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Tensor semistability from rectangular Specht lattices

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Abstract

We prove Bost's conjecture on the additivity of maximal slopes under Hermitian tensor products over the integers of any number field. For a subbundle with a geometrically semistable Plücker point, we use an invariant to detect a nonzero integral contraction between determinant components of tensor powers. The source and target are semistable by their symmetric group actions. We prove that a rectangular Specht lattice with a fixed number of rows and a boxes has logarithmic covolume per rank $O(\sqrt{a})$. Dividing the slope comparison by a and letting a tend to infinity yields the exact inequality. Weighted filtrations handle the geometrically unstable case. This argument was discovered while reconstructing Levent Alpöge's proof of the Riemann hypothesis.

1 Introduction

Bost formulated the tensor semistability problem for Hermitian vector bundles at Oberwolfach in 1997. Its equivalent formulations in terms of maximal slopes, semistability, and Harder–Narasimhan filtrations are Problems 1.3–1.5 of Bost and Chen [3, Sections 1.4 and 1.7]. Chen used geometric invariant theory to bound maximal slopes of tensor products [5]. André developed a perspective based on nef Hermitian lattices [2]. Bost and Chen obtained a bound with error $\frac{1}{2}\min\{\ell(\operatorname{rk} E), \ell(\operatorname{rk} F)\}$, where $\ell(d) = \sum_{j=2}^{d} 1/j$, and proved tensor semistability when $\operatorname{rk} E \operatorname{rk} F \leq 9$ [3, Theorems A and B]. Rémond proved tensor semistability when one of the metrized factors admits a multiplicity-free isometric group action [17]. Coulangeon proved exact additivity for further cases involving isodual or anti-isodual lattices [6, Theorems 4.6 and 4.8].

Theorem 1. *Let K be a number field and let E, F be nonzero Hermitian vector bundles on $\operatorname{Spec} \mathcal{O}_K$. Equip $E \otimes F$ with the Hilbert tensor product metrics at infinity. Then*

$$\widehat{\mu}_{\max}(E \otimes F) = \widehat{\mu}_{\max}(E) + \widehat{\mu}_{\max}(F). \tag{1}$$

In particular the tensor product of two semistable Hermitian vector bundles is semistable.

The main step is a slope bound for a subbundle whose Plücker point is geometrically semistable. Proposition 4 proves this bound for any finite number of factors, answering Bost and Chen's Problem 1.17 [3, Section 1.11]. The integral maps come from column alternation and duality. The James–Murphy branching formula then reduces their slope error to an average over standard tableaux. Section 6 uses the geometric bound to prove Theorem 1. The question of finding integral tensor maps with an error sublinear in the exponent arose during the reconstruction of the Lean archive accompanying Alpöge's announcement [1].

2 Slope conventions and contractions

Set $N = [K : \mathbb{Q}]$ and write $\ell_K(T) = N^{-1} \log \#T$ for a finite $\mathcal{O}_K$-module T. All degrees and slopes are normalized by N. A Hermitian lattice is a finite projective $\mathcal{O}_K$-module with

Hermitian metrics at the embeddings of K, compatible with complex conjugation. For a Hermitian line M and any $0 \neq v \in M$, its normalized degree is

$$\widehat{\deg}_{\mathrm{n}}(M) = \ell_K(M/\mathcal{O}_K v) - \frac{1}{N}\sum_{\sigma} \log \|v\|_\sigma.$$

The sum runs over all embeddings, so a complex conjugate pair contributes twice. The product formula gives independence of v. The degree of a lattice is the degree of its determinant line with the determinant metric, and its slope is that degree divided by its rank. The maximal slope is the maximum of the slopes of nonzero saturated sublattices with induced metrics. Semistability means $\widehat{\mu}_{\max}(M) = \widehat{\mu}(M)$. An exact sequence of lattices with subspace and quotient metrics has additive degree. Indeed, its determinant isomorphism is integral and is an isometry by orthogonal splitting at each embedding. In particular, a nonzero quotient of a semistable lattice has slope at least that of the lattice. The determinant identities also give $\widehat{\mu}(A^\vee) = -\widehat{\mu}(A)$ and $\widehat{\mu}(A \otimes B) = \widehat{\mu}(A) + \widehat{\mu}(B)$ for the Hilbert tensor product.

Lemma 2. *Let $f : A \to B$ be an $\mathcal{O}_K$-linear map of Hermitian lattices of positive generic rank with operator norm at most one at every embedding. Give $Q = A/\ker f$ the quotient metrics, and let $I \subset B$ be the saturation of $f(A)$. Then $\widehat{\mu}(Q) \leq \widehat{\mu}(I)$. Consequently, if A and B are semistable, then $\widehat{\mu}(A) \leq \widehat{\mu}(B)$. If f is generically injective, then $\widehat{\mu}(A) \leq \widehat{\mu}_{\max}(B)$.*

Proof. The kernel is saturated. The induced map $\bar{f} : Q \to I$ is contractive by taking the infimum of the norms of the lifts in A of each element of Q. Put $s = \operatorname{rk} Q = \operatorname{rk} I$. At each finite prime of $\mathcal{O}_K$, the elementary divisors of the full-rank inclusion $\bar{f}(Q) \subset I$ give the same length for $I/f(A)$ and $\det I/(\Lambda^s \bar{f})(\det Q)$. The line degree formula therefore gives

$$\widehat{\deg}_{\mathrm{n}}(Q) = \widehat{\deg}_{\mathrm{n}}(I) - \ell_K(I/f(A)) + \frac{1}{N}\sum_{\sigma} \log \|\Lambda^s \bar{f}_\sigma\| \leq \widehat{\deg}_{\mathrm{n}}(I). \tag{2}$$

Here the determinant operator norm is the product of the singular values of $\bar{f}_\sigma$ and is at most one. The slope inequalities now follow from the quotient and sublattice definitions. □

We will use the uniqueness of the Harder–Narasimhan filtration [3, Section 1.3]. In particular, a group of lattice isometries acting irreducibly on the generic fiber forces semistability. The first nonzero sublattice in the filtration is preserved by all these isometries, so irreducibility makes it the entire lattice. This is the symmetry argument used in [3, Section 1.8].

3 The covolume of the rectangular Specht lattice

For a partition λ of a, let $M_{\mathbb{Z}}^\lambda$ be the permutation lattice of row tabloids with its orthonormal tabloid basis. Let $S_{\mathbb{Z}}^\lambda \subset M_{\mathbb{Z}}^\lambda$ be the span of polytabloids, with the induced Euclidean norm. Write f^λ for the number of standard tableaux and $\mathrm{gd}(\lambda)$ for the Gram determinant of the standard polytabloid basis of $S_{\mathbb{Z}}^\lambda$ [12, pp. 29–33]. Thus $\operatorname{rk} S_{\mathbb{Z}}^\lambda = f^\lambda$ and $\operatorname{covol}(S_{\mathbb{Z}}^\lambda) = \mathrm{gd}(\lambda)^{1/2}$.

Lemma 3. *Fix $d \geq 1$ and let $R = (k^d)$ denote a rectangle of d rows and $k \geq 1$ columns. With $a = dk$,*

$$0 \leq \gamma_{d,k} := \frac{\log \mathrm{gd}(R)}{2f^R} \leq C_d\sqrt{a}$$

for a constant C_d depending only on d. If $d \geq 2$, then also $\gamma_{d,k} \geq c_d\sqrt{a}$ for some $c_d > 0$.

Proof. For $d = 1$, the single polytabloid is a unit vector and $\gamma_{1,k} = 0$. Suppose $d \geq 2$. For a partition λ with at most d rows, pad it with zeros and put $x_i(\lambda) = \lambda_i + d - i$. If $b = (j, \lambda_j)$ is a removable box, the James–Murphy branching scalar is

$$c(\lambda, b) = \prod_{i<j} \frac{x_i(\lambda) - x_j(\lambda) + 1}{x_i(\lambda) - x_j(\lambda)}. \tag{3}$$

The factor at row $i < j$ in [13] is $h_{i,\lambda_j}/(h_{i,\lambda_j} - 1)$, in the notation of [18, Section 3, equation (1)]. The column of b has height j, so $h_{i,\lambda_j} = \lambda_i - \lambda_j + j - i + 1$. Order the standard polytabloids by the row of the largest entry. The quotient at the row of b identifies with $S_{\mathbb{Z}}^{\lambda - b}$ by deleting that entry. Under this identification the quotient inner product is $c(\lambda, b)$ times the tabloid inner product for $\lambda - b$. Appendix A derives the quotient normalization directly. Orthogonally projecting each block of the ordered basis off the preceding blocks changes it by a block triangular matrix of determinant one. Thus

$$\operatorname{gd}(\lambda) = \prod_{b \text{ removable}} \operatorname{gd}(\lambda - b) c(\lambda, b)^{f^{\lambda - b}}. \tag{4}$$

In a uniform standard tableau of fixed shape λ, the probability that the largest entry is in b is $f^{\lambda - b}/f^{\lambda}$. Given this box, the remaining tableau is uniform of shape $\lambda - b$. Taking the logarithm in (4), dividing by f^{λ}, and iterating from R to the empty diagram therefore gives

$$\frac{\log \operatorname{gd}(R)}{f^R} = \sum_{t=1}^{a} \mathbb{E} \log c(\lambda(t), b_t). \tag{5}$$

Here $\lambda(t)$ is the shape of the entries $1, \ldots, t$ in a uniform standard tableau of shape R, and $b_t = \lambda(t) \setminus \lambda(t-1)$. Every term is nonnegative. The inequality $\log(1 + u) \le u$ for $u \ge 0$ gives the bound

$$\log c(\lambda, b) \le \sum_{1 \le i < j \le d} \frac{1}{x_i - x_j} \tag{6}$$

at a shape with coordinates x_i.

To bound the terms in (6), put $D = \binom{d}{2}$, $L = k + d - 1$ and $\Delta(x) = \prod_{i<j}(x_i - x_j)$. Each initial tableau of shape λ can be joined to every standard filling of the skew complement with labels $t+1, \ldots, a$, since all cross-boundary comparisons then hold. Rotating the complement by 180 degrees and replacing each label q by $a + 1 - q$ gives a tableau of the ordinary partition $\lambda^c = (k - \lambda_d, \ldots, k - \lambda_1)$. Hence the shape probability is $f^{\lambda} f^{\lambda^c}/f^R$. This is the rectangle version of the counting identity in [16, Section 2.1, equation (7)] for squares. The hook length formula [9] in Vandermonde form gives

$$f^{\lambda} = \frac{t! \Delta(x)}{\prod_i x_i!}, \qquad f^{\lambda^c} = \frac{(a-t)! \Delta(x)}{\prod_i (L - x_i)!}.$$

In the second formula the coordinates are $x_i(\lambda^c) = L - x_{d+1-i}(\lambda)$, so the Vandermonde is unchanged. At $\lambda = R$, the Vandermonde equals $\Delta_0 = \prod_{i<j}(j - i)$. Let X_i be the counts of each of d colors when $t + D$ objects are sampled uniformly without replacement from d piles of L objects, one color per pile. Their joint mass function is

$$H_t(x) = \frac{\prod_i \binom{L}{x_i}}{\binom{a+2D}{t+D}}.$$

Since $dL = a + 2D$, substituting the factorial formulas into $f^{\lambda} f^{\lambda^c}/f^R$ gives

$$\mathbb{P}(x(\lambda(t)) = x) = A_{d,k,t} \Delta(x)^2 H_t(x), \qquad A_{d,k,t} = \frac{\binom{a+2D}{t+D}}{\binom{a}{t} \Delta_0} \prod_{i=1}^{d} \frac{(k + d - i)!}{L!}, \tag{7}$$

on the strictly decreasing integer tuples x in $[0, L]$ with $\sum_i x_i = t + D$. Indeed, every such tuple corresponds to a partition in the rectangle by subtracting $d - i$ from coordinate i. We keep the fixed order of the coordinates and use H_t as a probability on all count tuples.

Put $s = 1 + \min(t, a-t)$. The factorial product in (7) is a product of D reciprocal integers, each at most $1/k$. The binomial ratio is

$$\frac{\prod_{\ell=1}^{2D}(a+\ell)}{\prod_{\ell=1}^{D}(t+\ell)(a-t+\ell)}.$$

Each pair $(t+\ell)(a-t+\ell)$ in the denominator is at least $sa/2$. The numerator is $O_d(a^{2D})$ and $a/k = d$. Consequently there is a constant C'_d such that $A_{d,k,t} \le C'_d s^{-D}$ for every t, including the two endpoints.

Let $M_t(X) = \max_i |X_i - (t+D)/d|$. The sample or its complement, whichever has size $m = \min(t, a-t) + D$, has the same absolute centered counts. For the count Y of one color in this smaller sample we have

$$\mathbb{E}e^{\theta Y} \le (1 - 1/d + e^{\theta}/d)^m.$$

The expectation on the left is the average of products of m distinct population weights, with weight e^{θ} for that color and 1 for the others. Maclaurin's inequality bounds it by the m-th power of the mean weight. This is also Hoeffding's sampling comparison [11, Theorem 4]. The second derivative of the logarithm of the one-trial moment is a tilted Bernoulli variance, at most $1/4$. Subtracting its linear term at zero gives $\mathbb{E}e^{\theta(Y-m/d)} \le e^{m\theta^2/8}$. Using both signs of θ and optimizing yields $\mathbb{P}(|Y - m/d| \ge u) \le 2e^{-2u^2/m}$. A union bound and the layer-cake formula yield, for each $p > 0$,

$$\mathbb{E}_{H_t} M_t(X)^p \le 2dp \int_0^{\infty} u^{p-1} e^{-2u^2/m}\, du \le C_{d,p} s^{p/2}. \tag{8}$$

Here $m \ge 1$ since $d \ge 2$.

For $i < j$, cancel a factor $x_i - x_j$ from $\Delta(x)^2$. Each of the $2D-1$ remaining factors on a decreasing tuple is at most $2M_t(x)$. Equations (7) and (8) give

$$\mathbb{E}\frac{1}{x_i(\lambda(t)) - x_j(\lambda(t))} \le C'_d s^{-D} \mathbb{E}_{H_t}(2M_t(X))^{2D-1} \le C''_d s^{-1/2}.$$

In the first bound we extended the sum to all count tuples after replacing the remaining factors by $(2M_t)^{2D-1}$. Summing through (6) and (5) yields $O_d(\sqrt{a})$, since each value of $\min(t, a-t)$ occurs at most twice.

For the lower bound, (7) and (8) with $p = 2D+2$ give $\mathbb{E}M_t(x(\lambda(t)))^2 \le C_d s$ for a constant C_d. By Markov's inequality the expected number of steps with $M_t(x(\lambda(t))) > B_d\sqrt{a}$ is at most $C_d a/B_d^2$. Choose $B_d \ge 1$ large enough that this is at most $(a-k)/2 = (1-1/d)a/2$. Each tableau has $a-k$ steps in rows $j > 1$. At each such step with $M_t(x(\lambda(t))) \le B_d\sqrt{a}$, the factor with $i = 1$ gives

$$\log c(\lambda(t), b_t) \ge \log\bigl(1 + (2B_d\sqrt{a})^{-1}\bigr) \ge (4B_d\sqrt{a})^{-1}.$$

The expected number of these steps is at least $(a-k)/2$. Substitution in (5) proves the lower bound. □

For example, the two standard polytabloids of shape $(2,2)$ have squared norms 4 and inner product 2. Their Gram determinant is 12, also obtained from (4) as $\mathrm{gd}(2,1) \cdot 2^2 = 3 \cdot 2^2$.

Relation to meander determinants. The exact rectangular path product also follows from Di Francesco's $SU(N)$ meander determinant at $N = d$, $n = k$ and $\beta = 2$ [7, Sections 4.3 and 4.5]. In that specialization $\mu_m(2) = m/(m+1)$. Write $x(t) = x(\lambda(t))$. For a step t in row j, the inverse square of the edge weight in [7, equation (4.35)] is

$$w_t^{-2} = \prod_{i<j} \frac{x_i(t) - x_j(t) + 1}{x_i(t) - x_j(t)} \prod_{\ell>j} \frac{x_j(t-1) - x_\ell(t-1) + 1}{x_j(t-1) - x_\ell(t-1)}.$$

Rotating a rectangular tableau by 180° and replacing each label q by $a+1-q$ gives another standard tableau. Its row $d+1-j$ step at time $a+1-t$ has first product equal to the second product of the original step. Thus over all tableaux and steps the two products contribute equally. Equations (4.31) and (4.36) of [7] multiply the inverse edge weights over all paths. After taking the logarithm and dividing by f^R, the result is the right-hand side of (5).

For $d = 2$, the determinant is $D_k(2)$ in the notation of Di Francesco, Golinelli and Guitter [8, equations (5.6) and (6.82)]. They obtain $\log D_k(2) \sim f^{(k^2)}\sqrt{\pi k}$, since their Catalan number c_k is $f^{(k^2)}$. In the normalization of Lemma 3 this gives

$$\gamma_{2,k} \sim \tfrac{1}{2}\sqrt{\pi k} \qquad (k \to \infty).$$

The upper bound for arbitrary fixed d in Lemma 3 is used in the slope comparison below.

4 Integral determinant components

Let W be a Hermitian lattice of rank d, and let a be a positive multiple of d. Set $k = a/d$ and define

$$\mathcal{B}_a(W) = (\det W)^{\otimes k} \otimes_{\mathbb{Z}} S_{\mathbb{Z}}^{(k^d)}.$$

The tensor power of the determinant line is taken over $\mathcal{O}_K$. On an open set where W has a basis $w_1, \dots, w_d$, write $\delta = w_1 \wedge \cdots \wedge w_d$. Send a row tabloid to the tensor word that has w_i in the positions labelled by the entries in row i. This is an injective map of free lattices on the tabloid basis and commutes with permutations of the labels. For a standard tableau T, send $\delta^{\otimes k} \otimes e_T$ to the image of its polytabloid e_T under this map. Equivalently, place the tensor

$$\mathrm{Alt}(w_1, \dots, w_d) = \sum_{\pi \in \mathfrak{S}_d} \mathrm{sgn}(\pi)\, w_{\pi(1)} \otimes \cdots \otimes w_{\pi(d)}$$

in each of the k columns, using the positions labelled by T from top to bottom. Under a basis change g, this alternating tensor changes by $\det g$, as does δ. The local maps thus glue to an $\mathcal{O}_K$-linear injection $j_W : \mathcal{B}_a(W) \to W^{\otimes a}$, equivariant for place permutations by $\mathfrak{S}_a$. The construction uses local bases of a projective module, so it applies to nonfree W.

At an embedding σ, we may use an orthonormal basis of W_σ. Then $\|\delta\|_\sigma = 1$ and the words corresponding to the tabloids are orthonormal. Thus j_W is an isometric embedding for the product of the determinant and Specht metrics on $\mathcal{B}_a(W)$. Its slope is

$$\widehat{\mu}(\mathcal{B}_a(W)) = a\widehat{\mu}(W) - \gamma_{d,a/d}. \tag{9}$$

Indeed, the line contributes degree $k\widehat{\deg}_{\mathrm{n}} W = a\widehat{\mu}(W)$ to the slope, and the determinant norm of a standard polytabloid basis of the Specht factor is $\mathrm{gd}((k^d))^{1/2}$ at every embedding. That basis is integral after tensoring with $\mathcal{O}_K$ and contributes $-\gamma_{d,k}$ to the normalized slope.

Over any algebraic closure of K, Schur–Weyl duality [10, Lectures 4 and 6] gives the decomposition of $W^{\otimes a}$ into $\mathrm{GL}(W)$-modules of highest weights λ with multiplicity spaces S^λ. The only weight with $\mathrm{SL}(W)$ invariants is (k^d), whose highest weight module is $(\det W)^{\otimes k}$. The image of j_W consists of invariants by column alternation and has the dimension of this entire summand. Thus, over $\overline{K}$,

$$j_W(\mathcal{B}_a(W)_{\overline{K}}) = (W_{\overline{K}}^{\otimes a})^{\mathrm{SL}(W_{\overline{K}})}.$$

The identity of generic spaces will be used to detect a nonzero integral map.

The symmetric group preserves the integral Specht lattice and its tabloid form. In characteristic zero, its action on the generic Specht module and its dual is absolutely irreducible [12]. It follows that $\mathcal{B}_a(W)$ and its dual are semistable for every W. A tensor product of several

such components or their duals also has an irreducible action, using a separate copy of $\mathfrak{S}_a$ on each factor. Indeed, after extending to $\overline{K}$, the endomorphism algebra of the external tensor representation that commutes with the product group is the tensor product of the commuting endomorphism algebras of the factors. Those algebras are one-dimensional, and complete reducibility gives irreducibility.

5 The contraction and the slope inequality

Proposition 4. *Let $E_1, \dots, E_s$ be Hermitian lattices of positive rank and let V be a nonzero sublattice of $\bigotimes_{i=1}^{s} E_i$ with the induced metrics. Suppose that the Plücker point of V_K is geometrically semistable for $\prod_{i=1}^{s} \mathrm{SL}(E_{i,K})$ with the usual projective linearization. Then*

$$\widehat{\mu}(V) \le \sum_{i=1}^{s} \widehat{\mu}(E_i).$$

Proof. Put $X = \bigotimes_i E_i$, let $u : V \hookrightarrow X$ be the inclusion, and write $r = \operatorname{rk} V$ and $d_i = \operatorname{rk} E_i$. Choose $\omega \in \det V_K \setminus \{0\}$. The invariant-polynomial criterion for projective semistability [15, Chapter 2] gives a homogeneous invariant J of positive degree on $\Lambda^r X_{\overline{K}}$, with $J((\Lambda^r u)(\omega)) \neq 0$. The polynomial

$$P(T) = J\big((\Lambda^r T)(\omega)\big), \qquad T \in \mathrm{Hom}_{\overline{K}}(V_{\overline{K}}, X_{\overline{K}}),$$

is homogeneous of degree $e = r \deg J > 0$. It is invariant under $\prod_i \mathrm{SL}(E_{i,\overline{K}})$ by invariance of J. It is invariant under $\mathrm{SL}(V_{\overline{K}})$ acting by precomposition with the inverse because this group fixes ω. In particular, $P(u) \neq 0$.

Take a to be any positive common multiple of $e, r, d_1, \dots, d_s$. In characteristic zero the symmetric multilinear form obtained by polarizing $P^{a/e}$ is equivariant with respect to linear changes of variables. It gives an invariant tensor

$$p_a \in \left(\bigotimes_{i=1}^{s} (E_{i,\overline{K}}^{\vee})^{\otimes a} \right) \otimes V_{\overline{K}}^{\otimes a}, \qquad \langle p_a, u^{\otimes a} \rangle = P(u)^{a/e}.$$

Each special linear group acts on its own tensor factor, so the full invariant space is the tensor product of the invariant subspaces in the displayed factors. To see this, take invariants one group at a time, using a basis of the factors on which that group acts trivially. The generic image formula for j_W therefore identifies p_a with an element of $\bigotimes_i \mathcal{B}_a(E_i^{\vee})_{\overline{K}} \otimes \mathcal{B}_a(V)_{\overline{K}}$.

The map of lattices used for the slope comparison is

$$\Phi_a \colon \mathcal{B}_a(V) \xrightarrow{j_V} V^{\otimes a} \xrightarrow{u^{\otimes a}} \bigotimes_i E_i^{\otimes a} \xrightarrow{\bigotimes_i j_{E_i^{\vee}}^{\vee}} \bigotimes_i \mathcal{B}_a(E_i^{\vee})^{\vee}. \tag{10}$$

The middle arrow includes the integral, isometric shuffle of tensor factors. The notation $j_{E_i^{\vee}}^{\vee}$ indicates the $\mathcal{O}_K$-linear dual of $j_{E_i^{\vee}}$, under the identification $\big((E_i^{\vee})^{\otimes a}\big)^{\vee} \simeq E_i^{\otimes a}$ by the slotwise pairing. In particular, all arrows are integral. At infinity the first two are isometries onto their images, and the last arrow is contractive, since each dual of an isometric embedding is contractive and the Hilbert tensor product preserves operator norm bounds.

For $v \in \mathcal{B}_a(V)_{\overline{K}}$ and $b_i \in \mathcal{B}_a(E_i^{\vee})_{\overline{K}}$ the evaluation is

$$\Phi_a(v)(\textstyle\bigotimes_i b_i) = \left\langle \bigotimes_i j_{E_i^{\vee}}(b_i),\, u^{\otimes a} j_V(v) \right\rangle.$$

By linearity this expresses $\langle p_a, u^{\otimes a} \rangle$ as a sum of evaluations of Φ_a after base change. Its nonzero value proves that Φ_a has positive generic rank. All coefficients used in polarizing $P^{a/e}$ occur in this test over $\overline{K}$. The lattice map Φ_a itself is the displayed composite over $\mathcal{O}_K$.

Both ends of Φ_a are semistable by their isometric actions. Lemma 2 and (9) give

$$a\widehat{\mu}(V) - \gamma_{r,a/r} \le a\sum_{i=1}^{s} \widehat{\mu}(E_i) + \sum_{i=1}^{s} \gamma_{d_i,a/d_i}.$$

The ranks $r, d_1, \dots, d_s$ are fixed. Thus the total error is $O(\sqrt{a})$ by Lemma 3. Dividing by a and taking the limit along the chosen common multiples proves the proposition. □

6 Reduction by weighted flags

We follow the weighted filtration method of Bost and Chen [3, Proposition 7.4], using the minimum Plücker weight for the product of the special linear groups. The optimization below allows either one of the two flags to be trivial.

Let M carry a decreasing flag by saturated sublattices $M_{\ge\lambda}$ with graded pieces M_λ equipped with quotient metrics. Write $M_{>\lambda}$ for the next smaller sublattice in the flag. A contractive, generically injective lattice map $g : U \to M$ induces a flag $U_{\ge\lambda} = g^{-1}(M_{\ge\lambda})$. Its sublattices are saturated because the corresponding quotients of M are torsion-free. On graded pieces the map $U_\lambda \to M_\lambda$ is injective and contractive. At each embedding, for $u \in U_{\ge\lambda,\sigma}$,

$$\inf_{z\in M_{>\lambda,\sigma}} \|g(u) - z\| \le \inf_{v\in U_{>\lambda,\sigma}} \|g(u - v)\| \le \inf_{v\in U_{>\lambda,\sigma}} \|u - v\|.$$

By additivity of degree and Lemma 2,

$$\widehat{\deg}_{\mathrm{n}}(U) \le \sum_{\lambda} \operatorname{rk}(U_\lambda)\widehat{\mu}_{\max}(M_\lambda), \tag{11}$$

where the sum is over nonzero pieces of U. In particular, an orthogonal direct sum of semistable lattices of one slope is semistable, by applying (11) to the flag of partial sums.

For weighted flags of E and F, denote their graded pieces by E_α and F_β. The tensor flag on $X = E \otimes F$ is defined by

$$X_{\ge\gamma} = \sum_{\alpha+\beta\ge\gamma} E_{\ge\alpha} \otimes F_{\ge\beta}.$$

Locally at a finite prime, saturated flags split into direct sums of free modules because their quotients are projective. Thus the $X_{\ge\gamma}$ are saturated, and the natural map from $\bigoplus_{\alpha+\beta=\gamma} E_\alpha \otimes F_\beta$ to X_γ is an integral isomorphism. At infinity choose the orthogonal splittings of the flags. Their Hilbert tensor decompositions show that the isomorphism is an isometry for the direct sum metrics.

On extending the number field, normalized degrees and maximal slopes are preserved [3, Proposition 1.2]. For a Galois extension, uniqueness makes the generic first Harder–Narasimhan subspace of the pullback Galois invariant. It descends to a subspace over K, and its saturated lattice is the pullback of the saturation over K, because the quotient is projective. With normalized degrees this gives the equality of maximal slopes in the Galois case. For a general finite extension, pullback cannot decrease maximal slope, and applying the Galois case to a Galois closure gives the reverse inequality.

For bases e_i of E_K and f_j of F_K with real weights x_i, y_j, the weight of $e_i \otimes f_j$ is $x_i + y_j$. For $w \ne 0$ on the Plücker line of V_K, let $w_V(x, y)$ be $1/r$ times the minimum weight of a nonzero coordinate of w in the exterior basis, where $r = \operatorname{rk} V$ and each coordinate has the sum of its r tensor basis weights. In a basis of V_K adapted to the induced decreasing filtration, the leading vectors of each weight in the corresponding graded piece of X_K are independent. Their exterior product over all weights is nonzero and occurs at the sum of their weights. All other terms use higher weights. Consequently

$$w_V(x, y) = \frac{1}{r}\sum_{\gamma} \gamma \dim_K V_{\gamma,K}, \tag{12}$$

where $V_{\gamma,K}$ are the induced generic graded pieces. In particular, w_V depends only on the two weighted flags.

Proof of Theorem 1. First suppose E, F are semistable. We rescale all the norms of each factor by the exponential of its slope, so both slopes become zero. This shifts the slopes of all tensor sublattices by the same constant. We prove that $E \otimes F$ is semistable by induction on $n + m$, where $n = \operatorname{rk} E$ and $m = \operatorname{rk} F$, simultaneously for all number fields. If either rank is one, tensoring by the corresponding Hermitian line shifts all slopes by its degree. Let $V \subseteq E \otimes F$ be any nonzero saturated sublattice. If its Plücker point is geometrically semistable, Proposition 4 gives $\widehat{\mu}(V) \leq 0$.

Suppose the point is geometrically unstable. By the Hilbert–Mumford criterion [15, Chapter 2], a one-parameter subgroup of $\mathrm{SL}(E_{\overline{K}}) \times \mathrm{SL}(F_{\overline{K}})$ sends w to zero in the limit at the origin. Its eigenbases over $\overline{K}$ give integer weights (x, y), with $\sum_i x_i = \sum_j y_j = 0$, for which every nonzero Plücker coordinate has positive weight. Thus $w_V(x, y) > 0$. We maximize w_V over all bases over $\overline{K}$ and all centered real weights of unit norm, using

$$\|(x, y)\|^2 = \frac{1}{n} \sum_i x_i^2 + \frac{1}{m} \sum_j y_j^2.$$

The set of supports of the Plücker coordinates realized by such bases is finite. For each support the objective is the minimum of finitely many linear forms divided by r. It attains its maximum on the centered unit sphere. Choose bases realizing a maximizing support, pass to a finite extension containing their coefficients, and take the saturated lattices in the two flags. By scalar extension invariance it suffices to work over this field. Write $c = w_V(x, y) > 0$ at the maximum and let E_α, F_β be the graded pieces for the first weights α, β. At least one of the flags is nontrivial because the weights are centered and have unit norm.

In each graded piece choose a basis adapted to its Harder–Narasimhan flag and lift these bases to the two original spaces. Their first weights give the same $w_V(x, y)$ by (12). Assign each basis vector its secondary weight p_i or q_j, the Harder–Narasimhan slope of its layer. Then $\sum_i p_i = \sum_\alpha \widehat{\deg}_{\mathrm{n}}(E_\alpha) = \widehat{\deg}_{\mathrm{n}}(E) = 0$, and $\sum_j q_j = 0$ as well. For the inner product corresponding to the chosen norm,

$$\langle (x, y), (p, q) \rangle = \frac{1}{n} \sum_\alpha \alpha \, \widehat{\deg}_{\mathrm{n}}(E_\alpha) + \frac{1}{m} \sum_\beta \beta \, \widehat{\deg}_{\mathrm{n}}(F_\beta) \leq 0. \tag{13}$$

In fact, for decreasing distinct $\alpha_1, \ldots, \alpha_l$, the first sum before division by n equals

$$\alpha_l \widehat{\deg}_{\mathrm{n}}(E) + \sum_{h<l} (\alpha_h - \alpha_{h+1}) \widehat{\deg}_{\mathrm{n}}(E_{\geq \alpha_h}).$$

The total degree is zero and all prefix degrees are nonpositive by semistability. The same reasoning applies to F.

We refine the first tensor flag as follows. At each γ, let V_γ be the lattice quotient for the flag on V induced from $X_{\geq \gamma}$. Its map to X_γ is an integral contraction. On X_γ, take the direct sum of the tensor filtrations from the Harder–Narasimhan flags of E_α and F_β for $\alpha + \beta = \gamma$. The secondary weights are the sums η of the slopes of their layers. Filter V_γ by pulling this flag back, and write $V_{\gamma,\eta}$ for the subquotients. The tensor flag calculation gives the ambient grade at (γ, η) as the orthogonal direct sum of the tensors of corresponding Harder–Narasimhan layers.

In the lifted bases on E_K, F_K, the pairs of weights of the tensor basis vectors are $(\gamma, \eta) = (x_i + y_j, p_i + q_j)$. For small positive ϵ, their ordering by $\gamma + \epsilon\eta$ is precisely the lexicographic ordering of these pairs. Thus on V_K the induced filtration for the perturbed weights first takes the γ-graded pieces and then their η-graded pieces. In particular, (12) gives

$$w_V(x + \epsilon p, y + \epsilon q) = c + \epsilon b, \qquad b = \frac{1}{r} \sum_{\gamma,\eta} \eta \, \operatorname{rk} V_{\gamma,\eta}.$$

After normalizing the perturbed weights to unit norm, maximality gives $(c+\epsilon b)/\|(x+\epsilon p, y+\epsilon q)\| \le c$. Taking the right derivative at zero yields $b \le c\langle (x,y),(p,q)\rangle \le 0$, by (13).

Every tensor in an ambient grade at (γ,η) has semistable factors and smaller total rank than $n+m$, since at least one first flag is nontrivial. The induction hypothesis gives semistability of each such tensor with slope η. The whole ambient grade is then semistable of slope η by the direct sum case of (11). Applying (11) to each map $V_\gamma \to X_\gamma$ with the secondary flag, and adding over γ, gives

$$\widehat{\deg}_{\mathrm{n}}(V) = \sum_{\gamma} \widehat{\deg}_{\mathrm{n}}(V_\gamma) \le \sum_{\gamma,\eta} \eta \operatorname{rk} V_{\gamma,\eta} = rb \le 0.$$

This completes the induction for semistable factors.

For arbitrary E, F, filter them by their Harder–Narasimhan flags. The induced tensor filtration has subquotients that are orthogonal direct sums of tensors of semistable layers. By the result just proved each subquotient is semistable of slope equal to its weight, the sum of the two corresponding slopes. Grouping all equal weights into one grade gives strictly decreasing slopes and hence the Harder–Narasimhan filtration of $E \otimes F$. Its first weight is $\widehat{\mu}_{\max}(E) + \widehat{\mu}_{\max}(F)$, which proves (1). □

7 Hermitian norm families over $\overline{\mathbb{Q}}$

Equation (1) also holds for Hermitian norm families in the sense of [3, Section 2]. By [3, Remark 2.1], each such family has approximating pure models on finite extensions. We work on a common field of definition for the two approximants and the two original families, and take the same Hermitian metrics at the infinite places for a family and its approximant.

To justify the limiting argument, let δ_v be a bound for the absolute log ratio of two norms at v. This bound also applies to each restriction, and its rank-r determinant norms have log ratio bounded by $r\delta_v$. At infinity this follows from the eigenvalues of one Hermitian metric relative to the other. At finite places the determinant norm is the quotient of the injective tensor power norm [3, Section 2.1.5], so the comparison follows by taking the tensor power and then the quotient. Consequently every subspace slope changes by at most the normalized distance

$$\sum_{v \in \Sigma_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]}\, \delta_v,$$

with $\mathbb{Q}_v$ denoting the completion below v. The bound is unchanged on further finite extensions and is uniform over all subspaces defined over $\overline{\mathbb{Q}}$. It thus applies to the maximal slope. For tensor products the local log ratio is bounded by the sum of the factor bounds, by the Hilbert norm at infinity and the injective tensor norm at finite places. For the latter, the operator norm description on $\operatorname{Hom}(E^\vee, F)$ gives the sum directly from the two norm comparisons. Pure models correspond to Hermitian lattices, whose normalized maximal slopes are preserved by base change. Applying Theorem 1 to the approximating pure models and taking the limit proves the stated equality.

A The normalization in Gram branching

Let $W = \mathbb{C}^d$ have its standard orthonormal basis $e_1, \dots, e_d$. Identify tabloids of shape λ with the tensor words of weight λ in $W^{\otimes a}$. A column of height h in a polytabloid is the alternation of $e_1, \dots, e_h$. The raising operators kill this alternating tensor. Acting on all columns shows that the polytabloids are highest weight vectors of weight λ. Schur–Weyl duality gives dimension f^λ for this highest weight space, so the standard polytabloids span it.

Let D_j be contraction against the jth basis vector in the last slot. For a standard tableau t with a in a removable box in row j, $D_j(e_t) = e_{t\setminus a}$. Indeed, the permutations of the column that fix a leave exactly the column alternation of the smaller tableau. All terms in e_t have last slot in rows at most j, since the column of a has height j. Let $U_{\le j}$ be the span of the standard polytabloids with a in rows at most j. It is the intersection of the Specht space with the tensors supported in these last slots. To see equality, take a nonzero linear combination and let h be the largest row of a among its nonzero coefficients. Applying D_h to the combination gives a nonzero linear combination of distinct standard polytabloids of weight $\lambda-\epsilon_h$, using the deletion identity. Hence a vector supported in last slots at most j lies in $U_{\le j}$. This description makes $U_{\le j}$ invariant under $\mathfrak{S}_{a-1}$. The deletion identity on the integral basis identifies the integral quotient at row j with $S^{\mu}_{\mathbb{Z}}$ for $\mu = \lambda - \epsilon_j$. Here ϵ_j is the jth coordinate vector.

For the first $a-1$ slots, the orthogonal Schur–Weyl summand of type S^{μ} is $S^{\mu} \otimes V_{\mu}$, where V_{μ} is the irreducible GL_d module of highest weight μ. Use the tabloid form on S^{μ} and choose the unitary invariant form on V_{μ} so that this decomposition is isometric. In this summand the highest weight space of weight μ is $S^{\mu} \otimes v$ for a unit highest weight vector v. With the last slot included, the branching summand for deletion of b is S^{μ} tensored with the highest weight line of weight λ in $V_{\mu} \otimes W$. This line has multiplicity one by the one-box Pieri rule [10, Section 6.1, equation (6.8)]. The pieces of the orthogonal branching filtration have distinct $\mathfrak{S}_{a-1}$ types, so this summand is the orthogonal representative of the quotient for b.

Write h for the orthogonal projection of $v \otimes e_j$ to that line. The adjoint of D_j on the full tensor space sends $s \otimes v$ to $s \otimes v \otimes e_j$. The adjoint of D_j restricted to the branching summand is obtained by projecting this vector, so it sends $s \otimes v$ to $s \otimes h$. Consequently D_j identifies the summand with $S^{\mu} \otimes v$ by a map whose singular values are all $\|h\|$. The quotient inner product is therefore the smaller tabloid form multiplied by $\|h\|^{-2}$.

We compute the projection using the operator

$$\Omega = \sum_{i,h=1}^{d} E_{ih} \otimes E_{hi} \quad \text{on } V_{\mu} \otimes W,$$

where E_{ih} are the matrix units acting in the indicated representations. The operator is self-adjoint, using $E_{ih}^{*} = E_{hi}$, and equals half the tensor Casimir minus half the sum of the factor Casimirs. Indeed, expanding the diagonal Casimir $\sum_{i,h} E_{ih}E_{hi}$ gives two cross terms, both equal to Ω. On a highest weight vector of weight ν, the Casimir has eigenvalue

$$C(\nu) = \sum_{i} \nu_i^2 + \sum_{i<h} (\nu_i - \nu_h) = \sum_{i} \nu_i(\nu_i + d + 1 - 2i),$$

because the commutators $[E_{ih}, E_{hi}]$ contribute $\nu_i - \nu_h$ for $i < h$. Thus Ω commutes with the diagonal action and on each Pieri summand of highest weight $\mu + \epsilon_i$ its eigenvalue is $(C(\mu+\epsilon_i) - C(\mu) - d)/2 = u_i := \mu_i + 1 - i$. The possible u_i are distinct.

Let $R_{ba}(z)$ be the entries of $(z-\Omega)^{-1}$ in the last slot and set $r_j(z) = \langle v, R_{jj}(z) v \rangle$, with the inner product linear in its second variable. The resolvent commutes with the diagonal action, hence

$$[E_{ih}, R_{ba}] = \delta_{ha} R_{bi} - \delta_{bi} R_{ha}.$$

Taking the jj entry of $(z-\Omega)R = 1$ and evaluating on v gives

$$(z - \mu_j + j - 1) r_j = 1 + \sum_{i<j} r_i.$$

For $i > j$, the expectation of $E_{ij}R_{ij}$ vanishes because $E_{ji}v = 0$. For $i < j$, commuting E_{ij} past R_{ij} gives $r_i - r_j$ because $E_{ij}v = 0$. If $s_j = 1 + \sum_{i \le j} r_i$, the recurrence gives $s_j = s_{j-1}(z - u_j + 1)/(z - u_j)$, starting with $s_0 = 1$. Thus

$$r_j(z) = \frac{1}{z - u_j} \prod_{i<j} \frac{z - u_i + 1}{z - u_i}.$$

The residue at u_j is the required squared projection norm. The projection is into the highest weight line because $v \otimes e_j$ has weight λ. Since $u_i - u_j = x_i(\lambda) - x_j(\lambda) + 1$, the residue is

$$\prod_{i<j} \frac{x_i(\lambda) - x_j(\lambda)}{x_i(\lambda) - x_j(\lambda) + 1}.$$

Its inverse is the branching scalar used in Lemma 3.

B A compactness reduction

Fix a number field K. We give another reduction of Theorem 1 to Proposition 4. Use the Euclidean norm of the underlying $\mathbb{Z}$-lattice given by summing the squared norms over all embeddings. For a Hermitian lattice L of rank d over $\mathcal{O}_K$, its covolume in this Euclidean space is

$$|\operatorname{disc} K|^{d/2} \exp(-[K:\mathbb{Q}]\widehat{\deg}_{\mathrm{n}}(L)).$$

For a free lattice, the formula is the Minkowski covolume of $\mathcal{O}_K^d$ times the determinant norm of an $\mathcal{O}_K$-basis, taken over all embeddings. The factor of two in the squared norm at a conjugate pair cancels the factor 2^{-1} per complex coordinate in the usual Minkowski covolume. In general, take a free sublattice of finite index. The index multiplies its covolume and subtracts its normalized logarithm from the degree relative to L, giving the same formula.

If L is semistable of degree zero, the saturation of $\mathcal{O}_K v$ has nonpositive degree for every $0 \neq v \in L$. The line degree formula then gives $\sum_\sigma \log \|v\|_\sigma \geq 0$, since the index term is nonnegative. By the arithmetic–geometric mean inequality, the shortest vector has length at least $\sqrt{[K:\mathbb{Q}]}$. Mahler's compactness theorem [4, Chapter V] therefore gives, along a subsequence of such lattices, $\mathbb{Z}$-bases in which the Gram matrices converge to a positive definite matrix.

To recover the $\mathcal{O}_K$-structure, fix a primitive integral element θ of K. Multiplication by θ has operator norm bounded by $\max_\sigma |\sigma(\theta)|$. Since the Gram matrices lie in a compact subset of the positive definite cone, the integer matrices for multiplication by θ have bounded entries. Passing to a subsequence makes that matrix constant. All the $\mathcal{O}_K$-actions now agree, since each element of K is a rational polynomial in θ. On the real span the limiting form still satisfies that the adjoint of multiplication by $u \in K \otimes_{\mathbb{Q}} \mathbb{R}$ is multiplication by its componentwise conjugate. This is a closed identity on the Gram matrix. It makes the idempotent summands for the real embeddings and complex pairs orthogonal, and makes the complex structure in each complex summand orthogonal. These summands thus give Hermitian metrics on the fixed projective module with the chosen convention for the squared norm. The degree and semistability inequalities pass to the limit by continuity of the metric on each fixed sublattice.

Comparison of the metrics by factors $e^{\pm\epsilon}$ compares all sublattice slopes up to ϵ simultaneously, so maximal slope is continuous. Stability is open among degree-zero lattices. In fact there are finitely many saturated sublattices of any bounded-below slope. The covolume formula bounds their covolumes, and each such sublattice is also saturated over $\mathbb{Z}$. Its primitive integral exterior vector in the appropriate exterior power of the underlying $\mathbb{Z}$-lattice has norm equal to that covolume, and there are finitely many bounded-norm vectors in each of these exterior powers. Thus for a stable lattice the slopes of proper sublattices have a strictly negative upper bound, which persists under small perturbations.

Suppose the tensor product of semistable lattices fails to be semistable, and choose ranks with minimal sum for which this occurs over K. By rescaling, take both factors of degree zero. For these ranks $\widehat{\mu}_{\max}(E \otimes F)$ attains a positive maximum on the pairs of semistable degree-zero lattices, by the compactness and continuity just proved. The tensor metrics vary continuously in each pair in the surviving integral bases. At such a maximum both factors are stable. Indeed, a proper slope-zero sublattice of a semistable factor and its quotient are

semistable of slope zero. Their tensors with the other factor are semistable by minimality of the rank sum. They form the two grades of a saturated filtration with the quotient metrics, which forces semistability of the full tensor by (11).

Fix a saturated sublattice V of rank r attaining the maximal slope of the maximizing tensor product. At an embedding σ, vary the norms on E by $\|z\|_t = \|\exp(tA)z\|$, where A is traceless Hermitian at a complex embedding and real symmetric at a real embedding. Vary the conjugate embedding conjugately. The determinant norms keep their value and the metrics stay stable for small $|t|$. Keeping the integral sublattice V fixed, its slope has a local maximum at zero. Let Π be the orthogonal projection onto V_σ and let $v_1, \dots, v_r$ be an orthonormal basis there. The derivative at zero of the log determinant of their Gram matrix under the variation is

$$2\sum_{\ell=1}^{r}\langle v_\ell, (A \otimes I)v_\ell\rangle = 2\,\mathrm{tr}\big(A\,\mathrm{tr}_{F_\sigma}\,\Pi\big).$$

At a conjugate pair the two equal embedding contributions add. In every allowed direction, stationarity gives $\mathrm{tr}(A\,\mathrm{tr}_{F_\sigma}\,\Pi) = 0$. The partial trace is Hermitian (real symmetric in the real case), and its trace is r, so it equals $(r/\operatorname{rk} E)I$. The same argument for F gives $\mathrm{tr}_{E_\sigma}\,\Pi = (r/\operatorname{rk} F)I$. At a real embedding these identities extend to the complexification.

Over $\mathbb{C}$ at σ, take a nonzero Plücker vector w for V. For any pair of traceless Hermitian operators A, B on the two factors, the function $t \mapsto \|(\exp(tA) \otimes \exp(tB)) \cdot w\|^2$, with the action on the exterior power, is a sum of real exponentials with nonnegative coefficients. This follows by expanding w in a tensor exterior basis of eigenvectors. Its derivative at zero is zero by the two partial trace identities, so convexity makes its value at zero minimal. By polar decomposition every special linear element is a unitary element times such a Hermitian exponential, and unitary elements preserve the norm. Thus the entire orbit of w has norm at least $\|w\|$. This is the norm minimization argument of Kempf and Ness [14]. Geometric instability over $\overline{K}$ would give a one-parameter subgroup sending w to zero, also after extending σ to $\overline{K}$. Hence the Plücker point is geometrically semistable. Proposition 4 now gives $\widehat{\mu}(V) \le 0$, contradicting the positive maximum. This gives tensor semistability for the fixed K, and the Harder–Narasimhan argument in Section 6 gives (1).

Declaration of AI use

OpenAI's ChatGPT assisted in developing and checking the mathematical arguments, searching the literature, and drafting and revising the manuscript. The author takes responsibility for the contents of the manuscript.

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