Upper bounds for nearly equal largest prime factors of consecutive integers

Yilin Liu

23 November 2026

Abstract

Let $p < q$ be the largest prime factors of n and $n+1$, in either order. For $X^{3/4+\eta} \le H \le X$, the number of $n \in (X, X+H]$ with $q/p \le 1+d$ is $O_\eta(dH\log(2+d\sqrt{X})/(\log X)^2)$, uniformly for $0 < d \le 1$. Imposing $|n/(pq)-1| \le w \le 1/2$ gives $O_\eta(wdH/(\log X)^2)$. For fixed window widths both estimates hold down to $H = X^{2/3+\eta}$. At $H = X$, matching lower bounds give the order of the first count whenever $d\sqrt{X} \ge (\log X)^{1+\delta}$ for fixed $\delta > 0$.

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1 Introduction

Write $P^+(n)$ for the largest prime factor of $n > 1$. The largest prime factors $p < q$ of $n, n+1$ are distinct since n and $n+1$ are coprime. Let $N_\sigma(X, H, d)$ count the integers $X < n \le X+H$ with $q \le (1+d)p$, where $\sigma = +1$ if $p = P^+(n)$ and $\sigma = -1$ otherwise. Let $B_\sigma(X, H, d, w)$ count those that also satisfy $|n/(pq)-1| \le w$. We take $0 < H \le X$, $0 < d \le 1$, $0 < w \le 1/2$, and put $L = \log X$, $Y = d\sqrt{X}$, and $r = H/X$.

For $s, v > 0$ define

$$\mathcal{R}_X(s, v) = \min\left\{s,\ X^{-1/6}v^{1/3},\ X^{-1/3}v^{-1/3}\right\}.$$

Theorem 1.1. *For each fixed $\eta > 0$, uniformly in both orientations,*

$$N_\sigma(X, H, d) \ll_\eta \frac{dH\log(2+Y)}{L^2} \qquad \text{if } r \ge X^\eta \mathcal{R}_X(d, d). \tag{1}$$

For $H = X$ the bound is absolute. For sufficiently large X the count is zero if $(X+H)d^2 \le 2$. If $H = X$ and $1 < Y \le \sqrt{2}$, one also has

$$N_\sigma(X, X, d) \ll \frac{1+(Y-1)\sqrt{X}}{\log^2(2+(Y-1)\sqrt{X})}. \tag{2}$$

Theorem 1.2. *For each fixed $\eta > 0$, uniformly in both orientations,*

$$B_\sigma(X, H, d, w) \ll_\eta \frac{wdH}{L^2} \qquad \text{if } r \ge X^\eta \mathcal{R}_X(\min(d, w), dw). \tag{3}$$

For $H = X$ the bound is absolute. For sufficiently large X the count is zero if

$$wd\sqrt{\frac{X+H}{(1+d)(1-w)}} < 1 - \sqrt{3/X}.$$

The last two terms in $\mathcal{R}_X(s, v)$ cross at $v = X^{-1/4}$. Their minimum is at most $X^{-1/4}$, giving the uniform threshold $H \ge X^{3/4+\eta}$. For fixed positive widths the last term gives $H \ge X^{2/3+\eta}$ after decreasing η in the estimates.

Theorem 1.3. *For each fixed $c > 0$, uniformly for $cL \le Y \le \sqrt{X}$,*

$$N_\sigma(X, X, d) \gg_c \frac{dX}{L^2}\log(2+Y/L), \tag{4}$$

also with the strict endpoint $q < (1+d)p$. Hence the order is $dX\log Y/L^2$ when $Y \ge L^{1+\delta}$ for fixed $\delta > 0$.

Erdős and Pomerance [2] used complementary cofactors to study the separation of the largest prime factors of consecutive integers. For fixed ratio windows about 1, Kim [6] obtained a lower bound, and Yang [9, Theorem 1.7 and Remark 1.8] proved a lower bound of order $X/\log X$ and proposed the matching upper bound. We obtain that order in each orientation. Our upper bound uses reciprocal congruences modulo differences of prime factors or cofactors. A change to product coordinates separates the stationary phases, allowing the spectral large sieve on unit intervals.

2 Sieve estimates

For $j \geq 1$, write $W_j(v) = (v/\varphi(v))^j$ for positive integers v. For $V \geq 1$ we use

$$\sum_{v \leq V} W_j(v) \ll_j V, \qquad \sum_{v \leq V} \frac{W_j(v)}{v} \ll_j \log(2V), \qquad \sum_{k \leq K} W_j(|hk \pm 1|) \ll_j K \quad (K \geq h \geq 2). \tag{5}$$

Indeed $W_j(v) = \sum_{l|v} g_j(l)$ with g_j multiplicative, supported on squarefree integers and $0 \leq g_j(\ell) \ll_j \ell^{-1}$ for primes ℓ. The sum $\sum_l g_j(l)/l$ converges. For the last bound, a divisor l is coprime to h and imposes one residue on k. The terms K/l give $O_j(K)$, and the errors sum to $O_j((\log(2hK))^{O_j(1)}) = O_j(K)$ by the Euler product over $\ell \leq 2hK$ and Mertens' estimates.

We record the uniform version of Selberg's upper sieve used below. For $\nu \in \{1, 2, 3\}$ and $D \geq 1$, retain primes $\ell > 2\nu$ with $\ell \nmid D$. Let $0 < g(\ell) \leq \nu/\ell$ with $g(\ell) = \nu/\ell + O_\nu(\ell^{-2})$, and extend g multiplicatively on squarefree products of the retained primes. There are weights λ_j, supported on those products $j \leq z$, such that

$$\lambda_1 = 1, \qquad |\lambda_j| \leq j, \qquad \sum_{i,j} \lambda_i \lambda_j g([i,j]) \ll_\nu \frac{W_\nu(D)}{(\log z)^\nu} \qquad (z \geq 2). \tag{6}$$

To see this, put $A(j) = \prod_{\ell|j}(1/g(\ell) - 1)$ and $G(z) = \sum_j 1/A(j)$, summing over the support. The quadratic form equals $\sum_j A(j)(\sum_{j|k} \lambda_k g(k))^2$. Choose the inner sum to be $\mu(j)/(G(z)A(j))$. Inversion gives $\lambda_1 = 1$ and $|\lambda_j| \leq 1/(g(j)A(j)) \leq 2^{\omega(j)} \leq j$. This choice gives the value $1/G(z)$. In the Euler product for $\sum_j 1/A(j)$ over retained primes up to z^{c_ν}, the weighted mean of $\log j$ is $\sum_{\ell \leq z^{c_\nu}} g(\ell) \log \ell$ with the sum restricted to retained primes. This is at most $\frac{1}{2} \log z$ for small fixed $c_\nu > 0$, by $\sum_{\ell \leq x} (\log \ell)/\ell \ll \log(2x)$. At least half the mass lies at $j \leq z$. The product is $\gg_\nu (\log z)^\nu / W_\nu(D)$, by $(1 - g(\ell))^{-1} = (1 - 1/\ell)^{-\nu}(1 + O_\nu(\ell^{-2}))$ and Mertens' estimates. This proves (6).

Lemma 2.1. *For $\nu \leq 3$ integral affine forms with nonzero slopes and pairwise nonzero determinants, let D be the absolute product of the slopes and determinants. Among integers in an interval of length $T \geq 2$, the number on which all forms are prime and exceed $T^{1/20}$ is*

$$\ll_\nu \frac{T W_\nu(D)}{\log^\nu(2T)}.$$

Proof. For a retained prime there are exactly ν distinct roots of the product of the forms. With $z = T^{1/20}$, the squared sum of the weights λ_j over j dividing that product majorizes the prime-tuple indicator. Expansion gives the main term $T/G(z)$ and error $\ll \sum_{i,j \leq z} |\lambda_i \lambda_j| \nu^{\omega([i,j])} \ll z^6$. This error is absorbed by the claimed bound. Bounded T is covered by trivial counting. □

In particular, for $P \geq 2$ and $0 < d \leq 1$, applying Lemma 2.1 to $p, p + h$ and summing over $1 \leq h \leq 2dP$ gives

$$\#\{P \leq p < 2P,\ p < q \leq (1 + d)p :\ p, q \text{ prime}\} \ll dP^2/\log^2(2P). \tag{7}$$

Lemma 2.2. *Let $m \geq 1$, $(c, m) = 1$, and*

$$F(x, y) = \chi((x - x_0)/U)\, \psi((y - f(x))/V),$$

where χ, ψ are fixed smooth nonnegative functions of compact support. On an interval containing the support in x, suppose either f is constant and $D = 0$, or

$$|f''(x)| \asymp DV/U^2, \qquad |f'''(x)| \ll DV/U^3.$$

For each residue pair modulo t with $(t, m) = 1$, the F-weighted count on $xy \equiv c \bmod m$ in that pair is

$$\frac{\varphi(m)}{(mt)^2} \iint F + O_\varepsilon\left(m^\varepsilon \left[\sqrt{m}\left(\sqrt{1+D} + \frac{U}{m\sqrt{1+D}}\right) + \frac{V}{m}\right]\right). \tag{8}$$

At primes $\ell \nmid m$, $\ell > 2\nu$, let a fraction $g(\ell) = \nu/\ell + O_\nu(\ell^{-2})$ of residue pairs is forbidden, with $0 < g(\ell) \leq \nu/\ell$. The count avoiding those pairs for $\ell \leq z$ is

$$\ll_\varepsilon \frac{UVW_\nu(m)}{m(\log z)^\nu} + z^8 \mathcal{E}_m,$$

where $\mathcal{E}_m$ is the error bound in (8) with its implied constant.

Proof. Put $M = mt$ and use Poisson summation on the two residue coordinates. The zero mode gives the stated integral, and each other mode k, l has coefficient $M^{-2}\widehat{F}(k/M, l/M)$ times a Kloosterman sum modulo m, with both indices multiplied by units and with a factor of modulus one. Write $S(k, l; m) = \sum^*_{u \bmod m} e((ku + l\bar{u})/m)$, where $e(t) = e^{2\pi i t}$ and the star restricts to units. We use the Weil–Estermann bound $\tau(m)\sqrt{m(k, l, m)}$ and the Ramanujan bound $|S(k, 0; m)| \leq (k, m)$ [3, 5].

Integration in y gives $V\widehat{\psi}(Vl/M)$. For the x-integral, let $B = DV|l|/M$ and scale x by U. The derivative of the f-term ranges over an interval of length $O(B)$, and the second derivative has size $\asymp B$ for $B > 0$. Frequencies within distance $O(1 + B)$ of the derivative range contribute $O(U/\sqrt{1 + B})$ each by the second-derivative bound. Outside this interval, two integrations by parts give quadratic decay in the frequency distance. Summing over $g \mid k$ gives

$$C\left(\frac{M}{g}\sqrt{1+B} + \frac{U}{\sqrt{1+B}}\right).$$

For $B = 0$ the same bound follows by Fourier decay. For $l \neq 0$, summing $V|\widehat{\psi}(Vl/M)|(1 + B)^{\pm 1/2}$ over $g \mid l$ gives $O((M/g)(1+D)^{\pm 1/2})$. For the plus sign use $(1+Dv)^{1/2} \leq (1+D)^{1/2}(1+v)^{1/2}$, and for the minus sign compare the decreasing majorant $(1 + v)^{-3}(1 + Dv)^{-1/2}$ with its integral. After division by M^2, the total is

$$C\left(\frac{\sqrt{1+D}}{g^2} + \frac{U}{Mg\sqrt{1+D}}\right).$$

Use $\sqrt{(k, l, m)} \leq \sum_{g|(k,l,m)} \sqrt{g}$ and sum over $g \mid m$. For $l = 0$, summing the cutoff transform over the nonzero multiples of g costs $O(M/g)$; expanding the Ramanujan bound over divisors gives $O(\tau(m)V/M)$. These estimates prove (8).

For the sieve, expand the square of the weights in (6). For $t = [i, j]$ the simultaneous forbidden conditions give $t^2 g(t)$ residue pairs. Their main terms sum to $\varphi(m)(\iint F)g(t)/m^2$, and their errors cost at most $t^2\mathcal{E}_m$. Since $\sum_{i,j\leq z} |\lambda_i\lambda_j|[i, j]^2 \leq z^8$, the stated bound follows. □

We use dimension two for the prime values

$$P_1 = (xy - c)/m, \qquad P_2 = ((x + m)y - c)/m, \qquad c = \pm 1. \tag{9}$$

For $\ell \nmid m$ their zero sets modulo ℓ are disjoint and each has $\ell - 1$ points. Thus $g(\ell) = 2(\ell-1)/\ell^2$. The same holds with x, y interchanged.

3 Determinant and progression estimates

Put $e = (1-\sigma)/2$ and write $n + e = ap$, $n + 1 - e = bq$. With $h = q - p$ and $k = a - b$,

$$bh - kp = \sigma, \qquad n = \frac{k}{h}p(p+h) + \frac{\sigma p}{h} - e. \tag{10}$$

For $p > 2$, the gap h is even, so $bh - \sigma > 0$, $k \geq 1$, and $(h, k) = 1$. Since $0 < t = h/p \leq d$ and $q \neq 2p$, (10) gives

$$hk < nt^2 = hk + (hk + \sigma)t - et^2 \leq 3hk. \tag{11}$$

Here $hk \geq 2$ makes the lower inequality strict in both orientations. In particular $hk < (X + H)d^2$. The case $p = 2$ requires $q = 3$ and $d \geq 1/2$, so the exclusion in Theorem 1.1 follows for large X. Also $2/d \leq p \ll dX$ for $p > 2$, by $h \leq dp$ and $1 \leq k \ll dX/p$. There are therefore $O(\log(2 + Y))$ admissible dyads for p.

If $H = X$ and $1 < Y \leq \sqrt{2}$, then $hk < 4$, hence $h = 2$ and $k = 1$. By (10), $n \geq p^2/2$, so the prime pair $p, p + 2$ has p in $[2/d, 2\sqrt{X}]$. Apply Lemma 2.1 on an interval of length $2 + O((Y - 1)\sqrt{X})$ to obtain (2).

3.1 Progressions in the two counts

On a dyad $p \asymp P$, write $A = X/P$ and take a gap dyad $h \asymp d_0 P$. Then $k \asymp d_0 A$ by (10). At fixed h, k, the congruence $kp \equiv -\sigma \bmod h$ puts p in one class modulo h, and $n'(p) \asymp A$ in (10). The progression parameter lies in an interval of length $O(r/d_0)$. If $d_0 \leq rX^{-\varepsilon}$, enlarge the interval to length $\asymp r/d_0$, which is $\gg X^{\varepsilon}$ and $O(P)$. Lemma 2.1 applies to $p, p + h$, with weight $W_2(h)$. Summing over h, k gives

$$\ll_{\varepsilon} \frac{rA}{L^2} \sum_{h \asymp d_0 P} W_2(h) \ll_{\varepsilon} \frac{rd_0 X}{L^2}. \tag{12}$$

When $r \geq X^{\eta} d$, choose $\varepsilon < \eta$ so this covers all gap dyads, since $d_0 \ll d$. In this range $p > 2$ for large X. Summing geometrically over d_0 and then over P proves the progression case of (1).

For the product band, $p \in [\sqrt{X/3}, 2\sqrt{X}]$, and

$$\frac{n}{pq} = \frac{k}{h} + \begin{cases} 1/(hq), & \sigma = +1, \\ -1/(hp), & \sigma = -1. \end{cases} \tag{13}$$

As $k \neq h$, this gives $1 - 1/p \leq wh$ and $|k - h| \ll wh$. The inequality

$$h \leq d\sqrt{\frac{X + H}{(1 + d)(1 - w)}}$$

follows from $h = t\sqrt{n/((1 + t)(n/(pq)))}$ and the monotonicity of $t/\sqrt{1 + t}$. This proves the exclusion in Theorem 1.2.

The integers $b - p$ and $k - h$ satisfy $(b - p)h - (k - h)p = \sigma$, so they have the same sign τ and neither is zero. Put $u_0 = |b - p|$, $j = |k - h|$, and $c = \tau\sigma$. Then

$$u_0 h - jp = c, \qquad q = p + h, \qquad n + 1 - e = (p + \tau u_0)q. \tag{14}$$

Since $1 \leq u_0 \leq wp + 1/q$, we have $u_0 \ll w\sqrt{X}$. In dyads $h \asymp d_0\sqrt{X}$, $u_0 \asymp w_0\sqrt{X}$, one has $j \asymp d_0 w_0 \sqrt{X}$.

If $d_0 \leq rX^{-\varepsilon}$, use the h, k progression above with $P \asymp \sqrt{X}$ and only $O(d_0 w_0 \sqrt{X})$ choices for $k = h + \tau j$. If $w_0 \leq rX^{-\varepsilon}$, fix u_0, j, τ instead. Now $u_0 q - (u_0 + j)p = c$ puts p in one

class modulo u_0, and $n'(p) \asymp \sqrt{X}$ in (14). Enlarge the parameter interval to length $\asymp r/w_0$, between X^ε and $O(\sqrt{X})$. The prime forms have slopes u_0, u_0+j and determinant of absolute value one. The weights $W_2(u_0)W_2(u_0+j)$ sum to $O(w_0\sqrt{X})$ at fixed j, by Cauchy–Schwarz and (5). In both cases the sum of the weights cancels the reciprocal gap scale from the progression length. Thus the band dyad contributes

$$\ll_\varepsilon \frac{rd_0w_0X}{L^2} \qquad (\min(d_0,w_0) \le rX^{-\varepsilon}). \tag{15}$$

If $r \ge X^\eta \min(d,w)$, all gap dyads satisfy this condition for $\varepsilon < \eta$. Summing over d_0, w_0 proves the progression case of (3).

3.2 The outer prime-factor ranges

Take a dyad $P \le p < 2P$, and assume $H \ge X^{2/3}$. For $X^{1/10} < P \le \sqrt{H}$, each prime pair gives at most $1 + H/(pq) \ll H/P^2$ integers by the Chinese remainder theorem. Equation (7) gives $O(dH/L^2)$ in the dyad.

For $2 \le P \le X^{1/10}$ every prime factor of a is at most $2P$. Multiply its prime factors, with multiplicity, until their product v is in $[X^{1/10}, 2PX^{1/10}]$. This is possible since $a \ge X/(2P)$. Set $\xi = 1/(2\log(2P))$ and define g_ξ multiplicatively by $g_\xi(\ell^j) = \ell^{j\xi} - \ell^{(j-1)\xi}$ for primes $\ell \le 2P$, and zero for other prime powers. Then

$$X^{\xi/10} \le v^\xi \le \sum_{\substack{j\mid a\\ j\le 2PX^{1/10}}} g_\xi(j), \qquad \sum_{j\ge1}\frac{g_\xi(j)}{j} = \prod_{\ell\le 2P}\frac{1-1/\ell}{1-\ell^{-1+\xi}} \ll 1.$$

The last estimate follows from $\ell^\xi - 1 \ll \xi\log\ell$ and $\sum_{\ell\le 2P}(\log\ell)/\ell \ll \log(2P)$. For a fixed pair p,q and an eligible j, the conditions $pj \mid n+e$ and $q \mid n+e+\sigma$ give at most $O(H/(pqj))$ integers since $pqj \ll X^{2/5}$. If $q \mid j$ the conditions are incompatible. Using (7), the dyad contributes

$$\ll \frac{dH}{\log^2(2P)}\exp\Big(-\frac{L}{20\log(2P)}\Big) \ll \frac{dH}{L^2}.$$

At the opposite end, put $A = X/P$ and suppose $A \le \sqrt{H}X^{-\varepsilon}$. Then $a, b \asymp A$, and $1 \le k \le 2bh/p \ll dA$ by (10). For fixed a, b, $(a,b) = 1$ and $ap + \sigma = bq$ gives two affine forms for p, q with slopes b, a and determinant of absolute value one. Their parameter interval has length $H/(ab) \asymp H/A^2 \gg X^{2\varepsilon}$, and the prime values are $\gg P$. Lemma 2.1 gives $O_\varepsilon(HW_2(a)W_2(b)/(A^2L^2))$. At each k the sum of $W_2(a)W_2(a-k)$ over the relevant $a \asymp A$ is $O(A)$ by Cauchy–Schwarz and (5). Summing over $k \ll dA$ gives $O_\eta(dH/L^2)$ in the dyad.

4 A sieve for product constraints

For $m = MK$, $x = X_1Ku$, and $y = X_2Kv$, take the nonnegative weight

$$F_m(x,y) = \chi(K,u,v)\,\Psi\left(\frac{K^2Q_1(u)Q_2(v)-1}{r}\right). \tag{16}$$

Here K, u, v lie in fixed compact positive intervals. Each $Q_i(t)$ is t or $t(t+\gamma_i)$, with bounded γ_i and with $t+\gamma_i$ in a fixed positive compact interval on the support. The smooth cutoffs have uniformly bounded derivatives and are supported where $K^2Q_1(u)Q_2(v) \asymp 1$. We count $xy \equiv c \bmod m$, $c = \pm1$, with either $x, x+m$ or the pair in (9) prime above $z = X^{c_0\varepsilon}$, for sufficiently small fixed $c_0 > 0$.

Lemma 4.1. *Suppose $X^{-1/3+\eta} \le r \le 1$, $X_1, X_2, M \le X^2$, and $M/X_i \ll 1$. If $M \gg 1$ and $X_i/M \ll r^{-1}X^{2\varepsilon}$ for $i = 1, 2$, the count with weight (16) is*

$$\ll_\varepsilon \frac{rX_1X_2}{L^2} + X^{O(\varepsilon)}\frac{M^{3/2}}{\sqrt{r}}. \tag{17}$$

If $M \ge X^{1/20}$ and $X_1X_2 \gg X^{3/4}$, it is also

$$\ll_{\eta,\varepsilon} \frac{rX_1X_2}{L^2} + X^{\varepsilon}\frac{\sqrt{X_1X_2}}{\sqrt{r}}. \tag{18}$$

Proof. At fixed K, solving $K^2Q_1(u)Q_2(v) = 1$ for v gives a graph with $-v' \asymp 1$ and $v'' \asymp 1$, by $(\log Q_i)' \asymp 1$ and $(\log Q_i)'' \asymp -1$. Its derivatives are uniformly bounded on the relevant compact intervals. The weight is majorized by a strip of thickness $O(r)$ in v, so Lemma 2.2 applies with $U \asymp X_1$, $V \asymp rX_2$, and $D \asymp 1/r$. The local densities outside m are $2/\ell$ for the affine pair and $2(\ell-1)/\ell^2$ for (9). Summing the main terms by (5) and the errors $O(X^{O(\varepsilon)}\sqrt{M}/\sqrt{r})$ per modulus proves (17).

For (18), sieve the product of the two prime values. Every squarefree sieve divisor D splits uniquely into D_1D_2, according to which value is divisible by each prime factor. The assignments are disjoint because a common prime divisor would divide c. Let $\alpha \equiv 0 \bmod D_1$, $\alpha \equiv 1 \bmod D_2$.

For the values (9) the sifted congruence is exactly

$$y(x + \alpha m) \equiv c \bmod mD.$$

This follows by reducing modulo mD_1 and mD_2, whose least common multiple is mD. Poisson summation gives $S(k, cl; mD)$ with the phase $e(-k\alpha/D)$, up to a choice of Fourier signs, and weight $(mD)^{-2}\widehat{F}_m(k/(mD), l/(mD))$.

For $x, x+m$ the restriction is $x = Dx' - \alpha m$ and $(m, D) = 1$. Poisson summation uses $x'y \equiv c\bar{D} \bmod m$ and the complete sum $S(k, cl\bar{D}; m)$. Lift it using

$$S(Dk, cl; mD) = c_D(l)S(k, cl\bar{D}; m), \qquad (m, D) = 1.$$

Here $c_D(l)$ is a Ramanujan sum, nonzero for squarefree D. The shift phase depends only on k, the multiplier $1/c_D(l)$ has modulus at most one, and Möbius inversion removes $(m, D) = 1$ at levels Dd_0, $d_0 \mid D$. In this case the weight is $D(mD)^{-2}\widehat{F}_m(k/(mD), Dl/(mD))$.

For the zero mode put $W(m) = m^{-1}\iint F_m$. Changing v into $K^2Q_1(u)Q_2(v)$ at fixed K, u shows

$$\int W \ll rX_1X_2, \qquad \sup|W| + \int |W'| \ll rX_1X_2/M.$$

The zero modes average $D^{-1}\varphi(mD)/(mD)$ or $D^{-1}\mathbf{1}_{(m,D)=1}\varphi(m)/m$ against $W(m)$. Both have mean $\zeta(2)^{-1}\prod_{\ell\mid D}(\ell+1)^{-1}$. Expanding $\varphi(n)/n = \sum_{a\mid n}\mu(a)/a$ gives $1 - \ell^{-2}$ at unrestricted primes and $(1-1/\ell)/\ell$ at primes dividing D. Partial summation in this expansion costs $O(\tau(D)L(\sup|W| + \int|W'|))$, since each congruence contributes $O(\tau(D))$ to the counting error and the reciprocal divisor sum is $O(L)$. Thus summing the assignments gives $g(\ell) = 2/(\ell+1)$ in (6). The zero-mode main term is $O_\varepsilon(rX_1X_2/L^2)$, and the error saves a power since $M \ge X^{1/20}$.

Let $\tilde{k}, \tilde{l}$ be the Fourier numerators at denominator mD, with effective frequencies $a_1 = \tilde{k}X_1/(MD)$, $a_2 = \tilde{l}X_2/(MD)$. Fix $B = X^{\varepsilon_0}$ with $\varepsilon_0 > 0$ small. For $\max(|a_1|, |a_2|) \le B$ away from the origin, $M/X_i \ll 1$ gives $O((DB)^2)$ modes with original indices $O(DB)$. The Weil bound and $\widehat{F}_m \ll rX_1X_2$ give relative error $M^{-1/2}X^{O(\varepsilon_0+c_0\varepsilon)}$, including modes on the axes.

For the other modes use $K^2Q_1(u)Q_2(v) = 1+rs$. At fixed K, s, $-dv/du \asymp 1$, so integration by parts in u discards incomparable frequencies. At fixed K, u, $\partial v/\partial s \asymp r$, which discards

$|a_2| > B/r$. In each discarded range repeated integration gives arbitrary power saving after summation. The remaining dyads have $|a_1|, |a_2| \asymp U$ with $B \ll U \ll B/r$.

Apply the inverse Kuznetsov formula at level D, or Dd_0 for the lifted affine case [1, Lemma 3]. The Bessel transform uses dm/m. After extracting $S(k', l'; mD)/(mD)$, the measure is $X_1X_2/(MD)\,du\,dv\,dK$, with an extra factor D in the affine case. Changing to $\Theta = K^2Q_1(u)Q_2(v) = 1 + rs$ gives a smooth amplitude of size $O(rX_1X_2/M)$. The product of the lifted indices has magnitude $|\tilde{k}\tilde{l}|$ in either case, so the phase and Bessel argument are

$$e(-a_1u - a_2v), \qquad \xi = \frac{4\pi\sqrt{|a_1a_2|}}{\sqrt{X_1X_2}}\sqrt{\frac{Q_1(u)Q_2(v)}{\Theta}}. \tag{19}$$

The Bessel argument is $O(U/\sqrt{X_1X_2}) \ll X^{-1/24+O(\varepsilon_0)}$.

Expand the inverse kernels using the series of $J_{\pm 2it}$ and $I_{\pm 2it}$. For real $|t| \geq 1$, Stirling's formula gives coefficients of size $O(|t|^{-1/2})$ after extracting $\xi^{\pm 2it}$, and the small Bessel argument permits truncation with arbitrary power saving by taking enough terms. At fixed Θ the two spatial phases are

$$-2\pi a_1 u \ \pm t\log Q_1(u), \qquad -2\pi a_2 v \ \pm t\log Q_2(v).$$

Unless $|t| \asymp U$ with compatible signs, one derivative is $\gg U + |t|$, giving arbitrary decay by integration by parts. For real $|t| \leq 1$, the differences of Bessel series cost at most $O(1+|\log\xi|)$. For imaginary t the powers have polynomial size in X and bounded logarithmic derivatives, so the same argument applies to the Fourier phases. For the holomorphic kernel $J_{k-1}(\xi)$, integrate by parts when $k \leq U^{1/2}$ and use $(C\xi)^{k-1}/\Gamma(k)$ otherwise. The ordinary spectral large sieve [1, Lemma 4] bounds the coefficient sums in these discarded ranges.

With compatible signs, $(\log Q_i)' = 2\pi|a_i|/|t|$ has a unique stationary solution since $(\log Q_i)'$ decreases from infinity to zero. The phase curvatures have size $\asymp U$, uniformly in γ_i. Applying stationary phase in each variable [7, Theorem 3] gives a finite expansion with arbitrary power-saving remainder. Each term has amplitude $O(rX_1X_2/(MU^{3/2}))$, from two factors $U^{-1/2}$ of stationary phase and the Bessel coefficient. Its phase separates into $\exp(i\Phi_{1,\tilde{k}}(t))\exp(i\Phi_{2,\tilde{l}}(t))$, with the powers $|a_i|^{\pm it}$ absorbed separately. The normalized amplitudes have bounded derivatives in the scaled indices. The stationary points depend smoothly on $a_i/U, t/U, \gamma_i$, stay in compact subsets of the positive domains, and may be evaluated against cutoffs extended by zero. Differentiation of the stationary actions gives bounds $O(\log X)$ for their first t-derivatives and $O_j(U^{1-j})$ for order $j \geq 2$.

Expand the amplitudes in Fourier series in the normalized indices, with summable bounds on the suprema of the coefficients over t. On each unit interval at $|t| \asymp U$, remove the phase values at its midpoint. The resulting exponentials, with a smooth extension to a bounded interval, have Fourier expansions in t of ℓ^1 norm bounded by powers of $\log X$, by the derivative bounds. The separated coefficient vectors have norm product

$$\ll X^{O(c_0\varepsilon)}\frac{MU}{\sqrt{X_1X_2}},$$

by the frequency spacings in a_1, a_2. For a nonempty dyad its length suffices for its lattice spacing. The lifted indices are $\ll D^{O(1)}U$ since $M/X_i \ll 1$.

At level q, Zhang's unit-interval large sieve [10, Theorem 2.2.1] bounds the cuspidal and continuous quadratic forms on coefficients supported at size N by $\ll_\varepsilon (qUN)^\varepsilon(N + qU)$ times the squared norm. See also Lam [8, Theorem 2.5] for the cuspidal form. In the normalization of [1, Lemma 3], the coefficients in these quadratic forms are $\sqrt{|n|}\rho(n)$, with spectral weight $1/\cosh(\pi t)$. The identity $W_{0,it}(4\pi|n|y) = 2\sqrt{|n|y}K_{it}(2\pi|n|y)$ converts Zhang's K-Bessel

coefficients. Dividing Zhang's estimate by the hyperbolic volume converts his probability-measure cusp basis to the area-measure basis and cancels the explicit volume multiplying his Eisenstein term. The Gaussian is bounded below on the unit interval. Thus Cauchy–Schwarz costs $UX^{O(\varepsilon_0+c_0\varepsilon)}$ times the norm product per unit interval. Summing $O(U)$ intervals with the transform size gives

$$\ll X^{O(\varepsilon_0+c_0\varepsilon)}\, r\sqrt{X_1X_2}\, U^{3/2}.$$

The coefficient norms are unchanged for negative indices, by parity for cusp forms and reflection of cusps for Eisenstein series. Summing dyads and sieve terms with small ε_0, c_0 proves the estimate. □

5 Applying the product sieve

Write $\omega = d$ for the full count and $\omega = dw$ for the band, and put

$$S = \min\{X^{-1/6}\omega^{1/3}, X^{-1/3}\omega^{-1/3}\}.$$

The progression cases were proved in Section 3. The remaining hypothesis is $r \geq X^{\eta}S$. By the exclusions in Theorems 1.1 and 1.2, a nonempty count has $\omega \gg X^{-1/2}$. Hence $r \gg X^{-1/3+\eta}$, which suffices for Lemma 4.1 with a smaller η. Throughout, ε is sufficiently small in terms of η.

5.1 The full ratio window

For $p \asymp P$, set $A = X/P$. Section 3 covers $P \leq \sqrt{H}$ and $A \leq \sqrt{H}X^{-\varepsilon}$. Equation (12) covers the gap dyads $h \asymp d_0P$ with $d_0 \leq rX^{-\varepsilon}$. In the remaining ranges set $R = \min(P, A)$ and $R' = X/R$. Use the scales

$$M = d_0R, \qquad X_1 = R, \qquad X_2 = d_0R'.$$

For $P \leq \sqrt{X}$ take $(m, x, y) = (h, p, k)$. The congruence is $xy \equiv -\sigma \bmod m$ and the prime forms are $x, x+m$. For $P > \sqrt{X}$ take $(m, x, y) = (k, b, h)$. The congruence is $xy \equiv \sigma \bmod m$ and the primes are (9). In both cases

$$\frac{n}{X} = K^2u(u+d_0)v + O\big(1/(d_0X)\big)$$

under the substitutions in (16). The two identities giving this model are (10) and $n = (b+k)(bh-\sigma)/k - e$. The error is $o(r)$ since $d_0 > rX^{-\varepsilon}$. Also $R \leq R'$ and $R^2 \geq HX^{-2\varepsilon}$, so the scales meet the hypotheses of Lemma 4.1. Choose its cutoffs to majorize the dyad and the interval for n. Here $p, q \geq P > \sqrt{H}$ exceed z.

5.2 The product band

Use d_0, w_0, u_0, j from (14) and the subsequent gap decomposition. Equation (15) covers $\min(d_0, w_0) \leq rX^{-\varepsilon}$. Otherwise take $(m, x, y) = (j, u_0, h)$ with

$$M = d_0w_0\sqrt{X}, \qquad X_1 = w_0\sqrt{X}, \qquad X_2 = d_0\sqrt{X}.$$

The prime forms are (9) with $c = \tau\sigma$. In the model variables, (14) gives $p = K\sqrt{X}\, uv - c/j$ and

$$\frac{n}{X} = K^2u(u+d_0)v(v+\tau w_0) + O\big(1/(d_0w_0X)\big).$$

The error is $o(r)$ since $d_0, w_0 > rX^{-\varepsilon}$ and $r \gg X^{-1/3+\eta}$. The range $b \asymp \sqrt{X}$ gives $v+\tau w_0 \asymp 1$, which is imposed on the support of the majorant. The scale conditions follow from $d_0, w_0 > rX^{-\varepsilon}$.

5.3 Summation

Write $\omega_0 = d_0$ or $d_0 w_0$, so $X_1 X_2 = \omega_0 X$ and $M \le \omega_0 \sqrt{X}$. Use (17) when $\omega_0 \le X^{-1/4}$ and (18) otherwise. In the latter range $M \gg X^{1/12-\varepsilon}$, so both size conditions in (18) hold. The error in either case is

$$\ll \frac{X^{O(\varepsilon)}}{\sqrt{r}} \min\{X^{3/4}\omega_0^{3/2}, X^{1/2}\sqrt{\omega_0}\},$$

with the terms crossing at $\omega_0 = X^{-1/4}$. Both the main term and this error sum geometrically over the gap dyads, since halving ω_0 multiplies each error term by at most $2^{-1/2}$. The main terms total $O_\eta(\omega H/L^2)$, per prime-factor dyad for the full count. The total error divided by ωH is $O(X^{O(\varepsilon)}(S/r)^{3/2})$, which saves a power of X. Finally the full count sums over $O(\log(2+Y))$ prime-factor dyads, including the possible dyad with $p = 2$. This proves the upper bounds.

6 The matching range

We count integers with two prescribed prime divisors and discard those for which a prescribed prime is smaller than the largest prime factor. Yang [9, §5] uses this decomposition for fixed d. Here the range of prime dyads varies with Y.

Let P be a power of two with $4 \le \sqrt{X}/P \le Y^\beta$, where $0 < \beta \le 1/1000$ is a small absolute constant. The prime number theorem gives $S \gg P/L$ primes in $[P, 2P)$. Divide this interval into at most $3/d$ intervals of length at most $dP/2$. Cauchy–Schwarz gives at least $(dS^2/3 - S)/2$ pairs in a common interval, all with $p < q < (1+d)p$ after ordering. Thus there are $\gg dP^2/L^2$ pairs whenever $dP \ge C_0 L$ for a large absolute C_0. As $pq \le X/4$, the congruences $p \mid n+e$, $q \mid n+1-e$ give at least $X/(2pq)$ integers $X < n \le 2X$ for each pair. The initial count is $\gg dX/L^2$.

In a discarded triple (n, p, q), one cofactor is $m\rho$ with ρ a larger prime and $m \le M = 3X/P^2$. For each choice of cofactor, the number of such tuples is

$$\ll \frac{dX \log(2M)}{L^3} + X^{4/5} \mathbf{1}_{\{d \ge X^{-1/100}\}}, \tag{20}$$

uniformly in β. For $d \ge X^{-1/100}$, fix m and the prime s on the side without ρ. The other prescribed prime v lies in an interval of length $O(dP)$, and $v\rho \equiv \pm m^{-1} \bmod s$ since $s \nmid m$. Sift both coordinates in Lemma 2.2 with f constant, second side $3X/(mP)$, $z = X^{1/1000}$, and local density $2/\ell - 1/\ell^2$. The main term is $O(dX/(mPL^2))$ per m, s. Summing over $m \le M$ and $O(P/L)$ primes s gives the first term of (20). The total error is

$$\ll PMz^8 P^{1/1000}(\sqrt{P} + 1 + M) \ll X^{4/5},$$

since $M \le 3Y^{2\beta} \le 3X^\beta$.

For $d < X^{-1/100}$, fix $h = q - p$, m, and $k = a - b$. With $s = q$ if $a = m\rho$, and $s = p$ if $b = m\rho$, (10) gives $hm\rho = ks + \sigma$ and $1 \le k \ll hM$. Since $(k, hm) = 1$, s lies in one class modulo hm. The forms $s, s \pm h, \rho$ have slopes hm, hm, k and absolute determinants $h^2 m, 1, |\sigma \mp kh|$, all nonzero as h is even. Enlarge the parameter interval to length $2P/(hm) \gg 1/(dM) \gg X^{9/1000}$. Lemma 2.1 bounds the contribution by

$$\ll \frac{P}{hmL^3} W_3(h) W_3(m) W_3(k) W_3(|\sigma \mp kh|).$$

The last two weights sum to $O(hM)$ over $k \ll hM$ by Cauchy–Schwarz and (5). Summing over $m \le M$ and $h \le 2dP$ proves (20).

Since $\log(2M) \le \log 6 + \beta L$, a sufficiently small β makes the first term of (20) smaller than a fixed fraction of the initial count. The second term is negligible for $dX \ge X^{99/100}$. Each surviving triple specifies the actual largest prime factors and so contributes a distinct integer in its orientation.

When Y/L exceeds a large absolute constant, use the dyads with $4 \le \sqrt{X}/P \le \min\{Y^\beta, Y/(C_0 L)\}$. There are $\gg_\beta \log(2 + Y/L)$ such dyads and all satisfy $dP \ge C_0 L$.

For $cL \le Y \le C_1 L$ with c, C_1 fixed, take $4 \le \sqrt{X}/P < 8$. Goldston, Pintz and Yıldırım [4, Theorem 1, (1.3)] give $\gg_c P/L$ consecutive prime pairs in this dyad with gap at most $\lambda_0 \log P$ for a fixed $0 < \lambda_0 < c/16$. They lie in the strict ratio window since $dP \ge cL/8$, and $P/L \gg_{C_1} dP^2/L^2$. Here $M = O(1)$ and $d < X^{-1/100}$, so (20) discards only $O(dX/L^3)$ tuples. This gives (4) in the remaining range.

For fixed $0 < d \le 1$, summing dyadic intervals gives the upper bound of order $x/\log x$ up to x, and the final dyadic interval supplies the lower bound. For fixed $d > 1$, the same per-dyad upper proof uses $q \asymp_d P$, $h \ll_d P$, and cofactors of order X/P, and one sums $O(\log X)$ prime-factor dyads. The lower bound follows by inclusion. For a fixed descending window $1 - \varepsilon \le P^+(n+1)/P^+(n) < 1$, take $d = \varepsilon/(1-\varepsilon)$ in the opposite orientation.

Acknowledgment. The author used OpenAI's GPT-6.2 Astra to formulate the results, develop and review the proofs, and draft the manuscript.

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