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CLOSED FORMULAS FOR MASUR-VEECH POLYNOMIALS

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Abstract. Andersen, Borot, Charbonnier, Delecroix, Giacchetto, Lewański and Wheeler attached to every (g,n) a polynomial $V\Omega_{g,n}^{\mathrm{MV}}(L_1,\dots,L_n)$ in boundary lengths. Its constant term is, up to an explicit normalising factor, the Masur–Veech volume of the principal stratum of quadratic differentials with n simple poles. We study these polynomials summed over any number of boundaries of length zero, with a weight X per such boundary, and put $\kappa = 1-\sqrt{1-X}$.

In genus zero we determine every coefficient in closed form. Equivalently, genus-zero Masur–Veech theory is genus-zero Witten–Kontsevich theory with the Zhukovsky map $\eta+\kappa/\eta$ replaced by $\eta+\kappa\cot\eta$, and with string equation $2\kappa-\kappa^2=X$. The one- and two-point functions live on the curve $Q=y+\kappa\cot y$ on the cylinder $\mathbb{C}/\pi\mathbb{Z}$. In genus one we determine all coefficients, for all boundary lengths, through a closed formula for the free energy.

Our main result concerns all genera. Equip the genus-zero curve with $\omega_{0,1}=-4\kappa^2\sin^{-2}y\,\mathrm{d}Q$ and $\omega_{0,2}=\big(\sin^{-2}(y_1-y_2)+\sin^{-2}(y_1+y_2)\big)\mathrm{d}y_1\mathrm{d}y_2$. Then topological recursion on it computes all Masur–Veech polynomials of every genus with at least one boundary. In particular each generating series $\sum_N h_{g,n+N}[d,0^N]X^N/N!$ is a polynomial in $\sqrt{1-X}$ and its inverse.

Consequences include a new proof of the genus-zero volume formula of Athreya, Eskin and Zorich, and a proof of the polynomiality conjecture of Andersen et al.

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1. Introduction

1.1. **Masur–Veech polynomials.** Let $\mathrm{MV}_{g,n}$ denote the Masur–Veech volume of the principal stratum of the moduli space of meromorphic quadratic differentials with n simple poles on curves of genus g, normalised as in [DGZZ21, ABC$^+$23]. Delecroix, Goujard, Zograf and Zorich

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[DGZZ21] expressed $\mathrm{MV}_{g,n}$ as a sum over stable graphs of genus g with n legs, with Kontsevich volumes at the vertices and ζ-values on the edges. Andersen, Borot, Charbonnier, Delecroix, Giacchetto, Lewański and Wheeler [ABC+23] refined this to polynomials in boundary lengths,

$$V\Omega^{\mathrm{MV}}_{g,n}(L_1,\dots,L_n) = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}\,\Gamma|} \int_{\mathbb{R}_+^{E_\Gamma}} \prod_{v\in V_\Gamma} V^{\mathrm{K}}_{h(v),k(v)}\big((\ell_e)_{e\ni v},(L_\lambda)_{\lambda\ni v}\big) \prod_{e\in E_\Gamma} \frac{\ell_e\,\mathrm{d}\ell_e}{\mathrm{e}^{\ell_e}-1}, \tag{1.1}$$

where $V^{\mathrm{K}}_{h,k}(L_1,\dots,L_k) = \int_{\overline{\mathcal{M}}_{h,k}} \exp\big(\sum_i L_i^2\psi_i/2\big)$ are Kontsevich's volume polynomials [Kon92]. They proved that these polynomials satisfy topological recursion [EO07] on the curve $x = z^2/2$, $y = -z$, with a bidifferential $\omega_{0,2}$ twisted by ζ-values. They also showed [ABC+23, (1.2)]

$$\mathrm{MV}_{g,n} = \frac{2^{4g-2+n}\,(4g-4+n)!}{(6g-7+2n)!}\, V\Omega^{\mathrm{MV}}_{g,n}(0,\dots,0). \tag{1.2}$$

For $(g,n)=(0,3)$ the factorials are not defined, and we use (1.2) only for $(g,n)\neq(0,3)$. Write $V\Omega^{\mathrm{MV}}_{g,n}(L) = \sum_d F_{g,n}[d_1,\dots,d_n]\prod_i L_i^{2d_i}/(2d_i+1)!$ and $\mathfrak{d}_{g,n} := 3g-3+n$.

In genus zero, the *one-row* coefficients $H_n[d] := F_{0,n}[d,0,\dots,0]$ are determined by a quadratic recursion [ABC+23, Cor. 5.1], and $\mathrm{MV}_{0,n} = \frac{2^{n-2}(n-4)!}{(2n-7)!}H_n[0]$. The authors write that they have not been able to solve this recursion, even using generating series. They note that a closed formula for $H_n[d]$ would give, by uniqueness, a new proof of the genus-zero volume formula

$$\mathrm{MV}_{0,n} = \pi^{2(n-3)}\,2^{5-n} \qquad (n\geq 4) \tag{1.3}$$

of Athreya, Eskin and Zorich [AEZ16], and they conjecture the shape of $H_n[d]$ for fixed d [ABC+23, Conj. 5.3].

In all genera, the volumes themselves (all boundary lengths zero) have been studied through intersection theory and generating series [CMS23, Kaz22, YZZ20]. Yang, Zagier and Zhang [YZZ20] obtained closed formulas: for $g\geq 2$ the genus-g volume series is a polynomial in $(1-X)^{-1/2}$ [YZZ20, Lemma 1]. Delecroix, Goujard, Zograf and Zorich asked for a direct derivation of (1.3) from the stable-graph formula [DGZZ21, arXiv version, Rem. 1.8]; Corollary 1.1 below gives one.

In this paper we study all coefficients, in all genera, summed over boundaries of length zero.

1.2. **Normalisation and generating series.** The powers of π can be removed (Lemma 2.1): $h_{g,n}[d] := \pi^{-2(\mathfrak{d}_{g,n}-|d|)}F_{g,n}[d]\in\mathbb{Q}$, with $h_{g,n}[d]=0$ unless $|d|\leq\mathfrak{d}_{g,n}$. We put $h_{g,n}:=0$ if $2g-2+n\leq 0$. For $d\in\mathbb{Z}^n_{\geq 0}$ let

$$G^{(g)}_d(X) := \sum_{N\geq 0} h_{g,n+N}[d,0^N]\,\frac{X^N}{N!}\in\mathbb{Q}[[X]], \qquad W_{g,n}(Q_1,\dots,Q_n) := \sum_d G^{(g)}_d(X)\prod_{i=1}^n Q_i^{-2d_i-2}. \tag{1.4}$$

Here X marks boundaries of length zero. Since $h_{g,n+N}[d,0^N]=0$ for $N<|d|-\mathfrak{d}_{g,n}$, we have

$$\mathrm{ord}_X\, G^{(g)}_d \;\geq\; |d|-\mathfrak{d}_{g,n}, \tag{1.5}$$

so each coefficient of X^N in $W_{g,n}$ is a polynomial in the Q_i^{-2}. Throughout,

$$\kappa := 1-\sqrt{1-X}, \qquad s := \sqrt{1-X} = 1-\kappa, \qquad X = 2\kappa-\kappa^2.$$

$W_{g,n}$ is the Laplace transform of the generating series of the polynomials $V\Omega^{\mathrm{MV}}$, in boundary lengths measured in units of π: $\int_0^\infty \mathrm{e}^{-Q\Lambda}\frac{\Lambda^{2d+1}}{(2d+1)!}\mathrm{d}\Lambda = Q^{-2d-2}$.

1.3. **Genus zero.** For integers $k, j \geq 0$ put

$$c_{k,j} := \left[t^{2j}\right]\,(t\cot t)^k \in \mathbb{Q}, \qquad t\cot t = \sum_{i\geq 0} \frac{(-4)^i B_{2i}}{(2i)!}\,t^{2i} = 1 - \frac{t^2}{3} - \frac{t^4}{45} - \cdots, \tag{1.6}$$

so that $c_{k,0} = 1$, $c_{k,1} = -k/3$ and $c_{k,2} = k(5k-7)/90$. For $d \geq 0$ define the polynomial

$$\rho_d(\kappa) := \operatorname*{Res}_{\eta=0} \big(\eta + \kappa\cot\eta\big)^{2d}\cot\eta\,\mathrm{d}\eta = \sum_{j=0}^{d} \binom{2d}{d+j} c_{d+j+1,j}\,\kappa^{d+j}. \tag{1.7}$$

Thus $\rho_0 = 1$, $\rho_1 = 2\kappa - \kappa^2$ and $\rho_2 = 6\kappa^2 - \frac{16}{3}\kappa^3 + \kappa^4$. The second expression follows by expanding $(\eta+\kappa\cot\eta)^{2d} = \sum_q \binom{2d}{q}\kappa^q\eta^{2d-2q}(\eta\cot\eta)^q$.

Theorem A (All genus-zero coefficients)**.** *For $n \geq 3$ and $d_1, \ldots, d_n \geq 0$, with $m := n - 3 - \sum_i d_i$,*

$$F_{0,n}[d_1,\ldots,d_n] = 2\,(n-3)!\,\pi^{2m}\prod_{i=1}^{n}(2d_i+1)\;\left[\kappa^{n-3}\right]\;\frac{\prod_{i=1}^n \rho_{d_i}(\kappa)}{(2-\kappa)^{n-2}}. \tag{1.8}$$

Equivalently, with $\kappa(X) = 1 - \sqrt{1-X}$,

$$F_{0,n}[d_1,\ldots,d_n] = \pi^{2m}\,\frac{\mathrm{d}^{\,n-3}}{\mathrm{d}X^{n-3}}\Big[(1-X)^{-1/2}\prod_{i=1}^{n}(2d_i+1)\,\rho_{d_i}\big(\kappa(X)\big)\Big]_{X=0}. \tag{1.9}$$

Since $\rho_d(\kappa) = O(\kappa^d)$, only $[\kappa^m]$ of $(2-\kappa)^{2-n}\prod_i \rho_{d_i}(\kappa)/\kappa^{d_i}$ matters. This is a finite sum of at most $m+1$ terms, and $F_{0,n}[d] = 0$ for $m < 0$. For instance, $F_{0,n}[0,\ldots,0] = \pi^{2m}(2m)!/(4^m m!)$. For $m = 0$ one recovers the Kontsevich values $F_{0,n}[d] = (n-3)!\prod_i (2d_i+1)!!/d_i!$.

The structure behind Theorem A is a deformation of genus-zero Witten–Kontsevich theory. Measure boundary lengths Λ in units of π. Let $x_1, \ldots, x_J$ be weights for boundaries of lengths $\Lambda_1, \ldots, \Lambda_J$, and let $G(\Lambda; X, \boldsymbol{x})$ be the generating series of the genus-zero polynomials with one distinguished boundary of length Λ (Section 2). Put

$$R(\Lambda;\kappa) := \operatorname*{Res}_{\eta=0} \cosh\big(\Lambda(\eta+\kappa\cot\eta)\big)\cot\eta\,\mathrm{d}\eta = \sum_{d\geq 0}\rho_d(\kappa)\frac{\Lambda^{2d}}{(2d)!}. \tag{1.10}$$

Theorem B (Genus-zero generating series)**.** *Let $\kappa = \kappa(X, \boldsymbol{x})$ be the formal solution with $\kappa(0,0) = 0$ of the* string equation

$$2\kappa - \kappa^2 = X + \sum_{j=1}^{J} x_j\,R(\Lambda_j;\kappa). \tag{1.11}$$

Then

$$\frac{\partial G(\Lambda)}{\partial X} = \frac{2}{\Lambda}\operatorname*{Res}_{\eta=0}\mathrm{e}^{\Lambda(\eta+\kappa\cot\eta)}\mathrm{d}\eta = 2\int_0^{\kappa} R(\Lambda;k)\,\mathrm{d}k. \tag{1.12}$$

For $n \geq 2$, if $x_2, \ldots, x_n$ are the weights of boundaries of lengths $\Lambda_2, \ldots, \Lambda_n$, the n-point functions are

$$\frac{\partial^{n-1}G(\Lambda_1)}{\partial x_2\cdots\partial x_n} = \frac{\partial^{n-2}}{\partial X^{n-2}}\Big[\,2\int_0^{\kappa}\prod_{i=1}^{n}R(\Lambda_i;k)\,\mathrm{d}k\Big]. \tag{1.13}$$

For Kontsevich volumes alone (no edges) the same statements hold with $\cot\eta$ replaced by $1/\eta$. Then $R(\Lambda;\kappa)=I_0(2\Lambda\sqrt{\kappa})$ and the string equation becomes $2\kappa = X + \sum_j x_j I_0(2\Lambda_j\sqrt{\kappa})$; that is, $u=2\kappa$ solves the genus-zero string equation of Witten–Kontsevich theory. In that case (1.13) is the classical formula

$$\langle\langle \tau_{a_1}\cdots\tau_{a_n}\rangle\rangle_0 = \partial_{t_0}^{n-3}\Big(u_{t_0}\prod_i \frac{u^{a_i}}{a_i!}\Big).$$

So genus-zero Masur–Veech theory is obtained by two replacements: $\operatorname{Res}\cosh(\Lambda(\eta+\kappa/\eta))\,\mathrm{d}\eta/\eta$ becomes $\operatorname{Res}\cosh(\Lambda(\eta+\kappa\cot\eta))\cot\eta\,\mathrm{d}\eta$, and 2κ becomes $2\kappa-\kappa^2$ in the string equation.

Corollary 1.1 (One row; Athreya–Eskin–Zorich)**.** *For $n\ge 3$, $0\le d\le n-3$ and $m=n-3-d$,*

$$H_n[d] = \pi^{2m}\,\frac{(2d+1)!}{2^d\,4^m}\sum_{j=0}^{\min(d,m)} 2^j\,c_{d+j+1,j}\,\frac{(2m+d-j)!}{(m-j)!\,(d-j)!\,(d+j)!}. \tag{1.14}$$

In particular $H_n[0]=\pi^{2m}\frac{(2m)!}{4^m m!}$, and the stable-graph formula (1.1)–(1.2) implies $\mathrm{MV}_{0,n}=\pi^{2(n-3)}2^{5-n}$ for all $n\ge 4$.

Corollary 1.2 (Polynomiality)**.** *Fix $k\ge 1$ and $d_1,\dots,d_k\ge 0$, and put $|d|=\sum_i d_i$. There is a polynomial $\mathcal{P}_{d_1,\dots,d_k}\in\mathbb{Q}[n]$ of degree exactly $|d|$, with leading coefficient $\prod_{i=1}^k(2d_i+1)\rho_{d_i}(1)>0$, such that for all $n\ge\max(k,3)$ with $m:=n-3-|d|\ge 0$,*

$$F_{0,n}[d_1,\dots,d_k,0,\dots,0] = \pi^{2m}\,\frac{(2m)!}{4^m\,m!}\,\mathcal{P}_{d_1,\dots,d_k}(n).$$

For $k=1$ this proves [ABC+23, Conj. 5.3]*: $H_n[d]=\frac{2d+1}{(2d-1)!!}P_d(n)\frac{(2m)!}{4^m m!}\pi^{2m}$ for all $n\ge d+3$. Here $P_d:=\frac{(2d-1)!!}{2d+1}\mathcal{P}_d$, with the convention $(-1)!!=1$, is*

$$P_d(n) = \big((2d-1)!!\big)^2\,d!\sum_{j=0}^{d} 2^j\,c_{d+j+1,j}\,\frac{m^{\underline{j}}\,(2m+1)^{\overline{d-j}}}{(d-j)!\,(d+j)!},\qquad m=n-3-d, \tag{1.15}$$

of degree exactly d with leading coefficient $(2d-1)!!\,\rho_d(1)>0$.

Here $x^{\underline{j}}=x(x-1)\cdots(x-j+1)$ and $x^{\overline{j}}=x(x+1)\cdots(x+j-1)$. The first values are $P_0=1$, $P_1=n-3$, $P_2=5n^2-34n+54$ and $P_3=\frac{3}{2}(32n^3-367n^2+1307n-1392)$. They agree with [ABC+23, Table 5], except that the version we consulted prints the constant term of P_2 as 52; the value 54 is forced by $H_5[2]=15$.

1.4. **The spectral curve.** Let $0<\kappa<1$, $\beta=\arcsin\sqrt{\kappa}\in(0,\frac{\pi}{2})$, $a=\beta+\sqrt{\kappa(1-\kappa)}$, and

$$P(y)=y+\kappa\cot y,\qquad \Psi(y)=-\frac{2\kappa^2}{\sin^2 y},\qquad K(y_1,y_2)=\frac{1}{\sin^2(y_1-y_2)}+\frac{1}{\sin^2(y_1+y_2)}.$$

There is an explicit Jordan domain $D_\kappa\ni 0$ (Lemma 4.1) that P maps biholomorphically onto $\widehat{\mathbb{C}}\setminus[-a,a]$, with $0\mapsto\infty$. Write y_0 for the inverse.

Theorem C (Spectral curve)**.** *Let $0<\kappa<1$ and $X=2\kappa-\kappa^2$. For $|Q_i|>a$ and $y_i=y_0(Q_i)$ the series below converge, and:*

(i) $W_{0,1}(Q)=2y_0^2-4\kappa\,(1-y_0\cot y_0)=2(Q^2-X)+\Psi(y_0)$;

(ii) *the two-point function is the periodic Bergman kernel in the coordinate y:*

$$W_{0,2}(Q_1,Q_2)=K(y_1,y_2)\,\frac{\mathrm{d}y_1}{\mathrm{d}Q_1}\frac{\mathrm{d}y_2}{\mathrm{d}Q_2}-\frac{1}{(Q_1-Q_2)^2}-\frac{1}{(Q_1+Q_2)^2};$$

(iii) *for $n \geq 3$,*

$$W_{0,n}(Q_1,\dots,Q_n) = \partial_X^{n-3}\Big[(1-X)^{-1/2}\prod_{i=1}^{n} e(Q_i)\Big], \qquad e(Q) = -\frac{1}{2}K(\beta,y)\frac{\mathrm{d}y}{\mathrm{d}Q}, \quad y = y_0(Q).$$

The branch points of P are $\pm\beta$ modulo π. With $\kappa = \sin^2\beta$ the cut endpoint is $a = \beta + \sin\beta\cos\beta$. At $X = 1$ ($\kappa = 1$, $a = \pi/2$) the cut touches its periodic images.

1.5. **Genus one.** Fix boundary lengths $\Lambda_1,\dots,\Lambda_J \geq 0$ with weights x_j, and let

$$F_g(X,\boldsymbol{x}) := \sum_{a,b}\frac{X^a}{a!}\prod_j \frac{x_j^{b_j}}{b_j!}\, V^{\flat}_{g,a+|b|}\big(0^a,\Lambda_1^{b_1},\dots,\Lambda_J^{b_J}\big),$$

where $V^{\flat}_{g,n}(\Lambda) := \sum_d h_{g,n}[d]\prod_i \Lambda_i^{2d_i}/(2d_i+1)!$ (Lemma 2.1). Derivatives in the x_j followed by $\boldsymbol{x} = 0$ give all coefficients: $\partial_{x_1}\cdots\partial_{x_n}F_g|_{\boldsymbol{x}=0} = \sum_N \frac{X^N}{N!}V^{\flat}_{g,n+N}(\Lambda_1,\dots,\Lambda_n,0^N)$.

Theorem D (Genus one)**.** *As formal power series in $(X,\boldsymbol{x})$, with $\kappa = \kappa(X,\boldsymbol{x})$ the solution of (1.11) and $\kappa_X = \partial_X\kappa$,*

$$F_1(X,\boldsymbol{x}) = \frac{1}{24}\log\big(2\kappa_X\big) + \frac{\kappa}{12} - \frac{1}{24}\log(1-\kappa). \tag{1.16}$$

Since every ∂_{x_j} acts through κ and its X-derivatives, by $\partial_{x_j}\kappa = R(\Lambda_j;\kappa)\,\kappa_X$, (1.16) determines all genus-one Masur–Veech polynomials. For instance, writing $R_i = R(\Lambda_i;\kappa)$ and $' = \partial_\kappa$, at $\boldsymbol{x} = 0$:

$$\partial_{x_1}F_1 = \frac{(4-2\kappa)R_1 + (1-\kappa)R_1'}{48(1-\kappa)^2}, \tag{1.17}$$

$$\partial_{x_1}\partial_{x_2}F_1 = \frac{\frac{1}{2}(1-\kappa)^2\big(R_1''R_2 + R_1'R_2' + R_1R_2''\big) + (1-\kappa)(\frac{5}{2}-\kappa)(R_1R_2)' + (3-\kappa)R_1R_2}{48(1-\kappa)^4}. \tag{1.18}$$

Corollary 1.3 (Genus one, one row)**.** *For $n \geq 1$ and $d \geq 0$,*

$$F_{1,n}[d,0^{n-1}] = \pi^{2(n-d)}\,\frac{(n-1)!\,(2d+1)}{24}\,\big[\kappa^{n-1}\big]\,\frac{(4-2\kappa)\rho_d(\kappa) + (1-\kappa)\rho_d'(\kappa)}{(1-\kappa)(2-\kappa)^n}.$$

Equivalently, $W_{1,1}(Q) = \big[(4-2\kappa) + (1-\kappa)\partial_\kappa\big]e(Q;\kappa)\big/\big(48(1-\kappa)^2\big)$. Here $e(Q;\kappa) = \sum_d(2d+1)\rho_d(\kappa)Q^{-2d-2}$ is the function of Theorem C(iii), and ∂_κ is taken at fixed Q.

At $\boldsymbol{x} = 0$, (1.16) reduces to the genus-one volume series $F_1(X) = \frac{\kappa}{12} - \frac{1}{12}\log(1-\kappa)$ of Yang, Zagier and Zhang [YZZ20]; this is the input to the proof of Theorem D. It gives $\mathrm{MV}_{1,n} = \pi^{2n}\big(\frac{2^{n-1}n!(n-1)!}{3(2n-1)!} + \frac{2^{1-n}n}{3(2n-1)}\big)$.

1.6. **All genera.** On the cylinder $\mathbb{C}/\pi\mathbb{Z}$ with coordinate y consider

$$\mathrm{d}Q = P'(y)\,\mathrm{d}y, \qquad \omega_{0,1} = 2\Psi\,\mathrm{d}Q = -\frac{4\kappa^2}{\sin^2 y}\,\mathrm{d}Q, \qquad \omega_{0,2}(y_1,y_2) = K(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2. \tag{1.19}$$

Since $P(y+\pi) = P(y)+\pi$, the function P is defined on the cylinder only modulo π, but $\mathrm{d}Q$ is a well-defined 1-form on it. The zeros of $\mathrm{d}Q$ are $\pm\beta$. Near each of them let σ be the local

involution, $P(\sigma(y)) = P(y)$, $\sigma(y) \neq y$; it is defined by any local branch of P. Topological recursion [EO07] defines, for $2g-2+n+1>0$ and $J=(y_1,\dots,y_n)$,

$$\begin{aligned}\omega_{g,n+1}(z_0,J) = \sum_{p=\pm\beta} \operatorname*{Res}_{y\to p} K^{\mathrm{TR}}(z_0,y)\Big[&\omega_{g-1,n+2}(y,\sigma(y),J)\\ &+ \sum_{h,I}{}' \,\omega_{h,1+|I|}(y,I)\,\omega_{g-h,1+n-|I|}(\sigma(y),J\setminus I)\Big],\end{aligned} \tag{1.20}$$

with $K^{\mathrm{TR}}(z_0,y) = \frac{1}{2}\int_{\sigma(y)}^{y}\omega_{0,2}(z_0,\cdot)\big/\big(\omega_{0,1}(y)-\omega_{0,1}(\sigma(y))\big)$, where $\sum'$ omits $(h,I)=(0,\emptyset)$ and (g,J).

Theorem E (Topological recursion)**.** *Let $n\geq 1$ and $2g-2+n>0$.*

(a) *For every $d\in\mathbb{Z}^n_{\geq 0}$, $G_d^{(g)}(X)$ is a polynomial in $s=\sqrt{1-X}$ and s^{-1} with rational coefficients. In particular the series (1.4) converges for $|X|<1$.*

(b) *For every $\kappa\in(0,1)$ and all $Q_1,\dots,Q_n$ with $|Q_i|>a$,*

$$\omega_{g,n}\big(y_0(Q_1),\dots,y_0(Q_n)\big) = W_{g,n}(Q_1,\dots,Q_n)\,\mathrm{d}Q_1\cdots\mathrm{d}Q_n, \qquad X=2\kappa-\kappa^2,$$

where the right side is a convergent series.

The initial data are genus-zero quantities. By Theorem C(ii), $\omega_{0,2}$ is the genus-zero two-point function plus $\frac{\mathrm{d}Q_1\mathrm{d}Q_2}{(Q_1-Q_2)^2}+\frac{\mathrm{d}Q_1\mathrm{d}Q_2}{(Q_1+Q_2)^2}$. Its first term is the Bergman kernel of the cylinder, and its second is minus the pull-back of that kernel under $y_2\mapsto -y_2$; in this way the periodic images of the cut are absorbed into the uniformisation. By Theorem C(i), $\omega_{0,1}=2\Psi\,\mathrm{d}Q$ is twice the π-periodic part of $W_{0,1}\,\mathrm{d}Q$; the factor 2 is shared between the two branch points (Lemma 11.1). As $X\to 0$ the branch points $\pm\beta$ merge with the pole $y=0$ of P. As $\kappa\to 1$ the branch points β and $\pi-\beta$ merge at $\pi/2$, where $P(\frac{\pi}{2}+\epsilon)=\frac{\pi}{2}+(1-\kappa)\epsilon-\frac{\kappa}{3}\epsilon^3+O(\epsilon^5)$ and $\omega_{0,1}/\mathrm{d}Q$ is even in ϵ. This is the $(3,2)$ local model. Correspondingly, the most singular part of the volume series at $X=1$ is governed by Painlevé I (Remark 10.5).

For $g=0$, Theorem E gives a second proof of Theorem C(iii). For $n=1$ and $d=0$ it contains the volume series $F_g'(X)=G_{(0)}^{(g)}(X)$ of [YZZ20]; part (a) agrees with their form $F_g\in\mathbb{Q}[s^{-1}]$ for $g\geq 2$.

1.7. **Strategy.** *Genus zero.* All stable graphs are trees. Cutting a tree at the vertex adjacent to the distinguished leg characterises the generating series as the unique solution of a fixed-point equation built from genus-zero Witten–Kontsevich theory, with effective times given by ζ-weighted moments of the series itself (Section 3). We pass to the Laplace transform $W(Q)=\sum_d G_d Q^{-2d-2}$. The weight $1/(\mathrm{e}^{\pi\lambda}-1)=\sum_{m\geq 1}\mathrm{e}^{-\pi m\lambda}$ becomes a sum over periodic images, and the fixed point becomes the one-cut singular integral equation

$$W_+(c)+W_-(c)+\sum_{m\neq 0} W(c+\pi m) = 2\,U_0(c), \qquad c\in(-a,a), \tag{1.21}$$

with the entire potential $U_0(Q)=Q^2-X-\sum_j x_j\cosh(\Lambda_j Q)$. We solve (1.21) for every such potential (Section 5). The solution is an explicit residue in the uniformising coordinate y of $P(y)=y+\kappa\cot y$, and the cut equation reduces to a residue identity over the fibres of P together with the string equation. A one-cut argument and an analytic-to-formal argument identify the solution with the tree series (Sections 6–7). An algebraic identity shows that the two-point function depends on the parameters only through κ. Together with the string equation this gives (1.13) (Section 8).

Genus one. A genus-one stable graph is either a genus-one Witten–Kontsevich vertex dressed by genus-zero trees, or a cycle of genus-zero beads. The first sum is governed by the endpoint a of the cut, the second by the two-point function, which depends only on κ (Section 10).

All genera. Both the Masur–Veech series and the outputs of topological recursion satisfy one system of *deformed loop equations*, and that system has only one solution in a class containing both (Section 11).

- On the Masur–Veech side, the recursion of [ABC+23] is equivalent to loop equations at $X = 0$, in which every correlator is dressed by its images $Q \mapsto Q + \pi m$. Summing over boundaries of length zero deforms them.
- On the side of topological recursion, the sum of a stable correlator over the whole fibre of Q vanishes. This turns the image-dressed correlators into half-differences between the two sheets that meet at the cut, and the deformed loop equations follow from the local loop equations at the branch points.
- A scaling argument shows that the outputs of topological recursion are meromorphic at $\kappa = 0$ with a controlled pole. A uniqueness argument in formal Laurent series in κ concludes.

The case $(g, n) = (1, 1)$ is taken from Theorem D.

Conventions. $[t^k]\, f$ is the coefficient of t^k in a series f. Square roots $\sqrt{Q^2 - c^2}$ $(c \geq 0)$ are taken on $\mathbb{C} \setminus [-c, c]$ with $\sqrt{Q^2 - c^2} \sim Q$ at ∞. For a function f on $\mathbb{C} \setminus [-a, a]$ and $c \in (-a, a)$, $f_\pm(c) := \lim_{\epsilon \downarrow 0} f(c \pm \mathrm{i}\epsilon)$. Residues $\mathrm{Res}_{\eta=0}$ are taken at isolated (possibly essential) singularities.

2. Set-up

2.1. **Kontsevich volumes in genus zero.** For $k \geq 3$ the genus-zero intersection numbers are $\langle \tau_{d_1} \cdots \tau_{d_k} \rangle_0 = \frac{(k-3)!}{d_1! \cdots d_k!}$ if $d_1 + \cdots + d_k = k - 3$ and 0 otherwise [Wit91, Kon92]. Hence

$$V^{\mathrm{K}}_{0,k}(L_1, \ldots, L_k) = \sum_{d_1 + \cdots + d_k = k-3} \langle \tau_{d_1} \cdots \tau_{d_k} \rangle_0 \prod_{i=1}^{k} \frac{L_i^{2d_i}}{2^{d_i} d_i!}, \tag{2.1}$$

a homogeneous polynomial of degree $2(k-3)$. In general $V^{\mathrm{K}}_{h,k}$ is homogeneous of degree $2\mathfrak{d}_{h,k}$.

2.2. **Removing π.** The moments

$$\int_0^\infty \frac{\lambda^{2j+1}\, \mathrm{d}\lambda}{\mathrm{e}^{\pi\lambda} - 1} = \frac{(2j+1)!\, \zeta(2j+2)}{\pi^{2j+2}} = \frac{2^{2j+1} |B_{2j+2}|}{2j+2} \in \mathbb{Q} \tag{2.2}$$

are rational. Let $V^\flat_{g,n}(\Lambda_1, \ldots, \Lambda_n)$ be defined by (1.1) with the edge measure $\ell\, \mathrm{d}\ell/(\mathrm{e}^\ell - 1)$ replaced by $\lambda\, \mathrm{d}\lambda/(\mathrm{e}^{\pi\lambda} - 1)$.

Lemma 2.1. $V\Omega^{\mathrm{MV}}_{g,n}(\pi\Lambda_1, \ldots, \pi\Lambda_n) = \pi^{2\mathfrak{d}_{g,n}} V^\flat_{g,n}(\Lambda_1, \ldots, \Lambda_n)$. *Consequently*

$$V^\flat_{g,n}(\Lambda_1, \ldots, \Lambda_n) = \sum_d h_{g,n}[d] \prod_i \frac{\Lambda_i^{2d_i}}{(2d_i+1)!}, \qquad h_{g,n}[d] := \pi^{-2(\mathfrak{d}_{g,n} - |d|)} F_{g,n}[d] \in \mathbb{Q}.$$

Proof. Substitute $\ell_e = \pi\lambda_e$ in the contribution of a stable graph Γ. Each edge gives $\ell_e \mathrm{d}\ell_e = \pi^2 \lambda_e \mathrm{d}\lambda_e$, and by homogeneity each vertex v gives $\pi^{2\mathfrak{d}_{h(v),k(v)}}$. Since $\sum_v (2h(v) - 2 + k(v)) = 2g - 2 + n$ and $\sum_v k(v) = n + 2|E|$, the total power is $2|E| + 2\sum_v \mathfrak{d}_{h(v),k(v)} = 2\mathfrak{d}_{g,n}$. Rationality follows from (2.2) and the rationality of the intersection numbers. □

2.3. **Stable trees.** A genus-zero stable graph with n legs is a finite tree T with n legs (half-edges not paired into edges), labelled by $\{1,\dots,n\}$ and attached to the vertices, such that every vertex v has valence $k(v)\ge 3$. Since legs are labelled, $\operatorname{Aut} T$ is trivial, and in genus zero (1.1) reads

$$V\Omega^{\mathrm{MV}}_{0,n}(L_1,\dots,L_n)=\sum_{T}\int_{\mathbb{R}_+^{E(T)}}\prod_{v\in V(T)}V^{\mathrm{K}}_{0,k(v)}\big((\ell_e)_{e\ni v},(L_i)_{i\ni v}\big)\prod_{e\in E(T)}\frac{\ell_e\,\mathrm{d}\ell_e}{\mathrm{e}^{\ell_e}-1}. \tag{2.3}$$

2.4. **Generating series in genus zero.** Fix $J\ge 0$ and lengths $\Lambda_1,\dots,\Lambda_J\ge 0$. Let $X,x_1,\dots,x_J$ be formal variables and set

$$G^{\mathrm{tree}}(\Lambda;X,\boldsymbol{x}):=\sum_{a,b_1,\dots,b_J\ge 0}\frac{X^a}{a!}\prod_{j=1}^{J}\frac{x_j^{b_j}}{b_j!}\,V^{\flat}_{0,1+a+|b|}\big(\Lambda,0^a,\Lambda_1^{b_1},\dots,\Lambda_J^{b_J}\big), \tag{2.4}$$

where 0^a means a entries 0 and $\Lambda_j^{b_j}$ means b_j entries Λ_j. Thus X marks boundaries of length zero and x_j boundaries of length Λ_j. Write $G^{\mathrm{tree}}=\sum_d G_d\,\Lambda^{2d}/(2d+1)!$. A tree contributing to $X^a\boldsymbol{x}^b$ has $1+a+|b|\ge d+3$ legs, so G_d has order at least $d+2$ in $(X,\boldsymbol{x})$. For $J=0$, $G_d(X)=G^{(0)}_{(d)}(X)$ in the notation of (1.4).

The multi-point coefficients are recovered as follows. Take $J=n-1$, label the lengths and weights $\Lambda_2,\dots,\Lambda_n$ and $x_2,\dots,x_n$, and let Λ_1 be the distinguished length. Then

$$\left.\frac{\partial^{\,n-1}G^{\mathrm{tree}}(\Lambda_1)}{\partial x_2\cdots\partial x_n}\right|_{\boldsymbol{x}=0,\,X=0}=V^{\flat}_{0,n}(\Lambda_1,\dots,\Lambda_n). \tag{2.5}$$

Let $\mathfrak{G}$ be the set of $G=\sum_d G_d\Lambda^{2d}/(2d+1)!$ with G_d of order at least $d+2$ in $\mathbb{R}[[X,\boldsymbol{x}]]$. For $G\in\mathfrak{G}$ and $k\ge 0$ put

$$t_k[G]:=X\delta_{k,0}+\sum_{j=1}^{J}x_j\frac{\Lambda_j^{2k}}{2^k k!}+\sum_{b\ge 0}u_{k,b}\,G_b,\qquad u_{k,b}:=\frac{1}{2^k k!\,(2b+1)!}\int_0^\infty\frac{\lambda^{2k+2b+1}\mathrm{d}\lambda}{\mathrm{e}^{\pi\lambda}-1}. \tag{2.6}$$

The sum over b converges in the $(X,\boldsymbol{x})$-adic topology, and $t_k[G]$ has order ≥ 1. Formally $t_k[G]=\int\frac{\lambda^{2k}}{2^k k!}\,\mathrm{d}\mu_G$ for $\mathrm{d}\mu_G:=X\delta_0+\sum_j x_j\delta_{\Lambda_j}+\lambda\,G(\lambda)\,\mathrm{d}\lambda/(\mathrm{e}^{\pi\lambda}-1)$.

3. The tree fixed point

3.1. **Genus-zero Witten–Kontsevich theory.** Let

$$F_0(t)=\sum_{k\ge 3}\frac{1}{k!}\sum_{d_1,\dots,d_k\ge 0}\langle\tau_{d_1}\cdots\tau_{d_k}\rangle_0\,t_{d_1}\cdots t_{d_k}$$

and $\langle\langle\tau_{d_1}\cdots\tau_{d_r}\rangle\rangle_0=\partial_{t_{d_1}}\cdots\partial_{t_{d_r}}F_0$. Give t_k the degree $k-1$. A monomial $t_{d_1}\cdots t_{d_r}$ of $\langle\langle\tau_d\rangle\rangle_0$ satisfies $\sum_i d_i=r-2-d$; so $\langle\langle\tau_d\rangle\rangle_0$ is homogeneous of degree $-d-2$, and has finitely many monomials with a given number $r\ge 2$ of factors.

Substitution. If f is a formal series in the t_k, homogeneous of some degree, and $t_k\mapsto s_k$ with s_k of order ≥ 1 in some formal variables, then the monomials of f with r factors contribute in order $\ge r$ and there are finitely many of them; so $f(s)$ is well defined, and substitution respects sums, products and the identities below.

Put $\varphi(w):=\sum_{k\ge 0}t_k w^k/k!$. Let $u(t)$ be the unique formal solution of $u=\varphi(u)$, given by Lagrange inversion as $u=\sum_{k\ge 1}\frac{1}{k}\left[w^{k-1}\right]\varphi(w)^k$; it is homogeneous of degree -1.

Lemma 3.1. *For every $d\ge 0$, $\langle\langle\tau_0\tau_d\rangle\rangle_0=\frac{u^{d+1}}{(d+1)!}$ and $\langle\langle\tau_d\rangle\rangle_0=\int_0^u\frac{w^{d+1}}{(d+1)!}\big(1-\varphi'(w)\big)\,\mathrm{d}w$, where $\varphi'(w)=\sum_{j\ge 1}t_j w^{j-1}/(j-1)!$.*

Proof. By the formula for $\langle\cdots\rangle_0$,

$$\langle\langle\tau_0\tau_d\rangle\rangle_0 = \sum_{k\geq 1}\frac{1}{k!}\sum_{d_1+\cdots+d_k=k-1-d}\frac{(k-1)!}{d!\,d_1!\cdots d_k!}\,t_{d_1}\cdots t_{d_k} = \frac{1}{d!}\sum_{k\geq 1}\frac{1}{k}\left[w^{k-1-d}\right]\varphi(w)^k.$$

Lagrange inversion for $u = z\varphi(u)$ gives $H(u) = H(0) + \sum_{k\geq 1}\frac{z^k}{k}\left[w^{k-1}\right]H'(w)\varphi(w)^k$; at $z=1$ (legitimate by homogeneity) and with $H(w) = w^{d+1}/(d+1)!$ this is the first identity. Call the right-hand side of the second identity R_d. Differentiating $u=\varphi(u)$ in t_0 gives $\partial_{t_0}u = (1-\varphi'(u))^{-1}$, so $\partial_{t_0}R_d = \frac{u^{d+1}}{(d+1)!} = \partial_{t_0}\langle\langle\tau_d\rangle\rangle_0$. At $t_0=0$ every monomial of $\langle\langle\tau_d\rangle\rangle_0$ would need $\sum d_i \geq r > r-2-d$, so $\langle\langle\tau_d\rangle\rangle_0|_{t_0=0}=0$; also $u|_{t_0=0}=0$, so $R_d|_{t_0=0}=0$. □

3.2. **Cutting the tree.** For $G\in\mathfrak{G}$ define $\mathcal{T}[G](\Lambda) := \sum_{d\geq 0}\frac{\Lambda^{2d}}{2^d d!}\langle\langle\tau_d\rangle\rangle_0\big|_{t=t[G]}$. By homogeneity this is well defined, and $\mathcal{T}[G]\in\mathfrak{G}$, since a monomial of $\langle\langle\tau_d\rangle\rangle_0$ has $r\geq d+2$ factors of order ≥ 1.

Proposition 3.2. *G^{tree} is the unique $G\in\mathfrak{G}$ with $G=\mathcal{T}[G]$.*

Proof. Fixed point. Let T be a stable tree contributing to (2.4), let v_0 be the vertex carrying the distinguished leg, of valence $k+1\geq 3$. Deleting v_0 and the distinguished leg leaves $k\geq 2$ half-edges at v_0; each is either a leg, or an edge e leading to a component T_e of $T\setminus v_0$. The legs S_e of T_e number at least two, and T_e, with the half-edge of e at its endpoint turned into a new leg, is a stable tree with leg set $\{*\}\sqcup S_e$. Conversely a set partition of the non-distinguished legs into $k\geq 2$ blocks, together with a stable tree on $\{*\}\sqcup S$ for every block S of size ≥ 2, determines T. Integrating out the edge lengths of the T_e first and summing with the exponential weights of (2.4), the exponential formula gives

$$G^{\mathrm{tree}}(\Lambda) = \sum_{k\geq 2}\frac{1}{k!}\int V^{\mathrm{K}}_{0,k+1}(\Lambda,\lambda_1,\dots,\lambda_k)\prod_{i=1}^{k}\mathrm{d}\mu_{G^{\mathrm{tree}}}(\lambda_i),$$

where a leg of length 0 contributes $X\delta_0$, a leg of length Λ_j contributes $x_j\delta_{\Lambda_j}$, and a block that is not a single leg contributes $\lambda\,G^{\mathrm{tree}}(\lambda)\mathrm{d}\lambda/(\mathrm{e}^{\pi\lambda}-1)$ (with its own exponential weight). Expanding $V^{\mathrm{K}}_{0,k+1}$ by (2.1) and using $t_d = \int\frac{\lambda^{2d}}{2^d d!}\mathrm{d}\mu$ gives $G^{\mathrm{tree}} = \mathcal{T}[G^{\mathrm{tree}}]$.

Uniqueness. If $G,G'\in\mathfrak{G}$ agree modulo terms of order $\geq N$, then so do all $t_k[G], t_k[G']$; every monomial of $\langle\langle\tau_d\rangle\rangle_0$ has at least two factors of order ≥ 1, so $\mathcal{T}[G]$ and $\mathcal{T}[G']$ agree modulo order $N+1$. Two fixed points agree modulo order 2, hence to all orders. □

Corollary 3.3. *For $G\in\mathfrak{G}$ put $\varphi_G(w) := \sum_k t_k[G]\,w^k/k!$ and let u_G (of order ≥ 1) be the unique solution of $u_G = \varphi_G(u_G)$. Then $G = G^{\mathrm{tree}}$ if and only if, for all $d\geq 0$,*

$$G_d = \frac{(2d+1)!!}{(d+1)!}\int_0^{u_G} w^{d+1}\big(1-\partial_w\varphi_G(w)\big)\,\mathrm{d}w. \tag{3.1}$$

Proof. Substitute $t=t[G]$ into Lemma 3.1; substituting into $u=\varphi(u)$ shows $u(t[G]) = u_G$ (the solution is unique because $v\mapsto\varphi_G(v)$ is an adic contraction). Since $\Lambda^{2d}/(2^d d!) = (2d+1)!!\,\Lambda^{2d}/(2d+1)!$, the right-hand side of (3.1) is the $\Lambda^{2d}/(2d+1)!$-coefficient of $\mathcal{T}[G]$. Conclude with Proposition 3.2. □

4. The curve $y+\kappa\cot y$

Fix $\kappa\in(0,1]$ and set

$$P(y) = P_\kappa(y) := y+\kappa\cot y, \qquad \beta := \arcsin\sqrt{\kappa}\in(0,\tfrac{\pi}{2}], \qquad a := \beta+\sqrt{\kappa(1-\kappa)}.$$

Then $P'(y) = 1 - \kappa/\sin^2 y$ vanishes exactly at the real points with $\sin^2 y = \kappa$ (if $\sin y = w \in [-1,1]$ then y is real: by $\sin(t+\mathrm{i}\vartheta) = \sin t\cosh\vartheta + \mathrm{i}\cos t\sinh\vartheta$, either $\vartheta = 0$, or $\cos t = 0$ and $|w| = \cosh\vartheta$, forcing $\vartheta = 0$), and $P(\beta) = \beta + \sin\beta\cos\beta = a$. Since $a = \beta + \frac{1}{2}\sin 2\beta$ and $\beta \mapsto \beta + \frac{1}{2}\sin 2\beta$ is increasing on $[0, \frac{\pi}{2}]$, $a < \pi/2$ for $\kappa < 1$ and $a = \pi/2$ for $\kappa = 1$. For $y = t + \mathrm{i}\vartheta$, $\cot y = \dfrac{\sin 2t - \mathrm{i}\sinh 2\vartheta}{\cosh 2\vartheta - \cos 2t}$, hence

$$\operatorname{Im} P(t+\mathrm{i}\vartheta) = \vartheta - \frac{\kappa\sinh 2\vartheta}{\cosh 2\vartheta - \cos 2t}. \tag{4.1}$$

Lemma 4.1. *Let $F(\vartheta) := \cosh 2\vartheta - \kappa\,\sinh(2\vartheta)/\vartheta$ for $\vartheta \neq 0$, $F(0) := 1 - 2\kappa$, and let $\vartheta_* > 0$ be defined by $\vartheta_*\tanh\vartheta_* = \kappa$.*

(a) *F is even and strictly increasing on $[0,\infty)$, with $F(\vartheta_*) = 1$. Hence $\tau(\vartheta) := \frac{1}{2}\arccos F(\vartheta)$ is defined and continuous for $|\vartheta| \le \vartheta_*$, strictly decreasing in $|\vartheta|$, with $\tau(0) = \beta$, $\tau(\pm\vartheta_*) = 0$. Put*

$$D = D_\kappa := \{t + \mathrm{i}\vartheta :\ |\vartheta| < \vartheta_*,\ |t| < \tau(\vartheta)\}.$$

Then D is a Jordan domain bounded by a piecewise real-analytic curve, invariant under $y \mapsto -y$ and $y \mapsto \bar y$, with $D \subset \{|\operatorname{Re} y| < \pi/2\}$ and $\partial D \cap \mathbb{R} = \{\pm\beta\}$.

(b) *P is real on ∂D and maps each of the arcs $\Gamma^\pm := \partial D \cap \{\pm\operatorname{Im} y \ge 0\}$ homeomorphically onto $[-a,a]$, with $P(\pm\beta) = \pm a$.*

(c) *$\operatorname{Im} P < 0$ on $D \cap \{\operatorname{Im} y > 0\}$, $\operatorname{Im} P > 0$ on $D \cap \{\operatorname{Im} y < 0\}$, and $P((0,\beta)) = (a,\infty)$. In particular $P(D\setminus\{0\}) \cap [-a,a] = \emptyset$.*

(d) *P maps $D\setminus\{0\}$ biholomorphically onto $\mathbb{C}\setminus[-a,a]$.*

Proof. (a) With $x = 2\vartheta > 0$ one has $\vartheta^2 F'(\vartheta) = \frac{x^2}{2}\sinh x - \kappa\,(x\cosh x - \sinh x)$, and

$$\frac{x^2}{2}\sinh x = \sum_{j\ge 1}\frac{x^{2j+1}}{2\,(2j-1)!}, \qquad x\cosh x - \sinh x = \sum_{j\ge 1}\frac{2j\,x^{2j+1}}{(2j+1)!}.$$

The ratio of the j-th coefficients is $(2j+1)/2 > 1$, so $F' > 0$ on $(0,\infty)$ for $\kappa \le 1$. Next, $F(\vartheta) = 1$ iff $2\sinh^2\vartheta = 2\kappa\sinh\vartheta\cosh\vartheta/\vartheta$ iff $\vartheta\tanh\vartheta = \kappa$, and $\cos 2\beta = 1 - 2\kappa$ gives $\tau(0) = \beta$. By (4.1), $\partial D = \{\pm\tau(\vartheta) + \mathrm{i}\vartheta\}$ lies in $\{\operatorname{Im} P = 0\}$ (for $\vartheta \neq 0$, $\cos 2t = F(\vartheta)$ means $\cosh 2\vartheta - \cos 2t = \kappa\sinh 2\vartheta/\vartheta$). Off $\mathbb{R}$ we have $P' \neq 0$, so $\{\operatorname{Im} P = 0\}$ is locally a real-analytic arc there; at $\pm\beta$ it consists, for $\kappa < 1$, of $\mathbb{R}$ and an orthogonal analytic arc, while for $\kappa = 1$ the points $\pm\pi/2$ are degenerate critical points, $P(\pm\frac{\pi}{2} + \epsilon) = \pm\frac{\pi}{2} - \frac{\epsilon^3}{3} + O(\epsilon^5)$, and ∂D_1 has corners with interior angle $2\pi/3$ there. Since $\tau(\vartheta) \le \beta \le \pi/2$, with $\tau(\vartheta) < \pi/2$ for $\vartheta \neq 0$, $D \subset \{|\operatorname{Re} y| < \pi/2\}$.

(b) On the open arc $\{\tau(\vartheta) + \mathrm{i}\vartheta : 0 < \vartheta < \vartheta_*\}$ we have $\operatorname{Im} P = 0$ and $P' \neq 0$, so P is strictly monotone along it; its endpoint values are $P(\beta) = a$ and $P(\mathrm{i}\vartheta_*) = \mathrm{i}(\vartheta_* - \kappa\coth\vartheta_*) = 0$. So this quarter-arc is mapped homeomorphically onto $[0,a]$, and the symmetries $P(-y) = -P(y)$, $P(\bar y) = \overline{P(y)}$ give the rest.

(c) If $\vartheta > 0$ and $|t| < \tau(\vartheta)$ then $0 \le 2|t| < 2\tau(\vartheta) \le \pi$ gives $\cos 2t > F(\vartheta)$, i.e. $0 < \cosh 2\vartheta - \cos 2t < \kappa\sinh 2\vartheta/\vartheta$, so $\operatorname{Im} P < 0$; the case $\vartheta < 0$ is symmetric. On $(0,\beta)$, $\sin^2 t < \kappa$, so $P' < 0$; with $P(0^+) = +\infty$ and $P(\beta) = a$ this gives $P((0,\beta)) = (a,\infty)$, and $P((-\beta,0)) = (-\infty,-a)$ by oddness.

(d) cot has no poles on $\overline{D}$ other than 0. Let $Q \in \mathbb{C}\setminus[-a,a]$. By (b), $P - Q$ does not vanish on ∂D and $P(\partial D) - Q$ is a closed curve inside the segment $[-a,a] - Q \not\ni 0$, so it has winding number zero about 0. By the argument principle $P - Q$ has as many zeros as poles in D, namely one. With (c), $P\colon D\setminus\{0\} \to \mathbb{C}\setminus[-a,a]$ is a holomorphic bijection. □

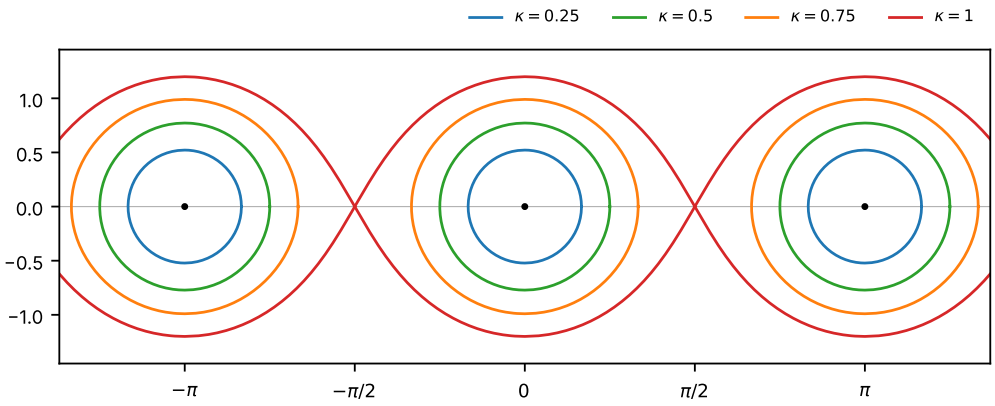

FIGURE 1. The domains D_κ of Lemma 4.1 and their translates by $\pm\pi$, for $\kappa = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. The map $y \mapsto y + \kappa \cot y$ sends each $D_\kappa + \pi m$ conformally onto the complement of $[\pi m - a, \pi m + a]$. At $\kappa = 1$ the closures touch at $\pm\pi/2$.

Lemma 4.2. *Let $\kappa \in (0,1)$ and let $y_0 \colon \widehat{\mathbb{C}} \setminus [-a,a] \to D$ be the inverse of $P|_{D\setminus\{0\}}$, extended by $y_0(\infty) = 0$.*

(a) *y_0 is holomorphic at ∞, odd, real on $\mathbb{R} \setminus [-a,a]$, and $y_0(Q) = \kappa Q^{-1} + O(Q^{-3})$.*
(b) *y_0 is bounded and extends continuously to both sides of $(-a,a)$, with $y_{0,+}(c) \in \Gamma^-$ and $y_{0,-}(c) \in \Gamma^+$ the two points of ∂D over c; it extends continuously to $\pm a$ with $y_0(\pm a) = \pm\beta$, and $|y_0'(Q)| = O(|Q \mp a|^{-1/2})$ near $\pm a$.*

Proof. (a) P has a simple pole with residue κ at 0; from $y_0(Q - y_0) = \kappa\, y_0 \cot y_0 = \kappa + O(y_0^2)$ we get $y_0 = \kappa/Q + O(Q^{-3})$; oddness and reality follow from the symmetries. (b) Let $c \in (-a,a)$ and $y^- \in \Gamma^-$ with $P(y^-) = c$. As $P'(y^-) \neq 0$, P maps a small disc B around y^- biholomorphically onto a neighbourhood N of c; $\partial D \cap B = \{\operatorname{Im} P = 0\} \cap B$ is a single arc cutting B into two components, and the one in D is mapped into, hence onto, $N \cap \{\operatorname{Im} Q > 0\}$ by Lemma 4.1(c). Since preimages in D are unique, $y_0 = (P|_B)^{-1}$ there, which gives $y_{0,+}(c) = y^-$; similarly for the lower side. Near β, $P(y) - a = \frac{1}{2}P''(\beta)(y-\beta)^2(1 + O(y-\beta))$ with $P''(\beta) = 2\kappa\cos\beta/\sin^3\beta > 0$. □

5. The planar problem for an entire potential

Throughout this section $\kappa \in (0,1)$ and U denotes an even entire function. Put

$$K_1(y,\eta) := \cot(y-\eta) - \cot(y+\eta) + 2\cot\eta. \tag{5.1}$$

Definition 5.1. For an even entire U define

$$\Phi_\kappa[U] := \operatorname*{Res}_{\eta=0} U(P(\eta)) \cot\eta \, d\eta, \tag{5.2}$$

$$\Psi_\kappa[U](y) := -\operatorname*{Res}_{\eta=0} U(P(\eta))\, K_1(y,\eta)\, d\eta \qquad (y \notin \pi\mathbb{Z}), \tag{5.3}$$

$$\mathcal{L}_\kappa[U](Q) := 2U(Q) + \Psi_\kappa[U]\big(y_0(Q)\big) \qquad (Q \notin [-a,a]). \tag{5.4}$$

In (5.3), for fixed $y \notin \pi\mathbb{Z}$ the function $\eta \mapsto K_1(y,\eta)$ is holomorphic near 0 except for the simple pole of $2\cot\eta$; so $\Psi_\kappa[U](y) = -\frac{1}{2\pi i}\oint_{|\eta|=\epsilon} U(P(\eta))K_1(y,\eta)d\eta$ for any $\epsilon < \operatorname{dist}(y, \pi\mathbb{Z})$. Hence $\Psi_\kappa[U]$ is holomorphic on $\mathbb{C} \setminus \pi\mathbb{Z}$, even, and π-periodic. We record:

$$\Phi_\kappa[1] = 1, \qquad \Phi_\kappa[Q^2] = 2\kappa - \kappa^2, \qquad \Phi_\kappa[\cosh(\Lambda\,\cdot)] = R(\Lambda;\kappa), \tag{5.5}$$

since $\operatorname{Res}\cot\eta = 1$, $\operatorname{Res}(\eta+\kappa\cot\eta)^2\cot\eta = 2\kappa\operatorname{Res}\eta\cot^2\eta+\kappa^2\operatorname{Res}\cot^3\eta = 2\kappa-\kappa^2$ ($c_{3,1}=-1$), and by (1.10).

Lemma 5.2. (a) *For $0<r<\pi/2$ and $|y_0(Q)|<r/2$,*

$$\mathcal{L}_\kappa[U](Q) = -\frac{1}{2\pi\mathrm{i}}\oint_{|\eta|=r} U(P(\eta))\,K_1\big(y_0(Q),\eta\big)\,\mathrm{d}\eta.$$

(b) *$\mathcal{L}_\kappa[U]$ is holomorphic on $\widehat{\mathbb{C}}\setminus[-a,a]$, even, vanishes at ∞ to order 2, is bounded near $[-a,a]$, and extends continuously to $(-a,a)$ from each side.*

(c) *$\mathcal{L}_\kappa[1]=0$.*

Proof. (a) Inside $|\eta|<r$ the integrand has singularities only at $\eta=0$ and at $\eta=\pm y$, $y=y_0(Q)$ (the other poles $\pm y+\pi\mathbb{Z}$, $\pi\mathbb{Z}$ lie outside). At $\eta=y$ the term $\cot(y-\eta)$ has residue -1, at $\eta=-y$ the term $-\cot(y+\eta)$ has residue -1; these contribute $-U(P(y))-U(P(-y)) = -2U(Q)$. Hence the right side equals $2U(Q)+\Psi_\kappa[U](y)$. (b) Holomorphy off the cut, boundedness and boundary behaviour follow from (5.4), Lemma 4.2 and the holomorphy of $\Psi_\kappa[U]$ on $\overline{D}\setminus\{0\}$. Near $Q=\infty$ use (a): the integral is holomorphic in y for $|y|<r/2$ and vanishes at $y=0$ since $K_1(0,\eta)=0$. It is even in y, since $K_1(-y,\eta)=K_1(y,\eta)$. So it is $O(y^2)$, and as $y_0(Q)=O(Q^{-1})$ is holomorphic at ∞, $\mathcal{L}_\kappa[U]=O(Q^{-2})$. (c) By (a), $-\frac{1}{2\pi\mathrm{i}}\oint K_1\,\mathrm{d}\eta = -(-1-1+2)=0$. □

The key fact is the following residue identity.

Lemma 5.3. *Let $c\in\mathbb{C}$, $h(y):=P(y)-c$, and let $\delta>0$ be so small that h has no zero in $0<|y|\le\delta$. Then*

$$\frac{1}{2\pi\mathrm{i}}\oint_{|y|=\delta}\Psi_\kappa[U](y)\,\frac{h'(y)}{h(y)}\,\mathrm{d}y = 2U(c).$$

Proof. Take $0<\epsilon<\delta$ so small that $|P(\eta)|>|c|+1$ for $|\eta|\le\epsilon$. Writing Ψ as an η-integral over $|\eta|=\epsilon$ and exchanging the order of integration over the two circles,

$$\frac{1}{2\pi\mathrm{i}}\oint_{|y|=\delta}\Psi\,\frac{h'}{h}\mathrm{d}y = -\frac{1}{2\pi\mathrm{i}}\oint_{|\eta|=\epsilon}U(P(\eta))\Big[\frac{1}{2\pi\mathrm{i}}\oint_{|y|=\delta}K_1(y,\eta)\frac{h'(y)}{h(y)}\mathrm{d}y\Big]\mathrm{d}\eta.$$

For fixed η the inner integrand has, inside $|y|<\delta$, the singularities $y=0$ (where $h'/h = -1/y+O(1)$ and $K_1(0,\eta)=0$, so the residue vanishes) and $y=\pm\eta$, with residues $h'(\eta)/h(\eta)$ and $-h'(-\eta)/h(-\eta)$. Since P is odd and P' even, the inner integral is $P'(\eta)\big[\frac{1}{P(\eta)-c}+\frac{1}{P(\eta)+c}\big]$. Substituting $Q=P(\eta)$, the circle $|\eta|=\epsilon$ is mapped to a closed curve winding once *clockwise* around $\pm c$ (P has a simple pole at 0), so

$$-\frac{1}{2\pi\mathrm{i}}\oint_{|\eta|=\epsilon}U(P(\eta))P'(\eta)\Big[\frac{1}{P(\eta)-c}+\frac{1}{P(\eta)+c}\Big]\mathrm{d}\eta = U(c)+U(-c) = 2U(c).$$ □

Proposition 5.4 (Cut equation)**.** *Let U be even and entire, $\kappa\in(0,1)$, and $W:=\mathcal{L}_\kappa[U]$. For $c\in(-a,a)$,*

$$W_+(c)+W_-(c)+\sum_{m\in\mathbb{Z}\setminus\{0\}}W(c+\pi m) = 2U(c)-2\Phi_\kappa[U], \tag{5.6}$$

where the series converges absolutely.

Proof. Since $a<\pi/2$, $|c+\pi m|\ge\pi-a>a$ for $m\neq0$, and absolute convergence follows from $W=O(Q^{-2})$. Fix c, let $h(y)=P(y)-c$ and, for $N\ge1$, $T\ge3$, let $R_{N,T}=\{|\operatorname{Re}y|<\pi(N+\frac12),\,|\operatorname{Im}y|<T\}$.

Zeros. On the vertical sides of $\partial R_{N,T}$, $\cot y=-\mathrm{i}\tanh(\operatorname{Im}y)$, and on the horizontal sides $|\cot y|\le\coth 3<1.01$; in both cases $|\kappa\cot y|\le1.01<|y-c|/2$. By Rouché's theorem for

meromorphic functions, h has, counted with multiplicity, one more zero than poles in $R_{N,T}$, i.e. $2N+2$ zeros. They are: the two points $y^{\pm} \in \Gamma^{\pm}$ over c, and for $0 < |m| \le N$ the points $\xi_m := y_0(c-\pi m) + \pi m \in D + \pi m$ (as $P(y+\pi m) = P(y) + \pi m$). These $2N+2$ points are distinct ($\overline{D} \subset \{|\operatorname{Re} y| \le \beta\}$, $\beta < \pi/2$) and lie in $R_{N,T}$ ($\vartheta_* < 2$); so they are all the zeros, and all are simple.

Rewriting. Write $\Psi = \Psi_\kappa[U]$. As Ψ is π-periodic, $W(c-\pi m) = 2U(c-\pi m) + \Psi(\xi_m)$, and $W_\pm(c) = 2U(c) + \Psi(y^\mp)$. Hence

$$S_N := W_+(c) + W_-(c) + \sum_{0<|m|\le N} W(c-\pi m) = 2U(c) + \sum_{|m|\le N} 2U(c-\pi m) + \sum_{\xi} \Psi(\xi),$$

the last sum over the $2N+2$ zeros of h.

Residues. The function $\Psi h'/h$ is holomorphic on a neighbourhood of $\overline{R_{N,T}}$ except at the zeros ξ of h (simple poles, residue $\Psi(\xi)$) and at the points πm, $|m| \le N$ (isolated singularities). At πm, by periodicity and $h(y+\pi m) = P(y) - (c - \pi m)$, Lemma 5.3 (with $c - \pi m$ and a small circle) shows that the residue is $2U(c-\pi m)$. Hence

$$\sum_{\xi} \Psi(\xi) = J_{N,T} - \sum_{|m|\le N} 2U(c-\pi m), \qquad J_{N,T} := \frac{1}{2\pi \mathrm{i}} \oint_{\partial R_{N,T}} \Psi \frac{h'}{h} \mathrm{d}y,$$

and $S_N = 2U(c) + J_{N,T}$; in particular $J_{N,T}$ does not depend on T.

The boundary integral. Represent $\Psi(y)$, $|\operatorname{Im} y| \ge 1$, by the integral over $|\eta| = \frac14$. There $\cot(y-\eta) - \cot(y+\eta) = \sin 2\eta / \big(\sin(y-\eta)\sin(y+\eta)\big) = O(\mathrm{e}^{-2|\operatorname{Im} y|})$ uniformly in $\operatorname{Re} y$ and η, so with $\Psi_\infty := -2 \operatorname{Res}_{\eta=0} U(P(\eta)) \cot \eta \, \mathrm{d}\eta = -2\Phi_\kappa[U]$ we have $|\Psi(y) - \Psi_\infty| \le C\mathrm{e}^{-2|\operatorname{Im} y|}$ for $|\operatorname{Im} y| \ge 1$. Being continuous and π-periodic, Ψ is therefore bounded on the line $\operatorname{Re} y = \frac{\pi}{2}$, and $M := \int_{\mathbb{R}} |\Psi(\frac{\pi}{2} + \mathrm{i}\vartheta) - \Psi_\infty| \, \mathrm{d}\vartheta < \infty$. Split $J_{N,T} = \Psi_\infty \cdot \frac{1}{2\pi\mathrm{i}} \oint_{\partial R_{N,T}} \frac{h'}{h}\mathrm{d}y + \frac{1}{2\pi\mathrm{i}} \oint_{\partial R_{N,T}} (\Psi - \Psi_\infty)\frac{h'}{h}\mathrm{d}y$. The first term is $\Psi_\infty \cdot (2N+2-(2N+1)) = \Psi_\infty$. In the second, $h' = 1 - \kappa/\sin^2 y$ is bounded on $\partial R_{N,T}$ and $|h| \ge |y-c|/2$ there, so $|h'/h| \le C'/N$ on the vertical sides and $|h'/h| \le C'/T$ on the horizontal ones; the vertical sides contribute at most $2MC'/N$ and the horizontal ones at most $2\pi(2N+1)C'C\mathrm{e}^{-2T}/T$. Letting $T \to \infty$ (recall $J_{N,T}$ does not depend on T) gives $|J_{N,T} - \Psi_\infty| \le C''/N$, hence $S_N \to 2U(c) - 2\Phi_\kappa[U]$. □

Remark 5.5. For $U = Q^2 - X$ with $X = 2\kappa - \kappa^2$ one finds $\Psi_\kappa[U](y) = -2\kappa^2/\sin^2 y$, so $\mathcal{L}_\kappa[U] = 2Q^2 - 2X - 2\kappa^2/\sin^2 y_0 = 2y_0^2 - 4\kappa(1 - y_0 \cot y_0)$, using $Q - y_0 = \kappa \cot y_0$. This is Theorem C(i).

6. From the cut equation to the fixed point

Lemma 6.1 (Abel inversion)**.** *Let $U(q) = \sum_{k\ge0} u_k q^{2k}$ converge for $|q| < r$ and $A(c) := \frac{1}{\pi}\int_0^\pi U(c\cos\vartheta)\,\mathrm{d}\vartheta$. Then for $0 \le q < r$, $U(q) = A(0) + q\int_0^q \frac{A'(c)}{\sqrt{q^2-c^2}}\,\mathrm{d}c$.*

Proof. $A(c) = \sum_k u_k \alpha_k c^{2k}$ with $\alpha_k = \binom{2k}{k} 4^{-k} = \frac{(2k-1)!!}{(2k)!!}$, and for $k \ge 1$, $q\int_0^q \frac{2k\alpha_k c^{2k-1}}{\sqrt{q^2-c^2}}\mathrm{d}c = 2k\alpha_k \frac{(2k-2)!!}{(2k-1)!!} q^{2k} = q^{2k}$. □

Lemma 6.2 (One-cut lemma)**.** *Let $0 < a < r$ and U even and holomorphic on $\{|Q| < r\}$. Let W be holomorphic on $\mathbb{C} \setminus [-a,a]$ with $W(Q) = O(1/Q)$ at ∞, bounded near $[-a,a]$, extending continuously to $(-a,a)$ from each side, with $W_+ + W_- = 2U$ there. With A as in Lemma 6.1, $A(a) = 0$ and, for $Q \notin [-a,a]$,*

$$W(Q) = \widetilde{W}(Q) := \int_0^a \Big(\frac{Q}{\sqrt{Q^2-c^2}} - 1\Big) A'(c)\,\mathrm{d}c. \tag{6.1}$$

Proof. $\widetilde{W}$ is holomorphic on $\mathbb{C}\setminus[-a,a]$, even, and $O(Q^{-2})$. As A is even and holomorphic, $A'(c)=c\,g(c^2)$ with g holomorphic on $\{|v|<r^2\}$; substituting $v=c^2$ and integrating by parts,

$$\widetilde{W}(Q)=Q\Big(g(0)\,Q-g(a^2)\sqrt{Q^2-a^2}+\int_0^{a^2}g'(v)\sqrt{Q^2-v}\,\mathrm{d}v\Big)-A(a)+A(0),$$

so $\widetilde{W}$ is bounded near $[-a,a]$ and extends continuously to $(-a,a)$ from each side. For $q\in(0,a)$, $\sqrt{Q^2-c^2}\to\sqrt{q^2-c^2}>0$ as $Q\to q$ if $c<q$, while for $c>q$ the two boundary values are $\pm\mathrm{i}\sqrt{c^2-q^2}$; by dominated convergence and Lemma 6.1, $\widetilde{W}_+(q)+\widetilde{W}_-(q)=2q\int_0^q\frac{A'(c)\,\mathrm{d}c}{\sqrt{q^2-c^2}}-2\int_0^a A'=2U(q)-2A(a)$, and by evenness on $(-a,0)$ too. With $R(Q)=\sqrt{Q^2-a^2}$ ($R_-=-R_+$), the function $E:=(W-\widetilde{W}-A(a))/R$ satisfies $E_+=E_-$ on $(-a,a)$, so it extends holomorphically across $(-a,a)$ (Morera); near $\pm a$ it is $O(|Q\mp a|^{-1/2})$, hence E is entire, and $E(Q)\to0$ at ∞. So $E\equiv0$, $W-\widetilde{W}\equiv A(a)$, and $Q\to\infty$ gives $A(a)=0$ and $W=\widetilde{W}$. □

Lemma 6.3 (Laplace transform and images)**.** *Let W be holomorphic and even for $|Q|>a$, $W=\sum_{d\ge0}G_dQ^{-2d-2}$, $G(\Lambda):=\sum_d G_d\Lambda^{2d}/(2d+1)!$. For every $b>a$, $|G(\Lambda)|\le C_b\mathrm{e}^{b|\Lambda|}$ and $W(Q)=\int_0^\infty\Lambda G(\Lambda)\mathrm{e}^{-Q\Lambda}\mathrm{d}\Lambda$ for $\operatorname{Re}Q>b$. If $a<\pi/2$ then for $|Q|\le a$*

$$\frac12\sum_{m\neq0}W(Q+\pi m)=\int_0^\infty\Lambda G(\Lambda)\frac{\cosh(Q\Lambda)}{\mathrm{e}^{\pi\Lambda}-1}\,\mathrm{d}\Lambda.$$

Proof. Cauchy's estimates give $|G_d|\le M_b\,b^{2d+2}$, hence the bound and, by dominated convergence, the Laplace representation. For $|Q|\le a$, $m\ge1$: $\operatorname{Re}(\pi m\pm Q)\ge\pi-a>a$; choose $b\in(a,\pi-a)$, use $W(Q-\pi m)=W(\pi m-Q)$, insert the Laplace representation and sum $\sum_{m\ge1}\mathrm{e}^{-\pi m\Lambda}=(\mathrm{e}^{\pi\Lambda}-1)^{-1}$ (Fubini). □

Proposition 6.4. *Let $U_0(Q)=Q^2-X-\sum_{j=1}^J x_j\cosh(\Lambda_jQ)$ with real X,x_j, and let $\kappa\in(0,1)$ satisfy the string equation $\Phi_\kappa[U_0]=0$. Let $W:=\mathcal{L}_\kappa[U_0]$, $W=\sum_d G_dQ^{-2d-2}$ for $|Q|>a$, $G(\Lambda):=\sum_d G_d\Lambda^{2d}/(2d+1)!$, and for $0\le w\le a^2/2$*

$$\varphi(w):=X+\sum_j x_jI_0\big(\Lambda_j\sqrt{2w}\big)+\int_0^\infty\lambda\,G(\lambda)\,I_0\big(\lambda\sqrt{2w}\big)\frac{\mathrm{d}\lambda}{\mathrm{e}^{\pi\lambda}-1}.\tag{6.2}$$

Then $\varphi(a^2/2)=a^2/2$ and, for every $d\ge0$, $G_d=\frac{(2d+1)!!}{(d+1)!}\int_0^{a^2/2}w^{d+1}\big(1-\varphi'(w)\big)\,\mathrm{d}w$.

Proof. By Lemma 6.3 the integral in (6.2) converges. Put $\mathcal{I}(Q):=\frac12\sum_{m\neq0}W(Q+\pi m)$, holomorphic and even on $|\operatorname{Re}Q|<\pi-a$, and $U:=U_0-\mathcal{I}$. Proposition 5.4 gives $W_++W_-=2U$ on $(-a,a)$, and Lemma 5.2(b) supplies the other hypotheses of Lemma 6.2 (with $r\in(a,\pi-a)$). Since $\frac1\pi\int_0^\pi\cosh(c\lambda\cos\vartheta)\mathrm{d}\vartheta=I_0(c\lambda)$ and by Lemma 6.3,

$$A(c)=\frac1\pi\int_0^\pi U(c\cos\vartheta)\mathrm{d}\vartheta=\frac{c^2}{2}-X-\sum_j x_jI_0(\Lambda_jc)-\int_0^\infty\lambda G(\lambda)\frac{I_0(c\lambda)\,\mathrm{d}\lambda}{\mathrm{e}^{\pi\lambda}-1}=\frac{c^2}{2}-\varphi\Big(\frac{c^2}{2}\Big).$$

So $A(a)=0$ is $\varphi(a^2/2)=a^2/2$, and $A'(c)=c\,(1-\varphi'(c^2/2))$. For $|Q|>a$ expand $Q/\sqrt{Q^2-c^2}-1=\sum_{k\ge1}\binom{2k}{k}4^{-k}c^{2k}Q^{-2k}$ in (6.1) and substitute $w=c^2/2$; comparing coefficients of Q^{-2d-2} with $\binom{2d+2}{d+1}2^{-d-1}=(2d+1)!!/(d+1)!$ gives the claim. □

7. The one-point function

Now let $X, \boldsymbol{x}$ be complex and small. Put $\Phi(\kappa; X, \boldsymbol{x}) := \Phi_\kappa[U_0] = 2\kappa - \kappa^2 - X - \sum_j x_j R(\Lambda_j; \kappa)$ (by (5.5)). Since Φ is holomorphic in $(\kappa, X, \boldsymbol{x})$ near 0 with $\partial_\kappa \Phi(0) = 2$, there is a unique holomorphic $\kappa(X, \boldsymbol{x})$ near 0 with $\kappa(0,0) = 0$ and $\Phi(\kappa(X, \boldsymbol{x}); X, \boldsymbol{x}) = 0$; its Taylor series is the formal solution of (1.11), and $\kappa(X, 0) = 1 - \sqrt{1 - X}$. For real $X, \boldsymbol{x}$ near 0, κ is real, and, since $\kappa = \frac{1}{2}(X + \sum_j x_j) + O(|X|^2 + |\boldsymbol{x}|^2)$, there is $r_0 > 0$ such that $\kappa \in (0, 1)$ on the nonempty open set $\Omega := \{0 < X < r_0,\ |x_j| < X/(4J + 4)\} \subset \mathbb{R}^{1+J}$.

Lemma 7.1. *Let $0 < |\kappa| < \kappa_0 := \frac{1}{22}$. For $|Q| > 0.6$ the equation $P_\kappa(y) = Q$ has exactly one solution $y = y_0(Q; \kappa)$ in $|y| < \frac{1}{2}$. It is simple, so $P'_\kappa(y_0) \neq 0$; y_0 is holomorphic in (Q, κ), extends holomorphically to $Q = \infty$ by $y_0(\infty; \kappa) = 0$, is odd in Q, and $y_0 = \kappa Q^{-1} + O(Q^{-3})$. If f is meromorphic on $|y| < 1$ and $f P'_\kappa$ is holomorphic on $0 < |y| < 1$, then $F(Q) := f(y_0(Q; \kappa))$ is holomorphic on $|Q| > 0.6$ and, for $R > 0.6$ and $k \geq 0$,*

$$\frac{1}{2\pi \mathrm{i}} \oint_{|Q|=R} F(Q)\, Q^k \, \mathrm{d}Q = -\frac{1}{2\pi \mathrm{i}} \oint_{|y|=\frac{1}{2}} f(y)\, P'_\kappa(y)\, P_\kappa(y)^k \, \mathrm{d}y. \tag{7.1}$$

For real $\kappa \in (0, \kappa_0)$, $y_0(\cdot; \kappa)$ coincides on $|Q| > 0.6$ with the inverse map of Lemma 4.2.

Proof. On $|y| = \frac{1}{2}$, $|\cot y| \leq \coth \frac{1}{2} < 2.2$, so $|P_\kappa(y)| \leq \frac{1}{2} + \frac{2.2}{22} = 0.6$. For $|Q| > 0.6$ the curve $P_\kappa(\{|y| = \frac{1}{2}\})$ does not wind around Q, so $P_\kappa - Q$ has as many zeros in $|y| < \frac{1}{2}$ as poles, namely one (the simple pole at 0). Hence $y_0(Q) = \frac{1}{2\pi \mathrm{i}} \oint_{|y|=\frac{1}{2}} \frac{y\, P'_\kappa(y)}{P_\kappa(y) - Q} \mathrm{d}y$, which is holomorphic in (Q, κ) and in $1/Q$ up to $Q = \infty$, where it vanishes. Oddness follows from $P_\kappa(-y) = -P_\kappa(y)$, and the expansion from $y_0(Q - y_0) = \kappa\, y_0 \cot y_0 = \kappa + O(y_0^2)$. Since $y_0(Q)$ is a simple root, $P'_\kappa(y_0) \neq 0$ and $F = (f P'_\kappa)(y_0)/P'_\kappa(y_0)$ is holomorphic. In (7.1) substitute $Q = P_\kappa(y)$: the circle $|Q| = R$ is the image of the closed curve $\gamma = y_0(\{|Q| = R\}) \subset \{0 < |y| < \frac{1}{2}\}$, which winds -1 times around 0 (y_0 is holomorphic in $1/Q$ with a simple zero at $Q = \infty$ and no other zero); as $f P'_\kappa P_\kappa^k$ is holomorphic on $0 < |y| < 1$, the integral over γ equals -1 times the integral over $|y| = \frac{1}{2}$. Finally, for real $\kappa \in (0, \kappa_0)$ we have $\beta < 0.22$ and $\vartheta_* < 0.22$, so $D_\kappa \subset \{|y| < \frac{1}{2}\}$, and the inverse of Lemma 4.2 is a solution in $|y| < \frac{1}{2}$. □

Definition 7.2. For $(X, \boldsymbol{x})$ near 0 let $\kappa = \kappa(X, \boldsymbol{x})$, and with $r_1 = \frac{1}{2}$, $r_2 = 1$,

$$\mathcal{W}(y) := -\frac{1}{2\pi \mathrm{i}} \oint_{|\eta|=r_2} U_0(P_\kappa(\eta))\, K_1(y, \eta)\, \mathrm{d}\eta \qquad (|y| < r_2),$$

$$G_d^{\mathrm{cl}} := -\frac{1}{2\pi \mathrm{i}} \oint_{|y|=r_1} \mathcal{W}(y)\, P_\kappa(y)^{2d+1} P'_\kappa(y)\, \mathrm{d}y,$$

and $G^{\mathrm{cl}}(\Lambda) := \sum_d G_d^{\mathrm{cl}} \Lambda^{2d}/(2d+1)!$.

The function $\mathcal{W}$ is holomorphic on $|y| < 1$, so the y-circle may be replaced by any circle $|y| = \epsilon < 1$. Whenever $0 < |\kappa| < \kappa_0$, Lemma 7.1 (with $f = \mathcal{W}$, $k = 2d + 1$) shows that G_d^{cl} is the coefficient of Q^{-2d-2} in the Laurent expansion of

$$W^{\mathrm{cl}}(Q) := \mathcal{W}\big(y_0(Q; \kappa)\big), \qquad |Q| > 0.6; \tag{7.2}$$

indeed W^{cl} is even ($\mathcal{W}$ is even since $K_1(-y, \eta) = K_1(y, \eta)$, and y_0 is odd) and holomorphic at ∞ with $W^{\mathrm{cl}}(\infty) = \mathcal{W}(0) = 0$, so its Laurent expansion is $\sum_d G_d^{\mathrm{cl}} Q^{-2d-2}$. For real $(X, \boldsymbol{x}) \in \Omega$ (shrinking r_0 so that $\kappa < \kappa_0$ on Ω), y_0 is the inverse of Lemma 4.2 and $|y_0(Q)| < \frac{1}{2}$, so Lemma 5.2(a) (with $r = 1$) gives $W^{\mathrm{cl}} = \mathcal{L}_\kappa[U_0]$ on $|Q| > 0.6$: G_d^{cl} is the coefficient of Q^{-2d-2} in

$\mathcal{L}_\kappa[U_0]$. Removing the residues at $\eta = \pm y$ as in Lemma 5.2(a), and noting that $U_0(Q)Q^{2d+1}\mathrm{d}Q$ is exact, one also gets the nested-residue formula

$$G^{\mathrm{cl}}(\Lambda) = \frac{1}{\Lambda}\operatorname*{Res}_{y=0}\operatorname*{Res}_{\eta=0} U_0(P(\eta))\,K_1(y,\eta)\,P'(y)\,\mathrm{e}^{\Lambda P(y)}\,\mathrm{d}\eta\,\mathrm{d}y \qquad (|\eta|<|y|). \tag{7.3}$$

Lemma 7.3. *There are $r>0$, $C>0$ and $b\in(0,\pi/4)$ such that for $|X|,|x_j|<r$: each G_d^{cl} is holomorphic, $|G^{\mathrm{cl}}(\Lambda)|\le C\mathrm{e}^{b|\Lambda|}$ for $\Lambda\in\mathbb{C}$, and G_d^{cl} has order at least $d+2$ in $(X,\boldsymbol{x})$.*

Proof. For $|\kappa|$ small the integrands in Definition 7.2 are holomorphic in $(\kappa,X,\boldsymbol{x})$ on the fixed contours (for $|y|=\frac12$ the poles $\pm y$ of $K_1(y,\cdot)$ lie inside, and its other poles $\pm y+\pi(\mathbb{Z}\setminus\{0\})$, $\pi(\mathbb{Z}\setminus\{0\})$ outside, the circle $|\eta|=1$), so G_d^{cl} is holomorphic. On $|y|=\frac12$, $|\cot y|\le\coth\frac12<2.2$, so $|P(y)|\le\frac12+2.2|\kappa|\le 0.6$, whence $|G_d^{\mathrm{cl}}|\le C'(0.6)^{2d}$ and the exponential bound with $b=0.6$. For the order, replace $(X,\boldsymbol{x})$ by $(tX,t\boldsymbol{x})$ with $|t|$ small, so that $|\kappa|\le C''|t|$, and use circles of radii $\epsilon\sqrt{|t|}$ for y and $2\epsilon\sqrt{|t|}$ for η (the integrals do not depend on the radii, by holomorphy). On these circles $|P|=O(\sqrt{|t|})$, $U_0(P(\eta))=O(|t|)$, $K_1=O(|t|^{-1/2})$ and $P'=O(1)$; the two contour lengths are $O(\sqrt{|t|})$ each. Hence $G_d^{\mathrm{cl}}=O(|t|^{1+1/2+1/2-1/2+(2d+1)/2})=O(|t|^{d+2})$. □

Theorem 7.4. *The Taylor series of G^{cl} at $(X,\boldsymbol{x})=0$ equals G^{tree}.*

Proof. Let r,b be as in Lemma 7.3. For $|X|,|x_j|<r$ define

$$t_k^{\mathrm{cl}} := X\delta_{k0} + \sum_j x_j\frac{\Lambda_j^{2k}}{2^k k!} + \frac{1}{2^k k!}\int_0^\infty \frac{\lambda^{2k+1}G^{\mathrm{cl}}(\lambda)\,\mathrm{d}\lambda}{\mathrm{e}^{\pi\lambda}-1}, \qquad \Phi^{\mathrm{cl}}(w) := \sum_k t_k^{\mathrm{cl}}\frac{w^k}{k!}.$$

By Lemma 7.3 and dominated convergence (expand $(\mathrm{e}^{\pi\lambda}-1)^{-1}=\sum_{m\ge1}\mathrm{e}^{-\pi m\lambda}$), t_k^{cl} is holomorphic with $|t_k^{\mathrm{cl}}|\le C_1^{k+1}k!$, and its Taylor series is $t_k[\widehat{G}]$, where $\widehat{G}$ is the Taylor series of G^{cl} (which lies in $\mathfrak{G}$ by Lemma 7.3). So Φ^{cl} is holomorphic for $|w|<w_0$ and its Taylor series is $\varphi_{\widehat{G}}$; for real parameters and $0\le w<w_0$, $\Phi^{\mathrm{cl}}(w)$ equals (6.2), by expanding $I_0(\lambda\sqrt{2w})=\sum_k(w/2)^k\lambda^{2k}/k!^2$.

The function $u:=a^2/2$ is holomorphic in $(X,\boldsymbol{x})$ near 0, since $a/\sqrt{\kappa}=\arcsin(\sqrt{\kappa})/\sqrt{\kappa}+\sqrt{1-\kappa}$ is holomorphic in κ, and $u(0)=0$. Consider

$$E_{-1} := u-\Phi^{\mathrm{cl}}(u), \qquad E_d := G_d^{\mathrm{cl}} - \frac{(2d+1)!!}{(d+1)!}\int_0^u w^{d+1}\big(1-\partial_w\Phi^{\mathrm{cl}}(w)\big)\mathrm{d}w.$$

They are holomorphic near 0. After shrinking r_0 so that $u<w_0$ on Ω, they vanish on Ω by Proposition 6.4; as Ω is a nonempty open subset of $\mathbb{R}^{1+J}$, they vanish identically. Taking Taylor series, $\widehat{u}$ solves $\widehat{u}=\varphi_{\widehat{G}}(\widehat{u})$, so $\widehat{u}=u_{\widehat{G}}$, and $\widehat{G}$ satisfies (3.1). By Corollary 3.3, $\widehat{G}=G^{\mathrm{tree}}$. □

From now on we identify $G=G^{\mathrm{tree}}=G^{\mathrm{cl}}$ (as holomorphic functions of $(X,\boldsymbol{x})$ near 0) and write $W=\sum_d G_dQ^{-2d-2}$; by (7.2), $W(Q)=\mathcal{W}_\kappa[U_0]\big(y_0(Q;\kappa)\big)$ for $|Q|>0.6$ whenever $0<|\kappa|<\kappa_0$, with the notation of Lemma 8.1.

8. Multi-point functions

Lemma 8.1. *Let $0<|\kappa|<\kappa_0$, U even and entire, $|Q|>0.6$, and $y=y_0(Q;\kappa)$ as in Lemma 7.1. Then*

$$\frac{\partial}{\partial\kappa}\Big[\mathcal{W}_\kappa[U]\big(y_0(Q;\kappa)\big)\Big]_{Q\text{ fixed}} = -\frac{2}{P'(y)}\frac{\partial\Phi_\kappa[U]}{\partial\kappa}, \qquad y=y_0(Q;\kappa),$$

where $\mathcal{W}_\kappa[U](y) := -\frac{1}{2\pi\mathrm{i}}\oint_{|\eta|=1}U(P_\kappa(\eta))K_1(y,\eta)\,\mathrm{d}\eta$ for $|y|<1$.

Proof. Write $\partial_\kappa P(\eta) = \cot\eta$ and $\partial_\kappa y_0 = -\cot y / P'(y)$. Differentiating under the integral and integrating by parts in η, using $\partial_y K_1(y,\eta) = -\partial_\eta\big[\cot(y-\eta)+\cot(y+\eta)\big]$,

$$\partial_\kappa\big[\mathcal{W}_\kappa[U](y_0)\big] = -\frac{1}{2\pi\mathrm{i}}\oint_{|\eta|=1} U'(P(\eta))\,B(y,\eta)\,\mathrm{d}\eta,$$

$$B(y,\eta) := \cot\eta\,K_1(y,\eta) - \frac{\cot y}{P'(y)}P'(\eta)\big(\cot(y-\eta)+\cot(y+\eta)\big).$$

Put $\alpha = \cot y$, $\alpha' = \cot\eta$, so $P'(y) = 1-\kappa(1+\alpha^2)$, $P'(\eta) = 1-\kappa(1+\alpha'^2)$, $\cot(y\mp\eta) = (\alpha\alpha'\pm 1)/(\alpha'\mp\alpha)$. A direct computation gives

$$\cot\eta\,K_1 = \frac{2\alpha'^2(\alpha^2+1)}{\alpha'^2-\alpha^2} + 2\alpha'^2, \qquad \cot(y-\eta)+\cot(y+\eta) = \frac{2\alpha(\alpha'^2+1)}{\alpha'^2-\alpha^2},$$

and then

$$\begin{aligned} P'(y)\,B &= \frac{2\alpha'^2(\alpha^2+1)P'(y) - 2\alpha^2(\alpha'^2+1)P'(\eta)}{\alpha'^2-\alpha^2} + 2\alpha'^2P'(y) \\ &= 2(1-\kappa) + 2\kappa\alpha^2\alpha'^2 + 2\alpha'^2P'(y) = 2\alpha'^2 + 2P'(\eta). \end{aligned}$$

Hence $B = \frac{2}{P'(y)}\big(\cot^2\eta + P'(\eta)\big)$, and

$$\partial_\kappa\big[\mathcal{W}_\kappa[U](y_0)\big] = -\frac{2}{P'(y)}\Big(\frac{1}{2\pi\mathrm{i}}\oint U'(P(\eta))\cot^2\eta\,\mathrm{d}\eta + \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}\big(U(P(\eta))\big)\Big) = -\frac{2}{P'(y)}\,\partial_\kappa\Phi_\kappa[U].$$

□

Proposition 8.2. *Let δU be even and entire, and let δ denote the derivative along a one-parameter family of potentials $U_0 + \epsilon\,\delta U$ at $\epsilon = 0$, with κ determined by the string equation. Then, at points where $0 < |\kappa| < \kappa_0$, for $|Q| > 0.6$,*

$$\delta W(Q) = \mathcal{W}_\kappa[\delta U]\big(y_0(Q)\big) + \frac{2}{P'(y_0(Q))}\,\Phi_\kappa[\delta U], \qquad \delta\kappa = -\frac{\Phi_\kappa[\delta U]}{\partial_\kappa\Phi_\kappa[U_0]}.$$

In particular δW depends on $(X,\boldsymbol{x})$ only through κ. For $\delta U = -1$ (i.e. $\delta = \partial_X$) this gives $\partial_X W = -2/P'(y_0) = -2\,y_0'(Q)$ and $\partial_X\kappa = 1/\partial_\kappa\Phi_\kappa[U_0]$; for $\delta U = -\cosh(\Lambda_j\,\cdot)$ (i.e. $\delta = \partial_{x_j}$), $\partial_{x_j}\kappa = R(\Lambda_j;\kappa)\,\partial_X\kappa$.

Proof. $W(Q) = \mathcal{W}_\kappa[U_0](y_0(Q;\kappa))$ is linear in U_0 for fixed κ, and y_0 depends only on (Q,κ). So

$$\delta W = \mathcal{W}_\kappa[\delta U](y_0) + \delta\kappa\cdot\partial_\kappa\big[\mathcal{W}_\kappa[U_0](y_0)\big],$$

and the string equation $\Phi_\kappa[U_0] = 0$ gives $\delta\kappa = -\Phi_\kappa[\delta U]/\partial_\kappa\Phi_\kappa[U_0]$. Insert Lemma 8.1. For $\delta U = -1$ use $\Phi_\kappa[1] = 1$ and $\mathcal{W}_\kappa[1](y) = -\frac{1}{2\pi\mathrm{i}}\oint_{|\eta|=1} K_1(y,\eta)\mathrm{d}\eta = -(-1-1+2) = 0$ for $|y| < 1$ (as in Lemma 5.2(c)); for $\delta U = -\cosh(\Lambda_j\cdot)$ use (5.5). □

Corollary 8.3. *Add to the J legs a leg of length $\Lambda' \geq 0$ and weight x', and put $G^{[2]}(\Lambda,\Lambda') := \partial_{x'}G(\Lambda)\big|_{x'=0}$, a holomorphic function of $(X,\boldsymbol{x})$ near 0. Then $G^{[2]}(\Lambda,\Lambda') = \mathcal{G}^{[2]}(\Lambda,\Lambda';\kappa(X,\boldsymbol{x}))$ for a function $\mathcal{G}^{[2]}$ holomorphic in κ near 0, and*

$$\partial_\kappa\mathcal{G}^{[2]}(\Lambda,\Lambda';\kappa) = 2R(\Lambda;\kappa)R(\Lambda';\kappa), \qquad \mathcal{G}^{[2]}(\Lambda,\Lambda';0) = 0.$$

The same holds without setting $x' = 0$, i.e. $\partial_{x'}G(\Lambda) = \mathcal{G}^{[2]}(\Lambda,\Lambda';\kappa)$ at every point near 0.

Proof. By Proposition 8.2 with $\delta U = -\cosh(\Lambda'\,\cdot)$, the Laurent coefficients of $\delta W = \partial_{x'}W$ are, by (7.1) with $fP' = \mathcal{W}_\kappa[\delta U]P' + 2\Phi_\kappa[\delta U]$,

$$-\frac{1}{2\pi\mathrm{i}}\oint_{|y|=\frac12}\Big(\mathcal{W}_\kappa[\delta U](y)\,P'(y) + 2\Phi_\kappa[\delta U]\Big)P(y)^{2d+1}\,\mathrm{d}y,$$

at points with $0<|\kappa|<\kappa_0$. This expression depends only on κ (and Λ') and is holomorphic for $|\kappa|<\kappa_0$; it defines $\mathcal{G}^{[2]}$, and $\partial_{x'}G(\Lambda)=\mathcal{G}^{[2]}(\Lambda,\Lambda';\kappa)$ holds where $\kappa\neq 0$, hence everywhere near 0 by continuity. Add a further leg of length $\Lambda''\geq 0$ and weight x'', and put $G^{[3]}(\Lambda,\Lambda',\Lambda''):=\partial_{x''}\partial_{x'}G(\Lambda)|_{x'=x''=0}$. By Proposition 8.2, $\partial_{x''}\kappa=R(\Lambda'';\kappa)\,\partial_X\kappa$, so

$$G^{[3]}(\Lambda,\Lambda',\Lambda'')=\partial_\kappa\mathcal{G}^{[2]}(\Lambda,\Lambda';\kappa)\,R(\Lambda'';\kappa)\,\partial_X\kappa. \tag{8.1}$$

$G^{[2]}$ and $G^{[3]}$ (with the added weights set to 0) are symmetric in their arguments, since their Taylor coefficients in $(X,\boldsymbol{x})$ are values of the symmetric polynomials $V_n^\flat$ (Theorem 7.4, (2.4)). Since $R(0;\kappa)=\Phi_\kappa[1]=1$ and $\partial_X\kappa\neq 0$ near 0, putting $\Lambda''=0$ and using the symmetry of $G^{[3]}$ twice gives $\partial_\kappa\mathcal{G}^{[2]}(\Lambda,\Lambda')=\partial_\kappa\mathcal{G}^{[2]}(\Lambda,0)\,R(\Lambda')=\partial_\kappa\mathcal{G}^{[2]}(0,0)\,R(\Lambda)R(\Lambda')$ at $\kappa=\kappa(X,\boldsymbol{x})$, hence (as $\kappa(X,0)$ takes all small values) as functions of κ. Finally $\mathcal{G}^{[2]}(0,0;\kappa)=\partial_X G_0$ is the coefficient of Q^{-2} in $\partial_X W=-2y_0'(Q)$, which is 2κ by Lemma 7.1; so $\partial_\kappa\mathcal{G}^{[2]}(0,0)=2$. At $(X,\boldsymbol{x})=0$ both κ and $G^{[2]}$ vanish (a tree has at least three legs), so $\mathcal{G}^{[2]}(\Lambda,\Lambda';0)=0$. □

Theorem 8.4. *For $n\geq 2$, legs of lengths $\Lambda_2,\dots,\Lambda_n$ and weights $x_2,\dots,x_n$ among the x_j, as holomorphic functions of $(X,\boldsymbol{x})$ near 0,*

$$\frac{\partial^{n-1}G(\Lambda_1)}{\partial x_2\cdots\partial x_n}=\frac{\partial^{n-2}}{\partial X^{n-2}}\Big[2\int_0^\kappa\prod_{i=1}^n R(\Lambda_i;k)\,\mathrm{d}k\Big],\qquad \kappa=\kappa(X,\boldsymbol{x}).$$

Also $\partial_X G(\Lambda)=2\int_0^\kappa R(\Lambda;k)\mathrm{d}k=\frac{2}{\Lambda}\operatorname{Res}_{\eta=0}\mathrm{e}^{\Lambda P_\kappa(\eta)}\mathrm{d}\eta$.

Proof. For $n=2$ this is Corollary 8.3. Write $\kappa_X=\partial_X\kappa$. If $H=f(\kappa)\,\kappa_X$ with f depending on κ and the Λ_i only, then by $\partial_{x_j}\kappa=R_j\kappa_X$ (with $R_j=R(\Lambda_j;\kappa)$) and $\partial_{x_j}\kappa_X=\partial_X(R_j\kappa_X)$,

$$\partial_{x_j}H=f'R_j\kappa_X^2+f\,\partial_X(R_j\kappa_X)=\partial_X\big(fR_j\kappa_X\big).$$

By (8.1) and Corollary 8.3, $G^{[3]}=2R_1R_2R_3\kappa_X=\partial_X\big[2\int_0^\kappa R_1R_2R_3\big]$; induction with the displayed rule (and $\partial_{x_j}\partial_X=\partial_X\partial_{x_j}$) gives the claim for all $n\geq 3$. The last statement is the case $n=2$, $\Lambda_2=0$, together with $\partial_\kappa\big(\frac{1}{\Lambda}\operatorname{Res}\mathrm{e}^{\Lambda P_\kappa}\big)=\operatorname{Res}\cot\eta\,\mathrm{e}^{\Lambda P_\kappa(\eta)}=R(\Lambda;\kappa)$ (the sinh-part has zero residue by parity) and vanishing at $\kappa=0$. □

Proof of Theorem B. This is Theorem 8.4 together with Theorem 7.4. □

Proof of Theorem A. Apply Theorem 8.4 with all n legs marked and evaluate at $\boldsymbol{x}=0$, $X=0$, using (2.5) and Lemma 2.1. At $\boldsymbol{x}=0$, $\kappa=1-\sqrt{1-X}$ and $2\kappa_X=(1-X)^{-1/2}$; for $n\geq 3$ the right-hand side of Theorem 8.4 is $\partial_X^{n-3}\big[(1-X)^{-1/2}\prod_i R(\Lambda_i;\kappa)\big]$. The coefficient of $\prod_i\Lambda_i^{2d_i}/(2d_i+1)!$ in $\prod_i R(\Lambda_i;\kappa)$ is $\prod_i(2d_i+1)\rho_{d_i}(\kappa)$, by (1.10). This gives (1.9). For (1.8), $\partial_X^{n-3}f|_{X=0}=(n-3)!\operatorname{Res}_{X=0}f\,X^{2-n}\mathrm{d}X$; substitute $X=\kappa(2-\kappa)$, $\mathrm{d}X=2(1-\kappa)\mathrm{d}\kappa$, $(1-X)^{-1/2}=(1-\kappa)^{-1}$. □

Remark 8.5 (Witten–Kontsevich)**.** Keeping only one-vertex trees replaces $1/(\mathrm{e}^{\pi\lambda}-1)$ by 0. Then the images disappear from (5.6), cot is replaced by $1/\eta$ in Sections 5–8, $\Phi_\kappa[Q^2]=2\kappa$, and $R(\Lambda;\kappa)=\operatorname{Res}\cosh(\Lambda(\eta+\kappa/\eta))\mathrm{d}\eta/\eta=\sum_q\binom{2q}{q}\kappa^q\Lambda^{2q}/(2q)!=I_0(2\Lambda\sqrt{\kappa})$. With $u=2\kappa$ the string equation is $u=X+\sum_j x_jI_0(\Lambda_j\sqrt{2u})$, the genus-zero string equation for the times $t_k=X\delta_{k0}+\sum_j x_j\Lambda_j^{2k}/(2^kk!)$, and Theorem 8.4 reduces to $\langle\langle\tau_{a_1}\cdots\tau_{a_n}\rangle\rangle_0=\partial_{t_0}^{n-3}\big(\partial_{t_0}u\prod_i u^{a_i}/a_i!\big)$.

9. One boundary, the spectral curve, and the critical point

9.1. **One row.** At $\boldsymbol{x}=0$ write

$$G(\Lambda;X)=\sum_{d\ge 0}G_d(X)\frac{\Lambda^{2d}}{(2d+1)!},\qquad G_d(X)=G^{(0)}_{(d)}(X)=\sum_{n\ge 3}h_{0,n}[d,0^{n-1}]\frac{X^{n-1}}{(n-1)!},$$
$$h_{0,n}[d,0^{n-1}]=\pi^{-2(n-3-d)}H_n[d].$$

Proposition 9.1. *Let $\kappa=1-\sqrt{1-X}$. Then*

$$G_d=4\sum_{j=0}^{d}\binom{2d+1}{d+1+j}c_{d+1+j,j}\Big(\frac{\kappa^{d+j+2}}{d+j+2}-\frac{\kappa^{d+j+3}}{d+j+3}\Big),$$

and for real $0<X<1$ and $|Q|>a$, $\sum_d G_dQ^{-2d-2}=2y_0^2-4\kappa(1-y_0\cot y_0)$, $y_0=y_0(Q)$; the series diverges for $|Q|<a$.

Proof. By Theorem 8.4, $\partial_X G_d=2\left[\Lambda^{2d}/(2d+1)!\right]\frac{1}{\Lambda}\operatorname{Res}\mathrm{e}^{\Lambda P_\kappa}=2\operatorname{Res}_{\eta=0}(\eta+\kappa\cot\eta)^{2d+1}\mathrm{d}\eta=2\sum_j\binom{2d+1}{d+1+j}c_{d+1+j,j}\kappa^{d+1+j}$, expanding as in (1.7). Integrate with $\mathrm{d}X=2(1-\kappa)\mathrm{d}\kappa$ and $G_d(0)=0$. For the resolvent: by Remark 5.5 (for $\boldsymbol{x}=0$ and $X=2\kappa-\kappa^2$ the string equation holds), $\mathcal{L}_\kappa[U_0]=2y_0^2-4\kappa(1-y_0\cot y_0)$ for every $\kappa\in(0,1)$. Its Laurent coefficients at ∞ are, as after Definition 7.2, $-\operatorname{Res}_{y=0}\big(2y^2-4\kappa+4\kappa y\cot y\big)P(y)^{2d+1}P'(y)\,\mathrm{d}y$, polynomials in κ; they agree with G_d for small κ by Theorem 7.4, hence for all κ, and the Laurent series converges on $|Q|>a$ because $\mathcal{L}_\kappa[U_0]$ is holomorphic there. The series diverges for $|Q|<a$ because $\mathcal{L}_\kappa[U_0]$ is not holomorphic at $\pm a$: $W'(Q)=4y_0+4(Q-y_0)y_0'(Q)$ with $Q-y_0\to\kappa\cot\beta\neq 0$ and $|y_0'|\to\infty$. □

Proof of Corollary 1.1. By Theorem A with $d_1=d$ and $n-1$ zeros, $H_n[d]=2(n-3)!\pi^{2m}(2d+1)\left[\kappa^{n-3}\right](2-\kappa)^{2-n}\rho_d(\kappa)$. Since $\rho_d=\sum_j\binom{2d}{d+j}c_{d+j+1,j}\kappa^{d+j}$ and $\left[\kappa^{m-j}\right](2-\kappa)^{-(m+d+1)}=\binom{2m+d-j}{m-j}2^{-(2m+d+1-j)}$,

$$H_n[d]=\pi^{2m}(2d+1)\sum_j\binom{2d}{d+j}c_{d+j+1,j}\frac{2\,(m+d)!\,(2m+d-j)!}{(m-j)!\,(m+d)!\,2^{2m+d+1-j}},$$

which is (1.14). For $d=0$ only $j=0$ contributes: $H_n[0]=\pi^{2m}(2m)!/(4^m m!)$, and by (1.2), for $n\ge 4$,

$$\mathrm{MV}_{0,n}=\frac{2^{n-2}(n-4)!}{(2n-7)!}\cdot\pi^{2(n-3)}\frac{(2n-6)!}{4^{n-3}(n-3)!}=\pi^{2(n-3)}\,2^{4-n}\frac{(2n-6)(n-4)!}{(n-3)!}=\pi^{2(n-3)}2^{5-n}.$$

□

9.2. **Polynomiality.**

Proposition 9.2. *Corollary 1.2 holds, with*

$$\mathcal{P}_{d_1,\dots,d_k}(n)=\prod_{i=1}^{k}(2d_i+1)\sum_{l=0}^{|d|}\pi_l\,2^{\,l-|d|}\,m^{\underline{l}}\,(2m+1)^{\overline{|d|-l}},\qquad m=n-3-|d|,\tag{9.1}$$

where $\sum_l\pi_l\kappa^l:=\prod_{i=1}^k\rho_{d_i}(\kappa)/\kappa^{d_i}$. Moreover, for $k\ge 3$,

$$G^{(0)}_{(d_1,\dots,d_k)}(X)=\sum_{n\ge k}h_{0,n}[d_1,\dots,d_k,0^{n-k}]\frac{X^{n-k}}{(n-k)!}=\frac{\mathrm{d}^{\,k-3}}{\mathrm{d}X^{k-3}}\Big[(1-X)^{-1/2}\prod_{i=1}^{k}(2d_i+1)\rho_{d_i}(\kappa)\Big];$$

for $k=1$, G_d is a polynomial in $s=\sqrt{1-X}$ of degree $2d+3$, divisible by $(1-s)^{d+2}$, with vanishing coefficient of s^1 and coefficient $\frac{4}{3}(2d+1)\rho_d(1)$ of s^3. Equivalently, $G_d(X)-(1-X)^{3/2}\widetilde{P}_d(X)$ is a polynomial of degree $\le d+1$ for a polynomial $\widetilde{P}_d$ of degree exactly d with $\widetilde{P}_d(1)=\frac{4}{3}(2d+1)\rho_d(1)$.

Proof. By (1.8) with $n-k$ zeros and $n-3=m+|d|$,

$$\frac{F_{0,n}[d_1,\dots,d_k,0^{n-k}]}{\pi^{2m}(2m)!/(4^m m!)}=2\,(m+|d|)!\,\frac{4^m m!}{(2m)!}\prod_i(2d_i+1)\sum_l \pi_l\left[\kappa^{m-l}\right](2-\kappa)^{-(m+|d|+1)},$$

and $\left[\kappa^{m-l}\right](2-\kappa)^{-(m+|d|+1)}=\binom{2m+|d|-l}{m-l}2^{-(2m+|d|+1-l)}$ (zero for $l>m$). The l-th term therefore equals $\prod_i(2d_i+1)\,\pi_l\,2^{l-|d|}\frac{m!}{(m-l)!}\frac{(2m+|d|-l)!}{(2m)!}$, which is (9.1). Each $m^{\underline{l}}(2m+1)^{\overline{|d|-l}}$ is a polynomial in m of degree $|d|$ with leading coefficient $2^{|d|-l}$, so $\mathcal{P}_{d_1,\dots,d_k}$ has degree $\le|d|$ and its coefficient of $m^{|d|}$ (equivalently of $n^{|d|}$) is $\prod_i(2d_i+1)\sum_l\pi_l=\prod_i(2d_i+1)\rho_{d_i}(1)$, which is positive by Proposition 9.3. For $k=1$, dividing (1.14) by $\frac{2d+1}{(2d-1)!!}\frac{(2m)!}{4^m m!}\pi^{2m}$ and using $(2d)!=2^d d!\,(2d-1)!!$ and $(2m+d-j)!/(m-j)!=\frac{(2m)!}{m!}m^{\underline{j}}(2m+1)^{\overline{d-j}}$ gives (1.15).

The generating function for $k\ge3$ is Theorem 8.4 at $\boldsymbol{x}=0$ (as in the proof of Theorem A). For $k=1$ and $\kappa=1-s$, Proposition 9.1 gives

$$\frac{\mathrm{d}G_d}{\mathrm{d}s}=-4s\sum_j\binom{2d+1}{d+1+j}c_{d+1+j,j}(1-s)^{d+1+j},$$

which vanishes at $s=0$; the coefficient of s^2 in it is $4\sum_j(d+1+j)\binom{2d+1}{d+1+j}c_{d+1+j,j}=4(2d+1)\rho_d(1)$, using $(d+1+j)\binom{2d+1}{d+1+j}=(2d+1)\binom{2d}{d+j}$. The coefficient of κ^{2d+3} in G_d is $-4c_{2d+1,d}/(2d+3)\ne0$, since $c_{2d+1,d}=\operatorname{Res}_{t=0}\cot^{2d+1}t\,\mathrm{d}t=\left[w^{2d}\right](1+w^2)^{-1}=(-1)^d$ (substitute $w=\tan t$). So the odd part of G_d in s is s^3 times a polynomial of degree exactly d in $s^2=1-X$, and the even part is a polynomial of degree $\le d+1$ in X. □

9.3. **The critical point.** At $\kappa=1$ the map $P_1(y)=y+\cot y$ is critical at $\pm\pi/2$, $P_1(\frac{\pi}{2}+\epsilon)=\frac{\pi}{2}-\frac{\epsilon^3}{3}+O(\epsilon^5)$, and Lemma 4.1 applies: P_1 maps $D_1\setminus\{0\}$ biholomorphically onto $\mathbb{C}\setminus[-\frac{\pi}{2},\frac{\pi}{2}]$ (Figure 1).

Proposition 9.3. *Let y_0 be the inverse of $P_1|_{D_1\setminus\{0\}}$ and $f(Q):=\tan y_0(Q)$. Then f is holomorphic on $\mathbb{C}\setminus[-\frac{\pi}{2},\frac{\pi}{2}]$, $f(Q)=\sum_{d\ge0}\rho_d(1)\,Q^{-2d-1}$ for $|Q|>\pi/2$, and*

$$\rho_d(1)=\int_{-\pi/2}^{\pi/2}c^{2d}\varrho(c)\,\mathrm{d}c,\qquad \varrho(c):=-\frac{1}{\pi}\operatorname{Im}f_+(c)>0.$$

In particular $\rho_d(1)>0$ for all d.

Proof. y_0 takes values in $D_1\subset\{|\operatorname{Re}y|<\pi/2\}$, where tan is holomorphic. For $R>\pi/2$, substituting $Q=P_1(y)$ as after Definition 7.2,

$$\frac{1}{2\pi\mathrm{i}}\oint_{|Q|=R}f(Q)Q^{2d}\mathrm{d}Q=-\operatorname*{Res}_{y=0}\tan y\,P_1(y)^{2d}P_1'(y)\,\mathrm{d}y=\operatorname*{Res}_{y=0}\cot y\,(y+\cot y)^{2d}\mathrm{d}y=\rho_d(1),$$

since $\tan y\,P_1'(y)=-\cot y$; even powers Q^{-2d-2} do not occur since f is odd. The arguments of Lemma 4.2(b) show that f has continuous boundary values on $(-\frac{\pi}{2},\frac{\pi}{2})$, $f_+(c)=\tan y^-$ with $y^-\in\Gamma^-$ over c. As $Q\to\pm\pi/2$, $y_0(Q)\to\pm\pi/2$ (every limit point $y^*\in\overline{D_1}$ has $P_1(y^*)=\pm\pi/2$, and $\pm\pi/2$ is the only such point by Lemma 4.1(b),(c)), and the cubic behaviour of P_1 gives $|y_0(Q)\mp\frac{\pi}{2}|\ge c_0|Q\mp\frac{\pi}{2}|^{1/3}$, hence $|f(Q)|\le C|Q\mp\frac{\pi}{2}|^{-1/3}$. Shrinking the circle

onto the segment (the small circles around $\pm\pi/2$ contribute $O(\epsilon^{2/3})$) and using $f_- = \overline{f_+}$, $\rho_d(1) = \frac{1}{2\pi\mathrm{i}}\int_{-\pi/2}^{\pi/2}(f_- - f_+)c^{2d}\mathrm{d}c = -\frac{1}{\pi}\int \operatorname{Im} f_+(c)\, c^{2d}\mathrm{d}c$. For $|c| < \frac{\pi}{2}$, $y^- = t - \mathrm{i}\vartheta$ with $\vartheta > 0$, and $\operatorname{Im}\tan(t-\mathrm{i}\vartheta) = -\sinh 2\vartheta/(\cos 2t + \cosh 2\vartheta) < 0$. □

Remark 9.4 (Asymptotics)**.** By Proposition 9.2, the generating series $G^{(0)}_{(d_1,\dots,d_k)}$ are singular only at $X = 1$, where the cut touches its images; the leading singularity is $\prod_i(2d_i+1)\rho_{d_i}(1)\,\partial_X^{k-3}(1-X)^{-1/2}$ for $k \geq 3$, and $\frac{4}{3}(2d+1)\rho_d(1)(1-X)^{3/2}$ for $k = 1$. Transfer theorems give, for example, $H_n[d] \sim (2d+1)\rho_d(1)\,\pi^{2(n-3-d)}(n-1)!/(\sqrt{\pi}\,n^{5/2})$ as $n \to \infty$; this is also immediate from Corollary 1.2 and $\frac{(2m)!}{4^m m!} \sim \frac{m!}{\sqrt{\pi m}}$.

9.4. **The two-point kernel.** Throughout this subsection $\boldsymbol{x} = 0$, $\kappa \in (0,1)$ and $X = 2\kappa - \kappa^2$. Let $e(Q;\kappa) := \sum_d (2d+1)\rho_d(\kappa)Q^{-2d-2}$, the Laplace transform $\int_0^\infty \Lambda R(\Lambda;\kappa)\mathrm{e}^{-Q\Lambda}\mathrm{d}\Lambda$ taken termwise, and

$$K(y_1,y_2) := \frac{1}{\sin^2(y_1-y_2)} + \frac{1}{\sin^2(y_1+y_2)}.$$

On $|\eta| = 1$ we have $|\cot\eta| \leq \coth 1 < 1.32$, so $|\eta + k\cot\eta| < 2.5$ for $|k| \leq 1.1$; hence $|\rho_d(k)| \leq 1.32 \cdot 2.5^{2d}$ for $|k| \leq 1.1$, and $e(Q;k)$ converges uniformly for $|Q| \geq 3$, $|k| \leq 1.1$.

Proposition 9.5. (a) *The series $e(Q;\kappa)$ converges for $|Q| > a$, and there $e(Q;\kappa) = -\frac{1}{2}K(\beta,y)/P'(y)$ with $y = y_0(Q)$.*

(b) *For $|Q_1|, |Q_2| > a$ the series $W_{0,2}(Q_1,Q_2) = \sum_{d_1,d_2} G^{(0)}_{(d_1,d_2)}(X)Q_1^{-2d_1-2}Q_2^{-2d_2-2}$ converges, and*

$$W_{0,2}(Q_1,Q_2) = \frac{K(y_1,y_2)}{P'(y_1)P'(y_2)} - \frac{1}{(Q_1-Q_2)^2} - \frac{1}{(Q_1+Q_2)^2}, \qquad y_i = y_0(Q_i).$$

(c) *For $n \geq 3$ and $|Q_i| > a$, the series $W_{0,n}(Q_1,\dots,Q_n) = \sum_d G_d^{(0)}(X)\prod_i Q_i^{-2d_i-2}$ converges and equals $\partial_X^{n-3}\big[(1-X)^{-1/2}\prod_i e(Q_i;\kappa(X))\big]$, with e given by (a).*

In particular Theorem C holds.

Proof. (a) For $|Q| > 3$, by uniform convergence on $|\eta| = 1$, $e(Q) = \frac{1}{2\pi\mathrm{i}}\oint_{|\eta|=1}\cot\eta\,\frac{1}{2}\big[(Q - P(\eta))^{-2} + (Q+P(\eta))^{-2}\big]\mathrm{d}\eta$, using $\sum_q(2q+1)z^{2q} = (1+z^2)/(1-z^2)^2$. Since $|P| < |Q|$ on $|\eta| = 1$, $P(\eta) \mp Q$ has exactly one zero in $|\eta| < 1$ (as many as poles); for $|Q|$ large this zero is $\pm y$, $y = y_0(Q)$, because $y_0(Q) \to 0$. Then, inside $|\eta| < 1$, the integrand is holomorphic at $\eta = 0$ (it is $O(\eta)$) and has double poles at $\pm y$, whose residues sum to $-\frac{1}{\sin^2 y\,P'(y)^2} - \frac{\cot y\,P''(y)}{P'(y)^3}$. With $t = \sin^2 y$, $P'(y) = 1 - \kappa/t$ and $P''(y) = 2\kappa\cot y/t$, this is $-\frac{1}{P'(y)}\cdot\frac{t-2\kappa t+\kappa}{(t-\kappa)^2}$, and $\frac{1}{2}K(\beta,y) = \frac{\sin^2 y+\kappa-2\kappa\sin^2 y}{(\sin^2 y-\kappa)^2}$ (from $\sin(y-\beta)\sin(y+\beta) = \sin^2 y - \kappa$ and $\sin^2(y-\beta)+\sin^2(y+\beta) = 1 - \cos 2y\cos 2\beta$). So (a) holds for $|Q|$ large. The right-hand side is holomorphic on $|Q| > a$ including ∞ (there $y_0 \in D_\kappa$, so $y_0 \neq \pm\beta$ and $P'(y_0) \neq 0$); hence its Laurent series at ∞, which is $e(Q)$, converges on $|Q| > a$ and represents it there.

(b) By Theorem 8.4 with $n = 2$ at $\boldsymbol{x} = 0$ (cf. (2.5)), $G^{(0)}_{(d_1,d_2)}(X) = 2\int_0^\kappa (2d_1+1)(2d_2+1)\rho_{d_1}(k)\rho_{d_2}(k)\,\mathrm{d}k$ as power series in X; the right-hand side is holomorphic in $|X| < 1$ (as $\kappa(X)$ is), so the series converges there and the identity holds for $0 < X < 1$. For $|Q_i| > 3$, uniform convergence allows summation under the integral: $W_{0,2} = 2\int_0^\kappa e(Q_1;k)e(Q_2;k)\,\mathrm{d}k$, where by (a) $e(Q_i;k)$ is given by the closed form, since $|Q_i| > 3 > a(k)$. Now fix Q_i. At fixed Q_i, $\partial_\kappa y_i = -\cot y_i/P'(y_i) = -\frac{1}{2}\big(\cot(y_i-\beta)+\cot(y_i+\beta)\big)$, since $\cot(y-\beta)+\cot(y+\beta) = \sin 2y/(\sin^2 y-\kappa)$. For $v(y) = \cot(y-b)$ one has the identity $\partial_{y_1}\big(v(y_1)\sin^{-2}(y_1-y_2)\big)+\partial_{y_2}\big(v(y_2)\sin^{-2}(y_1-y_2)\big) =$

$-\sin^{-2}(y_1-b)\sin^{-2}(y_2-b)$ (expand and use $\sin^2 A+\sin^2 B-2\sin A\sin B\cos(A-B)=\sin^2(A-B)$). Since $\partial_\kappa\big[f(y_1,y_2)\,y_1'(Q_1)\,y_2'(Q_2)\big]=\big[\partial_{y_1}(v_1 f)+\partial_{y_2}(v_2 f)\big]y_1'y_2'$ with $v_i=\partial_\kappa y_i$, applying the identity with $b=\pm\beta$ to both terms of K (to the second after $y_2\mapsto -y_2$, using that v is odd) gives $\partial_\kappa\big[K(y_1,y_2)y_1'y_2'\big]=\frac12 K(\beta,y_1)K(\beta,y_2)y_1'y_2'=2e(Q_1;\kappa)e(Q_2;\kappa)$. Both sides of the claimed formula tend to 0 as $\kappa\to 0$ (for fixed Q_i, $y_i=\kappa/Q_i+O(\kappa^2)$ together with its Q_i-derivative, by Lemma 7.1, the integral formula for y_0 in its proof being holomorphic at $\kappa=0$ as well; and $K(y_1,y_2)y_1'y_2'\to(Q_1-Q_2)^{-2}+(Q_1+Q_2)^{-2}$ by Möbius invariance of $\mathrm{d}y_1\mathrm{d}y_2/(y_1\mp y_2)^2$), and they have equal κ-derivatives. So (b) holds for $|Q_i|>3$. The right-hand side is holomorphic in $(1/Q_1,1/Q_2)$ on the polydisc $|Q_1|,|Q_2|>a$ (including ∞): y_0 is injective with values in $D_\kappa\subset\{|\operatorname{Re} y|<\pi/2\}$, so $\sin(y_1\mp y_2)=0$ only on $Q_1=\pm Q_2$, and there the double poles cancel against $(Q_1\mp Q_2)^{-2}$, as for any local biholomorphism. Hence its double Laurent series, which is $W_{0,2}$, converges there.

(c) By Proposition 9.2, $G^{(0)}_{(d_1,\dots,d_n)}(X)=\partial_X^{n-3}\big[(1-X)^{-1/2}\prod_i(2d_i+1)\rho_{d_i}(\kappa(X))\big]$, holomorphic on $|X|<1$. For $|Q_i|\ge 3$ the series $\sum_d(1-X)^{-1/2}\prod_i(2d_i+1)\rho_{d_i}(\kappa(X))Q_i^{-2d_i-2}$ converges uniformly for X in a complex neighbourhood of $[0,1-\delta]$ ($\delta>0$; there $|\kappa(X)|\le 1.1$), so it may be differentiated termwise; this gives (c) for $|Q_i|>3$. Since $y_0(Q;\kappa)$ is holomorphic in (Q,κ) on $\{|Q|>a(\kappa)\}$ (implicit function theorem, $P'(y_0)\neq 0$), the right-hand side of (c) is holomorphic in $1/Q_i$ on $|Q_i|>a$, and the extension follows as in (b). □

Remark 9.6 (Witten–Kontsevich at $t_2=\frac12$)**.** With $v:=2\kappa$ the relation $X=2\kappa-\kappa^2$ reads $v=X+v^2/4$, the genus-zero string equation for $t_0=X$, $t_2=\frac12$. By Lemma 3.1, $\langle\langle\tau_0\rangle\rangle_0=\frac{v^2}{2}-\frac{v^3}{6}=2\kappa^2-\frac43\kappa^3=G_0$ at these times: the genus-zero volumes are the genus-zero one-point function of Witten–Kontsevich theory at $t_0=X$, $t_2=\frac12$; equivalently $H_n[0]=\pi^{2m}\langle\tau_0^n\tau_2^m\rangle_0/(2^m m!)$ with $m=n-3$. Compare the intersection-theoretic formulas of Chen, Möller and Sauvaget [CMS23]. For non-zero boundary lengths this simple shift is replaced by $R(\Lambda;\kappa)$.

10. Genus one

10.1. **Dressed Witten–Kontsevich vertices.** Let $G=G^{\mathrm{tree}}$ be the genus-zero series of Section 2.4, let $t_k[G]$ be the effective times (2.6), $\varphi_G(w)=\sum_k t_k[G]w^k/k!$, and u_G the solution of $u_G=\varphi_G(u_G)$ (Corollary 3.3). Let $G^{[2]}(\Lambda,\Lambda')=\mathcal{G}^{[2]}(\Lambda,\Lambda';\kappa)=2\int_0^\kappa R(\Lambda;k)R(\Lambda';k)\,\mathrm{d}k$ be the genus-zero two-point function (Corollary 8.3). Let $\langle\langle\tau_{a_1}\cdots\tau_{a_r}\rangle\rangle_h$ denote the Witten–Kontsevich correlators, i.e. derivatives of the genus-h free energy $F_h^{\mathrm{WK}}(t)$. Recall $F_1^{\mathrm{WK}}=\frac{1}{24}\log\partial_{t_0}u$ with $u=\partial_{t_0}^2F_0^{\mathrm{WK}}$ [Wit91, DW90]. By Lemma 3.1, $\partial_{t_0}u=(1-\varphi'(u))^{-1}$, so

$$F_1^{\mathrm{WK}}(t[G])=-\frac{1}{24}\log\big(1-\varphi_G'(u_G)\big). \tag{10.1}$$

Define the *bead*

$$B(\lambda,\lambda'):=\sum_{a,b\ge 0}\langle\langle\tau_a\tau_b\rangle\rangle_0\big|_{t=t[G]}\frac{\lambda^{2a}\lambda'^{2b}}{2^a a!\,2^b b!},$$

the weighted sum over genus-zero trees with two marked half-edges at a common vertex. Let $w(\lambda)=\lambda/(\mathrm{e}^{\pi\lambda}-1)$. Products of kernels are taken in $L^2(\mathbb{R}_+,w\,\mathrm{d}\lambda)$, i.e. $(AwB)(\lambda,\lambda')=\int_0^\infty A(\lambda,\mu)w(\mu)B(\mu,\lambda')\mathrm{d}\mu$. Each Taylor coefficient of a product is a finite combination of the rational moments $\int_0^\infty\lambda^{2j+1}\mathrm{d}\lambda/(\mathrm{e}^{\pi\lambda}-1)$.

Proposition 10.1. *As formal power series in* $(X,\boldsymbol{x})$,

$$F_1=F_1^{\mathrm{WK}}(t[G])+\mathrm{Cyc},\qquad \mathrm{Cyc}:=\sum_{k\ge1}\frac{1}{2k}\operatorname{Tr}\big((Bw)^k\big)=\sum_{k\ge1}\frac{(-1)^{k-1}}{2k}\operatorname{Tr}\big((G^{[2]}w)^k\big),$$

where $G^{[2]}$ is the genus-zero two-point function.

Proof. A connected genus-one stable graph either has one vertex of genus one and is a tree, or has only genus-zero vertices and first Betti number one. In the first case, removing the genus-one vertex leaves legs and genus-zero trees, one attached through each edge. Exactly as in Proposition 3.2, the exponential formula sums these graphs to $F_1^{\mathrm{WK}}(t[G])$, where $t[G]$ accounts for legs of length 0, legs of length Λ_j, and edges to trees (measure $G(\lambda)w(\lambda)\mathrm{d}\lambda = \lambda G(\lambda)\mathrm{d}\lambda/(\mathrm{e}^{\pi\lambda}-1)$).

In the second case the graph has a unique cycle $v_1, e_1, v_2, \dots, v_k, e_k$ ($k \geq 1$; a self-loop if $k = 1$). Removing $e_1, \dots, e_k$ leaves k trees $T_i \ni v_i$, each with the two half-edges of the cycle at v_i. The weighted sum over such trees is B, again by the exponential formula. Each edge contributes $\lambda\,\mathrm{d}\lambda/(\mathrm{e}^{\pi\lambda}-1)$ (stripped normalisation). A graph with a k-cycle is obtained from exactly $2k$ sequences of beads (rotations and reversal) when $k \geq 3$. When $k \leq 2$ the count is $2k/|\mathrm{Aut}|$, with automorphism group of order 2 (flipping the self-loop, or exchanging the two parallel edges). In all cases the weight is $\frac{1}{2k}\operatorname{Tr}((Bw)^k)$.

The two-point function $G^{[2]}(\Lambda, \Lambda')$ sums over trees with two marked legs. Its vertices along the path between the marked legs are beads, joined by edges, so $G^{[2]} = \sum_{r\geq 1} B(wB)^{r-1}$, i.e. $G^{[2]} = B + Bw\,G^{[2]}$. Hence $(1-Bw)(1+G^{[2]}w) = 1$, so $-\operatorname{Tr}\log(1-Bw) = \operatorname{Tr}\log(1+G^{[2]}w)$ as formal series, which is the second expression. All series converge in the $(X, \boldsymbol{x})$-adic topology since B and $G^{[2]}$ have order ≥ 1. □

Lemma 10.2. *There is a power series* $\mathrm{cyc}(\kappa) \in \mathbb{R}[[\kappa]]$ *with* $\mathrm{cyc}(0) = 0$ *and* $\mathrm{Cyc} = \mathrm{cyc}(\kappa(X, \boldsymbol{x}))$. *Explicitly,* $\mathrm{cyc}(\kappa) = \frac{1}{2}\log\det(1+N_\kappa)$, *where N_κ is the integral operator on $L^2([0,\kappa])$ with kernel* $N(k,k') = 2\int_0^\infty R(\lambda;k)R(\lambda;k')\,w(\lambda)\,\mathrm{d}\lambda$.

Proof. By Corollary 8.3, $G^{[2]}$ depends on $(X, \boldsymbol{x})$ only through κ. Each $\operatorname{Tr}((\mathcal{G}^{[2]}(\kappa)w)^k)$ is a power series in κ of order $\geq k$ with coefficients finite combinations of moments. Writing $\mathcal{G}^{[2]} = 2\int_0^\kappa R_k \otimes R_k\,\mathrm{d}k$ and using $\operatorname{Tr}((AA^*)^k) = \operatorname{Tr}((A^*A)^k)$ gives the determinant form. For $0 \leq \kappa < 1$ the kernel N is finite and continuous on $[0,\kappa]^2$, because $|R(\lambda;k)| \leq C\mathrm{e}^{(a(\kappa)+\epsilon)\lambda}$ uniformly for $k \leq \kappa$ and $2a(\kappa) < \pi$. Indeed, $e(Q;k) = \sum_d (2d+1)\rho_d(k)Q^{-2d-2}$, the Laplace transform of $\lambda R(\lambda;k)$, is holomorphic in (Q,k) for $|Q| \geq r := a(\kappa)+\epsilon$. So Cauchy's estimates give $(2d+1)|\rho_d(k)| \leq Cr^{2d}$ uniformly for $k \leq \kappa$, and then $|R(\lambda;k)| \leq C\sum_d (r\lambda)^{2d}/(2d+1)! \leq C\mathrm{e}^{r\lambda}$. Moreover $N_\kappa = A^*A \geq 0$ is a Gram operator, hence trace class by Mercer's theorem. □

10.2. **The endpoint of the cut.** Recall from Sections 5–7 that for real parameters in the open set Ω (which has 0 in its closure), $W(Q) = \sum_d G_d Q^{-2d-2}$ equals $\mathcal{W}(y_0(Q))$, where $\mathcal{W} = 2U_0 \circ P + \Psi_\kappa[U_0]$ is holomorphic near $\overline{D_\kappa}$. Recall also that, with $A(c) = c^2/2 - \varphi_G(c^2/2)$, $W(Q) = \int_0^a \big(\frac{Q}{\sqrt{Q^2-c^2}} - 1\big)A'(c)\mathrm{d}c$ and $u_G = a^2/2$.

Lemma 10.3. $1 - \varphi_G'(u_G) = \dfrac{2\tan\beta}{\kappa_X\, a^{3/2} P''(\beta)^{1/2}}$. *Consequently*

$$F_1^{\mathrm{WK}}(t[G]) = \frac{1}{24}\log\big(2\kappa_X\big) + \frac{1}{24}\log\frac{\Theta(\kappa)}{2}, \qquad \Theta(\kappa) := \frac{a^{3/2}P''(\beta)^{1/2}}{2\tan\beta} = \frac{(a/\sqrt{\kappa})^{3/2}(1-\kappa)^{3/4}}{\sqrt{2}},$$

with Θ analytic at $\kappa = 0$, $\Theta(0) = 2$.

Proof. Write $A'(c) = c\,\mathfrak{a}(c^2)$; then $\mathfrak{a}(a^2) = A'(a)/a = 1 - \varphi'(a^2/2)$. The integration-by-parts form in the proof of Lemma 6.2 shows that, as $Q \downarrow a$,

$$W(Q) = W(a) - a\,\mathfrak{a}(a^2)\sqrt{2a}\,(Q-a)^{1/2} + O(Q-a).$$

On the other hand $P(y)-a=\frac12 P''(\beta)(y-\beta)^2(1+O(y-\beta))$ with $P''(\beta)=2\kappa\cos\beta/\sin^3\beta>0$, and $y_0((a,\infty))=(0,\beta)$. So $y_0(Q)-\beta=-\sqrt{2(Q-a)/P''(\beta)}\,(1+O((Q-a)^{1/2}))$ and $W(Q)=\mathcal{W}(\beta)-\mathcal{W}'(\beta)\sqrt{2/P''(\beta)}(Q-a)^{1/2}+O(Q-a)$. Comparing, $\mathfrak{a}(a^2)=\mathcal{W}'(\beta)/(a^{3/2}P''(\beta)^{1/2})$.

To compute $\mathcal{W}'(\beta)$ we use Lemma 8.1. In terms of $y=y_0(Q;\kappa)$ it reads $(\partial_\kappa\mathcal{W})(y)-\mathcal{W}'(y)\cot y/P'(y)=-2\,\partial_\kappa\Phi_\kappa[U_0]/P'(y)$, an identity of meromorphic functions of y near 0. It therefore holds up to $y=\beta$ for small κ. Multiplying by $P'(y)$ and letting $y\to\beta$ gives $\mathcal{W}'(\beta)\cot\beta=2\,\partial_\kappa\Phi_\kappa[U_0]=2/\kappa_X$. Both sides of the resulting identity are analytic in the parameters near 0, so it extends from Ω to a neighbourhood of 0, and the formula for F_1^{WK} follows from (10.1). For Θ use $\tan\beta=\sqrt{\kappa/(1-\kappa)}$, $P''(\beta)=2\sqrt{(1-\kappa)/\kappa}$, and that $a/\sqrt{\kappa}=\arcsin(\sqrt{\kappa})/\sqrt{\kappa}+\sqrt{1-\kappa}$ is analytic with value 2 at 0. □

10.3. **The genus-one volume series.** Yang, Zagier and Zhang [YZZ20, Cor. 2] prove that $\mathcal{H}(x,\epsilon)=\sum_{g,n}\epsilon^{2g-2}a_{g,n}x^n/n!$ satisfies

$$\epsilon\partial_\epsilon\partial_x\mathcal{H}+x\partial_x^2\mathcal{H}+\tfrac12\partial_x\mathcal{H}-\tfrac{\epsilon^2}{4}(\partial_x^2\mathcal{H})^2-\tfrac{\epsilon^2}{24}\partial_x^4\mathcal{H}=0,$$

where $\mathrm{MV}_{g,n}=2^{2g+1}\pi^{6g-6+2n}\frac{(4g-4+n)!}{(6g-7+2n)!}\,a_{g,n}$ [CMS23]. Comparing with (1.2) and Lemma 2.1, $a_{g,n}=2^{n+2g-3}h_{g,n}[0^n]$ for $(g,n)\neq(0,3)$. This also holds for $(0,3)$, since $a_{0,3}=1$ by the definition of $a_{g,n}$ in [YZZ20], and $h_{0,3}[0^3]=1$. Hence $\mathcal{H}_g(x)=2^{2g-3}F_g(2x)$, where $F_g(X):=F_g(X,0)$. The coefficient of ϵ^{2g-2} becomes

$$(X-\kappa)\,F_g''+\big(2g-\tfrac32\big)F_g'=\frac14\sum_{h=1}^{g-1}F_h''F_{g-h}''+\frac{1}{12}F_{g-1}'''',\qquad '=\frac{\mathrm{d}}{\mathrm{d}X},\tag{10.2}$$

where we used $F_0''=2\kappa$ (Proposition 9.1: $F_0'=G_0=2\kappa^2-\frac43\kappa^3$ and $\kappa'=\frac{1}{2(1-\kappa)}$).

Lemma 10.4. *For $g\geq1$, (10.2) determines F_g' uniquely as the solution analytic at $X=0$.*

Proof. $X-\kappa=\kappa(1-\kappa)=\frac{X}{2}+O(X^2)$, so near $X=0$ the homogeneous equation $(X-\kappa)\phi'+(2g-\frac32)\phi=0$ has solutions $\phi\propto\kappa^{-(4g-3)}$. These are singular at 0. □

Yang, Zagier and Zhang also give closed forms [YZZ20, Lemma 1]: in our normalisation $F_g\in\mathbb{Q}[s^{-1}]$ for $g\geq2$, of degree $5g-5$.

Remark 10.5. With $z=1-X$ one has $X-\kappa=\sqrt{z}-z$. The terms of (10.2) that are most singular at $z=0$ satisfy $\sqrt{z}\,F_g''=\frac14\sum_{h=1}^{g-1}F_h''F_{g-h}''+\frac{1}{12}F_{g-1}''''$. With $u_h=-\frac12F_h''$ and $u_0=\sqrt{z}$, this is the order-g part of the Painlevé I equation $u^2-\frac16u''=z$ (here $u_0=\sqrt{z}$ is the singular part of $-\frac12F_0''=-\kappa$). Hence, in the normalisation $u=-\frac12F''$, the coefficients of $s^{-(5g-5)}$ in F_g, $g\geq2$, are the genus-g free energies of Painlevé I; for instance $F_2=\frac{7}{2880}s^{-5}+\dots$.

10.4. **Proof of Theorem D.** By Proposition 10.1 and Lemmas 10.2, 10.3,

$$F_1(X,\boldsymbol{x})=\frac{1}{24}\log(2\kappa_X)+c(\kappa),\qquad c(\kappa):=\frac{1}{24}\log\frac{\Theta(\kappa)}{2}+\mathrm{cyc}(\kappa)\in\mathbb{R}[[\kappa]],\ c(0)=0.\tag{10.3}$$

So F_1 depends on $(X,\boldsymbol{x})$ only through κ and κ_X, and in this specific way. It remains to identify the one-variable series c. At $\boldsymbol{x}=0$ we have $\kappa=1-\sqrt{1-X}$ and $2\kappa_X=(1-\kappa)^{-1}$, so $F_1(X)=-\frac{1}{24}\log(1-\kappa)+c(\kappa)$. Since $X\mapsto\kappa(X)$ is invertible near 0, it suffices to know the volume series $F_1(X)$. By (10.2) and Lemma 10.4, F_1' is the unique solution analytic at $X=0$ of $(X-\kappa)F_1''+\frac12F_1'=\frac16\kappa''$. One checks that $F_1=\frac{\kappa}{12}-\frac{1}{12}\log(1-\kappa)$ solves it: with $\kappa'=\frac{1}{2(1-\kappa)}$ one finds $F_1'=\frac{2-\kappa}{24(1-\kappa)^2}$, $F_1''=\frac{3-\kappa}{48(1-\kappa)^4}$ and $\kappa''=\frac{1}{4(1-\kappa)^3}$. Hence $c(\kappa)=\frac{\kappa}{12}-\frac{1}{24}\log(1-\kappa)$. □

Remark 10.6 (A Fredholm determinant)**.** For $0 \le \kappa < 1$, with $a = \arcsin\sqrt{\kappa} + \sqrt{\kappa(1-\kappa)}$,

$$\tfrac{1}{2}\log\det(1+N_\kappa) = \frac{\kappa}{12} - \frac{7}{96}\log(1-\kappa) - \frac{1}{16}\log\frac{a}{2\sqrt{\kappa}}.$$

The left side is the sum over genus-one graphs whose cycle carries all the genus (Proposition 10.1). The identity is $\mathrm{cyc}(\kappa) = c(\kappa) - \frac{1}{24}\log\frac{\Theta}{2}$ together with Lemma 10.2, first as formal power series in κ. Rescaling $k = \kappa t$ turns N_κ into a kernel on $[0,1]^2$ that is analytic in κ near $[0,1)$. Since $N_\kappa \ge 0$, $\log\det(1+N_\kappa)$ is real-analytic on $[0,1)$, and so is the right-hand side, which gives the identity on $[0,1)$.

Proof of (1.17), (1.18) *and Corollary 1.3.* For (1.17), (1.18) apply $\partial_{x_j}\kappa = R_j\kappa_X$ and $\partial_{x_j}\partial_X^r\kappa = \partial_X^r(R_j\kappa_X)$ to (1.16), then set $\boldsymbol{x} = 0$, where $\kappa_X = \frac{1}{2(1-\kappa)}$, $\kappa_{XX} = \frac{1}{4(1-\kappa)^3}$, $\kappa_{XXX} = \frac{3}{8(1-\kappa)^5}$. The coefficient of $\Lambda^{2d}/(2d+1)!$ in R is $(2d+1)\rho_d$, and $\sum_d (2d+1)\rho_d Q^{-2d-2} = e(Q;\kappa)$ by Proposition 9.5(a). For the coefficient extraction use $[X^N]f = [\kappa^N]\,2(1-\kappa)f(\kappa)(2-\kappa)^{-N-1}$, from $X = \kappa(2-\kappa)$. For the formula for $\mathrm{MV}_{1,n}$ in the introduction expand $\kappa = \sum_{n\ge1}\frac{(2n-2)!}{n!\,(n-1)!\,2^{2n-1}}X^n$ and $-\log(1-\kappa) = -\frac{1}{2}\log(1-X)$, and use $\mathrm{MV}_{1,n} = \frac{2^{n+2}n!}{(2n-1)!}\pi^{2n}h_{1,n}[0^n]$, from (1.2). □

Remark 10.7. In Witten–Kontsevich theory $F_1 = \frac{1}{24}\log u_x$. Formula (1.16) has the same leading term, with u replaced by 2κ, plus a correction $\frac{\kappa}{12} - \frac{1}{24}\log(1-\kappa)$ depending only on κ. All of it comes from the cycles and from the change of the endpoint behaviour; the π-periodic images enter only through κ.

11. All genera: proof of Theorem E

Throughout this section $0 < \kappa < 1$, and P, Ψ, K, β, a, D_κ, y_0 are as in the introduction. For a meromorphic n-form ω on $(\mathbb{C}/\pi\mathbb{Z})^n$ we write $f = \omega/(\mathrm{d}Q_1\cdots\mathrm{d}Q_n)$ for the associated function of $y_1,\dots,y_n$; thus $f_{g,n}$, and $f_{0,2} = K(y_1,y_2)/(P'(y_1)P'(y_2))$. We write $\iota(y) = -y$, and for $k \ge 0$

$$b_k(y) := \partial_w^k K(y,w)\big|_{w=\beta} = \partial_w^k\Big(\frac{1}{\sin^2(y-w)} + \frac{1}{\sin^2(y+w)}\Big)\Big|_{w=\beta}.$$

By Theorem C(i),(ii) and Remark 5.5, the genus-zero one- and two-point functions are

$$\begin{aligned} W_{0,1}(Q) &= 2(Q^2 - X) + \Psi(y_0(Q)), \\ W_{0,2}(Q_1,Q_2) &= f_{0,2}\big(y_0(Q_1),y_0(Q_2)\big) - \frac{1}{(Q_1-Q_2)^2} - \frac{1}{(Q_1+Q_2)^2}. \end{aligned} \tag{11.1}$$

These are the input; Theorem E concerns $2g-2+n>0$.

11.1. Local structure.

Lemma 11.1 (Structure and folding)**.** *For $2g-2+n>0$:*

(a) *$\omega_{g,n}$ is a finite linear combination of the forms $\prod_i b_{k_i}(y_i)\mathrm{d}y_i$. In particular it is π-periodic and holomorphic away from $y_i \in \{\pm\beta\}+\pi\mathbb{Z}$, it is residue-free in each variable, and $|\omega_{g,n}/\mathrm{d}y_i| = O(\mathrm{e}^{-2|\mathrm{Im}\,y_i|})$ as $|\mathrm{Im}\,y_i| \to \infty$. Moreover $\iota_i^*\omega_{g,n} = -\omega_{g,n}$ in each variable, so $f_{g,n}$ is even in each y_i.*

(b) *In (1.20) the residues at $-\beta$ and at β are equal. Equivalently, $\omega_{g,n}$ is obtained by topological recursion with the single branch point β and with $\omega_{0,1}$ replaced by $\omega'_{0,1} := \Psi\,\mathrm{d}Q$.*

Proof. We induct on $2g-2+n$. Note that ι commutes with P up to sign, so $\iota\circ\sigma_\beta=\sigma_{-\beta}\circ\iota$; moreover $\iota^*\omega_{0,1}=-\omega_{0,1}$ and $\iota_i^*\omega_{0,2}=-\omega_{0,2}$ in each variable.

(b) Assume (a) for all correlators in the bracket of (1.20). The bracket $\mathcal{R}(y)$ is then a quadratic differential in y with $\iota^*\mathcal{R}=\mathcal{R}$, since each term is a product of two forms that are odd under ι. Write $\int_{\sigma(\iota y)}^{\iota y}\omega_{0,2}(z_0,\cdot)=\int_{\sigma(y)}^{y}\iota_2^*\omega_{0,2}(z_0,\cdot)=-\int_{\sigma(y)}^{y}\omega_{0,2}(z_0,\cdot)$. For the denominator, $\iota^*(\omega_{0,1}-\sigma^*\omega_{0,1})=-(\omega_{0,1}-\sigma^*\omega_{0,1})$. Hence $\iota^*\big(K^{\mathrm{TR}}(z_0,\cdot)\mathcal{R}\big)=K^{\mathrm{TR}}(z_0,\cdot)\mathcal{R}$, and the residue at $-\beta=\iota(\beta)$ equals the residue at β. Doubling one residue and halving $\omega_{0,1}$ have the same effect on K^{TR}; by induction the lower correlators of the two recursions coincide, hence so do the outputs.

(a) Expanding $\omega_{0,2}(z_0,w)$ in w around β gives

$$\tfrac{1}{2}\int_{\sigma(y)}^{y}\omega_{0,2}(z_0,\cdot)=\tfrac{1}{2}\sum_{k\geq 0}b_k(z_0)\,\frac{(y-\beta)^{k+1}-(\sigma(y)-\beta)^{k+1}}{(k+1)!}\,\mathrm{d}z_0.$$

Only finitely many k contribute to the residue at β. Together with (b) this gives the form of $\omega_{g,n+1}$ in z_0. In a spectator y_j, the bracket of (1.20) depends on y_j through lower correlators, which have the required form by induction, and through factors $\omega_{0,2}(y,y_j)$ and $\omega_{0,2}(\sigma(y),y_j)$. These we expand as $\omega_{0,2}(y,y_j)=\sum_k\frac{(y-\beta)^k}{k!}\,b_k(y_j)\,\mathrm{d}y\,\mathrm{d}y_j$; again only finitely many k contribute to the residue. The remaining properties hold for each $b_k\,\mathrm{d}y$: these are derivatives of cot, π-periodic, without residues, decaying like $\mathrm{e}^{-2|\operatorname{Im} y|}$, and odd under ι because b_k is even. Since $\iota^*\mathrm{d}Q=-\mathrm{d}Q$, $f_{g,n}$ is even. □

Proposition 11.2 (Local properties)**.** *Let $2g-2+n>0$ and $J=(y_1,\dots,y_{n-1})$ with $y_j\notin\{\pm\beta\}+\pi\mathbb{Z}$.*

(i) *$\omega_{g,n}$ is symmetric.*

(ii) (Linear loop equation.) *$f_{g,n}(y,J)+f_{g,n}(\sigma(y),J)$, for y near β, is a holomorphic function of $Q=P(y)$ near $Q=a$.*

(iii) (Quadratic loop equation.) *The function*

$$\begin{aligned}\mathfrak{B}_{g,n}(y;J):=\Psi(y)f_{g,n}(\sigma(y),J)+\Psi(\sigma(y))f_{g,n}(y,J)+f_{g-1,n+1}(y,\sigma(y),J)\\ +\sum{}''\,f_{h,1+|I|}(y,I)\,f_{g-h,n-|I|}(\sigma(y),J\setminus I),\end{aligned}\tag{11.2}$$

for y near β, is a holomorphic function of $Q=P(y)$ near $Q=a$. Here $\sum''$ runs over ordered splittings (h,I), $(g-h,J\setminus I)$ in which both factors are stable or equal to $f_{0,2}$; the term $f_{g-1,n+1}$ is present when $(g-1,n+1)$ is stable, and equals $f_{0,2}(y,\sigma(y))$ when $(g,n)=(1,1)$.

The same holds at $-\beta$, $Q=-a$, by Lemma 11.1(a).

Proof. (i) By Lemma 11.1(b), the restrictions of the $\omega_{g,n}$ to a neighbourhood of $(\beta,\dots,\beta)$ are the outputs of topological recursion on a *local* spectral curve with one simple branch point. This curve consists of Q near a, $\omega_{0,1}'=\Psi\mathrm{d}Q$, and the restriction of $\omega_{0,2}$ to a neighbourhood of (β,β). That restriction is symmetric, has a double pole with biresidue 1 on the diagonal, and is otherwise holomorphic; the term $\sin^{-2}(y_1+y_2)$ is regular there. For such local curves the outputs are symmetric [KS18, ABCO24]. Since the $\omega_{g,n}$ are meromorphic on $(\mathbb{C}/\pi\mathbb{Z})^n$, symmetry extends from the neighbourhood to all of $(\mathbb{C}/\pi\mathbb{Z})^n$.

(ii), (iii) We derive the loop equations directly from the folded recursion. Choose a local coordinate t at β with $P(y)-a=t^2$; then $\sigma(t)=-t$ and $\mathrm{d}Q=2t\,\mathrm{d}t$. Write $\Psi=\Psi^+{+}\Psi^-$ for the even and odd parts in t. Then $\Psi^-(t)=\Psi_1 t+O(t^3)$ with $\Psi_1\neq 0$, because $\Psi'(\beta)=4\kappa\cot\beta\neq 0$. Write $\omega_{0,2}=\big((t_0-t')^{-2}+H(t_0,t')\big)\mathrm{d}t_0\,\mathrm{d}t'$ with H holomorphic near $(0,0)$.

Fix J and let $\mathcal{R}(t) = r(t)\,\mathrm{d}t^2$ be the bracket of (1.20) for $\omega_{g,n}(\cdot, J)$; it contains the factors $\omega_{0,2}(y, y_j)$, and for $(g,n) = (1,1)$ it is $\omega_{0,2}(y, \sigma(y))$. By (i) for the lower correlators, $\sigma^*\mathcal{R} = \mathcal{R}$: reversing t exchanges each ordered splitting with its complement, and fixes the genus-reducing term. Hence r is even. It is meromorphic at 0.

For t_0 near 0 and $|t| < |t_0|$,

$$\int_{-t}^{t} \omega_{0,2}(t_0, \cdot) = \Big(\frac{2t}{t_0^2 - t^2} + h(t_0, t)\Big)\mathrm{d}t_0,$$

with h holomorphic and odd in t. With $\hat\rho(t) := r(t)/(4\Psi^-(t))$, which is odd, Lemma 11.1(b) gives

$$\omega_{g,n}(t_0, J) = \operatorname*{Res}_{t\to 0}\Big(\frac{1}{t_0^2 - t^2} + \frac{h(t_0,t)}{2t}\Big)\hat\rho(t)\,\mathrm{d}t\,\mathrm{d}t_0 = \varpi(t_0)\,\mathrm{d}t_0 + \upsilon(t_0)\,\mathrm{d}t_0,$$

where $\varpi(t_0) = \sum_{j\ge 0}\hat\rho_{-2j-1}t_0^{-2j-2}$ is even, $\hat\rho_i$ are the Laurent coefficients of $\hat\rho$, and υ is holomorphic near 0.

(ii) $\omega + \sigma^*\omega = (\upsilon(t_0) - \upsilon(-t_0))\,\mathrm{d}t_0$ is a σ-invariant holomorphic 1-form. Such a form is $t\,\ell(t^2)\mathrm{d}t = \frac12\ell(Q-a)\mathrm{d}Q$ with ℓ holomorphic. Since $\sigma^*\mathrm{d}Q = \mathrm{d}Q$, $f_{g,n}(y) + f_{g,n}(\sigma(y)) = \frac12\ell(Q-a)$.

(iii) With $f_{g,n}(t) = (\varpi(t) + \upsilon(t))/(2t)$ and $r(t)/\mathrm{d}Q^2 = \hat\rho(t)\Psi^-(t)/t^2$,

$$\mathfrak{B}_{g,n} = \frac{\Psi^-(t)}{t}\Big(\frac{\hat\rho(t)}{t} - \varpi(t)\Big) + \frac{\Psi(-t)\upsilon(t) - \Psi(t)\upsilon(-t)}{2t}.$$

Here $\hat\rho(t)/t - \varpi(t) = \sum_{i\ge1}\hat\rho_i t^{i-1}$ and $\Psi^-(t)/t$ are holomorphic, and the last numerator is holomorphic and odd. So $\mathfrak{B}_{g,n}$ is holomorphic and even in t, i.e. a holomorphic function of $Q - a = t^2$. □

Remark 11.3. In (iii) the function multiplying $f_{g,n}$ is Ψ, not 2Ψ, by Lemma 11.1(b). Statements (ii) and (iii) are the abstract loop equations of [BEO15, BS17] for the folded local curve; we have included the short derivation because our $\omega_{0,2}$ is not the Bergman kernel of a compact curve. We checked (ii) and (iii) numerically for $(g,n) = (1,2)$ at $\kappa = \frac{1}{20}$, on a circle of radius 0.04 around $Q = a$. The Laurent coefficients of negative order are below $3\cdot 10^{-39}$, while with 2Ψ in place of Ψ they are of order 10^{-1} (Appendix **??**).

11.2. **Fibres and traces.** Let $S_k = \{y : |\operatorname{Re} y - \pi k| < \frac{\pi}{2}\}$ and, for $Q \in \mathbb{C}\setminus[-a,a]$, $\xi_k(Q) := \pi k + y_0(Q - \pi k) \in D_\kappa + \pi k$.

Lemma 11.4 (Fibres)**.** *Let $|\operatorname{Re} Q| < \frac{\pi}{2}$ and $Q \notin [-a,a]$. The solutions $y \in \mathbb{C}$ of $P(y) = Q$ are the points $\xi_k(Q)$, $k \in \mathbb{Z}$, and one further point $\tilde y(Q) \in S_0 \setminus \overline{D_\kappa}$. All of them are simple. The function $\tilde y$ is holomorphic, $\tilde y(-Q) = -\tilde y(Q)$, $\tilde y(c \pm \mathrm{i}0) = y_0(c \mp \mathrm{i}0)$ for $c \in (-a,a)$, and $\tilde y(Q) \to \pm\beta$ as $Q \to \pm a$. Near $Q = \pm a$ the pair $\{y_0(Q), \tilde y(Q)\}$ is $\{y, \sigma(y)\}$ with y near $\pm\beta$.*

Proof. Fix k and $T > |\operatorname{Im} Q| + 3$, and let $R = S_k \cap \{|\operatorname{Im} y| < T\}$. Consider $H_\lambda(y) = y - Q + \lambda\kappa\cot y$ for $\lambda \in [0,1]$. On the vertical sides of R, $\cot y \in \mathrm{i}\mathbb{R}$, so $\operatorname{Re} H_\lambda = \pi k \pm \frac{\pi}{2} - \operatorname{Re} Q \neq 0$. On the horizontal sides, $|H_\lambda| \ge T - |\operatorname{Im} Q| - 2\kappa > 0$. Hence the winding number of $P - Q = H_1$ around ∂R equals that of $y - Q$, which is 1 if $k = 0$ and 0 otherwise. Inside R, $P - Q$ has exactly one pole, the simple pole at πk. So it has two zeros in S_0 and one in each S_k, $k \neq 0$.

Now $\xi_k(Q) \in D_\kappa + \pi k \subset S_k$ is a zero, since $Q - \pi k \notin [-a,a]$. For $k = 0$ the second zero $\tilde y$ is not in D_κ, because P is injective there. It is not on ∂D_κ, because $P(\partial D_\kappa) = [-a,a]$. It differs from $y_0(Q)$, because the zeros of P' in S_0 are $\pm\beta \in \partial D_\kappa$. The remaining claims follow from the implicit function theorem, continuity of zeros, $P(-y) = -P(y)$, and Lemma 4.2. As $Q \to c \in (-a,a)$ from above, the two zeros in S_0 tend to the two points of ∂D_κ over c; one of them is $y_0(c + \mathrm{i}0)$, so the other is $y_0(c - \mathrm{i}0)$. □

Lemma 11.5 (Trace formula)**.** *Let ω be a meromorphic 1-form on $\mathbb{C}/\pi\mathbb{Z}$ whose poles lie in a finite set $\Sigma \not\ni 0$ whose points have representatives with $|\operatorname{Re} p| < \frac{\pi}{2}$. Assume ω has no residues and $|\omega/\mathrm{d}y| \le C\mathrm{e}^{-2|\operatorname{Im} y|}$ for large $|\operatorname{Im} y|$. Let $f = \omega/\mathrm{d}Q$, and let Q be as in Lemma 11.4 with no point of the fibre in $\Sigma + \pi\mathbb{Z}$. Then $f(\xi_k(Q)) = O(k^{-2})$ and*

$$T[f](Q) := \sum_{k\in\mathbb{Z}} f(\xi_k(Q)) + f(\tilde{y}(Q)) = -\sum_{p\in\Sigma} \operatorname*{Res}_{y=p} \cot\big(P(y) - Q\big)\,\omega(y). \tag{11.3}$$

Proof. Near $y = 0$, ω is holomorphic and $P'(y) \sim -\kappa y^{-2}$, so $f = O(y^2)$; with $\xi_k - \pi k = y_0(Q - \pi k) = O(k^{-1})$ this gives the bound. Let $R_{N,T} = \{|\operatorname{Re} y| < \pi(N + \frac{1}{2}), |\operatorname{Im} y| < T\}$ and integrate $\omega(y)/(P(y) - Q)$ over its boundary. At $\pi\mathbb{Z}$ the integrand is holomorphic, because P has a pole there. Its other singularities in $R_{N,T}$ are the fibre points, with residue $f(\xi)$, and the points $p + \pi m$ with $p \in \Sigma$, $|m| \le N$. The integral over the horizontal sides is $O(N\mathrm{e}^{-2T}/T)$. On the vertical sides $|P - Q| \ge |\operatorname{Re}(P - Q)| \ge \pi N$, so their contribution is at most $\pi^{-1}N^{-1}\int_{\mathbb{R}} |\omega/\mathrm{d}y|(\pm\pi(N + \frac{1}{2}) + \mathrm{i}t)\,\mathrm{d}t = O(N^{-1})$. By Lemma 11.4 the fibre points in $R_{N,T}$, for T large, are ξ_k with $|k| \le N$ and $\tilde{y}$. Letting $T \to \infty$ and then $N \to \infty$ gives

$$\sum_k f(\xi_k) + f(\tilde{y}) = -\lim_{N\to\infty} \sum_{p\in\Sigma} \operatorname*{Res}_{y=p} \Big(\sum_{|m|\le N} \frac{1}{P(y) + \pi m - Q} \Big) \omega(y),$$

by periodicity of ω and $P(y + \pi m) = P(y) + \pi m$. The symmetric partial sums converge to $\cot(P(y) - Q)$ uniformly on small circles around p. □

Lemma 11.6 (Vanishing traces)**.** *Let $|\operatorname{Re} Q| < \frac{\pi}{2}$ and $Q \notin [-a, a]$.*

(a) *For $2g - 2 + n > 0$ and y_J off $\{\pm\beta\} + \pi\mathbb{Z}$, the trace of $f_{g,n}(\cdot, y_J)$ in its first variable vanishes: $T[f_{g,n}(\cdot, y_J)] \equiv 0$.*

(b) *For $y_2 \notin \pi\mathbb{Z} \cup \{\pm\beta\}$ with $|\operatorname{Re} y_2| < \frac{\pi}{2}$, and $Q \not\equiv \pm P(y_2)$ modulo π: $T[f_{0,2}(\cdot, y_2)](Q) = \sin^{-2}(Q - P(y_2)) + \sin^{-2}(Q + P(y_2))$.*

(c) *With $W_{0,1}$ as in (11.1): $\sum_{k\in\mathbb{Z}} W_{0,1}(Q - \pi k) = -\Psi(\tilde{y}(Q))$, and the series converges absolutely.*

Proof. (a) Lemma 11.1(a) puts $\omega = \omega_{g,n}(\cdot, y_J)$ in the setting of Lemma 11.5 with $\Sigma = \{\pm\beta\}$. The right side $R(Q)$ of (11.3) is meromorphic on $\mathbb{C}$ and π-periodic, with poles at most at $\pm a + \pi\mathbb{Z}$. As $\operatorname{Im} Q \to \pm\infty$, $\cot(P(y) - Q)$ tends to the constant $\pm\mathrm{i}$ uniformly for y near $\pm\beta$, so $R(Q) \to \pm\mathrm{i}\sum_p \operatorname{Res}_p \omega = 0$. Near $Q = a$ (off the cut) write

$$R(Q) = T[f](Q) = \big[f(y_0(Q)) + f(\tilde{y}(Q))\big] + \sum_{k\ne 0} f(\xi_k(Q)).$$

The sum over $k \ne 0$ converges uniformly near a and is holomorphic there, since $Q - \pi k$ stays away from $[-a, a]$. By Lemma 11.4 the bracket is $f(y) + f(\sigma(y))$, which by Proposition 11.2(ii) is a holomorphic function of Q near a. So R is bounded near a, and likewise near $-a$. Hence R is entire, periodic and tends to 0 at both ends, so $R \equiv 0$.

(b) Here $\Sigma = \{\pm y_2\}$. Write $g(y) = \cot(P(y) - Q)$. Since $\operatorname{Res}_{y=p} \sin^{-2}(y - p)\, g(y)\mathrm{d}y = g'(p)$, the right side of (11.3) is $-\big(g'(y_2) + g'(-y_2)\big)\mathrm{d}y_2$, and $g'(y) = -\sin^{-2}(P(y) - Q)P'(y)$. Using that P' is even and P is odd, this equals $\big(\sin^{-2}(Q - P(y_2)) + \sin^{-2}(Q + P(y_2))\big)P'(y_2)\mathrm{d}y_2$. Divide by $\mathrm{d}Q_2 = P'(y_2)\mathrm{d}y_2$.

(c) Repeat the proof of Lemma 11.5 with $\omega = \Psi\,\mathrm{d}Q$, whose poles lie on $\pi\mathbb{Z}$. Now the integrand $\Psi P'/(P - Q)$ is no longer regular at $\pi\mathbb{Z}$: it has poles of order 3 there. At $y = \pi m$ the residue of $\Psi P'/(P - Q)$ is $2(Q - \pi m)^2 - 2X$. Indeed, by periodicity it suffices to take

$m=0$ and $Q=c$. From $P=\kappa y^{-1}+(1-\frac{\kappa}{3})y+O(y^3)$ one gets

$$\frac{P'}{P-c}=-\frac{1}{y}\Big(1+\frac{c}{\kappa}y+\Big(\frac{c^2}{\kappa^2}-\frac{2(1-\kappa/3)}{\kappa}\Big)y^2+O(y^3)\Big),$$

and with $\Psi=-2\kappa^2(y^{-2}+\frac{1}{3}+O(y^2))$ the residue is $2c^2-4\kappa+2\kappa^2=2c^2-2X$. This is the case $U=Q^2-X$ of Lemma 5.3. The boundary terms vanish as before, since Ψ decays at $\pm\mathrm{i}\infty$. So

$$\sum_{|k|\le N}\big(\Psi(\xi_k)+2(Q-\pi k)^2-2X\big)+\Psi(\tilde y)\to 0.$$

Since Ψ is periodic, $\Psi(\xi_k(Q))=\Psi(y_0(Q-\pi k))$, so the k-th term equals $W_{0,1}(Q-\pi k)$ by (11.1). Finally $W_{0,1}=O(Q^{-2})$, which gives absolute convergence. □

11.3. **Two-sheet formulas.** Fix r_* with $a<r_*\le\frac{\pi}{2}$ and let $\mathcal{U}=\{|Q|<r_*\}\setminus[-a,a]$. Then $|Q+\pi m|\ge\pi-r_*>a$ for $Q\in\mathcal{U}$ and $m\neq 0$. For $Q\in\mathcal{U}$ write $u=y_0(Q)$, $v=\tilde y(Q)$, and for a function f of one variable

$$A[f](Q):=f(u)-f(v),\qquad S[f](Q):=f(u)+f(v);$$

operators in different variables are indicated by subscripts.

On the topological-recursion side define, for stable (g,n) and $y_j=y_0(Q_j)$,

$$W_{g,n}^{\mathrm{TR}}(Q,Q_J):=f_{g,n}(y_0(Q),y_J),$$

and dress any function $W(Q)$, holomorphic for $|Q|>a$, by its images:

$$\widehat{W}(Q):=W(Q)+\tfrac{1}{2}\sum_{m\neq 0}W(Q+\pi m).$$

For a function of two variables, $\widehat{\widehat{W}}$ denotes this operation in both. Put

$$\begin{aligned}\mathsf{w}(Q,Q')&:=\tfrac{1}{2}\big((Q-Q')^{-2}+(Q+Q')^{-2}\big),\\ \mathcal{K}(Q,Q')&:=\mathsf{w}(Q,Q')+\widehat{W}_{0,2}(Q,Q'),\qquad Y(Q):=-Q^2+X+\widehat{W}_{0,1}(Q),\end{aligned}\tag{11.4}$$

where the hat in $\widehat{W}_{0,2}$ acts on the first variable.

Corollary 11.7. *For $Q\in\mathcal{U}$, and spectators Q_j with $|Q_j|>r_*$:*

(a) $\widehat{W}_{g,n}^{\mathrm{TR}}(Q,Q_J)=\frac{1}{2}A[f_{g,n}(\cdot,y_J)](Q)$ *for* $2g-2+n>0$;
(b) $\widehat{\widehat{W}}_{g,n}^{\mathrm{TR}}(Q,Q',Q_J)=\frac{1}{4}A_1A_2[f_{g,n}(\cdot,\cdot,y_J)](Q,Q')$ *for* $2g-2+n>0$;
(c) $\mathcal{K}(Q,Q_j)=\frac{1}{2}A[f_{0,2}(\cdot,y_j)](Q)$;
(d) $Y(Q)=\frac{1}{2}A[\Psi](Q)$.

Proof. For $W(Q)=f(y_0(Q))$ with f π-periodic we have $W(Q+\pi m)=f(\xi_{-m}(Q))$. Hence

$$\widehat{W}=\tfrac{1}{2}W+\tfrac{1}{2}\sum_m W(\cdot+\pi m)=\tfrac{1}{2}f(u)+\tfrac{1}{2}\big(T[f]-f(v)\big).$$

Then (a) is Lemma 11.6(a). For (b), apply (a) in the second variable; this uses Lemma 11.6(a) for every fixed first argument. Then apply it in the first variable to $y\mapsto\frac{1}{2}A_2[f_{g,n}(y,\cdot,y_J)](Q')$, which again satisfies the hypotheses of Lemma 11.6(a).

For (c), by (11.1) and Lemma 11.6(b), and since $\sum_m(Q+\pi m\mp Q_j)^{-2}=\sin^{-2}(Q\mp Q_j)$,

$$\begin{aligned}\sum_m W_{0,2}(Q+\pi m,Q_j)&=T[f_{0,2}(\cdot,y_j)](Q)-f_{0,2}(v,y_j)-\sin^{-2}(Q-Q_j)-\sin^{-2}(Q+Q_j)\\&=-f_{0,2}(v,y_j).\end{aligned}$$

This holds for $Q \in \mathcal{U}$ with $Q \not\equiv \pm Q_j$ modulo π, where the hypothesis of Lemma 11.5 is met. Both ends are holomorphic on all of $\mathcal{U}$, because $W_{0,2}$ is regular on the diagonal and $f_{0,2}(v, y_j)$ is regular (see below). So the identity holds on $\mathcal{U}$. Hence $\widehat{W}_{0,2} = \frac{1}{2}W_{0,2} - \frac{1}{2}f_{0,2}(v, y_j) = \frac{1}{2}A[f_{0,2}(\cdot, y_j)] - \mathsf{w}$.

For (d), by Lemma 11.6(c), $\widehat{W}_{0,1} = \frac{1}{2}\big(2(Q^2 - X) + \Psi(u)\big) - \frac{1}{2}\Psi(v)$. □

In particular $\mathcal{K}(\cdot, Q_j)$ is holomorphic on $\mathcal{U}$. Indeed, $f_{0,2}(u, y_j)$ is singular only for $u = \pm y_j$, i.e. $Q = \pm Q_j \notin \mathcal{U}$. And $f_{0,2}(v, y_j)$ is regular, because $v \in S_0 \setminus \overline{D_\kappa}$ while $\pm y_j \in D_\kappa$, and $v \neq \pm\beta$.

11.4. **The deformed loop equations on the topological-recursion side.**

Definition 11.8. Let $2g - 2 + n > 0$ and $J = \{1, \dots, n-1\}$. Given correlators $W_{h,k}$ for $2h - 2 + k \le 2g - 2 + n$, together with Y and $\mathcal{K}$ as in (11.4), put

$$\mathcal{E}_{g,n}(Q; Q_J) := Y(Q)\,\widehat{W}_{g,n}(Q, Q_J) + Q^2\big(\widehat{W}_{g,n} - W_{g,n}\big)(Q, Q_J) + \mathcal{D}_{g,n}(Q; Q_J),$$

where, for $(g, n) \neq (1, 1)$,

$$\mathcal{D}_{g,n} := \tfrac{1}{2}\widehat{\widehat{W}}_{g-1,n+1}(Q, Q, Q_J) + \tfrac{1}{2}{\sum}'' \widehat{W}^{\natural}_{h,1+|I|}(Q, Q_I)\,\widehat{W}^{\natural}_{g-h,n-|I|}(Q, Q_{J\setminus I}).$$

The sum $\sum''$ is as in Proposition 11.2(iii), and $\widehat{W}^{\natural}_{0,2}(Q, Q_j) := \mathcal{K}(Q, Q_j)$, $\widehat{W}^{\natural} := \widehat{W}$ for stable factors. The first term is omitted if $(g-1, n+1)$ is unstable. For $(1, 1)$ put

$$\mathcal{D}_{1,1} := \tfrac{1}{2}\widehat{\widehat{W}}_{0,2}(Q, Q) + \tfrac{1}{8}Q^{-2} + \tfrac{1}{12}.$$

Thus in $\mathcal{D}_{g,n}$ the two orders of a splitting with a factor $(0, \{j\})$ together contribute

$$\mathcal{K}(Q, Q_j)\,\widehat{W}_{g,n-1}(Q, Q_{J\setminus j}).$$

For $(g, n) = (0, 3)$ we get $\mathcal{D}_{0,3} = \frac{1}{2}\sum'' = \mathcal{K}(Q, Q_1)\mathcal{K}(Q, Q_2)$. We write $\mathcal{E}^{\mathrm{TR}}_{g,n}$ when all stable correlators are the W^{TR}. The functions $W_{0,1}$, $W_{0,2}$ are always those of (11.1).

Proposition 11.9. *Let $2g - 2 + n > 0$, $(g, n) \neq (1, 1)$, and $|Q_j| > r_*$. Then $\mathcal{E}^{\mathrm{TR}}_{g,n}(\cdot; Q_J)$, a priori holomorphic on $\mathcal{U}$, extends to a holomorphic even function on $\{|Q| < r_*\}$ that vanishes to second order at $Q = 0$.*

Proof. Write $f = f_{g,n}(\cdot, y_J)$ and $F = f_{g-1,n+1}(\cdot, \cdot, y_J)$. Since $W^{\mathrm{TR}}_{g,n} = f(u) = \frac{1}{2}(A + S)[f]$, Corollary 11.7 gives on $\mathcal{U}$

$$\mathcal{E}^{\mathrm{TR}}_{g,n} = \tfrac{1}{4}A[\Psi]A[f] - \tfrac{1}{2}Q^2 S[f] + \tfrac{1}{8}A_1A_2[F]\big|_{Q'=Q} + \tfrac{1}{8}{\sum}'' A[f_1]A[f_2], \tag{11.5}$$

where f_1, f_2 denote the two factors of a splitting, with $f_{0,2}(\cdot, y_j)$ for a $(0, \{j\})$ factor. We now use the identity

$$A[g]A[h] = S[g]S[h] - 2\big(g(u)h(v) + g(v)h(u)\big),$$

and $A_1A_2[F]|_{Q'=Q} = S_1S_2[F]|_{Q'=Q} - 4F(u, v)$ for symmetric F. The cross terms of the $\sum''$ add up to $2\sum'' f_1(u)f_2(v)$, because the sum runs over ordered splittings. Hence

$$4\,\mathcal{E}^{\mathrm{TR}}_{g,n} = -2\,\mathfrak{B}_{g,n}(u; y_J) + \mathfrak{S}, \quad \mathfrak{S} := S[\Psi]S[f] + \tfrac{1}{2}S_1S_2[F]\big|_{Q'=Q} + \tfrac{1}{2}{\sum}'' S[f_1]S[f_2] - 2Q^2S[f], \tag{11.6}$$

with $\sigma(u)$ replaced by v in (11.2).

All terms of $\mathfrak{B}$ and $\mathfrak{S}$ are symmetric functions of the pair (u, v). We show that they extend holomorphically across $[-a, a]$.

Interior points of the cut. Let $c \in (-a, a)$, and let r_1, r_2 be the holomorphic branches of P^{-1} near c with $r_{1,2}(c) = y_0(c \pm \mathrm{i}0)$. By Lemma 11.4, $(u, v) = (r_1, r_2)$ above the cut and

$(u,v)=(r_2,r_1)$ below it. So any symmetric expression $H(u,v)$ equals $H(r_1,r_2)$ on both sides, provided H is holomorphic near $(r_1(c),r_2(c))$. This is the case for every term:

- $r_{1,2}(c)\in\partial D_\kappa\setminus\{\pm\beta\}$ avoids the poles $\pm\beta+\pi\mathbb{Z}$ of the functions $f_{h,k}$ and $f_{0,2}$ (which come from the forms and from the zeros of $\mathrm{d}Q$), and the poles $\pi\mathbb{Z}$ of Ψ;
- it also avoids $\pm y_j$, since $|Q_j|>r_*$;
- for $(g,n)\neq(1,1)$ no term $f_{0,2}(u,v)$ occurs.

Endpoints. By Proposition 11.2(iii), $\mathfrak{B}_{g,n}$ is holomorphic in Q near $\pm a$. For $\mathfrak{S}$:

- By Lemma 11.6(a), $S[f]=-\sum_{k\neq0}f(\xi_k(Q))$, which is holomorphic near $[-a,a]$.
- In the same way, $S_1S_2[F](Q,Q')=\sum_{k,l\neq0}F(\xi_k(Q),\xi_l(Q'))$. This series converges absolutely and locally uniformly, since $F(\xi_k,\xi_l)=O(k^{-2}l^{-2})$.
- $S[\Psi]$ is a symmetric function of the two roots near $\pm\beta$ of a function holomorphic there, hence holomorphic in Q. Also $S[f_{0,2}(\cdot,y_j)]=(\omega+\sigma^*\omega)/\mathrm{d}Q$ with $\omega=\omega_{0,2}(\cdot,y_j)$ holomorphic at $\pm\beta$. As in the proof of Proposition 11.2, this is a holomorphic function of Q. (Note that $f_{0,2}$ itself has a pole at $\pm\beta$, coming from the zero of $\mathrm{d}Q$.)

Therefore $\mathcal{E}^{\mathrm{TR}}_{g,n}$ extends holomorphically to $\{|Q|<r_*\}$.

Evenness. f is even and y_0 is odd, so $W^{\mathrm{TR}}_{g,n}$ is even in Q. The image sums, w and Y are even as well.

Value at 0. The boundary points over $Q=0$ are $u=y_0(0+\mathrm{i}0)$ and $v=y_0(0-\mathrm{i}0)=-u$. All correlators $f_{h,k}$, $f_{0,2}$ and Ψ are even in each variable. Hence every $A[\cdot]$ vanishes at $Q=0+\mathrm{i}0$, and so does $A_1A_2[F](0,0)=F(u,u)-2F(u,-u)+F(-u,-u)$. By (11.5), $\mathcal{E}^{\mathrm{TR}}_{g,n}(0)=0$. An even holomorphic function vanishing at 0 is $O(Q^2)$. □

Remark 11.10. For $(g,n)=(1,1)$ the argument breaks down twice. First, Corollary 11.7(b) fails for $(0,2)$, because $T[f_{0,2}]\neq0$. So $\widehat{\widehat{W}}_{0,2}(Q,Q)$ is not $\frac14A_1A_2[f_{0,2}]$ on the diagonal; it differs from a regularisation of it by explicit $\sin^{-2}$ terms, and (11.5) does not hold. Second, the term $f_{0,2}(u,v)$ in $\mathfrak{B}_{1,1}$ is singular at $Q=0$, where $v=-u$. This is why $(1,1)$ is treated separately. Numerically, $\mathcal{E}^{\mathrm{TR}}_{1,1}$ with $\mathcal{D}_{1,1}$ as in Definition 11.8 is also $O(Q^2)$ (Appendix **??**); we do not use this.

11.5. **Dependence on κ.** Let $2g-2+n>0$. For $\kappa\in(0,1)$ the function $W^{\mathrm{TR}}_{g,n}$ is holomorphic on $\{|Q_i|>a\}$, including $Q_i=\infty$, and of order Q_i^{-2} there. Write

$$W^{\mathrm{TR}}_{g,n}=\sum_m G^{\mathrm{TR}}_m(\kappa)\prod_i Q_i^{-2m_i-2};$$

only even powers occur by Lemma 11.1(a).

Lemma 11.11 (Scaling)**.** *Let $\mathfrak{d}=\mathfrak{d}_{g,n}$, with $\mathfrak{d}_{0,1}=-2$ and $\mathfrak{d}_{0,2}=-1$; in these two cases let G^{TR}_m denote the coefficients of $W_{0,1}$, $W_{0,2}$ in (11.1). There are $\delta,R_0,M>0$ and functions $\tilde C_m(\lambda)$, holomorphic on $|\lambda|<\delta$ and satisfying $|\tilde C_m(\lambda)|\le MR_0^{2|m|}$, such that for small $\kappa>0$*

$$G^{\mathrm{TR}}_m(\kappa)=\lambda^{|m|-\mathfrak{d}}\,\tilde C_m(\lambda),\qquad \lambda=\beta^2=\arcsin^2\sqrt{\kappa}.$$

In particular G^{TR}_m extends holomorphically to a punctured disc $0<|\kappa|<\delta'$, with a pole of order at most $\max(0,\mathfrak{d}-|m|)$ at 0, and $|G^{\mathrm{TR}}_m(\kappa)|\le M'(2R_0^2|\kappa|)^{|m|}|\kappa|^{-\mathfrak{d}}$ there, with $M'=2^{|\mathfrak{d}|}M$.

Proof. Put $y=\beta\tau$. Let $\psi(z)=\sin^2z/z^2$, $\varphi(z)=z\cot z$ and $\chi(z)=z^2/\sin^2z$; these are even and holomorphic near 0. Then

$$P(\beta\tau)=\beta\,p(\tau),\quad p(\tau)=\tau+\frac{\psi(\beta)\varphi(\beta\tau)}{\tau},\qquad \Psi(\beta\tau)=\lambda\tilde\Psi(\tau),\quad \tilde\Psi(\tau)=-\frac{2\psi(\beta)^2}{\tau^2\psi(\beta\tau)},$$

and $\omega_{0,2}$ becomes $\tilde\omega_{0,2} = \big(\chi(\beta(\tau_1-\tau_2))(\tau_1-\tau_2)^{-2} + \chi(\beta(\tau_1+\tau_2))(\tau_1+\tau_2)^{-2}\big)\mathrm{d}\tau_1\mathrm{d}\tau_2$. Each of these depends on β only through λ, and is holomorphic in (τ,λ) for τ in $U = \{0<|\tau|<4\}$ (off the diagonals in the case of $\tilde\omega_{0,2}$) and $|\lambda|<\delta$, if δ is small.

At $\lambda = 0$ the data become $p = \tau+\tau^{-1}$, $\tilde\Psi = -2\tau^{-2}$ and $\tilde\omega_{0,2} = ((\tau_1-\tau_2)^{-2} + (\tau_1+\tau_2)^{-2})\mathrm{d}\tau_1\mathrm{d}\tau_2$. For every λ the branch point is $\tau = 1$, and $p''(1) = 2+O(\lambda)\neq 0$. So the local involution $\tilde\sigma(\tau;\lambda)$ is holomorphic near $(1,0)$. The denominator of the kernel, divided by $(\tau-1)^2$, is holomorphic and nonvanishing near $(1,0)$; at $\lambda=0$ its value at $\tau=1$ is 16.

In the coordinate τ we have $\omega'_{0,1} = \Psi\,\mathrm{d}Q = \beta^3\tilde\Psi\,\mathrm{d}p$, and $\omega_{0,2} = \tilde\omega_{0,2}$. Topological recursion is homogeneous of degree $2-2g-n$ in $\omega_{0,1}$. So by Lemma 11.1(b),

$$\omega_{g,n} = \beta^{-3(2g-2+n)}\tilde\omega_{g,n},$$

where $\tilde\omega_{g,n}$ is the output of the folded recursion at $\tau=1$ for $(p, \tilde\Psi\,\mathrm{d}p, \tilde\omega_{0,2})$. Each residue at $\tau=1$ is a finite combination of Taylor coefficients of functions holomorphic in (τ,λ). By induction, $\tilde\omega_{g,n} = \sum_k \tilde c_k(\lambda)\prod_i \tilde b_{k_i}(\tau_i;\lambda)\mathrm{d}\tau_i$ with $\tilde c_k$ holomorphic on $|\lambda|<\delta$ and $\tilde b_k = \partial_w^k\big(\chi(\beta(\tau-w))(\tau-w)^{-2} + \chi(\beta(\tau+w))(\tau+w)^{-2}\big)|_{w=1}$.

For $|p|\geq R_0$ (with $R_0\geq 8$, say) and $|\lambda|<\delta$, the equation $\tau p = \tau^2 + \psi(\beta)\varphi(\beta\tau)$ has a unique solution $\tau_0(p;\lambda)$ with $|\tau_0|\leq 4/|p|$. It is holomorphic in $(1/p,\lambda)$, odd in p, and $\tau_0 = \psi(\beta)/p + O(p^{-3})$. For real β, $\beta\tau_0(Q/\beta)$ and $y_0(Q)$ are preimages of Q that are holomorphic for $|Q|>R_0\beta$ and agree to leading order at $Q=\infty$ (both are $\kappa/Q+O(Q^{-3})$). By uniqueness of the small root they coincide for large $|Q|$, hence on $|Q|>R_0\beta$.

Thus $\tilde\omega_{g,n}(\tau_0(p_1),\dots) = \Xi(p;\lambda)\prod_i \mathrm{d}p_i$, with Ξ holomorphic on $\{|p_i|>R_0\}\cup\{\infty\}$ and $|\lambda|<\delta$, even and $O(p_i^{-2})$ in each p_i. Write $\Xi = \sum_m \tilde C_m(\lambda)\prod_i p_i^{-2m_i-2}$; by Cauchy's estimates $|\tilde C_m|\leq MR_0^{2|m|+2n}$, and we absorb R_0^{2n} into M. Substituting $p_i = Q_i/\beta$, the coefficient of $\prod Q_i^{-2m_i-2}\mathrm{d}Q_i$ is $\beta^{-3(2g-2+n)}\beta^{-n}\beta^{2|m|+2n}\tilde C_m = \lambda^{|m|-\mathfrak{d}}\tilde C_m(\lambda)$. Finally, λ is a holomorphic function of κ near 0 with $\lambda = \kappa(1+O(\kappa))$. (For the Cauchy estimates we shrink δ and enlarge R_0 slightly, so that Ξ is bounded on the closed region.)

The unstable cases work in the same way. By (11.1) and Remark 5.5, $W_{0,1} = 2y_0^2 - 4\kappa(1-y_0\cot y_0)$. With $y_0=\beta\tau_0$ and $\kappa=\lambda\psi(\beta)$ this equals $\lambda\big(2\tau_0^2 - 4\psi(\beta)(1-\varphi(\beta\tau_0))\big)$. Here $(1-\varphi(z))/z^2$ is even and holomorphic, so $1-\varphi(\beta\tau_0)$ is $\lambda\tau_0^2$ times a holomorphic function of $\lambda\tau_0^2$. Hence the bracket is a holomorphic function of $(1/p,\lambda)$, even in p and of order p^{-2}. This gives $G^{\mathrm{TR}}_{(d)} = \lambda^{d+2}\tilde C_d(\lambda)$.

For $W_{0,2}$: $W_{0,2}\,\mathrm{d}Q_1\mathrm{d}Q_2 = \tilde\omega_{0,2}(\tau_0(p_1),\tau_0(p_2)) - \mathrm{d}p_1\mathrm{d}p_2\big((p_1-p_2)^{-2}+(p_1+p_2)^{-2}\big)$. This is holomorphic in $(1/p_1,1/p_2,\lambda)$ for $|p_i|>R_0$, including the diagonal and the antidiagonal, since both terms have the same double poles there with biresidue 1. This gives $G^{\mathrm{TR}}_{(d_1,d_2)} = \lambda^{d_1+d_2+1}\tilde C_{d_1,d_2}(\lambda)$. □

Lemma 11.12 (Rationality)**.** *For $\kappa\in(0,1)$, $G^{\mathrm{TR}}_m(\kappa)\in\mathbb{Q}[\kappa,\kappa^{-1},(1-\kappa)^{-1}]$, in the sense that there is a rational function in this ring which agrees with G^{TR}_m on $(0,1)$.*

Proof. Let $\gamma = \cot\beta$, so that $\kappa = (1+\gamma^2)^{-1}$, and let $\mathcal{R} = \mathbb{Q}[\gamma,\gamma^{-1},(1+\gamma^2)^{-1}]$. Expand at β in $\varepsilon = y-\beta$. The derivatives of cot at β are polynomials in γ, divisible by $1+\gamma^2$ from the first derivative on. Hence $P(\beta+\varepsilon)-P(\beta)\in\varepsilon^2\mathcal{R}[[\varepsilon]]$ with ε^2-coefficient γ, and $\Psi(\beta+\varepsilon)\in\mathcal{R}[[\varepsilon]]$ with $\Psi'(\beta) = 4\gamma(1+\gamma^2)^{-1}$. Also $\cot(2\beta) = (\gamma^2-1)/(2\gamma)$, so $\sin^{-2}(2\beta+\varepsilon)\in\mathcal{R}[[\varepsilon]]$.

It follows that $\sigma(\varepsilon)\in\mathcal{R}[[\varepsilon]]$, since solving for it only divides by γ. The inverse of the kernel denominator $(\Psi(\beta+\varepsilon)-\Psi(\beta+\sigma(\varepsilon)))P'(\beta+\varepsilon)$ lies in $\varepsilon^{-2}\mathcal{R}[[\varepsilon]]$: its leading coefficient is $16\gamma^2(1+\gamma^2)^{-1}$, whose inverse lies in $\mathcal{R}$. So all coefficients produced by the folded recursion lie in $\mathcal{R}$.

The Laurent coefficients at $Q=\infty$ are residues at $y=0$ of $b_k(y)P(y)^{2m+1}$ (one variable at a time). The Taylor coefficients of b_k at 0 are derivatives of $\sin^{-2}$ at $\pm\beta$, which lie in $\mathbb{Q}[\gamma]$, and $P(y)=y+(1+\gamma^2)^{-1}\cot y$. Hence $G_m^{\mathrm{TR}}=r(\gamma)$ with $r\in\mathcal{R}$.

Split r into its even and odd parts in γ. Since $\gamma^2=(1-\kappa)/\kappa$, we get $G_m^{\mathrm{TR}}(\kappa)=A(\kappa)+\sqrt{(1-\kappa)/\kappa}\,B(\kappa)$ with $A,B\in\mathbb{Q}[\kappa,\kappa^{-1},(1-\kappa)^{-1}]$. By Lemma 11.11, G_m^{TR} agrees for small $\kappa>0$ with a function that is single-valued and meromorphic on a punctured disc around $\kappa=0$. Analytic continuation around $\kappa=0$ changes the sign of the square root and fixes A and B. Hence $B\equiv 0$. □

11.6. **The Masur–Veech side.** Let $z_k=\zeta(2k)/\pi^{2k}\in\mathbb{Q}$, $\theta_{p,q}=\frac{(2p+2q+1)!}{(2p+1)!(2q+1)!}z_{p+q+1}$ and $\nu_j(p):=(2j+1)\theta_{j,p}=\binom{2p+2j+1}{2j}z_{p+j+1}$. Then

$$\eta_p(Q):=Q^{-2p-2}+\tfrac12\sum_{m\neq0}(Q+\pi m)^{-2p-2}=Q^{-2p-2}+\sum_{j\geq0}\nu_j(p)Q^{2j}\qquad(|Q|<\pi). \tag{11.7}$$

The twisted coefficients of [ABC+23], in our normalisation, are

$$\begin{aligned}B^{d_1}_{e,p}&=(2e+1)\delta_{p,d_1+e-1}+\delta_{d_1,0}\delta_{e,0}\,\theta_{0,p},\\ C^{d_1}_{p,q}&=\delta_{d_1,p+q+2}+(2q+3-2d_1)\theta_{q+1-d_1,p}+(2p+3-2d_1)\theta_{p+1-d_1,q}+\delta_{d_1,0}\theta_{0,p}\theta_{0,q},\end{aligned}$$

with $\theta_{j,\cdot}:=0$ for $j<0$. The recursion of [ABC+23, Thm. 1.2] reads, for $2g-2+n\geq2$ and $d=(d_1,d')$,

$$\begin{aligned}h_{g,n}[d_1,d']=&\sum_j\sum_p B^{d_1}_{d_j,p}h_{g,n-1}[p,d'_{\hat\jmath}]\\&+\tfrac12\sum_{p,q}C^{d_1}_{p,q}\Big(h_{g-1,n+1}[p,q,d']+\sum_{\text{stable}}h_{h,1+|I|}[p,d'_I]\,h_{g-h,n-|I|}[q,d'_{J\setminus I}]\Big),\end{aligned} \tag{11.8}$$

with $h_{0,3}[0,0,0]=1$, $h_{1,1}[0]=\frac{1}{12}$, $h_{1,1}[1]=\frac18$, and unstable h set to zero. (These coefficients are those of [ABC+23, Thm. 1.2] after the substitution of Lemma 2.1; for instance $F_{1,1}[0]=\zeta(2)/2=\pi^2/12$.)

Coefficient rules. For a formal series $\Phi(Q)$ we write $\left[Q^k\right]\Phi$ for its coefficients. From (11.7) we get, for $d_1\geq0$:

$$\left[Q^{-2d_1}\right]\eta_p=\delta_{p,d_1-1}+\delta_{d_1,0}\nu_0(p),\qquad\left[Q^{-2d_1}\right]\left(Q^2(\eta_p-Q^{-2p-2})\right)=0,$$
$$\left[Q^{-2d_1}\right]\eta_p\eta_q=C^{d_1}_{p,q},\qquad\left[Q^{-2d_1}Q'^{-2e-2}\right]\left(\mathsf{w}(Q,Q')\eta_p(Q)\right)=B^{d_1}_{e,p},$$

using $\mathsf{w}(Q,Q')=\sum_e(2e+1)Q^{2e}Q'^{-2e-2}$ for $|Q|<|Q'|$. The third rule uses $\nu_j(p)=(2j+1)\theta_{j,p}$ with $j=q+1-d_1$ or $j=p+1-d_1$, and $\nu_0(p)=\theta_{0,p}$.

Lemma 11.13 (Loop equations at $X=0$)**.** *Let $\mathcal{E}^0_{g,n}$ be $\mathcal{E}_{g,n}$ of Definition 11.8 formed with the polynomials $W^0_{h,k}:=W_{h,k}|_{X=0}$, with $X=0$ and with $W_{0,1}=W_{0,2}=0$. Thus $Y=-Q^2$ and $\mathcal{K}=\mathsf{w}$. Then (11.8), together with the stated values of $h_{0,3}$ and $h_{1,1}$, is equivalent to: for all $d_1\geq0$ and d',*

$$\left[Q^{-2d_1}Q_J^{-2d'-2}\right]\mathcal{E}^0_{g,n}=0.$$

Proof. By the rules above, the coefficient of $Q^{-2d_1}Q_J^{-2d'-2}$ is the difference of the two sides of (11.8). The term $-Q^2\widehat{W}$ contributes $-h_{g,n}[d_1,d']$; the term $Q^2(\widehat{W}-W)$ contributes nothing; the $\mathcal{K}=\mathsf{w}$ factors contribute the B-terms; and the products of hatted correlators contribute

the C-terms. For $(0,3)$ only ww remains, with $\left[Q^0Q_1^{-2}Q_2^{-2}\right]=1$. For $(1,1)$ only $\frac{1}{8}Q^{-2}+\frac{1}{12}$ remains. □

Lemma 11.14 (Deformed loop equations)**.** *Let $2g-2+n>0$, let $W_{h,k}$ be the formal series (1.4), and let $\mathcal{E}_{g,n}$ be as in Definition 11.8, expanded with (11.7) and with w expanded for $|Q|<|Q_j|$. Then for all $d_1\geq 0$ and d',*

$$\left[Q^{-2d_1}Q_J^{-2d'-2}\right]\mathcal{E}_{g,n}=0 \quad \text{in } \mathbb{Q}[[X]].$$

Each coefficient of X^N is a finite sum, by (1.5).

Proof. We claim that the coefficient of $X^N/N!$ in $\mathcal{E}_{g,n}$ is obtained from $\mathcal{E}^0_{g,n+N}$ by extracting the coefficient of Q_e^{-2} in each of N additional spectators Q_e. Given the claim, Lemma 11.13 gives the result.

To prove the claim, recall that products of exponential generating functions in X correspond to distributing the N labelled extra boundaries between the factors. Then compare $\mathcal{E}^0_{g,n+N}$ with $\mathcal{E}_{g,n}$ term by term:

- A split in $\mathcal{E}^0_{g,n+N}$ whose factor $(0,\{e_1,\dots,e_r\})$ contains only extra spectators is stable for $r\geq 2$. These terms produce $\widehat{W}_{0,1}\widehat{W}_{g,n}$; the factor $\frac{1}{2}$ cancels against the two orders.
- For $r=1$ the factor is $\mathsf{w}(Q,Q_e)$, whose Q_e^{-2}-coefficient is 1. There are N such terms, and they produce $X\widehat{W}_{g,n}$.
- Factors $(0,\{j\}\cup E)$ produce $\widehat{W}_{0,2}(Q,Q_j)$ when $E\neq\emptyset$, and $\mathsf{w}(Q,Q_j)$ when $E=\emptyset$; together they give $\mathcal{K}$.
- The genus-reducing term for $(1,1+N)$ with $N\geq 1$ produces $\frac{1}{2}\widehat{\widehat{W}}_{0,2}(Q,Q)$.
- The special terms of $\mathcal{E}^0_{0,3}$ and $\mathcal{E}^0_{1,1}$ occur only for $N=0$.
- All other terms are matched directly.

Together with $-Q^2\widehat{W}+Q^2(\widehat{W}-W)=-Q^2W$, this gives $\mathcal{E}_{g,n}$. □

11.7. **From analytic to formal identities.** We regard G_m^{TR} as an element of $\mathbb{C}((\kappa))$ via Lemma 11.11, and $G_m^{(g)}(X)$ as an element of $\mathbb{Q}[[\kappa]]$ via $X=2\kappa-\kappa^2$. The data $W_{0,1}$, $W_{0,2}$ are the same on both sides by (11.1).

Lemma 11.15. *Let $2g-2+n>0$ with $(g,n)\neq(1,1)$, and assume $G^{\mathrm{TR}}=G$ for all stable (h,k) with $2h-2+k<2g-2+n$. Then the identities of Lemma 11.14 hold in $\mathbb{C}((\kappa))$ with $G^{(g)}_{(\cdot,d')}$ replaced by $G^{\mathrm{TR}}_{(\cdot,d')}$ for (g,n).*

Proof. Fix a small real $\kappa>0$ and r with $a<r<r_*$. On the annulus $a<|Q|<r_*$, with $|Q_j|>r_*$, the function $\mathcal{E}^{\mathrm{TR}}_{g,n}$ is the product of absolutely convergent Laurent expansions:

- $W^{\mathrm{TR}}=\sum G_m^{\mathrm{TR}}\prod Q^{-2m-2}$ for $|Q|>a$;
- $\widehat{W}=\sum_i G_i\eta_i(Q)$, which converges absolutely for $a<|Q|<\pi-a$, because $|G_i|\leq Cr^{2i}$ and $\sum_j|\nu_j(i)||Q|^{2j}\leq\sum_{m\geq 1}(\pi m-|Q|)^{-2i-2}$;
- the expansion of $\mathsf{w}(Q,Q_j)$.

Hence its Laurent coefficients are the absolutely convergent series given by the coefficient rules. By Proposition 11.9 the coefficients of Q^{-2d_1}, $d_1\geq 0$, vanish. As functions of Q_J on $|Q_j|>r_*$ these coefficients have convergent expansions in the Q_j^{-2}, whose coefficients therefore vanish too.

Each such coefficient $\mathcal{F}(\kappa)$ is a series of products of the functions $G_\cdot$, $G_\cdot^{\mathrm{TR}}$ and of constants. The coefficients $C^{d_1}_{p,q}$, $B^{d_1}_{e,p}$ and $\nu_j(p)$ grow at most exponentially in p,q,e,j, since $z_k\leq\frac{1}{6}$ and

the binomial factors are at most $4^{p+q+e+j+2}$. By Lemma 11.11, which also covers the unstable data $G^{(0)}_{(q)}$ and $G^{(0)}_{(p,e)}$ with $\mathfrak{d} = -2, -1$, the series therefore converges locally uniformly on a punctured disc $0 < |\kappa| < \delta''$. So $\mathcal{F}$ is holomorphic there, and it vanishes on an interval, hence identically. Its Laurent coefficients at $\kappa = 0$ are therefore 0. They can be computed termwise, by uniform convergence on a circle. By (1.5) and Lemma 11.11, a term containing factors with free indices p (and q) has κ-order at least $p - C_0$ (resp. $p + q - C_0$) for a constant C_0. So only finitely many terms contribute to each power of κ, and the termwise result is exactly the formal identity. □

Lemma 11.16 (Uniqueness)**.** *Fix d'. Let $(\Delta_p)_{p\geq 0} \subset \mathbb{C}((\kappa))$ satisfy $\operatorname{ord}_\kappa \Delta_p \geq p - C_0$ for some constant C_0, and suppose that for all $d_1 \geq 0$*

$$\Delta_{d_1} = X\Delta_{d_1-1} + X\delta_{d_1,0}\sum_p \nu_0(p)\Delta_p + \sum_{p,q} C^{d_1}_{p,q}\, G^{(0)}_{(q)}\, \Delta_p \qquad (\Delta_{-1} := 0).$$

Then $\Delta_p = 0$ for all p.

Proof. If not, let $q_0 = \min_p \operatorname{ord}\Delta_p$; it exists because $\operatorname{ord}\Delta_p \geq -C_0$. Since $\operatorname{ord} X = 1$ and $\operatorname{ord} G^{(0)}_{(q)} \geq q+2 \geq 1$, every term on the right has order at least q_0+1. Each power of κ receives only finitely many contributions, so the right side has order at least $q_0 + 1$ for every d_1. This contradicts the choice of q_0. □

11.8. **The case** $(1, 1)$**.**

Proposition 11.17. *$G^{\mathrm{TR}}_{(d)} = G^{(1)}_{(d)}$ for all d.*

Proof. Write $\gamma = \cot\beta$, so that $\kappa = (1+\gamma^2)^{-1}$, and $\varepsilon = y - \beta$. Let t be the local coordinate of the proof of Proposition 11.2, $P(\beta+\varepsilon) - a = t^2$. We compare the principal parts at β of two meromorphic 1-forms on $\mathbb{C}/\pi\mathbb{Z}$, both odd under ι, decaying at $\pm i\infty$, with poles only at $\pm\beta$ and without residues.

The topological-recursion side. For $(1,1)$ the bracket is $\omega_{0,2}(y, \sigma(y))$. In the notation of the proof of Proposition 11.2, $r(t) = -\frac{1}{4t^2} - H(t,-t)$. With $\Psi^-(t) = \Psi_1 t + \Psi_3 t^3 + O(t^5)$ this gives $\hat\rho_{-3} = -1/(16\Psi_1)$ and $\hat\rho_{-1} = (\Psi_3/\Psi_1 - 4H(0,0))/(16\Psi_1)$, and so

$$\omega_{1,1} = \big(\hat\rho_{-3}t^{-4} + \hat\rho_{-1}t^{-2}\big)\mathrm{d}t + (\text{holomorphic near } \beta). \tag{11.9}$$

The local data are elementary to expand. From $\cot(\beta+\varepsilon) = \gamma - (1+\gamma^2)\varepsilon + \gamma(1+\gamma^2)\varepsilon^2 - \frac{1}{3}(1+\gamma^2)(1+3\gamma^2)\varepsilon^3 + O(\varepsilon^4)$ one gets

$$P(\beta+\varepsilon) - a = \gamma\varepsilon^2 - \big(\gamma^2 + \tfrac{1}{3}\big)\varepsilon^3 + \gamma\big(\gamma^2 + \tfrac{2}{3}\big)\varepsilon^4 - \big(\gamma^4 + \gamma^2 + \tfrac{2}{15}\big)\varepsilon^5 + O(\varepsilon^6),$$

so $t = \sqrt{\gamma}\,\varepsilon\,(1 + e_1\varepsilon + e_2\varepsilon^2 + e_3\varepsilon^3 + O(\varepsilon^4))$ with

$$e_1 = -\frac{3\gamma^2+1}{6\gamma}, \qquad e_2 = \frac{27\gamma^4 + 18\gamma^2 - 1}{72\gamma^2}, \qquad e_3 = -\frac{675\gamma^6 + 675\gamma^4 + 69\gamma^2 + 5}{2160\gamma^3}.$$

Inverting and inserting into $\Psi(\beta+\varepsilon) = -2\kappa^2(1+\cot^2(\beta+\varepsilon))$ gives

$$\Psi_1 = \frac{4\sqrt{\gamma}}{1+\gamma^2}, \qquad \Psi_3 = \frac{45\gamma^4 + 30\gamma^2 - 7}{18\,\gamma^{5/2}(1+\gamma^2)}.$$

For $H(0,0)$, write $y = \beta + \varepsilon(t)$ and use $K(y_1,y_2) = (y_1-y_2)^{-2} + \frac{1}{3} + \sin^{-2}(y_1+y_2) + O((y_1-y_2)^2)$ together with $\frac{y'(t_1)y'(t_2)}{(y(t_1)-y(t_2))^2} = \frac{1}{(t_1-t_2)^2} + \frac{1}{6}\{y;t\}(t_1) + O(t_1-t_2)$, where $\{y;t\}$ is the Schwarzian

derivative. This gives

$$H(0,0) = \tfrac{1}{6}\{y;t\}(0) + \big(\tfrac{1}{3} + \sin^{-2} 2\beta\big) y'(0)^2 = \frac{3\gamma^4 + 18\gamma^2 + 7}{24\gamma^3},$$

since $y'(0) = \gamma^{-1/2}$, $\{y;t\}(0) = -\frac{(1+\gamma^2)(3\gamma^2-1)}{4\gamma^3}$ and $\sin^{-2} 2\beta = \frac{(1+\gamma^2)^2}{4\gamma^2}$. Hence

$$\hat\rho_{-3} = -\frac{1+\gamma^2}{64\sqrt{\gamma}}, \qquad \hat\rho_{-1} = \frac{(1+\gamma^2)(9\gamma^4 - 186\gamma^2 - 91)}{4608\,\gamma^{7/2}}.$$

Substituting $t = t(\varepsilon)$ in (11.9), the principal part of $\omega_{1,1}/\mathrm{d}y$ at β is

$$\begin{aligned} &-\frac{1+\gamma^2}{64\gamma^2}\varepsilon^{-4} - \frac{(1+\gamma^2)(3\gamma^2+1)}{192\gamma^3}\varepsilon^{-3} - \frac{(1+\gamma^2)(2\gamma^2+1)}{48\gamma^4}\varepsilon^{-2} \\ &\quad = -\frac{1}{64(1-\kappa)}\,\varepsilon^{-4} - \frac{(3-2\kappa)\gamma}{192(1-\kappa)^2}\,\varepsilon^{-3} - \frac{2-\kappa}{48(1-\kappa)^2}\,\varepsilon^{-2}, \end{aligned} \tag{11.10}$$

using $1 + \gamma^2 = \kappa^{-1}$ and $\gamma^2 = (1-\kappa)/\kappa$. Since $b_0 = \varepsilon^{-2} + O(1)$, $b_1 = 2\varepsilon^{-3} + O(1)$ and $b_2 = 6\varepsilon^{-4} + O(1)$, equivalently $\omega_{1,1} = (A_0 b_0 + A_1 b_1 + A_2 b_2)\mathrm{d}y$ with

$$A_0 = -\frac{2-\kappa}{48(1-\kappa)^2}, \qquad A_1 = -\frac{(3-2\kappa)\gamma}{384(1-\kappa)^2}, \qquad A_2 = -\frac{1}{384(1-\kappa)}.$$

The genus-one side. By Corollary 1.3,

$$W_{1,1}\,\mathrm{d}Q = \frac{(4-2\kappa) + (1-\kappa)\partial_\kappa}{48(1-\kappa)^2}\big(e\,\mathrm{d}Q\big), \qquad e\,\mathrm{d}Q = -\tfrac{1}{2} b_0(y)\,\mathrm{d}y, \quad y = y_0(Q).$$

Here ∂_κ is taken at fixed Q and acts coefficientwise on the Laurent expansion. Using $\partial_\kappa \beta = (1+\gamma^2)/(2\gamma)$ and $\partial_\kappa y|_Q = -\cot y / P'(y)$, we get

$$\partial_\kappa(e\,\mathrm{d}Q) = -\tfrac{1}{2}\Big(\partial_\kappa\beta\; b_1(y) - \partial_y h(y)\Big)\mathrm{d}y, \qquad h := \frac{b_0 \cot y}{P'}.$$

Here $\cot y / P'(y) = \sin y \cos y / (\sin^2 y - \kappa)$ is regular at 0. One has $\cot(\beta+\varepsilon) = \gamma - (1+\gamma^2)\varepsilon + O(\varepsilon^2)$ and $P'(\beta+\varepsilon) = 2\gamma\varepsilon - (1+3\gamma^2)\varepsilon^2 + O(\varepsilon^3)$. Hence $h = \frac{1}{2}\varepsilon^{-3} + \frac{\gamma^2-1}{4\gamma}\varepsilon^{-2} + h_{-1}\varepsilon^{-1} + O(1)$. Moreover $h_{-1} = 0$: h is odd, so $h\,\mathrm{d}y$ has equal residues at $\pm\beta$, and their sum vanishes because $h \to 0$ at $\pm\mathrm{i}\infty$. The principal part of $W_{1,1}\mathrm{d}Q/\mathrm{d}y$ at β is therefore

$$-\frac{1}{96(1-\kappa)^2}\Big((4-2\kappa)\varepsilon^{-2} + (1-\kappa)\frac{3\gamma^2+1}{2\gamma}\,\varepsilon^{-3} + \tfrac{3}{2}(1-\kappa)\varepsilon^{-4}\Big).$$

With $(1-\kappa)(3\gamma^2+1) = \gamma^2(3-2\kappa)$ this is (11.10).

By ι-symmetry the principal parts at $-\beta$ agree as well. The difference of the two forms is a holomorphic 1-form on the cylinder that decays at both ends, hence zero. Comparing Laurent coefficients at $Q = \infty$ gives the claim. □

11.9. **Proof of Theorem E.** *Step 1: formal equality.* We show $G_m^{\mathrm{TR}} = G_m^{(g)}$ in $\mathbb{C}((\kappa))$ for all stable (g,n) with $n \geq 1$, by induction on $2g-2+n$. For $2g-2+n = 1$ the cases are $(1,1)$, which is Proposition 11.17, and $(0,3)$, which is covered by the generic argument below. Let $(g,n) \neq (1,1)$ and assume the claim for smaller $2g-2+n$; fix d'.

By Lemmas 11.14 and 11.15, both $G_{(\cdot,d')}$ and $G^{\mathrm{TR}}_{(\cdot,d')}$ satisfy the identities

$$\Big[Q^{-2d_1} Q_J^{-2d'-2}\Big]\,\mathcal{E}_{g,n} = 0.$$

By the induction hypothesis these identities involve the same lower data on both sides. They are affine in the unknowns $G_{(p,d')}$. By the coefficient rules, the unknowns enter only through $Y\widehat{W}_{g,n} + Q^2(\widehat{W}_{g,n} - W_{g,n})$, and the coefficient of Q^{-2d_1} of this expression is

$$-G_{(d_1,d')} + X\, G_{(d_1-1,d')} + X\delta_{d_1,0}\sum_p \nu_0(p) G_{(p,d')} + \sum_{p,q} C_{p,q}^{d_1} G_{(q)}^{(0)} G_{(p,d')}.$$

Hence $\Delta_p := G^{\mathrm{TR}}_{(p,d')} - G_{(p,d')}$ satisfies the system of Lemma 11.16, and $\operatorname{ord}\Delta_p \geq p + |d'| - \mathfrak{d}_{g,n}$ by (1.5) and Lemma 11.11. So $\Delta = 0$.

Step 2: part (a). By Step 1, G_m^{TR} has no pole at $\kappa = 0$. By Lemma 11.12 it therefore lies in $\mathbb{Q}[\kappa, (1-\kappa)^{-1}] = \mathbb{Q}[s, s^{-1}]$, where $\kappa = 1 - s$. So $G_m^{(g)}(X) = G_m^{\mathrm{TR}}(1 - \sqrt{1-X})$, an identity of formal power series. The right side is holomorphic for $|X| < 1$, which proves (a).

Step 3: part (b). For each $\kappa \in (0,1)$ the Laurent expansion of $\omega_{g,n}(y_0(Q_1), \dots)$ at $Q_i = \infty$ converges for $|Q_i| > a$, and its coefficients are $G_m^{\mathrm{TR}}(\kappa) = G_m^{(g)}(X)$ by Step 2. □

Statement on the use of AI

This paper was generated with substantial assistance from AI. In particular, the results were produced during the first two weeks of October 2026 using an agentic harness running ChatGPT 6 Pro (as available at that time), with the author extensively involved in the loop to guide, review, and verify the outputs. The text of this manuscript was then produced with the help of Claude Fable 5.2. The author takes full responsibility for the accuracy and integrity of the work, and plans to release a Lean formalization along with an updated exposition in the future.

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