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Chern-Pure Hyperkähler Manifolds and Tight Spherical Designs

Gerald Höhn

Department of Mathematics, Kansas State University

gerald@monstrous-moonshine.de

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Abstract

Let X be an irreducible compact hyperkähler manifold of complex dimension $2n$, with $b = b_2(X)$. Assume that every Chern class belongs to the subalgebra of $H^*(X, \mathbb{Q})$ generated by $H^2(X, \mathbb{Q})$. We prove that (n, b) is one of nine pairs; in particular, $n \in \{1, 2, 3, 5\}$ and $b \leq 24$. Under this assumption, the Riemann–Roch polynomial is a scalar multiple of the Gegenbauer polynomial $C_n^{b/2}$ after an affine change of variable. The same polynomial determines the inner products between distinct, non-antipodal points of a tight spherical $(2n+1)$-design in $\mathbb{R}^b$. Six of the nine pairs admit such a design, unique up to orthogonal transformation: the cross-polytope in $\mathbb{R}^{22}$ and the designs of 56, 552, 240, 4600, and 196560 points arising from E_8 and the Leech lattice. These are exactly the pairs realized by known Chern-pure hyperkähler manifolds, and exactly those for which the Riemann–Roch polynomial splits over $\mathbb{Q}$.

1 Introduction

Let X be an irreducible compact hyperkähler manifold, and write

$$\dim_{\mathbb{C}} X = 2n, \qquad b = b_2(X).$$

For the general theory, including deformations and period maps, we refer to Huybrechts [Huy99].

We call X *Chern-pure* when every Chern class belongs to the rational subalgebra of cohomology generated by $H^2(X, \mathbb{Q})$. This is condition (P) below. Under this condition, the Riemann–Roch polynomial is a scalar multiple of the Gegenbauer polynomial $C_n^{b/2}$ after an affine change of variable. The same polynomial occurs in the theory of tight spherical $(2n+1)$-designs in $\mathbb{R}^b$. We determine the possible pairs (n, b), identify the tight designs for these pairs, and compare the lattices of the known examples.

Our main result, Theorem 2.1, restricts (n, b) to the following nine pairs:

n	1	2	3	5
b	22	5, 7, 8, 23	5, 8, 23	24.

In particular, the complex dimension is at most ten.

The numerical lists in complex dimensions four and six are already known. Assuming that c_2 belongs to the subalgebra generated by H^2, Guan [Gua01], Main Theorem, part (5), obtained the fourfold list, and Fu–Wang–Wang–Zheng [FWWZ26], Theorem 1.3, obtained the sixfold list. By Lemma 2.3, this assumption is equivalent to (P) in these dimensions. Our proof recovers these cases within a uniform

characteristic-class calculation and settles the higher-dimensional possibilities under (P): it excludes $n = 4$ and $n \geq 6$, leaving only $(n, b) = (5, 24)$ for $n \geq 4$.

Table 1 displays the known realizations, the roots of the common Gegenbauer polynomial, and the corresponding designs.

(n, b)	Known realization	Zeros of $C_n^{b/2}$	Tight $(2n+1)$-design
$(1, 22)$	K3 surface	0	44 vertices of the cross-polytope
$(2, 5)$	open	$\pm 1/\sqrt{7}$	none
$(2, 7)$	Kum_2	$\pm 1/3$	56 minimal vectors of E_7^*
$(2, 8)$	open	$\pm 1/\sqrt{10}$	none
$(2, 23)$	$\mathrm{K3}^{[2]}$	$\pm 1/5$	552 vectors on the 276 equiangular lines
$(3, 5)$	open	$0, \pm 1/\sqrt{3}$	none
$(3, 8)$	OG_6	$0, \pm 1/2$	240 roots of E_8
$(3, 23)$	$\mathrm{K3}^{[3]}$	$0, \pm 1/3$	4600 minimal vectors of the shorter Leech lattice
$(5, 24)$	OG_{10}	$0, \pm 1/4, \pm 1/2$	196560 minimal vectors of the Leech lattice

Table 1: The nine possible parameter pairs under (P). "Open" means that no Chern-pure hyperkähler manifold with these parameters is known. The three design nonexistence statements are proved in Corollary 3.5.

The proof uses the square-root Todd identity, the Fujiki relations, and characteristic-class integrality. A valuation argument bounds n independently of b, and divisibility conditions leave finitely many pairs (n, b) to check by computer. The program is included in the appendix. A Bernoulli-prime obstruction then excludes all remaining cases with $n \geq 6$. The remaining low-dimensional cases follow from index identities and, in complex dimension four, Salamon's relation among Betti numbers [Sal96a, Sal96b].

Tight spherical designs of strength at least four are rare. In the range $4 \leq d \leq 24$, the classical restrictions leave exactly five parameter pairs for tight designs of odd strength at least five, and all five occur in the Chern-purity list under $d = b$ and $t = 2n + 1$. Section 3 begins with these restrictions. Theorem 3.1 gives the comparison: the five exceptional designs and the cross-polytope at $(n, b) = (1, 22)$ account for six of the nine pairs, with a unique configuration in each case. The other three are excluded by quadratic conjugacy of the eigenvalues of an integral Gram matrix. We construct the six designs from classical lattice models, prove the two root-system characterizations, and describe the classical uniqueness results for the other three higher-strength cases.

The common polynomial is the normalized Gegenbauer kernel $C_n^{b/2}(t)/C_n^{b/2}(1)$. On the hyperkähler side it is a centered and normalized Riemann–Roch polynomial. On the sphere, $C_n^{b/2}(t)$ itself is the reproducing kernel of homogeneous degree-n polynomial functions with respect to the normalized surface measure; dividing by $C_n^{b/2}(1)$ normalizes its diagonal to one. The extended Mukai space explains the parameter $b/2$ and the affine change of variable. Section 4 compares the Beauville–Bogomolov lattices of the known manifolds with the integer spans of the designs. It also relates the discriminant forms of the even sublattice of the shorter Leech lattice and the Beauville–Bogomolov lattice of $\mathrm{K3}^{[3]}$.

The roots give a further consequence. Under (P), the Riemann–Roch polynomial has distinct negative real roots in arithmetic progression, confirming [BS26], Conjecture 5.1(1), in this case. It splits over $\mathbb{Q}$ precisely for the six parameter pairs with designs in Table 1.

2 The possible parameters under Chern purity

Let X be an irreducible compact hyperkähler manifold, and put

$$\dim_{\mathbb{C}} X = 2n, \qquad b = b_2(X) \geq 3, \qquad \mathrm{SH}(X, \mathbb{Q}) = \langle H^2(X, \mathbb{Q}) \rangle.$$

We call X *Chern-pure* if

$$c_{2j}(X) \in \mathrm{SH}(X, \mathbb{Q}) \qquad (1 \leq j \leq n). \tag{P}$$

The odd Chern classes vanish. Condition (P) concerns characteristic classes; it does not require all cohomology to be generated by H^2.

Theorem 2.1 (Possible parameters)**.** *If X is Chern-pure, its parameters belong to the following list:*

n	1	2	3	5
$b_2(X)$	22	5, 7, 8, 23	5, 8, 23	24.

In particular, $\dim_{\mathbb{C}} X \leq 10$ *and* $b_2(X) \leq 24$.

2.1 The characteristic-class formula

Write $q(h) = B(h, h)$ for a positive rational normalization of the Beauville–Bogomolov form, of signature $(3, b-3)$. Let $C > 0$ be defined by

$$\int_X h^{2n} = C q(h)^n.$$

If G is the Gram matrix of B in a basis $x_1, \ldots, x_b$ of $H^2(X, \mathbb{Q})$, set

$$Q = q^\vee = \sum_{i,j} (G^{-1})_{ij} x_i x_j \in H^4(X, \mathbb{Q}).$$

For a deformation-invariant characteristic class $\alpha \in H^{4j}$, write $C(\alpha)$ for its generalized Fujiki constant:

$$\int_X \alpha h^{2n-2j} = C(\alpha) q(h)^{n-j}.$$

Define the positive constant

$$r = \frac{(2n-1) C(c_2)}{24 C}. \tag{1}$$

We use the Fujiki identities and Nieper's square-root Todd identity [Nie03]. In the present normalization, the latter is

$$\int_X \sqrt{\mathrm{td}_X}\, e^h = \left(1 + \frac{q(h)}{2r}\right)^n \int_X \sqrt{\mathrm{td}_X}. \tag{2}$$

For this normalization and the contraction formula below, see also [BS26], Proposition 2.4 and Theorem 2.5.

We use $(a)_j = a(a+1)\cdots(a+j-1)$ for a rising factorial and $a^{\underline{j}} = a(a-1)\cdots(a-j+1)$ for a falling factorial. The Gegenbauer polynomials are normalized by

$$C_n^\lambda(1) = \frac{(2\lambda)_n}{n!}.$$

Proposition 2.2. *Assume* (P)*, and put*

$$a = \frac{b}{2} + n - 1, \qquad u = \frac{r}{2}Q, \qquad S(U) = \sum_{j=0}^{n} \frac{U^j}{j!a^{\underline{j}}}.$$

Then

$$\sqrt{\mathrm{td}_X} = S(u), \tag{3}$$

$$c_2 = \frac{24r}{b+2n-2}Q = \frac{24}{a}u, \tag{4}$$

$$\mathrm{RR}_X(z) = (n+1)\frac{C_n^{b/2}(1+z/(4r))}{C_n^{b/2}(1)}, \tag{5}$$

$$\int_X u^n = \frac{(n+1)n!(b/2)_n^2}{(b)_n}. \tag{6}$$

Here $\mathrm{RR}_X(q(c_1(L))) = \chi(X, L)$ *for every line bundle* L*. Every Chern number is determined by* (n, b)*.*

Proof. Characteristic classes remain Hodge classes on every small deformation. Within $\mathrm{SH}^{4j}(X, \mathbb{Q})$, the classes that remain of type $(2j, 2j)$ on all small deformations form the line generated by Q^j; see [Bec23], Lemma 3.2. Thus (P) makes every component of $\sqrt{\mathrm{td}_X}$ a multiple of a power of Q.

Polarizing the Fujiki identity and contracting with the inverse form gives

$$C(Q^j) = C\prod_{i=1}^{j}\frac{b+2n-2i}{2n+1-2i}. \tag{7}$$

Comparison of the coefficients of $q(h)^{n-j}$ in (2), with (7), gives (3). Its degree-four component is $c_2/24$, proving (4).

For $t_j = [U^j]S(U)^2$, Vandermonde convolution gives

$$t_j = \frac{(b+2n-2j)_j}{j!(b/2+n-j)_j^2}, \qquad t_n = \frac{(b)_n}{n!(b/2)_n^2}. \tag{8}$$

Since $\int_X \mathrm{td}_X = \chi(X, \mathcal{O}_X) = n+1$, we have $\int_X u^n = (n+1)/t_n$, proving (6). Substitution in Hirzebruch–Riemann–Roch yields

$$\mathrm{RR}_X(z) = \frac{(n+1)n!}{(b)_n}\sum_{k=0}^{n}\frac{(b/2)_k(b+2k)_{n-k}}{k!(n-k)!}\left(\frac{z}{2r}\right)^k.$$

The Taylor formula

$$\frac{d^k}{dt^k}C_n^{\lambda}(t) = 2^k(\lambda)_k C_{n-k}^{\lambda+k}(t)$$

identifies this expression with (5).

Let $x_1, -x_1, \ldots, x_n, -x_n$ be the formal Chern roots and put $p_j = \sum_i x_i^{2j}$. The characteristic power series gives

$$\log\sqrt{\mathrm{td}_X} = -\sum_{j=1}^{n}\frac{B_{2j}}{2j(2j)!}p_j, \tag{9}$$

where B_{2j} denotes a Bernoulli number. Every coefficient is nonzero. Thus (3) determines all the p_j, and Newton's identities determine $c_{2j} = \gamma_j(n, b)u^j$. Equation (6) then determines every top Chern monomial without dependence on r. □

Why the Gegenbauer parameter is $b/2$. The extended Mukai space explains both the Gegenbauer parameter and the affine shift in (5). Let

$$E = \mathbb{Q}\alpha \oplus H^2(X,\mathbb{Q}) \oplus \mathbb{Q}\beta, \qquad (\alpha,\alpha) = (\beta,\beta) = 0, \quad (\alpha,\beta) = -1,$$

with H^2 orthogonal to α and β and carrying B. The extended Mukai space E has dimension $b+2$. The Verbitsky component identifies with the harmonic tensors $\ker(\Delta : \operatorname{Sym}^n E \to \operatorname{Sym}^{n-2} E)$; let π be harmonic projection, with the target of Δ understood as zero when $n = 1$. For $h \in H^2(X,\mathbb{Q})$, put

$$v_0 = \alpha + r\beta, \qquad v_h = \alpha + h + \bigl(r + q(h)/2\bigr)\beta.$$

Then

$$(v_0,v_0) = (v_h,v_h) = -2r, \qquad \frac{(v_0,v_h)}{(v_0,v_0)} = 1 + \frac{q(h)}{4r}. \tag{10}$$

By [Bec23], Section 2.2, Proposition 3.4 and equation (4.3), the projections of the Mukai vectors $\sqrt{\mathrm{td}_X}$ and $e^h\sqrt{\mathrm{td}_X}$ correspond to $\pi(v_0^n/n!)$ and $\pi(v_h^n/n!)$. The cited line-bundle formula extends to every h: cup product with h acts on E by $\alpha \mapsto h$, $x \mapsto B(h,x)\beta$ for $x \in H^2$, and $\beta \mapsto 0$; its exponential sends v_0 to v_h. Under (P) both Mukai vectors already belong to SH. For the induced invariant tensor pairing, the harmonic addition identity gives

$$\frac{\langle \pi(v_0^n), \pi(v_h^n)\rangle}{\langle \pi(v_0^n), \pi(v_0^n)\rangle} = \frac{C_n^{b/2}\bigl((v_0,v_h)/(v_0,v_0)\bigr)}{C_n^{b/2}(1)}.$$

The left side is $\mathrm{RR}_X(q(h))/(n+1)$: the cohomological Mukai pairing with $\sqrt{\mathrm{td}_X}$ is $\int_X e^h \,\mathrm{td}_X$, and its self-pairing is $\chi(X,\mathcal{O}_X) = n+1$. The parameter is $(\dim E - 2)/2 = b/2$. The addition identity is algebraic and therefore holds for the indefinite form on E.

Lemma 2.3 (A single-component criterion)**.** *Index the square-root Todd components by cohomological degree:*

$$T_j = (\sqrt{\mathrm{td}_X})_{4j}, \qquad m = \lfloor n/2 \rfloor.$$

Then condition (P) *is equivalent to* $T_m \in \mathrm{SH}^{4m}(X,\mathbb{Q})$.

Proof. For a Kähler class ω, write L_ω and Λ_ω for its Lefschetz operators. Jiang's identity ([Jia23], Corollary 3.20) is

$$\Lambda_\omega T_j = \lambda(\omega)^{-1} L_\omega T_{j-1}, \qquad \lambda(\omega) > 0.$$

The algebra $\mathrm{SH}(X,\mathbb{R})$ is a submodule for the semisimple Looijenga–Lunts–Verbitsky Lie algebra. Choose an invariant complement and write ε_j for the component of T_j in it. Projection commutes with the Lefschetz operators, so the same identity holds for the ε_j.

Suppose $\varepsilon_m = 0$. For $j = m, m-1, \ldots, 1$, L_ω is injective on $H^{4j-4}(X,\mathbb{R})$ because $4j - 4 < 2n$. Descending induction gives $\varepsilon_{j-1} = 0$. For $j = m+1, \ldots, n$, Λ_ω is injective on $H^{4j}(X,\mathbb{R})$ because $4j > 2n$. Ascending induction gives $\varepsilon_j = 0$. Thus all T_j belong to SH. Taking the logarithm, using the nonzero Bernoulli coefficients in (9), and applying Newton's identities recovers every Chern class in SH. Since $\mathrm{SH}(X,\mathbb{R}) \cap H^*(X,\mathbb{Q}) = \mathrm{SH}(X,\mathbb{Q})$, the conclusion also holds over $\mathbb{Q}$. The converse follows because the Todd components are polynomials in the Chern classes. □

Explicitly,

$$T_1 = \frac{c_2}{24}, \qquad T_2 = \frac{7c_2^2 - 4c_4}{5760}.$$

Thus $c_2 \in \mathrm{SH}^4$ suffices in complex dimensions four and six; in dimensions eight and ten the single test is $7c_2^2 - 4c_4 \in \mathrm{SH}^8$. The middle-component criterion is also noted in [Jia23], Remark 3.21(2). The descent

in our proof requires injectivity of L_ω on $H^{4j-4}(X,\mathbb{R})$, which the hard Lefschetz theorem guarantees when $4j-4<2n$. The unrestricted descent asserted in [BS26], Proposition 2.8, requires a further argument outside this range. We use the middle-component criterion proved above.

For later use, write $\ell_j=[U^j]\log S(U)$. Equations (9) and (8) give the tangent-bundle index

$$F_n(b):=\chi(X,T_X)=(n+1)\left(2n-\sum_{j=1}^{n}\frac{4j}{B_{2j}}\ell_j\frac{t_{n-j}}{t_n}\right). \tag{11}$$

2.2 An absolute cutoff from $\int_X c_2^n$

Set

$$d=b+2n-2\geq 2n+1,\qquad K_n=(n+1)!12^n.$$

From (4) and (6),

$$\begin{aligned} I_n(b):=\int_X c_2^n &= \frac{K_n\prod_{j=0}^{n-1}(b+2j)^2}{d^n\prod_{j=0}^{n-1}(b+j)}\\ &= \frac{K_n\prod_{k=1}^{n-1}(d-2k)^2}{d^{n-2}\prod_{j=0}^{n-1}(b+j)}. \end{aligned} \tag{12}$$

Consequently, for $n\geq 3$, its integrality implies

$$d^{n-2}\mid K_n\prod_{k=1}^{n-1}(d-2k)^2. \tag{13}$$

Lemma 2.4. *For $n\geq 3$, $d\geq 2n+1$, and every prime p,*

$$v_p\left(\prod_{k=1}^{n-1}(d-2k)\right)\leq (n-2)v_p(2)+v_p((n-2)!)+\lfloor\log_p(d-2)\rfloor.$$

Proof. Choose k_0 maximizing $v_p(d-2k_0)$. For $k\neq k_0$, the ultrametric inequality gives $v_p(d-2k)\leq v_p(2(k-k_0))$. The product of the differences has valuation at most

$$(n-2)v_p(2)+v_p((k_0-1)!(n-1-k_0)!)\leq (n-2)v_p(2)+v_p((n-2)!).$$

The removed factor is a positive integer at most $d-2$. □

Proposition 2.5. *If $n\geq 3$, $b\geq 3$, and $I_n(b)$ is integral, then $n\leq 575$.*

Proof. For $p\mid d$, reducing (13) modulo p shows that p divides $K_n2^{2n-2}((n-1)!)^2$. Hence $p\leq n+1$. Writing $e_p=v_p(d)$, Lemma 2.4 gives

$$(n-2)e_p\leq v_p(K_n)+2(n-2)v_p(2)+2v_p((n-2)!)+2\log_p d. \tag{14}$$

Multiplication by $\log p$ and summation over $p\mid d$ yields

$$(n-2-2\omega(d))\log d\leq \log\big((n+1)!48^n((n-2)!)^2\big),$$

where $\omega(d)$ counts the distinct prime divisors of d. Every prime other than 2 and 3 is congruent to 1 or 5 modulo 6, so $\omega(d)\leq\pi(n+1)\leq(n+1)/3+3$. Therefore, for $n>26$,

$$\log d\leq\frac{3((3n-3)\log(n+1)+n\log 48)}{n-26}.$$

For $n \geq 260$, the coefficients of $\log(n+1)$ and $\log 48$ are at most 10 and $10/3$, respectively. It follows that

$$d \leq 48^{10/3}(n+1)^{10} < 2^{19}(n+1)^{10}. \tag{15}$$

The strict inequality is equivalent to $3^{10} < 2^{17}$.

Suppose now $n \geq 576$, and put $L(n) = 19\log 2 + 10\log(n+1)$. Using $v_p(m!) \leq m/(p-1)$ and (15) in (14), we obtain

$$\begin{aligned} p \geq 5: \quad (n-2)e_p &\leq \frac{3n-3}{4} + \frac{2L(n)}{\log 5}, \\ (n-2)e_2 &\leq 7n - 7 + \frac{2L(n)}{\log 2}, \\ (n-2)e_3 &\leq \frac{5n-3}{2} + \frac{2L(n)}{\log 3}. \end{aligned}$$

At $n = 576$, the inequality $577 < 2^{10}$ gives $L(576) < 119\log 2$. The three right sides are respectively less than

$$\frac{1725}{4} + 119 < 574, \qquad 4025 + 238 < 8 \cdot 574, \qquad \frac{2877}{2} + 238 < 3 \cdot 574.$$

These inequalities persist for $n \geq 576$: the derivatives of the logarithmic terms are $20/((n+1)\log p)$, smaller than the respective linear margins $1/4$, 1, and $1/2$. Thus $e_p = 0$ for $p \geq 5$, $e_2 \leq 7$, and $e_3 \leq 2$. Hence $d \mid 2^7 3^2 = 1152$, contradicting $d \geq 2n+1 \geq 1153$. □

2.3 The finite calculation

Lemma 2.6. *For integers $n \geq 3$ and $b \geq 3$, $I_n(b)$ is integral exactly for the pairs in the following table:*

n	b	$d = b + 2n - 2$
3	5, 8, 23	9, 12, 27
4	3	9
5	24	32
7	4	16
21	8	48
22	6	48
23	4	48
31	4	64
45	8	96
46	6	96
47	4	96

Proof. Proposition 2.5 reduces the problem to $3 \leq n \leq 575$. For each prime $p \leq n+1$, put

$$h_p = v_p(2) + \lfloor \log_p(n-1) \rfloor, \qquad R_p = v_p(K_n) + 2(n-1)v_p(2) + 2v_p((n-1)!).$$

We first bound $e_p = v_p(d)$ by

$$E_p = \max\left\{ h_p, \left\lfloor \frac{R_p}{n-2} \right\rfloor \right\}. \tag{16}$$

Indeed, if $e_p > h_p$, then $v_p(d-2k) = v_p(2k)$ for every $1 \leq k < n$, and (13) supplies the second bound. Otherwise the first bound applies. Thus $d \mid D = \prod p^{E_p}$.

Let $A_p = v_p(K_n) + 2(n-2)v_p(2) + 2v_p((n-2)!)$. If $D < 2n+1$, there is no admissible d. Otherwise Lemma 2.4 and $d \le D$ permit the refinement

$$E_p \longleftarrow \min\left\{E_p,\ \left\lfloor \frac{A_p + 2\lfloor \log_p(D-2)\rfloor}{n-2} \right\rfloor\right\}. \tag{17}$$

Recompute D and repeat until the exponents stabilize or $D < 2n+1$. This process terminates, since the nonnegative exponents only decrease. In the stable case, enumerate every divisor $d \mid D$ with $d \ge 2n+1$, set $b = d - 2n + 2$, and test (12) exactly.

All bounds follow from integrality, and every remaining divisor is tested. The program in Appendix A gives the displayed rows. For $6 \le n \le 575$, it tests 3911 final divisors in total; the largest individual list has 361 divisors, at $n = 14$. The printed program checks the survivor table and these counts with assertions. □

2.4 The Bernoulli-prime obstruction

With the formal roots used in (9), Newton's identities are

$$p_j + c_2 p_{j-1} + \cdots + c_{2j-2}p_1 + jc_{2j} = 0.$$

Therefore each p_j is an integral polynomial in the Chern classes; these p_j are Newton polynomials, not Pontryagin classes. In particular,

$$\int_X p_6 c_2^{n-6} \in \mathbb{Z} \qquad (n \ge 6). \tag{18}$$

Write

$$\frac{S'(U)}{S(U)} = \sum_{k=0}^{n-1} g_k U^k \pmod{U^n}.$$

The coefficients of S satisfy $aS' - US'' = S \pmod{U^n}$. Consequently,

$$g_0 = \frac{1}{a}, \qquad (a-k)g_k = \sum_{i=0}^{k-1} g_i g_{k-1-i} \quad (1 \le k \le n-1). \tag{19}$$

Since $g_5 = 6\ell_6$, $B_{12} = -691/2730$, and $c_2 = (48/d)u$, equation (9) gives

$$p_6 = -\frac{2\cdot 12!}{B_{12}} g_5 u^6 = -\frac{2\cdot 12!}{B_{12}} g_5 \left(\frac{d}{48}\right)^6 c_2^6. \tag{20}$$

Proposition 2.7. *There is no Chern-pure irreducible compact hyperkähler manifold with $n \ge 6$.*

Proof. The eight pairs with $n \ge 6$ in Lemma 2.6 have $d \in \{16, 48, 64, 96\}$. The recurrence (19), evaluated modulo 691, gives

d	16	48	64	96
$g_5 \bmod 691$	275	296	647	121.

All denominators in the recurrence through g_5 are 691-adic units for these values of d. Thus g_5 is a 691-adic unit, and the scalar multiplying c_2^6 in (20) has 691-adic valuation -1. Every factorial and linear factor in (12) is a 691-adic unit: here $n+1 \le 48$ and $d \le 96$. Consequently $v_{691}(I_n(b)) = 0$, and

$$v_{691}\left(\int_X p_6 c_2^{n-6}\right) = -1.$$

This contradicts (18). □

2.5 The remaining dimensions

Proof of Theorem 2.1. Proposition 2.7 excludes $n \geq 6$. Lemma 2.6 gives $b \in \{5, 8, 23\}$ for $n = 3$ and $b = 24$ for $n = 5$. For $n = 4$, the only surviving value is $b = 3$. The tangent index (11) specializes to

$$F_4(b) = -\frac{40(5b^4 - 28b^3 + 111b^2 + 468b - 3348)}{(b+1)(b+3)(b+6)^2}, \qquad F_4(3) = \frac{80}{3},$$

contrary to index integrality.

For $n = 2$, Proposition 2.2 and $\int_X \mathrm{td}_X = 3$ give

$$\int_X c_2^2 = \frac{864b}{b+1}, \qquad e(X) = \int_X c_4 = 432 - \frac{2592}{b+1}. \tag{21}$$

Salamon's identity is

$$b_4 = 46 + 10b - b_3;$$

see [Sal96a], Theorem 1 and equation (7). Together with $e(X) = 2 + 2b + b_4 - 2b_3$, it gives

$$b_3 = \frac{4(b-23)(b-8)}{b+1}, \qquad b_4 - \binom{b+1}{2} = -\frac{(b-23)(b^2+13b-60)}{2(b+1)}. \tag{22}$$

Both quantities are nonnegative. For the second, the cup-product map $\mathrm{Sym}^2 H^2(X, \mathbb{Q}) \to H^4(X, \mathbb{Q})$ is injective: polarization of the Fujiki identity induces, up to a nonzero scalar, the form $\mathrm{tr}(A)\,\mathrm{tr}(B) + 2\,\mathrm{tr}(AB)$ on symmetric tensors identified with q-self-adjoint endomorphisms. This form is nondegenerate on the scalar and traceless parts, since $b + 2 \neq 0$. The inequalities in (22) exclude $b = 3$ and $b > 23$, and restrict the remaining possibilities to $b \in \{4, 5, 6, 7, 8, 23\}$. Integrality of $\int_X c_2^2$ is equivalent to $b + 1 \mid 864$, which excludes $b = 4$ and $b = 6$ and leaves $b \in \{5, 7, 8, 23\}$.

Finally, for $n = 1$, the Todd formula and $\chi(X, \mathcal{O}_X) = 2$ give $\int_X c_2 = 24$. Since X is simply connected, $e(X) = 2 + b_2(X)$, and hence $b = 22$. □

2.6 The roots of the Riemann–Roch polynomial

Corollary 2.8. *For every Chern-pure irreducible compact hyperkähler manifold, the roots of* RR_X *are distinct negative real numbers in arithmetic progression. Moreover,* RR_X *splits over* $\mathbb{Q}$ *if and only if*

$$(n, b) \in \{(1, 22),\ (2, 7),\ (2, 23),\ (3, 8),\ (3, 23),\ (5, 24)\}.$$

These are exactly the rows of Table 1 admitting a tight spherical $(2n+1)$*-design in dimension* b*.*

Proof. By Theorem 2.1, there are only the nine parameter pairs in Table 1. Their displayed Gegenbauer roots are distinct, lie in $(-1, 1)$, and form arithmetic progressions (with the usual vacuous interpretation for a singleton). Equation (5) sends a root t to $z = 4r(t-1)$. Since $r \in \mathbb{Q}_{>0}$, the resulting roots are negative and retain both the arithmetic-progression property and rationality. The only irrational root sets occur at $(2, 5)$, $(2, 8)$, $(3, 5)$. The design assertions follow from the constructions in Section 3.3 and Corollary 3.5. □

This proves [BS26], Conjecture 5.1(1), under (P). For $n \leq 3$, symmetry of the Gegenbauer roots already gives an arithmetic progression; the remaining case in the parameter list is $(n, b) = (5, 24)$. In the primitive integral Beauville–Bogomolov normalization, part (2) of that conjecture predicts negative even integer roots with successive difference two, which is stronger than the rationality statement above.

3 Tight spherical designs and the Chern-purity parameters

Let $d \geq 3$, and give S^{d-1} surface measure of total mass one. A finite nonempty subset $D \subset S^{d-1}$ is a *spherical t-design* if

$$\frac{1}{|D|} \sum_{x \in D} f(x) = \int_{S^{d-1}} f \, d\sigma \qquad (\deg f \leq t).$$

It is called *tight* if it attains the corresponding Fisher bound

$$|D| \geq \begin{cases} \binom{d+e-1}{e} + \binom{d+e-2}{e-1}, & t = 2e, \\ 2\binom{d+e-1}{e}, & t = 2e+1; \end{cases}$$

see [DGS77], Theorems 5.11 and 5.12. For odd strength, equality forces antipodality: every tight $(2e+1)$-design satisfies $D = -D$. We prove the odd bound and this equality statement in Theorem 3.2.

Tightness imposes severe restrictions. By the Bannai–Damerell theorem, for $d \geq 3$ the only possible strengths are

$$t \in \{1, 2, 3, 4, 5, 7, 11\},$$

and strength eleven requires $d = 24$; see [BD79, BD80] and [BB09], Theorem 2.16. Strengths one, two, and three give the elementary families of antipodal pairs, regular simplices, and cross-polytopes. For odd strength at least five, the dimension restrictions are as follows, with a and q integers:

t	Necessary dimension d	If $d \leq 24$
5	3 or $(2a+1)^2 - 2, \quad a \geq 1$	3, 7, 23
7	$3q^2 - 4, \quad q \geq 2$	8, 23
11	24	24.

See [NV12], Section 1, and [BB09], Section 2.4. Every entry in the last column is realized. The dimension-three case is the regular icosahedron; the other five are the E_8 and Leech configurations in Table 1, whose constructions and uniqueness are discussed below. The classification at strengths five and seven remains open in higher dimensions, with further arithmetic exclusions in [BMV04, NV12, Xu26]. Existence of a tight 4-design in dimension d is equivalent to existence of a tight 5-design in dimension $d+1$. The known strength-four examples for $d \geq 3$ have dimensions 6 and 22; see [BB09], Section 2.4.

The bound $b \leq 24$ in Theorem 2.1 puts the Chern-purity problem entirely within the dimension range of the table. Under $d = b$ and $t = 2n+1$, all five entries with $d \geq 4$ in the last column occur in the Chern-purity list. The icosahedron would give $(n, b) = (2, 3)$, which that list excludes. The additional design pair $(1, 22)$ belongs to the cross-polytope family. The following theorem compares the nine Chern-purity pairs with the design parameters.

Theorem 3.1 (Tight designs for the Chern-purity parameters)**.** *Let (n, b) be one of the nine pairs in Theorem 2.1. A tight spherical $(2n+1)$-design in $\mathbb{R}^b$ exists if and only if*

$$(n, b) \in \{(1, 22), (2, 7), (2, 23), (3, 8), (3, 23), (5, 24)\}.$$

For each of these six pairs, the design is unique up to orthogonal transformation. Its model and cardinality are given in Table 1.

We use d and e in the general kernel argument, and specialize to $d = b$ and $e = n$ for the Chern-purity parameters.

3.1 The Fisher bound and the common Gegenbauer kernel

Let $\mathcal{H}_j$ be the space of spherical harmonics of degree j, and put

$$h_j = \dim \mathcal{H}_j = \binom{d+j-1}{j} - \binom{d+j-3}{j-2}, \qquad Q_j(t) = \frac{d+2j-2}{d-2} C_j^{(d-2)/2}(t).$$

A binomial coefficient with negative lower index is zero. The addition formula identifies $Q_j(\langle u, v\rangle)$ with the reproducing kernel of $\mathcal{H}_j$, so $Q_j(1) = h_j$. The restrictions of homogeneous degree-e polynomials form the direct sum $\mathcal{H}_e \oplus \mathcal{H}_{e-2} \oplus \cdots$. Its reproducing kernel is

$$K_e(t) = \sum_{j=0}^{\lfloor e/2 \rfloor} Q_{e-2j}(t) = C_e^{d/2}(t), \qquad M_e := K_e(1) = \binom{d+e-1}{e}. \tag{23}$$

The identity follows by telescoping the usual Gegenbauer identities. The parameter $d/2$ therefore comes from the sum of all harmonics of the appropriate parity, rather than from $\mathcal{H}_e$ alone. This is the same parameter that appears in the normalized Riemann–Roch polynomial under (P).

Theorem 3.2 (Odd Fisher bound and its equality case)**.** *A spherical $(2e+1)$-design has at least $2M_e$ points. A design attaining this bound is antipodal, and its inner products between distinct points belong to*

$$\{-1\} \cup \{t : C_e^{d/2}(t) = 0\}.$$

Proof. Fix $u \in D$ and consider the nonnegative polynomial on the sphere

$$F_u(x) = (1 + \langle u, x\rangle) K_e(\langle u, x\rangle)^2.$$

Its degree is $2e+1$. Its integral is M_e, by parity and the reproducing identity, whereas its value at u is $2M_e^2$. The design identity gives $|D| \geq 2M_e$. Equality forces $(1+\alpha)K_e(\alpha)^2 = 0$ for every off-diagonal inner product α.

To prove antipodality, let $D' = D \cup (-D)$ and choose one point from each antipodal pair of D'. The kernels at distinct chosen points are nonzero and mutually orthogonal: changing a sign changes K_e only by the factor $(-1)^e$. There are at most M_e such kernels, so $|D'| \leq 2M_e = |D|$. Hence $D' = D$. □

This is the equality statement in [DGS77], Theorem 5.12. The following proposition gives its converse.

Proposition 3.3 (Evaluation-kernel criterion)**.** *Let $u_1, \ldots, u_{M_e}$ be points of S^{d-1} representing distinct antipodal pairs. Then $\{\pm u_i\}$ is a tight $(2e+1)$-design if and only if*

$$C_e^{d/2}(\langle u_i, u_j\rangle) = 0 \quad (i \neq j).$$

Equivalently, the functions $K_e(\langle u_i, -\rangle)/\sqrt{M_e}$ form an orthonormal basis of the restricted homogeneous polynomials of degree e.

Proof. Choose an orthonormal basis $f_1, \ldots, f_{M_e}$ of that polynomial space, and form the square matrix $E_{ij} = f_j(u_i)/\sqrt{M_e}$. The displayed kernel condition is $EE^{\mathsf{T}} = I$, equivalent to $E^{\mathsf{T}}E = I$. The latter says that the discrete average agrees with integration on every product $f_i f_j$. These products span the restrictions of homogeneous polynomials of degree $2e$. Multiplication by powers of $\|x\|^2$ includes every lower even degree, and antipodality gives every odd moment through degree $2e+1$. The reverse implication follows by applying the design identity to the products $f_i f_j$. □

3.2 The three irrational cases

The three remaining Chern-purity pairs violate the classical dimension restrictions above. We give a direct proof that isolates the role of irrational kernel zeros. For the two relevant degrees, these zeros are

e	zeros of $C_e^{d/2}$
2	$\pm\dfrac{1}{\sqrt{d+2}}$
3	$0,\ \pm\sqrt{\dfrac{3}{d+4}}$.

Their irrationality is incompatible with the second moment and the rank of the required Gram matrix in the three cases below.

Lemma 3.4 (An irrational-angle obstruction)**.** *Let $u_1, \ldots, u_m$ be points of S^{d-1} satisfying*

$$\sum_{i=1}^{m} u_i u_i^{\mathsf{T}} = \frac{m}{d} I_d, \qquad m > d.$$

Suppose that for some $q > 0$ with $q^2 \in \mathbb{Q}$ their off-diagonal inner products belong to $\{0, \pm 1/q\}$. If $q \notin \mathbb{Q}$, then $m = 2d$.

Proof. Write the Gram matrix as $G = I + S/q$, where S is an integral symmetric matrix. If U has rows u_i^{T}, the second moment gives $U^{\mathsf{T}}U = (m/d)I_d$. Hence $G = UU^{\mathsf{T}}$ has eigenvalues m/d and 0, with multiplicities d and $m - d$. The eigenvalues of S are therefore

$$\left(\frac{m}{d} - 1\right) q \quad (d \text{ times}), \qquad -q \quad (m - d \text{ times}).$$

Since the characteristic polynomial of S has integer coefficients, the quadratic conjugate q of $-q$ must also be an eigenvalue. The only positive eigenvalue displayed above is $(m/d - 1)q$, so $m = 2d$. □

For the icosahedron, $d = 3$ and $m = 6 = 2d$, so its irrational angles are consistent with the lemma. In the three cases needed here, the required number of antipodal pairs is larger than $2d$.

Corollary 3.5. *There is no tight 5-design in $\mathbb{R}^5$ or $\mathbb{R}^8$, and there is no tight 7-design in $\mathbb{R}^5$.*

Proof. Choose one vector from each antipodal pair of a hypothetical tight $(2e+1)$-design. Theorem 3.2 gives $m = \binom{d+e-1}{e}$ representatives and the kernel zeros above. Grouping the antipodal pairs in the second moment of the design gives the identity in Lemma 3.4. The three cases give

(e, d)	m	q	$2d$
$(2, 5)$	15	$\sqrt{7}$	10
$(2, 8)$	36	$\sqrt{10}$	16
$(3, 5)$	35	$\sqrt{3}$	10.

In each row q is irrational and $m \neq 2d$, a contradiction. □

3.3 The six lattice realizations

For $e = 1$, the kernel is $K_1(t) = dt$. The criterion in Proposition 3.3 says that the antipodal representatives form an orthonormal basis. Thus a tight 3-design is exactly a cross-polytope, settling $(n, b) = (1, 22)$, including uniqueness.

Let E_8 have minimum squared norm two, and let Λ be the Leech lattice with minimum squared norm four. For a positive definite lattice L, write $L_s = \{x \in L : (x,x) = s\}$. The configurations

$$(E_8)_2/\sqrt{2} \subset S^7, \qquad \Lambda_4/2 \subset S^{23}$$

have 240 and 196560 points. Integrality and the minimum squared norms give their non-antipodal angles $\{0, \pm 1/2\}$ and $\{0, \pm 1/4, \pm 1/2\}$, respectively. These are exactly the roots of C_3^4 and C_5^{12}, and their cardinalities are $2\binom{10}{3}$ and $2\binom{28}{5}$. Proposition 3.3 proves tightness.

Three further configurations arise from shell slices:

$$D_{7,5} = \left\{ \sqrt{\frac{2}{3}}(x - v/2) : x \in (E_8)_2,\ (x,v) = 1 \right\}, \qquad v \in (E_8)_2, \tag{24}$$

$$D_{23,7} = \left\{ \frac{x - v/2}{\sqrt{3}} : x \in \Lambda_4,\ (x,v) = 2 \right\}, \qquad v \in \Lambda_4, \tag{25}$$

$$D_{23,5} = \left\{ \sqrt{\frac{2}{5}}(x - v/2) : x \in \Lambda_4,\ (x,v) = 3 \right\}, \qquad v \in \Lambda_6. \tag{26}$$

The involution $x \mapsto v - x$ proves antipodality. Their unnormalized squared radii are $3/2$, 3, $5/2$. The first slice gives the 56 minimal vectors of E_7^*. The second gives the 4600 minimal vectors of the odd unimodular shorter Leech lattice. The third gives the 552 vectors on the 276 equiangular lines.

The slice sizes follow from the shell moments. For a spherical shell D of squared radius R that is a design of strength at least $2j$, the moment identity is

$$\sum_{x \in D} (x,v)^{2j} = |D| \frac{(R(v,v))^j (2j-1)!!}{d(d+2)\cdots(d+2j-2)}.$$

The preceding kernel argument gives strength seven for the E_8 shell and strength eleven for the Leech shell, so it supplies all moments used here. For an E_8 root v, the second moment gives 56 at each of the levels ± 1. For a Leech vector of squared norm four, the second and fourth moments give 4600 at each of the levels ± 2. For a Leech vector of squared norm six, the moments through degree six give

$$N_0 = 75900, \qquad N_{\pm 1} = 48600, \qquad N_{\pm 2} = 11178, \qquad N_{\pm 3} = 552,$$

where each signed level is counted separately. Higher levels are excluded by the minimum squared norm.

For two distinct, non-antipodal points of the last slice, represented by x and y in Λ_4, the inequalities

$$(x-y)^2 \geq 4, \qquad (v-x-y)^2 \geq 4$$

force $(x,y) \in \{1, 2\}$. After projection and normalization the angles are therefore $\pm 1/5$. The same calculation gives 0, $\pm 1/3$ in (25) and $\pm 1/3$ in (24). The kernel criterion proves tightness in all three cases. The lattice constructions and their uniqueness are treated in [BS81, GS75, CK07a].

3.4 Characterization and uniqueness

Two of the uniqueness statements follow from the classification of simply laced root systems. Scale a tight 7-design in dimension eight to squared norm two. Its Gram matrix is integral, so its integral span is an even lattice. The vectors of squared norm two form a simply laced root system of rank eight, containing the 240 design vectors. The ADE classification forces this root system to be E_8, and the design is its full root system.

For a tight 5-design in dimension seven, scale its vectors to squared norm $3/2$, and call the resulting set Y. Adjoin $v \perp Y$ with $v^2 = 2$. The vectors v and $y + v/2$ ($y \in Y$) have squared norm two and integral pairings, so generate an even rank-eight lattice. Its root system has a root v with at least 56 roots at inner product one with it. In the ADE list of rank at most eight, only E_8 has this property: the corresponding numbers in E_6, E_7, E_8 are 20, 32, 56, and those in A_r and D_r are $2(r-1)$ and $4(r-2)$. Thus the root system is E_8. The 56 vectors exhaust the slice at v, and transitivity on E_8 roots proves uniqueness of its projection. These arguments give the characterizations in [BS81], Theorems 7 and 11.

The remaining three uniqueness proofs use classical classification results, which we describe briefly.

- For $(d, t) = (23, 5)$, choose representatives of the 276 antipodal pairs. Their Gram matrix is $G = I + S/5$, where S is an integral Seidel matrix. The second moment gives $G^2 = 12G$, hence

  $$(S - 55I)(S + 5I) = 0.$$

  Goethals–Seidel classify the associated regular two-graph on 276 vertices [GS75]. Their proof reduces to the ternary Golay code. Uniqueness of its switching class determines G up to signs and permutation, and therefore determines the antipodal design up to orthogonal transformation.

- For $(d, t) = (24, 11)$, scaling to squared norm four gives an even integral lattice generated by the design. It has no roots: if z had $z^2 = 2$, all (x, z) would be in $\{0, \pm 1, \pm 2\}$, whereas the second and fourth design moments would give

  $$\sum_x (x, z)^2 = 65520, \qquad \sum_x (x, z)^4 = 60480,$$

  contradicting $k^4 \geq k^2$ for integers k. The remaining lattice and binary-code argument is [BS81], Theorem 15: the intersection numbers produce a coordinate frame, and the resulting binary code is the extended Golay code. The vectors are consequently the minimal vectors of the Leech lattice.

- For $(d, t) = (23, 7)$, write Y for the realization scaled to squared norm three. Adjoin $v \perp Y$ with $v^2 = 4$ and lift by $y \mapsto y + v/2$. The generated rank-24 lattice is even and contains the lifted slice of squared norm four. The intersection numbers and a coordinate frame reduce its determination to the Steiner system $S(3, 6, 22)$ and the associated binary code. Cohn–Kumar [CK07a], Section 3, Theorem 4, correct the argument in [BS81], Theorem 25: the lattice generated at this stage has index two in the Leech lattice, whereas Bannai–Sloane identified it with the Leech lattice itself. The correction preserves uniqueness of the projected slice.

Proof of Theorem 3.1. Corollary 3.5 excludes three of the nine possible pairs. Section 3.3 constructs the designs for the other six pairs. The cross-polytope criterion, the two root-system arguments, and the three classical uniqueness results just described characterize all six up to orthogonal transformation. □

4 The common polynomial and the associated lattices

4.1 The Gegenbauer polynomial

With the normalization of the Riemann–Roch polynomial used above, put

$$\kappa_X(t) = \frac{\mathrm{RR}_X(4r(t-1))}{n+1}.$$

For a manifold satisfying (P),

$$\kappa_X(t) = \frac{C_n^{b/2}(t)}{C_n^{b/2}(1)}, \qquad b = b_2(X). \tag{27}$$

The numerator is also the reproducing kernel for restrictions of homogeneous degree-n polynomials to S^{b-1}, with respect to the normalized surface measure. In particular,

$$C_n^{b/2}(1) = \binom{b+n-1}{n}.$$

The known hyperkähler examples in Table 1 satisfy (P). For the fourfolds, purity of c_2 suffices because c_4 is a top class. For the sixfolds, Lemma 2.3 gives (P) from purity of c_2. For the two O'Grady types and the other cases, see [BS26], Corollary 2.9, Examples 2.10–2.11 and Propositions 4.1–4.2.

For the five designs of strength at least five in Table 1, the respective normalized polynomials are

$$\frac{9t^2-1}{8}, \qquad \frac{25t^2-1}{24}, \qquad \frac{t(4t^2-1)}{3}, \qquad \frac{t(9t^2-1)}{8}, \qquad \frac{t(4t^2-1)(16t^2-1)}{45}.$$

These designs are unique up to orthogonal transformation, by Theorem 3.1.

To construct a tight design with this kernel, put $M_n = \binom{b+n-1}{n}$. Proposition 3.3 reduces the problem to finding a positive semidefinite $M_n \times M_n$ matrix of rank b, with diagonal entries one and off-diagonal entries among the roots of $C_n^{b/2}$. A Gram factorization gives M_n unit vectors; adjoining their negatives produces the design.

The number of antipodal pairs has a cohomological interpretation. Whenever the corresponding tight design D exists, it is antipodal by Theorem 3.2, so that $|D|/2$ counts its antipodal pairs:

$$\frac{|D|}{2} = \binom{b+n-1}{n} = \dim_{\mathbb{Q}} \mathrm{SH}^{2n}(X, \mathbb{Q}). \tag{28}$$

Indeed, Verbitsky's theorem identifies this middle-degree component with $\mathrm{Sym}^n H^2(X, \mathbb{Q})$; see [Bec23], Section 2.2, equation (2.1).

4.2 The Beauville–Bogomolov lattices and the design lattices

Let U be the even unimodular hyperbolic plane, and put

$$L_{\mathrm{K3}} = U^3 \oplus E_8(-1)^2.$$

We always use the primitive integral Beauville–Bogomolov form on $H^2(X, \mathbb{Z})$. The known lattices are listed in Table 2; see [Rap08], Introduction. Every lattice in that table is even and has signature $(3, b-3)$. For an even lattice L, write

$$A_L = L^*/L, \qquad q_L(\bar{x}) = (x, x) \pmod{2\mathbb{Z}}.$$

In the table, $\langle a/d \rangle$ denotes the quadratic form on $\mathbb{Z}/d\mathbb{Z}$ taking the value a/d on a specified generator.

For the positive definite lattices, we fix the scales as follows. Let Λ be the Leech lattice of minimum squared norm four, let J be the odd unimodular shorter Leech lattice of minimum squared norm three, and let M be the span of the 552 projected vectors of squared norm $5/2$ defined below. The lattice in Table 3 is in each case the integer span of the specified design vectors, rather than its saturation in an ambient lattice.

The Beauville–Bogomolov and design lattices have different signatures, and the determinants and parities usually differ as well. For instance, the O'Grady lattices have determinants 4 and 3, whereas the corresponding positive definite lattices E_8 and Λ are unimodular.

Type	$H^2(X,\mathbb{Z})$	$\lvert\det\rvert$	Discriminant form
K3	L_{K3}	1	0
Kum_2	$U^3 \oplus \langle -6\rangle$	6	$\langle -1/6\rangle$
$\mathrm{K3}^{[2]}$	$L_{\mathrm{K3}} \oplus \langle -2\rangle$	2	$\langle -1/2\rangle$
OG_6	$U^3 \oplus \langle -2\rangle^2$	4	$\langle -1/2\rangle^{\oplus 2}$
$\mathrm{K3}^{[3]}$	$L_{\mathrm{K3}} \oplus \langle -4\rangle$	4	$\langle -1/4\rangle$
OG_{10}	$L_{\mathrm{K3}} \oplus A_2(-1)$	3	$\langle -2/3\rangle$

Table 2: Primitive integral Beauville–Bogomolov lattices of the known examples satisfying (P).

b	t	Span of the scaled design	Squared radius	det	Integrality
22	3	$\mathbb{Z}^{22}$	1	1	odd integral
7	5	E_7^*	3/2	1/2	nonintegral
23	5	M	5/2	3/2	nonintegral
8	7	E_8	2	1	even integral
23	7	J	3	1	odd unimodular
24	11	Λ	4	1	even unimodular

Table 3: Positive definite design spans at their natural lattice scales. Scaling the vectors of E_7^* and M by $\sqrt{2}$ gives odd integral lattices of minimum squared norm 3 and 5, with determinants 2^6 and $3 \cdot 2^{22}$, respectively.

4.3 The two Leech hyperplane constructions

The two rank-23 design lattices arise as follows. Let $v \in \Lambda$ have squared norm m, where $m = 4$ or 6, and put

$$W_m = \Lambda \cap v^{\perp}.$$

The vector v is primitive: otherwise its division by an integer at least two would contradict the minimum of Λ. Since Λ is unimodular,

$$\det W_m = m, \qquad (A_{W_m}, q_{W_m}) \simeq (\mathbb{Z}/m\mathbb{Z}, q(\bar{1}) = -1/m \pmod{2\mathbb{Z}}). \tag{29}$$

The sign in this formula comes from the anti-isometry between the discriminant forms of primitive orthogonal complements in an even unimodular lattice; see [Nik80], Proposition 1.6.1.

For $m = 4$, the vectors

$$x - v/2, \qquad x \in \Lambda, \quad x^2 = 4, \quad (x, v) = 2,$$

have squared norm three and span J. Indeed, v and the 4600 adjacent minimal vectors generate the index-two sublattice $\widetilde{L} = \{x \in \Lambda : (x, v) \in 2\mathbb{Z}\}$; see [CK07b], proof of Lemma 9.5, p. 140. Its orthogonal projection to $v^{\perp}$ is exactly the displayed span. The kernel is $\mathbb{Z}v$, and therefore the determinant of the projected lattice is $\det \widetilde{L}/v^2 = 4/4 = 1$. The projected vectors have integral pairings and odd squared norms. This is the shorter Leech lattice; see also [Mar04], Corollary 5.6. A combination has even squared norm exactly when the sum of its coefficients is even. Such a combination belongs to $\Lambda \cap v^{\perp}$. Since the even sublattice of an odd unimodular lattice has index two, both sides have determinant four. Hence

$$J_{\mathrm{even}} = W_4. \tag{30}$$

For $m = 6$, the corresponding 552 vectors are

$$x - v/2, \qquad x \in \Lambda, \quad x^2 = 4, \quad (x, v) = 3.$$

They have squared norm $5/2$ and non-antipodal inner products $\pm 1/2$. Their span is M, with

$$M = W_6 + \mathbb{Z}(x_0 - v/2), \qquad [M : W_6] = 2, \qquad \det M = \frac{3}{2}, \tag{31}$$

where x_0 is any vector in the slice. Thus $\sqrt{2}M$ is the odd integral lattice of minimum squared norm five supporting the 276 equiangular lines; compare [Mar04], Section 5. The vectors of integral squared norm in M form precisely W_6: a vector in the other coset has squared norm in $5/2 + 2\mathbb{Z}$.

To verify (31), take the standard extended binary Golay code on 24 coordinates and let $\mathcal{O}$ be its 253 octads containing coordinate 1. With $\mathbf{1} = (1, \ldots, 1)$, the vectors

$$v_k = 4e_1 + 8e_k - \mathbf{1} \quad (2 \le k \le 24), \qquad w_O = 4\sum_{i \in O} e_i - 4e_1 - \mathbf{1} \quad (O \in \mathcal{O})$$

lie in the hyperplane $5z_1 + \sum_{i=2}^{24} z_i = 0$ and have squared norm 80, with mutual inner products ± 16. These are vectors from the indicated Leech slice: in the usual Golay-code coordinates, put $u = (5, 1, \ldots, 1)$ and $v = u/\sqrt{8}$. For each displayed vector z, the vector $x = (u + z)/\sqrt{32}$ is either of type $(4, 4, 0^{22})/\sqrt{8}$ or of type $(2^8, 0^{16})/\sqrt{8}$ with octad support, so belongs to Λ, and $x - v/2 = z/(4\sqrt{2})$. Thus their multiples by $1/4$ give one vector on each of the 276 lines at squared norm five. The 23 vectors $v_k/4$ have Gram matrix $4I_{23} + J_{23}$ and generate a sublattice of determinant $27 \cdot 4^{22}$. Writing c_O for the incidence vector of $O \setminus \{1\}$, the coordinates of $w_O/4$ in this basis are

$$\frac{1}{2}c_O - \frac{1}{6}\mathbf{1}.$$

Modulo the sublattice generated by the $v_k/4$, these coordinates generate a group of order $3 \cdot 2^{11}$: their 3-primary part is the constant line, and their 2-primary part is generated by the complements of the c_O. These generate the shortened binary Golay code, of dimension 11. Consequently

$$\det(\sqrt{2}M) = \frac{27 \cdot 4^{22}}{(3 \cdot 2^{11})^2} = 3 \cdot 2^{22}, \qquad \det M = \frac{3}{2}.$$

The dimension-11 binary rank assertion is checked in Appendix A.3 from the cyclic length-23 Golay generator $x^{11} + x^9 + x^7 + x^6 + x^5 + x + 1$, followed by the parity extension. The lattice $W_6 + \mathbb{Z}(x_0 - v/2)$ contains M, has index two above W_6, and hence has the same determinant $3/2$. This proves (31).

4.4 Discriminant forms

The Beauville–Bogomolov lattice of $\mathrm{K3}^{[3]}$ and J_{even} are both primitive orthogonal complements of a vector of squared norm four in an even unimodular lattice. On the design side, (30) gives

$$J_{\mathrm{even}} = v_\Lambda^\perp, \qquad v^2 = 4.$$

On the hyperkähler side, let e and f be a standard basis of U, with $e^2 = f^2 = 0$ and $(e, f) = 1$. Then

$$(e + 2f)^\perp_{L_{\mathrm{K3}} \oplus U} = L_{\mathrm{K3}} \oplus \mathbb{Z}(e - 2f) \cong L_{\mathrm{K3}} \oplus \langle -4 \rangle \cong H^2(\mathrm{K3}^{[3]}, \mathbb{Z}).$$

In each case, the anti-isometry for primitive orthogonal complements in an even unimodular lattice gives the finite quadratic form $\langle -1/4 \rangle$. On the design side, this uses the identification $J_{\mathrm{even}} = \Lambda \cap v^\perp$ obtained from the 4600-point configuration.

Moreover, the even indefinite lattice of signature $(3, 20)$ with this finite form is unique up to isometry, by [Nik80], Corollary 1.13.3: its rank is 23, whereas its discriminant group has length one. Thus $H^2(\mathrm{K3}^{[3]}, \mathbb{Z})$ is, up to isometry, the unique even lattice of signature $(3, 20)$ with the discriminant form of J_{even}.

The two fourfold cases give further coincidences of discriminant forms, with the rows interchanged. The dual E_7 of the Kum_2 design lattice E_7^* has form $\langle 3/2\rangle = \langle -1/2\rangle$, the discriminant form of the $\mathrm{K3}^{[2]}$ lattice. Likewise, the sublattice W_6 of vectors of integral squared norm in the $\mathrm{K3}^{[2]}$ design lattice M has form $\langle -1/6\rangle$, that of the Kum_2 lattice. Each comparison is between lattices of ranks 7 and 23.

4.5 Open questions

Do Chern-pure manifolds exist for any of the pairs $(2, 5)$, $(2, 8)$, and $(3, 5)$? For each of the six realized parameter pairs, must every Chern-pure manifold have the integral Beauville–Bogomolov lattice listed in Table 2?

We do not know a geometric explanation for the coincidence of the six pairs realized by known Chern-pure manifolds with the six design pairs.

A Exact finite computations

All programs in this appendix use exact integer or rational arithmetic and the standard library of Python 3.10 or later. The first two blocks are run consecutively; the Golay-code check is independent.

A.1 The parameter sieve

The following Python program implements the bounds (16)–(17) and tests every divisor of the final bound. It prints and checks the survivor table of Lemma 2.6, the 3911 higher-dimensional tests, and the maximum list size 361 at $n = 14$. The analytic exclusion for $n \geq 576$ is Proposition 2.5.

```
from fractions import Fraction
from itertools import product
from math import factorial, isqrt, prod

def vp(z, p):
    e = 0
    while z % p == 0:
        z //= p
        e += 1
    return e

def vf(z, p):
    e = 0
    while z:
        z //= p
        e += z
    return e

def logp(z, p):
    e = 0
    while z >= p:
        z //= p
        e += 1
    return e

def primes(m):
```

```
    return [p for p in range(2, m + 1)
            if all(p % q for q in range(2, isqrt(p) + 1))]

def bounds(n):
    E = {}
    for p in primes(n + 1):
        h = vp(2, p) + logp(n - 1, p)
        R = (vf(n + 1, p) + n * vp(12, p)
             + 2 * (n - 1) * vp(2, p) + 2 * vf(n - 1, p))
        E[p] = max(h, R // (n - 2))
    while True:
        D = prod(p ** e for p, e in E.items())
        if D < 2 * n + 1:
            return None
        new = {}
        for p, e in E.items():
            A = (vf(n + 1, p) + n * vp(12, p)
                 + 2 * (n - 2) * vp(2, p) + 2 * vf(n - 2, p))
            new[p] = min(e, (A + 2 * logp(D - 2, p)) // (n - 2))
        if new == E:
            return E
        E = new

def row(n):
    E = bounds(n)
    if E is None:
        return [], 0
    choices = [[p ** j for j in range(e + 1)]
               for p, e in E.items()]
    result, tested = [], 0
    K = factorial(n + 1) * 12 ** n
    for factors in product(*choices):
        d = prod(factors)
        b = d - 2 * n + 2
        if b < 3:
            continue
        tested += 1
        I = Fraction(K * prod((d - 2 * k) ** 2 for k in range(1, n)),
                     d ** (n - 2) * prod(b + j for j in range(n)))
        if I.denominator == 1:
            result.append(b)
    return sorted(result), tested

expected = {
    3: [5, 8, 23], 4: [3], 5: [24], 7: [4], 21: [8],
    22: [6], 23: [4], 31: [4], 45: [8], 46: [6], 47: [4]
}
rows, counts = {}, {}
for n in range(3, 576):
    bs, count = row(n)
    if bs:
        rows[n] = bs
        print(n, bs)
    if n >= 6:
        counts[n] = count
assert rows == expected
total = sum(counts.values())
assert total == 3911
assert max(counts.items(), key=lambda item: item[1]) == (14, 361)
print("Higher-dimensional divisor candidates:", total)
```

A.2 The characteristic-class checks

The following continuation computes the coefficients of $\log S$ independently from $S' = S(\log S)'$, verifies the four residues used in Proposition 2.7, and checks $F_4(3) = 80/3$. For the residue calculation, only $a = d/2$ and the first six coefficients are needed, so we may set $n = 6$ and $b = d - 10$.

```
def characteristic_coefficients(n, b):
    a = Fraction(b, 2) + n - 1
    s, ell = [Fraction(1)], [Fraction(0)]
    for j in range(1, n + 1):
        s.append(s[-1] / (j * (a - j + 1)))
        ell.append(s[j] - sum(Fraction(k, j) * ell[k] * s[j-k]
                              for k in range(1, j)))
    t = [sum(s[k] * s[j-k] for k in range(j + 1))
         for j in range(n + 1)]
    return ell, t

for d, residue in [(16, 275), (48, 296), (64, 647), (96, 121)]:
    ell, _ = characteristic_coefficients(6, d - 10)
    g5 = 6 * ell[6]
    assert g5.numerator * pow(g5.denominator, -1, 691) % 691 == residue

ell, t = characteristic_coefficients(4, 3)
B = [Fraction(1, 6), Fraction(-1, 30),
     Fraction(1, 42), Fraction(-1, 30)]
F = 5 * (8 - sum(4 * j * ell[j] * t[4-j] / (B[j-1] * t[4])
                 for j in range(1, 5)))
assert F == Fraction(80, 3)
print("691 residues and F_4(3) verified")
```

A.3 The Golay-code rank calculation

This check encodes coordinate 1 by the lowest bit and the parity coordinate by the highest bit. It row-reduces the complements of the c_O on the remaining 23 coordinates. Their rank 11, together with the constant 3-primary component, gives the index $3 \cdot 2^{11}$ and the determinant in (31).

```
generator = sum(1 << j for j in (0, 1, 5, 6, 7, 9, 11))
code = []
for message in range(1 << 12):
    c = 0
    for j in range(12):
        if (message >> j) & 1: c ^= generator << j
    code.append(c | ((c.bit_count() % 2) << 23))
assert len(set(code)) == 4096
assert {c.bit_count() for c in code} == {0, 8, 12, 16, 24}
octads = [c for c in code if c & 1 and c.bit_count() == 8]
assert len(octads) == 253
basis = {}
for c in octads:
    v = (((1 << 24) - 1) ^ c) >> 1
    while v:
        j = v.bit_length() - 1
        if j not in basis:
            basis[j] = v
            break
        v ^= basis[j]
```

```
assert len(basis) == 11
assert 27 * 4**22 == (3 * 2**11)**2 * (3 * 2**22)
print("Golay octads, binary rank, and determinant verified")
```

References

[BB09] E. Bannai and E. Bannai. A survey on spherical designs and algebraic combinatorics on spheres. *European J. Combin.*, 30:1392–1425, 2009. doi:10.1016/j.ejc.2008.11.007.

[BD79] E. Bannai and R. M. Damerell. Tight spherical designs, I. *J. Math. Soc. Japan*, 31:199–207, 1979. doi:10.2969/jmsj/03110199.

[BD80] E. Bannai and R. M. Damerell. Tight spherical designs, II. *J. London Math. Soc. (2)*, 21:13–30, 1980. doi:10.1112/jlms/s2-21.1.13.

[Bec23] T. Beckmann. Derived categories of hyper-Kähler manifolds: extended Mukai vector and integral structure. *Compos. Math.*, 159:109–152, 2023. doi:10.1112/S0010437X22007849.

[BMV04] E. Bannai, A. Munemasa, and B. Venkov. The nonexistence of certain tight spherical designs. *Algebra i Analiz*, 16(4):1–23, 2004. English translation, St. Petersburg Math. J. **16** (2005), 609–625. doi:10.1090/S1061-0022-05-00868-X.

[BS81] E. Bannai and N. J. A. Sloane. Uniqueness of certain spherical codes. *Canad. J. Math.*, 33:437–449, 1981. doi:10.4153/CJM-1981-038-7.

[BS26] T. Beckmann and J. Song. Second Chern class and Fujiki constants of hyperkähler manifolds. *Beitr. Algebra Geom.*, 2026. Published online 18 September 2026. doi:10.1007/s13366-026-00869-3; arXiv:2201.07767v3. Numbered references to this work follow arXiv v3.

[CK07a] H. Cohn and A. Kumar. Uniqueness of the $(22, 891, 1/4)$ spherical code. *New York J. Math.*, 13:147–157, 2007. Published article.

[CK07b] H. Cohn and A. Kumar. Universally optimal distribution of points on spheres. *J. Amer. Math. Soc.*, 20:99–148, 2007. arXiv:math/0607446.

[DGS77] P. Delsarte, J. M. Goethals, and J. J. Seidel. Spherical codes and designs. *Geom. Dedicata*, 6:363–388, 1977. doi:10.1007/BF03187604.

[FWWZ26] J. Fu, B. Wang, S. Wang, and Z. Zheng. A sharp bound for the second Betti number of hyperkähler sixfolds. arXiv:2610.15803v1, 2026. Theorem 1.3.

[GS75] J. M. Goethals and J. J. Seidel. The regular two-graph on 276 vertices. *Discrete Math.*, 12:143–158, 1975. doi:10.1016/0012-365X(75)90029-1.

[Gua01] D. Guan. On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four. *Math. Res. Lett.*, 8:663–669, 2001. Author PDF.

[Huy99] D. Huybrechts. Compact hyperkähler manifolds: basic results. *Invent. Math.*, 135(1):63–113, 1999. doi:10.1007/s002220050280; arXiv:alg-geom/9705025.

[Jia23] C. Jiang. Positivity of Riemann–Roch polynomials and Todd classes of hyperkähler manifolds. *J. Algebraic Geom.*, 32:239–269, 2023. doi:10.1090/jag/798; arXiv:2008.04685v3.

[Mar04] J. Martinet. Reduction modulo 2 and 3 of Euclidean lattices. II. In *Algebraic and Arithmetic Theory of Quadratic Forms*, volume 344 of *Contemp. Math.*, pages 265–273. Amer. Math. Soc., Providence, RI, 2004. Author's text.

[Nie03] M. A. Nieper. Hirzebruch–Riemann–Roch formulae on irreducible symplectic Kähler manifolds. *J. Algebraic Geom.*, 12:715–739, 2003. DOI.

[Nik80] V. V. Nikulin. Integral symmetric bilinear forms and some of their applications. *Math. USSR Izv.*, 14:103–167, 1980. Russian original, 1979. Publisher record.

[NV12] G. Nebe and B. Venkov. On tight spherical designs. *Algebra i Analiz*, 24(3):163–171, 2012. English translation, St. Petersburg Math. J. **24** (2013), 485–491. arXiv:1201.1830.

[Rap08] A. Rapagnetta. On the Beauville form of the known irreducible symplectic varieties. *Math. Ann.*, 340:77–95, 2008. arXiv:math/0606409. The introductory table lists the BBF lattices and Fujiki constants; Theorem 3.0.11 in the arXiv version treats the ten-dimensional O'Grady example.

[Sal96a] S. M. Salamon. Cohomology of Kähler manifolds with $c_1 = 0$. In P. de Bartolomeis, F. Tricerri, and E. Vesentini, editors, *Manifolds and Geometry (Pisa, 1993)*, volume XXXVI of *Sympos. Math.*, pages 294–310. Cambridge Univ. Press, Cambridge, 1996. arXiv:dg-ga/9502005. Theorem 1 and equation (7) refer to the arXiv version.

[Sal96b] S. M. Salamon. On the cohomology of Kähler and hyper-Kähler manifolds. *Topology*, 35(1):137–155, 1996. doi:10.1016/0040-9383(95)00006-2.

[Xu26] Z. Xu. Arithmetic nonexistence conditions for tight spherical 5-designs. Preprint, version 2, 30 September 2026. arXiv:2609.36725.