A computable Goldbach shift for a sequence of doubled squares

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Abstract

Althöfer, Chen, Perelli and Pintz ask for a particular shift $\beta \in [0,1]$ for which $2\lfloor (n+\beta)^3 \rfloor^2$ is a sum of two primes for every sufficiently large integer n. Conditional on their exceptional-set estimate for doubled squares, we specify such a shift by a deterministic algorithm that approximates it with certified error. The algorithm does not require the implied constant in the estimate.

Theorem 1. *Assume that the number of positive integers $m \le M$ for which $2m^2$ is not a sum of two primes is $O(M^{7/12})$. There is a fixed computable real $\beta \in [1/4, 3/4]$ such that $2\lfloor (n+\beta)^3 \rfloor^2$ is a sum of two primes for every sufficiently large integer n. The algorithm below specifies β without the implied constant.*

The hypothesis is supplied by Theorem 1 of Althöfer, Chen, Perelli and Pintz [1], with $\phi(m) = m^2$ and $\varepsilon = 1/12$. Their proof uses Li and Liu's fourth-moment bound ($\theta < 11/12$) and major-arc estimate ($\theta < 1$) [2, Theorem 1 and Proposition 4.1], where θ is a common upper bound on the real parts of nontrivial zeros of primitive Dirichlet L-functions. For $\theta \le 7/8$, ACPP [1, Sec. 2] cite the OpenAI preprint [3, Theorem 1.1] and its accompanying Lean scope note [4]. We use the resulting exceptional-set estimate as input, without reproving or independently verifying that analytic chain. Our construction gives a computable witness for the equation (5.5) question in this sense, not a simple closed-form value. It makes no claim about $\beta = 0$, rational shifts, or the separate requirement that every positive n work. Here computability means that, given $s \ge 0$, the algorithm returns a rational within 2^{-s-2} of the same real number. It does not give an explicit threshold for the eventual Goldbach property.

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1 Effective covers of the exceptional shifts

Call m bad if $2m^2$ is not a sum of two primes. Badness is decidable by finite trial division and enumeration of prime pairs. For $j \ge 0$, put $T = 2^j$ and consider the integer block $T \le n < 2T$. For every such n enumerate the bad m in $n^3 \le m < (n+1)^3$. Each yields the interval

$$[\, m^{1/3} - n,\ (m+1)^{1/3} - n \,).$$

These intervals cover exactly the bad shifts for the block in $[0,1)$. The excluded boundary value 1 will not be needed.

For any endpoint x in this construction let $g_j(x)$ be the largest grid point at most x on the grid of spacing $\delta_j = 2^{-4j-8}$. It is computable by exact comparisons of rational cubes with integers. Replace an interval with endpoints a, b by $(g_j(a) - \delta_j, g_j(b) + 2\delta_j)$, an open rational interval containing $[a,b]$. The union of the enlarged intervals is U_j. In particular, the construction never misses a bad shift that lies at an exact floor boundary.

Each original interval has length at most $1/(3T^2)$ by the mean value theorem. The integer ranges for the different n are disjoint, and in total contain at most $7T^3$ indices, all below $(2T)^3$. The number of bad indices among them is $O(T^{7/4})$ by hypothesis. Each enlargement adds less than $4\delta_j$. Hence

$$\operatorname{meas}(U_j) \ll T^{-1/4} + T^3\delta_j \ll 2^{-j/4}. \tag{1}$$

All union lengths here and below are computed exactly by sorting rational endpoints.

2 Choosing one shift without the implied constant

Set $b_j = 2^{-8-\lfloor j/8 \rfloor}$ and define $V_j = U_j$ if $\operatorname{meas}(U_j) \le b_j$, and $V_j = \emptyset$ otherwise. Equation (1) implies that $V_j = U_j$ for all sufficiently large j: indeed $2^{-j/4}/b_j \to 0$. No estimate for the last discarded block is

needed. The budgets satisfy

$$\sum_{j\ge 0} b_j = \frac{1}{16}, \qquad \sum_{j\ge J} b_j = 2^{-4-J/8} \quad (8 \mid J,\ J \ge 0).$$

Let $K = [0,1] \setminus \bigcup_{j\ge 0} V_j$, a closed set. The starting interval $I_0 = [1/4, 3/4]$ has intersection with K of measure at least $7/16$.

Starting with I_0, use the following rule at stage $s = 0, 1, \ldots$. Bisect I_s into two closed halves. Put $J = 16s + 8$. For each half H compute $\mathrm{meas}(H \setminus \bigcup_{j<J} V_j)$, and let I_{s+1} be the half with the larger value, choosing the left half in the event of a tie. To verify the rule, write $p_s = 2^{-2s-2}$. Inductively $\mathrm{meas}(K \cap I_s) \ge p_s$. The selected half has stage-J residual mass at least $p_s/2$, because the two residual masses sum to at least p_s (the shared endpoint has measure zero). The unprocessed tail has measure at most $2^{-4-J/8} = p_s/8$. Thus $\mathrm{meas}(K \cap I_{s+1}) \ge 3p_s/8 \ge p_{s+1}$, establishing the induction.

The nested nonempty compact sets $K \cap I_s$ have a common point. The interval lengths 2^{-s-1} tend to zero, so this point is unique; define it to be β. The midpoint of I_s is the promised rational approximation, obtained in finitely many exact operations. Finally, β avoids V_j for every j, hence U_j for every sufficiently large j. Those blocks cover all sufficiently large n, proving the claim.

The argument is an application of the constructive Borel–Cantelli principle (see Galatolo, Hoyrup and Rojas [5]). The construction is for computability, not for efficient digit calculation.

Range of the method. An exception bound $O(M^c)$ with $0 \le c < 2/3$ suffices after adjusting the budgets: the same cover has measure $O(2^{-(2-3c)j} + 2^{-j})$. Take an integer h with $1/h < \min(1, 2-3c)$ and an integer B with $h2^{1-B} \le 1/16$, and use $b_j = 2^{-B-\lfloor j/h \rfloor}$. The tail for $h \mid J$ is $h2^{1-B-J/h}$, so at each stage one can compute J with tail at most $p_s/8$. The rest of the argument is unchanged. This variant does not change the specific algorithm above.

First subdivision. The published verification through $4 \cdot 10^{18}$ [6] covers $2m^2$ for $2 \le m < 256^3$. Thus the only bad index in the first eight blocks is $m = 1$, whose covering interval has length greater than $b_0 = 1/256$. All eight retained unions are empty. The first comparison is a tie, so $\beta \in [1/4, 1/2]$. This is a consequence of the published computation, not a new verification of its range. The bracket does not settle $n = 1$, which works exactly when $\beta \ge 2^{1/3} - 1$ within the bracket.

AI disclosure. The human project lead selected the research question and directed the investigation. OpenAI models developed the construction and drafted and checked the text. Anthropic's Claude performed a separate line-by-line AI review and identified a correction to the interval enlargement bound. No independent human verification or formal proof checking is claimed.

References

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