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PRODUCTS OF TWO p-ADIC LOGARITHMS AT PRIME POWERS

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ABSTRACT. Let p be a prime, $k \geq 1$ and $x = p^k$. We prove that the p-adic number $\theta_p(x) = \log_p(1+x)\log_p(1-x)$ is irrational whenever $x \geq 79$; for instance $\log_3(82)\log_3(-80)$, $\log_2(129)\log_2(-127)$ and $\log_{79}(80)\log_{79}(-78)$ are irrational. A recent theorem of Kawashima on polylogarithms at algebraic points implies the irrationality of $\theta_p(p^k)$, together with much more, when $p^k > 4.5 \cdot 10^{11}$; the content of the present note is the explicit range, which contains all the small natural members. The proof reads Lysov's Hermite–Padé forms for $\log(1+1/z)\log(1-1/z)$ p-adically: the remainders are p-adically small by their contact order, a content lemma removes the primes p' with $\{n/p'\} \geq \frac{3}{4}$ from the denominators, the archimedean growth is controlled by the characteristic roots of the order-four recurrence satisfied by Lysov's forms, and nonvanishing follows by running the recurrence backwards to $\theta_p(x) \neq 0$. The proof uses the prime number theorem and two small rigorous computer certificates.

1. Introduction

For a prime p let $\log_p$ be the p-adic logarithm, $\log_p(1+u) = \sum_{m\geq 1}(-1)^{m+1}u^m/m$ for $u \in p\mathbb{Z}_p$.

Theorem 1. *Let p be a prime, $k \geq 1$ an integer and $x = p^k$. If $x \geq 79$, then*

$$\theta_p(x) := \log_p(1+x)\,\log_p(1-x) \notin \mathbb{Q}.$$

Since $\theta_p(-x) = \theta_p(x)$, the same holds for $x = -p^k$. The prime powers covered are every prime $p \geq 79$ (with $k = 1$) and $3^4 = 81$, $11^2 = 121$, $5^3 = 125$, $2^7 = 128$, $13^2 = 169$, $3^5 = 243$, $2^8 = 256$, $17^2 = 289$, $7^3 = 343$, and so on; the excluded prime powers are those below 79. For $79 \leq x \leq 98$ the proof uses a content lemma (Lemma 2.4); without it the same argument gives $x \geq 99$ (Remark 6.3).

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1.1. **Prior work and what is new.** *Archimedean products.* Hata [3] proved that $1, \log(1+1/q), \log(1-1/q)$ and $\log(1+1/q)\log(1-1/q)$ are linearly independent over $\mathbb{Q}$ for integers $q \geq 54$. Lysov [8, 9] found explicit type II Hermite–Padé approximants for $(\log(1+1/z), \log(1-1/z), \log(1+1/z)\log(1-1/z))$ and proved $\log(1+1/q)\log(1-1/q) \notin \mathbb{Q}$ for $q \geq 33$. In [7] we lowered this to $q \geq 22$, assuming the holonomy bound [2, Thm. 7.0.1] of Calegari, Dimitrov and Tang (an unrefereed preprint); the present note is the p-adic counterpart, and it uses Lysov's forms in the normalisation and with the lemmas of [7, §3].

p-adic values. Kawashima [5, Cor. 2.3] gives, for a number field K, a place v_0, pairwise distinct $\alpha_1, \dots, \alpha_m \in K^\times$, $\beta \in K$ with $|\beta|_{v_0} > H_{v_0}(\alpha)$ and an integer r, the linear independence over K of 1 and the products $\mathrm{Li}_{s_1}(\alpha_{i_1}/\beta)\cdots\mathrm{Li}_{s_k}(\alpha_{i_k}/\beta)$ with $|s| \leq r$, provided a quantity $V_{v_0}(\alpha,\beta)$ is positive. With $K=\mathbb{Q}$, $v_0 = p$, $m = r = 2$ (so $M = (m+1)^r - 1 = 8$), $\alpha = (1,-1)$ and $\beta = p^{-k}$ one finds $V_p = k\log p - (8\log 2 + 3\log 3 + 18)$. Hence the numbers 1, $\log_p(1\pm x)$, $\mathrm{Li}_2(\pm x)$, $\log_p(1\pm x)^2$, $\theta_p(x)$ are $\mathbb{Q}$-linearly independent, and in particular $\theta_p(x) \notin \mathbb{Q}$, as soon as $x = p^k > e^{26.84} \approx 4.5\cdot 10^{11}$. (This specialisation is ours; [5] states no p-adic example. As far as we know [5] is unrefereed.) Theorem 1 is the explicit range below this bound. The general theorems on p-adic values of G-functions (Bombieri [1], Väänänen and Xu [13]) give lower bounds for polynomials in the values at points ξ with $|\xi|_v < e^{-c\lambda(\log h)^{(4n-1)/4n}(\log\log h)^{1/4n}}$, $h(\xi) \leq h$ [13, Theorem, condition (3)], with constants depending on the system. For $\xi = p^k$ this size condition holds once p^k is large enough, but the resulting bound is not explicit, and we have not checked whether the hypotheses of [13] (a homogeneous system with algebraically independent solutions) cover $\log(1\pm z)$. In any case the large x are covered by [5].

Products of p-adic logarithms. In [6] we proved that $\log_p 2 \cdot \log_p(3+\sqrt{2})$ is not a p-adic linear form in logarithms of algebraic numbers, hence irrational, for every prime $p \equiv \pm 3 \pmod 8$. That argument transposes Diaz's complex-conjugation trick to a local Galois element and needs an argument moved by $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$; it says nothing about products of logarithms of rational numbers. Apart from [5] (and possibly results of the type of [13] for large x), we found no irrationality result for a product of two p-adic logarithms of rational numbers.

1.2. **Method and status.** For $n \geq 0$ let Q_n be Lysov's even polynomial of degree $6n$ and $P_{3,n}$ the polynomial part of $Q_n(z)\log(1+1/z)\log(1-1/z)$ at $z=\infty$. The remainder $Q_n f_3 - P_{3,n}$ vanishes to order $2n+1$ at $z=\infty$. At $z = 1/x$ this expansion converges p-adically, and the integers

$$N_n = \frac{T_n}{d_n}\, x^{6n}\big(a\,Q_n(1/x) - b\,P_{3,n}(1/x)\big) \qquad (\theta_p(x) = a/b)$$

are p-adically of size x^{-8n}, while their archimedean size is x^{6n} times the denominators T_n/d_n times the growth of the recurrence of Lysov's forms [7,

Prop. 3.4] at $w = x^{-2}$. The product formula gives the theorem once

$$M(x) := 6 - \left(3\log 2 - \frac{\pi}{2}\right) + \log \rho(x^{-2}) - 2\log x < 0,$$

where $\rho(w)$ is the largest modulus of a root of the characteristic quartic of that recurrence at w (Proposition 6.1); this holds exactly for integers $x \geq 79$ (Certificate 6.2). Nonvanishing of N_n for infinitely many n comes from the recurrence run backwards (Lemma 5.2).

The external inputs are: the prime number theorem; Lysov's contact orders, the denominator bound and the order-four recurrence, proved in [7, Lemmas 3.1, 3.3, Prop. 3.4] (the recurrence by exact creative-telescoping certificates); and two rigorous computations (Certificates 6.2 and 5.1), whose scripts are in the folder `certificates/` of the repository `https://github.com/telaviv/padic-log-products`. The proof was checked by an adversarial AI referee, and this note by a second one (Section 9); neither replaces a human expert's reading, and nothing is formalised.

2. Lysov's forms

For integers $n \geq 0$ and l put

$$c_{n,l} = \sum_k (-1)^{n+l} \binom{n+l}{l} \binom{n}{l-k} \binom{2n+k}{2k} \binom{n+k}{k} \binom{4n+2k}{2n+k}$$

(sum over $0 \leq k \leq 2n$ with $0 \leq l-k \leq n$), $Q_n(z) = \sum_{l=0}^{3n} c_{n,l} z^{2l} \in \mathbb{Z}[z]$, and $f_3(z) = \log(1+1/z)\log(1-1/z) = \sum_{M \geq 1} t_{2M} z^{-2M}$ with

$$t_{2M} = -\frac{A_{2M-1}}{M}, \qquad A_N = \sum_{i=1}^{N} \frac{(-1)^{i+1}}{i}$$

[7, (3.1)]. Let $P_{3,n}$ be the polynomial part of $Q_n f_3$ (a polynomial in z^2 of degree $\leq 6n-2$ in z) and $R_{3,n} = Q_n f_3 - P_{3,n} = \sum_{m \geq 1} r_{2m} z^{-2m}$, so $r_{2m} = \sum_l c_{n,l} t_{2l+2m}$. Put $T_n = \operatorname{lcm}(1, \ldots, 4n) \cdot \operatorname{lcm}(1, \ldots, 2n)$ and $d_n = \prod\{q \text{ prime} : \{n/q\} \geq \frac{3}{4}\}$; every prime of d_n is $\leq \frac{4}{3}n$, so $d_n \mid T_n$.

Fact 2.1 ([7, Lemma 3.1], after [9, Lemmas 7, 8])**.** *$R_{3,n}(z) = O(z^{-2n-1})$ as $z \to \infty$; that is, $r_{2m} = 0$ for $1 \leq m \leq n$.*

Fact 2.2 ([7, Lemma 3.3])**.** *$T_n P_{3,n} \in \mathbb{Z}[z]$.*

Fact 2.3 ([7, Prop. 3.4])**.** *There are $p_0, \ldots, p_4 \in \mathbb{Z}[n, w]$, each of degree 34 in n, such that for every integer $n \geq 0$ and every $z \in \mathbb{C}$, with $w = z^2$, $\sum_{i=0}^{4} p_i(n, z^2) X_{n+i}(z) = 0$ for $X \in \{Q, P_3\}$. Their leading coefficients in n satisfy $\sum_{j=0}^{4} \operatorname{lc}_n(p_j)(w)\lambda^j = 2^{59}(8w+1)(9w-1)(25w-16)\,\chi_w(-\lambda)$, where*

$$\begin{aligned}\chi_w(\lambda) = \lambda^4 &+ (16384w^3 - 20480w^2 + 5248w - 64)\,\lambda^3 \\ &+ (135168w^2 + 159744w + 1536)\,\lambda^2 \\ &+ (294912w - 16384)\,\lambda + 65536.\end{aligned}$$

In particular $\mathrm{lc}_n(p_4)(w) = 2^{59}(8w+1)(9w-1)(25w-16)$. We write $\rho(w)$ for the largest modulus of a root of χ_w. The explicit p_i are in `certificates/i1_L4_symbolic.json` accompanying this note (identical to the file of that name in the code of [7]).

Lemma 2.4 (content)**.** *For every $n \geq 1$: $d_n \mid c_{n,l}$ for all l, and $T_n P_{3,n} \in d_n\mathbb{Z}[z]$.*

The first assertion is [9, Lemma 9]; we give the proof, which combines it with the proof of Fact 2.2.

Proof. Fix a prime q with $\eta := \{n/q\} \geq \frac{3}{4}$; then $q \leq \frac{4}{3}n$. We use Kummer's theorem: $v_q\binom{A}{B}$ is the number of indices $i \geq 1$ with $\lfloor A/q^i \rfloor - \lfloor B/q^i \rfloor - \lfloor (A-B)/q^i \rfloor = 1$ (each such difference is 0 or 1). Consider a nonzero summand of $c_{n,l}$, indexed by k, and put $\xi = \{k/q\}$.

Level one. If $\eta + \xi \geq 1$, then the level-one difference for $\binom{n+k}{k}$ is 1. Otherwise $\xi < \frac{1}{4}$, so $2\eta + \xi \in [\frac{3}{2}, 2)$ and $\{(2n+k)/q\} = \{2\eta + \xi\} \geq \frac{1}{2}$, whence $\lfloor (4n+2k)/q \rfloor - 2\lfloor (2n+k)/q \rfloor = 1$: the level-one difference for $\binom{4n+2k}{2n+k}$ is 1. In either case the summand has $v_q \geq 1$, so $q \mid c_{n,l}$. This proves $d_n \mid c_{n,l}$.

P_3. By definition $P_{3,n}(z) = \sum_l c_{n,l} \sum_{m=1}^{l} t_{2m} z^{2(l-m)}$, so it suffices to show that each $T_n c_{n,l} A_{2m-1}/m$ with $1 \leq m \leq l \leq 3n$ has $v_q \geq 1$. We have $v_q(A_{2m-1}) \geq -e_2$ and $v_q(1/m) = -e_1$ with $e_1 = v_q(m)$, $e_2 = \lfloor \log_q(2m-1) \rfloor$, and $v_q(T_n) = \lfloor \log_q 4n \rfloor + \lfloor \log_q 2n \rfloor$. As in [7, proof of Lemma 3.3]:

- $e_1 \leq \lfloor \log_q 2n \rfloor + \delta_1$, where $\delta_1 = 1$ if $q^a \mid m$ for some $q^a \in (2n, 3n]$ and $\delta_1 = 0$ otherwise (since $m \leq 3n < 2 \cdot 2n$ at most one such power can occur beyond $2n$);
- $e_2 \leq \lfloor \log_q 4n \rfloor + \delta_2$, where $\delta_2 = 1$ if some $q^b \in (4n, 2m-1]$ and $\delta_2 = 0$ otherwise (since $2m - 1 < 6n < 2 \cdot 4n$).

If $\delta_1 = 1$, then for every nonzero summand $n < q^a$, $k \leq 2n < q^a$ and $n + k \geq l \geq m \geq q^a$, so the level-$a$ difference for $\binom{n+k}{k}$ is 1. If $\delta_2 = 1$, then $2n+k \leq 4n < q^b < 2m \leq 2l \leq 2n+2k$ and $4n+2k \leq 8n < 2q^b$, so the level-b difference for $\binom{4n+2k}{2n+k}$ is 1. Since $q \leq \frac{4}{3}n < 2n$, these levels satisfy $a, b \geq 2$, so they are different from the level-one carry found above. Kummer's count adds over levels and over the binomial factors, so every nonzero summand, hence $c_{n,l}$, has $v_q \geq 1 + \delta_1 + \delta_2$. Therefore

$$\begin{aligned} v_q\big(T_n c_{n,l} A_{2m-1}/m\big) &\geq \lfloor \log_q 4n \rfloor + \lfloor \log_q 2n \rfloor + 1 + \delta_1 + \delta_2 \\ &\quad - \big(\lfloor \log_q 2n \rfloor + \delta_1\big) - \big(\lfloor \log_q 4n \rfloor + \delta_2\big) \\ &= 1. \end{aligned}$$

Together with Fact 2.2 for the primes not dividing d_n, this gives $T_n P_{3,n} \in d_n\mathbb{Z}[z]$. □

Exact checks of Lemma 2.4 for every $1 \le n \le 300$ (the script `content_check.py` of our earlier pilot) and, independently, for $n = 4, 8, \dots, 64$, find no failure.

3. The p-adic remainder

Fix a prime p, $k \ge 1$ and $x = p^k$.

Lemma 3.1. *For every $n \ge 0$ the series $E_n := \sum_{m>n} r_{2m} x^{2m}$ converges in $\mathbb{Q}_p$,*

$$E_n = Q_n(1/x)\,\theta_p(x) - P_{3,n}(1/x),$$

and $v_p(E_n) \ge (2n+2)k - 2\log_p(8n+2)$.

Proof. For $|u|_p < 1$ the two logarithm series converge, and their Cauchy product converges to $\log_p(1+u)\log_p(1-u) = \sum_M t_{2M} u^{2M}$ (its coefficients are those of the formal product). Since $v_q(A_N) \ge -\lfloor \log_q N \rfloor$ for every prime q, we have

$$v_p(t_{2M}) \ge -v_p(M) - \lfloor \log_p(2M-1) \rfloor \ge -2\log_p(2M).$$

The identity $Q_n(z) f_3(z) - P_{3,n}(z) = R_{3,n}(z)$ holds in $\mathbb{Q}((1/z))$ (a Laurent polynomial times a power series in $1/z$). Substituting $1/z = x$ term by term is legitimate in $\mathbb{Q}_p$ because every series involved converges: the terms of $R_{3,n}(1/x)$ have $v_p(r_{2m}x^{2m}) \ge 2mk - 2\log_p(6n+2m) \to \infty$, using $c_{n,l} \in \mathbb{Z}$ and $l \le 3n$. By Fact 2.1, $r_{2m} = 0$ for $m \le n$. For $m \ge n+1$ the bound $2mk - 2\log_p(6n+2m)$ increases with m (its increment is $2k - 2\log_p \frac{6n+2m+2}{6n+2m} \ge 0$), so its minimum is at $m = n+1$, and the ultrametric inequality gives the claim. □

4. Archimedean growth

Lemma 4.1. *Let $w \in \mathbb{C}$ with $(8w+1)(9w-1)(25w-16) \ne 0$. Then every solution (u_n) of $\sum_{i=0}^{4} p_i(n,w) u_{n+i} = 0$ ($n \ge 0$) satisfies $\limsup_n |u_n|^{1/n} \le \rho(w)$.*

Proof. Since every p_i has degree 34 in n and $\mathrm{lc}_n(p_4)(w) \ne 0$, there is n_1 with $p_4(n,w) \ne 0$ for $n \ge n_1$, and $p_i(n,w)/p_4(n,w) \to a_i := \mathrm{lc}_n(p_i)(w)/\mathrm{lc}_n(p_4)(w)$. By Fact 2.3, $\sum_i a_i \lambda^i$ is a nonzero multiple of $\chi_w(-\lambda)$, so its roots have largest modulus $\rho(w)$. With $U_n = (u_n, \dots, u_{n+3})^T$ we have $U_{n+1} = C_n U_n$ for $n \ge n_1$, where the companion matrices C_n converge to the companion matrix C of $\sum_i a_i \lambda^i$, whose spectral radius is $\rho(w)$. For $\varepsilon > 0$ choose a vector norm whose operator norm satisfies $\|C\| \le \rho(w) + \varepsilon$ (from a Jordan basis with rescaled vectors). Then $\|C_n\| \le \rho(w) + 2\varepsilon$ for $n \ge n_2$, hence $\|U_n\| \le K(\rho(w) + 2\varepsilon)^n$. □

By Fact 2.3, $u_n = Q_n(1/x)$ and $u_n = P_{3,n}(1/x)$ are such solutions with $w = x^{-2}$, and $(8w+1)(9w-1)(25w-16) \ne 0$ at $w = x^{-2}$ for every integer $x \ge 4$.

5. Nonvanishing

Certificate 5.1. *$p_0(n,w) \neq 0$ for every integer $n \geq 0$ and every real $w \in (0, 79^{-2}]$.*

Verification (`cert_p0.py`, exact). Write $p_0(n,w) = \sum_{a\leq 34,\, b\leq 3} c_{a,b} n^a w^b$. Put $W_0 = 79^{-2}$, $\ell_b = c_{34,b}$ and $S_a = \sum_b |c_{a,b}| W_0^b$. One checks in exact rational arithmetic that $L := |\ell_0| - \sum_{b\geq 1} |\ell_b| W_0^b > 0$ and $L > \sum_{a<34} S_a/100^{34-a}$; hence $|p_0(n,w)| \geq n^{34} L - \sum_{a<34} n^a S_a > 0$ for all $n \geq 100$ and $w \in [0, W_0]$ (each n^{a-34} decreases in n). For $0 \leq n < 100$, the polynomial $p_0(n,\cdot) \in \mathbb{Z}[w]$, of degree ≤ 3, has no real root in $(0, W_0]$; this is checked by certified root isolation (FLINT/Arb [11, 4]).

Lemma 5.2. *Let $x = p^k \geq 79$. For every $m \geq 1$, at least one of $E_m, E_{m+1}, E_{m+2}, E_{m+3}$ is nonzero.*

Proof. By Fact 2.3 at $z = 1/x$ and Lemma 3.1, $E_n = Q_n(1/x)\theta_p(x) - P_{3,n}(1/x)$ is a $\mathbb{Q}_p$-combination with constant coefficients of two solutions of the recurrence with $w = x^{-2}$, so it is a solution for all $n \geq 0$. Suppose $E_m = \cdots = E_{m+3} = 0$. For $n = m-1, m-2, \ldots, 0$ the recurrence gives $p_0(n,w)E_n = -\sum_{i=1}^{4} p_i(n,w) E_{n+i} = 0$, and $p_0(n, x^{-2}) \neq 0$ by Certificate 5.1; hence $E_0 = 0$. But $Q_0 = 1$ and $P_{3,0} = 0$ (since $f_3 = O(z^{-2})$), so $E_0 = \theta_p(x)$. Finally $\theta_p(x) \neq 0$: the kernel of $\log_p$ on $1 + p\mathbb{Z}_p$ consists of roots of unity ([10, Ch. 5]), the only roots of unity in $\mathbb{Q}$ are ± 1, and $1 + x \geq 80$, $1 - x \leq -78$. □

6. Proof of Theorem 1

Proposition 6.1. *Let $x = p^k$ with $(8w+1)(9w-1)(25w-16) \neq 0$ at $w = x^{-2}$, and suppose*

$$M(x) := 6 - \Big(3\log 2 - \frac{\pi}{2}\Big) + \log \rho(x^{-2}) - 2\log x < 0.$$

If moreover the conclusion of Lemma 5.2 holds at x, then $\theta_p(x) \notin \mathbb{Q}$.

Proof. Suppose $\theta_p(x) = a/b$ with $a, b \in \mathbb{Z}$, $b \neq 0$. By Fact 2.2 and Lemma 2.4, $A_n := T_n x^{6n} Q_n(1/x)$ and $B_n := T_n x^{6n} P_{3,n}(1/x)$ lie in $d_n\mathbb{Z}$ (both are integer combinations of x^{6n-2l}, $l \leq 3n$, with coefficients in $d_n\mathbb{Z}$). So $N_n := (aA_n - bB_n)/d_n \in \mathbb{Z}$, and by Lemma 3.1 $N_n = b\,(T_n/d_n)\, x^{6n} E_n$ in $\mathbb{Q}_p$. By Lemma 5.2, $N_n \neq 0$ for infinitely many n; for those n, $|N_n| \cdot |N_n|_p \geq 1$. On the other hand,

$$|N_n| \leq (|a| + |b|) \frac{T_n}{d_n} x^{6n} \max\big(|Q_n(1/x)|, |P_{3,n}(1/x)|\big),$$

$$|N_n|_p = |b|_p\, |T_n/d_n|_p\, x^{-6n} p^{-v_p(E_n)} \leq x^{-6n} x^{-(2n+2)} (8n+2)^2,$$

using $|b|_p \le 1$, $d_n \mid T_n$ and Lemma 3.1. The factors $x^{\pm 6n}$ cancel:

$$0 \le \log(|a|+|b|) + \log\frac{T_n}{d_n} + \log\max\bigl(|Q_n(1/x)|, |P_{3,n}(1/x)|\bigr) - (2n+2)\log x + 2\log(8n+2).$$

Divide by n and let $n \to \infty$ along these n. By the prime number theorem, $\frac{1}{n}\log T_n = \frac{1}{n}(\psi(4n)+\psi(2n)) \to 6$. A prime $q \in \bigl(\frac{n}{t+1}, \frac{n}{t+3/4}\bigr]$, $t \ge 0$, has $\{n/q\} \ge \frac{3}{4}$; these intervals are disjoint, so for every K

$$\liminf_n \frac{1}{n}\log d_n \ge \sum_{t=0}^{K}\Bigl(\frac{1}{t+3/4} - \frac{1}{t+1}\Bigr)$$

by the prime number theorem at the $2(K+1)$ endpoints, and the right side tends to $\psi(1) - \psi(\frac{3}{4}) = 3\log 2 - \frac{\pi}{2}$ as $K \to \infty$ (here ψ is the digamma function). By Lemma 4.1, $\limsup \frac{1}{n}\log\max(|Q_n(1/x)|, |P_{3,n}(1/x)|) \le \log\rho(x^{-2})$. Hence $0 \le M(x)$, a contradiction. □

Certificate 6.2. *$M(x) < 0$ for every integer $x \ge 79$, and $M(x) > 0$ for every integer $2 \le x \le 78$.*

Verification (`cert_rho.py`). For $2 \le x \le 199$ the integer polynomial $x^6\chi_{1/x^2}$ has its four roots isolated in Arb balls [4]; with $\rho^{\pm}$ the maximum over the roots of the upper (resp. lower) endpoint of $|\lambda|$, one gets $M^+(x) < 0$ for $79 \le x \le 199$ and $M^-(x) > 0$ for $x \le 78$. For $x \ge 200$ put $W = x^{-2} \le 200^{-2}$; bounding each coefficient $a_i(W)$ of χ_W by $\sum_j |a_{i,j}|200^{-2j}$, where $a_i(W) = \sum_j a_{i,j}W^j$, one checks $120^4 > \sum_{i<4}|a_i(W)|120^i$ (the ratio is 0.651), so all roots satisfy $|\lambda| < 120$ by Cauchy's bound, and $M(x) \le 6 - (3\log 2 - \frac{\pi}{2}) + \log 120 - 2\log 200 < -0.31$. Selected values:

x	79	81	83	89	97
$M(x)$	-0.0259	-0.0814	-0.1356	-0.2901	-0.4800
x	121	125	128	169	
$M(x)$	-0.9645	-1.0354	-1.0870	-1.6885	

The exact threshold, the root of M, is $x^* = 78.0868$.

Proof of Theorem 1. For $x = p^k \ge 79$: $(8w+1)(9w-1)(25w-16) \ne 0$ at $w = x^{-2}$, Lemma 5.2 applies, and $M(x) < 0$ by Certificate 6.2. Apply Proposition 6.1. □

Remark 6.3 (without the content lemma). Without Lemma 2.4 one takes $d_n = 1$, and the constant $6 - (3\log 2 - \frac{\pi}{2})$ becomes 6. The same script certifies the resulting budget $M_0(x) = 6 + \log\rho(x^{-2}) - 2\log x$ to be negative for $99 \le x \le 399$ (Arb) and, by the same Cauchy bound, $M_0(x) \le 6 + \log 120 - 2\log 400 < -1.19$ for $x \ge 400$; and $M_0(x) > 0$ for $x \le 98$. So this weaker argument gives $\theta_p(x) \notin \mathbb{Q}$ for $x \ge 99$ (the threshold is 98.27).

7. Remarks

Remark 7.1 (the growth constant). At $w = 0$ the quartic is $\chi_0(\lambda) = (\lambda-16)^4$, and $\partial_w \chi_w(16)|_{w=0} = 2^{26}$. Hence for small w the roots are $\lambda \approx 16 + 2^{13/2} w^{1/4} e^{i\pi/4} i^j$, and $\rho(x^{-2})$ is substantially larger than 16 for the x of interest ($\rho \approx 24.93$ at $x = 81$). Replacing ρ by 16 would suggest the threshold $e^{4.13} \approx 62.3$. Lemma 4.1 is only an upper bound, but numerically the growth of $P_{3,n}(1/x)$ does approach ρ (at $x = 64$ it is 3.16 for $n = 40$ and still rising towards $\log \rho = 3.27$), so that value should not be attainable with these forms; this is a numerical observation.

Remark 7.2 (smaller x: a numerical observation). The following is numerical and is not claimed as a theorem. Deforming the exponents of Lysov's Rodrigues formula,

$$Q(t) = \mathfrak{d}^n\big[(t-1)^n t^n\, \mathfrak{d}^n[t^{\beta n}\mathfrak{D}^{\kappa n}[(t-1)^{\kappa n} t^{\gamma n}]]\big],$$

written in $t = z^2$, with $\mathfrak{d} : t^m \mapsto 2mt^{m-1}$, $\mathfrak{D} : t^m \mapsto (2m-1)t^{m-1}$ and $\beta n, \kappa n, \gamma n \in \mathbb{Z}$, keeps the contact order $2n+1$ for f_3 when $\beta \geq 1$, $\kappa \geq \beta + 1$ and $\gamma \geq \kappa$. We have only a proof sketch of this (the proof of [7, Lemma 3.1] appears to go through), and it is confirmed numerically. Lysov's forms are $(\beta, \kappa, \gamma) = (1, 2, 2)$. Scanning about 150 members in exact arithmetic, the best is $(1.5, 2.5, 3.5)$ (defined for even n). Its denominators are $\operatorname{lcm}(1, \ldots, 7n) \operatorname{lcm}(1, \ldots, 3n)$ (to within $O(\log n)$ in the exponent, measured for $n \leq 100$), and the upper envelope of its archimedean growth, estimated for $n \leq 400$, puts its threshold at $x^* \approx 49.4$. It would add $64, 67, 71, 73$ to Theorem 1. Measuring in nats per unit of excess contact for f_3 beyond the degree (that is, per n; for Lysov's forms this is exactly $M(x)$), it misses 49 by about 0.01, 32 by about 1.2, and 27 by about 1.7. Further families tried (Hata's forms [3], four-layer Rodrigues formulas, random three-layer ones) did no better. Stacking the holonomy bound of [2] on top would require a rank that we could neither establish nor exclude. Asymptotic net exponents in these units (negative is good; the first row is $M(x)$, the second numerical):

(β, κ, γ)	x^*	$x = 27$	32	49	64	67
$(1, 2, 2)$ (Lysov)	78.09	+2.44	+2.04	+1.05	+0.45	+0.34
$(1.5, 2.5, 3.5)$	49.4	+1.67	+1.17	+0.01	−0.52	−0.61

8. What is computer-assisted

- Fact 2.3 is from [7, Prop. 3.4]: exact creative-telescoping certificates and exact operator algebra. The polynomials p_i are shipped as `certificates/i1_L4_symbolic.json`.
- Certificate 5.1 (`certificates/cert_p0.py`): exact rational arithmetic and certified root isolation.
- Certificate 6.2 and Remark 6.3 (`certificates/cert_rho.py`): Arb ball arithmetic at 200 bits with rigorous endpoints, and an exact Cauchy bound.

- Sanity checks, not used in the proof: exact integer computation of N_n's two factors, of Lemma 2.4 and of Fact 2.1 for $n = 4, 8, \dots, 64$ at $x = 64, 79, 81$ (`scripts/exact_budget.py`); Lemma 2.4 for every $n \le 300$ (`content_check.py` of our earlier pilot, not included).

The certificate scripts use python-flint [12, 11] and run in about a second each. The folders `certificates/` and `scripts/` are in the repository `https://github.com/telaviv/padic-log-products`, tag `v1`.

9. Provenance and use of AI tools

Provenance. The p-adic reading of Lysov's forms was suggested by a survey of which general irrationality criteria still had untried targets; that survey's threshold (62.3) used the value 16 for the growth constant and is corrected by Remark 7.1. The external inputs are the prime number theorem and the results of [7, §3], which in turn use Lysov's explicit formula [9]; [5] is used only for comparison.

Use of AI tools. This work was carried out with substantial assistance from AI systems (Anthropic's Claude models), which were used to search the literature, to develop the proofs, to write the scripts and produce the certificates, to referee the arguments, and to draft this text. The referee reports were also produced by Claude models, working adversarially on separate copies of the files; they are not a substitute for a human expert's reading. The author directed the project and is responsible for its content. None of the results is verified in a proof assistant.

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