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First differences of real-rooted integer polynomials and simplicial complexes

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Abstract

Let $H(t) = \sum_{i=0}^{d} h_i t^i$ be a real-rooted polynomial with nonnegative integer coefficients and $h_0 = 1$. We prove that every strictly increasing initial segment of its coefficients has first differences forming the face-number vector of a simplicial complex. This answers Problem 2.10 in Brenti's list of open problems and Question 4.4 of Kubitzke and Welker, without symmetry, monicity, or degree restrictions. The proof combines a uniform additive estimate for the Kruskal–Katona operator with a continuous inequality for ultra-log-concave sequences. A classical trace bound for totally positive algebraic integers supplies the necessary initial-coefficient condition after the factors $1 + t$ are removed. These factors are then restored by a cone construction. In particular, the conclusion holds for products of any number of real-rooted irreducible cubic factors with nonnegative integer coefficients and constant term one.

1 Introduction

Real-rootedness imposes strong restrictions on coefficient sequences. For nonnegative coefficients, Newton's inequalities give ultra-log-concavity and hence unimodality. When the coefficients are integers and the constant term is one, it is natural to ask whether these numerical restrictions imply the existence of a combinatorial object with prescribed face numbers.

We use the convention that the face-number vector of a finite simplicial complex Δ includes the empty face:

$$(f_{-1}(\Delta), f_0(\Delta), \ldots, f_{k-1}(\Delta)), \qquad f_{-1}(\Delta) = 1.$$

Thus the entry with index i in a vector $(g_0, \ldots, g_k)$ counts faces of cardinality i. The complex $\{\varnothing\}$ has vector (1). The result of this paper is the following.

Theorem 1.1. *Let $H(t) = \sum_{i=0}^{d} h_i t^i \in \mathbb{Z}_{\geq 0}[t]$ be of degree d, have only real zeros, and satisfy $h_0 = 1$. If*

$$h_0 < h_1 < \cdots < h_k,$$

then there exists a finite simplicial complex with face-number vector

$$(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_k - h_{k-1}).$$

This is an affirmative answer to Brenti's Problem 2.10 [2] and to the earlier question of Kubitzke and Welker [8, Question 4.4]. All the hypotheses are retained from that question; in particular, H need not be symmetric or monic.

1.1 Related work

Bell and Skandera [1] proved that the coefficient sequence of a real-rooted polynomial with nonnegative integer coefficients and constant term one is an M-sequence, and asked whether it is the face-number vector of a simplicial complex. Their question concerns the original coefficients. Mu and Welker [9] obtained sufficient conditions for this coefficient-vector problem. The question of Kubitzke and Welker addressed here instead concerns the initial first differences.

A weaker first-difference conjecture, requiring Macaulay's inequalities, was recorded by Welker [13, Conjecture 8, p. 70]. Guo and Kang [5, Theorem 1.1] resolved Brenti's symmetric polytope question [2, Problem 2.9]: every monic palindromic real-rooted polynomial with nonnegative integer coefficients is the h-polynomial of a simplicial convex polytope. Ferudun's recent preprint [4, Theorems A and B] proves the M-sequence conclusion for initial differences without symmetry, under the weaker hypotheses $ih_i^2 \geq (i+1)h_{i-1}h_{i+1}$. It explicitly leaves the ordinary Kruskal–Katona strengthening open [4, Section 1.3 and Remark 6.3]. Macaulay growth allows repeated variables and is weaker than the squarefree growth required for ordinary simplicial complexes.

A public repository [3] records a computer-assisted proof of the precise initial-difference Kruskal–Katona assertion for all degrees at most 13, with no coefficient-height bound. This range already includes products of two cubic factors. Its proof uses a degree-sensitive Newton bound and exact finite certificates; the unrestricted-degree question is expressly outside its stated scope.

In a different setting, Zhang and Zhao [14, Theorem 1.2] prove Kruskal–Katona inequalities for nonnegative integral combinations of toric g-contribution polynomials with normalized constant term. Their theorem applies to that specified family, rather than arbitrary real-rooted polynomials.

Theorem 1.1 establishes the ordinary simplicial conclusion in every degree. Its proof uses the additive estimate for the Kruskal–Katona operator and the trace-dependent continuous inequality stated below, together with classical arithmetic input.

1.2 Notation and proof strategy

For a real x and an integer $j \geq 0$, we use

$$\binom{x}{j} = \frac{x(x-1)\cdots(x-j+1)}{j!}, \qquad \binom{x}{0} = 1.$$

For an integer $p \geq 1$ and a positive integer m, its canonical binomial expansion and its Kruskal–Katona raising are

$$m = \sum_{i=s}^{p} \binom{a_i}{i}, \qquad a_p > \cdots > a_s \geq s, \qquad K_p(m) = \sum_{i=s}^{p} \binom{a_i}{i+1}. \tag{1.1}$$

Set $K_p(0) = 0$. The Kruskal–Katona theorem [7, 6] says that a positive integer vector $(1, g_1, \ldots, g_k)$ is the face-number vector of a simplicial complex if and only if

$$g_{p+1} \leq K_p(g_p) \qquad (1 \leq p < k). \tag{1.2}$$

These are the inequalities for ordinary simplicial complexes. They differ from Macaulay's inequalities for multicomplexes.

Write $m = \binom{x}{p}$, with $x \geq p$, and set $F_p(m) = \binom{x}{p+1}$. Two estimates bridge Newton's inequalities and the discrete inequalities (1.2). The first is the purely combinatorial bound

$$K_p(m) \geq F_p(m) - \frac{m}{p+1}.$$

The second asserts that, for a positive ultra-log-concave sequence of order d with $h_0 = 1$, $h_d \geq 1$, and $h_1/d \geq 3/2$, two successive positive first differences $m = h_p - h_{p-1}$ and $h_{p+1} - h_p$, with $p \geq 3$, satisfy

$$h_{p+1} - h_p < F_p(m) - \frac{m}{p+1}.$$

The latter estimate is valid even for real coefficients.

The trace theorem in Section 2 gives $h_1/d \geq 3/2$ after removing all factors $1+t$. Sections 3 and 4 prove the two estimates. The first two ranks are handled directly in Section 5. A cone construction in Section 6 restores the factors $1+t$, completing the proof in Section 7.

2 The arithmetic reduction

We use the following classical consequence of Siegel's trace theorem. If $\alpha \neq 1$ is a totally positive algebraic integer of degree $e = [\mathbb{Q}(\alpha) : \mathbb{Q}]$, then

$$\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha) \geq \frac{3}{2}e. \tag{2.1}$$

Equality holds precisely for $\alpha = (3 \pm \sqrt{5})/2$. Here totally positive means that every conjugate is strictly positive. The result is Siegel's Theorem III [11]; see also [10, Theorem 4.8(2), p. 43] for this exact statement and its equality case. It also follows from Smyth's stronger trace estimate with five explicit exceptional minimal polynomials [12, Theorem, p. 2].

Lemma 2.1. *Let $H(t) \in \mathbb{Z}_{\geq 0}[t]$ be real-rooted with $H(0) = 1$. Write*

$$H(t) = (1+t)^r Q(t), \qquad r \geq 0, \qquad Q(-1) \neq 0,$$

where all factors $1+t$ have been removed. Then $Q \in \mathbb{Z}[t]$ and $Q(0) = 1$. If $e = \deg Q > 0$ and $Q(t) = \sum_{i=0}^{e} q_i t^i$, then every q_i is positive and

$$q_e \geq 1, \qquad q_1 \geq \frac{3}{2}e.$$

Proof. Take d to be the actual degree of H. Its nonnegative coefficients and constant term 1 imply that it has no nonnegative root. Hence

$$H(t) = \prod_{i=1}^{d}(1 + \lambda_i t), \qquad \lambda_i > 0.$$

Division by the monic polynomial $1+t$ preserves integer coefficients. After all its occurrences have been removed, Q is the product of the remaining factors $1+\lambda_i t$. Its coefficients are therefore positive throughout its support, its constant term is 1, and its leading coefficient is a positive integer.

Suppose that $e > 0$ and form

$$P(x) = x^e Q(-1/x) = \prod_{i=1}^{e}(x - \lambda_i).$$

The polynomial P is monic with integer coefficients, all its roots are strictly positive, and $P(1) = Q(-1) \neq 0$. Factor it as

$$P = \prod_j f_j^{b_j},$$

where the f_j are distinct monic irreducible polynomials in $\mathbb{Z}[x]$. Every root α_j of f_j is a totally positive algebraic integer, and $f_j \neq x-1$. Applying (2.1) to each factor, with its multiplicity, gives

$$q_1 = \sum_{i=1}^{e} \lambda_i = \sum_j b_j \operatorname{Tr}_{\mathbb{Q}(\alpha_j)/\mathbb{Q}}(\alpha_j) \geq \frac{3}{2} \sum_j b_j \deg f_j = \frac{3}{2} e.$$

The monicity required here is that of P, which follows from $Q(0) = 1$; no monicity assumption on Q is made. □

The weak inequality in Lemma 2.1 is necessary: the polynomial $1 + 3t + t^2$ has coefficient of t equal to $3/2$ times its degree.

3 A discrete approximation for the Kruskal–Katona operator

Recall the functions K_p and F_p from Section 1. The following estimate is independent of real-rootedness.

Theorem 3.1. *For all integers $p, m \geq 1$,*

$$F_p(m) - K_p(m) \leq \frac{m}{p+1}.$$

Equivalently,

$$m \leq \binom{p + 1 + (p+1)K_p(m)/m}{p}.$$

Proof. Convexity and two induction reductions. For $S_p(x) = \sum_{j=0}^{p-1} (x-j)^{-1}$, differentiation gives

$$F_p'(m) = \frac{x - p + 1/S_p(x)}{p+1}.$$

Its derivative with respect to x is positive, so F_p is strictly convex. Write

$$m = M + r, \quad M = \binom{a}{p}, \quad 0 \leq r < \binom{a}{p-1}, \quad a \geq p.$$

When $r = 0$, the assertion is immediate. Assume henceforth that $r > 0$. The chord between $\binom{a}{p}$ and $\binom{a+1}{p}$ has slope $(a - p + 1)/p$. Consequently, with

$$c = \frac{a-p+1}{p} - \frac{1}{p+1},$$

one has

$$F_p(m) - \frac{m}{p+1} - K_p(m) \leq -\frac{M}{p+1} + cr - K_{p-1}(r). \tag{3.1}$$

Here $K_{p-1}(0) = 0$. For $r > 0$, write $r = \binom{y}{p-1}$, where $p - 1 \leq y < a$. A second consequence is

$$F_p(m) - K_p(m) \leq \frac{r(a-y)}{p} + (F_{p-1}(r) - K_{p-1}(r)). \tag{3.2}$$

We will use the following product bound:

$$\frac{\binom{y}{p-1}}{\binom{a}{p-1}} = \prod_{j=0}^{p-2} \frac{y-j}{a-j} \le \left(\frac{y}{a}\right)^{p-1}. \tag{3.3}$$

Base ranks $p \le 3$. Rank one is exact. At rank two, writing $m = \binom{a}{2} + r$ gives

$$F_2(m) - K_2(m) \le \frac{r(a-r)}{2} \le \frac{m}{4} \le \frac{m}{3}.$$

The middle inequality is equivalent to $(a-2r)(a-2r-1) \ge 0$, since a, r are integers. At rank three, the rank-two quarter bound and (3.2) give the desired quarter bound when $a \ge 5$: indeed

$$\frac{r(a-y)}{3M} \le \frac{a}{3(a-2)} \left(\frac{2}{3}\right)^2 \le \frac{20}{81} < \frac{1}{4}.$$

For $a = 3$, $m \le 3$ and $K_3(m) = 0$, so $x < 4$ suffices. For $a = 4$, the six possibilities are

$$(m, K_3(m)) = (4,1), (5,1), (6,1), (7,2), (8,2), (9,3).$$

For the first three, $4 + 4K_3(m)/m \ge 14/3 > 9/2$ and $\binom{9/2}{3} = 105/16 > 6$; for the last three, $4 + 4K_3(m)/m \ge 5$ and $\binom{5}{3} = 10 > 9$. This proves the theorem through rank three.

Large top index: $a \ge 2p$, $p \ge 4$. Assume the theorem in lower ranks. Applying it to $r > 0$ in (3.1) shows that it is enough to prove

$$\binom{y}{p-1}((p+1)(a-y)+1) \le pM.$$

By (3.3), the ratio of the left side to pM is at most

$$\frac{(p+1)(a-y)+1}{a-p+1} \left(\frac{y}{a}\right)^{p-1}.$$

Maximizing the right side on the larger interval $0 \le y \le a$ gives at most

$$\frac{(p+1)a}{p(a-p+1)} \left(\frac{p-1}{p}\right)^{p-1} \left(1 + \frac{1}{(p+1)a}\right)^p.$$

For $a \ge 2p$ the first factor is at most 2. The middle factor is at most $(3/4)^3 = 27/64$ for $p \ge 4$: the function $(z-1)\log(1-1/z)$ decreases for $z > 1$, since its derivative is $\log(1-1/z) + 1/z < 0$. Finally,

$$(1+z)^p \le \frac{1}{1-pz} \quad (0 \le pz < 1)$$

follows by bounding each binomial coefficient by p^j and summing a geometric series. Here $pz \le 1/[2(p+1)] \le 1/10$, so the last factor is at most $10/9$. The displayed ratio is therefore at most $2(27/64)(10/9) = 15/16 < 1$.

Bounding the canonical tail. It remains to consider $p \le a < 2p$. For a rank-i digit define

$$f_i(A) = c\binom{A}{i} - \binom{A}{i+1} \qquad (A \ge i-1).$$

Since

$$f_i(A+1) - f_i(A) = \binom{A}{i-1}\left(c - \frac{A-i+1}{i}\right),$$

the maximum is attained at $A_i = \lfloor i(1+c) \rfloor$. Missing digits correspond to $A = i-1$ and give zero, so this independent digit maximum bounds every canonical tail. For $1 \le i \le p-1$,

$$A_i = \left\lfloor \frac{ai}{p} + \frac{i}{p(p+1)} \right\rfloor = \left\lfloor \frac{ai}{p} \right\rfloor.$$

The equality uses the fact that the fractional part of ai/p is an integer multiple of $1/p$, while $0 < i/[p(p+1)] < 1/p$. Thus it suffices to prove

$$S := \sum_{i=1}^{p-1} \left[c\binom{\lfloor ai/p \rfloor}{i} - \binom{\lfloor ai/p \rfloor}{i+1} \right] \le \frac{M}{p+1}. \tag{3.4}$$

If $a = p$, the original m lies between 1 and p, and $K_p(m) = 0$, $x < p+1$ prove the theorem directly. Put $k = a - p$, so $1 \le k \le p-1$ and

$$c = \frac{k}{p} + \frac{1}{p(p+1)}.$$

For $k = 1$, all $A_i = i$, whence

$$S = (p-1)c = 1 - \frac{2}{p(p+1)} < 1 = \frac{M}{p+1}.$$

Complementary-rank identity. Assume $k \ge 2$, and put

$$N_j = \left\lceil \frac{aj}{k} \right\rceil - 1, \quad L = \sum_{j=1}^{k-1} \binom{N_j}{j}, \quad R = \sum_{j=1}^{k-1} \binom{N_j}{j+1}.$$

The N_j strictly increase, so this is a canonical expansion and $R = K_{k-1}(L)$. Grouping the indices $0 \le i \le p-1$ according to $j = \lfloor ki/p \rfloor$, and applying the hockey-stick identity to each group, gives

$$\sum_{i=0}^{p-1} \binom{i + \lfloor ki/p \rfloor}{i} = \binom{a-1}{k} - R,$$
$$\sum_{i=0}^{p-1} \binom{i + \lfloor ki/p \rfloor}{i+1} = \binom{a-1}{k-1} - 1 - L.$$

For completeness, group j runs from $\lceil pj/k \rceil$ to $\lceil p(j+1)/k \rceil - 1$; the two sums over it equal the differences of $\binom{i+j+1}{j+1}$ and $\binom{i+j+1}{j}$, respectively. Pascal's identity telescopes the group boundaries to the two displayed formulas. As $pc - k = 1/(p+1)$, these formulas yield

$$S = \frac{M}{a(p+1)} + 1 - c + L - cK_{k-1}(L). \tag{3.5}$$

Therefore (3.4) is equivalent to

$$L - cK_{k-1}(L) \le \frac{a-1}{a}\frac{M}{p+1} + c - 1. \tag{3.6}$$

Bounding the smaller-rank expression. Here $k-1<p$. Write $L=\binom{y}{k-1}$. Since $N_{k-1}+1=\lceil a(k-1)/k\rceil\le a-2$, the canonical expansion implies $L<\binom{a}{k-1}$ and hence $y<a$. The induction hypothesis gives

$$K_{k-1}(L)\ge\frac{L}{k}(y-k).$$

Since $c>k/p$ and $a=p+k$, one has $k+k/c<a$. Consequently,

$$\begin{aligned}L-cK_{k-1}(L)&\le\frac{c}{k}\left(k+\frac{k}{c}-y\right)\binom{y}{k-1}\\&<\frac{c}{k}(a-y)\binom{y}{k-1}\\&\le\frac{M}{p+1}\,\frac{ca}{k}\left(\frac{k-1}{k}\right)^{k-1}.\end{aligned}$$

The last step uses the product bound with denominator $\binom{a}{k-1}=kM/(p+1)$, followed by maximizing $(a-y)(y/a)^{k-1}$ on $0\le y\le a$. We now verify that this upper bound is at most the right side of (3.6), apart from the small pairs in the final table.

The cases $k=2$ and $k=3$. For $k=2$, the right side of (3.6) minus the last upper bound is exactly

$$\frac{2p^3-11p^2-8p+12}{8p(p+1)}.$$

At $p=6+z$ its numerator is $2z^3+25z^2+76z\ge0$; the pairs $p=3,4,5$ are in the table. For $k=3$, the analogous difference is

$$\frac{15p^4+23p^3-326p^2-120p+360}{162p(p+1)}.$$

At $p=5+z$ its numerator is $15z^4+323z^3+2269z^2+5845z+3860>0$. Only $p=4$ remains for the table.

All $k\ge4$. For $p\ge k+1$ and $k\ge2$,

$$\frac{ca}{k}<2.$$

Indeed this is equivalent to $k(p+1)(p-k)>p+k$; the left minus right is increasing in p, and at $p=k+1$ equals $k^2-1>0$. Also $((k-1)/k)^{k-1}\le27/64$ for $k\ge4$, and $(a-1)/a\ge8/9$ since $a\ge9$. The right side of (3.6) minus the upper bound is thus greater than or equal to

$$\frac{13}{288}\frac{M}{p+1}-1.$$

Except when $(p,k)=(5,4)$, one has $M/(p+1)\ge30$, so this is positive. To see the stated lower bound, for $k=4,p\ge6$ it is at least $\binom{10}{4}/7=30$; for $k\ge5,p\ge k+1$ it is at least $\binom{11}{5}/7=66$. The latter minimum increases with k, because the ratio of consecutive minima is $2(2k+3)/(k+3)>1$.

Every remaining rational check. The columns give the entire extremal digit tuple and the exact margin $M/(p+1)-S$ in (3.4):

p	k	$(A_1,\dots,A_{p-1})$	c	S	$M/(p+1)$	$M/(p+1)-S$
3	2	$(1,3)$	3/4	2	5/2	1/2
4	2	$(1,3,4)$	11/20	12/5	3	3/5
5	2	$(1,2,4,5)$	13/30	83/30	7/2	11/15
4	3	$(1,3,5)$	4/5	26/5	7	9/5
5	4	$(1,3,5,7)$	5/6	83/6	21	43/6

The first row is already covered by the rank-three base case and is included to make the small-k classification complete. All bounds required by (3.1) now hold, closing induction on p and proving Theorem 3.1. □

4 A continuous bound for positive first differences

A positive sequence $(h_i)_{i=0}^d$ is *ultra-log-concave of order d* if its binomial normalization

$$a_i = \frac{h_i}{\binom{d}{i}} \qquad (0 \le i \le d)$$

is log-concave; that is, $a_i^2 \ge a_{i-1}a_{i+1}$ for $1 \le i \le d-1$. For every integer $p \ge 1$, the function $x \mapsto \binom{x}{p}$ is strictly increasing from zero to infinity on $(p-1, \infty)$.

Theorem 4.1. *Let $d \ge 4$ be an integer, and let $(h_i)_{i=0}^d$ be a positive real sequence that is ultra-log-concave of order d. Suppose that*

$$h_0 = 1, \qquad h_d \ge 1, \qquad h_1 \ge \frac{3}{2}d.$$

Fix an integer p with $3 \le p \le d-1$, and suppose that

$$m = h_p - h_{p-1} > 0, \qquad h_{p+1} - h_p > 0.$$

Let $x > p-1$ be the unique real number such that $m = \binom{x}{p}$. Then

$$h_{p+1} - h_p < \binom{x}{p+1} - \frac{m}{p+1}. \tag{4.1}$$

Neither integrality nor real-rootedness is assumed in this theorem. In particular, m may be any positive real number arising from the stated hypotheses. Its application to the Kruskal–Katona operator will require m to be a positive integer.

Proof. Put

$$B = \binom{d}{p-1}, \qquad C = \binom{d}{p}, \qquad \beta = \frac{B}{C} = \frac{p}{d-p+1}, \qquad R = \frac{3}{2}, \qquad v = h_{p-1}.$$

We first bound the next difference while keeping m fixed. Log-concavity of (a_i), applied between indices 0 and p and between indices 1 and p, respectively, gives

$$\left(\frac{v}{B}\right)^p \ge \left(\frac{m+v}{C}\right)^{p-1}, \qquad \left(\frac{v}{B}\right)^{p-1} \ge R\left(\frac{m+v}{C}\right)^{p-2}. \tag{4.2}$$

Indeed, these are the inequalities $a_{p-1}^p \ge a_0 a_p^{p-1}$ and $a_{p-1}^{p-1} \ge a_1 a_p^{p-2}$, using $a_0 = 1$ and $a_1 \ge R$. For fixed $m > 0$, the ratio of the left side to the right side in each inequality of (4.2) is a strictly increasing function of $v > 0$, with limits zero and infinity. Thus each inequality defines a positive lower endpoint for v. Let v_0 be the larger of the two endpoints. Then $v \ge v_0$, both inequalities hold at v_0, and at least one is an equality.

Define

$$q = \beta\left(1 + \frac{m}{v_0}\right) > \beta, \qquad L = \frac{v_0}{Bq^{p-1}}.$$

These definitions imply

$$\frac{v_0}{B} = Lq^{p-1}, \qquad \frac{m+v_0}{C} = Lq^p.$$

Consequently, the two inequalities at v_0 in (4.2) are exactly $L \geq 1$ and $Lq \geq R$. At least one is an equality, so

$$L = \max\{1, R/q\}. \tag{4.3}$$

The ultra-log-concavity inequality at index p yields

$$h_{p+1} - h_p \leq F(v), \qquad F(w) = \kappa \frac{(m+w)^2}{w} - m - w, \qquad \kappa = \frac{p(d-p)}{(p+1)(d-p+1)}.$$

Here $0 < \kappa < 1$, and

$$F'(w) = -\kappa \frac{m^2}{w^2} - (1-\kappa) < 0.$$

It follows that

$$0 < h_{p+1} - h_p \leq F(v) \leq F(v_0). \tag{4.4}$$

Moreover, $F(v) > 0$ implies

$$v < \frac{\kappa}{1-\kappa} m = \frac{p(d-p)}{d+1} m < (d-p)m. \tag{4.5}$$

We next check that passing from v to v_0 preserves the constraint at index d. For $w > 0$, set

$$A(w) = \frac{m+w}{C} \left(\beta \frac{m+w}{w} \right)^{d-p}.$$

The ratios a_{i+1}/a_i are nonincreasing. Hence

$$1 \leq a_d \leq a_p \left(\frac{a_p}{a_{p-1}} \right)^{d-p} = A(v).$$

On the other hand,

$$\frac{d}{dw} \log A(w) = \frac{w - (d-p)m}{w(m+w)} < 0 \qquad (0 < w \leq v),$$

where the last inequality follows from (4.5). Since $v_0 \leq v$, we obtain

$$Lq^d = A(v_0) \geq A(v) \geq 1. \tag{4.6}$$

For comparison, define

$$\widetilde{h}_0 = 1, \qquad \widetilde{h}_i = L \binom{d}{i} q^i \quad (1 \leq i \leq d).$$

Its binomial normalization is log-concave because $L \geq 1$. It satisfies $\widetilde{h}_1/d = Lq \geq R$ and $\widetilde{h}_d = Lq^d \geq 1$, and it has

$$\widetilde{h}_p - \widetilde{h}_{p-1} = m, \qquad \widetilde{h}_{p+1} - \widetilde{h}_p = F(v_0) > 0.$$

Thus, in view of (4.4), it remains to prove (4.1) with $F(v_0)$ on the left.

Set

$$t = \frac{\beta}{q-\beta} > 0, \qquad c = \frac{(p+1)(q-1)}{2}, \qquad Y = p+1+(p+1)\frac{F(v_0)}{m}.$$

Since $\beta = qt/(1+t)$, direct substitution gives

$$Y = \frac{p}{t} + 2p - q - t(q+1), \tag{4.7}$$

$$Y - p - 1 = (1+t)\left(\frac{p}{t} - q - 1\right) > 0, \tag{4.8}$$

$$m = \frac{Lq^p}{1+t}\binom{d}{p}. \tag{4.9}$$

In particular, $Y > p+1$ and $t < p/(q+1)$. Because $m = \binom{x}{p}$,

$$\binom{x}{p+1} - \frac{m}{p+1} = \frac{m(x-p-1)}{p+1}.$$

Therefore, it is enough to show $Y < x$, or equivalently $\binom{Y}{p} < m$.

For $0 \le i \le p-1$, put

$$\delta_i = q(d-i) - (Y-i) = (q-1)(p-i) + t(q+1), \qquad S = \sum_{i=0}^{p-1} \delta_i = pc + pt(q+1).$$

Every δ_i is positive. This is immediate when $q \ge 1$. If $0 < q < 1$, then (4.3) and (4.6) give $Rq^{d-1} \ge 1$, and hence

$$1 - q < -\log q \le \frac{\log R}{d-1} < \frac{1}{2(d-1)}. \tag{4.10}$$

Using $p \ge 3$ and $p \le d-1$, we have

$$p(1-q) < \frac{p}{2(d-1)} \le \frac{\beta}{2} < t(q+1),$$

so $\delta_i > 0$ in this case as well.

Since $Y > p+1$, all the following factors are positive, and expansion of the product gives the strict inequality

$$\frac{q^p\binom{d}{p}}{\binom{Y}{p}} = \prod_{i=0}^{p-1}\left(1 + \frac{\delta_i}{Y-i}\right) > 1 + \sum_{i=0}^{p-1}\frac{\delta_i}{Y-i} \ge 1 + \frac{S}{Y}. \tag{4.11}$$

The strict inequality uses $p \ge 3$ and $\delta_i > 0$ for every i. By (4.9), it now suffices to prove

$$1 + \frac{S}{Y} \ge \frac{1+t}{L}. \tag{4.12}$$

We consider three ranges of q.

Case 1: $q \ge R$. Here $L = 1$ and $c \ge (p+1)/4 \ge 1$. The identity

$$S - tY = p(c-1) + t[p(q-1)+q] + t^2(q+1) > 0$$

proves (4.12).

Case 2: $1 \le q \le R$. Here $L = R/q$. Define

$$E = t\{R(Y+S) - q(1+t)Y\}.$$

Substituting (4.7) and the expression for S yields the identity

$$\begin{aligned} E = (R-q)(p-qt) + \frac{3}{4}p(p-3)(q-1)t \\ + \left\{p\left(\frac{3}{2} - \frac{q}{2}\right) + (q-1)\left(2q + \frac{3}{2}\right)\right\} t^2 + q(q+1)t^3. \end{aligned} \tag{4.13}$$

Since $t < p/(q+1)$, the first summand is nonnegative. All the other summands are nonnegative, and the coefficients of t^2 and t^3 are positive. Thus $E > 0$, which implies (4.12).

Case 3: $0 < q < 1$. Again $L = R/q$, so we use (4.13). Inequality (4.10) and $p \le d-1$ imply $p(1-q) < 1/2$. Consequently, the coefficient of t^2 in (4.13) satisfies

$$p\left(\frac{3}{2} - \frac{q}{2}\right) + (q-1)\left(2q + \frac{3}{2}\right) > p - \frac{7}{4p} \ge \frac{p}{2}.$$

Also, $q(R-q) \le 9/16$ for $0 < q < 1$, and therefore

$$(R-q)(p-qt) \ge \frac{p}{2} - \frac{9}{16}t, \qquad \frac{3}{4}p(p-3)(q-1)t \ge -\frac{3}{8}(p-3)t.$$

Discarding the positive cubic term in (4.13) now gives

$$E > \frac{p}{2} - \frac{3p}{8}t + \frac{p}{2}t^2 = \frac{p}{2}\left[\left(t - \frac{3}{8}\right)^2 + \frac{55}{64}\right] > 0.$$

This proves (4.12) in the last case.

Combining (4.11), (4.12), and (4.9), we obtain $\binom{Y}{p} < m$. Hence $Y < x$, which proves the required strict inequality. □

5 The first two Kruskal–Katona inequalities

The first two ranks admit direct proofs without a trace hypothesis.

Lemma 5.1. *Let*

$$H(t) = \sum_{i=0}^{d} h_i t^i \in \mathbb{Z}_{\ge 0}[t]$$

be real-rooted with $h_0 = 1$ and $d = \deg H$, and put $h_i = 0$ for $i > d$ and $g_i = h_i - h_{i-1}$ for $i \ge 1$. If $g_2 > 0$, then $g_1 > 0$ and

$$g_2 < K_1(g_1). \tag{5.1}$$

If $g_2, g_3 > 0$, then

$$g_3 < K_2(g_2). \tag{5.2}$$

Proof. Factor $H(t) = \prod_{i=1}^{d}(1 + \lambda_i t)$ with $\lambda_i > 0$, and write $n = h_1 = \sum_i \lambda_i$. The hypotheses involving g_2 require $d > 0$. Since $\prod_i \lambda_i = h_d \ge 1$, the arithmetic–geometric mean inequality gives $n \ge d$.

Vertices and edges. Put $P_2 = \sum_i \lambda_i^2$. The Cauchy–Schwarz inequality gives $P_2 \geq n^2/d \geq n$, and therefore

$$g_2 = \frac{n^2 - P_2}{2} - n \leq \frac{n^2 - 3n}{2} = \binom{n-1}{2} - 1.$$

In particular, $g_2 > 0$ forces $n \geq 4$, so $g_1 = n - 1 > 0$ and $K_1(g_1) = \binom{n-1}{2}$. This proves (5.1).

Edges and triangles. Assume that $g_2, g_3 > 0$ and write

$$m = g_2 = \binom{r}{2} + s, \qquad r \geq 2, \qquad 0 \leq s < r.$$

The canonical expansion gives

$$K := K_2(m) = \binom{r}{3} + \binom{s}{2}.$$

The preceding estimate implies $m < \binom{n-1}{2}$, and hence $n \geq r + 2$. Adjoin $n - d$ zero parameters, and continue to denote the resulting n nonnegative parameters by $\lambda_1, \ldots, \lambda_n$. This leaves every coefficient of H unchanged. Since $h_1 = n$ and $h_2 = n + m$, Newton's inequality for these n parameters gives

$$h_3 \leq \frac{2(n-2)}{3n(n-1)} h_2^2,$$

and thus

$$g_3 \leq B(n, m) := \frac{2(n-2)(n+m)^2}{3n(n-1)} - n - m. \tag{5.3}$$

First suppose that $n \geq r + 3$. Regarding the first argument of B as a real variable $z \geq 5$, put $A_z = z^2 - 4z + 2 \geq 7$. Direct differentiation and completion of the square give

$$3\frac{\partial B(z, m)}{\partial z} = -1 + \frac{4}{A_z} - \frac{2A_z}{z^2(z-1)^2}\left(m - \frac{z^2}{A_z}\right)^2 < 0.$$

It follows that $g_3 \leq B(n, m) \leq B(r + 3, m)$. Set $x = r - 2 \geq 0$. The exact identity

$$3(r+3)(r+2)\big(K - B(r+3, m)\big) = As^2 + B_0 s + C$$

holds with

$$\begin{aligned} A &= (3x^2 + 23x + 48)/2, \\ B_0 &= -(4x^3 + 29x^2 + 81x + 84)/2, \\ C &= x^4 + 6x^3 + 21x^2 + 80x + 144. \end{aligned}$$

Moreover,

$$\begin{aligned} 4(4AC - B_0^2) = {} & 8x^6 + 96x^5 + 503x^4 + 2718x^3 \\ & + 14807x^2 + 43608x + 48240 > 0. \end{aligned}$$

Since $A > 0$, the expression

$$As^2 + B_0 s + C = A\left(s + \frac{B_0}{2A}\right)^2 + \frac{4AC - B_0^2}{4A}$$

is positive. Consequently $K > B(r+3, m) \geq g_3$.

It remains to treat $n = r + 2$. Put

$$u = r - 1 - s \geq 0, \qquad y_i = \lambda_i - 1 \quad (1 \leq i \leq n).$$

The padded parameters satisfy

$$\sum_i y_i = 0, \qquad h_2 = \binom{n}{2} - u, \qquad \sum_i y_i^2 = 2u.$$

Writing $e_3(y) = \sum_{i<j<\ell} y_i y_j y_\ell$, expansion around the all-ones vector gives

$$h_3 = \binom{n}{3} - (n-2)u + e_3(y),$$

and substitution of $n = r + 2$ and $s = r - 1 - u$ yields

$$K - g_3 = 2 + \frac{u + u^2}{2} - e_3(y).$$

As $\sum_i y_i = 0$, Newton's identities and the monotonicity of finite-dimensional ℓ^p norms give

$$3e_3(y) = \sum_i y_i^3 \leq \sum_i |y_i|^3 \leq \left(\sum_i y_i^2 \right)^{3/2} = (2u)^{3/2}.$$

Therefore

$$\begin{aligned} 6(K - g_3) &\geq 12 + 3u^2 + 3u - 4\sqrt{2}\, u^{3/2} \\ &= 12 + 3u \left(\sqrt{u} - \frac{2\sqrt{2}}{3} \right)^2 + \frac{u}{3} > 0. \end{aligned}$$

This proves (5.2) in the remaining case. The argument includes $u = 0$ and $n = 4$. When $r = 2$ and $m = 1$ or 2, one has $K = 0$, so the same strict inequality shows that these parameters cannot occur together with $g_3 > 0$. □

6 Restoring the factors $1 + t$

For a real-rooted polynomial $H(t) = \sum_{i=0}^{d} h_i t^i$ with nonnegative coefficients and $h_0 = 1$, set $h_i = 0$ outside $0 \leq i \leq d$, and set $g_i = h_i - h_{i-1}$ for $i \geq 0$. In particular $g_0 = 1$. If $d > 0$, all coefficients in the support are positive. Newton's inequalities imply that the ratios h_i/h_{i-1} strictly decrease, so there is a $k \geq 0$ such that

$$g_i > 0 \quad (1 \leq i \leq k), \qquad g_i \leq 0 \quad (i > k).$$

For $H = 1$ take $k = 0$. We call $(g_0, \ldots, g_k)$ the longest positive difference prefix, counting faces by their cardinality, with g_0 counting the empty face.

Lemma 6.1. *Let $H(t) \in \mathbb{Z}_{\geq 0}[t]$ be real-rooted with $H(0) = 1$. If its longest positive difference prefix is the face-number vector of a simplicial complex, then the same holds for $(1 + t)H(t)$.*

Proof. Let Δ realize $(g_0, \dots, g_k)$, and form its cone by adjoining a new vertex. For $0 \le i \le k$, the cone has $g_i + g_{i-1}$ faces of cardinality i, where $g_{-1} = 0$. It has g_k faces of cardinality $k+1$.

The differences of the coefficients of $(1+t)H(t)$ are

$$\widetilde{g}_i = g_i + g_{i-1}.$$

Thus $\widetilde{g}_i > 0$ for $1 \le i \le k$,

$$\widetilde{g}_{k+1} = g_k + g_{k+1} \le g_k,$$

and $\widetilde{g}_i \le 0$ for $i \ge k+2$. If $\widetilde{g}_{k+1} > 0$, retain exactly that many of the cone's faces of cardinality $k+1$; otherwise remove all faces of that cardinality. These are maximal faces, so deleting any collection of them preserves closure under taking subsets. All lower face numbers remain as required.

When $k = 0$, the realizing complex is $\{\varnothing\}$ and its cone has one vertex. The desired number of vertices is $1 + g_1 = h_1$. It is an integer between 0 and 1, since $h_1 \ge 0$ and $g_1 \le 0$. Keeping precisely this number of vertices therefore proves the assertion in this case as well. This includes $H = 1$, for which the new prefix has no vertex, and $H = 1 + t$, for which the new prefix has one vertex. □

7 Proof of the main theorem and products

Proof of Theorem 1.1. Remove all factors $1 + t$, writing $H = (1+t)^r Q$ as in Lemma 2.1. Suppose first that $Q \neq 1$, and write

$$Q(t) = \sum_{i=0}^{e} q_i t^i, \qquad b_0 = 1, \qquad b_i = q_i - q_{i-1} \quad (i \ge 1),$$

with $q_i = 0$ for $i > e$. All coefficients q_i, $0 \le i \le e$, are positive. Newton's inequalities imply that $q_i / \binom{e}{i}$ is log-concave. Moreover,

$$q_0 = 1, \qquad q_e \ge 1, \qquad q_1/e \ge 3/2.$$

Let $(1, b_1, \dots, b_\ell)$ be the longest positive difference prefix of Q. We verify (1.2) for each $1 \le p < \ell$. For $p = 1, 2$, Lemma 5.1 gives the required inequalities. For $p \ge 3$, both $m = b_p$ and b_{p+1} are positive, and $3 \le p \le e - 1$. Let $x \ge p$ be defined by $m = \binom{x}{p}$. Theorem 4.1 and Theorem 3.1 give

$$b_{p+1} < \binom{x}{p+1} - \frac{m}{p+1} \le K_p(m).$$

The Kruskal–Katona theorem therefore realizes the entire positive difference prefix of Q as the face-number vector of a simplicial complex. If $Q = 1$, this assertion holds with the complex $\{\varnothing\}$.

Apply Lemma 6.1 r times. This realizes the longest positive difference prefix of H. Its prefix of length $k+1$ is obtained by deleting all faces of cardinality greater than k, which preserves the simplicial-complex property. This gives the vector in the theorem. □

Corollary 7.1. *Let $H_1, \dots, H_s \in \mathbb{Z}_{\ge 0}[t]$ be real-rooted and satisfy $H_j(0) = 1$ for every j. Every strictly increasing initial coefficient segment of $H_1 \cdots H_s$ has first differences forming the face-number vector of a simplicial complex. In particular, this holds for the product of two irreducible cubic polynomials satisfying these hypotheses, and for multiplication by any further such cubic factor.*

Proof. The product has nonnegative integer coefficients, constant term one, and only real zeros. Apply Theorem 1.1. □

Irreducibility is thus unnecessary for the product conclusion. The arithmetic content of the argument lies in the trace reduction, while the two estimates in Sections 3 and 4 apply beyond polynomial factorization.

Remark 7.2. The proof is numerical in the sense of the Kruskal–Katona criterion: once the first differences are known, the usual colexicographic construction produces a realizing complex. It does not assert that this complex is pure, flag, shellable, or the boundary of a polytope.

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