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A PROOF OF THE OPEN ORBIT CONJECTURE

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Abstract. We prove the open orbit conjecture of Geiss, Leclerc and Schröer for every symmetric Cartan datum: the dual semicanonical and specialized dual canonical elements of an open-orbit component coincide with the same crystal label. We deduce the conormal-open conjecture for p-adic GL_N Vogan varieties and the reality of the simple symmetric quiver Hecke modules attached to rigid preprojective modules.

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1. Introduction

Let Q be a finite quiver without oriented cycles, with vertex set Q_0, and let Π be its preprojective algebra over $\mathbb{C}$. Multiple arrows are allowed. The underlying graph defines a symmetric generalized Cartan matrix and a Kac–Moody algebra. Write $\mathfrak{n}^-$ for its negative nilpotent part and $\mathcal{A}$ for the

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graded dual of $U(\mathfrak{n}^-)$. We compare its dual canonical and dual semicanonical bases on components containing an open orbit.

Lusztig introduced canonical bases of quantized enveloping algebras in [Lus90]. His subsequent construction by perverse sheaves on quiver representation spaces applies to symmetric Cartan matrices [Lus91]. Kashiwara developed the algebraic theory of crystal bases [Kas], and Kashiwara and Saito realized its crystal operators on irreducible components of nilpotent preprojective varieties [KS]. Lusztig's semicanonical basis uses constructible functions on those varieties and Euler-characteristic convolution [Lus00]. Thus intersection cohomology and constructible-function convolution lead to bases with a common geometric indexing set.

For a nonnegative integral dimension vector $\mathbf{d} = (d_i)_{i \in Q_0}$, let $\Lambda_{\mathbf{d}}$ be the variety of nilpotent Π-representations and put $G_{\mathbf{d}} = \prod_{i \in Q_0} \mathrm{GL}_{d_i}(\mathbb{C})$. Nilpotency means that all sufficiently long paths act by zero. The common crystal indexing set is

$$\coprod_{\mathbf{d}} \operatorname{Irr} \Lambda_{\mathbf{d}},$$

where Irr denotes irreducible components. For a component Z, write b_Z for the dual canonical element specialized at $v = 1$, and ρ_Z for the dual semicanonical element with the same label. The transition matrix has diagonal entries 1 by the comparison of bases of canonical type [Bau]. The two bases nevertheless differ in general. Among connected finite ADE types, Geiss, Leclerc and Schröer proved that they coincide precisely in types A_n with $n \leq 4$ [GLS05]. Kashiwara and Saito constructed a type-A example of reducible singular support [KS]. Leclerc exhibited imaginary dual canonical elements in types A_5 and D_4 whose squares contain several basis terms [Lec].

For finite dimensional nilpotent modules X, Y, evaluation of constructible functions defines an element $\varphi_X \in \mathcal{A}$. Its values on Chevalley words are Euler characteristics of varieties of composition-series flags. These evaluations satisfy $\varphi_{X \oplus Y} = \varphi_X \varphi_Y$, and their generic values on components give the dual semicanonical basis. The preprojective interpretation and the multiplication formulas were developed in [GLS05, GLS07]. A module is *rigid* if $\operatorname{Ext}^1_{\Pi}(X, X) = 0$. The preprojective Hom–Ext formula identifies rigid modules with open $G_{\mathbf{d}}$-orbits in irreducible components [GLS11]. If $X \in \Lambda_{\mathbf{d}}$ has an orbit open in Z, then $\rho_Z = \varphi_X$.

Cluster algebras provide a large family of such components. Fomin and Zelevinsky introduced them as an algebraic framework for canonical bases and total positivity [FZ]. Geiss, Leclerc and Schröer related mutation of rigid preprojective modules to cluster structures in finite ADE type [GLS06]. Buan, Iyama, Reiten and Scott constructed Frobenius categories attached to finite reduced Weyl group words, with stable categories of Calabi–Yau dimension two [BIRSc]. The subsequent unipotent coordinate-ring constructions of [GLS08, GLS11] place the associated cluster monomials in the dual semicanonical basis. A rigid object obtained as a sum of summands of a cluster-tilting object reached from a standard word object by successive mutations is called reachable; its character is a cluster monomial.

The open orbit conjecture of Geiss, Leclerc and Schröer asserts $\rho_Z = b_Z$ for every component containing an open orbit. It appears as Conjecture 23.1 in the version of [GLS08] dated April 5, 2008, and as Conjecture 18.1 in [GLS11]. Their quantum cluster structures [GLSQ] connected the reachable case to dual canonical membership of quantum cluster monomials. Kang, Kashiwara, Kim and Oh proved that membership for symmetric Cartan data by monoidal categorification [KKKO]. Qin proved its symmetrizable version through common triangular bases [Qin2], and McNamara gave an independent folding proof [McN]. Together with the semicanonical cluster-monomial theorem, these results establish the open orbit assertion for reachable rigid objects.

The finite-window statements of Hernandez Juan, posted on November 20, 2026, also imply the finite ADE case [HJ, Theorems 1.2–1.3]. To describe the connection, let c be a bipartite Coxeter element, h its Coxeter number, and w_0 the longest Weyl group element. For even $h > 2$, the reduced expression $w_0 = c^{h/2}$ [BKW, §2.2] is the word of a Hernandez–Leclerc window of level

$h/2-1$. The word and window Jacobian presentations agree [BIRS, DS]. The correspondence with two-term presilting complexes [AIR, Theorem 4.7] transfers a rigid object J to an object J' in the window category with the same presentation. Write T' for the standard window object. Applied to $J'[1]$, [HJ] has degree $\operatorname{ind}_{T'} J'$ and injective kernel $\operatorname{Hom}(T', J'[1])$, matching the index and Grassmannian module of J. The character formulas of [GLSCh, HJ] therefore differ only by a frozen monomial. The common triangular basis identifies simple characters with upper basis elements [Qin3, Remark A.11 and Theorem A.12], and multiplication by frozen minors clears negative frozen powers [Kim, Theorems 6.18–6.19]. The remaining even ranks of type A follow by extending modules by zero to the next odd rank; A_1 embeds in A_3 in the same way. The argument developed below treats every symmetric Cartan datum, including affine and indefinite type.

Theorem 1.1 (Open orbit conjecture)**.** *Let $Z \in \operatorname{Irr} \Lambda_{\mathbf{d}}$. If Z contains an open $G_{\mathbf{d}}$-orbit, then*

$$\rho_Z = b_Z.$$

Equivalently, the evaluation function of every finite dimensional nilpotent rigid Π-module is the specialized dual canonical basis element with the same crystal label.

In type A, the components have a conormal interpretation. Let E be a quiver representation space. For an orbit $S \subset E$, let $T^*_S E$ be its conormal bundle and set $Z_S = \overline{T^*_S E}$. For an orbit D, let IC_D be the perverse middle extension of the constant sheaf $\mathbb{Q}_D[\dim D]$ to E. Write $[Z_S]\,\mathrm{CC}(\mathrm{IC}_D)$ for the multiplicity of Z_S in its characteristic cycle. Deng and Xu studied the comparison between these multiplicities and canonical–semicanonical transition coefficients, proved the full comparison in type A_2, and established a local vanishing-cycle formula in arbitrary type A [DX]. An Adams–Barbasch–Vogan (ABV) packet is defined by the nonzero microlocal contributions of the intersection-cohomology complexes attached to representations; the p-adic construction is developed in [CFMMX]. We obtain the following consequence on components with an open orbit.

Theorem 1.2 (Conormal-open conjecture)**.** *Let E be a representation space of an equioriented type A quiver, or a finite product of such spaces. If Z_S contains an open base change orbit, then, for every orbit D,*

$$[Z_S]\,\mathrm{CC}(\mathrm{IC}_D) = \delta_{S,D},$$

where $\delta_{S,D}$ is 1 for $S = D$ and 0 otherwise. For every Vogan variety of p-adic GL_N, $N \geq 1$, and every conormal component containing an open centralizer orbit, the same characteristic-cycle assertion holds. The corresponding ABV packet is a singleton.

Theorem 1.2 proves the conormal-open assertion formulated as Conjecture 2.4.7 in Riddlesden's thesis [Rid1]. For parameters of Arthur type, equality of Arthur and ABV packets for GL_N was established by Cunningham and Ray, beginning with irreducible parameters [CR1, CR2], and by the combinatorial arguments of Riddlesden and Lo [Rid2, Lo]. Arthur packets for this group are singletons. Beyond Arthur type, Cunningham, Fiori and Kitt constructed a packet for GL_{16} containing two representations [CFK]. Theorem 1.2 fixes one conormal component with an open orbit and determines its multiplicity in every orbit intersection-cohomology complex.

For nilpotent modules X, Y with rigid direct sum, the upper global basis elements attached to X and Y multiply to one upper global basis element, up to a power of v. In the symmetric quiver Hecke categorification this product is a simple convolution. In particular, the simple module attached to a rigid X is *real*: its convolution square is simple. The type-A affine Hecke interpretation follows from Leclerc, Nazarov and Thibon [LNT]. Lapid and Mínguez [LM] prove the equivalence between rigidity and square-irreducibility for multiplicity-free multisegments, together with socle formulas. Section 8 derives the rigid-product assertions from Theorem 1.1 and nonnegative graded composition multiplicities.

The proof begins with the geometry of an open orbit. A grading in which forward arrows have degree zero and reverse arrows have degree one gives a weight-one action on each flag incidence

variety. Existence of this grading follows from [AO, Corollary 4.5]. We prove a flag-variety version of the weight-one Euler calculation in [HJ, Lemma 6.4], identifying flag evaluations with linear vanishing cycles. Word-complex parity and perverse concentration express the absolute values of the transition coefficients as local-system ranks. Hyperbolic restriction and Thom–Sebastiani then prove that these absolute coefficient rows multiply whenever the total direct sum is rigid.

In a finite word category, a cluster-tilting completion of a rigid object defines a Hall stability chamber. The Hall action sends the characters of its summands to monomials. On the quantum side, the common triangular basis and the theta-function results of Davison and Mandel [DM] give Laurent coefficients invariant under $v \mapsto v^{-1}$ in a generic chamber. We adapt the distant-direction argument of [HJ, §§3.3–3.5] to the full coefficient lattice of the word category: in any fixed finite box, each unwanted coefficient eventually has strictly negative highest degree in v. Laurentness and invariance force it to vanish. Comparing large direct sums inside the rigid completion, absolute-row multiplicativity and positivity eliminate every additional upper basis term.

Finally, an arbitrary nilpotent rigid module embeds in a sufficiently large extremal submodule of a locally nilpotent injective hull. Its quotient is rigid and belongs to a finite word category. Two extremal matrix coefficients preserve the entire relevant upper basis expansion, transferring the finite-word result to the original module. This construction is given in Section 4; the geometric identities occupy Section 3, and the main theorems are concluded in Section 8.

2. Conventions and two elementary reductions

2.1. **Preprojective varieties and evaluation functions.** We put $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the Kac–Moody algebra of the symmetric Cartan matrix

$$c_{ii} = 2, \qquad c_{ij} = -\#\{\text{edges between } i \text{ and } j\} \quad (i \neq j).$$

We fix a realization with simple roots α_i, simple coroots $\alpha_i^\vee$, fundamental weights ω_i, and Chevalley generators e_i, f_i. A dimension vector $\mathbf{d}$ is identified with $\sum_i d_i \alpha_i$. The enveloping algebra is taken over $\mathbb{Q}$, and

$$\mathcal{A} = \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} U(\mathfrak{n}^-)^*_{-\mathbf{d}}$$

is its graded dual. We extend scalars to $\mathbb{C}$ when interpreting this algebra as a coordinate algebra. Let q be the quantum-group parameter, and fix v with $q = v^2$. Quantum coordinate algebras are extended from $\mathbb{Q}(q)$ to $\mathbb{Q}(v)$. Classical specialization means $q = v = 1$ in the upper-basis integral graded-dual form. Its classical basis elements are denoted b_c, where c is the common crystal label, or b_Z when the corresponding component is specified.

Let $\overline{Q}$ be the double of Q, and write a^* for the reverse of $a \in Q_1$. We use the relation

$$\Pi = \mathbb{C}\overline{Q} / \Big(\sum_{a \in Q_1} (aa^* - a^* a) \Big).$$

The vertex components of the relation are understood. Let S_i be the one-dimensional simple module supported at vertex i. A module is nilpotent if every sufficiently long path acts by zero, and rigid if $\operatorname{Ext}^1_\Pi(X, X) = 0$. For a fixed graded vector space of dimension $\mathbf{d}$, set

$$E_{\mathbf{d}} = \bigoplus_{a : i \to j} \operatorname{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}), \qquad G_{\mathbf{d}} = \prod_i \mathrm{GL}_{d_i}(\mathbb{C}).$$

The arrowwise trace pairing identifies reverse arrows with $E_{\mathbf{d}}^*$. Thus a point of $\Lambda_{\mathbf{d}}$ is written $X = (x, y) \in T^* E_{\mathbf{d}}$, satisfying the moment relation and nilpotency. All irreducible components of $\Lambda_{\mathbf{d}}$ have dimension $\dim E_{\mathbf{d}}$.

For a color word $I = (i_1, \dots, i_h)$ of content $\mathbf{d}$, a flag of X means

$$X = X_0 \supset X_1 \supset \cdots \supset X_h = 0, \qquad X_{k-1}/X_k \simeq S_{i_k}.$$

Our top-first convention sets

$$\varphi_X(f_{i_1}\cdots f_{i_h}) = \chi\{\text{such } \Pi\text{-stable flags of } X\}.$$

Euler characteristics and Euler integrals are taken with compact supports. The varieties of full flags here are proper, so either Euler convention gives the same result. The Chevalley words span every homogeneous piece and hence determine φ_X. The usual construction of the dual semicanonical basis takes the generic value of this evaluation on a component: this is ρ_Z. On an open orbit it is the actual evaluation φ_X, and we write ρ_X for it.

The positive-half, sub-first convention of [GLS11] is obtained by the Chevalley identification followed by the word-reversing anti-involution. Both bases, their crystal labels, and their evaluation functions are transported by this map; see [KO, Proposition 3.19 and Remark 7.16].

Lemma 2.1. *A nilpotent module X has an orbit open in an irreducible component of $\Lambda_{\mathbf{d}}$ if and only if $\operatorname{Ext}^1_\Pi(X,X) = 0$.*

Proof. Put

$$q_Q(\mathbf{d}) = \sum_i d_i^2 - \sum_{a:i\to j} d_i d_j = \dim G_{\mathbf{d}} - \dim E_{\mathbf{d}}.$$

The preprojective Hom–Ext formula gives

$$\dim \operatorname{Ext}^1_\Pi(X,X) = 2\big(\dim \operatorname{End}_\Pi(X) - q_Q(\mathbf{d})\big);$$

see [GLS11, Lemma 2.1]. Since the automorphism group is open in the endomorphism space,

$$\dim G_{\mathbf{d}}X = \dim G_{\mathbf{d}} - \dim \operatorname{End}_\Pi(X).$$

The displayed dimensions are equal to $\dim E_{\mathbf{d}}$ exactly when the Ext group vanishes. An orbit of that dimension is open in its closure, which is an irreducible component of the equidimensional variety $\Lambda_{\mathbf{d}}$. Conversely, an orbit open in a component has that dimension. □

Lemma 2.2 (The common-label coefficient)**.** *In the expansion of ρ_Z in the specialized upper global basis, the coefficient of b_Z is* 1.

Proof. The canonical and semicanonical bases are bases of canonical type, with the same crystal parametrization. Their transition matrix is unitriangular for the string order, by [Bau, §§2.3–2.5, Proposition 2.6(ii)]. Each homogeneous piece is finite dimensional. Choose a linear extension of the string order on its labels. The inverse transpose matrix on the dual piece is unitriangular in the reverse order and has diagonal entries 1. This gives the assertion. □

2.2. **Direct sums and positivity.** Evaluation characters satisfy

$$\varphi_{X\oplus Y} = \varphi_X\varphi_Y. \tag{2.1}$$

Scale the summand Y and apply Euler localization to the flag variety of $X\oplus Y$. A fixed flag is the direct sum of a flag in X and a flag in Y, with their colors interleaved. The sum over interleavings is the coproduct of a Chevalley word, since $\Delta(f_i) = f_i\otimes 1 + 1\otimes f_i$. Its Euler characteristic is therefore $(\varphi_X\varphi_Y)(f_{i_1}\cdots f_{i_h})$. The spanning property of Chevalley words proves (2.1) for every X, Y.

The structure constants of the specialized upper global basis are nonnegative integers: they are the sums of the multiplicities in Lusztig's semisimple restriction functor [Lus91, Propositions 4.2(a), 10.23]. Section 3 proves the corresponding product identity for the absolute values of an open-orbit transition row.

3. Open orbits and vanishing cycles

Fix a graded vector space $D = \bigoplus_{i \in Q_0} D_i$ of dimension vector $\mathbf{d}$. Put

$$E = \operatorname{Rep}(Q, \mathbf{d}) = \bigoplus_{a : i \to j} \operatorname{Hom}(D_i, D_j), \qquad G = \prod_i \operatorname{GL}(D_i).$$

The trace pairing identifies E^* with the space of representations of the opposite quiver. Thus a point of the nilpotent variety $\Lambda_{\mathbf{d}}$ is written $X = (x, y) \in E \times E^*$, and its moment-map equation is

$$\langle [\xi, x], y \rangle = 0 \qquad (\xi \in \operatorname{Lie} G). \tag{3.1}$$

An open orbit means an orbit open in a component of $\Lambda_{\mathbf{d}}$. By Lemma 2.1, its module is rigid and its evaluation function is ρ_X.

Write $D^b_c(E)$ for the bounded derived category of rational constructible complexes on E. We use the normalization $\phi_f = R\Phi_f[-1]$. In particular, ϕ_f preserves perversity and ϕ_0 is the identity. Euler integration is taken with compact supports. For a bounded complex K with finite-dimensional cohomology, put

$$\chi(K) = \sum_j (-1)^j \dim H^j(K).$$

3.1. **The word identity.** Let $I = f_{i_1} \cdots f_{i_N}$ be a Chevalley word of content $\mathbf{d}$. In accordance with the top-first convention, let $\mathcal{F}_I$ be the variety of graded flags

$$D = F^0 \supset F^1 \supset \cdots \supset F^N = 0, \qquad \dim(F^{k-1}/F^k) = \alpha_{i_k}.$$

Define

$$\widetilde{\mathcal{F}}_I = \{(z, F) \in E \times \mathcal{F}_I : z_a(F^k_i) \subseteq F^k_j \text{ for } a : i \to j \text{ and all } k\}, \qquad \pi_I(z, F) = z.$$

The projection to $\mathcal{F}_I$ is a vector bundle, so that $\widetilde{\mathcal{F}}_I$ is smooth. The map π_I is proper. Let $\mathcal{F}_I(X)$ denote the closed subvariety of flags preserved by both x and y. By the definition of module evaluation,

$$\rho_X(I) = \chi(\mathcal{F}_I(X)). \tag{3.2}$$

The following grading is a special case of rigid-module gradability [AO, Corollaries 4.4–4.5]. We give its construction for the preprojective grading $\deg a = 0$, $\deg a^* = 1$.

Lemma 3.1. *If $X = (x, y)$ has an open orbit, there is a cocharacter $\lambda : \mathbb{C}^* \to G$ such that*

$$\lambda(t)x = x, \qquad \lambda(t)y = ty.$$

In particular, $f_y(x) = 0$, where $f_y(z) = \langle z, y \rangle$.

Proof. The action $(x, y) \mapsto (x, ty)$ preserves $\Lambda_{\mathbf{d}}$. Since $\mathbb{C}^*$ is connected, it preserves every irreducible component and hence the unique open orbit in the component containing X. The algebraic group

$$H = \{(g, t) \in G \times \mathbb{C}^* : gx = x,\ gy = ty\}$$

therefore maps onto $\mathbb{C}^*$. Its identity component also maps onto $\mathbb{C}^*$: otherwise the image of H would be a finite union of translates of a proper connected algebraic subgroup of $\mathbb{C}^*$. The resulting nontrivial character of H^0 is nontrivial on a maximal torus. Choosing a cocharacter of that torus and, if necessary, reversing it, we obtain a cocharacter γ of G such that $\gamma(t)x = x$ and $\gamma(t)y = t^m y$ for some integer $m > 0$.

Write $D_i = \bigoplus_{k \in \mathbb{Z}} D_i(k)$ for the γ-weight decomposition. Every x-arrow preserves the weight k, whereas every y-arrow sends weight k to weight $k + m$. Give $D_i(k)$ the new weight $\lfloor k/m \rfloor$. This defines an integral cocharacter λ with the required properties, because

$$\left\lfloor \frac{k+m}{m} \right\rfloor = \left\lfloor \frac{k}{m} \right\rfloor + 1 \qquad (k \in \mathbb{Z}).$$

Here the floor is used also for negative weights. Finally, invariance of the pairing gives $\langle x, y\rangle = \langle \lambda(t)x, \lambda(t)y\rangle = t\langle x, y\rangle$. Taking $t \neq 1$ gives $f_y(x) = 0$. □

Lemma 3.2. *If $X = (x, y)$ has an open orbit, then, for every Chevalley word I,*

$$\rho_X(I) = \chi\big((\phi_{f_y} R\pi_{I*}\mathbb{Q}_{\widetilde{\mathcal{F}}_I})_x\big). \tag{3.3}$$

The constant complex in this formula is unshifted.

Proof. Put $h = f_y\pi_I$. We first identify its critical points over x. For a graded flag F, let $E_F \subset E$ be the linear space of arrow maps preserving F. The image under $d\pi_I$ of the tangent space at (x, F) is

$$E_F + \{[\xi, x] : \xi \in \operatorname{Lie} G\}.$$

Indeed, the first summand consists of the vertical tangent directions, and the group action supplies all directions in the flag variety. The second summand pairs to zero with y by (3.1).

Choose a homogeneous basis $e_1, \dots, e_N$ adapted to F, with e_k of color i_k and $F^p = \langle e_{p+1}, \dots, e_N\rangle$. For an arrow $a : i \to j$, the allowed matrix entries of E_F send e_k to e_l with $l > k$. Equality cannot occur, since an arrow has distinct endpoint colors. Under the trace pairing, the dual entry is the reverse-arrow entry sending e_l to e_k. Orthogonality to every allowed entry thus excludes exactly the reverse-arrow entries that fail to preserve F. Therefore $E_F^\perp$ consists of reverse-arrow maps preserving the same flag F. This proves

$$\operatorname{Crit}(h) \cap \pi_I^{-1}(x) = \mathcal{F}_I(X). \tag{3.4}$$

This argument applies separately to each arrow, including parallel arrows.

Use the action $\lambda(t)^{-1}$ on $\widetilde{\mathcal{F}}_I$, where λ is given by Lemma 3.1. It preserves $\pi_I^{-1}(x)$ and satisfies $h(t \cdot u) = th(u)$. At a fixed point $u = (x, F)$, linearize the compact circle action in a holomorphic neighborhood of the smooth variety $\widetilde{\mathcal{F}}_I$ and choose an invariant Hermitian ball. The circle action identifies the local fibers as their values run once around zero. Since the weight is exactly one, this continuation closes at the identity. The local Milnor monodromy consequently acts as the identity on the cohomology of the Milnor fiber $\mathrm{MF}_u(h)$.

If the germ of h at u is nonzero and critical, it belongs to the square of the maximal ideal. A'Campo's Lefschetz theorem, in the form stated in [GLN, Theorem 0.1], applies to such a germ and gives zero Lefschetz number for its local monodromy. It follows that $\chi(\mathrm{MF}_u(h)) = 0$. If the germ is identically zero, the nearby fiber is empty and has the same Euler characteristic. At a noncritical point the local Milnor fiber is contractible. The vanishing-cycle triangle and our normalization therefore give, at every fixed point,

$$\chi\big((\phi_h\mathbb{Q})_u\big) = 1 - \chi\big(\mathrm{MF}_u(h)\big) = \mathbf{1}_{\mathrm{Crit}(h)}(u). \tag{3.5}$$

Both the local Euler function on the left and the critical locus are invariant under the torus. For an invariant constructible function α on an algebraic $\mathbb{C}^*$-variety Y, Euler integration gives

$$\int_Y \alpha\, d\chi_c = \int_{Y^{\mathbb{C}^*}} \alpha\, d\chi_c.$$

To see this, stratify the complement of the fixed locus into invariant constructible pieces admitting geometric quotients, and refine by the values of α. The quotient fibers are nontrivial torus orbits $\mathbb{C}^*/\mu_r$, of Euler characteristic zero. Apply this identity to $Y = \pi_I^{-1}(x)$ and then to its critical locus. Equations (3.4) and (3.5) show that the Euler integral of $\phi_h\mathbb{Q}$ over this fiber is $\chi(\mathcal{F}_I(X))$. Proper commutation of vanishing cycles with $R\pi_{I*}$ identifies this integral with the right-hand side of (3.3). Now use (3.2). □

3.2. **The absolute transition row.** Let P_c be the simple perverse sheaf corresponding to the lower global basis label c in Lusztig's construction. Set $r_I = \dim \widetilde{\mathcal{F}}_I$. The decomposition theorem gives a decomposition

$$L_I := R\pi_{I*}\mathbb{Q}[r_I] \simeq \bigoplus_{c,k} P_c[k]^{\oplus a_{Ick}}, \qquad a_{Ick} \in \mathbb{N}. \tag{3.6}$$

Every label occurs in such a full-word complex. Indeed, the flag-generated definition allows blocks of a single color, and refining these blocks gives a flag bundle whose direct image contains the constant complex as a direct summand. At $v = 1$, Lusztig's realization gives

$$b_c(I) = \sum_k a_{Ick}; \tag{3.7}$$

see [Lus91, §§3, 10].

Lemma 3.3. *For each label c there is a parity $\kappa_c \in \mathbb{Z}/2\mathbb{Z}$ such that*

$$a_{Ick} \neq 0 \quad \Longrightarrow \quad k - r_I \equiv \kappa_c \pmod 2.$$

Proof. For two words I, J of content $\mathbf{d}$, consider

$$Z_{IJ} = \widetilde{\mathcal{F}}_I \times_E \widetilde{\mathcal{F}}_J.$$

The diagonal G-orbits in $\mathcal{F}_I \times \mathcal{F}_J$ are indexed by the relative positions of the two flags at each vertex. Above such an orbit, the projection from Z_{IJ} is the vector bundle of arrow maps preserving both flags. Each double-flag orbit projects to the first flag variety with a Schubert cell as fiber. Bruhat cells in the first flag variety trivialize this affine bundle. It follows that each stratum has Borel–Moore homology in even degrees only. Order the finitely many relative positions so that successive unions are closed. Their inverse images give a closed filtration of Z_{IJ}, and the Borel–Moore exact sequences imply

$$H^{\mathrm{BM}}_{2j+1}(Z_{IJ}, \mathbb{Q}) = 0 \qquad (j \in \mathbb{Z}). \tag{3.8}$$

Proper adjunction and exceptional base change, followed by the smoothness of the two incidence varieties, give

$$\mathrm{Hom}^p_{D^b_c(E)}(L_I, L_J) \simeq H^{\mathrm{BM}}_{r_I + r_J - p}(Z_{IJ}, \mathbb{Q}). \tag{3.9}$$

More explicitly, the derived Hom complex is $R\Gamma(Z_{IJ}, \omega_{Z_{IJ}}[-r_I - r_J])$, where $\omega_{Z_{IJ}}$ denotes the dualizing complex. If $P_c[k]$ occurs in L_I and $P_c[l]$ occurs in L_J, the identity of P_c, together with the two summand maps, gives a nonzero element of $\mathrm{Hom}^{k-l}(L_I, L_J)$. By (3.8) and (3.9), $r_I + r_J - k + l$ is even. This is equivalent to $k - r_I \equiv l - r_J \pmod 2$, as required. □

For $y \in E^*$ write

$$G_y = \{g \in G : gy = y\}, \qquad L_y = (T_y(Gy))^\perp \subset E.$$

If $X = (x, y)$ has an open orbit, the moment equation places GX in the conormal bundle to $Gy \subset E^*$. This bundle is irreducible of dimension $\dim E$, equal to $\dim GX$. The orbit GX is therefore open in that bundle. Its fiber over y is

$$O_X = G_y x, \tag{3.10}$$

which is open in the entire linear space L_y. Moreover, differentiating $\lambda(t)y = ty$ shows that $y \in T_y(Gy)$, and hence $L_y \subseteq f_y^{-1}(0)$.

Lemma 3.4. *Suppose that $X = (x, y)$ has an open orbit, and write $s_X = \dim L_y$. There is a local system $\mathcal{L}_{X,c}$ on O_X, possibly zero, such that*

$$(\phi_{f_y} P_c)|_{O_X} \simeq \mathcal{L}_{X,c}[s_X].$$

If $n_{X,c} = \operatorname{rank} \mathcal{L}_{X,c}$, then the transition coefficient $m_{X,c} = [b_c]\rho_X$ satisfies

$$m_{X,c} = (-1)^{\kappa_c + s_X} n_{X,c}, \qquad n_{X,c} = |m_{X,c}|. \tag{3.11}$$

Proof. First, equivariance implies

$$\operatorname{supp}(\phi_{f_y} P_c) \subseteq L_y. \tag{3.12}$$

Here the vanishing-cycle complex is extended by zero from $f_y^{-1}(0)$ to E. To verify the assertion at a point $z \notin L_y$, choose $\xi \in \operatorname{Lie} G$ for which $\langle [\xi, z], y\rangle \neq 0$. The local analytic action map

$$(t, w) \longmapsto \exp(t\xi) w$$

is smooth. Equivariance makes the pullback of P_c constant in the t-direction. By the inverse function theorem, $(t, w) \mapsto (f_y(\exp(t\xi)w), w)$ is a local coordinate change near $(0, z)$. In these coordinates the function is the first projection, and the complex is pulled back from the second factor. Its vanishing cycles vanish. Smooth base change proves (3.12).

The complex $\phi_{f_y} P_c$ is perverse. By (3.12), it can be regarded as a perverse sheaf on the closed subspace L_y. Its restriction to the open subset O_X is still perverse, and G_y-equivariance makes its cohomology sheaves local systems there. Since O_X is smooth of dimension s_X, the restriction is a local system shifted by s_X. In particular,

$$\chi\big((\phi_{f_y} P_c)_x\big) = (-1)^{s_X} n_{X,c}. \tag{3.13}$$

Applying Lemma 3.2 to (3.6), and using Lemma 3.3, gives

$$\begin{aligned} \rho_X(I) &= \sum_{c,k} a_{Ick} (-1)^{k - r_I} \chi\big((\phi_{f_y} P_c)_x\big) \\ &= \sum_c (-1)^{\kappa_c} \chi\big((\phi_{f_y} P_c)_x\big) b_c(I). \end{aligned}$$

Chevalley words span the enveloping algebra. Thus this identity of evaluations identifies the coefficients of ρ_X in the specialized upper basis. Equation (3.13) proves (3.11). □

3.3. **Rigid direct sums.** Define the absolute row of a rigid module X by

$$a_X = \sum_c |m_{X,c}| b_c = \sum_c n_{X,c} b_c.$$

Proposition 3.5. *If U, V, and $U \oplus V$ are rigid nilpotent Π-modules, then*

$$a_{U \oplus V} = a_U a_V. \tag{3.14}$$

Proof. Choose graded spaces D_U, D_V and write

$$x_0 = x_U \oplus x_V, \qquad y_0 = y_U \oplus y_V, \qquad D = D_U \oplus D_V.$$

Consider the cocharacter $\eta(t) = 1_{D_U} \oplus t 1_{D_V}$. It fixes both x_0 and y_0 and gives a decomposition

$$E = E_{<0} \oplus E_0 \oplus E_{>0}, \qquad E_0 = \operatorname{Rep}(Q, \dim U) \times \operatorname{Rep}(Q, \dim V).$$

In block notation its action on an arrow is

$$\begin{pmatrix} u & b \\ c & v \end{pmatrix} \longmapsto \begin{pmatrix} u & t^{-1} b \\ tc & v \end{pmatrix}.$$

Thus the attracting space for $t \to 0$ is $E^+ = E_0 \oplus E_{>0}$, whose arrow maps preserve D_V. Let $i : E^+ \hookrightarrow E$ be the inclusion and $p : E^+ \to E_0$ the limit map. The two factors of p are the quotient D_U and the subspace D_V, in that order.

The functor $p_! i^*$ is Lusztig's unnormalized restriction functor. By [Lus91, Proposition 4.2(a)], it has a finite direct-sum decomposition

$$p_! i^* P_c \simeq \bigoplus_{a,b,k} (P_a \boxtimes P_b)[k]^{\oplus r^k_{ab;c}}, \qquad r^k_{ab;c} \in \mathbb{N}. \tag{3.15}$$

The usual quantum normalization changes this functor by a shift depending only on the two dimension vectors. Hence [Lus91, Proposition 10.23] gives, after specialization at $v = 1$,

$$[b_c](b_a b_b) = \sum_k r^k_{ab;c}. \tag{3.16}$$

The block description also identifies this attracting functor in [FLX, Proposition 2.10].

Put $h = f_{y_0}$ and define $h^0(z_U, z_V) = f_{y_U}(z_U) + f_{y_V}(z_V)$ on E_0. The function h is η-invariant and $h|_{E^+} = h^0 p$. Write i_0, p_0 for the restrictions

$$i_0 : E^+ \cap h^{-1}(0) \hookrightarrow h^{-1}(0), \qquad p_0 : E^+ \cap h^{-1}(0) \longrightarrow (h^0)^{-1}(0).$$

Descombes' commutation theorem gives

$$(p_0)_! i_0^* \phi_h P_c \simeq \phi_{h^0}(p_! i^* P_c). \tag{3.17}$$

The action is algebraic, h is invariant, and P_c is η-equivariant, so [Des, Proposition 3.2] applies with the compactly supported attracting functor $p_! i^*$.

We compute the total cohomology dimension of both sides at x_0. The left stalk is

$$R\Gamma_c(x_0 + E_{>0},\, (\phi_h P_c)|_{x_0 + E_{>0}}).$$

By (3.12), its possible support is the affine space

$$F = x_0 + (E_{>0} \cap L_{y_0}).$$

Since $U \oplus V$ is rigid, $O = G_{y_0} x_0$ is open in L_{y_0}. Its complement is closed in L_{y_0} and invariant under η, because $\eta(\mathbb{C}^*) \subseteq G_{y_0}$. Every point of F tends to x_0 as $t \to 0$. If such a point belonged to the complement, closedness would force x_0 to belong to it. Consequently, $F \subseteq O$.

Put $s_0 = \dim L_{y_0}$ and $r = \dim F$. By Lemma 3.4, the restriction to F is a local system of rank $n_{U\oplus V,c}$ shifted by s_0. This local system is constant, since F is a complex affine space. Thus the left stalk of (3.17) is isomorphic to

$$\mathbb{Q}^{\oplus n_{U\oplus V,c}}[s_0 - 2r],$$

and its total cohomology dimension is $n_{U\oplus V,c}$.

For the right stalk, the perverse-normalized Thom–Sebastiani isomorphism yields

$$(\phi_{h^0}(P_a \boxtimes P_b))_{(x_U, x_V)} \simeq (\phi_{f_{y_U}} P_a)_{x_U} \otimes (\phi_{f_{y_V}} P_b)_{x_V}. \tag{3.18}$$

This is the restriction of Thom–Sebastiani to $f_{y_U}^{-1}(0) \times f_{y_V}^{-1}(0)$ inside $(h^0)^{-1}(0)$; both points lie in these zero fibers. See [Des, §3.1, (3.4)–(3.5)]. By Lemma 3.4, the two factors in (3.18) each have cohomology in a single degree, of dimensions $n_{U,a}$ and $n_{V,b}$. The direct-sum decomposition (3.15) therefore shows that the total dimension of the right stalk is

$$\sum_{a,b,k} r^k_{ab;c}\, n_{U,a} n_{V,b}.$$

Comparing the two sides gives

$$n_{U\oplus V,c} = \sum_{a,b} \left(\sum_k r^k_{ab;c} \right) n_{U,a} n_{V,b}.$$

Together with (3.16), this is the coefficient identity for (3.14). □

4. Finite word categories and extremal matrix coefficients

Extremal highest weight modules give the ambient basis support for a finite word category and a coefficient-preserving reduction to such a category.

4.1. **Extremal submodules.** Let W be the Weyl group, with simple reflections s_i. For a dominant integral weight $\lambda = \sum_i \lambda_i \omega_i$, let I_λ be the injective hull of $\bigoplus_i S_i^{\lambda_i}$ in the category of locally nilpotent Π-modules. Local nilpotency means that each vector is annihilated by all paths of sufficiently large length. Fix an identification

$$I_\lambda = \bigoplus_i I_{\omega_i}^{\lambda_i}.$$

The corrected [ST, Theorem 4.15] provides this hull and the following description.

For each $\sigma \in W$, there is a unique submodule

$$I_{\lambda,\sigma} \subset I_\lambda, \qquad \dim I_{\lambda,\sigma} = \lambda - \sigma\lambda.$$

Uniqueness holds inside the fixed hull. These finite nilpotent submodules form a directed system under Bruhat order and their union is I_λ; see [ST, Propositions 4.9, 4.11–4.12 and Lemma 4.14].

Let $L(\lambda)$ be the integrable highest weight module, with highest vector u_λ. Write $u_{\sigma\lambda}$ for its normalized extremal vector and

$$L_\sigma(\lambda) = U(\mathfrak{n}^+)u_{\sigma\lambda}$$

for the corresponding Demazure module. The quiver Grassmannian realization gives

$$L(\lambda) \ \subset \ \bigoplus_{\mathbf{d}} \mathrm{CF}(\mathrm{Gr}_\Pi(\mathbf{d}, I_\lambda)), \tag{4.1}$$

Here $\mathrm{Gr}_\Pi(\mathbf{d}, I_\lambda)$ parametrizes submodules of dimension $\mathbf{d}$, CF denotes rational constructible functions, and the Chevalley operators are Euler integrals over one-step submodule incidences. The function u_λ is 1 at the zero submodule. By [ST, Corollaries 6.3–6.4], incidences with fixed dimensions lie in a sufficiently deep finite extremal submodule and are finite dimensional. The realization and its operator normalization are [ST, (6.2), Proposition 6.7 and Theorem 6.8].

Fix a reduced expression $\sigma = s_{i_1} \cdots s_{i_r}$ and put

$$\sigma_{\le k} = s_{i_1} \cdots s_{i_k}, \qquad V_k = I_{\omega_{i_k}, \sigma_{\le k}} \subset I_{\omega_{i_k}}.$$

For each vertex j occurring in the expression, let $k_j = \max\{k : i_k = j\}$ and define

$$P_\sigma = \bigoplus_j V_{k_j}, \qquad \mathcal{C}_\sigma = \mathrm{Fac}(P_\sigma), \qquad T = \bigoplus_{k=1}^{r} V_k.$$

Here $\operatorname{add} M$ consists of summands of finite direct sums of M, and $\operatorname{Fac} M$ consists of quotients of those finite direct sums. Extremal uniqueness and [GLS11, §2.4 and Corollary 9.2] identify the V_k with the standard iterated-socle modules. In the convention of that paper, $\mathcal{C}_\sigma = \mathcal{C}^{\mathrm{GLS}}_{\sigma^{-1}}$. The category is extension closed and Frobenius; its projective-injectives are $\operatorname{add} P_\sigma$, and its stable category is Hom-finite and 2-Calabi–Yau. The object T is basic cluster-tilting and defines the standard word seed [GLS11, Theorems 2.8–2.10].

The last-occurrence summands satisfy

$$\dim V_{k_j} = \omega_j - s_{i_1} \cdots s_{i_{k_j}} \omega_j = \omega_j - \sigma\omega_j.$$

Every reflection following k_j fixes ω_j, which gives the last equality. Embed each copy of V_{k_j} in the corresponding copy of I_{ω_j} in the fixed decomposition of I_λ. Their direct sum has dimension $\lambda - \sigma\lambda$. For an omitted vertex, $\sigma\omega_j = \omega_j$ and its extremal submodule is zero. Uniqueness inside I_λ therefore gives

$$I_{\lambda,\sigma} = \bigoplus_j V_{k_j}^{\lambda_j}. \tag{4.2}$$

4.2. **The two evaluation maps.** Let F belong to a dual weight space of $L(\lambda)$ and extend it by zero to the other weights. Define two functionals on $U(\mathfrak{n}^-)$ by

$$A_\lambda(F)(u) = F(uu_\lambda), \tag{4.3}$$

$$B_{\lambda,\sigma}(F)(u) = F(\tau(u)u_{\sigma\lambda}), \tag{4.4}$$

where τ is the anti-isomorphism satisfying

$$\tau(f_{i_1}\cdots f_{i_h}) = e_{i_h}\cdots e_{i_1}.$$

Write $*$ for the word-reversing anti-involution and $\vee$ for the Chevalley isomorphism $f_i \mapsto e_i$. Then $\tau(u) = (u^*)^\vee$. The same symbol $*$ denotes its induced involution on crystal labels. For a submodule $K \subset I_{\lambda,\sigma}$, let F_K be evaluation at K in (4.1).

Lemma 4.1 (Endpoint flag identities)**.** *For every such K,*

$$A_\lambda(F_K) = \varphi_K, \qquad B_{\lambda,\sigma}(F_K) = \varphi_{I_{\lambda,\sigma}/K}.$$

Both identities have coefficient 1 *with the normalizations above.*

Proof. The lowering operators in (4.1) integrate over submodules with a prescribed simple quotient. Thus evaluation of $f_{i_1}\cdots f_{i_h}u_\lambda$ at K is the Euler characteristic of its top-first flags, proving the first identity.

The function representing $u_{\sigma\lambda}$ is the indicator of the single point $I_{\lambda,\sigma}$. The divided-power order is

$$u_{\sigma\lambda} = f_{i_1}^{(a_1)}\cdots f_{i_r}^{(a_r)}u_\lambda, \qquad a_k = \langle s_{i_{k+1}}\cdots s_{i_r}\lambda, \alpha_{i_k}^\vee\rangle.$$

Reducedness and dominance give $a_k \geq 0$. The operators act from the right, so the intermediate extremal submodules correspond to suffixes of the reduced expression. Every such suffix is below σ in Bruhat order, and extremal uniqueness places its module inside $I_{\lambda,\sigma}$. At a step of color i and length a, the quotient is supported at i and is semisimple. Its full flag variety has Euler characteristic $a!$. The denominator in $f_i^{(a)}$ cancels this factor. Induction from $u_\lambda(0) = 1$ gives the value 1 at the unique endpoint.

The raising operators in (4.4) integrate over supermodules of K terminating at this endpoint. The assignment $K' \mapsto K'/K$ identifies their incidences with flags of $I_{\lambda,\sigma}/K$. For two colors, the order is explicit:

$$(e_je_iu_{\sigma\lambda})(K) = \int_{\substack{K\subset K'\subset I_{\lambda,\sigma}\\ K'/K\simeq S_j,\ I_{\lambda,\sigma}/K'\simeq S_i}} 1\, d\chi.$$

The quotient flag has top i and bottom j. Iterating the same incidence calculation gives the top-first order for every word and proves the second identity. □

Let $\mathcal{B}(\lambda)$ index the specialized lower global basis $G_\lambda^{\mathrm{low}}(b)$ of $L(\lambda)$, and let ξ_b be its dual functional: $\xi_b(G_\lambda^{\mathrm{low}}(b')) = \delta_{b,b'}$. Let $\mathcal{B}_\sigma(\lambda)$ be the lower basis labels in $L_\sigma(\lambda)$.

Lemma 4.2 (Upper basis compatibility)**.** *There are injective maps of labels j_λ and $j_{\sigma\lambda}^\vee$ on $\mathcal{B}(\lambda)$ and $\mathcal{B}_\sigma(\lambda)$, respectively, such that*

$$A_\lambda(\xi_b) = b_{j_\lambda(b)}, \tag{4.5}$$

$$B_{\lambda,\sigma}(\xi_b) = \begin{cases} b_{*j_{\sigma\lambda}^\vee(b)}, & b \in \mathcal{B}_\sigma(\lambda),\\ 0, & b \notin \mathcal{B}_\sigma(\lambda). \end{cases} \tag{4.6}$$

All displayed nonzero coefficients are exactly 1.

Proof. Let $(\ ,\ )_\lambda$ be the symmetric contravariant form with $(u_\lambda, u_\lambda)_\lambda = 1$ and with e_i adjoint to f_i. The vector $G^{\mathrm{up}}_\lambda(b)$ representing ξ_b satisfies $\xi_b(z) = (G^{\mathrm{up}}_\lambda(b), z)_\lambda$. Consequently

$$\begin{aligned} A_\lambda(\xi_b)(u) &= (G^{\mathrm{up}}_\lambda(b), uu_\lambda)_\lambda, \\ B_{\lambda,\sigma}(\xi_b)(u) &= (G^{\mathrm{up}}_\lambda(b), \tau(u)u_{\sigma\lambda})_\lambda \\ &= (u_{\sigma\lambda}, uG^{\mathrm{up}}_\lambda(b))_\lambda. \end{aligned}$$

These are the specializations of the integral quantum matrix coefficients in [KO, Definitions 3.40, 3.43 and Propositions 3.44, 3.46]. The second uses $\tau(u) = (u^*)^\vee$, which supplies the star in (4.6). The label maps are injective by [KO, Propositions 3.16, 3.44]. Demazure compatibility of the lower basis, [KO, Proposition 3.34], gives the displayed kernel. All identities specialize in the integral graded dual, and the nonzero coefficients in [KO, Proposition 3.46] are 1. □

To specify the subalgebra in the next statement, let

$$\begin{aligned} \mathcal{A}_{\sigma,v} &= A_q[N_-(\sigma)] \otimes_{\mathbb{Q}(q)} \mathbb{Q}(v) \\ &= (*U^-_q(\sigma)) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(v) \end{aligned}$$

in the notation of [KO, Definition 3.26]. Here $U^-_q(\sigma)$ is the quantum unipotent subalgebra of the negative half defined in [KO, §3.3]. Let $\mathcal{A}_\sigma$ be the specialization at $v = 1$ of the upper-basis integral graded-dual form of $\mathcal{A}_{\sigma,v}$. It is a subalgebra of $\mathcal{A}$, by [KO, Proposition 3.27, Definition 3.28 and Theorem 3.29]; its quantum cluster structure is [GLSQ].

Lemma 4.3 (Ambient support)**.** *For every $J \in \mathcal{C}_\sigma$, the expansion of φ_J in the ambient specialized upper global basis is supported on the upper basis subset of the nonlocalized word algebra $\mathcal{A}_\sigma$. Hence the absolute value of that expansion also belongs to $\mathcal{A}_\sigma$.*

Proof. The factor-category description supplies a surjection $I_{\mu,\sigma} \twoheadrightarrow J$ for some dominant μ. Let K be its kernel. The second identity of Lemma 4.1 gives $\varphi_J = B_{\mu,\sigma}(F_K)$. Expand F_K in the finite dual weight basis $\{\xi_b\}$. By (4.6), every surviving upper label is $*j^\vee_{\sigma\mu}(b)$ for a Demazure label. Their union is the basis subset of the word subalgebra by [KO, Definition 3.28 and Theorem 3.48], in the same transpose convention. Its compatibility with the ambient upper basis is also [Kim, Theorems 4.22, 4.26]. Taking absolute values changes only the scalar coefficients in this finite basis expansion. □

4.3. **A rigid quotient containing all relevant labels.**

Proposition 4.4. *Let X be any finite dimensional nilpotent rigid Π-module of dimension $\mathbf{d}$. There are λ, σ and an embedding*

$$0 \longrightarrow X \longrightarrow V = I_{\lambda,\sigma} \longrightarrow Y \longrightarrow 0$$

such that $Y \in \mathcal{C}_\sigma$ is rigid and

$$L(\lambda)_{\lambda-\mathbf{d}} \subset L_\sigma(\lambda). \tag{4.7}$$

The nonzero upper basis coefficients of φ_X and φ_Y are in bijection, with identical signed values.

Proof. Write $\operatorname{soc} X$ for the sum of the simple submodules of X. Choose $\lambda_i \geq \dim \operatorname{Hom}_\Pi(S_i, X)$, embed $\operatorname{soc} X$ into $\operatorname{soc} I_\lambda$, and extend the embedding to $X \to I_\lambda$ by injectivity in the locally nilpotent category. Its kernel intersects $\operatorname{soc} X$ trivially. Every nonzero submodule of a finite-length module has nonzero socle, so the map is injective.

The directed union of the extremal submodules contains X, so $X \subset I_{\lambda,\sigma_0}$ for some σ_0. Choose a basis of the finite-dimensional space $L(\lambda)_{\lambda-\mathbf{d}}$. The Demazure modules form a directed union equal to $L(\lambda)$, so its finitely many basis vectors lie in one $L_{\sigma_1}(\lambda)$. Choose a common Bruhat upper bound σ of σ_0, σ_1. Both $X \subset I_{\lambda,\sigma}$ and (4.7) then hold. This is the whole-weight-space construction of [ST, Lemma 4.14].

By (4.2), $V \in \operatorname{add} P_\sigma$ and its quotient Y belongs to $\mathcal{C}_\sigma$. This category is extension closed: every ambient extension of two of its objects remains in the category. Its relative and ambient first extension groups therefore agree. Relative projective-injectivity gives

$$\operatorname{Ext}^1_\Pi(V,Y) = \operatorname{Ext}^1_\Pi(Y,V) = 0. \tag{4.8}$$

Write $\iota : X \to V$ and $\pi : V \to Y$ for the maps. For any $f : X \to Y$, the vanishing of $\operatorname{Ext}^1_\Pi(X,X)$ gives a lift $h : X \to V$ with $\pi h = f$. The second vanishing in (4.8) extends h to $k : V \to V$ with $k\iota = h$. Thus πk extends f, and

$$\operatorname{Hom}_\Pi(V,Y) \longrightarrow \operatorname{Hom}_\Pi(X,Y)$$

is surjective. The long exact sequence for $\operatorname{Hom}_\Pi(-,Y)$ and the first vanishing in (4.8) now give $\operatorname{Ext}^1_\Pi(Y,Y) = 0$.

Finally expand $F_X = \sum_b c_b \xi_b$ in the dual of the weight space in (4.7). By Demazure compatibility every label in that weight belongs to $\mathcal{B}_\sigma(\lambda)$. Lemmas 4.1 and 4.2 give

$$\varphi_X = \sum_b c_b b_{j_\lambda(b)}, \qquad \varphi_Y = \sum_b c_b b_{*j^\vee_{\sigma\lambda}(b)}.$$

Both label maps are injective on this whole weight space, so the nonzero signed coefficients correspond with their multiplicities. □

5. Finite word categories and a Hall chamber

Fix the reduced expression for σ, the category $\mathcal{C}_\sigma$, and its standard cluster-tilting object T from Section 4. Order the indecomposable summands as

$$T = T_1 \oplus \cdots \oplus T_{n+f},$$

with the n nonprojective summands first and the f projective-injective summands last. Their coordinates are called mutable and frozen, respectively. Put

$$E_T = \operatorname{End}_\Pi(T)^{\mathrm{op}}, \qquad A = \operatorname{End}_{\underline{\mathcal{C}_\sigma}}(T)^{\mathrm{op}}.$$

Here $\underline{\mathcal{C}_\sigma}$ is the quotient by morphisms factoring through $\operatorname{add} P_\sigma$, the projective-injective objects. Its suspension $J[1]$ is represented by the cokernel of an embedding $J \hookrightarrow I$ with $I \in \operatorname{add} P_\sigma$. Write $D_{\mathrm{vec}} V = \operatorname{Hom}_{\mathbb{C}}(V,\mathbb{C})$. The stable category is Hom-finite and 2-Calabi–Yau:

$$D_{\mathrm{vec}} \operatorname{Hom}_{\underline{\mathcal{C}_\sigma}}(U,V) \simeq \operatorname{Hom}_{\underline{\mathcal{C}_\sigma}}(V,U[2])$$

functorially in U,V. In particular A is finite dimensional. An object $R \in \mathcal{C}_\sigma$ is cluster-tilting when

$$\operatorname{add} R = \{J \in \mathcal{C}_\sigma : \operatorname{Ext}^1_{\mathcal{C}_\sigma}(R,J) = 0\} = \{J \in \mathcal{C}_\sigma : \operatorname{Ext}^1_{\mathcal{C}_\sigma}(J,R) = 0\}.$$

The object T has this property [GLS11, Theorems 2.8–2.10]. Inside a Hom space of the stable category, an object denotes its image in that category.

Put $H_T = (\dim \operatorname{Hom}_\Pi(T_i,T_j))_{i,j}$. This is the Cartan matrix of E_T, which has finite global dimension [GLSCh, Section 1.4]. Finite projective resolutions express the simple classes in the projective classes, so H_T is invertible over $\mathbb{Z}$. Let S_i^T be the simple head of the projective E_T-module $\operatorname{Hom}_\Pi(T,T_i)$. The extended exchange matrix B, with all rows and the first n columns, and the skew-symmetric matrix L are

$$\begin{aligned} B_{ij} &= \dim \operatorname{Ext}^1_{E_T}(S_j^T,S_i^T) - \dim \operatorname{Ext}^1_{E_T}(S_i^T,S_j^T), \qquad j \le n, \\ L_{ij} &= \dim \operatorname{Hom}_\Pi(T_i,T_j) - \dim \operatorname{Hom}_\Pi(T_j,T_i). \end{aligned}$$

Thus B_{ij} counts arrows $j \to i$ minus arrows $i \to j$ in the quiver of E_T. We identify the full exponent lattice with $\mathsf{M} = \mathbb{Z}^{n+f}$, and write a_{mut} for the first n coordinates of $a \in \mathsf{M}$. The vectors ε_i, $1 \le i \le n$, denote the standard basis of $\mathbb{Z}^n$. The compatibility constant is 2. To verify it, let D_T be the matrix

whose j-th column is $\dim T_j$, and let $C = (c_{ij})$ be the symmetric Cartan matrix. The Ringel matrix of E_T is $H_T^{-\mathsf{T}}$, and its mutable columns form B [GLSCh, Section 1.4 and Section 2.2(vii)]:

$$B = H_T^{-\mathsf{T}} \begin{pmatrix} \mathrm{Id}_n \\ 0 \end{pmatrix}.$$

The two exchange exact sequences at each mutable summand give $D_T B = 0$ [KO, Proposition 7.10(4)]. Since T is rigid, the preprojective Hom–Ext formula [GLS11, Lemma 2.1] gives $H_T + H_T^{\mathsf{T}} = D_T^{\mathsf{T}} C D_T$. Thus $L = 2H_T - D_T^{\mathsf{T}} C D_T$, and

$$B^{\mathsf{T}} L = (2\,\mathrm{Id}_n, 0). \tag{5.1}$$

In particular, B is injective. Set $x_i = \varphi_{T_i}$ and $x^a = \prod_{i=1}^{n+f} x_i^{a_i}$ for $a \in \mathsf{M}$, viewed in the Laurent field of the initial seed.

For $J \in \mathcal{C}_\sigma$, choose a short exact sequence

$$0 \longrightarrow T^{(1)} \longrightarrow T^{(0)} \longrightarrow J \longrightarrow 0, \qquad T^{(0)}, T^{(1)} \in \operatorname{add} T, \tag{5.2}$$

whose last map is a right $\operatorname{add} T$-approximation: every map from an object of $\operatorname{add} T$ to J factors through it. To construct it, choose $0 \to K \to I \to J \to 0$ in $\mathcal{C}_\sigma$ with $I \in \operatorname{add} P_\sigma$, and adjoin an evaluation map $T^{\oplus m} \to J$ from a basis of $\mathrm{Hom}_\Pi(T, J)$. The resulting epimorphism from $T^{(0)} = I \oplus T^{\oplus m}$ is a right approximation. Its kernel fits into

$$0 \longrightarrow K \longrightarrow T^{(1)} \longrightarrow T^{\oplus m} \longrightarrow 0,$$

so extension closure places $T^{(1)}$ in $\mathcal{C}_\sigma$. The long exact Hom sequence gives $\mathrm{Ext}^1_{\mathcal{C}_\sigma}(T, T^{(1)}) = 0$; hence $T^{(1)} \in \operatorname{add} T$. Define the full index

$$g_J = \mathrm{ind}_T\, J = [T^{(0)}] - [T^{(1)}] \in \mathsf{M},$$

where brackets denote multiplicity vectors in the ordered summand basis. Applying $\mathrm{Hom}_\Pi(T, -)$ to (5.2) gives a length-one projective resolution of $\mathrm{Hom}_\Pi(T, J)$. Writing $h_J = (\dim \mathrm{Hom}_\Pi(T_k, J))_k$ as a column vector gives

$$H_T g_J = h_J, \qquad g_J = H_T^{-1} h_J.$$

The displayed vector is the extended index of [GLSCh, (6.7)–(6.10)]. In particular the index is independent of the sequence. Direct sums of sequences prove its additivity; passing to the stable category gives its mutable projection.

The finite-dimensional A-module

$$W_J = \mathrm{Ext}^1_{\mathcal{C}_\sigma}(T, J) = \mathrm{Hom}_{\underline{\mathcal{C}_\sigma}}(T, J[1])$$

has dimension vector in $\mathbb{N}^n$, indexed by the mutable summands. The standard object T is reachable, so the character theorem [GLSCh, (1.3) and Theorem 4] applies to every $J \in \mathcal{C}_\sigma$ and gives

$$\varphi_J = x^{g_J} \sum_{d \in \mathbb{N}^n} \chi(\mathrm{Gr}_d(W_J)) x^{Bd}. \tag{5.3}$$

Here $\mathrm{Gr}_d(W_J)$ is the projective variety of submodules of dimension vector d. The sum is finite, and its constant coefficient is one. If J is rigid, then $\varphi_J = \rho_J$. For a projective-injective summand J one has $W_J = 0$ and $(g_J)_{\mathrm{mut}} = 0$, while its full character is the corresponding frozen monomial.

By Lemma 4.3, the ambient upper-basis expansion of every φ_J is finite and belongs to the nonlocalized word algebra $\mathcal{A}_\sigma$. We use its frozen localization in the triangular-basis and scattering calculations below.

5.1. **The polynomial potential of the initial seed.**

Proposition 5.1. *The algebra A is the ordinary Jacobian algebra of a finite polynomial potential on the mutable standard-word quiver. This potential is nondegenerate. The initial seed is injective-reachable, and its classical stability scattering diagram agrees with its classical cluster scattering diagram.*

Proof. The standard object T is dual to the standard object in the submodule category used in [BIRS]; see [GLS11, Section 2.4]. This duality reverses endomorphisms, so A is the stable endomorphism algebra in [BIRS, Section 6, Theorems 6.4–6.6]. Those results identify it with the completed Jacobian algebra of the standard word potential. The same opposite-algebra presentation appears in [GLSCh, Section 6.1 and Lemma 6.2].

Write Γ for its mutable quiver. Horizontal arrows join successive occurrences of one vertex and point in one positional direction. Nonhorizontal arrows join different vertex types and point in the other positional direction. Thus Γ has neither loops nor opposite pairs of arrows. The construction keeps separate labels for parallel arrows of Q. Every term of the potential consists of two nonhorizontal arrows and a nonempty horizontal path, and therefore has length at least three. The horizontal subquiver is acyclic. These observations apply also after simultaneous reversal of the quiver and potential.

Give horizontal arrows degree zero and all other arrows degree one. The potential is a finite polynomial homogeneous of degree two, and its derivative ideal is homogeneous. Let H_0 be the largest length of a horizontal path, with $H_0 = 0$ for the empty quiver. A path with m nonhorizontal arrows has at most $m + 1$ horizontal stretches, so its length ℓ satisfies

$$m \le \ell \le m + (m+1)H_0. \tag{5.4}$$

Every homogeneous piece of the ordinary path algebra is therefore finite dimensional, and the arrow-length and nonhorizontal-degree filtrations are cofinal.

Let J_{cyc} be the ordinary derivative ideal. The degree-m piece of its closure in the completed path algebra is exactly $(J_{\mathrm{cyc}})_m$. Indeed, sufficiently long paths have no degree-m component by (5.4), and taking a homogeneous component preserves this ideal. Writing $(\mathbb{C}\Gamma)_m$ for the degree-m path space, we obtain

$$\widehat{\mathbb{C}\Gamma}/\overline{J_{\mathrm{cyc}}} \simeq \prod_{m\ge 0} (\mathbb{C}\Gamma)_m/(J_{\mathrm{cyc}})_m.$$

The left side is the finite-dimensional algebra A. Infinitely many nonzero factors on the right would contain an infinite-dimensional direct sum. Thus only finitely many factors survive, the product equals the direct sum, and the ordinary and completed Jacobian algebras coincide.

The potential is rigid by [BIRS, Theorem 6.5], and hence nondegenerate. For every finite reduced word, [Qin2, Theorem 8.2.2 and Lemma 9.1.3] provides a mutation sequence from the standard seed to an injective seed. The mutable leading degrees of its variables are the negative coordinate vectors up to permutation, with their frozen corrections. The initial seed is thus injective-reachable. Qin's comparison theorem [Qin1, Theorems 1.2.4 and 6.3.1], applied to this finite-dimensional nondegenerate Jacobian algebra, identifies the classical stability and cluster diagrams. □

5.2. **The chamber of an arbitrary rigid completion.** All Hom spaces in presentation triangles in the next proof are taken in $\underline{\mathcal{C}_\sigma}$. For $\theta \in \mathbb{R}^n$, a nonzero A-module U is θ-semistable if $\theta \cdot \dim U = 0$ and $\theta \cdot \dim U' \le 0$ for every submodule $U' \subset U$. Stability chambers are connected open regions of weights with no nonzero semistable module.

Lemma 5.2. *Let $M \in \mathcal{C}_\sigma$ be rigid. There is a cluster-tilting object $R \in \mathcal{C}_\sigma$ containing every indecomposable summand of M and every projective-injective summand of T. Let $R_1, \dots, R_n$ be its*

nonprojective summands, and put

$$g_i = (g_{R_i})_{\mathrm{mut}}, \qquad \mathscr{K}_R = \left\{ \sum_{i=1}^{n} p_i g_i : p_i > 0 \right\} \subset \mathbb{R}^n.$$

The g_i form an integral basis. The cone $\mathscr{K}_R$ lies in an open stability chamber with torsion-free class

$$\mathcal{F} = \mathrm{Sub}(W_R) = \{U : U \hookrightarrow W_R^k \text{ for some } k \in \mathbb{N}\}.$$

For every $J \in \operatorname{add} R$, classical scattering transport from the initial positive chamber to this chamber sends φ_J to x^{g_J}.

Proof. Take the basic stable object with the same indecomposable summands as the image of M, and write it as $U \oplus P[1]$, where $P \in \operatorname{add} T$ and U has no summand in $\operatorname{add} T[1]$. Put $\mathsf{H} = \mathrm{Hom}_{\underline{\mathcal{C}_\sigma}}(T, -)$, and write τ_A for the Auslander–Reiten translation of A-modules. The correspondence [AIR, Theorem 4.1] gives the τ_A-rigid pair $(\mathsf{H}U, \mathsf{H}P)$:

$$\mathrm{Hom}_A(\mathsf{H}U, \tau_A \mathsf{H}U) = 0, \qquad \mathrm{Hom}_A(\mathsf{H}P, \mathsf{H}U) = 0.$$

Choose an idempotent $e \in A$ with $\operatorname{add}(Ae) = \operatorname{add}(\mathsf{H}P)$. The second equality implies $e\mathsf{H}U = 0$, and [AIR, Proposition 2.3(a)] makes $\mathsf{H}U$ a τ-rigid module over $A/\langle e \rangle$. Its Bongartz completion over this quotient is a τ-tilting module V containing $\mathsf{H}U$ [AIR, Theorem 2.10]. Proposition 2.3(a) of the same paper returns the support τ-tilting pair $(V, \mathsf{H}P)$ over A, and Theorem 4.1 lifts it to a stable cluster-tilting object containing $U \oplus P[1]$.

Choose representatives of its nonprojective summands containing those of M, and adjoin one copy of each indecomposable projective-injective. Stable isomorphisms between nonprojective indecomposables lift to isomorphisms in this Krull–Schmidt category: an endomorphism factoring through projectives cannot be invertible, since its inverse would make the object a summand of a projective. It therefore lies in the radical of the local endomorphism ring, so lifts of stable inverse maps have invertible composites. Since stable first extensions are the first extensions of the Frobenius category, the resulting object R is cluster-tilting and contains every required summand. The indices of its nonprojective summands form an integral basis by [AIR, Corollary 5.2].

Fix $J \in \operatorname{add} R$ and a triangle

$$T_1' \longrightarrow T_0' \longrightarrow J \longrightarrow T_1'[1], \qquad T_0', T_1' \in \operatorname{add} T.$$

Applying $\mathrm{Hom}(T, -)$ to the shifted triangle and using the 2-Calabi–Yau pairing gives the injective copresentation

$$0 \longrightarrow W_J \longrightarrow I_1 \xrightarrow{\alpha} I_0, \qquad I_j = D_{\mathrm{vec}} \mathrm{Hom}_{\underline{\mathcal{C}_\sigma}}(T_j', T) \quad (j = 0, 1). \tag{5.5}$$

The leading zero follows from $\mathrm{Hom}(T, T_0'[1]) = 0$. The functorial identification

$$\mathrm{Hom}_A(W_R, D_{\mathrm{vec}} \mathrm{Hom}(T_j', T)) = D_{\mathrm{vec}} \mathrm{Hom}(T_j', R[1])$$

shows that $\mathrm{Hom}_A(W_R, I_1) \to \mathrm{Hom}_A(W_R, I_0)$ is surjective. Its dual is the injection $\mathrm{Hom}(T_0', R[1]) \to \mathrm{Hom}(T_1', R[1])$, whose kernel vanishes because $\mathrm{Hom}(J, R[1]) = 0$.

For $U \subset W_R^k$, a map $U \to I_0$ extends to W_R^k, by injectivity of I_0, and then lifts to I_1. Taking dimensions in (5.5) gives

$$\dim \mathrm{Hom}_A(U, W_J) = -(g_J)_{\mathrm{mut}} \cdot \dim U. \tag{5.6}$$

Here $\dim \mathrm{Hom}_A(U, D_{\mathrm{vec}} \mathrm{Hom}(T_j', T))$ is the pairing of $\dim U$ with the multiplicity vector of T_j'. For $J = R$, put $C_\alpha = \operatorname{im} \alpha$. Every map $U \to C_\alpha$, viewed in I_0, lifts to I_1. The exact sequence

$$0 \longrightarrow W_R \longrightarrow I_1 \longrightarrow C_\alpha \longrightarrow 0$$

therefore implies

$$\mathrm{Ext}_A^1(U, W_R) = 0 \quad \text{for every } U \in \mathcal{F}. \tag{5.7}$$

The class $\mathcal{F}$ is closed under submodules and extensions. To prove the latter, take an extension with ends $U', U'' \in \mathcal{F}$, choose embeddings $U' \hookrightarrow W_R^a$ and $U'' \hookrightarrow W_R^b$, and extend the first map across the middle object E using (5.7). Combined with $E \to U'' \hookrightarrow W_R^b$, it gives an embedding $E \hookrightarrow W_R^{a+b}$.

For an arbitrary finite-dimensional A-module E, choose among its submodules K with $E/K \in \mathcal{F}$ one of minimal length. These submodules form a nonempty set closed under intersection, because

$$E/(K_1 \cap K_2) \;\hookrightarrow\; E/K_1 \oplus E/K_2.$$

Thus K is contained in every admissible kernel. If $K \to W_R$ had nonzero image, quotienting E by its kernel would give an extension of two objects of $\mathcal{F}$, and hence a smaller admissible kernel. It follows that $\mathrm{Hom}_A(K, W_R) = 0$. Since every object of $\mathcal{F}$ embeds in a finite sum of W_R's, every map from K to an object of $\mathcal{F}$ vanishes. We obtain the torsion pair

$$\mathcal{T} = \{U : \mathrm{Hom}_A(U, W_R) = 0\}, \qquad \mathcal{F} = \mathrm{Sub}(W_R). \tag{5.8}$$

Let $\theta = \sum_i p_i g_i$ with all $p_i > 0$. For $0 \neq U \in \mathcal{F}$, equation (5.6) gives

$$\theta(\dim U) = -\sum_i p_i \dim \mathrm{Hom}_A(U, W_{R_i}) < 0.$$

Strictness follows from a nonzero component of an embedding into a finite sum of W_R's. Write $I_{j,i}$ for the term I_j of (5.5) with $J = R_i$. For $0 \neq U \in \mathcal{T}$, the kernel of $\mathrm{Hom}_A(U, I_{1,i}) \to \mathrm{Hom}_A(U, I_{0,i})$ is $\mathrm{Hom}_A(U, W_{R_i}) = 0$, so $g_i \cdot \dim U \geq 0$. The pairings cannot all be zero since the g_i are a basis. Hence $\theta(\dim U) > 0$. The torsion decomposition now gives

$$\mathcal{F} = \{U : \theta(\dim U') \leq 0 \text{ for every submodule } U' \subset U\}.$$

Indeed, an object outside $\mathcal{F}$ has a nonzero torsion submodule of positive θ-value. A nonzero semistable object would belong to $\mathcal{F}$ and have θ-value zero, which is impossible. This works on the entire full-dimensional open cone $\mathscr{K}_R$, placing it in a chamber with the stated torsion pair. We compute the transport. Use the subobject-first convention for Hall multiplication. Let $1_{\mathcal{F}}$ be the Hall element of all objects of $\mathcal{F}$, and let $\mathrm{Gr}(W_J)$ record submodules of W_J, with their specified inclusions. The stack of maps

$$\{(U, h) : U \in \mathcal{F},\ h : U \to W_J\}$$

has two descriptions. Its kernel and image belong to $\mathcal{F}$. Recording the kernel and then the image with inclusion gives

$$\mathrm{Map}(\mathcal{F}, W_J) = 1_{\mathcal{F}} * \mathrm{Gr}(W_J).$$

Conversely these extension data reconstruct the map and its automorphisms. Over the stack of U's of dimension d, the maps form a vector bundle of rank $-(g_J)_{\mathrm{mut}} \cdot d$, by (5.6). This also follows on representation atlases from the linear intertwining equations and their constant-rank stratification.

Write t^2 for the Hall weight of an affine line, and let σ_c^{H} multiply charge-d terms by $t^{2c \cdot d}$. The descriptions give

$$1_{\mathcal{F}} * \mathrm{Gr}(W_J) = \sigma^{\mathrm{H}}_{-(g_J)_{\mathrm{mut}}}(1_{\mathcal{F}}). \tag{5.9}$$

In the extended Hall algebra [Bri, Section 10.4], let a_d be homogeneous of dimension vector d, and let z^g be the formal monomial with exponent $g \in \mathsf{M}$. Then

$$a_d z^g = t^{2g_{\mathrm{mut}} \cdot d} z^g a_d, \qquad z^g a = \sigma^{\mathrm{H}}_{-g_{\mathrm{mut}}}(a) z^g.$$

Consequently (5.9) gives both identities

$$1_{\mathcal{F}}^{-1} z^{g_J} 1_{\mathcal{F}} = \mathrm{Gr}(W_J) z^{g_J}, \tag{5.10}$$

$$1_{\mathcal{F}} \big(\mathrm{Gr}(W_J) z^{g_J}\big) 1_{\mathcal{F}}^{-1} = z^{g_J}. \tag{5.11}$$

Equation (24) and the proof of Theorem 10.1 of [Bri] identify positive-to-θ transport with conjugation by $1_{\mathcal{F}}$. Thus (5.11) is the positive-to-θ identity for every integral full index g_J.

The Grassmannian element is regular and finite in dimension vector. Hall wall-crossing and Euler integration [Bri, Sections 11.1–11.3] therefore apply degree by degree to the adjoint action. After integration and the substitution $d \mapsto Bd$, the original function is exactly (5.3). All coordinates of g_J and Bd, including frozen coordinates, are retained. One can first use independent charge variables and then make the substitution. On full monomials the charge d acts by

$$\partial_d(x^a) = (a_{\mathrm{mut}} \cdot d)x^{a+Bd}.$$

Let B_{mut} be the first n rows of B. This matrix is skew-symmetric, so direct composition gives

$$\begin{aligned}[\partial_d, \partial_e](x^a) &= ((a_{\mathrm{mut}} \cdot e)d^{\mathsf{T}}B_{\mathrm{mut}}e - (a_{\mathrm{mut}} \cdot d)e^{\mathsf{T}}B_{\mathrm{mut}}d)x^{a+B(d+e)} \\ &= (d^{\mathsf{T}}B_{\mathrm{mut}}e)\partial_{d+e}(x^a).\end{aligned}$$

This is the charge Lie action in [Qin1, Appendix A.2], so the substitution intertwines the scattering actions. Injectivity of B prevents identification of charges. Proposition 5.1 identifies this classical transport with classical cluster transport, proving the lemma. □

6. A finite-box estimate for the upper global basis

We now use the localized ordinary quantum word algebra. Its based torus has multiplication

$$X^u X^w = v^{L(u,w)} X^{u+w}, \qquad u, w \in \mathsf{M}, \tag{6.1}$$

and bar involution $\overline{v} = v^{-1}$, $\overline{X^u} = X^u$, reversing multiplication. Fix a positive integer κ sufficient for the root extensions below. Coefficients lie in $\mathbb{Q}[v^{\pm 1/\kappa}]$, or in its fraction field when specified, and degrees are measured in v. For $d \in \mathbb{N}^n$, put $Y^d = X^{Bd}$. Compatibility gives

$$L(Bd, a) = 2a_{\mathrm{mut}} \cdot d, \qquad L(a, Bd) = -2a_{\mathrm{mut}} \cdot d. \tag{6.2}$$

A series is *pointed at* a if it is

$$X^a + \sum_{0 \neq d \in \mathbb{N}^n} q_d X^{a+Bd}.$$

The vector d is its charge; a remains a full exponent. The cone $B\mathbb{N}^n$ is pointed because B is injective.

The localized upper global basis, with its standard bar normalization, is the common triangular basis of this ordinary quantum cluster algebra [Qin2, Theorem 9.5.1]. Write $b[a]$ for its element pointed at $a \in \mathsf{M}$. At $v = 1$ it specializes to the localized upper basis; the subset coming from $\mathcal{A}_{\sigma,v}$ specializes to the ambient nonlocalized upper basis of $\mathcal{A}_\sigma$. Qin's standard elements I_a are pointed at the same full exponents. In the positive-charge completion, [Qin2, (6.1), Definition 6.1.1 and Lemma 6.2.4] gives

$$b[a] = I_a + \sum_{0 \neq j \in \mathbb{N}^n} c_{a,j} I_{a+Bj}, \qquad c_{a,j} \in v^{-1}\mathbb{Z}[v^{-1}]. \tag{6.3}$$

Only finitely many terms contribute to a fixed charge box.

The estimate below adapts the distant-direction argument of [HJ, Sections 3.3–3.5] to the full word lattice.

6.1. Laurent coefficients at a generic point.

We place the full-coefficient diagram in $\mathsf{M}_{\mathbb{R}} = \mathsf{M} \otimes_{\mathbb{Z}} \mathbb{R}$. A wall of charge d is contained in

$$\{a \in \mathsf{M}_{\mathbb{R}} : d \cdot a_{\mathrm{mut}} = 0\}.$$

The initial walls are invariant under translations in the frozen coordinate subspace. The scattering construction preserves these translations: intersections of wall supports and the outgoing supports retain the same translation directions. Equivalently, (6.2) identifies the frozen subspace with the annihilator of all charges, which is parallel to every wall in [DM, proof of Proposition 3.18]. We choose lifts of mutable reference points and paths to $\mathsf{M}_{\mathbb{R}}$. Their wall crossings depend only on the mutable projections, while the associated monomials have exponents in M.

At a generic reference point, monomial expansions are taken in the charge completion. The next lemma concerns each coefficient of such an expansion.

Lemma 6.1. *At every generic point of the quantum cluster scattering diagram, each coefficient of the transported $b[a]$ in based monomials is a bar-invariant Laurent polynomial in a fixed root of v.*

Proof. The theta functions have Laurent polynomial coefficients at arbitrary generic points, and those coefficients are bar invariant for the cluster initial data [DM, Theorem 3.9, Proposition 3.12 and Lemma 4.3]. The ordinary quantum cluster algebra is contained in the middle algebra [DM, Proposition 4.14]. This middle algebra is the module spanned by the relevant theta functions over a Laurent polynomial coefficient ring. After expressing the parameter in v and taking the fixed root extension, this ring is $\mathbb{Q}[v^{\pm 1/\kappa}]$. The ordinary Laurent form is the one identified in [Qin2, Theorems 9.2.3 and 9.2.5]. The Laurent form of each $b[a]$ therefore has a finite expression

$$b[a] = \sum_{\ell=1}^{s} p_\ell(v)\vartheta_\ell, \qquad p_\ell(v) \in \mathbb{Q}[v^{\pm 1/\kappa}],$$

with a finite index set. The theta functions are linearly independent and bar invariant. Since $b[a]$ is bar invariant, $\overline{p_\ell} = p_\ell$. At every generic point, a fixed based-monomial coefficient is thus a finite sum of products of bar-invariant Laurent polynomials.

The parameter conventions agree with (6.1). In the compatibility convention (47) of [DM], take

$$B_1(\varepsilon_i) = B\varepsilon_i, \qquad \Lambda_{\mathrm{DM}} = -L/2, \qquad t_{\mathrm{DM}} = v^{-2}.$$

Then $\Lambda_{\mathrm{DM}}(a, B\varepsilon_i) = a_i$, and their based multiplication is exactly $v^{L(u,w)}X^{u+w}$. Thus the parameter v in this identification is the one used in the triangular expansion (6.3). □

6.2. **The estimate.** For $d, e \in \mathbb{N}^n$, write $d \le e$ for coordinatewise inequality, and $d < e$ for $d \le e$ with $d \neq e$. Set

$$[0, e] = \{d \in \mathbb{N}^n : 0 \le d_i \le e_i \text{ for every } i\}.$$

The complement of this box is an additive ideal of $\mathbb{N}^n$. Modulo that ideal, all charge computations are finite.

Lemma 6.2. *Assume $n > 0$. Fix $e \in \mathbb{N}^n$ and an integral vector $\nu \in \mathscr{K}_R$ such that*

$$\nu_i \neq 0 \ (1 \le i \le n), \qquad \nu \cdot d \neq 0 \ (0 \neq d \in [0, e]). \tag{6.4}$$

Choose a lift $\widetilde{\nu} \in \mathsf{M}$. There is a generic point θ in the same Hall chamber, arbitrarily close to ν, with the following property. For every finite set $\mathscr{H} \subset \mathsf{M}$, there is N_0 such that, for $N \ge N_0$ and $h \in \mathscr{H}$, transport of $b[N\widetilde{\nu} + h]$ to θ has leading coefficient one and zero coefficient at

$$N\widetilde{\nu} + h + Bd, \qquad 0 \neq d \in [0, e].$$

The same statement holds in the classical chart after $v = 1$.

Proof. The case $e = 0$ is immediate. Work first in the positive-charge completion over $\mathbb{Q}(v^{1/\kappa})$. For charge series define

$$\sigma_c(Y^d) = v^{-4c_{\mathrm{mut}}\cdot d}Y^d. \tag{6.5}$$

When $c \in \mathbb{Z}^n$, its frozen part is understood to be zero in this formula. Equation (6.2) implies, for every charge series G,

$$GX^a = X^a\sigma_{-a}(G). \tag{6.6}$$

Traverse the line

$$s \longmapsto \nu + s(1, \dots, 1)$$

from large positive to large negative s, using a lift to $\mathsf{M}_{\mathbb{R}}$. A small generic perturbation allows its reference point at $s = 0$ to be a fully generic θ in the Hall chamber, while preserving all signs in

(6.4). Indeed, truncate the diagram above total charge $\sum_i e_i$. Only finitely many relevant walls and joints remain. Avoiding joints and preserving the finitely many nonzero pairings is an open condition, in which one can choose a fully generic point.

Let Ad_F denote quantum transport from the positive chamber to θ, and let $\mathrm{Ad}_{T'}$ denote transport from θ to the negative chamber. Choose constant-one unit lifts in the rational charge completion and put

$$H = T'F.$$

Our convention is $\mathrm{Ad}_F(G) = FGF^{-1}$. Such lifts are supplied by the quantum wall factors. Their individual coefficients may be rational in v.

A wall normal is a positive charge d. The wall can meet the unperturbed line only at

$$s = -\frac{\nu \cdot d}{\sum_i d_i}.$$

It can be crossed before the reference point only if $\nu \cdot d < 0$, and afterward only if $\nu \cdot d > 0$. The perturbation preserves this for boxed charges. Products, powers and inverse series preserve these strict signs, since pairings add. Thus

$$[Y^d]F \neq 0, \quad 0 \neq d \leq e \implies \nu \cdot d < 0, \tag{6.7}$$

$$[Y^d]T' \neq 0, \quad 0 \neq d \leq e \implies \nu \cdot d > 0. \tag{6.8}$$

The series H and H^{-1} need only have nonnegative charges.

We calculate I_a. Put $a^- = [-a_{\mathrm{mut}}]_+$, taking positive parts coordinatewise. The reachable injective seed has negative-permutation mutable leading degrees. Its appropriate normalized cluster monomial has full leading exponent w with $w_{\mathrm{mut}} = -a^-$. The frozen coordinates of this monomial are included in w. At the reachable negative chart it is the monomial X^w [DM, Corollary 4.13]; at the initial chart it is $H^{-1}X^wH$.

In [Qin2, (6.1)], the standard product puts the positive initial factor on the left and the injective factor on the right. Its initial factor has exponent $a - w$ with mutable part $[a_{\mathrm{mut}}]_+$; arbitrary frozen powers are allowed after localization. Dividing out the leading Weyl scalar gives

$$\begin{aligned} I_a &= v^{-L(a-w,w)} X^{a-w} H^{-1} X^w H \\ &= X^a \sigma_{a^-}(H^{-1}) H. \end{aligned} \tag{6.9}$$

Frozen directions pair trivially with Bd, explaining their absence from σ; they remain in a. Conjugating by F and using $HF^{-1} = T'$ yields

$$\mathrm{Ad}_F(I_a) = X^a \sigma_{-a}(F) \sigma_{a^-}(H^{-1}) T'. \tag{6.10}$$

Let $a = N\widetilde{\nu} + h$, with h fixed. For large N, each coordinate of a_{mut} has the sign of the corresponding coordinate of ν; hence

$$a^- = N[-\nu]_+ + \text{a fixed vector}.$$

Fix $0 \neq d \leq e$. A contribution to the coefficient at $a+Bd$ in (6.10) comes from charges $d_1+d_2+d_3 = d$ in F, H^{-1}, T'. The two σ-factors contribute

$$4a_{\mathrm{mut}} \cdot d_1 - 4a^- \cdot d_2$$

to the parameter exponent. Converting X^aY^d to the based monomial X^{a+Bd} contributes $-2a_{\mathrm{mut}} \cdot d$. The internal products of the three charge monomials have phases independent of N. The coefficient of N in the total exponent is therefore

$$2(\nu \cdot d_1 - |\nu| \cdot d_2 - \nu \cdot d_3). \tag{6.11}$$

Here $2[-\nu]_+ + \nu = |\nu|$ coordinatewise.

If $d_1 \neq 0$, its contribution is strictly negative by (6.7). If $d_2 \neq 0$, its contribution is strictly negative since all $\nu_i \neq 0$. If $d_3 \neq 0$, use (6.8). At least one charge is nonzero, so every nonzero contribution has strictly negative slope in N.

For a rational function, let $\deg_\infty$ be numerator degree minus denominator degree, with its rational-valued version after root extension, and put $\deg_\infty 0 = -\infty$. There are finitely many boxed decompositions $d = d_1 + d_2 + d_3$. Degrees of the rational coefficients of F, H^{-1}, T', their internal phases and the fixed offset contribute a bounded amount independent of N. For each of the finitely many nonzero summands, its degree has the form $\beta_s - N\lambda_s$ with $\lambda_s > 0$. Take N larger than all β_s/λ_s and the coordinate-sign thresholds above. Then

$$\deg_\infty [X^{a+Bd}] \operatorname{Ad}_F(I_a) < 0 \quad (0 \neq d \leq e) \tag{6.12}$$

for all sufficiently large N. This bound is uniform for finitely many offsets, including their translates $h + Bj$ with $j \leq e$.

Apply Ad_F to (6.3). Only $j \leq d$ can contribute at $a + Bd$. For $j = 0$, use (6.12). For $0 < j < d$, use the same estimate with offset $h + Bj$ and residual charge $d - j$, together with $\deg_\infty c_{a,j} \leq -1$. For $j = d$, the leading coefficient of the transported standard element is one, and $c_{a,d}$ itself has degree at most -1. These bounds suffice even though the triangular coefficients can depend on N. Degree of a finite sum does not exceed the maximum degree of its terms. Every nonleading boxed coefficient of $\operatorname{Ad}_F(b[a])$ thus has strictly negative degree for large N.

Lemma 6.1 says that this coefficient is a bar-invariant Laurent polynomial. If nonzero, its largest and smallest exponents would be negatives of one another. Its largest exponent could not be strictly negative. The coefficient therefore vanishes exactly. Constant-one transports and nonnegative charges preserve the leading coefficient one.

Finally, specializing these Laurent coefficients at $v = 1$ gives the classical scattering expansion by [DM, Proposition 4.8]. This proves the classical assertion and the uniform bound for the finite set $\mathscr{H}$. □

Remark 6.3. Vectors satisfying (6.4) exist in $\mathscr{K}_R$: remove the finitely many coordinate and charge hyperplanes, choose a rational point in the remaining open set, and multiply by a positive integer. The reference point and the bound on N are chosen for the prescribed box.

7. Elimination in a finite word category

Lemma 7.1. *Suppose an element pointed at g has a finite expansion in basis elements with distinct full leading exponents, each supported in its own translate of $B\mathbb{N}^n$. Every basis element occurring in this expansion has its leading exponent in $g + B\mathbb{N}^n$, and the coefficient at g is one.*

In particular, if two such basis elements have leading exponents g and h, then their product at $v = 1$ has coefficient one at the basis element with leading exponent $g + h$.

Proof. Within a coset of $B\mathbb{Z}^n$, put $u \preceq w$ if $w - u \in B\mathbb{N}^n$. This is a partial order since B is injective. If the finite expansion had a nonzero term with leading exponent outside $g + B\mathbb{N}^n$, choose a minimal such exponent. An exponent inside the cone cannot contribute to it. Another outside exponent could contribute only if it were strictly smaller, contradicting minimality. Different cosets cannot interact. Its nonzero leading term would survive in the pointed element, a contradiction. At g, only the basis element with that leading exponent can contribute, and both leading coefficients are one.

At $v = 1$, the product of two pointed elements is pointed at the sum of their leading exponents, with coefficient one. Apply the first assertion to its finite basis expansion. □

Theorem 7.2. *Every rigid object M in every finite word category satisfies $\rho_M = b_M$, where b_M is the specialized upper global basis element with the same crystal label.*

Proof. If $n = 0$, the stable initial cluster-tilting object is zero. Its cluster-tilting property forces $\underline{\mathcal{C}_\sigma} = 0$. Every object of $\mathcal{C}_\sigma$ is thus a sum of projective-injectives. Its character is a frozen cluster

monomial by (5.3), and hence is the corresponding upper basis element. Assume from now on that $n > 0$.

By Lemma 4.3, ρ_M has a finite expansion in the nonlocalized ambient upper basis:

$$\rho_M = \sum_c m_{M,c} b_c. \tag{7.1}$$

After frozen localization, these basis elements lie in the common triangular basis and have distinct full leading exponents. The character formula is pointed at g_M. Lemma 7.1 therefore gives

$$\rho_M = b[g_M] + \sum_{0 \neq e \in \mathbb{N}^n} m_{M,e}\, b[g_M + Be], \tag{7.2}$$

with finite support.

Suppose that $m_{M,e} \neq 0$ for some $e \neq 0$. Complete M to the rigid object R of Lemma 5.2. Its mutable indices form a basis, so their positive cone is full dimensional. Avoiding the finitely many hyperplanes in (6.4) for the box $[0, e]$, choose rational positive coefficients and multiply by a common positive integer. We obtain a direct sum

$$P = \bigoplus_i R_i^{p_i} \ \oplus \ \text{projective-injective summands}, \qquad p_i \in \mathbb{Z}_{>0},$$

such that $\nu = (g_P)_{\mathrm{mut}}$ satisfies those conditions. Add positive multiplicities of all projective-injective summands needed by M; this changes only the frozen coordinates of g_P.

For all sufficiently large integers N, there is a complement Y with

$$P^{\oplus N} \cong M \oplus Y. \tag{7.3}$$

All three objects are rigid and belong to $\operatorname{add} R$, and $g_Y = Ng_P - g_M$. Choose a generic θ as in Lemma 6.2 for this single box. Writing Φ_θ for classical transport from the positive chart, Lemma 5.2 gives

$$\Phi_\theta(\rho_{P^{\oplus N}}) = x^{Ng_P}. \tag{7.4}$$

For large N, the finite-box lemma gives exactly the same boxed coefficients for $b[Ng_P]$.

We claim that the upper basis row of $P^{\oplus N}$ has zero coefficient at every nonzero offset $d \leq e$. Its finite expansion has the cone form (7.2). Transport keeps the leading coefficient of every basis element equal to one, and its corrections have nonnegative charge. Compare the coefficient at $Ng_P + Bd$, using an order extending the coordinatewise partial order on $[0, e]$. The leading element $b[Ng_P]$ contributes zero there by the finite-box lemma. Any other element that contributes has offset at most d. Those at strictly smaller nonzero offsets have already been eliminated. The sole remaining contribution is the row coefficient at offset d, with leading multiplier one. Equation (7.4) forces it to vanish. Offsets outside or incomparable with the box cannot enter this comparison. Induction gives

$$m_{P^{\oplus N},d} = 0 \qquad (0 \neq d \leq e). \tag{7.5}$$

For the absolute rows defined in Section 3, Proposition 3.5 and the direct-sum decomposition (7.3) give

$$a_{P^{\oplus N}} = a_M a_Y. \tag{7.6}$$

All row coefficients in this identity are nonnegative. The specialized upper-basis structure constants are nonnegative too: they are sums of the restriction multiplicities in Lusztig's construction.

The offending element $b[g_M + Be]$ has a positive coefficient in a_M. The leading element $b[g_Y]$ has coefficient one in a_Y, by Lemma 7.1. Both factors belong to the nonlocalized ambient upper basis, since they occur in the finite rows of M and Y. Their product lies in $\mathcal{A}_\sigma$ and has a finite

ambient upper-basis expansion. After localization, Lemma 7.1 applied to this finite expansion gives coefficient one at the leading upper element, whose full exponent is

$$g_M + Be + g_Y = Ng_P + Be.$$

Consequently $b[Ng_P + Be]$ is itself an ambient nonlocalized upper element. The nonnegative ambient structure constants apply, so the right side of (7.6) has a strictly positive coefficient at that exponent. On the left, that coefficient is $|m_{P^{\oplus N},e}| = 0$ by (7.5), a contradiction.

It follows that $\rho_M = b[g_M]$. The same-label unit diagonal in Lemma 2.2 identifies this single surviving element with the upper global basis element carrying the crystal label of the component containing M. □

8. The global theorem and its consequences

8.1. **Proof of the open orbit conjecture.**

Proof of Theorem 1.1. Choose X in the open orbit of Z. It is rigid by Lemma 2.1. Proposition 4.4 gives a rigid quotient $Y \in \mathcal{C}_\sigma$ and a bijection between the signed upper basis coefficients of φ_X and φ_Y. Theorem 7.2 makes φ_Y a single upper basis element with coefficient 1. The coefficient bijection gives the same statement for $\varphi_X = \rho_Z$. Lemma 2.2 identifies that element as b_Z. For a disconnected graph, the module varieties are products and both bases are tensor products, so the result holds componentwise. □

8.2. **Characteristic cycle rows in type A.** Let Q be an equioriented type A quiver and put $E = E_{\mathbf{d}}$. Its $G_{\mathbf{d}}$-orbits are finite in number. The conormal closures $Z_S = \overline{T^*_S E}$ are the components of $\Lambda_{\mathbf{d}}$. The stabilizer of a representation is the group of units of its endomorphism algebra, a nonempty open subset of a complex vector space. It is connected, so the irreducible equivariant local system on each orbit is the constant rank-one system. The Lusztig simple perverse sheaves are therefore IC_D, and their labels are Z_D [Lus91].

For fixed S, D, write

$$n_{S,D} = [Z_S]\,\mathrm{CC}(\mathrm{IC}_D).$$

Perverse normalization makes these multiplicities nonnegative [DX, Proposition 2.5(7)]. The principal component has multiplicity $n_{S,S} = 1$, the rank of the constant local system.

Proposition 8.1. *If Z_S has an open $G_{\mathbf{d}}$-orbit and*

$$\rho_{Z_S} = \sum_D m_{S,D} b_{Z_D},$$

then $|m_{S,D}| = n_{S,D}$ for every D.

Proof. Let $\widehat{S}$ be the dual orbit in E^*, characterized by $Z_S = \overline{T^*_{\widehat{S}} E^*}$ under $T^*E = E \times E^* = T^*E^*$. The regular conormal locus is

$$(T^*_S E)_{\mathrm{reg}} = T^*_S E \setminus \bigcup_{S' \neq S} Z_{S'}.$$

It is nonempty, invariant and open in Z_S, and is contained in $S \times \widehat{S}$ [DX, §4, equation (4)]. Its intersection with the open orbit is nonempty and invariant, so that entire orbit lies in this locus. Choose $X = (x, y)$ on it.

Since $y \in \widehat{S}$, the conormal fiber has dimension $s_X = \dim L_y = \dim E - \dim \widehat{S}$. The local formula of [DX, Proposition 4.6] gives

$$\chi((\phi_{f_y}\,\mathrm{IC}_D)_x) = (-1)^{\dim E - \dim \widehat{S}} n_{S,D}. \tag{8.1}$$

By Lemma 3.4, the left side is $(-1)^{s_X}|m_{S,D}|$. The exponents agree, hence $|m_{S,D}| = n_{S,D}$. □

Proof of Theorem 1.2 in type A. Theorem 1.1 gives $m_{S,D} = \delta_{S,D}$. Proposition 8.1 gives the characteristic cycle multiplicities. For a finite product of type A spaces, orbits and conormal bundles are products, intersection cohomology complexes are external products, and characteristic cycle multiplicities multiply [DX, Proposition 2.5(1)]. Orbit dimensions add, so an orbit open in the product conormal projects to an open orbit in every factor. Applying the result to each factor proves the product assertion. □

8.3. **The GL_N Vogan variety.** Let F be a nonarchimedean local field of characteristic zero, let W_F be its Weil group, and write $|\cdot| : W_F \to \mathbb{R}_{>0}$ for the absolute value character. For a semisimple infinitesimal parameter $\eta : W_F \to \mathrm{GL}(U)$ with $\dim U = N$, the Vogan variety and its group are

$$V_\eta = \{z \in \mathrm{End}_{\mathbb{C}}(U) : \eta(w) z \eta(w)^{-1} = |w| z \text{ for all } w\}, \qquad H_\eta = Z_{\mathrm{GL}(U)}(\eta).$$

Decompose the semisimple representation as

$$U = \bigoplus_\tau \tau \otimes U_\tau,$$

where the τ are pairwise nonisomorphic irreducible Weil representations. Schur's lemma gives

$$H_\eta \simeq \prod_\tau \mathrm{GL}(U_\tau), \qquad V_\eta \simeq \bigoplus_\tau \mathrm{Hom}(U_\tau, U_{\tau|\cdot|}). \tag{8.2}$$

Only the finitely many nonzero multiplicity spaces occur.

If $\tau \simeq \tau|\cdot|^k$, determinants give $|\cdot|^{k \dim \tau} = 1$. The absolute value of a Frobenius element differs from 1, so $k = 0$. Every finite support in (8.2) is therefore a disjoint union of finite equioriented type A lines, with zero multiplicity gaps separating components. On each twist line the cotangent trace pairing is the usual multiplicity-space pairing times the nonzero constant $\dim \tau$. Rescaling the reverse coordinates identifies the conormal geometry with the type A geometry.

Completion of the proof of Theorem 1.2. The isomorphism (8.2) respects the group action and conormal varieties, so the product result proves the characteristic cycle assertion. The geometric local Langlands correspondence labels the irreducible representations with infinitesimal parameter η by the complexes IC_D; write π_D for the representation with this label. Let ϕ_S be the Langlands parameter corresponding to S. The ABV occurrence criterion is

$$\Pi^{\mathrm{ABV}}_{\phi_S} = \{\pi_D : [Z_S]\,\mathrm{CC}(\mathrm{IC}_D) \neq 0\};$$

see [Rid1, §2.4] and [CFK, proof of Proposition 3.2.1]. The characteristic cycle formula makes this set $\{\pi_S\}$, proving [Rid1, Conjecture 2.4.7]. □

8.4. **Rigid products and real simple modules.** For a rigid module X, let c_X be the crystal label of the component $\overline{G_{\dim X} X}$. Let $G_q^{\mathrm{up}}(c)$ denote the integral upper global basis in the standard quantum-group parameter q. We use a quiver Hecke algebra over a field k of characteristic zero with symmetric defining polynomials: $Q_{ii} = 0$, and for $i \neq j$, $Q_{ij}(u, w)$ is a nonzero homogeneous polynomial of degree $-c_{ij}$ in $u - w$, satisfying $Q_{ij}(u, w) = Q_{ji}(w, u)$. Let L_c be the finite-dimensional graded self-dual simple with label c, and write $L\langle a\rangle$ for grading shift by a, so that $[L\langle a\rangle] = q^a[L] = v^{2a}[L]$ in the graded Grothendieck group. The upper basis corresponds to these simple classes and multiplication to convolution, denoted $\circ$ [KKKO, Theorems 2.1.2, 2.1.4].

Corollary 8.2. *If X, Y and $X \oplus Y$ are nilpotent rigid preprojective modules, then for some integer a,*

$$G_q^{\mathrm{up}}(c_X) G_q^{\mathrm{up}}(c_Y) = q^a G_q^{\mathrm{up}}(c_{X \oplus Y}), \qquad L_{c_X} \circ L_{c_Y} \simeq L_{c_{X \oplus Y}}\langle a\rangle,$$

In particular L_{c_X} is real for every rigid X: its convolution square is simple.

Proof. By Theorem 1.1 and (2.1),

$$b_{c_X} b_{c_Y} = \rho_X \rho_Y = \rho_{X \oplus Y} = b_{c_{X \oplus Y}}.$$

Write the quantum multiplication as

$$G_q^{\mathrm{up}}(c_X) G_q^{\mathrm{up}}(c_Y) = \sum_c p_c(q) G_q^{\mathrm{up}}(c), \qquad p_c(q) \in \mathbb{N}[q, q^{-1}].$$

The coefficients are graded composition multiplicities in the quiver Hecke category. At $q = 1$ the displayed classical identity says $p_c(1) = 0$ unless $c = c_{X \oplus Y}$, and that the remaining value is 1. Nonnegativity forces every other polynomial to vanish and the remaining polynomial to be q^a for an integer a. Hence the convolution has a single composition factor counted with total multiplicity one, which means it is simple with the indicated integer grading shift. Taking $Y = X$ proves reality, since $X \oplus X$ is rigid whenever X is. □

For type A affine Hecke algebras over $\mathbb{C}$, let the parameter $t \in \mathbb{C}^*$ have infinite multiplicative order. The integral-spectrum category consists of finite-dimensional modules whose commuting Bernstein generators have eigenvalues in $t^{\mathbb{Z}}$. Theorems 12–13 of [LNT] identify simple multisegment classes with the specialized upper basis and parabolic induction with multiplication. Hence a multisegment whose preprojective component has an open orbit gives a simple affine Hecke module whose induction square is simple.

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