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Springer's 1951 conjecture on inverse coefficients settled: true at every odd order, false at every even order from four on

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Abstract

Let $g(z) = z + b_1 z^{-1} + b_2 z^{-2} + \cdots$ be analytic and one-to-one in $|z| > 1$, and let $G(w) = w + B_1 w^{-1} + B_2 w^{-2} + \cdots$ be its inverse near infinity. Fix $n \geq 1$, and let q be the least prime divisor of $n+1$. In 1951 Springer conjectured that, among all such g, the modulus of the inverse coefficient B_n is largest when g maps $|z| > 1$ onto the complement of q straight slits of equal length that issue from the origin at equal angles. At odd orders $n = 2m+1$ the conjecture reads $|B_{2m+1}| \leq C_m$, where $C_m = (2m)!/(m!\,(m+1)!)$ is a Catalan number, with equality only for the maps $z + e^{i\theta}/z$. This case was known for $m \leq 7$ and for odd functions g, and further cases through $m = 16$ have been reported. We prove this inequality and the equality statement for every m and every g. More precisely, B_{2m+1} lies in the closed disk with center $-2^{-m} C_m\, b_1^{m+1}$ and radius $(1 - 2^{-m})C_m$. The proof follows the inverse function along a Loewner evolution and bounds its coefficients, by induction, by those of the inverses of the slit maps $z + t/z$. For $n = 2$ the conjecture is a classical theorem of Schiffer and Golusin. For every even $n \geq 4$ the conjecture is false. When $n+1$ is not divisible by 3, this failure is due to Chang, Schiffer and Schober (1981) and Kuznetsov (1988). We establish the failure at the remaining orders $n = 8, 14, 20, \ldots$, and we give explicit counterexamples for every even $n \geq 4$. For even $n \geq 4$ the maximum of $|B_n|$ is, to our knowledge, unknown, but it is at least a positive constant times $2^n n^{-3/2}$, whereas the conjectured bound is only $2/(n+1)$ when $n+1$ is prime. These results are verified in Lean.[1]

Keywords: univalent functions, inverse coefficients, Springer's conjecture, Loewner differential equation, Catalan numbers.
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1 Introduction

1.1 Springer's conjecture

Let $\Delta = \{z \in \mathbb{C} : |z| > 1\}$ and let Σ_0 be the class of functions

$$g(z) = z + \sum_{k=1}^{\infty} b_k z^{-k} \tag{1}$$

that are analytic and univalent (one-to-one) in Δ. They have a simple pole at infinity and are called meromorphic univalent functions. The class is denoted Σ' in [14, 28]. Each $g \in \Sigma_0$ maps Δ

[1] Lean 4 verifies Theorems 1.2–1.4; the odd-order results assume existence of the normalized Loewner evolution in the disk, and only the equality characterization in Theorem 1.2 also assumes the exterior area theorem: https://github.com/david44hello/paper_springer-1951-inverse-coefficients/tree/v2.20/lean

conformally onto a domain containing a neighborhood of infinity, and its inverse has an expansion

$$G(w) = w + \sum_{n=1}^{\infty} B_n w^{-n} \tag{2}$$

near infinity. The *inverse coefficients* B_n are polynomials in the b_k. By Lagrange's formula, nB_n is the coefficient of z^{-1} in $-g(z)^n$. For instance, $B_1 = -b_1$, $B_2 = -b_2$, $B_3 = -(b_3 + b_1^2)$ and $B_4 = -(b_4 + 3b_1b_2)$. Problem 6.4 of Hayman's collection of research problems [10] asks for the exact bounds for $|b_n|$ and $|B_n|$ over Σ_0. For univalent functions in the unit disk, the sharp bounds for the inverse coefficients were found by Loewner [18] in 1923, in the paper that introduced the Loewner differential equation.

For an integer $q \geq 2$ the function $g_q(z) = z(1 + z^{-q})^{2/q}$ belongs to Σ_0. It maps Δ onto the complement of q radial slits of length $2^{2/q}$ that issue from the origin at equal angles (Figure 2). For $q = 2$ it is the map $g_2(z) = z + 1/z$ onto the complement of the segment $[-2, 2]$. Springer [32] computed the inverse coefficients of g_q. The coefficient of order n vanishes unless q divides $n + 1$, and if $n + 1 = qk$ its modulus is $\frac{1}{n}\binom{2n/q}{k}$, where $\binom{x}{k} = x(x-1)\cdots(x-k+1)/k!$. Springer showed that, for fixed n, this modulus is largest when q is the least prime divisor of $n + 1$. Springer also proved that, when q divides $n + 1$, the nth coefficient of the inverse of g_q has the smallest real part among the inverse functions with the same first $n - 1$ coefficients [32, p. 438], and stated the following conjecture.

Conjecture 1.1 (Springer [32, p. 438])**.** *Let $n \geq 1$, let q be the least prime divisor of $n + 1$, and put $k = (n + 1)/q$. Then every $g \in \Sigma_0$ satisfies*

$$|B_n| \leq \sigma_n := \frac{1}{n}\binom{2n/q}{k}, \tag{3}$$

with equality only for the functions $e^{i\alpha} g_q(e^{-i\alpha} z)$, α real.

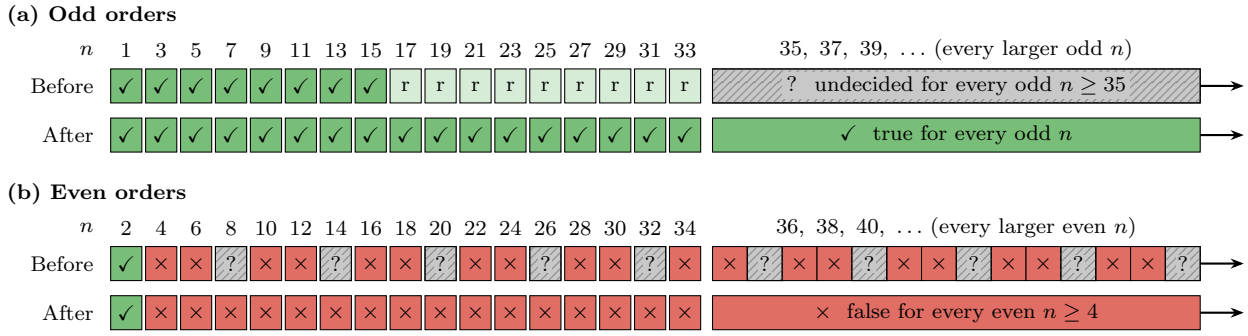


Figure 1: In a green cell (✓) Springer's conjectured bound for $|B_n|$ (Conjecture 1.1) is known to hold, and in a red cell (×) it is known to fail. A pale green cell (r), at odd n from 17 to 33, marks a proof that has been published or announced; for $n = 27$ it is known to us only from later papers (Section 1.2). In a gray hatched cell (?) the conjecture was, to our knowledge, undecided. The rows show the state before and after this paper, for odd orders (a) and for even orders (b). Each row continues for all larger n, as its arrow indicates. (a) Before this paper the bound was proved for odd $n \leq 15$ and undecided for every odd $n \geq 35$; it holds for every odd n (Theorem 1.2). (b) Before this paper the bound was known to fail at the even $n \geq 4$ for which $n + 1$ is not divisible by 3 [3, 16] (Section 1.3), and it was undecided at the other even $n \geq 8$; it fails for every even $n \geq 4$ (Theorem 1.4).

In Springer's notation, S is the class of the inverse functions $\phi = G$, b_n is our B_n, and $-c_n^{(q)}$ is the modulus of the nth coefficient of the inverse ϕ_q of g_q. Thus $-c_n^{(q)} = \sigma_n$ when q is the least prime divisor of $n+1$. In Springer's words [32, p. 438]: "This leads us to the general conjecture that for functions in the class S, $|b_n| \le -c_n^{(q)}$ where q is the least prime divisor of $n+1$, equality holding only when $\phi(w) = e^{i\alpha}\phi_q(e^{-i\alpha}w)$." The conjecture thus concerns every order n, even and odd.

For odd n we have $q = 2$, and σ_n is a Catalan number. Throughout the paper

$$C_m = \frac{1}{m+1}\binom{2m}{m} = \frac{(2m)!}{m!\,(m+1)!} \qquad (m \ge 0)$$

denotes the mth Catalan number. Its first values are $1, 1, 2, 5, 14, 42$. For $n = 2m+1$ one finds $\sigma_{2m+1} = C_m$, and the odd case of Conjecture 1.1 reads

$$|B_{2m+1}| \le C_m \qquad (m \ge 0), \tag{4}$$

with equality only for $g(z) = z + e^{i\theta}/z$, θ real. Springer wrote this case out separately [32, (95)], and it is in this form, as the inequality $|B_{2N-1}| \le (2N-2)!/(N!\,(N-1)!)$ with $N = m+1$, that the conjecture is usually quoted [14, 27, 28] and recorded in the problem collection [10, Update 6.4].

In this paper we prove that Conjecture 1.1 holds for every odd n. For even $n \ge 4$ the conjecture is false. When $n+1$ is not divisible by 3, this is known from work of Chang, Schiffer and Schober [3] and of Kuznetsov [16] (Section 1.3). Thus Conjecture 1.1, as a statement about all n, has been known to be false since 1981. What remained open were the odd orders, for which the conjecture is usually quoted, and the even orders $8, 14, 20$, and so on. We establish the failure at these even orders, and we give explicit counterexamples for every even $n \ge 4$. Since the case $n = 2$ is classical, the conjecture is now decided for every n. Table 1 and Figure 1 compare these results with what was known.

1.2 Odd orders

When Conjecture 1.1 was made, the cases $n = 1$ and $n = 2$ were known. The case $n = 1$ is the inequality $|b_1| \le 1$ of the area theorem, and the case $n = 2$ is the inequality $|b_2| \le \frac{2}{3}$ of Schiffer [26]

Table 1: Springer's conjectured bound for $|B_n|$ (Conjecture 1.1) in the class Σ_0: earlier results and the results of this paper.

Order n	Best result before this paper	This paper
$n = 1$ and $n = 2$	True, with the equality statement: the area theorem, and Schiffer [26] and Golusin [8].	No change: both cases are classical.
n odd, $n \ge 3$	True for $n \le 15$ [32, 14, 27, 28, 24], with bounds in terms of $\|b_1\|$ for $5 \le n \le 15$ [27, 28, 24], and, for every odd n, when $g(-z) = -g(z)$ [2, 20]. Proofs have been published or announced for odd n from 17 to 33 (Section 1.2) and, according to the review of [38], for every odd n under sign conditions on the b_k.	True for every odd n, with the equality statement (Theorem 1.2). A disk for B_n in terms of b_1, for every odd n (Theorem 1.3; see the comparison with earlier bounds in Section 9).
n even, $n \ge 4$	False when $n+1$ is not divisible by 3: for $n \ge 6$ by [3, Section 8], and for $n = 4$ by [16] (see Section 1.3 for [22]). When $n+1$ is divisible by 3, that is, for $n = 8, 14, 20, \ldots$, no earlier result is known to us (Section 1.3).	False for every even $n \ge 4$, by explicit functions (Theorem 1.4). To our knowledge, this is new when $n+1$ is divisible by 3, and the exact maximum of $\|B_n\|$ is not known.

and Golusin [8]. Springer proved $|B_3| \le 1$ in the same paper [32]. Kubota [14] proved (4) for $m = 2, 3, 4$, that is, $|B_5| \le 2$, $|B_7| \le 5$ and $|B_9| \le 14$. Schober [27] proved it for $m \le 6$, and in [28] added the case $m = 7$, that is, $|B_{15}| \le 429$, as the case $k = 1$ of a theorem on functions with a k-quasiconformal extension [28, Corollary, p. 413], which includes the equality statement. The case $m = 7$ was also proved, with the equality statement, by Ren [24], in a note received in April 1979, which refers to an earlier proof by the same author for $m = 5$ and $m = 6$ [23]. The proofs in [27] rest on exponentiated Grunsky inequalities, and Schober expressed the belief that these inequalities "may lead to a solution of the conjecture [...] for all n" [27, p. 116]. For odd functions, that is, when $g(-z) = -g(z)$, inequality (4) holds for every m. This goes back to a paper of Bazilevich [2] from 1937, fourteen years before Springer's conjecture. Using Loewner's method, Bazilevich determined, for every $p \ge 2$, the largest modulus of each inverse coefficient among the functions in Σ_0 with p-fold symmetry, and found that it is attained by g_p [2, formula (7), pp. 691–692]. For $p = 2$ this formula gives the Catalan numbers. The result was proved again by Netanyahu [20, Corollary, p. 340], to whom Schober attributes it [27, p. 116]. Schober [27, Theorem 2] proved the inequality for B_{2N-1} under the weaker assumption that the even coefficients $B_2, B_4, \dots$ of index at most $N - 2$ vanish. Further cases have been reported in a series of papers and short notes, among them $|B_{17}|$ in [47], $|B_{19}|$ and $|B_{21}|$ in [11] and again in [41], $|B_{23}|$ in [43], $|B_{25}|$ in [36], $|B_{29}|$ in [45, 40], $|B_{31}|$ in [46], and $|B_{33}|$ in [37]. The note [40] names Wang and Cheng among those who proved the cases $m \le 13$, which include $|B_{27}|$. The review of [38] in zbMATH (Zbl 0781.30013) records (4) as proved for all $m \le 16$. Under additional assumptions the inequality for B_{2N-1} is reported for $N = 10$ when the coefficients b_k are real [34], for $N = 10, \dots, 13$ [4] and for $N = 17$ and $N = 18$ [7] when $b_2 = 0$, and, according to the same review of [38], for every N when all the b_k are positive, or when all terms in the expression of B_{2N-1} as a polynomial in the b_k are real and of the same sign. See also [5, 25, 39, 42, 44, 48]. To our knowledge, no proof for all odd orders has been available in the full class Σ_0.

Theorem 1.2. *Let $g \in \Sigma_0$ and let B_n be its inverse coefficients* (2)*. For every $m \ge 0$,*

$$|B_{2m+1}| \le C_m = \frac{(2m)!}{m!\,(m+1)!}.$$

If equality holds for some $m \ge 0$, then $g(z) = z + e^{i\theta}/z$ for some real θ; conversely, for these functions equality holds for every m.

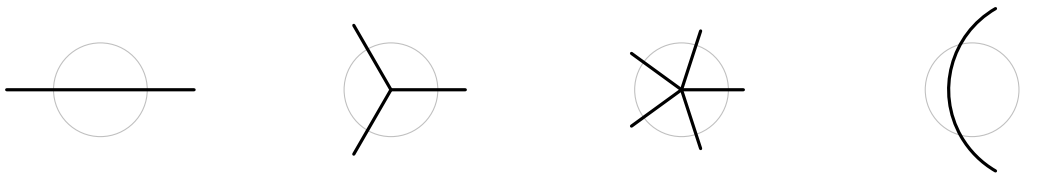
Theorem 1.2 follows from a sharper inequality, in which the first coefficient b_1 of g appears.

Theorem 1.3. *Let $g \in \Sigma_0$. For every $m \ge 0$,*

$$\left|B_{2m+1} + 2^{-m} C_m\, b_1^{m+1}\right| \le \left(1 - 2^{-m}\right) C_m. \tag{5}$$

Consequently $|B_{2m+1}| \le C_m - 2^{-m} C_m \left(1 - |b_1|^{m+1}\right)$.

Since $|b_1| \le 1$, with equality only for $g(z) = z + b_1/z$, Theorem 1.3 implies Theorem 1.2 (Section 7). By (5), B_{2m+1} lies in a closed disk whose center is determined by b_1 and which is contained in the disk $|B_{2m+1}| \le C_m$. For a given b_1 this disk is in general not the exact range of B_{2m+1} (see question (2) in Section 9). For $m = 0$ inequality (5) is the identity $B_1 = -b_1$. For $m = 1$ it is Springer's inequality $|B_3 + \frac{1}{2}B_1^2| \le \frac{1}{2}$ [32, (71)], from which Springer deduced $|B_3| \le 1$. In terms of g it reads $|b_3 + \frac{1}{2}b_1^2| \le \frac{1}{2}$ and is the Grunsky inequality for the coefficient $\gamma_{22} = b_3 + \frac{1}{2}b_1^2$ [6, §4.3]. For odd g, its consequence $|B_3| \le \frac{1}{2} + \frac{1}{2}|b_1|^2$ is already in Bazilevich's paper [2, (9_1)]. For $m \ge 2$ we have not found the disk (5) in the literature. Its consequence $|B_{2m+1}| \le C_m - 2^{-m} C_m (1 - |b_1|^{m+1})$,



(a) $q = 2$ every odd n | (b) $q = 3$ $n = 2$ | (c) $q = 5$ $n = 4$: $|B_4| = \frac{2}{5}$ | (d) circular arc $n = 4$: $|B_4| = \frac{3}{4}$

Figure 2: The sets omitted by Springer's maps and by $g(z) = z - 3/(4z-2)$, the map of Theorem 1.4(a). The unit circle is drawn in gray for scale. (a)–(c) The radial slits omitted by $g_q(z) = z(1 + z^{-q})^{2/q}$ for $q = 2, 3, 5$. Conjecture 1.1 predicts that these maps maximize $|B_n|$ at the orders n shown. (d) The circular arc omitted by the map of Theorem 1.4(a). Its fourth inverse coefficient has modulus $\frac{3}{4}$, compared with $\frac{2}{5}$ for the map in (c), so the prediction fails at $n = 4$.

however, is weaker than bounds in terms of $|b_1|$ that are contained in the earlier proofs for $2 \le m \le 7$ and in some of the later ones. The comparison is made in Section 9.

Problem 6.6 of [10] quotes from [32] a lower bound for $|B_{2N-1}|$ that is incompatible with Theorem 1.2 for large N. The discrepancy is an extraneous factor e in Springer's asymptotic formula [32, (96)] for the number C_{N-1} (Section 9).

1.3 Even orders

For even n the number q in Conjecture 1.1 is odd, and the conjectured extremal maps are the functions g_q with three or more slits. For $n = 2$, where $q = 3$, the conjecture is the classical inequality $|b_2| \le \frac{2}{3}$. This is the only even order at which it holds. In this subsection and in Section 8 even orders are written $n = 2m$.

Theorem 1.4. *For every even $n \ge 4$ there is a function $g \in \Sigma_0$ with $|B_n| > \sigma_n$; thus Conjecture 1.1 is false for every even $n \ge 4$. More precisely:*

(a) the function $g(z) = z - \dfrac{3}{4z-2}$, which maps Δ onto the complement of a circular arc, has $|B_4| = \frac{3}{4}$, whereas $\sigma_4 = \frac{2}{5}$;

(b) for every $m \ge 3$ the function $g(z) = z + \left(1 - \frac{1}{m}\right)z^{-1} + \frac{1}{2m}z^{-2}$ has

$$|B_{2m}| \;\ge\; \frac{1}{4m}\binom{2m}{m}\left(1 - \frac{1}{m}\right)^{m-1} > \sigma_{2m}.$$

For most even orders the failure of Conjecture 1.1 was already known. Chang, Schiffer and Schober [3, Section 8] proved, by means of the second variation, that for $q \ge 5$ the function g_q does not maximize $|B_n|$, with one possible exception, the case $q = 5$ and $n = 4$: "it is still an open question whether $|B_4| \le \frac{2}{5}$" [3, p. 223]. The conjecture is therefore false whenever the least prime divisor q of $n + 1$ is at least 5 and $n \ne 4$. Kuznetsov then showed that g_q does not maximize $|B_n|$ for any $q \ge 4$ and any n [16, Lemma 5], which includes the case $n = 4$. For even n we have $q \ge 5$ exactly when $n + 1$ is not divisible by 3. The conjecture was therefore known to fail at the orders $4, 6, 10, 12, 16, 18$, and so on. Neither [3] nor [16] treats the case $q = 3$, to which the even orders $8, 14, 20$, and so on, belong. For these orders we have found no earlier result. The proof in [16] is variational: it shows that suitable functions close to g_q have a larger coefficient.

To our knowledge, Theorem 1.4 is therefore new at the orders $8, 14, 20$, and so on. At the even orders $n \geq 6$ with $n+1$ not divisible by 3 the theorem adds explicit functions and explicit values. At $n = 4$ it gives an explicit function. Qiu and Huang [22] study B_4 for functions with real coefficients, a class containing this function, and an explicit example may already appear in their paper. The maps in (b) belong to a family of starlike functions studied by Mogra, Reddy and Juneja [19, Theorem 5]. What we add is the comparison of their inverse coefficients with σ_n, which we have not found in the literature. A partial positive result was also known. If $b_1 = \cdots = b_{m-1} = 0$, then $B_{2m} = -b_{2m}$, and in this case $|b_{2m}| \leq 2/(2m+1)$ by a theorem of Jenkins and Duren, as recorded in [13, p. 267] (see also [16, pp. 93–94]). The modulus for $q = 2m+1$ is $2/(2m+1)$, and σ_{2m} is at least this value. Hence (3) holds for these functions at the order $n = 2m$. Every counterexample at the order $2m$ therefore has a nonzero coefficient among $b_1, \ldots, b_{m-1}$.

The analogous conjecture for the coefficients of g itself, namely $|b_n| \leq 2/(n+1)$ with equality for g_{n+1}, holds for $n = 1$ and $n = 2$. Bazilevich [2, (10)] disproved it for odd $n \geq 3$, and it is now known to be false for every $n \geq 3$ [3, 13, 35]; see [17, p. 71]. This does not decide Conjecture 1.1, because B_n also contains products of the lower coefficients. For example, the function with $b_1 \geq 0$ that maximizes $\operatorname{Re} b_4$ in [13] has $|B_4| = \frac{5533}{16384} < \frac{2}{5}$. Further results of Kubota concern the coefficients b_4 and b_5 of g. When b_1 is real, Kubota determined the sharp bound for $\operatorname{Re} b_4$ [13] and, more generally, for $\operatorname{Re}(b_4 + \alpha b_1 b_2)$ with $0 \leq \alpha \leq 1$ [15], whereas $-B_4 = b_4 + 3b_1 b_2$ corresponds to $\alpha = 3$. For functions with real coefficients Kubota also proved the sharp inequality $b_5 \leq \frac{1}{3} + \frac{4}{507}$ [12].

If $n+1$ is prime, then $q = n+1$ and $\sigma_n = 2/(n+1)$, whereas by (b) the maximum of $|B_n|$ over Σ_0 is at least a positive constant times $2^n n^{-3/2}$ for even n. To our knowledge, the exact maximum of $|B_n|$ is not known for any even $n \geq 4$. Already for $n = 4$ the value $\frac{3}{4}$ is not the largest possible (Remark 8.1).

By the result of Bazilevich [2] and Netanyahu [20, Corollary, p. 340] quoted in Section 1.2, among the functions in Σ_0 with p-fold symmetry, $g(e^{2\pi i/p} z) = e^{2\pi i/p} g(z)$, where $p \geq 2$, the map g_p maximizes the modulus of every inverse coefficient. This shows why Springer's maps are the natural candidates among symmetric maps. Since the moduli of the inverse coefficients of g_p are at most σ_n, a function with $|B_n| > \sigma_n$ cannot have p-fold symmetry for any $p \geq 2$.

1.4 Method of proof

The proof of Theorem 1.3 has three steps. In the first (Section 3) we assume that g omits a disk and use Loewner's parametric representation [18] to construct a family of maps $Y_t(w) = w + P(t) - H_t(w)$, defined near infinity, that starts at a translation $w \mapsto w - w_0$ and ends at the inverse function G. We follow their Laurent coefficients at infinity. The series H_t satisfies a differential equation driven by probability measures on the unit circle. This equation is not linear: its right-hand side is an average of reciprocals $1/(w - H_t - a)$. In the second step (Section 4) we attach to H a family of series $K_u(H)$, $0 \leq u \leq 1$, in a new variable x, defined by residues. At $u = 0$ the series $K_u(H)$ is the first coefficient of H, and at $u = 1$ it records all the coefficients of H. Along the evolution this family satisfies a *linear* system of differential equations in t, with an integral over the parameter, in which the equation for the coefficient of x^{2n} involves only coefficients of lower order. In the third step (Sections 5 and 6) we replace the measure on the circle by a random choice between $+1$ and -1. The system can then be solved exactly, apart from one additional term caused by the change of the constant term $P(t)$. The solution is given by the series attached to the slit maps $z + t/z$. Since the derivative of each coefficient is determined by coefficients of lower order from the same family, an induction on n and a convexity argument show that the coefficients of this explicit solution bound those of the general one. We deduce Theorems 1.2 and 1.3 in Section 7. The even orders are treated in Section 8 by direct computation, and in Section 9 we discuss the results and state some open questions.

2 Preliminaries

2.1 Notation

We write $\mathbb{D} = \{|z| < 1\}$ and $\mathbb{T} = \{|z| = 1\}$. For a formal Laurent series in w that contains only finitely many positive powers of w, res denotes its coefficient of w^{-1}. For a formal power series in x, $[x^m]$ denotes its coefficient of x^m. We denote by $\mathcal{H}$ the set of formal series $H = \sum_{j\ge1} h_j w^{-j}$ with complex coefficients, and we call $H \in \mathcal{H}$ *odd* if $h_j = 0$ for all even j. A *random sign with mean* ω, where $-1 \le \omega \le 1$, is a random variable χ that takes the value 1 with probability $\frac{1+\omega}{2}$ and the value -1 with probability $\frac{1-\omega}{2}$. Thus $\mathbf{E}_\chi f(\chi) = \frac{1+\omega}{2} f(1) + \frac{1-\omega}{2} f(-1)$ for every function f on $\{1, -1\}$.

We shall use the area theorem [6, §2.2]: every $g \in \Sigma_0$ satisfies

$$\sum_{k=1}^{\infty} k|b_k|^2 \le 1. \tag{6}$$

In particular $|b_1| \le 1$, and $|b_1| = 1$ forces $b_k = 0$ for all $k \ge 2$, that is, $g(z) = z + b_1/z$.

2.2 Two families of series

For a complex number τ let

$$S_\tau(w) = \frac{w - \sqrt{w^2 - 4\tau}}{2} = \sum_{m=0}^{\infty} C_m \, \tau^{m+1} w^{-(2m+1)}, \tag{7}$$

where the square root is the branch that behaves like w at infinity. The expansion is the generating function of the Catalan numbers. The series S_τ is the solution in $\mathcal{H}$ of the quadratic equation $S(w - S) = \tau$. Since $w - S_\tau(w)$ is the inverse of $z \mapsto z + \tau/z$ near infinity, the inverse coefficients of

$$g(z) = z + \frac{b_1}{z} \qquad \text{are} \qquad B_{2m+1} = -C_m \, b_1^{m+1}, \quad B_{2m} = 0. \tag{8}$$

This proves the last assertion of Theorem 1.2. We shall also use the series

$$F_\tau(w) = w - \sqrt{w^2 - 2\tau} = \sum_{m=0}^{\infty} 2^{-m} C_m \, \tau^{m+1} w^{-(2m+1)}, \tag{9}$$

whose expansion follows from (7) because $F_\tau = 2S_{\tau/2}$. Both S_τ and F_τ are odd elements of $\mathcal{H}$. Differentiating the relations $S(w - S) = \tau$ and $(w - F)^2 = w^2 - 2\tau$ gives

$$\frac{\partial S_\tau}{\partial \tau} = \frac{1}{w - 2S_\tau}, \qquad \frac{\partial S_\tau}{\partial w} = -\frac{S_\tau}{w - 2S_\tau}, \qquad \frac{\partial F_\tau}{\partial \tau} = \frac{1}{w - F_\tau}. \tag{10}$$

2.3 Loewner's parametric representation

We need the following classical consequence of Loewner's theory [18]; see [21, Chapter 6] and [9, Theorem B], which refers to [21, Problem 3, p. 164]. In [9], Theorem B gives the family φ_ℓ, with the time parameter $\ell/\ell_0 \in [0, 1]$, and Section 2.2 there recalls why such a family satisfies a differential equation of the form (11).

Theorem 2.1 (Loewner, Kufarev, Pommerenke)**.** *Let $\psi : \mathbb{D} \to \mathbb{D}$ be analytic and univalent, with $\psi(0) = 0$ and $\psi'(0) = \rho \in (0, 1)$, and put $\ell_0 = \log(1/\rho)$. Then there is a function $p : \mathbb{D} \times [0, \ell_0] \to \mathbb{C}$*

such that $p(\zeta,\cdot)$ is measurable for every $\zeta\in\mathbb{D}$, and $p(\cdot,\ell)$ is analytic with $\operatorname{Re}p(\cdot,\ell)>0$ and $p(0,\ell)=1$ for almost every ℓ, with the following property. For every $\zeta\in\mathbb{D}$ the initial value problem

$$\frac{d}{d\ell}\varphi_\ell(\zeta)=-\varphi_\ell(\zeta)\,p\big(\varphi_\ell(\zeta),\ell\big),\qquad \varphi_0(\zeta)=\zeta, \tag{11}$$

has a unique absolutely continuous solution on $[0,\ell_0]$, each φ_ℓ is a univalent map of $\mathbb{D}$ into itself with $\varphi_\ell(0)=0$ and $\varphi_\ell'(0)=e^{-\ell}$, and $\varphi_{\ell_0}=\psi$.

By the Herglotz representation, for almost every ℓ there is a Borel probability measure μ_ℓ on $\mathbb{T}$ with

$$p(\zeta,\ell)=\int_{\mathbb{T}}\frac{1+\xi\zeta}{1-\xi\zeta}\,d\mu_\ell(\xi)\qquad(\zeta\in\mathbb{D}). \tag{12}$$

The moments $\int\xi^k\,d\mu_\ell$ with $k\ge1$ are, up to a factor 2, the Taylor coefficients of $p(\cdot,\ell)$, and those with $k\le-1$ are their complex conjugates. They are therefore measurable functions of ℓ. Since every continuous function f on $\mathbb{T}$ is a uniform limit of trigonometric polynomials, $\ell\mapsto\int f\,d\mu_\ell$ is measurable for every such f.

3 An evolution equation at infinity

Let $g\in\Sigma_0$, and suppose that g *omits a closed disk*: there are $w_0\in\mathbb{C}$ and $\rho\in(0,1)$ such that $|g(z)-w_0|>\rho$ for all $z\in\Delta$. We put $s=\rho^2$. The next lemma, illustrated in Figure 3, connects the translation $w\mapsto w-w_0$ with the inverse function G by a family of maps Y_t.

Lemma 3.1. *Under these assumptions there exist a number $R_0>0$, functions Y_t $(s\le t\le1)$ analytic in $R_0<|w|<\infty$ with expansions*

$$Y_t(w)=w+P(t)-H_t(w),\qquad H_t(w)=\sum_{j=1}^{\infty}h_j(t)w^{-j}, \tag{13}$$

and Borel probability measures μ_t on $\mathbb{T}$ $(s\le t\le1)$, such that the following holds.

(i) $Y_s(w)=w-w_0$ and $Y_1=G$; thus $P(s)=-w_0$, $H_s=0$, $P(1)=0$ and $h_j(1)=-B_j$ for all $j\ge1$.

(ii) The functions P and h_j $(j\ge1)$ are absolutely continuous on $[s,1]$, and for every continuous function f on $\mathbb{T}$ the function $t\mapsto\int f\,d\mu_t$ is measurable.

(iii) For almost every $t\in[s,1]$,

$$P'(t)=-\frac{1}{\sqrt{t}}\int_{\mathbb{T}}\xi\,d\mu_t(\xi),\qquad \frac{\partial H_t}{\partial t}=\int_{\mathbb{T}}\frac{\xi^2}{w-H_t-a_t(\xi)}\,d\mu_t(\xi),\qquad a_t(\xi)=\sqrt{t}\,\xi-P(t), \tag{14}$$

where the second equation is understood coefficientwise in powers of $1/w$.

In (14), the reciprocal is the series $1/(w-H-a)=\sum_{k\ge0}(H+a)^kw^{-k-1}$. Hence its coefficient of w^{-j} is a polynomial in a and $h_1,\dots,h_{j-1}$, and its coefficient of w^{-1} is 1. In particular the first coefficient $h_1(t)$ satisfies

$$h_1'(t)=\int_{\mathbb{T}}\xi^2\,d\mu_t(\xi),\qquad h_1(s)=0,\qquad h_1(1)=-B_1=b_1. \tag{15}$$

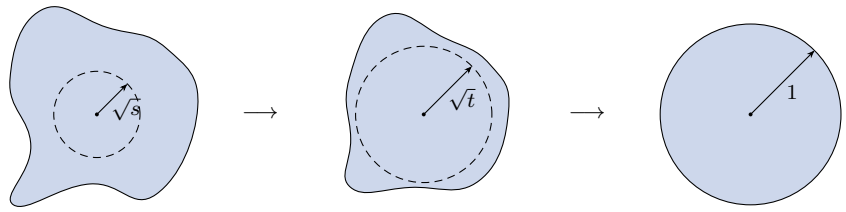


Figure 3: The family Y_t of Lemma 3.1 (schematic). Shaded: the set that Y_t omits, that is, the complement of the image of $g(\Delta)$ under Y_t. It contains the disk of radius $\sqrt{t}$ about the origin, whose boundary is dashed. At $t = s$ the shaded set is a translate of the set omitted by g. As t increases the dashed disk grows, and at $t = 1$ it fills the shaded set, which is then the closed unit disk. Each Y_t is conformal on $g(\Delta)$ and satisfies $Y_t(w) = w + O(1)$ at infinity.

Proof. The function $\psi(\zeta) = \rho/(g(1/\zeta) - w_0)$ is analytic and univalent in $\mathbb{D}$, satisfies $|\psi| < 1$ because $|g - w_0| > \rho$ in Δ, and has the expansion $\psi(\zeta) = \rho\zeta + O(\zeta^2)$ at the origin. Let p, φ_ℓ and μ_ℓ be as in Theorem 2.1 and (12), with $\ell_0 = \log(1/\rho)$. As ℓ runs from ℓ_0 down to 0, the number $t = e^{-2\ell}$ runs from s up to 1. We change the parameter to $t = e^{-2\ell} \in [s, 1]$, write μ_t for the measure μ_ℓ with $\ell = -\frac{1}{2}\log t$ (on the exceptional null set, where μ_ℓ is not defined, we put $\mu_t = \delta_1$), and define

$$Y_t(w) = \frac{\sqrt{t}}{\varphi_\ell\big(1/G(w)\big)}.$$

This formula defines a conformal map of $g(\Delta)$ into $\{|y| > \sqrt{t}\}$, because φ_ℓ is univalent, maps $\mathbb{D}$ into itself and vanishes only at the origin. Below we use only its expansion at infinity. Since $\varphi_\ell(\zeta) = e^{-\ell}\zeta + O(\zeta^2)$ and $1/G(w) = w^{-1} + O(w^{-3})$, the function Y_t is analytic near infinity with $Y_t(w) = w + O(1)$. This gives the expansion (13). For $t = 1$ we have $\varphi_0(\zeta) = \zeta$ and $Y_1 = G$. For $t = s$ we have $\varphi_{\ell_0} = \psi$ and $g(G(w)) = w$, so $\psi(1/G(w)) = \rho/(w - w_0)$ and $Y_s(w) = w - w_0$. This proves (i).

To prove (iii) fix w and write $\varphi = \varphi_\ell(1/G(w))$, so that $Y_t = \sqrt{t}/\varphi$. By (11) and (12), and because $d\sqrt{t}/d\ell = -\sqrt{t}$,

$$\frac{\partial Y_t}{\partial \ell} = -\frac{\sqrt{t}}{\varphi} + \frac{\sqrt{t}}{\varphi}\int_{\mathbb{T}} \frac{1+\xi\varphi}{1-\xi\varphi}\, d\mu_t(\xi) = Y_t \int_{\mathbb{T}} \frac{2\xi\varphi}{1-\xi\varphi}\, d\mu_t(\xi) = 2\sqrt{t}\int_{\mathbb{T}} \frac{\xi\, Y_t}{Y_t - \sqrt{t}\,\xi}\, d\mu_t(\xi).$$

Since $dt/d\ell = -2t$ and $Y/(Y - \sqrt{t}\,\xi) = 1 + \sqrt{t}\,\xi/(Y - \sqrt{t}\,\xi)$, this gives, for almost every t,

$$\frac{\partial Y_t}{\partial t} = -\int_{\mathbb{T}} \Big(\frac{\xi}{\sqrt{t}} + \frac{\xi^2}{Y_t - \sqrt{t}\,\xi}\Big) d\mu_t(\xi). \tag{16}$$

Inserting $Y_t = w + P - H_t$ and comparing the constant terms and the terms of negative order in w, we obtain (14).

It remains to justify the passage to coefficients and to prove (ii). The functions $\varphi_\ell/e^{-\ell}$ are univalent in $\mathbb{D}$ and normalized at the origin, so the Koebe distortion theorem gives $|\varphi_\ell(\zeta)| \geq \rho|\zeta|/4$ for all $\ell \in [0, \ell_0]$. Since $g(\Delta)$ contains a neighborhood of infinity, we can fix R_1 such that the set $\{|w| \geq R_1\}$ lies in $g(\Delta)$, where G is analytic and $1/G(w) \in \mathbb{D}$. Then $Y_t(w) - w$ is analytic in $|w| \geq R_1$, including the point at infinity, and bounded on the circle $|w| = R_1$ uniformly in t. By the maximum principle there is M with $|Y_t(w) - w| \leq M$ for all $|w| \geq R_1$ and all $t \in [s, 1]$. We

choose $R_0 > \max(R_1, 2M+2)$. Then $|Y_t(w) - \sqrt{t}\,\xi| \geq R_0 - M - 1$ on $|w| = R_0$, so the right-hand side of (16) is bounded on this circle, uniformly in t. For each fixed w on this circle, $t \mapsto Y_t(w)$ is absolutely continuous, by (11) and because φ_ℓ does not vanish at $1/G(w)$. Hence $Y_t(w) - Y_s(w)$ is the integral over $[s,t]$ of the right-hand side of (16). For almost every t this right-hand side equals $-\frac{Y_t}{2t}\big(p(\sqrt{t}/Y_t, \ell) - 1\big)$, by (12). It is measurable in t for fixed w, because $p(\zeta, \ell)$ is measurable in ℓ and analytic in ζ and $\ell \mapsto \varphi_\ell(1/G(w))$ is continuous. It is continuous in w, hence jointly measurable. The coefficients $P(t)$ and $h_j(t)$ are Cauchy integrals of Y_t over the circle. Taking Cauchy integrals of this identity and using Fubini's theorem, we see that each coefficient is Lipschitz in t and that its derivative is, for almost every t, the corresponding Cauchy integral of the right-hand side of (16). Since there are countably many coefficients, one null set of values of t serves for all of them. On $|w| = R_0$ the expansion of $1/(Y_t - \sqrt{t}\,\xi)$ in powers of $1/w$ converges uniformly in $\xi \in \mathbb{T}$, so these integrals are the coefficients displayed in (14). The measurability in (ii) was noted after (12). □

Remark 3.2. The evolution that leads to the slit map is the following. If $\mu_t = \frac{1}{2}(\delta_1 + \delta_{-1})$ for all t, then $P' = 0$, and a short computation with (10) shows that $P \equiv 0$ and $H_t = S_t$ solve (14). Here the map $Y_t = w - S_t$ is the inverse of $z + t/z$. It maps the complement of the segment $[-2\sqrt{t}, 2\sqrt{t}]$ onto $\{|y| > \sqrt{t}\}$. This solution does not start from $H_s = 0$, since $S_s \neq 0$. In Section 6 every evolution (14) with $H_s = 0$ will be compared with it.

4 A family of residues

The coefficients of $H \in \mathcal{H}$ are collected in the exponential generating function $\operatorname{res}\big(e^{xw}H\big) = \sum_{m\geq 0} h_{m+1}x^m/m!$. Along the evolution (14) its derivative involves the residue of $e^{xw}/(w - H - a)$, which depends nonlinearly on the coefficients of H. For example, for $a = 0$ and $H = h_1/w$ it equals $\sum_{k\geq 0} h_1^k x^{2k}/(2k)!$. To obtain a linear evolution we enlarge the family of functionals. The derivative of $\operatorname{res}(e^{xw}H)$ in a direction L is $\operatorname{res}(e^{xw}L)$. In this derivative we replace xw by $uxw + (1-u)xH$, with a parameter $u \in [0,1]$. The functional K_u of Definition 4.1 below vanishes at $H = 0$, and its derivative in a direction L is $\operatorname{res}(e^{uxw+(1-u)xH}L)$. The reason for this choice is the identity $uxw + (1-u)xH = uxJ + uax + xH$ with $J = w - H - a$. In the direction $L = 1/J$, the relation $(e^{uxJ} - 1)/J = x\int_0^u e^{vxJ}\,dv$ removes the reciprocal, and what remains is an integral of residues of the same form, with v in place of u. Lemma 4.3 below expresses the derivative of each member of the resulting family through members of the same family, so that all of them can be followed together along the evolution.

Definition 4.1. For $H \in \mathcal{H}$ and $0 \leq u \leq 1$ let

$$K_u(H) = \operatorname{res}\Big(e^{uxw}\int_0^H e^{(1-u)x\eta}\,d\eta\Big) = \sum_{j,l\geq 0} \frac{u^j(1-u)^l}{j!\,(l+1)!}\,\operatorname{res}\big(w^j H^{l+1}\big)\,x^{j+l}, \tag{17}$$

a formal power series in x. Here the integral stands for the series $\sum_{l\geq 0}((1-u)x)^l H^{l+1}/(l+1)!$, and the residue is taken coefficientwise in x: each coefficient of a power of x is a Laurent series in w with only finitely many positive powers.

Since H^{l+1} begins with the power $w^{-(l+1)}$, the residue $\operatorname{res}(w^j H^{l+1})$ vanishes for $l > j$, and for $l \leq j$ it is a polynomial in $h_1, \ldots, h_{j+1}$ with non-negative integer coefficients. Hence $[x^m]K_u(H)$ is a polynomial in u and $h_1, \ldots, h_{m+1}$. For example,

$$[x^0]K_u(H) = h_1, \qquad [x^1]K_u(H) = u\,h_2, \qquad [x^2]K_u(H) = \tfrac{1}{2}u^2 h_3 + \tfrac{1}{2}u(1-u)h_1^2.$$

We record the properties of K_u that will be used.

Lemma 4.2. *Let $H \in \mathcal{H}$ and $0 \le u \le 1$.*

(i) $K_0(H) = h_1$ *and* $K_1(H) = \operatorname{res}(e^{xw}H) = \sum_{m\ge 0} h_{m+1}x^m/m!$.

(ii) $\operatorname{res}\big(e^{uxw+(1-u)xH}\big) = (1-u)\,x\,K_u(H)$.

(iii) If H is odd, then $K_u(H)$ contains only even powers of x.

(iv) If $h_j \ge h_j^* \ge 0$ for all j, where $H^* = \sum h_j^* w^{-j}$, then $[x^m]K_u(H) \ge [x^m]K_u(H^*) \ge 0$ for all m.

Proof. (i) and (iv) are read off from (17). For (ii), note that $(1-u)x\int_0^H e^{(1-u)x\eta}d\eta = e^{(1-u)xH} - 1$, and that $\operatorname{res}(e^{uxw}) = 0$ because e^{uxw} contains no negative power of w. For (iii), if H is odd then w^jH^{l+1} contains only powers of w of the parity of $j+l+1$, so its residue vanishes unless $j+l$ is even. □

For $L \in \mathcal{H}$ we write

$$\mathrm{d}K_u(H)[L] = \frac{d}{d\varepsilon}K_u(H+\varepsilon L)\Big|_{\varepsilon=0} = \operatorname{res}\big(e^{uxw+(1-u)xH}\,L\big)$$

for the derivative of K_u at H in the direction L. The next lemma shows that, in the direction of a reciprocal $1/(w-H-a)$, the derivative of K_u is a linear expression in the residues $K_v(H)$ with $v \le u$.

Lemma 4.3. *Let $H \in \mathcal{H}$, $a \in \mathbb{C}$ and $0 \le u \le 1$. Then*

$$\mathrm{d}K_u(H)\Big[\frac{1}{w-H-a}\Big] = e^{uax} + x^2\int_0^u (1-v)\,e^{(u-v)ax}\,K_v(H)\,dv. \tag{18}$$

Moreover

$$\mathrm{d}K_u(H)\Big[\frac{\partial H}{\partial w}\Big] = -u\,x\,K_u(H). \tag{19}$$

Proof. Put $J = w-H-a$. Since $uxw + (1-u)xH = uxJ + uax + xH$, we have

$$\mathrm{d}K_u(H)\Big[\frac{1}{J}\Big] = \operatorname{res}\Big(e^{uax+xH}\,\frac{e^{uxJ}}{J}\Big) = \operatorname{res}\Big(\frac{e^{uax+xH}}{J}\Big) + x\int_0^u \operatorname{res}\big(e^{uax+xH}e^{vxJ}\big)\,dv,$$

where we used $(e^{uxJ}-1)/J = x\int_0^u e^{vxJ}dv$. In the first term, $e^{xH} = 1 + O(w^{-1})$ and $1/J = w^{-1} + O(w^{-2})$, so the residue equals e^{uax}. In the second term, $uax+xH+vxJ = (u-v)ax + vxw + (1-v)xH$, and by Lemma 4.2(ii) the residue equals $e^{(u-v)ax}(1-v)xK_v(H)$. This proves (18). For (19), the residue of the derivative with respect to w of $e^{uxw}\int_0^H e^{(1-u)x\eta}d\eta$ is zero. This derivative equals $ux\,e^{uxw}\int_0^H e^{(1-u)x\eta}d\eta + e^{uxw+(1-u)xH}\,\partial H/\partial w$, and taking residues gives (19). □

All identities in this section are identities between formal power series in x whose coefficients are polynomials in finitely many of the h_j. Consequently, if H_t is the family of Lemma 3.1, each coefficient of $K_u(H_t)$ is absolutely continuous in t, and (14) and (18) give, for almost every $t \in [s,1]$ and all $u \in [0,1]$,

$$\frac{\partial}{\partial t}K_u(H_t) = \int_{\mathbb{T}} \xi^2\Big(e^{ua_t(\xi)x} + x^2\int_0^u (1-v)\,e^{(u-v)a_t(\xi)x}\,K_v(H_t)\,dv\Big)d\mu_t(\xi). \tag{20}$$

This system is linear in the series $K_v(H_t)$. Because of the factor x^2, the coefficient of x^m on the right involves, besides the measures μ_t and the function P, only the coefficients of x^j with $j \le m-2$ of the series $K_v(H_t)$, $0 \le v \le u$.

5 Two explicit solutions

For the series S_τ and F_τ of Section 2.2 and complex τ, put

$$U_u(\tau) = K_u(S_\tau), \qquad V_u(\tau) = K_u(F_\tau) \qquad (0 \le u \le 1).$$

Their coefficients are polynomials in u and τ. By Lemma 4.2(i) and the expansions (7) and (9),

$$[x^0]U_u(\tau) = [x^0]V_u(\tau) = \tau, \qquad [x^{2m}]U_1(\tau) = \frac{C_m\,\tau^{m+1}}{(2m)!}, \qquad [x^{2m}]V_1(\tau) = \frac{2^{-m}C_m\,\tau^{m+1}}{(2m)!}. \tag{21}$$

Lemma 5.1. *Let $0 \le u \le 1$.*

(i) The series $U_u(\tau)$ and $V_u(\tau)$ contain only even powers of x. For real $\tau \ge 0$ the coefficients of $V_u(\tau)$ and of $U_u(\tau) - V_u(\tau)$ are non-negative.

(ii) For all complex τ,

$$\frac{\partial V_u}{\partial \tau}(\tau) = 1 + x^2 \int_0^u (1-v)\, V_v(\tau)\, dv. \tag{22}$$

(iii) Let $t > 0$ and let χ be a random sign with mean $\omega \in [-1, 1]$. Then

$$\mathbf{E}_\chi\Big[e^{u\sqrt{t}\,\chi x} + x^2 \int_0^u (1-v)\, e^{(u-v)\sqrt{t}\,\chi x}\, U_v(t)\, dv\Big] = \frac{\partial U_u}{\partial t}(t) + \frac{\omega\, u\, x}{\sqrt{t}}\, U_u(t). \tag{23}$$

Proof. (i) Since the series S_τ and F_τ are odd, the first statement follows from Lemma 4.2(iii). For $\tau \ge 0$ the coefficients of S_τ dominate those of F_τ, which are non-negative, by (7) and (9). The second statement now follows from Lemma 4.2(iv).

(ii) By the chain rule and (10), $\partial V_u/\partial\tau = \mathrm{d}K_u(F_\tau)[1/(w - F_\tau)]$, and (22) is (18) with $H = F_\tau$ and $a = 0$.

(iii) Write $S = S_t$. By (18) with $H = S$ and $a = \sqrt{t}\,\chi$, the left-hand side of (23) equals $\mathrm{d}K_u(S)[L]$ with $L = \mathbf{E}_\chi[1/(w - S - \sqrt{t}\,\chi)]$. Using $(w-S)^2 - t = (w-S)(w-2S)$, which follows from $S(w-S) = t$, we find

$$L = \frac{w - S + \omega\sqrt{t}}{(w-S)^2 - t} = \frac{1}{w-2S}\Big(1 + \frac{\omega\sqrt{t}}{w-S}\Big) = \frac{1}{w-2S}\Big(1 + \frac{\omega S}{\sqrt{t}}\Big) = \frac{\partial S}{\partial t} - \frac{\omega}{\sqrt{t}}\,\frac{\partial S}{\partial w},$$

by (10). Now $\mathrm{d}K_u(S)[\partial S/\partial t] = \partial U_u/\partial t$ by the chain rule, and $\mathrm{d}K_u(S)[\partial S/\partial w] = -uxU_u(t)$ by (19). □

By (22), the chain rule and (15),

$$\frac{\partial}{\partial t} V_u\big(h_1(t)\big) = \int_{\mathbb{T}} \xi^2 \Big(1 + x^2 \int_0^u (1-v)\, V_v\big(h_1(t)\big)\, dv\Big) d\mu_t(\xi)$$

for almost every t. That is, the series $V_u(h_1(t))$ satisfy (20) with every $a_t(\xi)$ replaced by 0. Thus F_τ with $\tau = h_1(t)$ is the evolution in which the measures μ_t enter only through $h_1'(t) = \int \xi^2\, d\mu_t$. This is why $V_u(h_1(t))$ is subtracted from $K_u(H_t)$ in Proposition 6.1 below, and it is what produces the center in (5). Equation (23) says that the series $U_u(t)$ satisfy (20) with $P \equiv 0$ and with the measure on the circle replaced by a random sign, up to the last term, which will account for the change of the constant term $P(t)$.

6 The comparison

We return to the setting of Lemma 3.1: $g \in \Sigma_0$ omits the closed disk with center w_0 and radius $\rho = \sqrt{s}$, and $Y_t = w + P(t) - H_t$, $h_1(t)$ and μ_t are as in that lemma.

Proposition 6.1. *For all $n \geq 0$, $t \in [s,1]$, $u \in [0,1]$ and $c \in \mathbb{C}$,*

$$\Big|[x^{2n}]\Big(e^{cx}K_u(H_t) - V_u\big(h_1(t)\big)\Big)\Big| \leq [x^{2n}]\Big(e^{|c-uP(t)|\,x}\,U_u(t) - V_u(t)\Big). \tag{24}$$

The exponentials e^{cx} and $e^{|c-uP(t)|x}$ multiply only $K_u(H_t)$ and $U_u(t)$, not the series V_u. On the left, V_u is evaluated at the complex number $h_1(t)$, and on the right at the real number t. For $c = 0$, $u = 1$ and $t = 1$, inequality (24) is equivalent to inequality (5) with $m = n$ for functions that omit a disk. Section 7 gives the details and removes this restriction by a dilation. The exponent $|c - uP(t)|$ on the right is chosen so that its derivative with respect to t produces the last term of (23). The proof is by induction on n, and the additional parameters u and c are needed for it. Indeed, by (26) below, the coefficient of x^{2n} of the derivative with respect to t of $e^{cx}K_u(H_t) - V_u(h_1(t))$ involves, besides an explicit term, only coefficients of x^{2n-2} of series of the same form, with other values $v \leq u$ and $c + (u-v)a_t(\xi)$ of the two parameters. In the induction step, the modulus of each term will be bounded by a convex function of $|Z + \beta\xi|^2$, where $Z = c - uP(t)$ and $\beta \geq 0$, and the following elementary inequality will replace the integral over $\xi \in \mathbb{T}$ by an average over two points.

Lemma 6.2. *Let Φ be a convex function on $[0,\infty)$, let μ be a probability measure on $\mathbb{T}$, let $Z \in \mathbb{C}$ and $r = |Z|$, and put $\omega = \mathrm{Re}\big(\overline{Z}\int \xi\, d\mu(\xi)\big)/r$ if $r > 0$ and $\omega = 0$ if $r = 0$. Then, for every $\beta \geq 0$,*

$$\int_{\mathbb{T}} \Phi\big(|Z + \beta\xi|^2\big)\, d\mu(\xi) \leq \mathbf{E}_\chi\, \Phi\big((r + \beta\chi)^2\big),$$

where χ is a random sign with mean ω.

Proof. Let $\xi \in \mathbb{T}$ and put $\omega_\xi = \mathrm{Re}(\overline{Z}\xi)/r$ if $r > 0$ and $\omega_\xi = 0$ if $r = 0$. Then $|\omega_\xi| \leq 1$ and

$$|Z + \beta\xi|^2 = r^2 + \beta^2 + 2\beta r\omega_\xi = \frac{1+\omega_\xi}{2}(r+\beta)^2 + \frac{1-\omega_\xi}{2}(r-\beta)^2,$$

so convexity gives $\Phi(|Z+\beta\xi|^2) \leq \frac{1+\omega_\xi}{2}\Phi((r+\beta)^2) + \frac{1-\omega_\xi}{2}\Phi((r-\beta)^2)$. Integrating with respect to μ and using $\int \omega_\xi\, d\mu(\xi) = \omega$ proves the lemma. □

Note that the mean ω in Lemma 6.2 does not depend on β or on Φ. Hence one random sign serves for every scale β and every convex function at once. This is what will allow us to treat all terms of the derivative together.

Proof of Proposition 6.1. We use induction on n. For $n = 0$ both sides of (24) vanish, since $[x^0]K_u(H_t) = h_1(t) = [x^0]V_u(h_1(t))$ and $[x^0]U_u(t) = t = [x^0]V_u(t)$ by (21). Let $n \geq 1$ and suppose that (24) holds with $n-1$ in place of n, for all t, u and c. Fix $u \in [0,1]$ and $c \in \mathbb{C}$, write $Z(t) = c - uP(t)$ and $r(t) = |Z(t)|$, and put

$$\lambda(t) = [x^{2n}]\Big(e^{cx}K_u(H_t) - V_u\big(h_1(t)\big)\Big), \qquad \Lambda(t) = [x^{2n}]\Big(e^{r(t)x}\,U_u(t) - V_u(t)\Big).$$

Both functions are absolutely continuous on $[s,1]$. At $t = s$ we have $H_s = 0$ and $h_1(s) = 0$, so $\lambda(s) = 0$, while $\Lambda(s) \geq [x^{2n}](U_u(s) - V_u(s)) \geq 0$ by Lemma 5.1(i). It therefore suffices to prove that

$$|\lambda'(t)| \leq \Lambda'(t) \qquad \text{for almost every } t \in [s,1], \tag{25}$$

for then $|\lambda(t)| \le \int_s^t |\lambda'| \le \Lambda(t) - \Lambda(s) \le \Lambda(t)$.

The derivative of λ. By (20), and by (22) and (15), for almost every t,

$$\lambda'(t) = \int_{\mathbb{T}} \xi^2\, [x^{2n}]\Big(e^{(c+ua)x} - 1 + x^2 \int_0^u (1-v)\Big(e^{(c+(u-v)a)x} K_v(H_t) - V_v\big(h_1(t)\big)\Big)dv\Big) d\mu_t(\xi), \quad (26)$$

where $a = a_t(\xi) = \sqrt{t}\,\xi - P(t)$. After multiplication by x^2, the coefficient of x^{2n} of the inner integral is the integral of the coefficient of x^{2n-2} of the expression $e^{c'x}K_v(H_t) - V_v(h_1(t))$ with $c' = c + (u-v)a$, and the induction hypothesis applies to this coefficient. Since $c' - vP(t) = Z(t) + (u-v)\sqrt{t}\,\xi$ and $c + ua = Z(t) + u\sqrt{t}\,\xi$, we obtain, using $|\xi| = 1$,

$$|\lambda'(t)| \le \int_{\mathbb{T}} \Big(\Psi\big(|Z + u\sqrt{t}\,\xi|^2\big) + \int_0^u (1-v)\,\Phi_v\big(|Z + (u-v)\sqrt{t}\,\xi|^2\big)\,dv\Big) d\mu_t(\xi), \quad (27)$$

where $Z = Z(t)$ and, for $y \ge 0$,

$$\Psi(y^2) = [x^{2n}]\big(e^{yx} - 1\big) = \frac{y^{2n}}{(2n)!}, \qquad \Phi_v(y^2) = [x^{2n-2}]\big(e^{yx}U_v(t) - V_v(t)\big) \quad (0 \le v \le u).$$

Because $U_v(t)$ contains only even powers of x and has non-negative coefficients (Lemma 5.1(i)), $[x^{2n-2}](e^{yx}U_v(t)) = \sum_{k=0}^{n-1} \frac{y^{2k}}{(2k)!}[x^{2n-2-2k}]U_v(t)$ is a polynomial in y^2 with non-negative coefficients. Thus Ψ and all the Φ_v are convex functions of $y^2 \in [0,\infty)$.

Reduction to a random sign. Let $\omega = \omega(t)$ be the number associated with $\mu = \mu_t$ and $Z = Z(t)$ in Lemma 6.2, and let χ be a random sign with mean ω. Exchanging the integrals over v and ξ in (27) and applying that lemma for each fixed v, with $\beta = u\sqrt{t}$ and $\beta = (u-v)\sqrt{t}$, we obtain averages of $\Psi(y^2)$ and $\Phi_v(y^2)$ with $y = r + \beta\chi$. This number may be negative. However, the coefficient of an even power of x in e^{yx} or in $e^{yx}U_v(t)$ is unchanged when y is replaced by $-y$, so the formulas that define Ψ and Φ_v remain valid for it. We get

$$\begin{aligned}
|\lambda'(t)| &\le \mathbf{E}_\chi\,[x^{2n}]\Big(e^{(r+u\sqrt{t}\,\chi)x} - 1 + x^2\int_0^u (1-v)\big(e^{(r+(u-v)\sqrt{t}\,\chi)x}U_v - V_v\big)dv\Big) \\
&= [x^{2n}]\Big(e^{rx}\,\mathbf{E}_\chi\Big[e^{u\sqrt{t}\,\chi x} + x^2\int_0^u (1-v)e^{(u-v)\sqrt{t}\,\chi x}U_v\,dv\Big] - 1 - x^2\int_0^u (1-v)V_v\,dv\Big) \\
&= [x^{2n}]\Big(e^{rx}\Big(\frac{\partial U_u}{\partial t} + \frac{\omega\,u\,x}{\sqrt{t}}U_u\Big) - \frac{\partial V_u}{\partial t}\Big), \qquad (28)
\end{aligned}$$

where $r = r(t)$, the series U_v, V_v and their derivatives are evaluated at t, and the last step uses (23) and (22).

The derivative of Λ. The coefficient $[x^{2n}](e^{rx}U_u(t))$ is a polynomial in r^2 and t, and $r(t)^2 = |c - uP(t)|^2$ is absolutely continuous. By (14),

$$\frac{d}{dt}\,r(t)^2 = -2u\,\mathrm{Re}\big(\overline{Z(t)}\,P'(t)\big) = \frac{2u}{\sqrt{t}}\,\mathrm{Re}\Big(\overline{Z(t)}\int_{\mathbb{T}}\xi\,d\mu_t(\xi)\Big) = \frac{2u\,\omega(t)\,r(t)}{\sqrt{t}}$$

for almost every t. When $r(t) = 0$ both sides vanish. For $r > 0$ the derivative of $[x^{2n}](e^{rx}U_u)$ with respect to r^2 is $\frac{1}{2r}[x^{2n}](xe^{rx}U_u)$, and for $r = 0$ we have $[x^{2n}](xU_u) = 0$, because U_u contains only even powers of x. It follows that

$$\Lambda'(t) = [x^{2n}]\Big(e^{rx}\Big(\frac{\omega\,u\,x}{\sqrt{t}}U_u + \frac{\partial U_u}{\partial t}\Big) - \frac{\partial V_u}{\partial t}\Big)$$

for almost every t, which is the right-hand side of (28). This proves (25) and the proposition. □

7 Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.3. Suppose first that g omits a closed disk, and apply Proposition 6.1 with $t = 1$, $u = 1$, $c = 0$ and $n = m$. Below, H_1 is the series H_t at $t = 1$, while the subscript of K_1, U_1 and V_1 is the parameter $u = 1$. By Lemma 3.1(i) we have $P(1) = 0$, $h_j(1) = -B_j$ and $h_1(1) = b_1$, so Lemma 4.2(i) and (21) give

$$[x^{2m}]K_1(H_1) = -\frac{B_{2m+1}}{(2m)!}, \qquad [x^{2m}]V_1(b_1) = \frac{2^{-m}C_m\, b_1^{m+1}}{(2m)!},$$
$$[x^{2m}]\big(U_1(1) - V_1(1)\big) = \frac{(1-2^{-m})C_m}{(2m)!}.$$

Thus (24) is (5), divided by $(2m)!$.

Now let $g \in \Sigma_0$ be arbitrary. For $R > 1$ the function $g_R(z) = g(Rz)/R$ belongs to Σ_0, and it omits the set $\{g(z)/R : 1 < |z| \le R\}$, which has interior points. Hence it omits a closed disk of radius less than one. Its coefficients are $b_k R^{-k-1}$, and its inverse is $G(Rw)/R$, with coefficients $B_n R^{-n-1}$. Applying (5) to g_R and multiplying by R^{2m+2}, we obtain

$$\big|B_{2m+1} + 2^{-m}C_m\, b_1^{m+1}\big| \le \big(1 - 2^{-m}\big)C_m\, R^{2m+2},$$

and (5) follows as $R \to 1$. The last statement of the theorem follows from the triangle inequality. □

Proof of Theorem 1.2. By Theorem 1.3 and the inequality $|b_1| \le 1$ of (6),

$$|B_{2m+1}| \le C_m - 2^{-m}C_m\big(1 - |b_1|^{m+1}\big) \le C_m.$$

The inequality $|b_1| \le 1$ also follows without the area theorem. If g omits a closed disk, then integrating (15) over $[s, 1]$ gives $|b_1| = |h_1(1)| \le 1 - s$, because the μ_t are probability measures. Applied to the functions g_R of the preceding proof, this gives $|b_1| \le R^2$ for every $R > 1$. If $|B_{2m+1}| = C_m$ for some m, then $|b_1| = 1$, and (6) gives $g(z) = z + b_1/z$. Conversely, these functions satisfy $|B_{2m+1}| = C_m$ for every m by (8). □

Remark 7.1. The proof gives more for functions that omit a disk. In the proof of Proposition 6.1 we discarded the initial value $\Lambda(s) \ge 0$. Keeping it, with $u = 1$, $c = 0$ and $r(s) = |P(s)| = |w_0|$, we find that every $g \in \Sigma_0$ that omits the closed disk with center w_0 and radius $\rho < 1$ satisfies

$$\big|B_{2m+1} + 2^{-m}C_m\, b_1^{m+1}\big| \le \big(1 - 2^{-m}\big)C_m - \Big(\sum_{k=0}^{m}\binom{2m}{2k}C_{m-k}\,|w_0|^{2k}\rho^{2(m-k+1)} - 2^{-m}C_m\,\rho^{2m+2}\Big).$$

For $w_0 = 0$, sharp bounds of $|B_{2m+1}|$ in terms of ρ are due to Tammi [33] for $m \le 2$ and, on a range of ρ, for $m = 3$ and $m = 4$ (see question (3) in Section 9).

8 Even orders

In this section we prove Theorem 1.4. The proof is elementary and does not use the earlier results for the even orders with $q \ge 5$ [3, 16] that are described in Section 1.3. Recall that nB_n is the coefficient of z^{-1} in $-g(z)^n$.

Proof of Theorem 1.4(a). Let $0 < \kappa < 1$ and $g(z) = z - (1-\kappa^2)/(z-\kappa)$. For $\kappa = \frac{1}{2}$ this is the function in the statement. If $g(z) = g(z')$ with $z \neq z'$, then $(z-\kappa)(z'-\kappa) = -(1-\kappa^2)$, that is, $z' = (\kappa z - 1)/(z-\kappa)$. Since $|z-\kappa|^2 - |\kappa z - 1|^2 = (1-\kappa^2)(|z|^2 - 1)$, the point z' lies in $\mathbb{D}$ whenever $z \in \Delta$, and in $\mathbb{T}$ whenever $z \in \mathbb{T}$. Hence g is univalent in Δ, and $g(\Delta)$ does not meet $g(\mathbb{T})$. Conversely, let $w \in \mathbb{C} \setminus g(\mathbb{T})$. The equation $g(z) = w$ reads $(z-\kappa)^2 + (\kappa - w)(z-\kappa) = 1-\kappa^2$. The product of the values of $z-\kappa$ at its two roots is $-(1-\kappa^2) \neq 0$, so neither root is the pole κ, and the roots are a point z and the point $z' = (\kappa z - 1)/(z-\kappa)$. Neither root lies on $\mathbb{T}$, so by the identity above exactly one of them lies in Δ. Thus $g(\Delta) = \mathbb{C} \setminus g(\mathbb{T})$. The expansion of g is $g(z) = z - (1-\kappa^2)\sum_{k\geq 1} \kappa^{k-1} z^{-k}$, so $g \in \Sigma_0$ with $b_k = -(1-\kappa^2)\kappa^{k-1}$. The two critical points $z = \kappa \pm i\sqrt{1-\kappa^2}$ of g lie on $\mathbb{T}$. Since $g(z) - (1-\kappa^2)/\kappa = z(\kappa z - 1)/(\kappa(z-\kappa))$ has modulus $1/\kappa$ on $\mathbb{T}$, the function g maps $\mathbb{T}$ onto an arc of the circle with center $(1-\kappa^2)/\kappa$ and radius $1/\kappa$, covering each endpoint once and every other point of the arc twice (Figure 2(d)). From $B_4 = -(b_4 + 3b_1 b_2)$ we obtain

$$-B_4 = -(1-\kappa^2)\kappa^3 + 3(1-\kappa^2)^2\kappa = \kappa(1-\kappa^2)(3-4\kappa^2), \tag{29}$$

which equals $\frac{3}{4}$ for $\kappa = \frac{1}{2}$. On the other hand, for $n = 4$ we have $q = 5$ and $k = 1$, so $\sigma_4 = \frac{1}{4}\binom{8/5}{1} = \frac{2}{5}$. □

Remark 8.1. The value $\frac{3}{4}$ is not the largest in this family. The right-hand side of (29) is largest for $\kappa^2 = (21 - \sqrt{201})/40$, that is, $\kappa = 0.4129\ldots$, where it equals $0.7939\ldots$. We do not know whether this is the maximum of $|B_4|$ over Σ_0.

Proof of Theorem 1.4(b). Let $m \geq 3$ and $g(z) = z + \beta z^{-1} + \gamma z^{-2}$ with $\beta = 1 - \frac{1}{m}$ and $\gamma = \frac{1}{2m}$. For distinct $z, z' \in \Delta$,

$$\frac{g(z) - g(z')}{z - z'} = 1 - \frac{\beta}{zz'} - \frac{\gamma(z+z')}{z^2 z'^2},$$

and the two subtracted terms have moduli less than β and 2γ. Since $\beta + 2\gamma = 1$, the quotient does not vanish, and $g \in \Sigma_0$. Since all coefficients of g are non-negative, every term in the expansion of $g(z)^{2m}$ has a non-negative coefficient. The terms that consist of m factors z, $m-1$ factors βz^{-1} and one factor γz^{-2} contribute $\frac{(2m)!}{m!\,(m-1)!}\beta^{m-1}\gamma\, z^{-1} = \binom{2m-1}{m}\beta^{m-1} z^{-1}$. Hence

$$2m\,|B_{2m}| \;\geq\; \binom{2m-1}{m}\Big(1 - \frac{1}{m}\Big)^{m-1} = \frac{1}{2}\binom{2m}{m}\Big(1 - \frac{1}{m}\Big)^{m-1} > \frac{1}{6}\binom{2m}{m}, \tag{30}$$

because $(1 - \frac{1}{m})^{-(m-1)} = (1 + \frac{1}{m-1})^{m-1} < e < 3$. This is the first inequality in (b).

To compare with σ_{2m}, let $q \geq 3$ be the least prime divisor of $2m+1$ and $k = (2m+1)/q$. Then $4m/q = 2k - 2/q$ lies between $k-1$ and $2k$, and $\binom{x}{k}$ increases with x for $x \geq k-1$, so

$$2m\,\sigma_{2m} = \binom{2k - 2/q}{k} < \binom{2k}{k}.$$

Suppose first that $m \neq 4$. Then $m - k \geq 2$. Indeed, for $m = 3$ we have $q = 7$ and $k = 1$, and for $m \geq 5$ we have $m - k \geq m - \frac{2m+1}{3} = \frac{m-1}{3} > 1$, and $m - k$ is an integer. Since $\binom{2l}{l}/\binom{2l-2}{l-1} = 4 - \frac{2}{l} \geq 3$ for $l \geq 2$, applying this for $l = k+1, \ldots, m$ we get $\binom{2m}{m} \geq 9\binom{2k}{k}$, and (30) gives $2m|B_{2m}| > \frac{3}{2}\binom{2k}{k} > 2m\,\sigma_{2m}$. For $m = 4$ we have $q = 3$ and $k = 3$, so $\sigma_8 = \frac{1}{8}\binom{16/3}{3} = \frac{130}{81}$, while (30) gives $|B_8| \geq \frac{35}{8}\big(\frac{3}{4}\big)^3 = \frac{945}{512} > \frac{130}{81}$. □

9 Discussion

Springer's conjecture thus holds at every odd order and, by the classical theorem of Schiffer and Golusin, at order two. It fails at every even order $n \geq 4$: by [3, 16] when $n+1$ is not divisible by 3, and by the explicit functions of Theorem 1.4 in every case. At odd orders the inverse coefficient B_{2m+1} lies in the disk (5), which touches the circle $|B_{2m+1}| = C_m$ only when $|b_1| = 1$.

The bound $|B_{2m+1}| \leq C_m - 2^{-m} C_m (1 - |b_1|^{m+1})$ that follows from Theorem 1.3 is weaker than bounds in terms of $|b_1|$ that are contained in earlier proofs. Schober's first paper [27, pp. 114–115] has $|B_5| \leq 1 + |b_1|^2$, $|B_7| \leq \frac{3}{2} + \frac{7}{2}|b_1|^2$, $|B_9| \leq 7 + 7|b_1|^2$ and similar bounds for B_{11} and B_{13}. Schober's second paper [28, pp. 414–418] has, in the full class, $|B_9| \leq 4 + 9|b_1|^2 + |b_1|^4$ and, from its inequality (9) with $k = 1$, $|B_{15}| \leq \frac{2467}{6} + \frac{107}{6}|b_1|^2$, which for $|b_1| < 1$ is smaller than the bound $|B_{15}| \leq 415 + 14|b_1|^2$ of Ren [24, Chinese original, p. 195]. Each of these bounds is smaller than the bound for $|B_{2m+1}|$ obtained from (5), for every $|b_1| < 1$, and by the triangle inequality they imply (5) for these m when $|b_1|$ is below a threshold depending on m (for $m = 2$, when $|b_1| \leq (\sqrt{5} - 1)/2$). Bounds of this kind also appear in, or follow from, the four notes [43, 36, 40, 46] on the orders 23, 25, 29 and 31. At the orders 23 and 31 the bound obtained from (5) is the smaller one for every $|b_1| < 1$. At the order 29 the earlier bound is the smaller one for every $|b_1| < 1$, and at the order 25 it is the smaller one when $|b_1| \leq 0.72$ and the larger one when $0.73 \leq |b_1| < 1$. For $m \geq 17$ we know of no earlier bound in terms of b_1 in the full class. The paper [7] treats $m = 17$ when $b_2 = 0$.

Theorem 1.2 answers the part of Problem 6.4 of [10] that concerns the inverse coefficients of odd order. Problem 6.6 of [10] asks for the orders of magnitude of the inverse coefficients. At the odd orders the maxima are the numbers in Springer's formula [32, (95)], and with $N = m + 1$ we have

$$\max_{g \in \Sigma_0} |B_{2N-1}| = \frac{(2N-2)!}{N!\,(N-1)!} \sim \frac{2^{2N-2}}{\sqrt{\pi N^3}} \qquad (N \to \infty), \tag{31}$$

because $(2N-2)!/(N!\,(N-1)!) = \frac{1}{N}\binom{2N-2}{N-1}$ and $\binom{2k}{k} \sim 4^k/\sqrt{\pi k}$. Springer's formula (96), obtained by applying Stirling's formula to the factorials in (95), contains an extraneous factor e on the right-hand side. For $N = 10$, for example, the number in (95) is 4862, the asymptotic expression in (31) is about 4677, whereas Springer's formula (96) gives about 12713. Problem 6.6 of [10] states a lower bound with the right-hand side of (96): it says that $|B_{2N-1}| > (1 - \varepsilon)\, 2^{2N-2} e/\sqrt{\pi N^3}$ is possible for all sufficiently large N. With the factor e removed, the statement follows from $g(z) = z + 1/z$ and (31), and Theorem 1.2 shows that the constant cannot be improved.

For even n, the order of magnitude of $\max_{g \in \Sigma_0} |B_n|$ is, to our knowledge, not known. Theorem 1.4(b) gives a lower bound of the order $2^n n^{-3/2}$. The only upper bound known to us is Springer's estimate $|B_n| \leq 2^n/n$ [32, (94)], which holds for every n and is larger by a factor of the order $\sqrt{n}$. Baernstein and Schober [1] deduced Loewner's bounds from estimates of integral means and obtained, by the same method, sharp bounds for the inverse coefficients of meromorphic univalent functions in the unit disk with a pole at a prescribed point. For the class Σ_0 they give no bound and describe Springer's problem as one that "remains open" [1, p. 78].

We end with three questions.

(1) *Even orders.* What is the maximum of $|B_n|$ over Σ_0 for even $n \geq 4$, and which functions attain it? Theorem 1.4 and the results of [3, 16] show only that the maximum exceeds σ_n. Already for $n = 4$ we do not know the maximum. The circular arc maps give $|B_4| = \kappa(1 - \kappa^2)\,|3 - 4\kappa^2|$, which is about 0.794 for $\kappa \approx 0.413$ (Remark 8.1).

(2) *The exact region at odd orders.* What is the exact range of B_{2m+1} over the functions $g \in \Sigma_0$ with a prescribed first coefficient b_1? Theorem 1.3 places this range in a disk with center

$-2^{-m}C_m b_1^{m+1}$ and radius $(1-2^{-m})C_m$, which is not the exact range in general. For example, for $m=2$ and $b_1=0$ we have $B_5=-(b_5+2b_2^2)$, and the Grunsky inequality $|b_5+b_2^2|\le\frac{1}{3}$ together with $|b_2|\le\frac{2}{3}$ gives $|B_5|\le\frac{7}{9}$, whereas (5) gives $\frac{3}{2}$ and Schober's bound $1+|b_1|^2$ [27, p. 114] gives 1. The bound $\frac{7}{9}$ is attained by the three-slit map g_3, and the functions $e^{i\alpha}g_3(Re^{-i\alpha}z)/R$ with $R\ge1$ and α real, together with the identity map, show that the exact range is the closed disk of radius $\frac{7}{9}$.

(3) *Omitted disks and quasiconformal extensions.* For functions $g\in\Sigma_0$ that omit the closed disk with center w_0 and radius ρ, Remark 7.1 improves (5). For $w_0=0$ the sharp bounds of the first odd inverse coefficients are known. Let g_ρ be the map defined by $g_\rho+\rho^2/g_\rho=z+1/z$, which maps Δ onto the exterior of the disk $|w|\le\rho$ minus two radial slits. Tammi [33] proved that, among the functions in Σ_0 that omit this disk, g_ρ maximizes $|B_1|$, $|B_3|$ and $|B_5|$ for every ρ, $|B_7|$ at least for $\rho\le0.8569$, and $|B_9|$ at least for $\rho\le0.7689$. For g_ρ one finds $|B_3|=1-\rho^2$ and $|B_5|=(1-\rho^2)(2-\rho^2)$. For g_ρ, equality holds in Remark 7.1 when $m=1$ but not when $m=2$; at $m=2$ the sharp bound of $|B_5|$ is Tammi's. Tammi also showed that $|B_2|$ is maximized by the corresponding map with three slits. What are the sharp bounds for the remaining orders and radii, for $w_0\neq0$, and in terms of b_1? In a different normalization, where the omitted disk is centered at the origin and the constant term of the function is left free, Siejka and Tammi [31] determined the sharp bounds for all inverse coefficients, extending the theorem of Bazilevich [2, p. 692] and Netanyahu [20] on non-vanishing functions (see also Siejka [29, 30]). Since a translation changes the inverse coefficients, their bounds do not apply directly to the class Σ_0. Schober [28] obtained bounds for $B_3,\dots,B_{15}$ in the subclasses of functions with a k-quasiconformal extension, where $0<k<1$ (the case $k=1$ is the full class). Does the method of this paper yield sharp bounds in these subclasses for every odd order?

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