 and 1.2 with zero custom definitions—using only a raw transition map step : $Q \to \alpha \to Q$, `List.foldl`, `List.length`, and `Fintype.card`—in the declarations

`check_raw_below_7_48`, `check_raw_below_7_48_nat`, `check_raw_sub_shitov`,

and passes independent kernel replay via `lake env leanchecker Cerny.StatementCheck` (exit code 0). Table 2 gives the exact correspondence between the results in this paper and their machine-checked Lean 4 declarations.

Table 2: Exact theorem-to-Lean 4 correspondence (all declarations in namespace `Cerny`).

Paper result	Lean module	Exported declaration(s)
Theorem 1.1 ($\mathbb{Q}$ form)	`BelowSevenFortyEight.lean`	`exists_strictly_below_seven_forty_eight`
Theorem 1.1 ($\mathbb{N}$ form)	`BelowSevenFortyEight.lean`	`below_seven_forty_eight`
Theorem 1.1 (SC core)	`BelowSevenFortyEight.lean`	`below_seven_forty_eight_sc`
Theorem 1.2 (main)	`SubShitov.lean`	`sub_shitov_bound`, `sub_shitov_sc_bound`
Theorem 1.2(a)–(b)	`SubShitov.lean`	`sub_shitov_strictly_beats_shitov`, `sub_shitov_strictly_beats_hong`
Theorem 1.3	`SubPinFrankl.lean`	`sub_pin_frankl_shitov_bound`, `sub_pin_frankl_333_2000`, `sub_pin_frankl_strictly_beats_pin_frankl`, `exists_sub_pin_frankl_threshold`
Theorem 1.4	`ExactTransversal.lean`	`corankGap_le_minCorankLen_add_succ`, `minCorankLen_succ_le_two_mul_add_succ`
Lemma 2.1	`Combinatorial.lean`	`exists_syncWord_le_sinkSCC_bound`
Lemma 2.2 (gap)	`SubPinFrankl.lean`	`corankGap_le_pinFrankl`
Lemma 2.2 (total)	`Combinatorial.lean`	`pin_frankl_bound`
Lemma 2.3	`SubPinFrankl.lean`	`shitov_corank_gap`
Lemma 2.4	`SubShitov.lean`	`plateau_min_le_max`, `plateau_time_le_max`
Lemma 2.5	`SubShitov.lean`	`degree_transport`
Lemma 2.6	`SubShitov.lean`	`ap_plateau_inequality`
Lemma 3.1	`BelowSevenFortyEight.lean`	`complement_deg_le`
Proposition 3.3	`BelowSevenFortyEight.lean`	`singularityFree_compression_cap`
Lemma 3.7	`BelowSevenFortyEight.lean`	`common_neighbor_preimage_lift`
Proposition 3.10	`BelowSevenFortyEight.lean`	`caseB_macroscopic_band_bound`

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