Sharp dimension bounds for right-angled hyperbolic polytopes

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Abstract

We determine the largest dimensions in which right-angled hyperbolic polytopes of finite volume can exist. The dimension is at most eight in general, and at most four when all vertices lie at infinity. Both bounds are attained, improving the previously known upper bounds of twelve and six. Our proofs use elementary properties of hyperbolic polytopes together with bounds for configurations of points on spheres.

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1 Introduction

Right-angled hyperbolic polytopes provide fundamental domains for reflection groups. Potyagailo and Vinberg [5] construct finite-volume examples through dimension eight. Dufour [2] proves that their dimension is at most twelve. For polytopes whose vertices are all ideal, the regular right-angled 24-cell gives an example in dimension four, while the upper bound six was established in [3, Corollary 7]. We close these two gaps. The compact case already has the sharp dimension bound four; see [5, Section 2].

A polytope is *entirely ideal* if all its vertices are ideal. Throughout, hyperbolic polytopes have finitely many facets and finite volume, and the curvature is normalized to -1. A right-angled polytope has dihedral angle $\pi/2$ at every ridge. Finite and ideal vertices are allowed in any combination.

Theorem 1 (All vertex types)**.** *A finite-volume right-angled hyperbolic polytope has dimension at most eight.*

Theorem 2 (Entirely ideal polytopes)**.** *An entirely ideal finite-volume right-angled hyperbolic polytope has dimension at most four.*

Finite-volume right-angled examples in dimension eight are recalled in [5]. The regular ideal right-angled 24-cell gives the four-dimensional example. Thus both bounds are sharp.

1.1 Outline of the proofs

Both arguments begin with the same geometric observation. Projecting a finite interior point to a facet of maximum unit-normal height gives an interior point of that facet and decreases every inherited height by a controlled factor. Repetition produces a low-dimensional face whose outward normals determine a spherical code.

For Theorem 1, three projections in dimension nine give a six-dimensional right-angled face whose facets determine a spherical code with pairwise scalar products strictly below $1/4$. An explicit cubic sum of tensor squares bounds the cardinality of this code, and hence the number of facets, by 26. This contradicts the known minimum of 27 facets in dimension six.

For Theorem 2, one projection in dimension five gives an entirely ideal four-dimensional facet. Its facets determine a code on S^3 with pairwise scalar products strictly below $1/2$,

so Musin's kissing-number theorem bounds its number of facets by 24. The minimum-facet theorem [3, Theorem 4] forces equality and identifies the facet, up to hyperbolic isometry, with the regular ideal right-angled 24-cell. The heights at two opposite facets then contradict the strict height bound from the projection, excluding the equality case.

The projection is proved in Section 2; the two contradictions follow in Sections 3 and 4. Figures 1 and 2 summarize the two reductions.

1.2 Previous work

We recall three results used in the proofs.

Proposition 3 (Six-dimensional facet minimum)**.** *A six-dimensional finite-volume right-angled hyperbolic polytope has at least* 27 *facets.*

This follows from Lemma 1 and Proposition 4 of Dufour [2]; see also [1, Section 4].

Proposition 4 (Musin)**.** *A finite subset of $S^3 \subset \mathbb{R}^4$ with pairwise products at most* $1/2$ *between distinct members has cardinality at most* 24*.*

This is the Main theorem of [4, p. 4].

Proposition 5 ([3, Theorem 4])**.** *An entirely ideal right-angled hyperbolic four-polytope has at least* 24 *facets. Equality holds only for the regular ideal right-angled* 24*-cell, up to hyperbolic isometry.*

The last two propositions are needed only for Theorem 2. The spherical bound used for Theorem 1 is proved directly in Section 3.

For a right-angled polytope in the Lorentz model, distinct outward unit facet normals have nonpositive products, including facets that do not share a ridge. As explained in [6, p. 42, §2.1], nonobtuse ridge angles imply this condition: nonadjacent supporting planes do not intersect, and the off-diagonal Gram entries are nonpositive. Faces of dimension at least three inherit right angles: every intrinsic ridge has a finite relative-interior point, where the incident walls are orthogonal. In the closure in the Klein model, a face has vertices among those of the original polytope, so entire ideality is inherited. Triangulation into finite or ideal simplices gives finite volume.

2 Facet projections

Use Lorentz signature $(-,+,\dots,+)$, future unit timelike points, and outward *unit* spacelike normals. Thus

$$P = \{x \in \mathbb{H}^n : \langle x, m\rangle \le 0 \text{ for every facet normal } m\}, \qquad \langle m, m'\rangle \le 0 \quad (m \ne m').$$

All heights below refer to this fixed unit-normal scaling. They are the hyperbolic sines of distances to the corresponding walls.

Lemma 6 (Farthest-facet projection)**.** *Let q be a finite relative-interior point of a right-angled polytope P, with $q^2 = -1$. Set $t(m) = -\langle q, m\rangle > 0$, choose a facet normal n with largest height $t = t(n)$, and put*

$$q' = \frac{q + tn}{\sqrt{1+t^2}}. \tag{1}$$

Then q' lies in the relative interior of the facet $Q = P \cap n^\perp$. Each facet of Q is induced by a facet of P with unit normal m_i orthogonal to n. This normal remains unit in $n^\perp$ and has height

$$0 < t_i' = -\langle q', m_i\rangle = \frac{t(m_i)}{\sqrt{1+t^2}} \le \frac{t}{\sqrt{1+t^2}} < 1. \tag{2}$$

Proof. The numerator in (1) has square $-1-t^2$ and product zero with n. Its product with q is $-1-t^2<0$, so normalization gives a future unit point in $n^\perp$. For every other facet normal m,

$$\langle q', m\rangle = \frac{\langle q, m\rangle + t\langle n, m\rangle}{\sqrt{1+t^2}} < 0. \tag{3}$$

The first term in the numerator is negative and the second is nonpositive. Thus q' lies in the relative interior of Q.

Each facet of Q is a ridge of P shared with a unique other facet, whose unit normal we denote by m_i. Right angles give $\langle n, m_i\rangle = 0$, so restriction to $n^\perp$ preserves its unit norm. Formula (2) now follows from the maximum choice of t. □

If all heights at q are at most A, monotonicity of $x/\sqrt{1+x^2}$ gives a new bound $A/\sqrt{1+A^2}$. Start with the finite maximum A_0 at any finite interior point, and let A_j be the maximum height after j projections. Repeating the construction intrinsically gives

$$A_j \le \frac{A_0}{\sqrt{1+jA_0^2}} < \frac{1}{\sqrt{j}} \qquad (j \ge 1), \tag{4}$$

whenever the selected faces still have dimension at least three. The formula follows by induction, or by adding one to the inverse square of the previous bound.

We will use the following translation from heights to spherical codes. At a unit point q of a d-dimensional face, choose q as its time axis and write the intrinsic unit normals as

$$m_i = \big(t_i, \sqrt{1+t_i^2}\, u_i\big), \qquad u_i \in S^{d-1} \subset \mathbb{R}^d. \tag{5}$$

For distinct facets, $\langle m_i, m_j\rangle \le 0$ gives

$$u_i \cdot u_j \le \frac{t_i t_j}{\sqrt{(1+t_i^2)(1+t_j^2)}}. \tag{6}$$

The vectors u_i are distinct: if $u_i = u_j$, the Lorentz product of the two unit normals would be positive. Consequently the number of code points is exactly the number of facets of the selected face.

3 A bound for spherical codes in dimension six

Lemma 7. *A finite set of unit vectors in $\mathbb{R}^6$ whose distinct elements have inner product strictly less than $1/4$ has at most 26 members.*

Proof. Consider the cubic polynomial

$$F(x) = (x+\tfrac{1}{2})^2(x-\tfrac{1}{4}) = x^3 + \tfrac{3}{4}x^2 - \tfrac{1}{16}. \tag{7}$$

We express this polynomial as a sum of tensor inner products.

Use Euclidean tensor inner products. For a unit vector u define the rank-two tensor

$$A(u) = u \otimes u - I/6$$

and the symmetric rank-three tensors

$$S(u)_{abc} = u_a\delta_{bc} + u_b\delta_{ac} + u_c\delta_{ab}, \qquad T(u) = u^{\otimes 3} - S(u)/8.$$

For $x = u \cdot v$, contraction gives

$$\langle A(u), A(v)\rangle = x^2 - \tfrac{1}{6}, \qquad \langle T(u), T(v)\rangle = x^3 - \tfrac{3}{8}x. \tag{8}$$

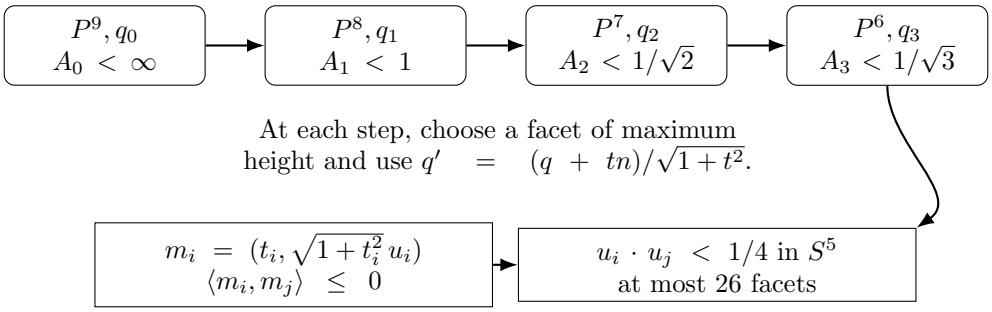

Figure 1: Three successive facet projections and the resulting spherical code. The arrows represent applications of the projection formulas. The code has one element for each facet of the six-dimensional face, so the bound of 26 contradicts the minimum of 27 facets.

Each cross contraction of a cubic tensor with S is $3x$; the remaining contraction is

$$\langle S(u), S(v) \rangle = (3 \cdot 6 + 6)x = 24x.$$

The scalar identity therefore becomes

$$\begin{aligned} F(u \cdot v) = {} & \tfrac{1}{16} + \tfrac{3}{8}(u \cdot v) \\ & + \tfrac{3}{4}\langle A(u), A(v) \rangle + \langle T(u), T(v) \rangle. \end{aligned}$$

Thus, for $u_1, \ldots, u_N$,

$$\begin{aligned} \sum_{i,j} F(u_i \cdot u_j) &= \frac{N^2}{16} + \frac{3}{8}\|\sum_i u_i\|^2 \\ &\quad + \frac{3}{4}\|\sum_i A(u_i)\|^2 + \|\sum_i T(u_i)\|^2 \\ &\geq \frac{N^2}{16}. \end{aligned} \tag{9}$$

On the other hand, $F(1) = 27/16$, and $F(x) \leq 0$ for $x < 1/4$. The diagonal terms and (9) give

$$\frac{N^2}{16} \leq \sum_{i,j} F(u_i \cdot u_j) \leq \frac{27N}{16},$$

so $N \leq 27$. If $N = 27$, equality throughout forces each off-diagonal polynomial value to vanish. Since $u_i \cdot u_j < 1/4$ strictly, the only available root is $-1/2$. Four such vectors would already give

$$\|u_1 + u_2 + u_3 + u_4\|^2 = 4 + 12(-\tfrac{1}{2}) = -2,$$

which is impossible. Hence $N \leq 26$. □

Proof of Theorem 1. Suppose first that a right-angled nine-polytope exists. Three projections from a finite interior point produce a six-face R and a relative-interior point at which every facet height is strictly less than $1/\sqrt{3}$, by (4). The inherited face is right-angled and has finite volume. For its intrinsic normals, (6) now gives

$$u_i \cdot u_j < \frac{1}{4} \qquad (i \neq j)$$

in $S^5 \subset \mathbb{R}^6$. Lemma 7 gives at most 26 facets for R, contradicting Proposition 3. Every higher-dimensional polytope has a nine-face inheriting finite volume and right angles, giving the same contradiction. □

4 The ideal bound from one projection

We first record a property of the regular ideal right-angled 24-cell. In the standard Lorentz model of the regular ideal right-angled 24-cell, its outward unit normals are

$$(1, \epsilon e_i + \delta e_j), \qquad 1 \le i < j \le 4, \quad \epsilon, \delta \in \{1, -1\}, \tag{10}$$

and the center is $(1, 0)$.

Lemma 8. *No future unit timelike point of the regular ideal right-angled 24-cell has all its unit-normal heights strictly less than one. More precisely, if every height is at most one, the point is the center and all the heights equal one.*

Proof. The normals in (10) come in opposite spatial pairs $(1, r), (1, -r)$. Write the point as (a, w), with $a > 0$ and $a^2 - \|w\|^2 = 1$. The two corresponding heights are $a - r \cdot w$ and $a + r \cdot w$. If both are at most one, their sum gives $a \le 1$. Future unit timelikeness gives $a \ge 1$. Thus $a = 1, w = 0$, and every height equals one. Strict inequalities on all heights are impossible. □

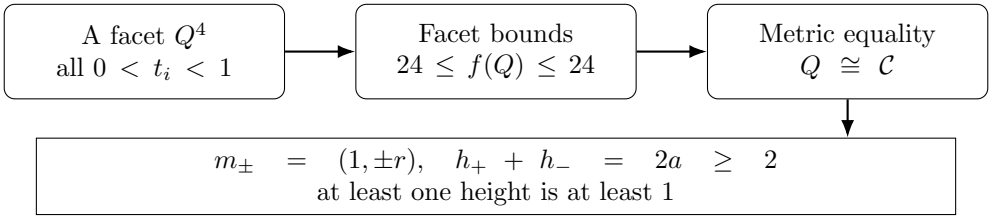

Figure 2: The ideal case. Musin's spherical bound and the facet theorem [3, Theorem 4] identify the projected facet with the regular ideal 24-cell. The heights at two opposite facets then contradict the strict height bound from the projection.

Proof of Theorem 2. Assume an entirely ideal right-angled five-polytope exists. A single farthest-facet projection from a finite interior point gives a four-dimensional facet Q at whose relative-interior point all heights satisfy $0 < t_i < 1$. It inherits finite volume, entire ideality and right angles. By (6), the associated unit vectors in S^3 have pairwise products strictly below $1/2$. Proposition 4 bounds its facet number by 24; Proposition 5 gives the reverse bound and identifies Q with the regular ideal 24-cell. Lemma 8 contradicts the strict heights $t_i < 1$.

There is no entirely ideal right-angled five-polytope. In any higher dimension a five-face would inherit all three hypotheses and give the same contradiction. The ideal right-angled 24-cell attains dimension four. □

5 Supplementary verification

Exact Python checks verify the 24-cell coordinates in (10), their Gram products and opposite pairs, and the tensor identities (8)–(9). The polynomial identities are checked over $\mathbb{Q}$ modulo the unit-sphere equations. The code and verification records are available in the accompanying repository [7].

Selected Lean 4 modules verify the internal logical deductions for the projection, the cubic bound and the dimension bounds. The geometric statements, face inheritance and cited theorems are supplied as explicit premises.

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