Sign changes of Kloosterman sums on moduli with at most three prime factors

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Abstract

For each sign, we prove that $\gg X/\log X$ squarefree moduli in $(X, 2X]$ with at most three prime factors have a normalized Kloosterman sum of that sign with magnitude bounded below by a positive constant. The bound extends to short arithmetic progressions, uniformly for nonzero coefficients in a small power range. The proof combines Sato–Tate marginals for complementary blocks of prime factors with a centered linear divisor weight.

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1 Introduction

Write $e(t) = e^{2\pi it}$. For integers A, B and a positive modulus n, set

$$S(A, B; n) = \sideset{}{^*}\sum_{x \bmod n} e\big((Ax + B\bar{x})/n\big), \qquad K_{A,B}(n) = \frac{S(A, B; n)}{\sqrt{n}}.$$

The sum is over units, and $\bar{x}$ is the inverse of x mod n. The substitution $x \mapsto -x$ shows that $S(A, B; n)$ is real. For $n > 1$, let $P^-(n)$ be its least prime factor and let $\omega(n)$ count its distinct prime factors.

Theorem 1.1. *There are positive constants $c, \kappa, \epsilon, \sigma, \delta, \alpha$ such that for every sufficiently large X, every $X^{1-\sigma} \le Y \le X$, every positive integer $m \le X^{\delta}$, every reduced class $a \bmod m$, every pair of nonzero integers A, B with $|AB| \le X^{\alpha}$, and each sign $s \in \{-1, 1\}$, at least $cY/(\varphi(m)\log X)$ integers n satisfy*

$$X < n \le X + Y, \qquad n \equiv a \pmod m, \qquad \mu^2(n) = 1,$$
$$\omega(n) \le 3, \qquad P^-(n) > X^{\epsilon}, \qquad sK_{A,B}(n) > \kappa.$$

Fouvry and Michel [3] initiated the study of these signs on almost primes. Matomäki [6] and Xi [10] developed lower bounds for absolute moments from Sato–Tate marginals. Xiao and Li [9, Theorem 1.1] give $\gg X/\log X$ moduli of each sign with at most four prime factors in $(X, 2X]$. For two prime factors, Drappeau and Maynard [1, Corollary 1.3] obtain sign changes given sufficiently close exceptional zeros.

We partition the prime factors to minimize the difference between the two sums of logarithms. A periodic-coefficient estimate and the multiplicative large sieve give second moments for the two parts. A linear divisor weight then separates the contributions from at most three factors, with its quadratic and cubic terms accounting for the two parity biases of an exceptional character.

2 Moments on prime boxes

For a nonzero integer C write $K_C(n) = K_{1,C}(n)$ and $\mathrm{Kl}(t; p) = K_{1,t}(p)$. The Weil bound gives $|K_C(n)| \le \tau(n)$. For squarefree $n = p_1 \cdots p_r$, the Chinese remainder theorem gives

$$K_C(n) = \prod_{i=1}^{r} Z_i, \qquad Z_i = \mathrm{Kl}\left(C\overline{n/p_i}^{\,2}; p_i\right). \tag{1}$$

The Sato–Tate law ν on $[-2,2]$ has density $(2\pi)^{-1}\sqrt{4-z^2}$. Its second moment is 1, and its first absolute moment is $\rho = 8/(3\pi)$.

Consider disjoint prime intervals $I_i \subset [P_i, 2P_i]$ in decreasing order, with a prescribed unit class modulo m in each. Put $X_0 = \prod_i P_i \asymp X$, $b_i = \log P_i / \log X_0$, and $b(I) = \sum_{i\in I} b_i$. We take r fixed and $\mathbf{b}$ in compact subsets of $b_1 > \cdots > b_r > 0$, away from the equality cases below. In each such subset the nonconstant Chebyshev tests below have sums $O(X^{1-\eta})$ for some $\eta > 0$, uniformly in the residue restrictions and C with $p_i \nmid C$. Thus the limiting assertions hold on boxes containing $\gg X^{1-\eta/2}$ tuples. Expectations are uniform averages over one box.

Lemma 2.1. *The coordinates $(Z_i)_{i\in I}$ have limiting law $\nu^{\otimes|I|}$ if*

$$b(I) < \tfrac{1}{2} \quad \text{or} \quad \tfrac{1}{2}b_j < b(I) - b_j < \tfrac{1}{2}(1-b_j) \quad \text{for some } j \in I. \tag{2}$$

Proof. By polynomial approximation it suffices to test products of $U_k(Z_i/2)$, where U_k is the Chebyshev polynomial of the second kind and $\int U_k(z/2)\,\mathrm{d}\nu(z) = 0$ for $k \geq 1$. Over $\mathbb{F}_p^\times$, the function $U_k(\mathrm{Kl}(C/u^2;p)/2)$ is the trace of the k-th symmetric power of the inverse-square pullback of Katz's Kloosterman sheaf, up to a constant sign. Pullback by the finite étale map $u \mapsto C/u^2$ preserves the connected geometric monodromy SL_2 for $p > 2$. For $k \geq 1$ the symmetric power is geometrically irreducible of rank $k+1$ and bounded conductor. Its complete sums against every multiplicative character are $O_k(\sqrt{p})$, uniformly for $C \not\equiv 0 \bmod p$, by the trace formula and Deligne's bound [11, Lemma 3.2, (3.3)].

For the first inequality in (2), put the degree-zero coordinates in the complement and write $q = \prod_{i\in I} p_i \asymp Q$, $u = n/q \asymp M$ for the resulting blocks. Chinese remaindering gives multiplicative Fourier coefficients $O(q^{-1/2})$ for the test as a function of $u \bmod q$. The principal coefficient gives a saving of order $Q^{-1/2}$. A character of conductor $\ell \mid q$ agrees with its inducing primitive character on the support of u, since the prime intervals are disjoint. In a range $\ell \asymp D$ there are $O(Q/D)$ possible q for each conductor. The coefficient is $O(D^{-1/2})$. With γ_u the indicator of the product of the complementary primes, Cauchy–Schwarz and the multiplicative large sieve [5, Theorem 25.15] give the bound

$$\frac{Q}{D^{3/2}} \sum_{\ell\asymp D} \sideset{}{^{\mathrm{prim}}}\sum_{\chi \bmod \ell} \left|\sum_u \gamma_u \chi(u)\right| \ll \frac{Q}{\sqrt{D}}(M+D^2)^{1/2}\|\gamma\|_2 \ll QM\sqrt{D^{-1}+D/M}.$$

Here $\|\gamma\|_2 \ll \sqrt{M}$ by unique factorization in disjoint intervals. Each nonprincipal conductor contains a prime from the box, and Q/M is smaller than a fixed negative power of X by (2). The $O(\log X)$ conductor ranges therefore give a power saving.

For the second inequality in (2), a test of degree zero at j reduces to the first case. Otherwise fix $p = p_j$ and put $v = \prod_{i\in I\setminus\{j\}} p_i \asymp N$, $u = \prod_{i\notin I} p_i \asymp M$, including all degree-zero coordinates in these products. Each of the other tests in I is periodic in u modulo its own prime divisor of v. The estimate of Fouvry and Michel [3, Proposition 7.2] for a bounded coefficient periodic in $u \bmod v$, with the remaining test at p, gives

$$\ll MN\left(p^{-1/8} + p^{1/8}N^{-1/4} + \sqrt{N/M}\right).$$

Both $p^{1/2} < N$ and $N < M$ hold with power margins.

For general C, the four off-diagonal pullbacks in [3, §7.2] have arguments

$$C[du_i(u_i-z)]^{-2}, \qquad C[dz(u_i-z)]^{-2} \qquad (i=1,2),$$

where $p \nmid du_1u_2(u_1-u_2)$. Over the algebraic closure, d is replaced by $d/\sqrt{C}$, so the loci of nontrivial projective wild inertia are still $\{u_1\}, \{0,u_1\}, \{u_2\}, \{0,u_2\}$. The separable pullbacks

retain geometric monodromy SL_2. The distinct loci rule out rank-one twist identifications, so the same Goursat–Kolchin–Ribet argument gives product monodromy. The tensor of the four nontrivial symmetric powers has no invariant and bounded conductor, giving the complete bound $O_k(\sqrt{p})$. The Weil bound handles the diagonal, and summation over p_j gives the saving. □

Suppose $b_1 < 1/2$. Let I be the heavier part of a partition minimizing $|2b(I) - 1|$, and write $t = b(I) > 1/2$. Moving any $j \in I$ to the complement shows $b_j \geq 2t - 1$, with equality excluded by the compact cuts. Since $|I| \geq 2$, the smallest b_j in I also satisfies $t - b_j \geq b_j$. Thus I satisfies the second criterion of (2), and its complement satisfies the first. Cauchy–Schwarz, using the two product marginals, bounds the mean of $|K_C|$ by $1 + o(1)$. For $b_1 > 1/2$, apply Weil at p_1 and use the product marginal on its complement. In all,

$$\mathbb{E}|K_C| \leq \begin{cases} 1 + o(1), & b_1 < 1/2, \\ 2\rho^{r-1} + o(1), & b_1 > 1/2 \end{cases} \qquad (r \geq 2). \tag{3}$$

For two and three prime factors use the regions

$$\mathcal{D}_2 = \{b_2 > 3b_1/4\}, \qquad \mathcal{D}_3 = \{b_2 > b_1/2\}, \qquad \mathcal{S} = \{b_3 > b_1/2\} \subset \mathcal{D}_3.$$

On $\mathcal{D}_2$, Lemma 2.1 gives the law of Z_2. To test Z_1, fix $p_1 = p$ and sum over p_2. Expanding a residue-class indicator modulo m gives additive phases with total coefficient norm one. For each phase write $e(vp_2/m) = e(kp_2/p)e(\theta p_2)$ with $|\theta| \leq 1/(2p)$. The first twist preserves the bounded conductor and geometric irreducibility of rank $k' + 1 \geq 2$ for a test of degree k'. At endpoints $y \asymp P_2$, the prime trace estimate [2, Theorem 1.5, (1.3)] is $O_{k',\gamma}(y(1+p/y)^{1/12}p^{-\gamma/2})$ for $\gamma < 1/24$. It saves a power for $P_2 > p^{3/4}$ with a power margin. Differencing the endpoints and applying partial summation to $e(\theta p_2)$ costs $O(1 + P_2/p)$.

On $\mathcal{D}_3$, the pair $\{2, 3\}$ satisfies (2) with $j = 3$, since $b_3/2 < b_2 < b_1$. For Z_1, fix p_1 and use the bilinear trace estimate [2, Theorem 1.17, (1.6)] over p_2, p_3. The term $p_1^{1/4}P_2^{-1/2}(\log p_1)^{1/2}$ saves a power because $b_2 > b_1/2$. The other terms, $p_1^{-1/4}$ and $P_3^{-1/2}$, do so as well. This gives the law of Z_1 and the product law for Z_2, Z_3 as separate marginals.

Matomäki's rearrangement bound [6, Lemma 2.3] lower-bounds the mean of a product of two nonnegative variables by the integral of their oppositely ordered quantiles. Xi's numerical lower bounds for the cosine normalization [10, §4.2], multiplied by 4 and 8, give

$$\mathbb{E}|K_C| \geq c_2 - o(1) \quad \text{on } \mathcal{D}_2, \qquad \mathbb{E}|K_C| \geq c_3 - o(1) \quad \text{on } \mathcal{D}_3, \qquad (c_2, c_3) = (0.44416, 0.28448). \tag{4}$$

On $\mathcal{S}$, also $\{1, 3\}$ satisfies (2) with $j = 1$ because $b_1/2 < b_3 < b_2$. In a joint weak limit, each of Z_1, Z_2 consequently has law ν conditional on Z_3. Applying the two-variable rearrangement bound to their conditional coupling and integrating against $|Z_3|$ gives $\mathbb{E}|K_C| \geq c_2\rho - o(1)$ there.

3 The linear divisor weight

On the ordered simplex $\Delta_r = \{b_1 > \cdots > b_r > 0,\ \sum_i b_i = 1\}$ put

$$\mathrm{d}\mu_r = \frac{\mathrm{d}b_2 \cdots \mathrm{d}b_r}{\prod_i b_i}, \qquad D_r H(\mathbf{b}) = \sum_{I \subseteq \{1,\ldots,r\}} (-1)^{|I|} H(b(I)).$$

Take μ_1 to have unit mass at $b_1 = 1$. Following the logarithmic centering in Drappeau and Maynard [1, §2 and (4.5)], set

$$H = \frac{43H_2 + 17H_3}{60}, \qquad H_j(t) = (1/2 - t)_+^j. \tag{5}$$

If $r \equiv j \bmod 2$, pairing complementary subsets shows that $2D_rH_j$ is the r-th difference of the polynomial $(1/2-t)^j$. Thus $D_rH_j = 0$ when also $r > j$. For $b_1 > 1/2$ the subsets containing index 1 contribute zero, and the other terms form a difference of order $r-1$ of the same polynomial. In that region $D_rH_j = 0$ for $r \geq j+2$.

For $r \leq 3$ the differences are nonnegative. In rank one $D_1H_j = H_j(0)$. In ranks two and three the differences are

	H_2		H_3
$r=2$	b_1b_2		$1/8-(1/2-b_2)^3$
$r=3$	$\begin{cases} 2b_2b_3, \\ 1/2-\sum_i b_i^2, \end{cases}$	$\begin{matrix} b_1 \geq 1/2, \\ b_1 \leq 1/2 \end{matrix}$	$3b_1b_2b_3.$

Define the lower contributions

$$E_j = c_2 \int_{\mathcal{D}_2} D_2H_j \,\mathrm{d}\mu_2, \qquad O_j = c_3 \int_{\mathcal{D}_3} D_3H_j \,\mathrm{d}\mu_3 + (c_2\rho - c_3) \int_{\mathcal{S}} D_3H_j \,\mathrm{d}\mu_3.$$

On $\mathcal{D}_2$ we have $0 < x = 1/2 - b_2 < 1/14$. The quotient of the cubic weight by b_1b_2 is $x + 1/(4(1/2+x))$, so

$$E_2 = \frac{c_2}{14}, \qquad E_3 = c_2\left(\frac{1}{392} + \frac{1}{4}\log(8/7)\right). \tag{6}$$

For the quadratic triple weight, the part of $\mathcal{D}_3$ with $b_1 > 1/2$ contributes $2\log(4/3) - 1/2$ to the integral. Indeed, with $z = b_3$, the inner integral of $2/(1-z-b_2)$ over $(1-z)/3 < b_2 < 1/2-z$ is $2\log(4(1-z)/3)$, which is integrated over $0 < z < 1/4$. Every triple with $b_1 < 1/2$ belongs to $\mathcal{D}_3$, since $b_2 \geq (1-b_1)/2$. There set $x_i = 1-2b_i$, with elementary symmetric functions e_2, e_3. The quotient $D_3H_2/\prod b_i$ is $4e_2/(e_2-e_3) \geq 4$. By symmetry and the Jacobian $1/4$, this region contributes at least

$$\frac{1}{6}\int_{\substack{x_i>0 \\ \sum_i x_i = 1}} (1+3e_3)\,\mathrm{d}x_2\,\mathrm{d}x_3 = \frac{7}{80},$$

using $e_2 \leq 1/3$. The (b_2, b_3)-areas of $\mathcal{D}_3$ and $\mathcal{S}$ are $1/24$ and $1/120$. Integrate from $\max(b_3, (1-b_3)/3)$ or $\max(b_3, 1-3b_3)$ to $(1-b_3)/2$, respectively, wherever the interval is nonempty. The region $\mathcal{S}$ lies in $b_1 < 1/2$. Consequently

$$\begin{aligned} O_2 &\geq c_3(2\log(4/3) - 33/80) + (c_2\rho - c_3)/30, \\ O_3 &= c_3/8 + (c_2\rho - c_3)/40. \end{aligned} \tag{7}$$

Write $V_{r,j} = \int_{\Delta_r} |D_rH_j| \,\mathrm{d}\mu_r$. Above three factors the quadratic weight contributes only on odd r in $b_1 \leq 1/2$. The cubic weight contributes only on even r, and only for $r = 4$ can it contribute in $b_1 > 1/2$. There $D_4H_3 = 6b_2b_3b_4$, and its integral is $\frac{1}{2}\int_{1/2}^1 (1-x)^2\,\mathrm{d}x/x = (\log 2 - 5/8)/2$. Since $2\rho^3 < 1.224$, the upper bounds in (3) give the following costs for $r \geq 4$. The inequalities are proved in Appendix A.

$$T_2 = \sum_{\substack{r \geq 5 \\ r \text{ odd}}} V_{r,2} < 0.0642, \tag{8}$$

$$T_3 = \sum_{\substack{r \geq 4 \\ r \text{ even}}} V_{r,3} + 0.224\,(\log 2 - 5/8)/2 < 0.0958. \tag{9}$$

For parity multipliers $1+w$ and $1-w$, the lower contribution from $r \le 3$ minus the upper contribution from $r \ge 4$ is at least

$$(1+w)\frac{43E_2+17(E_3-T_3)}{60}+(1-w)\frac{43(O_2-T_2)+17O_3}{60} > 0.00022 \qquad (-1 \le w \le 1). \tag{10}$$

Indeed the two coefficients exceed 0.00011 and 0.00013, respectively, by (6)–(9). For the rational comparisons use $\rho > 28/33$ and three terms each of $2\operatorname{arctanh}(1/7)$ and $2\operatorname{arctanh}(1/15)$. The disjoint parities above three factors allow the absolute values in (8) and (9) to be used directly for H.

4 Intervals and progressions

Let $J = (X, X+Y]$, $L = \log X$, and $M_0 = Y/(\varphi(m)L)$. Set $C = AB$. We will choose $\alpha < \epsilon$, so $P^-(n) > X^\epsilon$ ensures $(n, AB) = 1$. On those integers, substituting $y = Ax \bmod n$ gives $K_{A,B}(n) = K_C(n)$.

4.1 Smoothing and signed cancellation

Fix a smooth cutoff ψ with values in $[0,1]$, zero for $v \le 2$ and one for $v \ge 3$. For $0 < h < 1/8$, set $H_{j,h}(t) = H_j(t)\psi((1/2-t)/h)$ on $t \ge 0$ and $H_h = (43H_{2,h} + 17H_{3,h})/60$. Use a common compactly supported extension on the left for each pair $H_j, H_{j,h}$.

Define

$$f(u) = \frac{2|\sin(u/2)|}{u} \quad (u > 0), \qquad F(s) = \int_0^\infty e^{-su} f(u)\,\mathrm{d}u \quad (s > 0).$$

Then $F(s) \le 1/s$. Integrating over successive periods of $2|\sin(u/2)|$ gives

$$-F'(s) = \frac{\coth(\pi s)}{s^2+1/4}, \qquad -sF'(s) \le 4/\pi. \tag{11}$$

The inequality follows from $x\coth x \le 1 + x^2/3$, by comparing coefficients in $x\cosh x \le (1+x^2/3)\sinh x$, and $\pi^2 < 12$.

Lemma 4.1. *For $j = 2, 3$,*

$$\sum_{r\ge1}\int_{\Delta_r} |D_r(H_j - H_{j,h})|\,\mathrm{d}\mu_r \ll h^{j-4/\pi}.$$

Proof. The defect is a translate and rescaling of $h^j v_+^j(1-\psi(v))$, so its Fourier transform is $O(h^{j+1}(1+h|t|)^{-j-1})$. The absolute finite-difference factor at frequency $t > 0$ is $\prod_i 2|\sin(tb_i/2)|$. On the full simplex its integral against $\prod_i b_i^{-1}\,\mathrm{d}b_2\cdots\mathrm{d}b_r$ is $tf^{*r}(t)$, by $u_i = tb_i$, with convolution on $\mathbb{R}_+$. The ordered integral is $1/r!$ times this. Thus Fourier inversion bounds the sum by

$$\ll h^{j+1}\int_0^\infty \frac{tG(t)}{(1+ht)^{j+1}}\,\mathrm{d}t, \qquad G(t) = \sum_{r\ge1} f^{*r}(t)/r!.$$

By (11), $F(1/T) \le (4/\pi)\log T + O(1)$ for $T \ge 1$. The Laplace transform of G is $e^{F(s)} - 1$, so $\int_0^T G(t)\,\mathrm{d}t \le e(e^{F(1/T)} - 1) \ll T^{4/\pi}$. The range $t \le 1/h$ and the dyadic ranges above it give the stated bound, as $j > 4/\pi$. □

Set $W_h(n) = \sum_{d|n}\mu(d)H_h(\log d/\log n)$ for $n \in (X, 2X]$. Nonzero terms have $d \le (2X)^{1/2-2h} \le X^{1/2-h}$ for large X.

Lemma 4.2. *For fixed $0 < h < 1/8$, $m \le X^{h/8}$, $(a,m) = 1$, and $1 \le |C| \le X^{h/100}$,*

$$\sum_{d \le X^{1/2-h}} \sup_{X \le U \le V \le 2X} \left| \sum_{\substack{U < n \le V \\ d \mid n,\ n \equiv a \bmod m}} \mu^2(n) K_C(n) \right| \ll_h X^{1-h/6}.$$

Proof. Expand $\mu^2(n) = \sum_{v^2 \mid n} \mu(v)$. For $v \le X^{h/5}$ put $q = [d, v^2]$. A nonempty congruence condition requires $(q,m) = 1$ and has the form $n \equiv qc \bmod qm$, with $c \equiv a\bar{q} \bmod m$. Steiner's estimate [7, Theorem 18] for prefix sums of $S(1, C; n)/n$ is bounded, up to $X^{o(1)}$, by

$$|C| \left(X^{1/6} q^{-1/3} + m^{1/2} (1 + X/(qm)^2)^{2\vartheta} \right), \qquad 0 \le \vartheta \le 7/64.$$

In that theorem the two spectral coefficient factors are $O(|C|^{7/12})$ and $O(|C|^{1/2})$, and the exceptional-spectrum factor is $O(|C|^{3\vartheta/2}(1 + X/(qm)^2)^{2\vartheta})$. The first coefficient 1 makes the common divisor of the two coefficients equal to one. Partial summation to $K_C(n)$ costs $\sqrt{X}$. Since $q \ge d$, summation over $d \le D = X^{1/2-h}$, for each v, gives at most

$$|C| X^{1/2+o(1)} \left(X^{1/6} D^{2/3} + m^{1/2} D + m^{1/2} X^{2\vartheta} D^{1-4\vartheta} \right).$$

Before the factors $|C|$ and $X^{h/5}$ from summing over v, the saving is at least $h/2$, since $1 - 4\vartheta \ge 9/16$ and $m^{1/2} \le X^{h/16}$. For $v > X^{h/5}$ the Weil and divisor bounds give at most $X^{o(1)} \sum_{X^{h/5} < v \le \sqrt{2X}} (2X/v^2 + 1) \ll X^{1-h/5+o(1)}$. Both contributions satisfy the lemma. □

Each coefficient $H_h(\log d/\log t)$ on $[X, 2X]$ has bounded supremum and total variation, by monotonicity of $\log d/\log t$ and smoothness of H_h. For $\delta < h/8$, $\delta + \sigma < h/6$, and $\alpha < h/100$, Lemma 4.2 thus gives $o(M_0)$ for $\sum_{n \in J,\ n \equiv a \bmod m} \mu^2(n) W_h(n) K_C(n)$.

The absolute contribution from $p \mid n$ with $p \le X^\epsilon$ is $O_h(\epsilon M_0)$. To prove this, Fourier inversion in W_h multiplies the factors $1 - e^{it \log p/\log n}$ over $p \mid n$ on squarefree integers. With $u = 1 + |t|$, their product times $|K_C(n)|$ is bounded by the multiplicative function f_u given by

$$f_u(p) = 4 \min(1, u \log p/L), \qquad f_u(p^k) = 0 \quad (k \ge 2).$$

Extract $p \le X^\epsilon$, $p \nmid m$, and apply Shiu's theorem [5, Theorem 20.3] to the remaining factor in the class $a\bar{p} \bmod m$ and interval of length Y/p. Here $0 \le f_u \le \tau_4$, the endpoint is $(X+Y)/p$, and $Y/(pm) \ge ((X+Y)/p)^{1/2}$ for large X when $\epsilon, \delta, \sigma < 1/10$. The bound is

$$\ll \frac{M_0}{p} \exp\left(\sum_{\ell \le 2X,\ \ell \text{ prime}} \frac{f_u(\ell)}{\ell} \right) \ll \frac{M_0}{p} u^4.$$

The first inequality uses Mertens' estimate on the excluded local factors in Shiu's exponential. For the second, split the prime sum at $X^{1/u}$. The part below the split is $O(1)$ by $\sum_{\ell \le z} (\log \ell)/\ell \ll \log z$, and the part above is at most $4 \log u + O(1)$ by Mertens' theorem. If $X^{1/u} < 2$, then $u \gg L$ gives the same bound. Now $\sum_{p \le X^\epsilon} f_u(p)/p \ll u\epsilon$, and integration against $|\widehat{H}_h(t)|$ gives the assertion. In particular, on the set $\mathcal{N}$ of squarefree $n \in J$ in the class $a \bmod m$ with $P^-(n) > X^\epsilon$, we have

$$\left| \sum_{n \in \mathcal{N}} W_h(n) K_C(n) \right| \le (O_h(\epsilon) + o(1)) M_0. \tag{12}$$

4.2 Prime counts in boxes

For fixed ϵ there are at most $R = \lceil 2/\epsilon \rceil$ prime factors in $\mathcal{N}$. Partition the prime variables into bins $(P, P(1+\Delta)]$ with $\Delta = X^{-\xi}$, and take $\sigma < \xi$. For each m, denote its possible exceptional real character and zero by χ_* and β, in the sense of Thorner and Zaman [8, Theorem 1.1]. In a reduced class u mod m, for any fixed tolerance $\zeta > 0$ the count in a bin is

$$\frac{B_P}{\varphi(m)}(1 - \chi_*(u)t_P)(1 + O(\zeta)), \qquad B_P = \int_P^{P(1+\Delta)} \frac{\mathrm{d}v}{\log v}, \qquad t_P = \frac{1}{P\Delta}\int_P^{P(1+\Delta)} v^{\beta-1}\,\mathrm{d}v. \tag{13}$$

The exceptional term is absent when there is no such zero. To apply the estimate at $P \gg X^{\epsilon}$, use $1 - \chi_*(u)t_P \gg m^{-1/2}$ by [8, (1.2)]. The weighted length divided by $\varphi(m)$ exceeds $(P(1+\Delta))^{32/37+\eta_0}$ if $\xi + 3\delta/2 < \epsilon(1 - 32/37 - \eta_0)$, for fixed sufficiently small $\eta_0 > 0$. The denominator in the exponential error is $\log m + O(L^{2/3}(\log L)^{1/3})$, giving tolerance ζ after decreasing δ. Division by $\log P$ removes the prime logarithms with a relative error $o(1)$. For $m = 1$ count the unit class modulo 2.

A box with product comparable to X has $\gg X^{1-r(\xi+3\delta/2)}L^{-r}$ tuples for every prescribed pattern of unit classes. Taking $R(\xi + 3\delta/2) < \eta/2$ makes this sufficient for the uniform trace estimates of Section 2 on each compact region there. Summing the modeled counts over unit patterns $u_1 \cdots u_r = a \bmod m$ gives the density factor

$$\frac{1}{\varphi(m)^{r-1}} \sum_{u_1\cdots u_r = a} \prod_i (1 - \chi_*(u_i)t_{P_i}) = 1 + (-1)^r \chi_*(a) \prod_i t_{P_i}.$$

Every nonempty proper subset in the expansion has character average zero, by varying a slot in the subset against one outside it. The full subset contributes $\chi_*(a)$. The errors in (13) cost $O_R(\zeta)$ in this factor, since each slot multiplier is at most two. Moreover $\prod_i t_{P_i} = X^{\beta-1} + O_R(1/L)$, by averaging $(\prod_i v_i)^{\beta-1}$ and using $\sup_{0\le t\le 1} tX^{-t} \ll 1/L$ with $\prod_i v_i \asymp X$. Write $w = \chi_*(a)X^{\beta-1}$, with $w = 0$ when the exceptional term is absent.

For decreasing prime factors put $\mathbf{b}(n) = (\log p_i / \log n)_{i=1}^r$. The change of variables $z = \prod_i v_i$, $b_i = \log v_i / \log z$, gives

$$\prod_i \frac{\mathrm{d}v_i}{\log v_i} = \frac{\mathrm{d}z}{\log z}\,\mathrm{d}\mu_r. \tag{14}$$

Each squarefree integer has one decreasing tuple. Summing $\prod_i B_{P_i}$ corresponds to integrating (14), giving total mass $O_\epsilon(Y/L)$ for the boxes meeting J. The boxes crossing an endpoint lie in product strips of width $O_R(X\Delta)$, so their class count is $O_{R,\epsilon}(X\Delta/(\varphi(m)L)) = o(M_0)$. The cutoff is $b_r > \epsilon + O_\epsilon(1/L)$. At fixed ϵ, strips about the comparison hyperplanes have arbitrarily small μ_r-measure. In such strips count by (13), allowing all tuples in coincident bins, and use $|K_C| \le 2^R$.

Off these strips, freeze the weights within each bin and sum (13). A fixed bounded, piecewise continuous weight in $\mathbf{b}(n)$ then contributes its integral over $\Delta_r \cap \{b_r > \epsilon\}$ times $M_0(1+(-1)^r w)$, up to an arbitrarily small fixed error times M_0 and $o(M_0)$. For a nonnegative weight multiplied by $|K_C|$, insert the corresponding bounds of Section 2. In particular the ordinary count satisfies $|\mathcal{N}| \ll_\epsilon M_0$.

4.3 Counting each sign

On $\mathcal{N}$ let $W(n) = D_r H(\mathbf{b}(n))$, where $r = \omega(n)$. The smoothed weight $W_h(n)$ is the same finite difference with H_h. Since the box means of $|K_C|$ are at most $2 + o(1)$, including the

Weil bound at $r = 1$, and $1 + (-1)^r w \le 2$, the comparison applied to $|D_r(H - H_h)|$ costs at most

$$4M_0 \sum_{r \ge 1} \int_{\Delta_r} |D_r(H - H_h)| \, \mathrm{d}\mu_r$$

up to the fixed tolerance and $o(M_0)$. Lemma 4.1 bounds this by $O(h^{2-4/\pi} M_0)$.

Choose h first to make the smoothing error small compared to (10). Then choose ϵ to make $O_h(\epsilon)$ in (12) and the loss from $b_r \le \epsilon$ in the lower integrals small. The latter follows from (6) and the bounded triple quotients $D_3 H_j / \prod b_i$, since $0 \le D_3 H_2 \le 2 b_2 b_3$ and $b_1 \ge 1/3$. Fix the compact regions and the tolerance next. Finally choose the exponents $\xi, \delta, \sigma, \alpha > 0$ satisfying the previous strict inequalities and $\alpha < \epsilon$.

Write $A_{\le 3}$ and $A_{\ge 4}$ for the sums of $|W K_C|$ on $\mathcal{N}$ with the indicated numbers of prime factors, and S for the full sum of $W K_C$ on $\mathcal{N}$. By (10), (12), and the bounds for smoothing and transfer,

$$A_{\le 3} - A_{\ge 4} - |S| \gg M_0.$$

Let $S_{\le 3}$ be the signed sum over at most three prime factors. On these integers $W \ge 0$, and the weighted absolute mass of sign s is

$$\tfrac{1}{2}(A_{\le 3} + s S_{\le 3}) \;\ge\; \tfrac{1}{2}(A_{\le 3} - A_{\ge 4} - |S|) \;\gg\; M_0.$$

Here $|S_{\le 3}| \le A_{\ge 4} + |S|$. The bound $|\mathcal{N}| \ll_\epsilon M_0$, together with boundedness of W on the first three ranks, permits discarding $|K_C| \le \kappa$ for a sufficiently small fixed $\kappa > 0$. Each remaining summand on these ranks is bounded by a constant, by $|K_C| \le 8$. This proves Theorem 1.1.

A Bounds for the simplex integrals

If $r \ge j + 1$ has the parity of $j + 1$, Fourier inversion gives $D_r H_j$, up to a sign, as

$$\frac{j!}{\pi} \int_0^\infty t^{-j-1} \prod_i 2 \sin(t b_i / 2) \, \mathrm{d}t. \tag{15}$$

Indeed the $(j+1)$-st distributional derivative of H_j is a signed atom of magnitude $j!$ at $1/2$. The differences cancel the centering phase and annihilate polynomials of degree at most j. Taking absolute values and integrating on the simplex gives $t f^{*r}(t)/r!$ for the absolute sine product, as in Lemma 4.1. The Laplace integral for t^{-j} and Tonelli's theorem yield

$$V_{r,j} \le \frac{j}{\pi r!} \int_0^\infty s^{j-1} F(s)^r \, \mathrm{d}s. \tag{16}$$

Convergence follows from $F(s) \le 1/s$ and (11).

A.1 Three exact integrals

Inclusion and exclusion gives $D_r H_j / (j! \prod b_i)$ as the $(r - 1 - j)$-th derivative at zero of the density of the sum of independent uniforms on $[-b_i/2, b_i/2]$. The density is even and nonincreasing on the positive half-line because convolution with a centered uniform preserves this property. Its value and second derivative give $D_4 H_3 \ge 0$, $D_5 H_2 \le 0$, and $D_6 H_3 \le 0$. Integrate the signed kernel in (15) at these pairs to obtain equality in (16) with $F(s)$ replaced by $2 \arctan(1/(2s))$.

Let $B_k = \int_0^{\pi/2} u^k \cot u \, \mathrm{d}u$. Substituting $s = \frac{1}{2} \cot u$ and integrating by parts gives

$$V_{4,3} = \frac{3B_1 - B_3 - \pi^3/16}{3\pi}, \qquad V_{5,2} = \frac{80 B_3 - \pi^5/8}{60\pi}, \qquad V_{6,3} = \frac{10 B_3 - B_5 - \pi^5/64}{15\pi}.$$

For $k = 1, 3, 5$, integrate $u^k \cot u$ by parts and use the Abel limit of the Fourier series for $\log(2 \sin u)$ to obtain

$$B_1 = \frac{\pi \log 2}{2}, \qquad B_3 = \frac{\pi^3 \log 2}{8} - \frac{9\pi}{16}\zeta(3), \qquad B_5 = \frac{\pi^5 \log 2}{32} - \frac{15\pi^3}{32}\zeta(3) + \frac{225\pi}{64}\zeta(5).$$

These give $V_{4,3} < 0.081300$, $V_{5,2} < 0.035710$, and $V_{6,3} < 0.004908$, using $3.14159 < \pi < 3.14160$ from Machin's formula, eight terms of the arctanh series for $\log 2$, and the first 100 terms of the zeta series with their integral-test remainders. Also $(\log 2 - 5/8)/2 < 0.034074$.

A.2 Quadratic second moments

For odd $r \geq 7$ put $s = 4/r$ and

$$I_r = \int_0^1 \frac{((1+u)^s - (1-u)^s)^r}{u^3}\,\mathrm{d}u, \qquad p_r = \frac{r}{\mathrm{B}(4-s, s)} \int_0^{1/2} v^{3-s}(1-v)^{s-1}\,\mathrm{d}v.$$

Apply Cauchy–Schwarz to $r!V_{r,2}$ on the full simplex with squared factors $\prod_i b_i^{s-1}$ and $|D_r H_2|^2 \prod_i b_i^{-1-s}$. The first may be integrated over $\max b_i \leq 1/2$, since $D_r H_2$ vanishes elsewhere. Its integral is $(1 - p_r)\Gamma(s)^r/6$, since the excluded events $b_i > 1/2$ are disjoint and $1 - b_i$ has beta law $\mathrm{B}(4-s, s)$ under the Dirichlet probability density.

For the second integral, put $x_i = b_i/b_r$ for $i < r$, $x_r = 1$, and $T = \sum_i x_i$. The Jacobian is T^{-r} and the centered quadratic difference has degree two, so the total radial factor is $T^{rs-4} = 1$. In the square of (15) write the two frequencies as t, ut with $0 < u < 1$, and double the integral. The substitution $y_i = tx_i$ gives $r - 1$ copies of the integral below and leaves $t^{-5+(r-1)s}\,\mathrm{d}t = t^{-1-s}\,\mathrm{d}t$, giving the last copy. The integral in question is

$$\int_0^\infty 4 \sin(y/2) \sin(uy/2) y^{-1-s}\,\mathrm{d}y = 2^{1-s}(-\Gamma(-s)) \cos(\pi s/2)((1+u)^s - (1-u)^s).$$

The integral with absolute values is $O_s(u^s)$ by $2|\sin(uy/2)| \leq \min(2, uy)$, so Fubini applies since $rs = 4$. The second squared factor integrates to $8/\pi^2$ times the integral of the r-th power of this kernel against $\mathrm{d}u/u^3$. Euler's reflection formula gives

$$V_{r,2}^2 \leq (1 - p_r)\frac{\pi^{r-2} I_r}{12(r!)^2 s^r \sin^r(\pi s/2)}. \tag{17}$$

The substitution $x = ((1-u)/(1+u))^{1/r}$ gives the positive series

$$I_r = 32r \int_0^1 \frac{x^{r-1}(1-x^4)^r}{(1-x^{2r})^3}\,\mathrm{d}x = 8r\,r! \sum_{k \geq 0} \frac{\binom{k+2}{2}}{\prod_{i=0}^r (r(2k+1)/4 + i)}.$$

The sum over $k \geq 40$ before multiplication by $8r\,r!$ is at most $2(2/r)^{r+1} 39^{2-r}/(r-2)$ by $\binom{k+2}{2} \leq 2k^2$ and the integral test. For p_r, expand $(1-v)^{s-1}$ in the beta integral to get

$$p_r \geq \frac{6r \sin(\pi s)\, 2^{s-4}}{\pi(1-s)(2-s)(3-s)} \sum_{k=0}^{13} \frac{(1-s)_k}{2^k k!(4-s+k)},$$

where $(1-s)_k$ denotes the rising factorial. Inserting these bounds in (17) gives the following upper bounds. Use six terms of the alternating sine series, with the lower endpoint for π in positive terms and the upper endpoint in negative terms. For 2^s, use eight terms of the exponential series at s times the six-term lower bound for $\log 2$.

r	7	9	11	13	15	17
$10^6 V_{r,2}$ upper bound	16030	6800	3026	1382	641	301

A.3 The two tails

Let $R_m(v) = \sum_{r \ge m,\ r \equiv m \bmod 2} v^r/r!$. By (16), the two tails are bounded by

$$\frac{j}{\pi} \int_0^\infty s^{j-1} R_m(F(s))\, \mathrm{d}s, \qquad (j,m) \in \{(2,19),(3,8)\}.$$

Put $b = 1/16$, $\pi_- = 3.14159$ and $P_8(v) = \sum_{k=1}^{8} v^k/k!$. Split each dyadic interval in $[b, 16]$ into eight equal cells. At their endpoints bound $-F'(x)$ by

$$A(x) = \frac{1 + 2/P_8(2\pi_- x)}{x^2 + 1/4}.$$

Since $-F'$ is the Laplace transform of the nonnegative function $2|\sin(u/2)|$, it is convex. Starting with $U(16) = 1/16$, set

$$U(x) = U(y) + \lceil (y - x)(A(x) + A(y))/2 \rceil_9$$

on successive cells $[x, y]$ from right to left. The ceiling is to a multiple of 10^{-9}. The trapezoid bound gives $F(x) \le U(x)$, and the recurrence gives $U(b) < 9/2$. On $[0, 9/2]$,

$$R_m(v) \le Q_m(v) := \sum_{\substack{m \le r \le 36 \\ r \equiv m \bmod 2}} \frac{v^r}{r!} + \frac{(9/2)^{37}}{37!\,(1 - (9/2)/38)}.$$

Both F and R_m are convex, and R_m is increasing. Thus the piecewise linear interpolant L_m of the endpoint values $Q_m(U(x))$ majorizes $R_m(F(s))$ on $[b, 16]$.

For $0 < s < b$, use $F(s) \le U(b) + (4/\pi_-)\log(b/s)$. With $u = U(b)$ and $q = 4/(j\pi_-)$, termwise integration after $s = be^{-t}$ gives the upper bound

$$\int_0^b s^{j-1} R_m(F(s))\, \mathrm{d}s \le \frac{b^j}{j(1-q^2)} \left(\sum_{k=0}^{m} \frac{u^k q^{m-k}}{k!} + \frac{u^{m+1}}{(m+1)!\,(1 - u/(m+2))} \right) =: B_{j,m}.$$

In fact the integrated r-th term is $(b^j/j)\sum_{k=0}^{r} u^k q^{r-k}/k!$. Summing first over r of the chosen parity gives the formula, with the exponential tail for $k > m$ bounded geometrically. On $s \ge 16$, $F(s) \le 1/s$ gives $R_m(F(s)) \le 2s^{-m}/m!$.

The resulting rational comparisons are

$$\frac{j}{\pi_-} \left(B_{j,m} + \int_b^{16} s^{j-1} L_m(s)\, \mathrm{d}s + \frac{2 \cdot 16^{j-m}}{m!(m-j)} \right) < \begin{cases} 0.00027, & (j,m) = (2,19), \\ 0.00192, & (j,m) = (3,8). \end{cases}$$

The middle integral is a sum over the 64 cells. With the finite contributions this gives (8) and (9).

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