Extremal and near-extremal chiral CFTs

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Abstract

How large can the gap in the spectrum of Virasoro primaries of a chiral conformal field theory be? Modular invariance bounds this gap by $\lfloor c/24 \rfloor + 1$. Theories that saturate the bound, called extremal, have no primaries below the black hole threshold, and Witten proposed them as the building blocks of holographic duals of pure three-dimensional gravity. Beyond $c = 40$, however, no extremal theory is known, and no chiral CFT with a primary gap larger than two was previously known to exist. In this paper we begin a systematic exploration of the space of chiral CFTs with large gap, using orbifolds of lattice and free-fermion theories by large finite groups. We construct three examples. At $c = 64$ we find an extremal theory, with gap three. At $c = 96$ we find a near-extremal theory with gap four, whose lightest primaries lie exactly at the black hole threshold; if holomorphic CFTs define classes in topological modular forms, no theory at this central charge has a larger gap. At $c = 128$ we complete the Barnes–Wall orbifold proposed by Keller, Roberts and Roberts, which has gap four, by proving that its anomaly vanishes. We determine its partition function and show that its automorphism group is the finite simple group $E_8(2)$, whose smallest nontrivial irreducible representation is realized by the weight-four primaries. The main difficulties are to show that the orbifold groups are free of anomalies, for which level matching is in general not sufficient, and, at $c = 64$ and $c = 96$, to find a single choice of discrete torsion that removes all light twisted-sector states; at $c = 128$ every choice of torsion gives gap four, and we single out the one that keeps the lightest twisted states. All three theories are unitary. We explain the tools we use to overcome these difficulties, and discuss the implications for three-dimensional gravity and the obstacles to constructing theories with even larger gaps.

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1 Introduction

1.1 Chiral CFTs, three-dimensional gravity and the primary gap

Chiral conformal field theories in two dimensions are, in a sense, the most symmetric possible CFTs: every state is a descendant of the vacuum under the full chiral algebra. In mathematical language these are holomorphic (or self-dual) vertex operator algebras.[1] The torus partition function of such a theory is modular invariant up to a phase, and its central charge c is a multiple of eight. The simplest examples are the lattice theories built from even unimodular lattices, and the most famous is the Monster CFT $V^\natural$ of Frenkel, Lepowsky and Meurman at $c = 24$ [1].

The states of a chiral CFT are organized into representations of the Virasoro algebra. The vacuum representation, which contains the stress tensor and its descendants, is present in every theory; all other states are descendants of nonvacuum Virasoro primaries. We define the *primary gap* μ of a theory to be the smallest conformal weight of a nonvacuum Virasoro primary. The stress tensor does not count towards the gap: below weight μ the spectrum coincides with that of the Virasoro vacuum module. This paper is about the question:

> *How large can the primary gap of a chiral CFT be, and what do theories with a large gap look like?*

An important motivation for this question comes from three-dimensional gravity. Pure gravity in AdS_3 has no local degrees of freedom, and its asymptotic symmetries form two copies of the Virasoro algebra with central charge $c = 3\ell/2G$ [2]. A holographic dual of pure gravity should therefore contain only the states that gravity itself provides: the vacuum and its Virasoro descendants, which describe the boundary gravitons, together with the BTZ black holes [3]. In a holomorphic theory the black holes correspond to states with $h \geq c/24$, so all nonvacuum primaries should lie at or above this threshold. Primaries with $0 < h < c/24$ would instead describe massive particles, such as perturbative states or conical defects in the bulk. Witten proposed that pure gravity is dual to a CFT that factorizes holomorphically, with $c = 24k$, and whose holomorphic factor is an *extremal* CFT: it has no primaries below weight $k+1$, which is the largest gap allowed by modular invariance [4]. Such a theory has the sparsest possible spectrum: its partition function is uniquely determined, and all of its nonvacuum primaries lie above the black hole threshold. For $k = 1$ the Monster CFT is an example. Avramis, Kehagias and Mattheopoulou suggested extending this proposal to $c = 8m$ [5]. The question is sharpest in chiral gravity, a version of topologically massive gravity whose conjectured dual is a single chiral CFT [6]. There the sum over classical geometries reproduces the extremal partition function, so that constructing the quantum theory amounts to constructing an extremal CFT [7]. Benjamin, Dyer, Fitzpatrick, Maloney and Perlmutter proposed a modification of Witten's proposal, in which a theory with $c = 24k$ may also have primaries exactly at the black hole threshold $h = k$, interpreted as small black holes with vanishing classical horizon area [8]. They called such theories *near-extremal*.

[1] In this paper we will use "chiral CFT" and "holomorphic CFT" interchangeably.

More broadly, the gravitational path integral does not by itself produce a consistent spectrum in ordinary Einstein gravity [9–11], and it has been suggested that the dual of pure gravity, if it exists, may be an ensemble of CFTs rather than a single theory [12], as is the case for a related theory of abelian gauge fields in AdS_3 [13, 14]. A sparse spectrum of light operators is a basic property of CFTs with weakly coupled gravitational duals [15], and the gap of non-chiral CFTs has been studied extensively using the modular bootstrap [16–20]. Chiral CFTs with a large gap are also natural places to look for large finite symmetry groups, as the example of the Monster shows.

For chiral CFTs, modular invariance is very restrictive. The partition function is fixed by its first $\lfloor c/24 \rfloor + 1$ coefficients, and this implies that $\mu \leq \lfloor c/24 \rfloor + 1$ for every c [21,22]. Theories that saturate this bound were called extremal by Höhn, who showed that they exist for $c \leq 40$: lattice theories at $c = 8$ and 16, the Monster CFT at $c = 24$, and $\mathbb{Z}_2$ orbifolds of extremal lattice theories at $c = 32$ and $c = 40$ [21]. For $c \geq 48$, where the extremal bound requires a gap of at least three, no extremal theory is known. Such theories, if they exist, have been studied in detail: their genus-two partition functions are consistent with modular invariance [23,24]; an extremal theory at $c = 48$ cannot have the Monster as its symmetry group [22,25]; a conjecture that would have ruled out extremal theories for $k \geq 42$ [26] turned out to be false [25]; and the numerical bootstrap gives evidence against their existence for $k \geq 20$ [27]. Lin and Pei showed that if holomorphic CFTs define classes in topological modular forms, as conjectured by Segal, Stolz and Teichner, then there are no extremal theories at $c = 48$, 72, 96, 120, and infinitely many other values of c [28]. Supersymmetric analogues of extremal theories have been studied as well [8,22,29].

Even a more modest question has remained open: are there chiral CFTs with primary gap larger than two at all? Since the extremal bound forces $\mu \leq 2$ for $c < 48$, such theories must have $c \geq 48$. A natural way to look for them is to start with a theory that has few light states, such as the lattice theory of a lattice with large minimal norm, and to remove the remaining light states with an orbifold. Gemünden and Keller studied orbifolds of lattice theories at $c = 48$ and $c = 72$ along these lines [30, 31]. They found theories without currents and with very few weight-two primaries, but none with gap larger than two. Keller, Roberts and Roberts (KRR) then proposed a theory with $c = 128$ and gap four: an orbifold of the lattice theory of the Barnes–Wall lattice by an extraspecial group of order 2^{15} [32]. Its existence depended on a conjecture that a certain anomaly vanishes. For non-chiral theories, Ji and Tian recently constructed infinite families of CFTs with Virasoro primary gap three and four [33].

1.2 Results

In this paper we begin a systematic exploration of the space of chiral CFTs with large gap. We construct three theories, summarized in Table 1:

- At $c = 64$ we construct a theory with gap three. Since $\lfloor 64/24 \rfloor + 1 = 3$, this theory is extremal, and to our knowledge it is the first extremal chiral CFT with $c \geq 48$. All of its nonvacuum primaries lie above the black hole threshold $c/24 = 8/3$,

c	$\lfloor c/24 \rfloor + 1$	gap	parent theory	orbifold group G	$\lvert G \rvert$
64	3	3	$V_{D_{64}^+}$ (128 free fermions)	$\mathbb{F}_2^{14} \rtimes \mathbb{F}_2^7$ (Gold code)	2^{21}
96	5	4	$Z^{\otimes 3}$, with $Z = V_{\mathrm{BW}_{32}}/\mathbb{Z}_2$	$\mathbb{F}_2^{24} \rtimes \mathbb{Z}_9$	$9 \cdot 2^{24}$
128	6	4	$V_{\mathrm{BW}_{128}}$	2_+^{1+14} (extraspecial)	2^{15}

Table 1: The three theories constructed in this paper. The second column is the extremal bound on the gap. Here D_{64}^+ and BW_n denote the even unimodular lattices described in the text, and V_L is the lattice theory of a lattice L.

and its partition function is exactly the one proposed in [5], following [4], for the holomorphic factor of the dual of pure gravity at this central charge.

- At $c = 96$ we construct a theory with gap four, one unit below the extremal value $\lfloor 96/24 \rfloor + 1 = 5$. Its lightest primaries sit exactly at the black hole threshold $c/24 = 4$, so it is a near-extremal theory in the sense of [8], with $k = 4$. Near-extremal theories with $k \leq 2$ are easy to find (they are the theories at $c = 24$ with currents and the non-extremal theories at $c = 48$ without currents), but this is the first one we know of with $k > 2$. If the conjecture used in [28] holds, an extremal theory at $c = 96$ does not exist, and gap four is the largest possible. Numerical bootstrap studies have given evidence against near-extremal theories with $k \geq 4$ [27]; we comment on this tension in Section 7.2.

- At $c = 128$ we complete the KRR construction by proving that its anomaly vanishes, and we show that the resulting theories are unitary. For the most symmetric choice of discrete torsion we also determine the full partition function and show that the automorphism group is the finite simple group $E_8(2)$, with the weight-four primaries forming the smallest nontrivial irreducible representation of this group. We also show that no orbifold of this theory by a group of its automorphisms has gap larger than four.

To our knowledge these are the first chiral CFTs with a primary gap larger than two. Table 2 summarizes what we determine about each of them. At $c = 64$ and $c = 96$ our arguments show that suitable choices of discrete torsion exist without singling out a particular one. The numbers of suitable torsion classes are not numbers of inequivalent theories, since different classes may give isomorphic theories. We return to the gravitational interpretation of these results in Section 7.2.

1.3 Strategy and main difficulties

All three theories are orbifolds, and they are constructed by following the same steps:

1. *Choose a parent theory V.* It should have relatively few light states, for instance because it is the lattice theory of a lattice with a large minimal norm. It should also

c	what is determined
64	unitary; extremal, so the partition function is fixed; suitable torsion classes exist, but none is singled out
96	unitary; at least $349 \cdot 2^{26}$ of the 2^{36} torsion classes are suitable; the partition function is fixed up to the weight-four dimension d_4
128	unitary; every torsion class gives gap four; for the distinguished class $\epsilon_{\max}$ the partition function and the automorphism group $E_8(2)$ are determined

Table 2: The status of the three theories. Counts refer to discrete torsion classes; different classes may give isomorphic theories.

have a large automorphism group, so that a large subgroup can be used to project out the remaining light states. Sometimes it is convenient to describe the parent in terms of free fermions rather than free bosons.

2. *Choose an orbifold group* $G \subset \mathrm{Aut}(V)$. This involves two technical points. First, G must act by genuine automorphisms of V. For instance, a symmetry of a lattice has to be lifted to an automorphism of the lattice CFT, and different lifts can lead to different orbifolds. Second, the action of G must be free of anomalies, as we discuss below.

3. *Check that the light untwisted states are projected out*, so that the G-invariant subalgebra V^G has no light primaries.

4. *Check that the twisted sectors do not reintroduce light states.* Often it suffices to show that the ground state energies of the twisted sectors are large. Sometimes this fails for a few sectors, and one must show that a suitable choice of discrete torsion projects out their light states.

5. *Construct and analyze the theory*, ideally by computing its partition function and symmetry group.

Steps 1 and 3, and the first part of step 4, are relatively standard, although the choice of V and G requires care. The two main obstacles are the anomaly in step 2 and the choice of discrete torsion in step 4.

Anomalies. The anomaly of the G action is a class $\alpha \in H^3(G, U(1))$. It measures the failure of the twisted sectors to fit together consistently, and a chiral orbifold exists only if $\alpha = 0$. This requirement is special to chiral theories. If we regard V as the left-moving half of the non-chiral theory $V \otimes \overline{V}$, then G acting diagonally on left- and right-movers is always free of anomalies, and the corresponding orbifold always exists. More generally, for a group acting on both left- and right-movers the anomalies of the two sides can cancel,

which is exploited in [33]. A chiral orbifold gauges G acting on the left-movers alone, and no such cancellation is available. For a cyclic group the anomaly is determined by the ground state energy of the twisted sector, through the familiar level-matching condition. For a noncyclic group, however, the anomaly need not vanish even if its restriction to every cyclic subgroup does. This is precisely what made the KRR construction conditional: the cyclic tests leave one class in $H^3(G, U(1))$ undetermined.

Our main tool for computing anomalies is the use of *auxiliary symmetries*: symmetries of the parent theory that are not part of the orbifold group, but that nevertheless constrain α. They enter in two ways. If an auxiliary symmetry normalizes G, then α must be invariant under conjugation by it, which can shrink the space of possible anomalies enough that cyclic subgroups detect it. This is how we compute the anomaly at $c = 64$, using an auxiliary symmetry of order 127. Alternatively, a subgroup $K \subset G$ may also be contained in a different symmetry group K' of the parent; then the restriction of α to K equals the restriction of the anomaly of K', which may be easier to compute. At $c = 128$ the undetermined class is detected by a quaternion subgroup Q_8, which is also contained in a generalized quaternion group of order 16 whose anomaly is controlled by a cyclic subgroup of order eight. At $c = 96$ the situation is different. There the orbifold group is built from an elementary abelian 2-group, for which cyclic subgroups do detect the anomaly. The anomaly is the cubic invariant of the exceptional group $E_6(2)$, and we choose the orbifold group so that this cubic cancels between three copies of a $c = 32$ theory.

Discrete torsion. Once the anomaly vanishes, the orbifolds of V by G are labeled by discrete torsion $\epsilon \in H^2(G, U(1))$ [34], which determines which states survive the projection in each twisted sector. At $c = 64$ and $c = 96$, a small number of twisted sectors contain states below the target gap, and we need a *single* choice of ϵ that removes all of them at once; the projections cannot be adjusted sector by sector. We solve this problem by counting. For each light twisted state, the choices of ϵ for which it survives form an affine subspace whose dimension we can compute, and an averaging argument shows that some ϵ avoids all of these subspaces. At $c = 128$ the issue is the opposite one: we show that there is a unique choice of ϵ that keeps all of the lightest twisted states, and this choice gives the theory with the largest symmetry.

Unitarity. An orbifold of a unitary theory is not automatically unitary, since the positive form has to be compatible with the operator products of the twisted sectors. By a theorem of Carpi, Gaudio, Giorgetti and Hillier, it suffices to find a subalgebra that is *completely* unitary, in the sense that the canonical forms on all fusion products of its modules are positive [35]. At $c = 128$ and $c = 64$ the orbifold groups become abelian after dividing by their centers. The sectors twisted by central elements then form a subalgebra whose representations are all simple currents, and its complete unitarity follows from results of Gui. At $c = 96$ the orbifold group has trivial center, and we use the theory of conformal nets instead.

Symmetry. At $c = 128$ the lattice description hides most of the symmetry of the orbifold theory. We show that the same theory can be obtained as an orbifold of four copies of a $c = 32$ theory. This second description makes new automorphisms manifest, and together with an analysis of the centralizer of one involution it allows us to identify the full automorphism group as $E_8(2)$, a finite simple group of order about 3.4×10^{74}. The 545,925,250 weight-four primaries form its smallest nontrivial irreducible representation. The $c = 128$ theory thus combines a large primary gap with an exceptional finite symmetry group, as the Monster CFT does at $c = 24$.

Obstacles to larger gaps. Gap four appears twice in Table 1, and this is not a coincidence. In both cases the orbifold contains a weight-four primary that is invariant under every symmetry of a certain type. For orbifolds of lattice theories by groups that preserve the free bosons, this primary is a universal quartic expression in the bosons (Section 2.5); for orbifolds of tensor products by groups that permute the factors, it is built from the stress tensors of the individual factors (Section 5.7). To reach gap five one needs symmetries that do not preserve these structures. The $E_8(2)$ symmetry of the $c = 128$ theory is of this type, but we show that no orbifold of that theory by a subgroup of $E_8(2)$ has gap five (Section 7.3).

1.4 Outline

In Section 2 we review holomorphic orbifolds and explain the general tools that we use. Sections 3, 4 and 5 contain the three constructions, each organized along the steps listed above. We begin with the $c = 128$ theory, where the analysis of the untwisted and twisted sectors is simplest and the anomaly argument is cleanest. In Section 6 we return to this theory: we single out the torsion class that keeps all of the lightest twisted states, compute its partition function, and determine its automorphism group, using the $c = 32$ theory introduced in Section 5. In Section 7 we summarize what we have learned about the landscape of chiral CFTs with large gap, discuss the relation of our results to three-dimensional gravity, and describe the obstructions to larger gaps. Technical details are collected in the appendices. Several steps of our arguments rely on finite computations, mostly linear algebra over $\mathbb{F}_2$ and exact integer arithmetic. They are described in the text and in Appendix E, and the code is provided as ancillary files.

Note added

While this paper was in preparation, the preprint [36] appeared, which contains similar results about the theory at $c = 64$. There an extremal holomorphic VOA of central charge 64 is constructed as a simple current extension of a fixed-point subalgebra of $Z \otimes Z$, where Z is the $c = 32$ theory of Section 5.1. This subalgebra appears to coincide with the algebra D of Proposition 6.6.

2 Holomorphic orbifolds and the primary gap

In this section we collect the general facts that we use, roughly in the order of the five steps listed in the introduction. We state results in the language of vertex operator algebras (VOAs) where this is needed for precision, but most of the discussion can be read in physical terms.

2.1 Chiral CFTs, characters and the primary gap

By a chiral CFT we mean a holomorphic VOA $V = \bigoplus_{n\geq 0} V_n$: a strongly rational VOA whose only irreducible module is V itself. In particular $V_0 = \mathbb{C}\mathbf{1}$ is spanned by the vacuum. The central charge is a multiple of eight, $c = 8m$, and the character (the torus partition function)

$$Z_V(\tau) = \mathrm{Tr}_V\, q^{L_0 - c/24} = q^{-c/24}\sum_{n\geq 0} d_n q^n, \qquad d_n = \dim V_n, \qquad q = e^{2\pi i\tau}, \tag{2.1}$$

satisfies $Z_V(-1/\tau) = Z_V(\tau)$ and $Z_V(\tau+1) = e^{-2\pi i c/24} Z_V(\tau)$ [37]. Hence $\eta^c Z_V$ is a holomorphic modular form of weight $4m$, and can be expanded in the basis $E_4^{m-3i}\Delta^i$, $0 \leq i \leq \lfloor m/3\rfloor$, where $\Delta = \eta^{24}$. Dividing by η^c gives

$$Z_V = \sum_{i=0}^{\lfloor m/3\rfloor} a_i\, X^{m-3i}, \qquad X = \frac{E_4}{\eta^8} = q^{-1/3}\left(1 + 248q + 4124q^2 + 34752q^3 + \cdots\right). \tag{2.2}$$

Here $X = j^{1/3}$ is the character of the E_8 lattice theory. Since $X^{m-3i} = q^{-c/24+i}(1+O(q))$, the coefficients a_i, and hence the whole partition function, are determined by the first $\lfloor c/24\rfloor + 1$ coefficients $d_0, \ldots, d_{\lfloor c/24\rfloor}$. When c is divisible by 24 it is often more convenient to write Z_V as a polynomial in $J = j - 744 = q^{-1} + 196884q + \cdots$.

A holomorphic VOA has a nondegenerate invariant bilinear form, and therefore decomposes into a direct sum of irreducible highest-weight representations of the Virasoro algebra. For $c > 1$ the Verma module with positive highest weight h is irreducible, while the vacuum representation is the quotient of the Verma module with $h = 0$ by the submodule generated by $L_{-1}\mathbf{1}$. Let $\mathrm{Prim}_h(V)$ be the space of Virasoro primaries of weight $h > 0$ and $N_h = \dim \mathrm{Prim}_h(V)$. Then

$$q^{c/24} Z_V(\tau)\prod_{n\geq 1}(1-q^n) = 1 - q + \sum_{h\geq 1} N_h q^h. \tag{2.3}$$

The *primary gap* $\mu(V)$ is the smallest h with $N_h \neq 0$. If N_h vanishes for $h \leq \lfloor c/24\rfloor$, then $d_0, \ldots, d_{\lfloor c/24\rfloor}$ agree with those of the vacuum Virasoro module and (2.2) fixes Z_V completely. Using this one can show that [21, 22]

$$\mu(V) \leq \left\lfloor \frac{c}{24}\right\rfloor + 1. \tag{2.4}$$

A theory is *extremal* if equality holds [21]. When $c = 24k$ is divisible by 24, we follow [8] and call a theory *near-extremal* if $\mu(V) = k$. Recall from the introduction that in a holographic interpretation the black hole threshold lies at $h = c/24$. A chiral CFT therefore has no primaries below this threshold if and only if it is extremal, or c is divisible by 24 and it is near-extremal; in the latter case it has primaries exactly at the threshold.

For the three central charges of this paper, these observations give the following. At $c = 64$ an extremal theory has the unique partition function [5]

$$Z = X^8 - 1984X^5 + 705057X^2 = q^{-8/3}\left(1 + q^2 + 278512\,q^3 + \cdots\right). \tag{2.5}$$

At $c = 96$, a theory with $\mu \geq 4$ has $(d_0, d_1, d_2, d_3) = (1, 0, 1, 1)$, which fixes all coefficients but the constant term:

$$Z = J^4 - 787535\,J^2 - 85975039\,J + 74069025264 + d_4. \tag{2.6}$$

The weight-four dimension d_4 is not determined by modularity. An extremal theory would have $d_4 = 2$, the number of vacuum descendants at weight four, while a near-extremal theory has $d_4 > 2$. At $c = 128$, the condition $\mu \geq 4$ fixes four of the six coefficients in (2.2); the partition function is then determined by d_4 and d_5.

2.2 Orbifolds and twisted sectors

Let G be a finite group of automorphisms of V. The G-invariant states form a subalgebra V^G, which contains the stress tensor. To obtain a new holomorphic theory one must add twisted sectors. For every $g \in G$ there is a unique irreducible g-twisted V-module $V(g)$ [38], which we call the g-twisted sector. If g has order n, the conformal weights in $V(g)$ lie in $\rho_g + \frac{1}{n}\mathbb{Z}_{\geq 0}$, where the *ground state energy* ρ_g is the lowest conformal weight of $V(g)$. For $g \neq 1$ we have $\rho_g > 0$ in all our examples. The centralizer $C_G(g)$ acts on $V(g)$, in general only projectively.

All groups in this paper are solvable. For solvable G the subalgebra V^G is again strongly rational [39]. Its irreducible modules are the isotypic components of the twisted sectors under the centralizers, and its representation category is that of the twisted Drinfeld double of G [40–43]:

$$\mathrm{Rep}(V^G) \simeq \mathcal{Z}\big(\,\mathrm{Vec}_G^{\alpha}\,\big). \tag{2.7}$$

Here $\alpha \in H^3(G, U(1))$ is the anomaly of the G action, discussed in Section 2.3. If $\alpha = 0$, then for every choice of discrete torsion ϵ (Section 2.4) there is a holomorphic orbifold theory

$$(V/G)_\epsilon = \bigoplus_{[g]} V(g)^{(\epsilon)}, \tag{2.8}$$

where the sum runs over the conjugacy classes of G, and $V(g)^{(\epsilon)} \subset V(g)$ is the subspace on which $C_G(g)$ acts by a character determined by ϵ. Its partition function is

$$Z_{(V/G)_\epsilon}(\tau) = \frac{1}{|G|}\sum_{\substack{g,h\in G\\ gh=hg}} \epsilon(g,h)\,Z_{g,h}(\tau), \qquad Z_{g,h}(\tau) = \mathrm{Tr}_{V(g)}\;\phi_g(h)\,q^{L_0 - c/24}, \tag{2.9}$$

where the ϕ_g are a coherent choice of linear actions of $C_G(g)$ on $V(g)$, which exists because $\alpha = 0$. That $(V/G)_\epsilon$ is a holomorphic VOA, with local and associative operator products, follows from the theory of VOA extensions: it corresponds to a commutative algebra object of maximal dimension (a Lagrangian algebra) in (2.7), and such algebra objects give holomorphic extensions of V^G [41, 44, 45].

Three simple observations reduce the computation of the gap of $(V/G)_\epsilon$ to finite problems:

1. The orbifold contains V^G with the same stress tensor, so every nonvacuum primary of V^G is a primary of $(V/G)_\epsilon$. In particular $\mu((V/G)_\epsilon) \le \mu(V^G)$.

2. A state of weight n in $(V/G)_\epsilon$ lies either in V^G or in a twisted sector $V(g)$ with $\rho_g \le n$. Since all weights of $(V/G)_\epsilon$ are integers, a twisted sector with $\rho_g > n-1$ cannot contribute states of weight less than n.

3. Ground state energies can be computed explicitly for the twists that we need.

For lattice theories the last point works as follows. Let $V = V_L$ and let $\hat{g}$ be a lift to $\mathrm{Aut}(V_L)$ of an isometry g of L of order n. If m_j is the multiplicity of the eigenvalue $e^{2\pi ij/n}$ of g on $L \otimes \mathbb{C}$, the twisted oscillators have vacuum energy

$$\rho_g^{\mathrm{osc}} = \sum_{j=1}^{n-1} \frac{j(n-j)}{4n^2}\, m_j. \tag{2.10}$$

The ground state energy $\rho_{\hat{g}}$ is ρ_g^{osc} plus the minimum of $|\lambda|^2/2$ over a coset of the projection of L onto the fixed space of g. The coset depends on the lift. For a *standard* lift, which acts trivially on the states e^β with β in the fixed lattice L^g, the coset contains zero and $\rho_{\hat{g}} = \rho_g^{\mathrm{osc}}$ [46–48]. For permutation orbifolds, if $V = U^{\otimes n}$ and $g = (h^{-1}, 1, \ldots, 1)\sigma$ with σ a cyclic permutation of the n factors, then g^n acts as h^{-1} on each factor and [49, 50]

$$\rho_g = \frac{c_U}{24}\left(n - \frac{1}{n}\right) + \frac{\rho_{h^{-1}}}{n}, \tag{2.11}$$

where c_U is the central charge of U. The first term is the ground state energy of U on an n-sheeted cover.

2.3 The anomaly

It is useful to think of the g-twisted sector as the space of states on a circle crossed by a symmetry defect line labeled by g. Two defect lines labeled g and h can be joined into a line labeled gh, and the junction is unique up to a phase. Reassociating three lines gives a phase $\omega(g,h,k)$. Consistency requires ω to be a 3-cocycle, and rephasing the junctions changes it by a coboundary. The class

$$\alpha = [\omega] \in H^3(G, U(1)) \cong H^4(G, \mathbb{Z}) \tag{2.12}$$

is the anomaly of the G action [40, 51, 52]. Mathematically, it is the associator of the G-graded category formed by the twisted sectors [42], and it is the class appearing in (2.7). The twisted sectors can be combined into a consistent orbifold theory (2.8) if and only if $\alpha = 0$.

For cyclic groups the anomaly can be read off from the twisted sector. Let g have order n as an automorphism of V. Then $H^3(\langle g\rangle, U(1)) \cong \mathbb{Z}_n$, and the restriction of α to $\langle g\rangle$ is given by the *type* [48]

$$t_g = n^2\rho_g \mod n. \tag{2.13}$$

Thus $\alpha|_{\langle g\rangle} = 0$ if and only if $\rho_g \in \frac{1}{n}\mathbb{Z}$, which is the familiar level-matching condition. Two details will matter below: n is the order of g acting on V, which can be smaller than the order of the transformation from which g is constructed, and ρ_g must be computed for the actual lift of g.

For noncyclic groups the anomaly is not determined by these data in general, because the restriction map

$$H^3(G, U(1)) \longrightarrow \bigoplus_{C\le G \text{ cyclic}} H^3(C, U(1)) \tag{2.14}$$

need not be injective. It is injective for elementary abelian 2-groups $G \cong \mathbb{F}_2^r$ [53]. In this case the anomaly is equivalent to the function $t : G \to \mathbb{F}_2$, $x \mapsto t_x$, which is a polynomial of degree at most three in the coordinates of x with no constant term; every such polynomial arises. The function t also controls the projective action in the twisted sectors: in $V(x)$, two elements $y, z \in G$ commute up to the sign $(-1)^{H_x(y,z)}$, where

$$H_x(y, z) = \sum_{\varepsilon\in\mathbb{F}_2^3} t(\varepsilon_1 x + \varepsilon_2 y + \varepsilon_3 z) \tag{2.15}$$

is the trilinear form obtained by polarizing t. This follows by computing the projective 2-cocycle on the centralizer from a trilinear representative of α.

In general we use the following three principles. The first two express what we mean by the use of *auxiliary symmetries*: symmetries of V that are not part of G, but constrain its anomaly.

(A1) *Invariance.* If a symmetry $N \subset \mathrm{Aut}(V)$ normalizes G, then α is invariant under the action of N on $H^3(G, U(1))$ by conjugation, since conjugation by an automorphism of V maps the G action to an equivalent one.

(A2) *Naturality.* If $K \subset G$ is also contained in another finite group $G' \subset \mathrm{Aut}(V)$, then $\alpha_G|_K = \alpha_{G'}|_K$: both are the anomaly of the action of K.

(A3) *Sylow subgroups.* α vanishes if and only if its restriction to a Sylow p-subgroup vanishes for every prime p dividing $|G|$, because restriction followed by transfer is multiplication by the index.

2.4 Discrete torsion and light twisted states

Suppose that $\alpha = 0$. The coherent choices of the actions ϕ_g in (2.9) then form a torsor over $H^2(G, U(1))$ [34]: fixing one reference choice, any other is obtained from a 2-cocycle β, and multiplies $\phi_g(h)$ by the discrete torsion phase

$$\epsilon(g,h) = \frac{\beta(g,h)}{\beta(h,g)}, \qquad gh = hg. \tag{2.16}$$

For fixed g this is a character of $C_G(g)$. In (2.8), the subspace $V(g)^{(\epsilon)}$ consists of the states on which every $h \in C_G(g)$ acts, in the reference normalization, by $\epsilon(g,h)^{-1}$. Changing ϵ therefore changes which states survive in every twisted sector simultaneously. It is important that the projections in different twisted sectors are not independent: if one chooses them sector by sector, the result is in general not a consistent theory.

At $c = 64$ and $c = 96$ a few twisted sectors contain states below the desired gap. We will reduce the problem of removing them to the following form. In each such sector the light states form a space T on which a subgroup $S \subseteq C_G(g)$ acts through characters. The light states survive the projection if and only if $\epsilon(g,\cdot)$ restricted to S equals one of these characters (up to the inversion above). For the groups we consider, the torsion classes form a vector space over $\mathbb{F}_2$ and $\epsilon(g,\cdot)|_S$ depends linearly on ϵ, so for each character that occurs, the choices of torsion that select it form an affine subspace, or the empty set. If several characters occur, the bad choices form a union of such subspaces. In our applications there is a single condition for each light sector: either the light space is one-dimensional ($c = 64$), or the characters that occur are all extensions of a single character of the subgroup of S that acts by scalars, and the light states survive exactly when this character is selected ($c = 96$). The characters by which S acts depend on the reference choice, which is not canonical and which we never compute. Our arguments are therefore designed to show that good choices of ϵ exist for every possible value of these reference characters.

These conditions can be handled by counting. Suppose that in the i-th light sector the bad choices of torsion form an affine subspace of codimension r_i, or the empty set. A uniformly random torsion class lies in it with probability at most 2^{-r_i}, and therefore

$$\mathbb{P}(\text{some light state survives}) \leq \sum_i 2^{-r_i}, \tag{2.17}$$

without any assumption of independence between different sectors. If the right-hand side is smaller than one, a suitable torsion class exists, and in fact at least a fraction $1 - \sum_i 2^{-r_i}$ of all torsion classes is suitable; this is how we argue at $c = 96$. At $c = 64$ this bound is borderline, and we refine it in two ways. We first fix part of the torsion so that one family of light sectors is removed, and we then use the fact that the remaining bad events overlap: if the expected number of surviving light states is at most one, but some choices of torsion retain two of them, then other choices must retain none.

2.5 A universal quartic primary

There is a simple obstruction to obtaining a gap larger than four from a single orbifold of a lattice theory. Let V_L be the lattice theory of a lattice of rank d, and let a^i_n, $1 \le i \le d$, be the oscillators of the free bosons, normalized so that $[a^i_m, a^j_n] = m\delta^{ij}\delta_{m+n,0}$. Define the weight-four states

$$A = \sum_i a^i_{-3}a^i_{-1}\mathbf{1}, \qquad B = \sum_i (a^i_{-2})^2\mathbf{1}, \qquad C = \Big(\sum_i (a^i_{-1})^2\Big)^2\mathbf{1}, \tag{2.18}$$

and

$$R_d = 4A - 3B - \frac{6}{d+2}\,C. \tag{2.19}$$

Using $[L_m, a^i_n] = -n\,a^i_{m+n}$, one finds

$$\begin{aligned} &L_1 : (A,B,C) \mapsto (3T, 4T, 0), && L_2 : (A,B,C) \mapsto \big(3S, 0, 2(d+2)S\big), \\ &L_3 : (A,B,C) \mapsto (0,0,0), && L_4 : (A,B,C) \mapsto (3d, 4d, 0)\,\mathbf{1}, \end{aligned} \tag{2.20}$$

where $T = \sum_i a^i_{-2}a^i_{-1}\mathbf{1}$ and $S = \sum_i (a^i_{-1})^2\mathbf{1}$. Hence R_d is a Virasoro primary. The three oscillator patterns $3+1$, $2+2$ and $1+1+1+1$ are mutually orthogonal, and

$$\|R_d\|^2 = 120\,d + \frac{288\,d}{d+2} > 0, \tag{2.21}$$

so $R_d \neq 0$. Since R_d is invariant under the full orthogonal group acting on the oscillators, and momentum phases do not act on the zero-momentum Fock space, we obtain:

Proposition 2.1. *Let G be a finite group of automorphisms of V_L that preserves the Heisenberg subalgebra generated by the $a^i_{-1}\mathbf{1}$. Then $R_d \in V_L^G$, and every orbifold $(V_L/G)_\epsilon$ has a primary of weight four. In particular $\mu((V_L/G)_\epsilon) \le 4$.*

Note that a primary built only from the modes a^i_{-1} corresponds to a harmonic polynomial, and the orthogonal group fixes no harmonic polynomial of positive degree. So R_d must involve higher modes: it is primary only because the terms A and B cancel the contribution of C. For $d = 1$, R_1 is proportional to the familiar weight-four primary of the $\mathbb{Z}_2$ orbifold of a single free boson. An analogous obstruction for orbifolds of tensor products is described in Section 5.7.

3 Warm-up: the Barnes–Wall orbifold at $c = 128$

We begin with the theory proposed by Keller, Roberts and Roberts [32]. Its untwisted and twisted sectors are easy to analyze, and they show that the orbifold, if it exists, has gap four. The difficulty is the anomaly. Level matching holds for every cyclic subgroup, but this leaves one class in H^3 undetermined. In Section 3.5 we show that this class vanishes as well, using an auxiliary symmetry of order eight. This establishes the existence of the

theory. In Section 3.6 we show that the resulting theories are unitary, by an argument that we also use at $c = 64$. We return to the theory in Section 6, where we single out the discrete torsion that keeps all of the lightest twisted states, compute the partition function, and determine the automorphism group.

Theorem 3.1. *Let $P = 2_+^{1+14}$ act on the lattice theory $V = V_{\mathrm{BW}_{128}}$ by the lifts constructed in [32]. The anomaly of this action vanishes, so every choice of discrete torsion ϵ gives a holomorphic orbifold $(V/P)_\epsilon$ of central charge 128. Each of these orbifolds is unitary, has no states of weight one, its states of weight two and three are the stress tensor and its derivative, and its primary gap is exactly four.*

3.1 Parent theory: the Barnes–Wall lattice BW_{128}

Label an orthonormal basis of $\mathbb{R}^{128}$ by e_x, $x \in \mathbb{F}_2^7$, and identify a subset $I \subseteq \{0, \dots, 6\}$ with its indicator vector in $\mathbb{F}_2^7$. Following [32, App. A], the Barnes–Wall lattice BW_{128} is spanned by the vectors

$$w_I = 2^{\lceil (7-|I|)/2 \rceil - 2} \sum_{x \subseteq I} e_x, \qquad I \subseteq \{0, \dots, 6\}. \tag{3.1}$$

The Gram matrix $(w_I, w_J) = 2^{k_I + k_J + |I \cap J|}$, with $k_I = \lceil (7 - |I|)/2 \rceil - 2$, is integral, its diagonal entries are 8 or 16, and its determinant is one. So BW_{128} is even and unimodular. Its minimal norm is 8, and it has 1,260,230,400 vectors of norm 8 [54, 55]. The lattice theory $V = V_{\mathrm{BW}_{128}}$ has central charge 128. It has 128 currents $a^i_{-1}\mathbf{1}$, but no momentum states of weight less than four.

3.2 Orbifold group: the extraspecial group 2_+^{1+14}

For $0 \le a \le 6$ let

$$X_a\, e_x = e_{x+e_a}, \qquad Z_a\, e_x = (-1)^{x_a} e_x. \tag{3.2}$$

These signed permutations preserve BW_{128} [32]. Together with $z = -1$ they generate the extraspecial group $P = 2_+^{1+14}$ of order 2^{15} (denoted $E(7)$ in [32]), which is the real Pauli group on seven qubits. The quotient $A = P/\langle z \rangle \cong \mathbb{F}_2^{14}$, with coordinates $(u, v) \in \mathbb{F}_2^7 \oplus \mathbb{F}_2^7$ recording the X and Z parts, carries the quadratic form $q(u, v) = u \cdot v$ and its polar form b. An element of P that projects to $a \in A$ squares to $z^{q(a)}$, and two elements commute up to $z^{b(a,a')}$. Apart from 1 and z, the group P has 8255 conjugacy classes $\{g, zg\}$ of noncentral involutions, labeled by the nonzero a with $q(a) = 0$, and 8128 classes of elements of order four, labeled by $q(a) = 1$. The group P is normalized by a much larger group of lattice automorphisms: the full automorphism group of BW_{128} is the real Clifford group, which contains P as a normal subgroup with quotient an orthogonal group of dimension 14 over $\mathbb{F}_2$ [56, 57].

A lattice isometry acts naturally on the oscillators, but its action on the momentum states e^β requires a choice of phases, because the vertex operators of a lattice theory involve a cocycle. Two lifts of the same isometry can differ by a character of the lattice,

and they can have different twisted sectors. KRR constructed a lift of all of P to $\mathrm{Aut}(V)$ with respect to an explicit cocycle [32, Thm. 3.5], and we use this lift throughout. Every element of P then acts by a standard lift in the sense of Section 2.2. For elements without fixed vectors this is automatic; for the remaining involutions it is a finite check, described in Appendix B.2.

3.3 Untwisted sector

Since BW_{128} has minimal norm 8, all states of weight at most three are oscillator states. The central element z changes the sign of every oscillator, so only states with an even number of oscillators are invariant. At weight two these are the states $\sum_{ij} c_{ij} a^i_{-1} a^j_{-1}\mathbf{1}$, and at weight three the states $\sum_{ij} c_{ij} a^i_{-2} a^j_{-1}\mathbf{1}$. The representation of P on $\mathbb{R}^{128}$ is irreducible, being the tensor product of seven copies of the two-dimensional real Pauli representation, so the only invariant bilinear form is the metric. Hence

$$V_1^P = 0, \qquad V_2^P = \mathbb{C}\omega, \qquad V_3^P = \mathbb{C}L_{-1}\omega. \tag{3.3}$$

At weight four, V^P contains the universal primary R_{128} of Proposition 2.1, since P preserves the Heisenberg subalgebra. Every orbifold of V by P therefore has $\mu \le 4$.

3.4 Twisted sectors

Every nonidentity Pauli matrix is traceless. Hence a noncentral involution has eigenvalues $+1$ and -1 on $\mathbb{R}^{128}$, each with multiplicity 64, and an element of order four has eigenvalues $\pm i$, each with multiplicity 64. For standard lifts, (2.10) gives the ground state energies in Table 3.

element	number	order	eigenvalues	ρ_g	type $n^2\rho_g \bmod n$
z	1	2	$(-1)^{128}$	8	0
noncentral involution	16510	2	$1^{64}, (-1)^{64}$	4	0
order four	16256	4	$i^{64}, (-i)^{64}$	6	0

Table 3: Twisted sectors of the $c = 128$ orbifold.

All nontrivial twisted sectors have $\rho_g \ge 4$. By the observations of Section 2.2, every orbifold $(V/P)_\epsilon$ that exists agrees with V^P below weight four, and it contains R_{128}. Its primary gap is therefore exactly four, for every choice of discrete torsion. This was the conditional result of [32]; it remains to show that the anomaly vanishes.

3.5 The anomaly

Theorem 3.2. *The anomaly $\alpha \in H^3(P, U(1))$ of the action of P on V vanishes.*

The last column of Table 3 shows that α restricts trivially to every cyclic subgroup. We first show that this leaves exactly one possibility besides $\alpha = 0$, and then use an auxiliary symmetry to exclude it.

What cyclic subgroups miss. Let $\pi : P \to A$ be the projection, and let x_i, y_i, $0 \le i \le 6$, be the coordinates on A dual to X_i, Z_i, so that $q = \sum_i x_i y_i$. The integral cohomology of P in degree four is [32, 58]

$$H^4(P, \mathbb{Z}) \cong (\mathbb{Z}/2)^{454} \oplus \mathbb{Z}/4. \tag{3.4}$$

Let $\beta : H^1(A, \mathbb{F}_2) \to H^2(A, \mathbb{Z})$ be the Bockstein homomorphism, and define

$$\Theta = \sum_{i=0}^{6} \beta(x_i)\,\beta(y_i) \in H^4(A, \mathbb{Z}), \qquad \eta = \pi^*\Theta \in H^4(P, \mathbb{Z}). \tag{3.5}$$

The class η is nonzero; in fact it is twice a generator of the $\mathbb{Z}/4$ summand in (3.4) (Appendix B.1). Nevertheless it restricts to zero on every cyclic subgroup. On a subgroup generated by an involution over $a \in A$, it restricts to $q(a)u^2 = 0$, where u generates $H^2(\mathbb{Z}_2, \mathbb{Z})$. On a cyclic subgroup of order four, every class $\beta(x_i)$ or $\beta(y_i)$ restricts to an even multiple of the generator t of $H^2(\mathbb{Z}_4, \mathbb{Z})$, so every product restricts to a multiple of $4t^2 = 0$.

Lemma 3.3. *The kernel of the restriction map* $H^4(P, \mathbb{Z}) \to \bigoplus_{C \le P \text{ cyclic}} H^4(C, \mathbb{Z})$ *is* $\{0, \eta\}$.

The proof, given in Appendix B.1, reduces to an elementary statement about Boolean polynomials that vanish on the quadric $q = 0$. Together with Table 3 it shows that $\alpha \in \{0, \eta\}$. The class η is also invisible to a second natural test. In each twisted sector, the obstruction to linearizing the projective action of the centralizer is the class in $H^2(C_P(g), U(1))$ obtained from α. For the representative $\omega_\eta(g, h, k) = (-1)^{\sum_i x_i(g) y_i(h) y_i(k)}$ of η, this class is represented by $c_g(h, k) = (-1)^{\sum_i x_i(g) y_i(h) y_i(k)}$, which is the coboundary of $b_g(h) = i^{\sum_i x_i(g) y_i(h)}$. So every twisted sector admits a linear centralizer action even if $\alpha = \eta$, and no test that looks at one twisted sector at a time can detect η.

A quaternion subgroup detects η. Let

$$I = X_0 Z_0, \qquad S = Z_0 X_1 Z_1. \tag{3.6}$$

Both square to z, and $IS = zSI$, so $K = \langle I, S \rangle$ is a quaternion group Q_8. In coordinates s, t on $\pi(K) \cong \mathbb{F}_2^2$ dual to I, S we have $x_0 = s$, $y_0 = s + t$ and $x_1 = y_1 = t$, so that q restricts to the anisotropic form $s^2 + st + t^2$. Writing $U = \beta(s)$ and $W = \beta(t)$, the integral cohomology ring of Q_8 satisfies $U^2 = W^2 = 0$ and $UW = 4u$, where u generates $H^4(Q_8, \mathbb{Z}) \cong \mathbb{Z}/8$ [59]. Therefore

$$\eta|_K = U^2 + UW + W^2 = 4u \neq 0. \tag{3.7}$$

It thus suffices to show that $\alpha|_K = 0$. Since K is not cyclic, level matching cannot show this directly. Instead we embed K in a different group of automorphisms of V, which is not contained in P.

An auxiliary symmetry of order eight. Define

$$R = \tfrac{1}{2}(1+I)(X_2+Z_2). \tag{3.8}$$

Since $I^2=-1$, $(X_2+Z_2)^2=2$, and I commutes with X_2+Z_2, we find $R^2=I$, $R^4=-1$ and $R^8=1$. One also checks that $RSR=S$, *i.e.* $S^{-1}RS=R^{-1}$. Thus $\langle R,S\rangle$ is the generalized quaternion group Q_{16}, and it contains $K=\langle R^2,S\rangle$ as a subgroup of index two. The matrix R is orthogonal, and although it contains a factor $\frac{1}{2}$, it preserves BW_{128}, because $(1+I)(X_2+Z_2)$ is divisible by two on the lattice. More importantly, R preserves the specific cocycle used to define the KRR lifts. It therefore has a lift $\hat{R}\in\mathrm{Aut}(V)$ that acts on momentum states without additional phases, and the same construction applied to X_0,Z_0,X_1,Z_1 reproduces the KRR lifts (Appendix B.2). Consequently

$$\hat{R}^2=\hat{I}, \qquad \langle\hat{R},\hat{S}\rangle\cong Q_{16}, \qquad \langle\hat{R}^2,\hat{S}\rangle=\langle\hat{I},\hat{S}\rangle=K \tag{3.9}$$

as groups of automorphisms of V. The group Q_{16} is not contained in P, and we do not need a common finite group containing both.

The characteristic polynomial of R is $(t^4+1)^{32}$: each primitive eighth root of unity occurs with multiplicity 32. In particular R has no fixed vectors, its lift is standard, and (2.10) gives

$$\rho_{\hat{R}}=\frac{1}{4\cdot 8^2}\sum_{k=1,3,5,7}32\,k(8-k)=\frac{11}{2}, \qquad 8^2\rho_{\hat{R}}=352\equiv 0 \bmod 8. \tag{3.10}$$

So the anomaly α_{16} of the Q_{16} action restricts trivially to $C_8=\langle\hat{R}\rangle$.

Proof of Theorem 3.2. The group $H^4(Q_{16},\mathbb{Z})\cong\mathbb{Z}/16$ is generated by the second Chern class c_{16} of the two-dimensional representation $Q_{16}\subset SU(2)$. Its restrictions generate $H^4(Q_8,\mathbb{Z})\cong\mathbb{Z}/8$ and $H^4(C_8,\mathbb{Z})\cong\mathbb{Z}/8$; this follows from the Gysin sequence, since each of these groups acts freely on $S^3\subset\mathbb{C}^2$. Hence the restriction maps to C_8 and to Q_8 have the same kernel $\{0,8c_{16}\}$. Since $\alpha_{16}|_{C_8}=0$, we have $\alpha_{16}\in\{0,8c_{16}\}$, and therefore $\alpha_{16}|_K=0$. By naturality (A2) and (3.9), $\alpha|_K=\alpha_{16}|_K=0$. Comparing with (3.7) excludes $\alpha=\eta$, so $\alpha=0$. □

Note that we did not need to show that $\alpha_{16}=0$, and indeed the argument does not decide this.

Since the anomaly vanishes, the orbifolds of V by P are labeled by $H^2(P,U(1))$. The untwisted sector (3.3) and the ground state energies of the twisted sectors show that all of them have the properties stated in Theorem 3.1; the weight-four primary R_{128} of Proposition 2.1 shows that the gap is exactly four. The choice of torsion only affects the spectrum from weight four on. We return to it in Section 6.

3.6 Unitarity

The parent V is unitary, and the orbifolds $(V/P)_\epsilon$ are unitary as well. This does not follow directly from unitarity of V. Since the argument uses tools that are rather different from

the rest of the paper, we only outline it here; the details are in Appendix D, where we formulate it for general orbifolds and also apply it at $c = 64$.

By a theorem of Carpi, Gaudio, Giorgetti and Hillier, a simple CFT-type extension of a regular VOA $\mathcal{C}$ is unitary if $\mathcal{C}$ is completely unitary [35]. Complete unitarity requires positivity not only of the irreducible modules of $\mathcal{C}$ but also of the canonical forms on their fusion products. For $\mathcal{C} = V^P$ the first condition holds. Every irreducible V^P-module occurs in a twisted sector of V [60], and since P acts by standard lifts of lattice isometries, these twisted sectors carry positive forms compatible with their twisted vertex operators [61]. The second condition is known for orbifolds by cyclic groups [62], but not for the nonabelian group P. We avoid it by passing to an intermediate algebra.

1. Every discrete torsion class is inflated from $A = P/Z(P)$. As a result, the sector twisted by the central element z contributes its P-invariant states to $(V/P)_\epsilon$, and together with V^P they form a subalgebra B, which is a $\mathbb{Z}_2$ simple current extension of V^P.

2. The representation category of B is the untwisted Drinfeld double of the abelian group A, so all irreducible B-modules are simple currents.

3. Using positivity results of Gui for fusion products involving simple currents [62], and adapting his arguments [63, 64], one shows that every irreducible B-module is unitarizable. Since all of them are simple currents, B is completely unitary.

The theorem of [35], applied to $B \subset (V/P)_\epsilon$, then gives a positive form on $(V/P)_\epsilon$ that is compatible with its operator products.

4 An extremal CFT at $c = 64$

Our second example is extremal, so all of its nonvacuum primaries lie above the black hole threshold $c/24 = 8/3$. Its parent is the lattice theory of D_{64}^+, which we describe in terms of 128 free fermions. The orbifold group is built from a binary code, the Gold code, which acts on the fermions by sign changes, together with the translations of the fermion labels. Two new issues arise compared to $c = 128$. First, the symmetries are naturally defined on the fermions, and their lifts to automorphisms of the bosonic theory must be chosen with care. Second, some twisted sectors now contain states of weight two, and they have to be removed by a single choice of discrete torsion.

Theorem 4.1. *There is a unitary holomorphic VOA W of central charge 64, obtained as an orbifold of $V_{D_{64}^+}$ by a group of order 2^{21}, with*

$$W_1 = 0, \qquad W_2 = \mathbb{C}\omega, \qquad \mu(W) = 3. \tag{4.1}$$

Its partition function is the extremal function (2.5). It has 278,511 *primaries of weight three, of which* 2667 *lie in the untwisted sector.*

4.1 Parent theory: 128 free fermions

Let

$$D_{64} = \Big\{(n_1,\dots,n_{64}) \in \mathbb{Z}^{64} : \sum_i n_i \in 2\mathbb{Z}\Big\}, \qquad D_{64}^+ = D_{64} \cup (s + D_{64}), \qquad s = \big(\tfrac{1}{2},\dots,\tfrac{1}{2}\big)\,. \tag{4.2}$$

Since $|s|^2 = 16$, the lattice D_{64}^+ is even and unimodular. Its lattice theory $V = V_{D_{64}^+}$ has $c = 64$ and many light states: the 64 oscillators and the 8064 momentum states of the roots $\pm e_i \pm e_j$ give 8128 currents, which generate the affine algebra $\mathfrak{so}(128)$ at level one.

For our purposes it is better to describe V in terms of 128 real fermions ψ_x of weight $\frac{1}{2}$. Pairing them into complex fermions and bosonizing gives [65]

$$V = \mathcal{F}_{\mathrm{NS}}^{\mathrm{even}} \oplus \mathcal{F}_{\mathrm{R}}^{+}. \tag{4.3}$$

The first summand is the part of the Neveu–Schwarz Fock space with even fermion number. The second is one chirality of the Ramond sector; its ground states have weight $128/16 = 8$ and correspond to the momenta of minimal norm 16 in $s + D_{64}$. The individual fermions are not states of V, but their bilinears $\psi_x\psi_y$ are the currents. All states of weight less than eight lie in $\mathcal{F}_{\mathrm{NS}}^{\mathrm{even}}$, so the low-lying spectrum can be analyzed with free fermions. The Ramond summand nevertheless matters, because it determines how symmetries of the fermions act on V.

We label the fermions by the elements of the finite field

$$F = \mathbb{F}_{128} = \mathbb{F}_2[\theta]/(\theta^7 + \theta + 1), \tag{4.4}$$

whose additive group we identify with $\mathbb{F}_2^7$ by writing an element in the basis $1, \theta, \dots, \theta^6$. When convenient we write elements of F as integers between 0 and 127, whose binary digits are these coefficients. Let $\mathrm{tr} : F \to \mathbb{F}_2$, $\mathrm{tr}(z) = \sum_{i=0}^{6} z^{2^i}$, be the field trace. The pairing $(y, z) \mapsto \mathrm{tr}(yz)$ is nondegenerate, so every $\mathbb{F}_2$-linear function on F has the form $x \mapsto \mathrm{tr}(\ell x)$.

Note that V contains 128 commuting Ising models, generated by the stress tensors of the individual fermions. Their differences are weight-two primaries, which are invariant under any group of sign changes. To remove them, the orbifold group must also permute the fermions.

4.2 Orbifold group: the Gold code

The choice of group is guided by the untwisted sector. A sign change $\psi_x \mapsto (-1)^{f(x)}\psi_x$ multiplies a monomial in fermion modes by $(-1)^{\sum_{x\in S} f(x)}$, where S is the set of labels that occur an odd number of times. So the monomial is invariant under the sign changes by all words f of a binary code if and only if S lies in the dual code. The currents $\psi_x\psi_y$ have $|S| = 2$, and the weight-two states built from two or four distinct fermions have $|S| = 2$ or 4. To remove all of these, we want a code whose dual has no words of weight two or four. The only invariant states of weight two are then the stress tensors of the 128 individual

fermions, and a group of permutations that acts transitively on the labels leaves only their sum, the total stress tensor. The Gold code, together with the translations of the field with 128 elements, has these properties, as we show in Section 4.3.

A binary code of length 128 is a subspace of the space of functions $f : F \to \mathbb{F}_2$. The Gold code [66] is

$$\mathcal{C} = \left\{ f_{a,\ell,e}(x) = \mathrm{tr}(ax^3 + \ell x) + e : \ a, \ell \in F, \ e \in \mathbb{F}_2 \right\}. \tag{4.5}$$

It has dimension 15 and weight enumerator

$$1 + 8128\, y^{56} + 16510\, y^{64} + 8128\, y^{72} + y^{128}. \tag{4.6}$$

To see this, note that for $a \neq 0$ the function $x \mapsto \mathrm{tr}(ax^3)$ is a quadratic form over $\mathbb{F}_2$, since squaring is $\mathbb{F}_2$-linear. Its polar form $\mathrm{tr}(a(x^2y + xy^2))$ has radical $\{0, r\}$ with $ar^3 = 1$, so it has rank six, and the sums $\sum_x (-1)^{\mathrm{tr}(ax^3+\ell x)}$ take the values 0 and ± 16. This gives the three weights 56, 64 and 72; the words with $a = 0$ have weights 0, 64 and 128.

Codewords act by sign changes, $D_f \psi_x = (-1)^{f(x)} \psi_x$, and elements of F act by translations, $T_v \psi_x = \psi_{x+v}$. Translations preserve the code, since

$$f_{a,\ell,e}(x + v) + f_{a,\ell,e}(x) = \mathrm{tr}\left(k(a; v)\, x\right) + \mathrm{tr}(av^3 + \ell v), \qquad k(a; v) = av^2 + (av)^{64}. \tag{4.7}$$

The sign change by the all-ones word acts trivially on V, as we explain below, and we define

$$H = (\mathcal{C}/\langle \mathbf{1} \rangle) \rtimes F, \qquad |H| = 2^{21}. \tag{4.8}$$

Writing $g = (a, \ell, v) \in F^3$ for the element $D_{f_{a,\ell,0}} T_v$, the group law reads

$$(a, \ell, v)\,(d, m, u) = \big(a + d,\ \ell + m + k(d; v),\ v + u\big). \tag{4.9}$$

The center and the commutator subgroup coincide, $Z(H) = [H, H] = \{(0, \ell, 0)\} \cong \mathbb{F}_2^7$, and $H/Z(H) \cong F \oplus F$ with coordinates (a, v).

Lifting to automorphisms of V. The formulas above define H as a group of orthogonal transformations of the fermions. Such transformations act on $\mathcal{F}_{\mathrm{NS}}^{\mathrm{even}}$ directly, but to act on the Ramond summand they must be lifted to the spin group. Let γ_x be the generators of the Clifford algebra, ordered by their integer labels. The volume element $\Omega = \gamma_0 \gamma_1 \cdots \gamma_{127}$ acts trivially on $\mathcal{F}_{\mathrm{R}}^{+}$, while the central element $-1 \in \mathrm{Spin}(128)$ acts on it by -1. The group acting faithfully on V is therefore $\mathrm{Spin}(128)/\langle \Omega \rangle$. In particular the all-ones sign change, which is $-1 \in O(128)$, has the lift Ω that acts trivially on V. The preimage of H in $\mathrm{Spin}(128)/\langle \Omega \rangle$ is a central extension of H by $\{\pm 1\}$, and we have to show that it splits and choose a splitting.

For this we use an auxiliary symmetry. Multiplication of the labels by the primitive element θ, $\psi_x \mapsto \psi_{\theta x}$, is a permutation of order 127 (a Singer cycle), and it has a unique lift to $\mathrm{Aut}(V)$ of order 127. It normalizes H:

$$(a, \ell, v) \longmapsto (\theta^{-3} a,\ \theta^{-1} \ell,\ \theta v). \tag{4.10}$$

The class of the extension in $H^2(H, \mathbb{F}_2)$ is therefore invariant under this action. In Appendix A.1 we show that the invariant part of $H^2(H, \mathbb{F}_2)$ is one-dimensional, and that it is detected on the subgroup $E \cong \mathbb{Z}_2^2$ generated by T_1 and the sign change by the linear codeword $f_0(x) = (\text{coefficient of } \theta^0 \text{ in } x)$. On E the extension splits: the lifts

$$\hat{D} = \prod_{j=0}^{63} \gamma_{2j+1}, \qquad \hat{T}_1 = \prod_{j=0}^{63} \frac{\gamma_{2j} - \gamma_{2j+1}}{\sqrt{2}} \tag{4.11}$$

both square to one and commute modulo Ω, since the product of the even-labeled and the odd-labeled blades is $+\Omega$. Hence the extension splits. The splittings form an affine space over $\mathrm{Hom}(H, \mathbb{F}_2)$, on which the Singer cycle acts without fixed vectors. Averaging over the 127 conjugates of any splitting therefore gives a unique splitting that is invariant under the Singer cycle. This is the lift of H that we use. As we will see, the choice matters: for the lightest twisted sectors, the other lift has a different ground state energy and a nonvanishing anomaly.

4.3 Untwisted sector

A monomial in fermion modes acting on the vacuum transforms under D_f by $(-1)^{\sum_{x\in S} f(x)}$, where $S \subseteq F$ is the set of labels that occur an odd number of times. It is invariant under all sign changes if and only if S lies in the dual code $\mathcal{C}^\perp$, *i.e.*

$$|S| \equiv 0 \bmod 2, \qquad \sum_{x\in S} x = 0, \qquad \sum_{x\in S} x^3 = 0. \tag{4.12}$$

No two-element set satisfies these conditions, and neither does any four-element set: writing it as $\{a, b, c, a+b+c\}$, the cubic condition would require

$$a^3 + b^3 + c^3 + (a+b+c)^3 = (a+b)(a+c)(b+c) = 0, \tag{4.13}$$

which is impossible for distinct elements. So the nonempty sets in $\mathcal{C}^\perp$ have at least six elements.

Since every fermion mode has weight at least $\frac{1}{2}$, a state of weight at most three involves at most six modes. At weight one, all states $\psi^x_{-1/2}\psi^y_{-1/2}\mathbf{1}$ with $x \neq y$ have $|S| = 2$, so all 8128 currents are projected out. At weight two, the only states with $S \in \mathcal{C}^\perp$ are the states $\psi^x_{-3/2}\psi^x_{-1/2}\mathbf{1}$ with $S = \emptyset$, which are permuted transitively by the translations; their sum is the stress tensor. At weight three, the states with $S = \emptyset$ give $L_{-3}\mathbf{1}$, and the remaining invariants come from the six-fermion states $\psi^{x_1}_{-1/2}\cdots\psi^{x_6}_{-1/2}\mathbf{1}$ with $\{x_1, \ldots, x_6\} \in \mathcal{C}^\perp$. These states are Virasoro primaries, because all their labels are distinct. By the MacWilliams identity applied to (4.6), $\mathcal{C}^\perp$ contains 341,376 sets of size six. Translations act on them freely, because a translation by $v \neq 0$ that fixes a six-element set divides it into three pairs $\{x, x+v\}$, whose total sum is $v \neq 0$. Each orbit of 128 sets gives one invariant state. Hence

$$V_1^H = 0, \qquad V_2^H = \mathbb{C}\omega, \qquad V_3^H = \mathbb{C}L_{-3}\mathbf{1} \oplus \mathrm{Prim}_3(V^H), \tag{4.14}$$

where $\mathrm{Prim}_3(V^H)$ has dimension $341376/128 = 2667$. An example is the set $S = \{0, 1, 2, 6, 104, 109\}$, whose cubes are $0, 1, 8, 120, 115, 2$; summing the translates of the corresponding six-fermion state gives an explicit invariant primary. In particular every orbifold of V by H has gap at most three.

4.4 Twisted sectors

Every element $g \in H$ lies in a maximal torus of $\mathrm{Spin}(128)/\langle\Omega\rangle$, so it is conjugate to $\exp(2\pi i\, h_0)$ for a vector h in the Cartan subalgebra $\mathbb{R}^{64}$. The g-twisted sector is then the lattice module $V_{D_{64}^+ + h}$, with ground state energy

$$\rho_g = \min_{p \in D_{64}^+} \frac{|p+h|^2}{2}. \tag{4.15}$$

Both cosets of D_{64} in D_{64}^+ contribute to the minimum. This is where the choice of lift matters: changing the lift of g by the central element -1 shifts h by a vector in the nontrivial coset of D_{64} in $\mathbb{Z}^{64}$.

For a signed translation $D_f T_v$ with $v \neq 0$, the relevant datum is the derivative $\delta_v f(x) = f(x+v) + f(x)$, which by (4.7) is either zero, one, or a nonconstant (and hence balanced) affine function. On each pair $\{x, x+v\}$, the transformation $D_f T_v$ squares to $(-1)^{\delta_v f(x)}$. This determines the eigenvalues, and (4.15) gives the ground state energies in Table 4. For example, a sign change by a word of weight w has $\rho_g = \min(w, 128-w)/16$. For a nonconstant derivative one can take $h = (0^{16}, (\frac{1}{2})^{16}, (\frac{1}{4})^{32})$, so that $\rho_g = 3$.

element of H	order on V	ρ_g
sign change, weight 56 or 72	2	7/2
sign change, weight 64	2	4
signed translation, $\delta_v f = 0$	2	4
signed translation, $\delta_v f$ nonconstant	4	3
signed translation, $\delta_v f = 1$	2	2

Table 4: Twisted sectors of the $c = 64$ orbifold. All types $n^2\rho_g \bmod n$ vanish.

The last row deserves comment. A signed translation with $\delta_v f = 1$ has order four as a transformation of the fermions, but its square is the all-ones sign change, which acts trivially on V. So it acts on V as an involution. For the lift chosen above, it is conjugate to the shift $h = (\frac{1}{4})^{64}$, and the ground states are the two states $e^{\pm h}$ of weight 2; its type $2^2 \cdot 2 \bmod 2$ vanishes. The other lift has $\rho_g = \frac{9}{4}$ and a nonvanishing type (Appendix A.2).

Only the elements in the last row can produce states of weight less than three. They form 254 conjugacy classes. For $v \neq 0$ put $a_v = v^{-3}$. Then (4.7) is constant if and only if $a \in \{0, a_v\}$, which gives two classes for each v, with representatives

$$g_0(v) = (0, v^{-1}, v), \qquad g_1(v) = (a_v, 0, v). \tag{4.16}$$

In both twisted sectors the weight-two space is spanned by e^h and e^{-h}, and a central element of H exchanges these two states, so at most one linear combination survives the projection onto states invariant under the center. If such a line exists, the image of the centralizer $C_H(g_i(v))$ in $H/Z(H) = F \oplus F$, namely

$$S_0(v) = \{(e\, a_v, u) : e \in \mathbb{F}_2,\ u \in F\}, \qquad S_1(v) = \{(a_v(t^2 + t + e), vt) : e \in \mathbb{F}_2,\ t \in F\}, \tag{4.17}$$

acts on it by a character, which we write additively as a linear map $\lambda_i(v) : S_i(v) \to \mathbb{F}_2$. The images of the representatives themselves, $w_0(v) = (0, v)$ and $w_1(v) = (a_v, v)$, satisfy $\lambda_i(v)(w_i(v)) = 0$, because the state has integral weight. The characters $\lambda_i(v)$ depend on the reference normalization of the twisted sectors, and we will not need to compute them.

4.5 The anomaly

Theorem 4.2. *The anomaly of the action of H on V vanishes.*

Proof. By Table 4, the anomaly restricts trivially to every cyclic subgroup, but for the nonabelian group H this does not suffice by itself. We use the Singer cycle (4.10) as an auxiliary symmetry. By (A1), the anomaly lies in the invariant subgroup $H^4(H, \mathbb{Z})^{C_{127}}$. In Appendix A.1 we compute this subgroup using the Lyndon–Hochschild–Serre spectral sequence of the central extension $Z(H) \to H \to F \oplus F$, and find

$$H^4(H, \mathbb{Z})^{C_{127}} \cong \mathbb{Z}/2, \qquad \text{detected by restriction to } \langle g_* \rangle, \tag{4.18}$$

where $g_* = (1, 0, 1) = g_1(1)$ is one of the light involutions above. Its lift has $\rho_{g_*} = 2$ and hence type zero. So the anomaly vanishes. □

The Singer cycle is not part of the orbifold group, and it need not survive as a symmetry of the orbifold theory; it is only used to constrain the anomaly.

4.6 Discrete torsion

It remains to remove the 254 weight-two lines of the light twisted sectors with a single choice of discrete torsion. We use torsion classes inflated from the abelian quotient $H/Z(H) = F \oplus F$. Let Λ be an alternating $\mathbb{F}_2$-bilinear form on F, which has 21 parameters, and let M be an arbitrary $\mathbb{F}_2$-bilinear form on F, which has 49 parameters. Choosing a bilinear form Λ_+ with $\Lambda_+ + \Lambda_+^T = \Lambda$, the function $\xi(g, h) = (-1)^{\Lambda_+(v,u) + M(a,u)}$, for $g = (a, \ell, v)$ and $h = (d, m, u)$, is a 2-cocycle on H. Its commuting-pair phase is $\epsilon(g, h) = (-1)^{\beta((a,v),(d,u))}$ with

$$\beta\big((a, v), (d, u)\big) = \Lambda(v, u) + M(a, u) + M(d, v). \tag{4.19}$$

By Section 2.4, the weight-two line in the sector of $g_i(v)$ survives if and only if

$$\beta\big(w_i(v), \cdot\big)\big|_{S_i(v)} = \lambda_i(v). \tag{4.20}$$

If some of the 254 lines do not exist, we may add artificial ones with arbitrary characters satisfying $\lambda_i(v)(w_i(v)) = 0$; this only makes the following problem harder.

Proposition 4.3. *For every choice of the characters $\lambda_i(v)$, there exist Λ and M such that (4.20) fails for all $i \in \{0,1\}$ and all $v \in F^\times$.*

Proof. We choose Λ and M at random and show that the probability that all 254 lines are removed is positive.

The first family. For $i = 0$ the condition (4.20) becomes

$$\Lambda(v,\cdot) = f_v, \qquad M(a_v, v) = s_v, \tag{4.21}$$

where the linear form f_v on F, with $f_v(v) = 0$, and $s_v \in \mathbb{F}_2$ are determined by $\lambda_0(v)$. For fixed $v \neq 0$ the map $\Lambda \mapsto \Lambda(v,\cdot)$ is onto the six-dimensional space of linear forms that vanish at v, so the first condition holds with probability 2^{-6}. The expected number of v for which it holds is $127/64 < 2$. Hence there is a Λ for which it holds for at most one v, say $v = r$. We fix such a Λ. If r exists we restrict to M satisfying the affine condition

$$\ell_r(M) := M(a_r, r) = 1 + s_r, \tag{4.22}$$

which removes the line for $v = r$. All lines of the first family are now removed, and M ranges over an affine space of dimension 48 (or 49 if r does not exist).

The second family. With Λ fixed, the condition (4.20) for $i = 1$ becomes a system of seven linear equations for M,

$$L_v(M)(t) := M(a_v, vt) + M\big(a_v(t^2+t), v\big) = q_v(t), \qquad t \in F, \tag{4.23}$$

where q_v depends on Λ and $\lambda_1(v)$. (The component of $S_1(v)$ labeled by e gives the equation for $t = 1$ again, by the condition $\lambda_1(v)(w_1(v)) = 0$.) After rescaling the two arguments of M by a_v and v, the equation for t corresponds to the tensor $1 \otimes t + (t^2+t) \otimes 1$. For $t \notin \{0,1\}$ this tensor has rank two, because $t^2 + t$ is neither 0 nor 1 (its trace vanishes, while $\mathrm{tr}(1) = 1$). For $t = 1$ it is $1 \otimes 1$. Hence L_v has rank seven, and the only functional of rank one in its row space is $M \mapsto M(a_v, v)$. It follows that $\mathrm{rank}(L_v, \ell_r) = 8$ for $v \neq r$.

Let Y be the number of v for which (4.23) holds. If r does not exist, then $\mathbb{E}Y = 127/128 < 1$, so $Y = 0$ for some M. If r exists, the 126 values $v \neq r$ each contribute 2^{-7} to $\mathbb{E}Y$, and $v = r$ contributes 0 or 2^{-6}, so $\mathbb{E}Y \leq 1$. This bound alone does not exclude that $Y = 1$ for every M, and we need one more input. For any random variable with values in $\mathbb{Z}_{\geq 0}$ we have $\mathbb{P}(Y = 0) = 1 - \mathbb{E}Y + \mathbb{E}\big[(Y-1)\mathbf{1}_{Y\geq 2}\big]$. Since $\mathbb{E}Y \leq 1$ and $(Y-1)\mathbf{1}_{Y\geq 2} \geq \mathbf{1}_{Y\geq 2}$, this gives

$$\mathbb{P}(Y = 0) \geq \mathbb{P}(Y \geq 2). \tag{4.24}$$

In words: if the average number of surviving lines is at most one, but some choices of M retain two of them, then other choices must retain none. In Appendix A.3 we show that

$$\mathrm{rank}\left(\ell_r, L_{\theta r}, L_{(\theta+1)r}\right) = 15. \tag{4.25}$$

Hence, conditional on (4.22), the systems (4.23) for $v = \theta r$ and $v = (\theta+1)r$ hold simultaneously with probability 2^{-14}, whatever their right-hand sides. Therefore $\mathbb{P}(Y = 0) \geq \mathbb{P}(Y \geq 2) \geq 2^{-14} > 0$. □

We emphasize that this argument does not assume that the events for different v are independent. It shows that a good choice of torsion exists, uniformly in the unknown characters $\lambda_i(v)$, but it does not single out a particular choice.

4.7 The orbifold theory

Let ϵ be a discrete torsion class as in Proposition 4.3, and let $W = (V/H)_\epsilon$. By Theorem 4.2, and since H is a 2-group and hence solvable, W is a holomorphic VOA. By (4.14) and Table 4, and since the light twisted states are removed,

$$W_1 = 0, \qquad W_2 = \mathbb{C}\omega. \tag{4.26}$$

The 2667 untwisted primaries of weight three show that $\mu(W) = 3$. So W is extremal, and its partition function is (2.5):

$$Z_W = X^8 - 1984X^5 + 705057X^2 = q^{-8/3}\left(1 + q^2 + 278512\,q^3 + \cdots\right). \tag{4.27}$$

Subtracting the vacuum descendant $L_{-3}\mathbf{1}$, there are 278,511 primaries of weight three. Since 2667 of them lie in the untwisted sector, the twisted sectors contribute 275,844. Modular invariance fixes this total; we have not computed the contributions of individual twisted sectors. Because W is extremal, Höhn's analogue of Venkov's theorem for extremal lattices applies to it: its spaces of fixed conformal weight form conformal designs [67]. W is the first example beyond $c = 40$ to which this result applies.

Unitarity. The theory W is unitary, by the argument outlined in Section 3.6; see Proposition D.1. The inputs are as follows. Every element of H is conjugate into a maximal torus of the group $\mathrm{Spin}(128)/\langle\Omega\rangle$ acting on V, so its twisted sector is a shifted lattice module, which is unitary [61]. The torsion cocycle ξ above does not depend on the central coordinate ℓ, so the sectors twisted by central elements of H form a subalgebra $B \subset W$. It is a simple current extension of V^H by $Z(H) \cong \mathbb{F}_2^7$, and its representation category is the untwisted Drinfeld double of the abelian group $H/Z(H)$. As at $c = 128$, this implies that B is completely unitary, and the theorem of [35], applied to $B \subset W$, gives a positive form on W that is compatible with its original operator products.

Finally, we note what the construction does not determine. It shows that suitable torsion classes exist without singling one out, and we have not determined the automorphism group of W. It contains the quantum symmetry group $\mathrm{Hom}(H, U(1)) \cong \mathbb{F}_2^{14}$; the Singer cycle, which was crucial for the anomaly computation, need not survive as a symmetry of W.

5 A near-extremal CFT at $c = 96$

Our third example has gap four at $c = 96$, one unit below the extremal value. Since $c/24 = 4$, its lightest primaries lie exactly at the black hole threshold, and it is near-extremal in the sense of Section 2.1. The parent theory is three copies of a $c = 32$ theory Z, the $\mathbb{Z}_2$ orbifold of the lattice theory of the Barnes–Wall lattice BW_{32}, whose automorphism group contains the exceptional group $E_6(2)$. The new feature is the anomaly. The natural binary group of symmetries of Z is anomalous, and its anomaly is the cubic invariant of

$E_6(2)$. We construct a subgroup of three copies of this group on which the three cubics cancel. The orbifold group also contains an element of order nine that permutes the three copies. It removes the relative stress tensors, and the internal twist that it carries cancels the anomaly of the cyclic permutation of the three copies.

Theorem 5.1. *There is a group $G \cong \mathbb{F}_2^{24} \rtimes \mathbb{Z}_9$ of automorphisms of $Z^{\otimes 3}$ with vanishing anomaly. Its 2^{36} discrete torsion classes include at least $349 \cdot 2^{26}$, about 34%, for which the orbifold $W = (Z^{\otimes 3}/G)_\epsilon$ is a unitary holomorphic VOA of central charge 96 with*

$$W_1 = 0, \qquad W_2 = \mathbb{C}\omega, \qquad W_3 = \mathbb{C}L_{-1}\omega, \qquad \mu(W) = 4. \tag{5.1}$$

In particular W is near-extremal. Its partition function is (2.6) with $d_4 \geq 3$.

5.1 Parent theory: three copies of the $c = 32$ Barnes–Wall orbifold

Let $N = \mathrm{BW}_{32}$ be the Barnes–Wall lattice of rank 32. It is even and unimodular with minimal norm four [54, 55], so its lattice theory V_N has 32 currents but no momentum states of weight one. Let θ be the standard lift of -1, which acts as -1 on the currents. Its twisted sector has ground state energy $32/16 = 2$, and the corresponding orbifold

$$Z = V_N^+ \oplus V_N^{T,+} \tag{5.2}$$

is a holomorphic theory with $Z_1 = 0$. Here V_N^+ is the θ-invariant subalgebra and $V_N^{T,+}$ is the integral-weight part of the θ-twisted sector. Its partition function is

$$Z_Z = X^4 - 992\,X = q^{-4/3}\left(1 + 139504\,q^2 + 69332992\,q^3 + \cdots\right), \tag{5.3}$$

so Z is an extremal theory at $c = 32$ [21]. Shimakura showed that [68]

$$\mathrm{Aut}(Z) \cong 2^{27}.E_6(2), \tag{5.4}$$

where the normal subgroup $A \cong \mathbb{F}_2^{27}$ is elementary abelian and $E_6(2) = \mathrm{Aut}(Z)/A$ acts on it through one of its two 27-dimensional irreducible representations. Since $\mathrm{Aut}(Z)$ is finite, it preserves the positive invariant form of Z [69].

Involutions in A. The nonzero elements of A form three orbits under $E_6(2)$ [70], and they have different twisted sectors, as shown in Table 5. One representative of the first orbit is the quantum $\mathbb{Z}_2$ symmetry of (5.2), which acts as -1 on $V_N^{T,+}$. Its twisted sector contains V_N^-, and in particular the 32 states $a^i_{-1}\mathbf{1}$ of weight one. Representatives of the other two orbits are lifts of a Pauli element of order four in $\mathrm{Aut}(N)$. Their twining functions on Z are

$$\left(\frac{\eta(2\tau)}{\eta(4\tau)}\right)^{16} \pm 256\left(\frac{\eta(\tau)\eta(4\tau)}{\eta(2\tau)^2}\right)^{16}, \tag{5.5}$$

and their modular transformations give the ground state energies $\frac{3}{2}$ and $\frac{7}{4}$ and the ground state degeneracies in Table 5. The size of the first orbit follows from the structure of the centralizer of the quantum symmetry [68], and the sizes of the other two then follow from the traces on Z_2 listed in the table, since the average of these traces over A equals $\dim Z_2^A = 1$.

rank	orbit size	ρ_a	ground state degeneracy	$\mathrm{Tr}_{Z_2}\, a$
1	139,503	1	32	8432
2	69,193,488	3/2	512	240
3	64,884,736	7/4	8192	-272

Table 5: The three orbits of nonzero elements of $A \cong \mathbb{F}_2^{27}$, labeled by their rank in the Albert algebra.

The anomaly of A. For an involution the type is $4\rho_a \bmod 2$, which takes the values $0, 0, 1$ on the three orbits. Since A is an elementary abelian 2-group, its anomaly is completely determined by this function (Section 2.3). The function has a natural interpretation. The 27-dimensional representation of $E_6(2)$ can be identified with the split Albert algebra over $\mathbb{F}_2$, the exceptional Jordan algebra of 3×3 Hermitian matrices over the octonions, and $E_6(2)$ is the group of linear maps preserving its cubic norm N [70,71]. The three orbits consist of the elements of rank one, two and three, and N vanishes on the first two and equals one on the third. Hence:

The anomaly of A is given by the cubic norm of the Albert algebra.

By (2.15), the commutator form in the a-twisted sector is the polarization $H_a(b,c) = \sum_{\varepsilon \in \mathbb{F}_2^3} N(\varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c)$. Its rank is 10, 18 and 26 on the three orbits, in agreement with the ground state degeneracies 2^5, 2^9 and 2^{13}.

We will use the following explicit model of the Albert algebra [71]. Let $\mathbb{F}_4 = \{0, 1, \zeta, \zeta^2\}$ with $\zeta^2 + \zeta + 1 = 0$ and conjugation $\bar{x} = x^2$, and write $B^\dagger = \bar{B}^T$. The Hermitian 3×3 matrices over $\mathbb{F}_4$ form a nine-dimensional space $\mathrm{Herm}_3(\mathbb{F}_4)$ over $\mathbb{F}_2$, and

$$A \cong \mathrm{Herm}_3(\mathbb{F}_4) \oplus M_3(\mathbb{F}_4), \qquad N(a,B) = \det a + \mathrm{Tr}_{\mathbb{F}_4/\mathbb{F}_2}(\zeta \det B) + \mathrm{tr}(aBB^\dagger). \tag{5.6}$$

A nonzero element (a, B) has rank one if and only if

$$a^\# = BB^\dagger, \qquad \zeta B^\# = B^\dagger a, \tag{5.7}$$

where $M^\#$ denotes the adjugate matrix. These are the equations for the vanishing of the gradient of N, computed before N is reduced to a function on $\mathbb{F}_2^{27}$. It has rank three if $N(a,B) = 1$, and rank two otherwise.

5.2 Orbifold group

The design of the orbifold group is dictated by three requirements. First, since $Z_1 = 0$, every state of $Z^{\otimes 3}$ of weight at most three lies in a single factor, so the projection must remove the weight-two and weight-three primaries of each factor separately. Second, the sectors twisted by binary elements should have ground state energy at least three, which by Table 5 holds if every such element acts nontrivially on all three factors. Third, the anomalies $N(x_1) + N(x_2) + N(x_3)$ of the three factors must cancel. We meet the second

and third requirements by taking $L = \{(x, Tx, T^2x) : x \in K\} \subset A^3$ for a subspace $K \subset A$ and a linear map T of K. Every nonzero element of L is then nontrivial in all three factors, and the anomaly vanishes on L if $N(x) + N(Tx) + N(T^2x) = 0$ for all $x \in K$. The first requirement only involves K. A state of Z that transforms under A by a character $\chi \in A^*$ is invariant under K if and only if χ lies in the annihilator $K^\perp$, and we will see in Section 5.3 that it suffices that all nonzero elements of $K^\perp$ have rank three. Finally, the relative stress tensors $\omega_i - \omega_j$ must be removed by an element that permutes the factors and normalizes L. We now construct K and T, with $\dim K = 24$ and T of order nine.

The subgroup L. Let $\mathbb{F}_{64} = \mathbb{F}_2[\alpha]/(\alpha^6 + \alpha + 1)$, and let $\nu = \alpha^7$, which has order nine, so that $\zeta = \nu^3 \in \mathbb{F}_4$. We regard $\mathbb{F}_{64}$ as a three-dimensional vector space over $\mathbb{F}_4$ with the Hermitian form $\langle x, y\rangle = \mathrm{Tr}_{\mathbb{F}_{64}/\mathbb{F}_4}(xy^8)$. The elements 1, $\alpha + \alpha^2$ and $1 + \alpha + \alpha^3$ form an orthonormal basis, and in this basis multiplication by ν is given by the unitary matrix

$$V = \begin{pmatrix} 0 & 0 & 1+\zeta \\ 1 & 0 & 0 \\ 0 & 1+\zeta & 0 \end{pmatrix}, \qquad V^3 = \zeta I. \tag{5.8}$$

For $b \in \mathbb{F}_{64}$, the $\mathbb{F}_4$-linear map $A_b(z) = bz^4 + b^2z^{16}$ is self-adjoint. The maps A_b form a six-dimensional subspace $A_6 \subset \mathrm{Herm}_3(\mathbb{F}_4)$, and $\det A_b = \mathrm{Tr}_{\mathbb{F}_4/\mathbb{F}_2}(b^{21})$. Let Σ be the set of the 21 nonzero b with $\det A_b = 0$. Finally, let J_6 be an $\mathbb{F}_2$-linear map of $\mathbb{F}_{64}$ with $J_6^2 + J_6 + 1 = 0$, such that Σ, $J_6\Sigma$ and $J_6^2\Sigma$ partition $\mathbb{F}_{64}^\times$; an explicit choice is given in Appendix E.1. We define

$$K = A_6 \oplus M_3(\mathbb{F}_4) \subset A, \qquad T(A_b, B) = (A_{J_6 b}, BV), \qquad L = \{(x, Tx, T^2x) : x \in K\} \subset A^3. \tag{5.9}$$

Both K and L have dimension 24 over $\mathbb{F}_2$. The map T has order nine, and $T^3 = h$ with $h(a, B) = (a, \zeta B)$. Each of the three projections $L \to A$ is injective with image K, so a nonzero element of L is nontrivial in all three copies.

Proposition 5.2. *$N(x) + N(Tx) + N(T^2x) = 0$ for all $x \in K$. Hence the anomaly of A^3 restricts trivially to L.*

Proof. The three terms in (5.6) cancel separately. Let $x = (A_b, B)$. For $b \neq 0$, exactly one of b, $J_6 b$, $J_6^2 b$ lies in Σ, so the three determinants $\det A_b$, $\det A_{J_6 b}$, $\det A_{J_6^2 b}$ sum to zero. The second terms involve $\det(BV^k) = \zeta^k \det B$, and $1 + \zeta + \zeta^2 = 0$. For the third terms, $(BV^k)(BV^k)^\dagger = BB^\dagger$ since V is unitary, while $A_b + A_{J_6 b} + A_{J_6^2 b} = 0$ since $J_6^2 + J_6 + 1 = 0$. The anomaly of the product action of A^3 is $N(x_1) + N(x_2) + N(x_3)$, and since L is an elementary abelian 2-group its anomaly is determined by the restriction of this function. □

As a check, we have also evaluated the cubic on all 2325 vectors of K with at most three nonzero coordinates, which determine a polynomial of degree three in 24 variables.

The element of order nine. To remove the relative stress tensors, the orbifold group must permute the three factors. The cyclic permutation σ itself is not suitable. By (2.11) its twisted sector has ground state energy $\frac{32}{24}\big(3-\frac{1}{3}\big)=\frac{32}{9}$, which is above three, so it would not introduce light states; but its type is $3^2\cdot\frac{32}{9}=32\equiv 2 \bmod 3$, so σ is anomalous. We therefore combine σ with an automorphism $\hat{h}$ of order three of one factor. The twisted sector of $(\hat{h}^{-1},1,1)\sigma$ then has ground state energy $\frac{32}{9}+\frac{1}{3}\rho_{\hat{h}}$, which equals four if $\rho_{\hat{h}}=\frac{4}{3}$, and this element of order nine then has type $81\cdot 4\equiv 0 \bmod 9$. So the internal twist cures the anomaly of the permutation, and as a side effect it raises the ground state energy of its twisted sector to four. We take $\hat{h}$ to be a lift of the map $h=T^3$ of (5.9), which is an element of the quotient $E_6(2)$ of (5.4), and we construct such a lift, together with its twisted sector, from the lattice. Label an orthonormal basis of $\mathbb{R}^{32}$ by e_x, $x\in\mathbb{F}_2^5$, and for $a,b\in\mathbb{F}_2^5$ let $P(a,b)e_x=(-1)^{b\cdot x}e_{x+a}$. Choose three pairs (P_i,Q_i) of such elements with $P_i^2=Q_i^2=-1$ and $P_iQ_i=-Q_iP_i$, such that different pairs commute; an explicit choice is given in Appendix E.1. The isometry

$$t=\prod_{i=1}^{3}\tfrac{1}{2}\left(-1+P_i+Q_i+P_iQ_i\right) \tag{5.10}$$

preserves BW_{32} and has order three, with characteristic polynomial $(X-1)^8(X^2+X+1)^{12}$; its fixed lattice is isometric to $\sqrt{3}E_8$. The standard lift of t commutes with θ, so it induces an automorphism of Z of order three, which we also call t. By (2.10), $\rho_t^{\mathrm{osc}}=24\cdot\frac{(1/3)(2/3)}{4}=\frac{4}{3}$, and the same value holds for $-t$; so the twisted sector of t on Z has ground state energy at least $\frac{4}{3}$, with equality for the standard lift.

The image of t in $E_6(2)$ is conjugate to h. The group $E_6(2)$ has three classes of elements of order three, which fix 651, 120 and 21 rank-one elements of the Albert algebra, respectively [72], and h fixes 21 of them (those with $B=0$ and $a=uu^\dagger$). On the other hand, the weight-two states of Z other than the stress tensor transform under A by the 139503 characters of rank one, each with multiplicity one, and t permutes these lines. It acts trivially on each line that it fixes. Indeed, $\mathrm{Aut}(Z)$ preserves the positive invariant form of Z, and hence commutes with the antilinear involution that defines the compact real form of Z. Each character line therefore has a real structure preserved by t, and a real cube root of unity is one. So the number of rank-one characters fixed by t is $\mathrm{Tr}_{Z_2}\,t-1$. A direct computation gives $\mathrm{Tr}_{Z_2}\,t=6+16=22$, where 6 comes from the oscillators and 16 from the ground states of the inversion-twisted sector. So t fixes 21 rank-one characters, and its image is conjugate to h. By the Schur–Zassenhaus theorem, all lifts of order three of a given element of $E_6(2)$ are conjugate under A, so we may assume that t lifts h, and we write $\hat{h}=t$ for this lift.

Now define on $Z^{\otimes 3}$

$$g=(\hat{h}^{-1},1,1)\,\sigma,\qquad \sigma(u\otimes v\otimes w)=w\otimes u\otimes v. \tag{5.11}$$

Then g normalizes L, acting on the coordinate x in (5.9) as T^{-1}, and $g^3=(\hat{h}^{-1},\hat{h}^{-1},\hat{h}^{-1})$, so g has order nine. The orbifold group is

$$G=L\rtimes\langle g\rangle\cong\mathbb{F}_2^{24}\rtimes\mathbb{Z}_9,\qquad |G|=9\cdot 2^{24}. \tag{5.12}$$

5.3 Untwisted sector

To determine which states of Z survive the projection onto K-invariants, we need to know how the light states of Z transform under A, not just how many there are. The states of Z can be chosen to transform under A by characters $\chi \in A^*$, and A^* is the dual of the Albert module, with the corresponding notion of rank. At weight two, Z_2 consists of the stress tensor, with trivial character, and one state for each of the 139503 characters of rank one. At weight three, Z_3 contains each character of rank zero, one and two exactly once, and no character of rank three. To see this, let $d = \dim Z_3 = 69332992$. The weight-three traces of the three types of involutions are 126976, -4096 and 4096, and averaging their first and second powers over A gives

$$\frac{d + \sum_i n_i t_i}{2^{27}} = 1, \qquad \frac{d^2 + \sum_i n_i t_i^2}{2^{27}} = d, \tag{5.13}$$

where n_i and t_i are the orbit sizes and traces. If m_χ are the multiplicities of the characters, these equations say that the trivial character occurs once and that $\sum_\chi m_\chi^2 = \sum_\chi m_\chi$, so all m_χ are zero or one. Since L_{-1} maps Z_2 injectively into Z_3, all characters of rank at most one occur. The remaining 69,193,488 characters form a union of orbits, so they must be exactly the characters of rank two.

A state with character χ is invariant under K if and only if χ lies in the annihilator $K^\perp \subset A^*$, which is three-dimensional. In suitable coordinates it consists of the Hermitian operators of multiplication by the elements of $\mathbb{F}_8 \subset \mathbb{F}_{64}$, so its seven nonzero elements are invertible and have rank three. None of them occur in Z_2 or Z_3, and therefore Z^K has dimensions $(1, 0, 1, 1)$ at weights $0, 1, 2, 3$.

Since $Z_1 = 0$, every state of $Z^{\otimes 3}$ of weight less than four lies in a single factor. Each projection of L onto a factor has image K, so the only L-invariant states of weight at most three are the stress tensors ω_i of the three factors and their derivatives. The element g permutes them cyclically. Hence the fixed-point subalgebra $\mathcal{C} = (Z^{\otimes 3})^G$ satisfies

$$\mathcal{C}_0 = \mathbb{C}\mathbf{1}, \qquad \mathcal{C}_1 = 0, \qquad \mathcal{C}_2 = \mathbb{C}\omega, \qquad \mathcal{C}_3 = \mathbb{C}L_{-1}\omega, \qquad \omega = \omega_1 + \omega_2 + \omega_3. \tag{5.14}$$

5.4 Twisted sectors

Sectors twisted by elements of L. For a nonzero $y = (x, Tx, T^2x) \in L$, the y-twisted sector of $Z^{\otimes 3}$ is the tensor product of the twisted sectors of Z for the three components, all of which are nonzero. By Table 5, $\rho_y \geq 3$, with equality if and only if all three components have rank one. Sectors with $\rho_y > 3$, such as $\rho_y = \frac{7}{2}$, cannot contribute states of weight at most three, since the weights of the orbifold are integers.

Lemma 5.3. *Exactly 567 nonzero elements of L have three components of rank one. They form 63 orbits of size nine under conjugation by g.*

Proof. Let $x = (A_b, B)$, so that the k-th component of (x, Tx, T^2x) is $(A_{J_6^k b}, BV^k)$. Multiplying the second equation in (5.7) by B and using the first gives $\zeta \det B = \det a$ for every element (a, B) of rank one. If $b \neq 0$, two of the three Hermitian components are

invertible, and the k-th component would have to satisfy $\zeta^{k+1} \det B = 1$ for two different values of k modulo three, which is impossible. Hence $b = 0$, and (5.7) reduces to $B^{\#} = 0$ and $BB^\dagger = 0$. The nonzero solutions are $B = uv^\dagger$ with $u, v \neq 0$ and $v^\dagger v = 0$. There are 63 choices for u and 27 isotropic vectors v (those with exactly two nonzero entries), and each B arises from three pairs (u, v), giving $63 \cdot 27/3 = 567$ elements. Right multiplication by V permutes them freely in orbits of length nine. □

Permutation sectors. An element of Lg^k with $3 \nmid k$ is conjugate under L to g^k, since $T^k - 1$ is invertible on K. Its cube acts on each factor by $\hat{h}^{\mp 1}$, whose twisted sector has ground state energy at least $\frac{4}{3}$, so (2.11) gives

$$\rho \geq \frac{32}{24}\left(3 - \frac{1}{3}\right) + \frac{1}{3} \cdot \frac{4}{3} = \frac{32}{9} + \frac{4}{9} = 4. \tag{5.15}$$

Without the internal twist, the ground state energy would be $\frac{32}{9}$. This would still be above three, but, as explained in Section 5.2, the permutation would then be anomalous.

The cosets Lg^3 and Lg^6 require a separate argument, because $T^3 - 1 = h - 1$ has an 18-dimensional image with kernel A_6. Up to conjugation, these cosets contain 64 elements each, labeled by $b \in \mathbb{F}_{64}$, which act on the three factors by $a_k \hat{h}^{\mp 1}$ with $a_k = A_{J_6^k b}$. For $b = 0$ each factor contributes at least $\frac{4}{3}$. For $b \neq 0$ we need the ground state energy of the product $a_k \hat{h}^{\mp 1}$, which does not follow from those of a_k and $\hat{h}$. Let $z_0 \in A$ be the quantum involution of Z, whose fixed-point algebra is V_N^+. Both a_k and $\hat{h} = t$ commute with z_0, so they preserve V_N^+. The group A acts on V_N^+ through the Pauli group 2^{1+10} of BW_{32} [68], so a_k acts on V_N^+ either trivially or as a noncentral Pauli element r; since a_k is fixed by h, r commutes with t. Two automorphisms of Z that agree on V_N^+ differ at most by z_0, so $a_k \hat{h}^{\mp 1}$ is induced by a lift of the lattice isometry $t^{\mp 1}$ or $rt^{\mp 1}$, and as for t itself, its twisted sector on Z has ground state energy at least the smaller of those of the two lifts $\pm t^{\mp 1}$ or $\pm rt^{\mp 1}$. For $\pm t$ this is $\frac{4}{3}$, as shown above. Conjugation by t permutes the three Pauli elements P_i, Q_i, P_iQ_i of each quaternion pair in (5.10) cyclically, so the Pauli elements that commute with t are those that commute with the three pairs. They form a group 2^{1+4}, and over $\mathbb{C}$ the lattice decomposes as $\mathbb{C}^4 \otimes \mathbb{C}^8$, where this group acts on the first factor and the three quaternion pairs act on the second. Correspondingly $t = 1 \otimes t_8$, where t_8 has eigenvalues $1, 1, \zeta, \zeta, \zeta, \bar{\zeta}, \bar{\zeta}, \bar{\zeta}$, and $r = r_4 \otimes 1$, where r_4 has eigenvalues $1, 1, -1, -1$ or $i, i, -i, -i$. In both cases (2.10) gives the oscillator ground state energy $\frac{4}{3}$ for rt and for $-rt$; for instance, if $r^2 = -1$ then $\pm rt$ has characteristic polynomial $\Phi_4(X)^4 \Phi_{12}(X)^6$, and the energy is $8 \cdot \frac{3}{64} + 6 \cdot \frac{11+35+35+11}{576} = \frac{4}{3}$. Momentum contributions are nonnegative. Hence every factor contributes at least $\frac{4}{3}$, and $\rho \geq 4$ for these sectors as well.

In summary, the only twisted sectors that can contribute states of weight at most three are the 567 sectors of Lemma 5.3. Each of them has a ground space of dimension $32^3 = 2^{15}$ at weight three.

5.5 The anomaly

Theorem 5.4. *The anomaly of the action of G on $Z^{\otimes 3}$ vanishes.*

Proof. By (A3) it suffices to consider the Sylow subgroups L and $\langle g\rangle$. The restriction to L vanishes by Proposition 5.2. For $\langle g\rangle \cong \mathbb{Z}_9$ we need the exact ground state energy of the g-twisted sector, not just a lower bound, and the most convenient way to obtain it is to work with the lattice theory $V_N^{\otimes 3}$ before the inversions are gauged. Let $K_\theta \cong \mathbb{Z}_2^3$ be generated by the inversions of the three factors, and let $\tilde{g} = (\tilde{t}^{-1}, 1, 1)\sigma$, where $\tilde{t}$ is the standard lift of t to V_N. These generate a group $K_\theta \rtimes \mathbb{Z}_9$. The anomaly of K_θ vanishes, since it acts as a product and each inversion has ground state energy 2. The standard lift $\tilde{t}^{-1}$ acts trivially on its fixed lattice, so its twisted sector has ground state energy exactly $\frac{4}{3}$, and (2.11) gives $\rho_{\tilde{g}} = 4$. Hence $81\rho_{\tilde{g}} \equiv 0 \bmod 9$, and the anomaly of $\langle\tilde{g}\rangle$ vanishes. By (A3), the anomaly of $K_\theta \rtimes \mathbb{Z}_9$ vanishes.

Gauging the normal subgroup K_θ turns $V_N^{\otimes 3}$ into $Z^{\otimes 3}$, on which $\tilde{g}$ induces an automorphism conjugate to g. We can choose the trivialization of the anomaly of $K_\theta \rtimes \mathbb{Z}_9$ to restrict to the standard one on K_θ: the two differ by a $\mathbb{Z}_9$-invariant class in $H^2(K_\theta, U(1))$, which is a 2-group on which multiplication by nine is invertible. The induced action of $\mathbb{Z}_9$ on $Z^{\otimes 3}$ is then anomaly-free, so $\alpha|_{\langle g\rangle} = 0$. □

5.6 Discrete torsion

Since $|L|$ and 9 are coprime, the discrete torsion group is

$$H^2(G, U(1)) \cong H^2(L, U(1))^{\mathbb{Z}_9} \cong \left(\bigwedge\nolimits^2 L^*\right)^T, \tag{5.16}$$

the space of alternating bilinear forms on L that are invariant under T. Solving the 276 linear equations for invariance shows that it has dimension 36.

Fix one of the 567 light fluxes $v = (v_1, v_2, v_3) \in L$. Since its orbit under g has nine elements, its centralizer in G is L. The ground space of the v-twisted sector is the tensor product of three irreducible projective representations of A of dimension 32. An element $y \in L$ acts on the i-th factor through its i-th component $E_i(y) \in A$, and by (2.15) it acts by a scalar on that factor if and only if $E_i(y)$ lies in the radical of the commutator form H_{v_i}. Let

$$K_v = \{y \in L : \ H_{v_i}(E_i y, z) = 0 \text{ for all } z \in A \text{ and } i = 1, 2, 3\} \tag{5.17}$$

be the subgroup acting by scalars on the whole ground space. Outside K_v the trace of y on at least one factor vanishes. By Fourier analysis on L, the characters of L that occur in the ground space are therefore precisely the extensions of a single character λ_v of K_v, each with multiplicity $2^{15}/2^{24-\dim K_v}$. An L-invariant ground state thus exists if and only if λ_v is trivial. Changing the discrete torsion by β multiplies λ_v by $y \mapsto (-1)^{\beta(v,y)}$, so the torsion classes for which the ground states of v survive are the solutions of

$$\Gamma_v(\beta) = \lambda_v, \qquad \Gamma_v : \left(\bigwedge\nolimits^2 L^*\right)^T \to K_v^*, \quad \Gamma_v(\beta) = \beta(v, \cdot)|_{K_v}. \tag{5.18}$$

This set is either empty or an affine subspace of dimension $36 - \operatorname{rank}\Gamma_v$. The nine fluxes in a g-orbit give the same condition. Computing K_v and $\operatorname{rank}\Gamma_v$ for all 567 fluxes gives the following.

number of orbits	$\dim K_v$	multiplicity of λ_v	$\operatorname{rank}\Gamma_v$
36	9	1	6
24	11	4	8
3	13	16	10

By the counting principle (2.17), whatever the values of the λ_v, the number of torsion classes for which some light ground state survives is at most $36 \cdot 2^{30} + 24 \cdot 2^{28} + 3 \cdot 2^{26}$. This proves:

Proposition 5.5. *At least*

$$2^{36} - 36 \cdot 2^{30} - 24 \cdot 2^{28} - 3 \cdot 2^{26} = 349 \cdot 2^{26} \tag{5.19}$$

of the 2^{36} discrete torsion classes remove all twisted states of weight at most three.

This is a count of torsion classes; different classes may well give isomorphic theories.

5.7 The orbifold theory

Let ϵ be one of the torsion classes of Proposition 5.5, and let $W = (Z^{\otimes 3}/G)_\epsilon$. Since G is solvable and its anomaly vanishes, W is a holomorphic VOA, and by (5.14) and the analysis of the twisted sectors, $W_n = \mathcal{C}_n$ for $n \le 3$. In particular W has no primaries of weight one, two or three.

A quartic primary. Products of the stress tensors of the individual factors give a weight-four primary. Let $L_n^{(i)}$ be the Virasoro modes of the i-th factor, each of central charge $c_0 = 32$, and let

$$A_4 = \sum_i \big(L_{-2}^{(i)}\big)^2 \mathbf{1}, \qquad B_4 = \sum_i L_{-4}^{(i)} \mathbf{1}, \qquad D_4 = \sum_{i<j} L_{-2}^{(i)} L_{-2}^{(j)} \mathbf{1}. \tag{5.20}$$

These states are invariant under every automorphism that permutes the three factors and preserves their stress tensors, so they lie in $\mathcal{C}$. A combination $Q = aA_4 + bB_4 + dD_4$ is annihilated by L_1 if and only if $3a+5b = 0$, and by L_2 if and only if $(c_0+8)a+6b+c_0 d = 0$; the conditions from L_3 and L_4 then hold automatically. For $c_0 = 32$ we obtain the primary

$$Q = 80A_4 - 48B_4 - 91D_4. \tag{5.21}$$

The norms $\|(L_{-2})^2\mathbf{1}\|^2 = c_0(c_0+8)/2 = 640$, $\|L_{-4}\mathbf{1}\|^2 = 5c_0 = 160$ and $\langle (L_{-2})^2\mathbf{1}, L_{-4}\mathbf{1}\rangle = 3c_0 = 96$ in a single factor, together with $\|L_{-2}^{(i)}L_{-2}^{(j)}\mathbf{1}\|^2 = (c_0/2)^2 = 256$, give

$$\|Q\|^2 = 3\left(80^2 \cdot 640 + 48^2 \cdot 160 - 2 \cdot 80 \cdot 48 \cdot 96\right) + 3 \cdot 91^2 \cdot 256 = 17{,}541{,}888 > 0. \tag{5.22}$$

So Q is a nonzero primary of weight four in $\mathcal{C} \subset W$, and $\mu(W) = 4$. This is the analogue of Proposition 2.1 for tensor products: any orbifold of three copies of a theory by a group that permutes the factors and preserves their stress tensors contains such a primary, and so has gap at most four.

Partition function. The partition function of W is (2.6). The two vacuum descendants at weight four and the primary Q show that $d_4 \geq 3$, but we have not determined d_4. In particular we do not know the number of weight-four primaries, which may depend on the choice of discrete torsion.

Unitarity. The theory W is unitary, with a positive-definite invariant form compatible with its original operator products. As at $c = 64$, the essential point is to show that the fixed-point subalgebra $\mathcal{C}$ is completely unitary, after which the theorem of [35] applies. Here we use the theory of conformal nets, including recent results that are so far only available as preprints. The theory Z is strongly local, with a holomorphic conformal net [73], on which its automorphisms act unitarily [69]. The conformal net of $\mathcal{C}$ is the fixed-point net of $Z^{\otimes 3}$, and each of its irreducible representations gives an irreducible unitary $\mathcal{C}$-module [74]. Applying this first to the fixed-point nets of cyclic subgroups of G, and using the spin–statistics theorem [75], one finds that G also acts on the conformal net of $Z^{\otimes 3}$ without anomaly. By the orbifold theory of holomorphic conformal nets [52,76], the fixed-point net then has at least as many irreducible representations as $\mathcal{C}$ has irreducible modules, so every irreducible $\mathcal{C}$-module is unitarizable. Complete unitarity then follows from Tener's criterion [77], using bounds on localized vertex operators obtained from the analytic results of [78, 79]. We give more details in Appendix D.

6 The Barnes–Wall theory and its symmetry group $E_8(2)$

We now return to the orbifolds of Section 3. They all have gap four, but they differ in which twisted states of weight four survive, and keeping all of them leads to the most symmetric theory. For this theory we compute the partition function and determine the automorphism group. The second step uses the theory Z of Section 5.1: the same theory can be obtained as an orbifold of four copies of Z, and this second construction makes symmetries visible that the lattice construction hides.

Theorem 6.1. *There is a unique choice $\epsilon_{\max}$ of discrete torsion for which the orbifold of Theorem 3.1 contains all ground states of the twisted sectors of noncentral involutions. The theory $W = (V/P)_{\epsilon_{\max}}$ has partition function*

$$Z_W = X\big(j^5 - 3968\, j^4 + 5346369\, j^3 - 2798220719\, j^2 + 470623467780\, j - 11824040348928\big). \tag{6.1}$$

Its automorphism group is the finite simple group $E_8(2)$. The 545,925,250 primaries of weight four form the irreducible representation of $E_8(2)$ of smallest nontrivial dimension, and there are no primaries of weight five.

6.1 Discrete torsion with maximal spectrum

The discrete torsion classes of the orbifolds of V by P form the group

$$H^2(P, U(1)) \cong \mathrm{Alt}(A)/\langle b \rangle, \tag{6.2}$$

the alternating bilinear forms on A modulo the commutator form b. All of these orbifolds have gap four, but they differ in which twisted states survive. For a noncentral involution g, the ground states of $V(g)$ have weight four and form an irreducible representation T_g of a finite Heisenberg group of the twisted lattice, of dimension 2^{16} [32]. These states do not affect the gap, but they account for most of the weight-four spectrum, and keeping all of them leads to the most symmetric theory.

Proposition 6.2. *There is a unique class $\epsilon_{\max} \in H^2(P, U(1))$ for which $(V/P)_{\epsilon_{\max}}$ contains the full ground spaces T_g of all 8255 classes of noncentral involutions.*

We define $W = (V/P)_{\epsilon_{\max}}$. The proof has three steps.

(i) Scalar action. The centralizer $C_P(g)$ acts on T_g by scalars. To see this, one checks that the lifted centralizer elements commute, up to scalars, with the twisted lattice operators that act irreducibly on T_g, so that Schur's lemma applies. For the representatives $g = Z_0$ and $g = X_0 Z_0$ (the latter of order four) this is a finite computation with the lattice cocycle. Lifts of elements of the lattice normalizer of P (the cyclic shift of the seven qubits, a controlled-NOT, a controlled-Z, and a two-qubit Hadamard transformation) then transport the result to all other noncentral elements. We can therefore normalize the centralizer action so that $C_P(g)$ acts trivially on the ground states. A coherent normalization as in (2.9) differs from this one by a character of $C_P(g)$. This character is trivial on g, because the ground states have integral weight, and on z, so it has the form $h \mapsto (-1)^{\ell_v(w)}$ with $v = \pi(g)$, $w = \pi(h)$ and ℓ_v a linear form on $v^\perp/\langle v\rangle$. For the torsion class of an alternating form Λ, the ground space T_g survives if and only if $\Lambda(v, w) = \ell_v(w)$ for all $w \in v^\perp$.

(ii) Reciprocity. Let g be a noncentral involution and h a commuting element of order four. In the normalization of step (i), their twisted twining functions can be computed directly from the oscillators:

$$T^+(g, h; \tau) = 2^{16} \left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{32}, \qquad T^+(h, g; \tau) = 2^{32} \left(\frac{\eta(\tau)}{\eta(\tau/2)}\right)^{32}, \tag{6.3}$$

and they satisfy $T^+(g, h; -1/\tau) = T^+(h, g; \tau)$ without any additional sign. In a coherent normalization, on the other hand, the modular S transformation maps $Z_{g,h}$ to $Z_{h,g^{-1}}$ without a phase, because the anomaly vanishes. Comparing the two gives

$$\ell_v(w) = \ell_w(v) \qquad \text{whenever } b(v, w) = 0 \text{ and } q(v) \neq q(w). \tag{6.4}$$

(iii) From local to global. On a nondegenerate split quadratic space of dimension at least six, every family of linear forms $\ell_v \in (v^\perp/\langle v\rangle)^*$ satisfying (6.4) comes from a single alternating form: $\ell_v(w) = \Lambda(v, w)$ for all orthogonal v, w, with Λ unique modulo b (Appendix B.3).

Taking $\epsilon_{\max}$ to be the class of this Λ keeps all the T_g, and in fact all the ground states of the noncentral twisted sectors. Conversely, if two alternating forms both keep all T_g, their difference vanishes on $v \times v^\perp$ for every singular v. Writing the difference as $b(Mv, w)$, every singular vector is an eigenvector of M; this forces M to be a multiple of the identity, so the difference is a multiple of b. This proves Proposition 6.2.

6.2 Partition function

By (2.2), the partition function of W is determined by $d_0, \dots, d_5$. Equation (3.3) gives $(d_0, d_1, d_2, d_3) = (1, 0, 1, 1)$. Knowing d_4 as well would not suffice, because the basis element X in (6.1) first contributes at weight five; the weight-five spectrum has to be computed.

Proposition 6.3. $\dim W_4 = \dim W_5 = 545{,}925{,}252$, *and* $W_5 = L_{-1}W_4$. *In particular* W *has no primaries of weight five.*

Proof. Write $H = \mathbb{C}^{128}$ for the oscillator representation of P. At weight four, W_4 receives three contributions.

- *Oscillators.* Only even numbers of oscillators are invariant under z. The mode patterns $3+1$ and $2+2$ each contribute one invariant (the metric), and the pattern $1+1+1+1$ contributes $\dim(\mathrm{Sym}^4 H)^P$. Since every noncentral element of P has $\det(1-tg)^{-1} = (1 \mp t^2)^{-64}$, averaging over P gives

$$\dim(\mathrm{Sym}^4 H)^P = \frac{2\binom{131}{4} + (2^{15}-2)\binom{65}{2}}{2^{15}} = 2795, \tag{6.5}$$

  for a total of 2797.

- *Momentum states.* P acts on the 1,260,230,400 vectors of norm 8 with orbits of size 256, and the stabilizer of a vector β acts trivially on e^β. This gives 4,922,775 invariant states, one for each orbit.

- *Twisted sectors.* The ground states of the 8255 noncentral involution sectors contribute $8255 \cdot 2^{16} = 540{,}999{,}680$ states. All other twisted sectors start at weight six or higher.

The total is 545,925,252. At weight five the corresponding contributions are as follows.

- *Oscillators.* The even patterns $4+1$, $3+2$ and $2+1+1+1$ give $1+1+\dim(H \otimes \mathrm{Sym}^3 H)^P = 2797$ invariants. Since L_{-1} commutes with P and is injective on states of positive weight h in the unitary theory V ($\|L_{-1}v\|^2 = 2h\|v\|^2 + \|L_1 v\|^2$), these are exactly the derivatives of the weight-four oscillator invariants.

- *Momentum states.* BW_{128} has no vectors of norm 10, since a Barnes–Wall lattice of minimal norm m has no vectors of norm strictly between m and $3m/2$ [80]. So the weight-five momentum states are of the form $a_{-1}e^\beta$ with $|\beta|^2 = 8$. The stabilizer of β in P is isomorphic to $\mathbb{F}_2^7$ and acts on H by its regular representation, so the only invariant direction is β itself. Each orbit contributes one state, $\beta_{-1}e^\beta = L_{-1}e^\beta$.

- *Twisted sectors.* Let g be a noncentral involution, with $H = H_+ \oplus H_-$ its eigenspaces. Excitations of relative energy one above T_g with zero momentum come either from H_+ at mode -1, which has no invariants under the centralizer, or from $\mathrm{Sym}^2 H_-$ at mode $-\frac{1}{2}$, which has exactly one invariant, the metric. So they form a single copy of T_g, which must be $L_{-1}T_g$, since $L_1 L_{-1} t = 8t$ for $t \in T_g$. States with nonzero momentum in the fixed lattice cost at least two units of energy.

Hence $W_5 = L_{-1}W_4$. □

With $(d_0, \ldots, d_5) = (1, 0, 1, 1, 545925252, 545925252)$, solving (2.2) gives (6.1), and (2.3) then gives all primary multiplicities. The first few are listed in Table 6.

h	d_h	N_h
1, 2, 3	0, 1, 1	0, 0, 0
4	545,925,252	545,925,250
5	545,925,252	0
6	76,322,319,758,924	76,321,227,908,420
7	51,396,383,027,047,674	51,320,060,161,363,500
8	198,735,599,368,523,610,832	198,684,126,663,176,804,235

Table 6: State and primary multiplicities of W at $c = 128$.

The absence of primaries at weight five does not increase the gap, which remains four. The theory is far from extremal, since $\lfloor 128/24 \rfloor + 1 = 6$.

6.3 The automorphism group

Theorem 6.4. $\mathrm{Aut}(W) \cong E_8(2)$. *The space* $\mathrm{Prim}_4(W)$ *is the irreducible representation of* $E_8(2)$ *of smallest nontrivial dimension,* 545,925,250*, and* $\mathrm{Prim}_6(W)$ *is the irreducible representation of dimension* 76,321,227,908,420.

Here $E_8(2)$ is the finite group of Lie type E_8 over the field with two elements. It is simple, of order

$$|E_8(2)| = 2^{120} \prod_{d \in \{2,8,12,14,18,20,24,30\}} (2^d - 1) \approx 3.4 \times 10^{74}. \tag{6.6}$$

It should not be confused with the compact Lie group E_8; indeed W has no currents. The dimension of $\mathrm{Prim}_4(W)$ can be written as $\sum_e 2^e$, where e runs over the exponents $1, 7, 11, 13, 17, 19, 23, 29$ of E_8: it is the dimension of the reflection representation in the principal series of $E_8(2)$ [81].

The lattice construction makes only a small part of this symmetry visible. The group of characters $\mathrm{Hom}(P, U(1)) \cong \mathbb{F}_2^{14}$ acts on W, by multiplying the states in the g-twisted sector with the value of the character on g; these are the quantum (or dual) symmetries of the orbifold. Elements of the normalizer of P in $\mathrm{Aut}(V)$ also lift to automorphisms of W: by uniqueness they preserve $\epsilon_{\max}$, so they map W, regarded as an algebra object in $\mathrm{Rep}(V^P)$, to an isomorphic one [44]. We call the group generated by these automorphisms the inherited symmetry group H_{inh}. A computation of invariants shows that $\mathrm{Prim}_4(W)^{H_{\mathrm{inh}}} = \mathbb{C}R_{128}$: up to scale, the universal quartic is the only weight-four primary fixed by H_{inh}. The remaining symmetries become visible in a second construction of W.

The proof of Theorem 6.4 rests on three intermediate results: the equivalence of the two constructions (Proposition 6.5), the structure of a rational VOA D of central charge 64 that appears in an orbifold of W (Proposition 6.6), and the centralizer of an involution (Proposition 6.7). We sketch the proof in six steps; the group-theoretic part of the fifth step is given in Appendix C.

Step 1: a second construction of W. Let $N = \mathrm{BW}_{32}$ and let $Z = V_N^+ \oplus V_N^{T,+}$ be the $c = 32$ orbifold of V_N by the lift of -1. Its automorphism group is $2^{27}.E_6(2)$, with a normal subgroup $A_Z \cong \mathbb{F}_2^{27}$ [68]; we discuss Z in detail in Section 5.1. On $Z^{\otimes 4}$ consider the group $J = \Delta A_Z \times \mathbb{F}_2^2$, where ΔA_Z acts diagonally and $\mathbb{F}_2^2$ permutes the four factors regularly. Since J is an elementary abelian 2-group, its anomaly is determined by cyclic subgroups: a diagonal element has four times the type of an element of A_Z, and an element with nontrivial permutation part is conjugate to a pair of transpositions, whose twisted sector has ground state energy $2 \cdot \frac{32}{16} = 4$. So J is anomaly-free. The sectors twisted by elements (a, p) with $p \neq 0$ have one-dimensional ground spaces of weight four, and there is a unique discrete torsion that keeps all of them. Let W' be the corresponding orbifold.

Proposition 6.5. *$W' \cong W$, by an isomorphism that maps the universal quartic R_{128} of W to that of W'.*

Sketch of proof. Both theories are obtained from $V_N^{\otimes 4}$ by gauging the same group in two stages, in different orders, as indicated in the diagram

$$\begin{array}{ccccc} V_N^{\otimes 4} & \xrightarrow{R_{16}} & V_{\mathrm{BW}_{128}} & \xrightarrow{\quad P \quad} & W \\ \Big\downarrow{\scriptstyle E_0} & & & & \Big\downarrow{\scriptstyle \cong} \\ Z^{\otimes 4}/\mathbb{Z}_2 & \xrightarrow[\mathbb{Z}_2]{} & Z^{\otimes 4} & \xrightarrow[J]{\quad\quad} & W' \end{array} \tag{6.7}$$

in which each arrow denotes the orbifold by the indicated group. Let $N_0 \subset N$ be the sublattice of index 2^{16} with $N_0 \cong \sqrt{2}N$ that appears in Shimakura's description of A_Z, and let $R_{16} \cong \mathbb{F}_2^{16}$ be the group of diagonal lattice phases that act trivially on V_{L_0}, where $L_0 = \{(n_1, \ldots, n_4) \in N^4 : \sum_i n_i \in N_0\}$. Gauging R_{16} turns $V_N^{\otimes 4}$ into the lattice theory of $L = L_0 + \{(u, u, u, u)/2 : u \in N_0\}$, and an explicit integral basis change shows that $L \cong \mathrm{BW}_{128}$. Under this isomorphism the diagonal Pauli group of the four copies, the even numbers of factor inversions and the regular permutations of the factors together generate the Pauli group P. Gauging first the group $E_0 \cong \mathbb{F}_2^3$ of even numbers of factor inversions instead gives the $\mathbb{Z}_2$ orbifold of $Z^{\otimes 4}$ by its diagonal quantum symmetry. Undoing this last orbifold, the remaining group acts on $Z^{\otimes 4}$ as J. The two routes are compared as follows.

1. The actions of the two descriptions agree on the common subalgebra V_{L_0}, including the cocycle phases. This is a finite computation (Appendix E): comparing the 14 generators on a basis of L_0 gives a linear system over $\mathbb{F}_2$ for a global sign correction, which is solved by the trivial correction.

2. W is an orbifold of $V_N^{\otimes 4}$ by the full group $G_0 = R_{16} \times P$. Indeed, by (2.7) the holomorphic extensions of $\mathcal{C} = (V_N^{\otimes 4})^{G_0}$ are labeled by conjugacy classes of pairs (H, ψ), where $H \subseteq G_0$ and ψ trivializes the anomaly on H [45, Cor. 2], and the space of $\mathcal{C}$-module maps from $V_N^{\otimes 4}$ to the extension labeled by (H, ψ) has dimension $[G_0 : H]$, since the untwisted part of that extension is the permutation representation of G_0 on G_0/H. For W this space is one-dimensional, because the only $\mathcal{C}$-module that W and $V_N^{\otimes 4}$ have in common is $\mathcal{C}$ itself: the other irreducible $\mathcal{C}$-modules in W lie in sectors twisted by nontrivial elements of G_0, and these are disjoint from the untwisted ones [60]. Hence $H = G_0$.

3. In a sector twisted by an element of E_0 that inverts two factors, the ground states of W are selected by a character χ of R_{16}. For $\chi \neq 0$, a suitable Pauli element of order four acts on the selected ground states with both signs, since its sign is given by a nondegenerate quadratic function. By step (i) of Section 6.1, however, the centralizer acts on the ground states of W by scalars (the Pauli elements of the two descriptions induce the same automorphism of $\mathcal{C}$, so they agree up to a scalar on this sector), so $\chi = 0$. In each of the six sectors twisted by such elements, the normalization of W and that of the standard orbifold by E_0 therefore both act trivially on the selected ground states. Since two normalizations differ by a character of the centralizer, they agree on the whole sector. Fusing complementary sectors of this kind, one finds that the discrete torsion of W, relative to that of the standard first stage, is inflated from G_0/E_0, and such a class only affects the second stage. This fixes the intermediate orbifold by E_0, together with the action of the remaining group, uniquely.

4. The ground states of W in the sectors corresponding to the mixed sectors of J are all retained, because $\epsilon_{\max}$ keeps the full ground space T_g for every noncentral involution, and T_g, of dimension 2^{16}, is the sum of the one-dimensional ground spaces of the corresponding 2^{16} twisted sectors of $V_N^{\otimes 4}$, so that none of them is lost. By uniqueness of the torsion choice in the definition of W', the second stage reproduces W'.

The isomorphism restricts to the identity on $\mathcal{C}$, which contains the universal quartic by Proposition 2.1, since G_0 preserves the free bosons; so it identifies the two universal quartics. □

Step 2: symmetries that move the quartic. Any automorphism f of Z acts diagonally on $Z^{\otimes 4}$ and normalizes J. By uniqueness it preserves the torsion choice, and therefore lifts to an automorphism of $W' \cong W$ [82]. The universal quartic R_{32} of the 32 bosons of V_N is not invariant under $\mathrm{Aut}(Z)$. If it were, its zero mode would commute with the action of $\mathrm{Aut}(Z)$ on $\mathrm{Prim}_2(Z)$, which is irreducible, and would therefore act on it by a scalar. Instead it has two different eigenvalues, $\frac{384}{17}$ on the traceless quadratic oscillator states and $\frac{20}{17}$ on the momentum states of norm four. Since

$$R_{128} = \tfrac{5627}{5915} \sum_{i=1}^{4} R_{32}^{(i)} + (\text{terms built from the four stress tensors}), \tag{6.8}$$

a suitable diagonal f moves R_{128}. Combined with $\mathrm{Prim}_4(W)^{H_{\mathrm{inh}}} = \mathbb{C}R_{128}$, this shows that no weight-four primary is invariant under all of $\mathrm{Aut}(W)$.

Step 3: the centralizer of an involution. The quantum symmetries corresponding to a totally singular plane in A give three commuting involutions $t_1, t_2, t_3 = t_1 t_2$ of W, which are conjugate in H_{inh}. Let t be one of them. The orbifold $U = W/\langle t\rangle$ has two commuting Virasoro subalgebras of central charge 64 at weight two, $U_2 = \mathbb{C}\omega_+ \oplus \mathbb{C}\omega_-$, and the commutant of either one is a rational VOA D of central charge 64. (This theory is not holomorphic and should not be confused with the theory of Section 4.)

Proposition 6.6. *$D \cong (Z\otimes Z)^{\Delta A_Z \times \langle s\rangle}$, where s exchanges the two factors. All irreducible D-modules are simple currents. They form a group $\mathrm{Irr}(D) \cong \mathbb{F}_2^{56}$, on which the conformal weight modulo one defines a nondegenerate quadratic form $r \mapsto 2h_r \bmod 2$ of plus type. The theory U is the diagonal extension*

$$U = \bigoplus_{r\in \mathrm{Irr}(D)} M_r \otimes M_r, \tag{6.9}$$

and W is recovered from U as the orbifold by the involution S that exchanges the two tensor factors. Finally, $\mathrm{Aut}(D) \cong E_7(2)$.

Sketch of proof. The description of D and of U follows from the four-copy construction of Step 1, by comparing lattice descriptions of D and identifying the extension of $D\otimes D$; we omit the details. The automorphism group of D can be read off from its modules. The irreducible D-modules whose lowest weight space is one-dimensional and has weight two form a set $\mathcal{P}$ of 277,347,807 elements. The stabilizer of one of them in $\mathrm{Aut}(D)$ is $2^{27}{:}E_6(2)$, acting on $\mathcal{P}$ with orbits of sizes 1, 279,006, 142,851,072 and 134,217,728. The action of $\mathrm{Aut}(D)$ on $\mathcal{P}$ is faithful, so $\mathrm{Aut}(D)$ is finite. There is an automorphism of D of order 11: realizing D as a fixed-point subalgebra of the $\mathbb{Z}_2$ orbifold of $V_{\mathrm{BW}_{64}}$ by a group 2^{12} coming from the Pauli group 2_+^{1+12}, it is induced by an element of order 11 in the normalizer of this Pauli group in $\mathrm{Aut}(\mathrm{BW}_{64})$. Since 11 does not divide the order of the stabilizer, and only the union of all four orbits has a size divisible by 11, $\mathrm{Aut}(D)$ acts transitively on $\mathcal{P}$. Comparing the possible block sizes with $|\mathcal{P}|$ shows that the action is primitive. Since $|\mathcal{P}| = 3^5\cdot 11\cdot 19\cdot 43\cdot 127$ is odd, not a prime power and not a proper power, the O'Nan–Scott theorem [83], the perfectness of the stabilizer and the solvability of outer automorphism groups of finite simple groups (a consequence of the classification) then force $\mathrm{Aut}(D)$ to be simple, of order $277347807\cdot 2^{27}\cdot |E_6(2)| = |E_7(2)|$. Since a finite simple group is determined by its order, apart from the pair A_8, $L_3(4)$ and the pairs $B_n(q)$, $C_n(q)$ with $n\geq 3$ and q odd [84], we conclude that $\mathrm{Aut}(D) \cong E_7(2)$. □

Proposition 6.7. *The centralizer of t in $\mathrm{Aut}(W)$ is*

$$C_{\mathrm{Aut}(W)}(t) \cong 2_+^{1+56}{:}E_7(2). \tag{6.10}$$

In particular it is finite, of order $2^{57}|E_7(2)|$.

Sketch of proof. Every automorphism of U preserves or exchanges $\omega_\pm$, since $\omega_\pm/2$ are the two primitive idempotents of the algebra U_2 with product $a \cdot b = a_{(1)}b$. Hence every automorphism acts on $D \otimes D$, and since $\mathrm{Aut}(D)$ acts faithfully on $\mathrm{Irr}(D)$ and automorphisms that preserve the diagonal extension lift to it [82], this gives an exact sequence $1 \to \mathrm{Irr}(D)^* \to \mathrm{Aut}(U) \to E_7(2) \times \mathbb{Z}_2 \to 1$. Since U and W are orbifolds of each other, and automorphisms lift along the corresponding $\mathbb{Z}_2$ simple current extensions [82], $C_{\mathrm{Aut}(W)}(t)/\langle t\rangle \cong \mathrm{Aut}(U)/\langle S\rangle$. The extension of $\mathrm{Irr}(D)^* \cong \mathbb{F}_2^{56}$ by $\langle t\rangle$ is determined by the mixed anomaly between S and the quantum symmetries of U, which can be computed from the S-twisted sectors: for $k \in \mathrm{Irr}(D)$, the sector twisted by S composed with the corresponding quantum symmetry has ground state energy $4 + h_k/2$, so its type is $2h_k \bmod 2$, and gauging S turns this mixed anomaly into a central extension with this squaring map [85]. Since the quadratic form has plus type, the extension is the extraspecial group 2_+^{1+56}. The group $E_7(2)$ acts faithfully on $\mathrm{Irr}(D)^* \cong \mathbb{F}_2^{56}$, so some composition factor is nontrivial; since the only nontrivial irreducible representation of $E_7(2)$ in characteristic two of dimension at most 56 is the 56-dimensional one [86], $\mathrm{Irr}(D)^*$ is this module. Comparing with the centralizer of a root involution in $E_8(2)$ [87], and using $H^2(E_7(2), \mathbb{Z}_2) = 0$, which holds since $E_7(2)$ is perfect with trivial Schur multiplier [88], one obtains (6.10). □

Step 4: finiteness. Since W is C_2-cofinite, it is finitely generated [89], so $\mathrm{Aut}(W)$ is an algebraic group [90]. By Proposition 6.7, the centralizers of t_1, t_2 and t_3 are finite. Each t_i therefore acts as -1 on the Lie algebra of $\mathrm{Aut}(W)$. This is impossible for $t_3 = t_1t_2$ unless the Lie algebra vanishes, so $\mathrm{Aut}(W)$ is finite.

Step 5: recognition. From Proposition 6.7 and the fact that t is conjugate in $\mathrm{Aut}(W)$ to the two other involutions of Step 3, which commute with it, standard arguments show that $\mathrm{Aut}(W)$ is a simple group, that its Sylow 2-subgroups have order 2^{120}, and that its order is divisible by the primes 19, 73 and 127 dividing $|E_7(2)|$ (Appendix C). By the classification of finite simple groups, $E_8(2)$ is the only simple group with these properties.

Step 6: representations. The smallest nontrivial degree of a complex irreducible representation of $E_8(2)$ is 545,925,250, attained by a unique representation, and the next smallest is 76,321,227,908,420 [91]. A representation of dimension 545,925,250 is therefore either irreducible or trivial. The space $\mathrm{Prim}_4(W)$ is not trivial, since the quantum symmetries act on the twisted ground states by nontrivial signs. So $\mathrm{Prim}_4(W)$ is the minimal representation; in particular it has no invariant vectors, in agreement with Step 2. For $\mathrm{Prim}_6(W)$, whose dimension is 76,321,227,908,420 by Table 6, the only alternative to irreducibility would be a sum of copies of the minimal representation and at least 332,033,170 trivial representations. This is ruled out by an explicit bound: the symmetries of the four-copy construction fix a subspace of W_6 of dimension at most 76,430,266.

Remark 6.8 (Graded traces). Since $E_8(2)$ acts on every W_n, the graded traces $T_g(\tau) = \mathrm{Tr}_W\, g\, q^{L_0 - 16/3}$ are defined for all $g \in E_8(2)$. For the quantum symmetries they can be

computed from the decomposition of $\mathrm{Prim}_4(W)$ under $\mathrm{Hom}(P, U(1))$. For an involution g corresponding to a singular or a nonsingular vector of A, respectively,

$$T_g = q^{-16/3}\left(1 + q^2 + q^3 + 9054340\,q^4 + 9054340\,q^5 + O(q^6)\right),$$
$$T_g = q^{-16/3}\left(1 + q^2 + q^3 + 665732\,q^4 + 665732\,q^5 + O(q^6)\right). \tag{6.11}$$

The corresponding values 9,054,338 and 665,730 of the character of $\mathrm{Prim}_4(W)$ agree with the character table of $E_8(2)$ [91]. Unlike the McKay–Thompson series of the Monster, these functions are not Hauptmoduln: they are invariant under $\tau \mapsto \tau + 3$ but no smaller translation, and have a pole of order 16 at the cusp. It would be interesting to understand whether the T_g have other special properties. We note that $E_8(2)$ itself cannot be used as an orbifold group: its anomaly restricts nontrivially to a cyclic subgroup of order three (see the proof of Proposition 7.1).

7 Discussion

7.1 The landscape of chiral CFTs with large gap

Table 7 summarizes what is known about the largest primary gap of chiral CFTs at selected central charges up to $c = 128$, and compares it with the extremal bound and with the black hole threshold $c/24$. Below $c = 48$ the extremal bound is attained by known theories. Above it, the examples of this paper are, to our knowledge, the only known theories with a gap larger than two. At $c = 64$ the bound is attained; at $c = 96$ it is attained up to one unit, which is optimal if the conjecture of [28] holds; and at $c = 128$ our example falls two units short of it.

The three constructions illustrate different ways of dealing with the two main obstacles. At $c = 128$ the anomaly is invisible to cyclic subgroups, and it is detected by a quaternion subgroup that also lies in a larger group of auxiliary symmetries. At $c = 64$ the anomaly is reduced to a single detectable bit by invariance under an auxiliary normalizer, while the light twisted states are removed by a counting argument in a space of discrete torsion choices. At $c = 96$ the anomaly is a cubic form, which we cancel across three copies, and an internal twist cancels the anomaly of the permutation of the copies. In all three cases the arguments reduce to finite computations, mostly linear algebra over $\mathbb{F}_2$, once the parent theory and the group are fixed. This suggests that a more systematic search could map out a substantial part of the landscape. First steps in this direction suggest that gap four is common at larger central charge, while larger gaps are hard to reach. For example, further computations indicate that the construction of Section 3 extends to Barnes–Wall lattices of larger rank and, using orbifolds of tensor products of copies by permutations combined with internal symmetries, to every central charge that is a multiple of 128, always with gap exactly four.

Non-chiral CFTs with Virasoro primary gap three and four were recently constructed by Ji and Tian [33], using groups whose anomalies cancel between left- and right-movers.

c	$c/24$	$\lfloor c/24 \rfloor + 1$	largest known gap	example
24	1	2	2	Monster CFT $V^\natural$ [1]
32	4/3	2	2	$\mathbb{Z}_2$ orbifolds of extremal lattice theories [21]
40	5/3	2	2	$\mathbb{Z}_2$ orbifolds of extremal lattice theories [21]
48	2	3	2	lattice orbifolds [30]; extremal theory excluded*
64	8/3	3	3	Section 4
72	3	4	2	lattice orbifolds [30]; extremal theory excluded*
96	4	5	4	Section 5; extremal theory excluded*
128	16/3	6	4	Section 3 and [32]

Table 7: The largest known primary gap of chiral CFTs at selected central charges, compared with the black hole threshold $c/24$ and the extremal bound. An asterisk indicates that the extremal theory is excluded if the conjecture used in [28] holds.

These theories do not seem to lead directly to chiral theories with a large gap: under natural assumptions on the extension, further computations indicate that the chiral algebra of their example at $c = 48$ has no holomorphic extension with gap three.

The $c = 128$ example also shows that the lattice construction of a theory can hide most of its symmetry. Character degrees are useful for identifying representations once the automorphisms have been constructed, but they do not construct automorphisms. In our case the additional symmetries came from a second construction of the same theory, obtained by gauging the same group in a different order.

7.2 Relation to three-dimensional gravity

As explained in the introduction, a dual of pure gravity should have no primaries with $0 < h < c/24$, which would describe massive particles creating conical defects [4, 11], and by Section 2.1 a chiral CFT has this property if and only if it is extremal, or $c = 24k$ and it is near-extremal. Comparing the second and fourth columns of Table 7, our three theories differ in an instructive way.

The theory at $c = 64$ is extremal, so all of its nonvacuum primaries lie strictly above the threshold $c/24 = 8/3$. It has exactly the spectrum that [5], extending the proposal of [4], associated with pure gravity at this central charge: below weight three there are only boundary gravitons, and above it the primaries account for the black hole states. It is the first theory beyond $c = 40$ with this property.

The theory at $c = 96$ is near-extremal with $k = 4$: it has no primaries below the

threshold $c/24 = 4$, but it has $d_4 - 2 \geq 1$ primaries exactly at it. In [8] such states were interpreted as small black holes, whose horizon area vanishes classically, and a formula for their number was proposed based on their one-loop entropy. Our construction does not determine d_4, and it would be interesting to compute it and compare it with this proposal. If holomorphic CFTs define classes in topological modular forms, the argument of [28] requires d_4 to be divisible by three. Bae, Lee and Lee studied near-extremal theories with the numerical conformal bootstrap, using the four-point function of a primary of weight k, and found evidence that they do not exist for $k \geq 4$ [27]. The theory of Section 5 is a unitary near-extremal theory with $k = 4$, and its weight-four primary Q in (5.21) satisfies the assumptions of that analysis. It would be interesting to understand the origin of this discrepancy.

The theory at $c = 128$, finally, has primaries of weight four below the threshold $c/24 = 16/3$. In a gravitational dual they would describe massive particles rather than black holes. A theory at $c = 128$ without such states would have to be extremal, with gap six.

At the central charges considered here $c/24$ is of order one, so a putative bulk dual is far from the semiclassical regime. Nevertheless, the existence of extremal and near-extremal theories at finite c is a precise test of the proposals of [4, 5, 7, 8]. A dual of semiclassical pure gravity would require such theories at arbitrarily large central charge, and hence a gap that grows linearly with c. Our constructions do not produce such families: as we explain next, single orbifolds of lattice theories or of tensor products by groups of the type we use have gap at most four.

7.3 Obstructions to larger gaps

The gap four in two of our examples is explained by two universal weight-four primaries. For orbifolds of a lattice theory by a group that preserves the free bosons, the quartic R_d of Proposition 2.1 survives. For orbifolds of a product of three copies of a theory by a group that preserves the set of stress tensors of the factors, the primary Q of (5.21) survives. To obtain gap five from such constructions one needs symmetries that do not preserve these structures.

The $E_8(2)$ symmetry of the $c = 128$ theory W is of this kind: it acts irreducibly on the weight-four primaries, and in particular moves R_{128}. One might therefore hope that an orbifold of W by a suitable subgroup of $E_8(2)$ has gap five. This is not the case.

Proposition 7.1. *Every proper subgroup of* $E_8(2) = \mathrm{Aut}(W)$ *fixes a nonzero vector in* $\mathrm{Prim}_4(W)$*, and the anomaly of* $E_8(2)$ *itself is nonzero. Hence no orbifold of* W *by a group of automorphisms has a gap larger than four.*

Sketch of proof. The anomaly of $E_8(2)$ restricts nontrivially to a cyclic subgroup of order three, so $E_8(2)$ cannot be gauged. For example, the permutation of the labels $\mathbb{F}_2^7$ that acts as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ on two coordinates and fixes the others preserves BW_{128} and normalizes P. Its standard lift has order three and 32 three-cycles on the 128 coordinates, hence $\rho = \frac{32}{9}$ and type two on V. Since its order is prime to $|P|$, it induces an automorphism

of W with the same type. Its trace on $\mathrm{Prim}_4(W)$ is 141,985, the value of the minimal character on the class 3A. Since any orbifold of W by H contains W^H, it suffices to show that every proper subgroup H fixes a weight-four primary. Let $R = \mathrm{Prim}_4(W)$. As shown in Section 6.3, R is the reflection representation in the principal series of $E_8(2)$. Let $v \in R$ be the vector fixed by the maximal parabolic subgroup whose Levi factor is $E_7(2)$. Using the explicit description of the Hecke algebra action on the vectors fixed by a Borel subgroup [81], one finds that the normalized matrix coefficient $\phi(g) = (v, gv)/(v, v)$ satisfies $\phi(g) \equiv 1 \bmod 331$ for every $g \in E_8(2)$, as a congruence in the ring of rational numbers whose denominators are prime to 331. Here $331 = \Phi_{30}(2)$ is the order of a Coxeter torus. If $331 \nmid |H|$, then averaging over H gives $(v, P_H v)/(v, v) \equiv 1 \bmod 331$, where P_H is the projection onto H-invariant vectors, so $P_H v \neq 0$. If 331 divides $|H|$ and H is proper, then H is contained in the normalizer of a Coxeter torus, of order 9930 [92]. Since the minimal value of the character of R is -6014, we then have $\dim R^H \geq (545925250 - 6014(|H| - 1))/|H| > 0$. □

A simpler and more general obstruction comes from torus one-point functions. Suppose that a weight-four primary v of a holomorphic theory V satisfies $\mathrm{Tr}_{V_n} o(v) \neq 0$ for some n. If H is a finite group of automorphisms of V, then $\mathrm{Tr}_{V_n} o(hv) = \mathrm{Tr}_{V_n} o(v)$ for $h \in H$, so the average of v over H is a nonzero H-invariant primary of weight four, and no orbifold of V by H has gap larger than four. This applies also to symmetries that are not visible in the construction of V. Further computations indicate that it applies to the theories of Section 5 and to the orbifolds of Section 3 with discrete torsion other than $\epsilon_{\max}$. In the latter case the zero mode of R_{128} acts by the same nonzero scalar on all twisted ground states of weight four, so its trace on the weight-four space, which vanishes for W, becomes nonzero when some of these states are projected out. For W itself all such traces vanish: by Theorem 6.4, $v \mapsto \mathrm{Tr}_{W_n} o(v)$ is an $E_8(2)$-invariant linear form on $\mathrm{Prim}_4(W)$, which is a nontrivial irreducible representation. This is why Proposition 7.1 requires a different argument.

Proposition 7.1 and the trace argument only exclude a single further orbifold by a group of automorphisms. They do not exclude orbifolds of other parent theories, sequences of orbifolds that pass through different theories, or more general constructions such as gauging non-invertible symmetries.

7.4 Open questions

Our results raise a number of questions. The most obvious is whether there are chiral CFTs with gap five or larger at any central charge, and how they could be constructed. Closer to the examples of this paper, it would be interesting to know whether gap three can be reached at $c = 48$, 56 or 72. At $c = 48$ and $c = 56$ such a theory would be extremal, and at $c = 48$ it is excluded if the conjecture of [28] holds. At $c = 72$ it would be near-extremal, with $k = 3$. Among the central charges that are multiples of 24, the first one beyond $c = 24$ at which the test of [28] allows an extremal theory is $c = 144$, where such a theory would have gap seven.

One route to gap five at $c = 128$ that does not use orbifolds starts from the principal W-algebra of type E_8 at central charge 128 [93]. Its strong generators have weights $2, 8, 12, 14, 18, 20, 24$ and 30, the degrees of the basic invariants of the Weyl group of E_8, so its vacuum module contains no Virasoro primaries below weight eight. In the free-field realization with background charge Q, where $Q^2 = -10$ so that $c = 8 - 12Q^2 = 128$, the Fock modules with real momentum p have lowest weight $5 + p^2/2 \geq 5$. A holomorphic extension of the W-algebra by such modules would therefore have gap at least five. Computations indicate that modular invariance does not exclude this: the partition functions allowed for gap five appear to decompose into characters of the W-algebra and of these Fock modules with nonnegative integer multiplicities. Whether such an extension exists depends on the operator products of the modules, not only on their characters, and remains open.

We have determined the automorphism group of the $c = 128$ theory but not of the other two. At $c = 96$ we have not determined d_4, the number of states of weight four, and it would be interesting to know whether it depends on the choice of discrete torsion and how it compares with the counting of small black hole states in [8]. For the $c = 128$ theory it would be interesting to understand the graded traces of $E_8(2)$ more fully, in analogy with monstrous moonshine [38, 94].

Finally, it would be interesting to understand whether the techniques developed here can be pushed to larger central charge, and whether they lead to families of theories whose gap grows with c, as a dual of semiclassical pure gravity would require.

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Statement of AI Use

GPT-6 Astra and Claude Opus 5.5 were used extensively in the preparation of all aspects of this paper.

A The Gold orbifold: cohomology, lifts and a rank computation

A.1 Invariant cohomology of the Gold group

In this appendix we compute the part of the low-degree cohomology of the group H of (4.8) that is invariant under the Singer cycle (4.10). This proves the statements used in the construction of the lift and in Theorem 4.2.

We use the mod-two Lyndon–Hochschild–Serre spectral sequence of the central extension $Z(H) \to H \to F \oplus F$. Its E_2-page is the polynomial algebra on generators of bidegree $(1,0)$, the coordinates on the quotient, and of bidegree $(0,1)$, the coordinates on the center. We extend scalars to F, where the action of the Singer cycle is diagonal. Since the Singer cycle has odd order, taking invariants commutes with taking cohomology, and we can work with the invariant parts of the pages. As bases of the coordinate forms over F we use the Frobenius twists

$$A_i = a^{2^i}, \qquad T_i = v^{2^i}, \qquad Z_i = \ell^{2^i}, \qquad i \in \mathbb{Z}/7. \tag{A.1}$$

They are linearly independent, because a nonzero linearized polynomial of degree at most 64 cannot vanish on all of F. The Singer cycle acts on them with weights $-3 \cdot 2^i$, 2^i and -2^i modulo 127.

The differential $d_2 Z_i$ is the i-th Frobenius component of the extension cocycle $(d, v) \mapsto dv^2 + (dv)^{64}$ from (4.9), which gives

$$d_2 Z_i = e_i = A_i T_{i+1} + A_{i-1} T_{i-1}. \tag{A.2}$$

Enumerating all monomials of total degree at most three and keeping those of total weight zero modulo 127, one finds that the invariant parts of the E_2-page in positive total degree at most three are spanned by (indices modulo seven)

$$\begin{aligned}
&(1,1): \quad T_i Z_i, \\
&(3,0): \quad P_i = A_i T_i T_{i+1}, \\
&(2,1): \quad U_i = A_i T_{i+2} Z_i, \quad V_i = T_i^2 Z_{i+1}, \\
&(1,2): \quad W_i = T_{i+1} Z_i^2.
\end{aligned}$$

In particular there are no invariants in bidegrees $(0,2)$ and $(0,3)$.

Degree two. We have $d_2(T_i Z_i) = P_i + P_{i-1}$, and these elements span a six-dimensional space. So the invariant part of $H^2(H, \mathbb{F}_2)$ has dimension at most one. It has dimension exactly one: restoring the constant codeword gives the extension cocycle

$$\eta\big((a,\ell,v),(d,m,u)\big) = \operatorname{tr}(dv^3 + mv), \tag{A.3}$$

which is invariant under the Singer cycle, and its restriction to the subgroup E of Section 4 is the product of the two degree-one classes of $E \cong \mathbb{Z}_2^2$, which is nonzero. Hence restriction to E detects the invariant part of $H^2(H, \mathbb{F}_2)$, as claimed in Section 4.

Degree three. Let $S_i = A_i^2 T_{i+1} T_{i+2}$ and $C_i = A_i T_i^3$. Then

$$d_2 U_i = S_i + A_i A_{i-1} T_{i-1} T_{i+2}, \qquad d_2 V_i = C_i + A_{i+1} T_i^2 T_{i+2}. \tag{A.4}$$

These are 14 independent elements, and the S_i remain independent modulo them, since each relation involving S_i contains its own monomial $A_i A_{i-1} T_{i-1} T_{i+2}$. The W_i are closed

under d_2. The next differential is determined by the transgression of the Steenrod square: $d_3 \operatorname{Sq}^1 z = \operatorname{Sq}^1(d_2 z)$ for a central class z of degree one, modulo d_2-boundaries [95]. Extending Sq^1 linearly over F, we have $\operatorname{Sq}^1 Z_{i+1} = Z_i^2$, and similarly for A and T, so that

$$d_3 W_i = T_{i+1} \operatorname{Sq}^1(e_{i+1}) \equiv S_i + S_{i-1} + C_{i+1} + C_{i-1}. \tag{A.5}$$

The coefficient of S_i shows that a vector in the kernel has constant coefficients, and the sum of all seven images vanishes, so the rank is six. The invariant classes that survive in total degree three are therefore one class in bidegree $(3,0)$ and one in bidegree $(1,2)$. No further differentials can reach or leave them, since the only other potential source, in bidegree $(0,2)$, has no invariants. So the invariant part of $H^3(H, \mathbb{F}_2)$ is two-dimensional. It is spanned by the inflation of $p(a,v) = \operatorname{tr}(av^3) = \sum_i P_i$ and by $\operatorname{Sq}^1 \eta$. The class p restricts nontrivially to the subgroup generated by $g_* = (1,0,1)$, which is an involution in H since $\operatorname{tr}(1) = 1$, and it vanishes on E. The class $\operatorname{Sq}^1 \eta$ restricts to E as $t^2 z + t z^2 \neq 0$, where t, z are the degree-one classes of E.

Degree four. The Bockstein homomorphism of $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{F}_2 \to 0$ maps $H^3(H, \mathbb{F}_2)$ onto the subgroup of elements of order two in $H^4(H, \mathbb{Z})$, also after taking invariants. The class $\operatorname{Sq}^1 \eta$ is the reduction of the integral class $\beta\eta$, so its Bockstein vanishes. The Bockstein of p restricts to the generator of $H^4(\langle g_* \rangle, \mathbb{Z}) \cong \mathbb{Z}/2$. Hence the invariant elements of order two in $H^4(H, \mathbb{Z})$ form a group of order two, detected by restriction to $\langle g_* \rangle$. Since H is a 2-group, $H^4(H, \mathbb{Z})$ is a finite 2-group. If x were an invariant element of order 2^j with $j \geq 2$, then $2^{j-1}x$ would be an invariant element of order two with trivial restriction to $\langle g_* \rangle$, which is impossible. This proves (4.18).

A.2 The lift of the light involutions

Here we justify the statements of Section 4 about the involutions with $\delta_v f = 1$, for the lift chosen in Section 4. The Singer cycle permutes the 254 classes (4.16) in two orbits, with representatives $g_* = g_1(1) = (1,0,1)$ and $g_0(1) = (0,1,1)$, so it suffices to consider these two.

For a subset S of an ordered orthonormal basis, we write γ_S for the ordered product of the corresponding Clifford generators. Two such products multiply to $\pm\gamma_{S \triangle T}$, with sign $(-1)^{\#\{(s,t) \in S \times T \colon s > t\}}$, and permuting the basis changes the sign by the parity of the permutation of the ordered set. This makes all sign computations below finite.

For the sign changes we use the codewords $f_{2^i,0,0}$ and $f_{0,2^i,0}$, $0 \leq i < 7$, as a basis, and lift each to γ_S with S its support. These lifts square to one and commute, because all codewords have weights divisible by eight, so that pairwise intersections of supports are even; the all-ones word lifts to Ω. For the translations we use the orthonormal basis $\varepsilon_\ell = 2^{-7/2} \sum_x (-1)^{\operatorname{tr}(\ell x)} \gamma_x$, $\ell \in F$, in which the translation T_v is the sign change supported on $\{\ell : \operatorname{tr}(\ell v) = 1\}$, and lift in the same way. Let s_0 be either of these two lifts, and C the lift of the Singer cycle. Then

$$s(u) = \prod_{j=0}^{126} C^j \, s_0(C^{-j} u) \, C^{-j} \tag{A.6}$$

is invariant under the Singer cycle, and it is a homomorphism because all factors lie over the same element and differ by signs. By uniqueness, these are the restrictions of the splitting chosen in Section 4. With $\mu = \theta$ in our polynomial basis, a direct computation of the parities shows that the averaging does not change the lifts of the codewords $f_{1,0,0}$ and $f_{0,1,0}$, nor the lift of the translation T_1. The latter agrees with $\hat{T}_1$ in (4.11), since the determinant of the change of basis between the Fourier basis and the coordinate basis, restricted to the relevant 64 vectors, is $+1$.

Now let f be a codeword with $\delta_v f = 1$, so that each pair $\{x, x+v\}$ contains exactly one element of the support of f. In the product of $\gamma_{\mathrm{supp}\, f}$ with $\hat{T}_1$, the scalar component equals 2^{-32} times $(-1)^{\#\{\text{odd labels in the support of } f\}}$. This exponent is even, because the restriction of f to the odd labels is a polynomial of degree at most two in six binary variables, and such a function has even weight. So the scalar component is $+2^{-32}$. On each pair the transformation is a rotation by a quarter turn, and the lift with positive scalar component is conjugate to $\exp(2\pi i\, h_0)$ with $h = (\frac{1}{4})^{64}$. Its twisted sector has the two ground states $e^{\pm h}$ of weight 2. The other lift differs by the central element -1, which shifts h by a vector in the nontrivial coset of D_{64} in $\mathbb{Z}^{64}$. Then one coordinate is at distance $\frac{3}{4}$ rather than $\frac{1}{4}$ from the nearest point of either coset, so that $\rho = 2 + \frac{1}{2}\big((\frac{3}{4})^2 - (\frac{1}{4})^2\big) = \frac{9}{4}$, and the type is $4 \cdot \frac{9}{4} \equiv 1 \bmod 2$.

A.3 The rank computation

The only nontrivial rank needed in the proof of Proposition 4.3 is (4.25). For $r = 1$, order the entries $M(\theta^i, \theta^j)$, $0 \le i, j < 7$, row by row as coordinates $0, \dots, 48$, and compute the 15 rows of ℓ_1, L_θ and $L_{\theta+1}$ by evaluating each bilinear expression on the elementary matrices. The 15 columns

$$0, 28, 29, 31, 32, 33, 35, 36, 42, 43, 44, 45, 46, 47, 48 \tag{A.7}$$

give an invertible minor. Encoding each row in hexadecimal, with the first of these columns as the least significant bit, the rows are

```
0001  0284  0480  0a08  1010
20a0  4000  03c7  03c1  0586
0fc7  18c1  30c6  6007  40c0
```

The integer determinant of this binary matrix is $+1$, so the minor is invertible over $\mathbb{F}_2$. Rescaling M by the field multiplications $a \mapsto r^{-3}a$ and $v \mapsto rv$ shows that the rank is the same for every $r \neq 0$.

B The Barnes–Wall orbifold: anomaly and torsion

B.1 The cyclic kernel

In this appendix we prove Lemma 3.3. We write $A = P/\langle z \rangle \cong \mathbb{F}_2^{2m}$ with $m = 7$, with coordinates x_i, y_i and quadratic form $q = \sum_i x_i y_i$.

The inflation image. By the Lyndon–Hochschild–Serre spectral sequence [58], the image X of the inflation map $H^4(A,\mathbb{Z}) \to H^4(P,\mathbb{Z})$ fits into an exact sequence $0 \to X \to H^4(P,\mathbb{Z}) \to \mathbb{F}_2 \to 0$, and

$$X = H^4(A,\mathbb{Z})/\{\beta(fq) : f \in A^*\}. \tag{B.1}$$

By (3.4), X is the subgroup of elements of order at most two.

Classes in X. Evaluating cyclic types on the subgroups of order two identifies $H^4(A,\mathbb{Z})$ with the space of Boolean functions on A given by polynomials of degree one, two and three without constant term: the classes $\beta(v_i^3)$, $\beta(v_i)\beta(v_j)$ and $\beta(v_iv_jv_k)$, where the v_i run over the $2m$ coordinates, evaluate to v_i, v_iv_j and $v_iv_jv_k$. These functions are linearly independent, and their number $2m + \binom{2m}{2} + \binom{2m}{3} = 469$ is the dimension of $H^4(A,\mathbb{Z})$. Inflated classes restrict trivially to the cyclic subgroups of order four of P, since such a restriction factors through the map $H^4(\mathbb{Z}_2,\mathbb{Z}) \to H^4(\mathbb{Z}_4,\mathbb{Z})$ induced by $\mathbb{Z}_4 \to \mathbb{Z}_2$, which sends the generator u^2 to $(2t)^2 = 4t^2 = 0$. The cyclic subgroups of order two that are not central are generated by involutions over the nonzero singular vectors of A. So an element of X restricts trivially to all cyclic subgroups if and only if the corresponding polynomial vanishes on the quadric $q = 0$.

Lemma B.1. *Let $m \geq 3$, and let Π be a squarefree Boolean polynomial of degree at most three in the variables x_i, y_i, $1 \leq i \leq m$, with no constant term. Then Π vanishes at all points with $q = 0$ if and only if*

$$\Pi = q\left(c + \sum_i \lambda_i x_i + \sum_i \mu_i y_i\right) \tag{B.2}$$

in the Boolean ring, for some $c, \lambda_i, \mu_i \in \mathbb{F}_2$. The $2m+1$ polynomials q, qx_i, qy_i are linearly independent.

Proof. Evaluating at points with all $y_i = 0$, or all $x_i = 0$, shows that Π has no terms involving only x's or only y's. Evaluating at $x = e_i$, $y = e_j$ with $i \neq j$ removes the mixed quadratic terms x_iy_j with $i \neq j$, and evaluating at $x = e_i + e_j$, $y = e_k$ for distinct i, j, k, and at the transposed points, removes the cubic terms without a pair x_iy_i. So, with $t_i = x_iy_i$,

$$\Pi = \sum_i t_i\Big(a_i + \sum_{j\neq i} b_{ij}x_j + \sum_{j\neq i} c_{ij}y_j\Big). \tag{B.3}$$

Evaluating at the singular point $x = y = e_i + e_j$ gives $a_i + a_j + b_{ij} + b_{ji} + c_{ij} + c_{ji} = 0$. Adding x_k with $k \notin \{i,j\}$ keeps the point singular and gives $b_{ik} = b_{jk}$, so $b_{ik} = \lambda_k$ does not depend on i; similarly $c_{ik} = \mu_k$. The previous equation then says that $a_i + \lambda_i + \mu_i = c$ does not depend on i. Using $t_ix_i = t_iy_i = t_i$ in the Boolean ring, this gives the stated form. Conversely, such a polynomial vanishes on $q = 0$. If it vanishes identically, its cubic coefficients force $\lambda_i = \mu_i = 0$, and then its quadratic coefficients force $c = 0$. □

The inflation relations $\beta(fq)$, $f \in A^*$, evaluate to the functions fq, which span the $2m$-dimensional space of the qx_i and qy_i. Hence the elements of X that restrict trivially to all cyclic subgroups form the line spanned by the image of q, which is η.

Classes outside X. Let $a \notin X$. Then a has order four, and $2a$ is a nonzero element of X. It restricts trivially to every cyclic subgroup: to those of order two because $H^4(\mathbb{Z}_2, \mathbb{Z})$ is killed by two, and to those of order four because every element of X does. By the previous paragraph $2a = \eta$; in particular η is twice a generator of the $\mathbb{Z}/4$ summand in (3.4). By (3.7), the restriction of a to $K \cong Q_8$ is then $2u$ or $6u$, and its restriction to a cyclic subgroup of order four in K is nonzero, since u restricts to a generator of $H^4(\mathbb{Z}_4, \mathbb{Z})$. So no class outside X restricts trivially to all cyclic subgroups, which completes the proof of Lemma 3.3.

B.2 Lattice coordinates and lifts

Index the basis (3.1) by integers $0 \le i < 128$ through the binary representation of I, let $\mathrm{wt}(i)$ be the number of binary digits equal to one, and let $p(i) = (7 - \mathrm{wt}(i)) \bmod 2$. The Gram matrix of [32] is

$$G_{ij} = 2^{(p(i)+p(j)+6-\mathrm{wt}(i\oplus j))/2}, \tag{B.4}$$

where $\oplus$ is the bitwise sum. The cocycle used to define the KRR lifts is $\varepsilon(\alpha, \beta) = (-1)^{\alpha H \beta^T}$, where H is the lower half of G: $H_{ij} = G_{ij}$ for $i > j$, $H_{ii} = G_{ii}/2$, and $H_{ij} = 0$ for $i < j$. Acting on row vectors, the Pauli generators are given by

$$w_i X_b = \begin{cases} -w_i + 2^{p(i)} w_{i+2^b}, & i[b] = 0, \\ w_i, & i[b] = 1, \end{cases} \qquad w_i Z_b = \begin{cases} w_i, & i[b] = 0, \\ -w_i + 2^{p(i)} w_{i-2^b}, & i[b] = 1, \end{cases} \tag{B.5}$$

where $i[b]$ is the binary digit of i at position b, and position 0 is the least significant one.

The matrix R. Modulo two, write each Pauli generator as $1 + F_j$. The formulas above show that F_j maps the span of the basis vectors with $p = 0$ into the span of those with $p = 1$, and annihilates the latter. So $F_j F_k \equiv 0$ for all j, k, and hence $1 + I \equiv F_{X_0} + F_{Z_0}$ and $X_2 + Z_2 \equiv F_{X_2} + F_{Z_2}$ have product zero modulo two. So R is an integral matrix. It is orthogonal, and $R^{-1} = R^7$ is integral as well, so R is an automorphism of BW_{128}.

Preservation of the cocycle. Each of $R, S, I, X_0, Z_0, X_1, Z_1$ changes only the three lowest binary digits, so it is block diagonal with respect to the 16 blocks of 8 consecutive basis vectors. Within a block $\mathrm{wt}(i \oplus j) \le 3$, so that 4 divides G_{ij}, while $G_{ii} \in \{8, 16\}$; hence the diagonal blocks of H are even. The off-diagonal blocks of H are those of G or zero. For a block-diagonal isometry M this implies $MHM^T \equiv H \bmod 2$. Consequently $e^\alpha \mapsto e^{\alpha M}$, together with the induced action on the oscillators, is an automorphism of V without additional phases, and these automorphisms multiply like the matrices. For X_0, Z_0, X_1, Z_1 they coincide with the KRR lifts. This proves (3.9).

Standardness. The KRR lift of an element $g \in P$ with matrix M_g acts on e^α as $e^{\alpha M_g}$ times a sign $(-1)^{q_g(\alpha)}$, where q_g is an explicit quadratic function whose polarization is $(\alpha, \beta) \mapsto \alpha(H - M_g H M_g^T)\beta^T \bmod 2$. The lift is standard if q_g vanishes on the fixed lattice of M_g. Since the polarization vanishes on the kernel of $M_g - 1$ modulo two, q_g is

linear there, and it suffices to check that it vanishes on a basis of this kernel. We have checked this for all involutions in P. Elements of order four and the central element z have no fixed vectors, so their lifts are automatically standard.

B.3 From local to global

Here we prove step (iii) in the proof of Proposition 6.2. Let A be a nondegenerate quadratic space over $\mathbb{F}_2$ of plus type and dimension $2m \geq 6$, with polar form b. Suppose that for every nonzero $v \in A$ we are given a linear form ℓ_v on $v^\perp$ with $\ell_v(v) = 0$, such that (6.4) holds. We show that there is an alternating form Λ with $\ell_v(w) = \Lambda(v, w)$ whenever $b(v, w) = 0$, unique up to adding a multiple of b.

For $m = 3$ this is a finite problem in linear algebra. There are 63 nonzero vectors, and the forms ℓ_v are described by 378 coordinates, subject to 63 conditions that make them well defined on $v^\perp$ and 63 conditions $\ell_v(v) = 0$; the reciprocity conditions (6.4) add 420 linear equations. The solution space has dimension 14, and it is spanned by the forms $\ell_v = \Lambda(v, \cdot)$ obtained from the 15-dimensional space of alternating forms Λ, modulo b.

For general m, note first that any pair of orthogonal vectors lies in a nondegenerate subspace of dimension six and plus type. By the case $m = 3$, the reciprocity $\ell_v(w) = \ell_w(v)$ therefore holds for all orthogonal pairs, not only for those with $q(v) \neq q(w)$. Choose a hyperbolic basis e_i, f_i of A. For orthogonal pairs of basis vectors, define Λ by these reciprocal values, and let $d_i = \Lambda(e_i, f_i)$, which is still to be determined. Let

$$C_{ij} = \ell_{e_i+e_j}(f_i + f_j) + \Lambda(e_i, f_j) + \Lambda(e_j, f_i). \tag{B.6}$$

Applying the case $m = 3$ to the span of three hyperbolic pairs gives $C_{ij} = d_i + d_j$, and hence $C_{ij} + C_{jk} + C_{ik} = 0$. We may therefore set $d_1 = 0$ and $d_i = C_{1i}$. The resulting form Λ agrees with the ℓ_v, up to a multiple of b, on every six-dimensional subspace spanned by three hyperbolic pairs of the basis. If u has at most two nonzero coordinates, then $u^\perp$ is spanned by vectors each of which lies, together with u, in such a subspace; hence $\ell_u = \Lambda(u, \cdot)$. Finally, for arbitrary v, the vectors with at most two nonzero coordinates span $v^\perp$, and reciprocity gives $\ell_v(u) = \ell_u(v) = \Lambda(v, u)$. The only freedom in the construction is to replace all d_i by $d_i + 1$, which adds b to Λ.

C From the involution centralizer to $E_8(2)$

In this appendix we complete Step 5 of the proof of Theorem 6.4. Let $\mathcal{A} = \mathrm{Aut}(W)$, which is finite by Step 4, let t be the involution of Step 3, and let $C = C_{\mathcal{A}}(t) \cong 2_+^{1+56}{:}E_7(2)$ and $Q = O_2(C) \cong 2_+^{1+56}$. We use two facts. First, $E_7(2)$ acts irreducibly on $Q/Z(Q) \cong \mathbb{F}_2^{56}$, which implies that C is perfect, that $C_C(Q) = Z(Q) = \langle t \rangle$, and that the normal subgroups of C are 1, $\langle t \rangle$, Q and C. Second, the three quantum symmetries t, s, st of Step 3 commute and are conjugate in $\mathcal{A}$; in particular $s \in C \setminus \langle t \rangle$.

The Sylow 2-subgroup. Let S be a Sylow 2-subgroup of C. It contains Q, and its center centralizes Q, so $Z(S) = \langle t \rangle$. If S were not a Sylow 2-subgroup of $\mathcal{A}$, then, since a proper subgroup of a 2-group is properly contained in its normalizer, $N_{\mathcal{A}}(S)$ would contain a 2-group strictly larger than S. But $N_{\mathcal{A}}(S)$ fixes the unique central involution t of S, so it lies in C, a contradiction. Hence $|\mathcal{A}|_2 = |C|_2 = 2^{57} \cdot 2^{63} = 2^{120}$.

Simplicity. Suppose $M \neq 1$ is a normal 2-subgroup of $\mathcal{A}$. Then $M \subseteq S$, and M meets $Z(S)$ nontrivially, so $t \in M$, and also $s \in M$ by conjugacy. Since $M = M \cap C$ is a normal 2-subgroup of C containing s, we have $M = Q$, and then $\langle t \rangle = Z(M)$ is normal in $\mathcal{A}$, contradicting the fact that t is conjugate to $s \neq t$. Next suppose $M \neq 1$ is a normal subgroup of odd order. Since C has no nontrivial normal subgroup of odd order, $C_M(t) = 1$, so t acts on M without nontrivial fixed points, and the same holds for s and st. A fixed-point-free automorphism of order two of a finite group makes it abelian and acts by inversion; but then st would act trivially. So $\mathcal{A}$ has no nontrivial solvable normal subgroup.

Let M be a minimal normal subgroup of $\mathcal{A}$, so $M \cong T_1^r$ for a nonabelian simple group T_1. Since $C_M(t)$ is normal in C, there are four cases. If $C_M(t) = 1$, then t acts on M without fixed points and M is abelian, a contradiction. If $C_M(t) = \langle t \rangle$, then a Sylow 2-subgroup of M is $\langle t \rangle$, by the same normalizer argument as above, and Burnside's transfer theorem shows that M has a normal 2-complement, contradicting that M is perfect. If $C_M(t) = Q$, then Q is a Sylow 2-subgroup of M, and since $Z(Q)$ has order two we must have $r = 1$. Then $C_{\mathcal{A}}(M) \subseteq C_C(Q) = \langle t \rangle \subseteq M$, so $C_{\mathcal{A}}(M) = 1$ and $\mathcal{A}$ embeds into $\mathrm{Aut}(M)$. Since $\mathrm{Out}(M)$ is solvable and C is perfect, this gives $C \subseteq M$, a contradiction. In the remaining case $C \subseteq M$, so S is a Sylow 2-subgroup of M, and the Frattini argument gives $\mathcal{A} = M\, N_{\mathcal{A}}(S)$. Since $N_{\mathcal{A}}(S) \subseteq C \subseteq M$, we find $\mathcal{A} = M$, and minimality forces $r = 1$. Hence $\mathcal{A}$ is simple.

Recognition. We now know that $\mathcal{A}$ is a finite simple group with $|\mathcal{A}|_2 = 2^{120}$, whose order is divisible by 19, 73 and 127, since $2^{18} - 1 = 3^3 \cdot 7 \cdot 19 \cdot 73$ and $2^{14} - 1 = 3 \cdot 43 \cdot 127$ divide $|E_7(2)|$. We use the classification of finite simple groups. The 2-part of $|A_n|$ is $2^{n - s_2(n) - 1}$, where $s_2(n)$ is the number of binary digits of n equal to one; it jumps from 2^{119} at $n = 126, 127$ to 2^{126} at $n = 128$, so it never equals 2^{120}. The orders of the sporadic groups have 2-parts at most 2^{46}. For a simple group L of Lie type in odd characteristic, the centralizer $C_L(x)$ of an involution x is solvable modulo its layer, *i.e.* the subgroup generated by its quasisimple subnormal subgroups. To see this, write $L = \mathbf{G}^F/Z$ with $\mathbf{G}$ simply connected and F a Steinberg endomorphism, and lift x to $u \in \mathbf{G}^F$. Since $u^2 \in Z$ has order prime to the characteristic, u is semisimple, so $\mathbf{H} = C_{\mathbf{G}}(u)$ is connected and reductive [96]. The commutator with u embeds the inverse image of $C_L(x)$ modulo $\mathbf{H}^F$ into Z, and $\mathbf{H}^F$ modulo the image of the rational points of the simply connected cover of $[\mathbf{H}, \mathbf{H}]$ is abelian: it is built from the rational points of a torus and a Galois cohomology group of the central kernel. That image is a product of quasisimple groups, up to the solvable exception $SL_2(3)$ and the group ${}^2G_2(3)$, which has a normal subgroup $PSL_2(8)$ of index three [97]. Our centralizer C has trivial layer but the nonsolvable

quotient $C/Q \cong E_7(2)$, so $\mathcal{A}$ is not of this type. Finally, for a group of Lie type over $\mathbb{F}_q$ with $q = 2^f$, the 2-part of the order is q^N, where N is the number of positive roots. The condition $fN = 120$ leaves the cases in Table 8, and in all of them except $E_8(2)$ one of the primes 19, 73, 127 does not divide the order. The Suzuki and Ree groups would require even values of f, which do not occur. Hence $\mathcal{A} \cong E_8(2)$.

type	f	missing prime	type	f	missing prime
A_1	120	19	D_5, ${}^2D_5$	6	127
A_2, ${}^2A_2$	40	19	D_6, ${}^2D_6$	4	19
$B_2 = C_2$	30	19	A_{15}	1	19
A_3, ${}^2A_3$	20	19	${}^2A_{15}$	1	73
A_4	12	127	G_2	20	19
${}^2A_4$	12	19	F_4	5	19
D_4, ${}^2D_4$	10	19	${}^3D_4$	10	19
A_5, ${}^2A_5$	8	19	E_8	1	none

Table 8: Groups of Lie type in characteristic two whose order has 2-part 2^{120}, and a prime among 19, 73, 127 that does not divide their order.

D Unitarity

In this appendix we give more details on the unitarity of the theories of Sections 3, 4 and 5. In all cases the final step is the theorem of Carpi, Gaudio, Giorgetti and Hillier [35, Thm. 4.7]: a simple CFT-type extension of a regular, completely unitary VOA $\mathcal{C}$ admits a unitary structure extending that of $\mathcal{C}$. Its proof conjugates the multiplication of the extension by a positive invertible map Σ to a unitary one. Pulling back the positive form along Σ gives a positive invariant form for the original multiplication, so the unitary structure is on the theory that we constructed, with the operator products selected by our choice of discrete torsion.

D.1 Orbifolds by groups with abelian central quotient

The theories at $c = 128$ and $c = 64$ are covered by the following result.

Proposition D.1. *Let V be a unitary holomorphic VOA, and let H be a finite group of unitary automorphisms of V whose anomaly vanishes. Assume that*

(a) for every $h \in H$ the h-twisted V-module is unitarizable,

(b) the quotient $K = H/Z(H)$ is abelian, and

(c) the discrete torsion ϵ can be represented by a 2-cocycle ξ that is inflated from K.

Then the orbifold $W = (V/H)_\epsilon$ is unitary.

Proof. Let $\mathcal{C} = V^H$. By (b) the group H is nilpotent, so $\mathcal{C}$ is regular [39, Cor. 5.25], and since the anomaly vanishes, $\mathrm{Rep}(\mathcal{C}) \simeq \mathcal{Z}(\mathrm{Vec}_H)$ by (2.7).

1. *Every irreducible $\mathcal{C}$-module is unitarizable.* Every irreducible $\mathcal{C}$-module occurs in a twisted module of V [60, Thm. 3.3], so this follows from (a) by restricting the form [62, Prop. 10.9].

2. *The central sectors.* By (c) we have $\xi(z,g) = \xi(g,z) = 1$ for $z \in Z(H)$ and $g \in H$. Hence the sectors of W twisted by central elements form a subalgebra B, which is a simple current extension of $\mathcal{C}$ by $Z(H)$, with $\mathcal{C} = B^{\widehat{Z(H)}}$. It is strongly rational [98, Thm. 4.14] and carries a unitary structure extending that of $\mathcal{C}$ [62, Thm. 11.5]. Under $\mathrm{Rep}(\mathcal{C}) \simeq \mathcal{Z}(\mathrm{Vec}_H)$, the algebra B corresponds to the algebra $A_Z = \bigoplus_{z\in Z(H)}[z,1]$ of central fluxes with trivial charge, and $\mathrm{Rep}(B)$ is braided equivalent to the category of local A_Z-modules [44,99]. Such a module is an H-graded, H-equivariant vector space with isomorphisms t_z from degree g to degree zg satisfying $t_z t_{z'} = t_{zz'}$, and locality means that $Z(H)$ acts trivially. Identifying x with $t_z x$ gives a K-graded, K-equivariant vector space, and the relative tensor product and the braiding descend to those of $\mathcal{Z}(\mathrm{Vec}_K)$. So $\mathrm{Rep}(B) \simeq \mathcal{Z}(\mathrm{Vec}_K)$, and since K is abelian, all irreducible B-modules are simple currents.

3. *Modules of simple current extensions.* Every irreducible B-module is unitarizable. This is a localized form of results of Gui [63, 64, 100]. Their published statements assume more than we know about $\mathcal{C}$, in particular that it is completely unitary, but the parts of their proofs that we use are algebraic, and they need positivity of the canonical forms only on the fusion products that actually occur. Here these always have a simple current as one factor, so they are irreducible, and they are positive by [62, Thm. 8.6]. In particular, the simple currents of B form a unitary ribbon fusion category [62, Prop. 11.4], in which B is a standard Q-system [64, Thm. 2.21]. For an irreducible B-module N, regarded as a $\mathcal{C}$-module, the free module $B \boxtimes N$ carries a canonical positive form. The action of B on it is unitary, because the associators $(J \boxtimes J') \boxtimes X \to J \boxtimes (J' \boxtimes X)$ are unitary for simple currents J, J' and irreducible X: their coefficients are braid coefficients [63, Prop. 7.4], and the arguments of [63, Props. 7.1, 7.2 and 7.5] show that these have modulus one, using orthonormal bases only for fusion products with a simple current factor. The same localization gives the compatibility of the tensor product of morphisms with adjoints [63, (7.24)–(7.27)], and the identification of the categorical adjoint with the adjoint of the vertex operators [64, Thm. 1.11]. Since N is a quotient of $B \boxtimes N$, and the orthogonal projection onto the complement of the kernel commutes with the action of B by the Frobenius property of B [64, Lemma 2.26], the module N inherits a positive form, which is invariant for the vertex operators of B [64, Thm. 2.30].

4. *Complete unitarity.* By step 3, every irreducible B-module is unitarizable. Since all of them are simple currents, the fusion product of any two irreducible modules

is irreducible and unitarizable, and [62, Thm. 8.6] shows that the canonical forms on all fusion products are positive. So B is completely unitary, and [35, Thm. 4.7] applies to $B \subset W$.

□

D.2 The theories at $c = 128$ and $c = 64$

At $c = 128$, every element of P acts on $V = V_{\mathrm{BW}_{128}}$ by a standard lift of a lattice isometry (Appendix B.2). These lifts preserve the positive form of V, and their twisted modules are unitary [61, Sec. 3.2 and Lemma 3.8]; see also [62, Rem. 10.10]. The quotient $P/Z(P) = A \cong \mathbb{F}_2^{14}$ is abelian, and every class in $H^2(P, U(1)) \cong \mathrm{Alt}(A)/\langle b \rangle$ is inflated from A. So Proposition D.1 applies to every orbifold $(V/P)_\epsilon$, and in particular to $W = (V/P)_{\epsilon_{\max}}$.

At $c = 64$, the code words in H have even weight and the translations are even permutations, so H lies in the image of the connected group Spin(128), which acts on $V = V_{D_{64}^+}$ by unitary inner automorphisms. Hence every element of H is conjugate, by a unitary automorphism of V, into a maximal torus, and its twisted module is a shifted lattice module, which is unitary [61, Sec. 3.2 and Lemma 3.8]. The quotient $H/Z(H) \cong F \oplus F$ is abelian, and the cocycle ξ of Section 4.6 is inflated from it, since it does not depend on the central coordinate ℓ. So Proposition D.1 applies to the theories of Proposition 4.3.

D.3 The theory at $c = 96$

Here Proposition D.1 does not apply: the group G has trivial center, and unitarity of the twisted modules of $Z^{\otimes 3}$ is not known a priori. We use conformal nets instead. Let $\mathcal{C} = (Z^{\otimes 3})^G$. This part of the argument uses recent results on the relation between unitary VOAs and conformal nets, some of which are so far only available as preprints [73, 74, 78, 79]; theorem numbers refer to the arXiv versions listed in the bibliography.

1. *The parent.* The theory Z is framed, so it is a unitary extension of $L(\frac{1}{2}, 0)^{\otimes 64}$ [35, Thm. 4.7]. Since $L(\frac{1}{2}, 0)^{\otimes 64}$ satisfies Condition I$^+$ of [73] (Ex. 6.19 and Thm. 6.18(1) there), so does Z (Thm. 6.18(2)). Hence Z is strongly local (Thm. 6.2), and its conformal net $\mathcal{A}_Z$ is completely rational with the vacuum as its only irreducible representation (Thm. 6.14), that is, holomorphic. Since $\mathrm{Aut}(Z)$ is finite, it preserves the positive form [69, Thm. 5.21]. So G acts unitarily on $Z^{\otimes 3}$, and for every subgroup $H \subseteq G$ the fixed-point algebra $(Z^{\otimes 3})^H$ is strongly local, with conformal net $(\mathcal{A}_Z^{\otimes 3})^H$ [69, Cor. 8.2, Thm. 7.1 and Prop. 7.6]. Since H is solvable, $(Z^{\otimes 3})^H$ is also regular [39, Cor. 5.25].

2. *From representations of nets to modules.* Let $H \subseteq G$, and let K be an irreducible representation of $(\mathcal{A}_Z^{\otimes 3})^H$. Since $(Z^{\otimes 3})^H$ is strongly local, it is the unitary VOA associated with this net [74, Thm. 8.9]. The rotation generator L_0 acts on K with discrete spectrum, and the finite-energy vectors of K form a unitary $(Z^{\otimes 3})^H$-module M, a priori with possibly infinite-dimensional weight spaces, on which L_0 is the zero

mode of the conformal vector, and which is characterized by $\pi_I(Y(v,f)) = Y^M(v,f)$ for smeared vertex operators localized in an interval I [74, Thm. 9.5, Cors. 9.6 and 9.7, Rem. 9.8]. The orthogonal complement of a submodule of M is again a submodule, and the orthogonal projection onto it commutes with the closures of the smeared vertex operators, and hence with the net. Since K is irreducible, M is irreducible, and since $(Z^{\otimes 3})^H$ is regular, M is an ordinary module. Inequivalent representations give inequivalent modules, since an isomorphism of irreducible unitary modules is a multiple of a unitary map [101, Prop. 2.19], which then intertwines the representations of the net. On the other hand, every irreducible $(Z^{\otimes 3})^H$-module occurs in a twisted module of $Z^{\otimes 3}$, and is labeled by a conjugacy class $[u]$ of H and an irreducible projective representation of $C_H(u)$ [60, Thm. 3.3], so there are at most $\sum_{[u]} |\operatorname{Irr}(C_H(u))|$ of them.

3. *The anomaly on the net.* The action of G on $\mathcal{A}_Z^{\otimes 3}$ is also anomaly-free. Its anomaly is the associator of the category of twisted sectors (solitons) of $\mathcal{A}_Z^{\otimes 3}$, which has exactly one simple object of each degree $u \in G$ [76, Cor. 4.7], so it restricts to subgroups in the natural way. By (A3) and the injectivity of (2.14) for elementary abelian 2-groups, it therefore suffices to show that it vanishes on each cyclic subgroup $H = \langle h \rangle$ of L and of $\langle g \rangle$. Let n be the order of h. Since $H^2(\mathbb{Z}_n, U(1)) = 0$, the net $(\mathcal{A}_Z^{\otimes 3})^H$ has exactly n^2 irreducible representations, whatever its anomaly [76], and $(Z^{\otimes 3})^H$ has at most n^2 irreducible modules. So the map of step 2 is a bijection, and it preserves L_0. Since H acts on $Z^{\otimes 3}$ without anomaly (Proposition 5.2 and Theorem 5.4), all these modules have conformal weights in $\frac{1}{n}\mathbb{Z}$. By the spin–statistics theorem [75], the statistics phases $e^{2\pi i L_0}$ of all irreducible representations of the net are therefore n-th roots of unity. On the other hand, the representation category of $(\mathcal{A}_Z^{\otimes 3})^H$ is $\mathcal{Z}(\mathrm{Vec}^{\omega}_{\mathbb{Z}_n})$, where ω is the anomaly of H on the net [52, Prop. 2.2.1], and if ω has type t, the representations in the sector twisted by h have statistics phases $e^{2\pi i(t/n^2 + j/n)}$ with $j \in \mathbb{Z}$. Hence $t = 0$.

4. *Strong unitarity.* By step 3, the representation category of $\mathcal{A}_{\mathcal{C}} = (\mathcal{A}_Z^{\otimes 3})^G$ is $\mathcal{Z}(\mathrm{Vec}_G)$ [52, Prop. 2.2.1], so $\mathcal{A}_{\mathcal{C}}$ has exactly $\sum_{[u]} |\operatorname{Irr}(C_G(u))|$ irreducible representations. By step 2, this is at least the number of irreducible $\mathcal{C}$-modules, so the map of step 2 is a bijection: every irreducible $\mathcal{C}$-module is unitarizable, and comes from a representation of $\mathcal{A}_{\mathcal{C}}$.

5. *Complete unitarity.* We use Tener's criterion [77, Thm. 4.4 and Cor. 4.6], for the system of generalized annuli of [101, Def. 3.18]. These annuli act by bounded operators on every positive-energy representation of the Virasoro algebra, and they agree with the corresponding operators of [78], since both are bounded and they agree on finite-energy vectors. The criterion has two requirements. First, $\mathcal{C}$ must have bounded localized vertex operators [77, Defs. 2.15–2.17]: for a pair (B, A) of annuli localized in an interval I, the preimages of the finite-energy vectors under A and B^* are dense [74, Lemmas 4.15 and 7.9], the map $v \otimes \xi \mapsto B\, Y(s^{L_0} v, z) A \xi$ extends to a bounded operator from the Hilbert space tensor product $H_{\mathcal{C}} \otimes H_{\mathcal{C}}$ to

$H_{\mathcal{C}}$ [79, Thm. 3.6 and Cor. A.1], and these operators lie in $\mathcal{A}_{\mathcal{C}}(I)$ [74, Lemma 6.8]. Second, every irreducible $\mathcal{C}$-module M must carry a representation π^M of the net generated by these operators, with $\pi^M_I(B\,Y(v,z)A) = B\,Y^M(v,z)A$ [77, Def. 2.18]. It is obtained by restricting the representation of $\mathcal{A}_{\mathcal{C}}$ from step 4 [74, Thm. 9.5 and Lemma 6.8]. Hence the canonical forms on all fusion products are positive definite [77, Prop. 3.16], $\mathcal{C}$ is completely unitary, and [35, Thm. 4.7] applies to $\mathcal{C} \subset W$.

E Finite computations

Several steps of our arguments rely on finite computations, all of which use exact arithmetic: linear algebra over $\mathbb{F}_2$ or over the integers, and arithmetic in finite fields. The table below lists them, together with their results. We distinguish computations that are *essential* inputs to a proof from those that *check* a statement that is also proved in the text. The code is provided as ancillary files. Their README lists, for each line of the table, the script that performs the computation, together with an independent reimplementation.

E.1 Explicit data for the construction at $c = 96$

The map J_6 of Section 5.2 can be taken to be the $\mathbb{F}_2$-linear map of $\mathbb{F}_{64}$ whose matrix in the basis $1, \alpha, \ldots, \alpha^5$ has the columns $32, 41, 63, 42, 6, 33$, written as binary integers. Writing the Pauli element $P(a,b)$ of Section 5.2 as the integer $a + 32b$, the three quaternion pairs (P_i, Q_i) in (5.10) can be taken to have the labels $(33, 67)$, $(35, 66)$ and $(132, 268)$.

E.2 List of computations

claim	computation and result	role
$c = 64$		
weight enumerator (4.6)	enumeration of the 2^{15} words of $\mathcal{C}$	check
341376 six-element sets in $\mathcal{C}^\perp$	MacWilliams transform of (4.6)	check
App. A.1	invariant monomials and ranks of the differentials over $\mathbb{F}_2$	essential
App. A.2	Clifford sign computations for the lifts (4.11)	essential
$\operatorname{rank}(L_v, \ell_r) = 8$ for $v \neq r$	ranks over $\mathbb{F}_2$ for all pairs	check
(4.25)	ranks over $\mathbb{F}_2$ for all r; certificate in App. A.3	essential
partition function (2.5)	expansion in q	check

claim	computation and result	role
$c = 96$		
properties of J_6	$J_6^2 + J_6 + 1 = 0$, and Σ, $J_6\Sigma$, $J_6^2\Sigma$ partition $\mathbb{F}_{64}^\times$	essential
Proposition 5.2	the cubic vanishes on the 2325 vectors of K with at most three nonzero coordinates	check
the isometry t	t preserves BW_{32}, has order three, characteristic polynomial $(X-1)^8(X^2+X+1)^{12}$ and fixed lattice $\sqrt{3}E_8$	essential
$\mathrm{Tr}_{Z_2}\, t = 22$	6 from the oscillators, 16 from the inversion-twisted ground states	essential
Lemma 5.3	enumeration of the elements of L with three rank-one components	check
$\dim H^2(G, U(1)) = 36$	276 linear equations over $\mathbb{F}_2$	essential
K_v and $\mathrm{rank}\,\Gamma_v$	linear algebra over $\mathbb{F}_2$ for all 567 light fluxes, giving the table in Section 5.6	essential
$\|Q\|^2$	17,541,888	check

claim	computation and result	role
$c = 128$		
standardness of the KRR lifts	the lift of every element of P acts trivially on its fixed lattice	essential
App. B.2	integrality of R, preservation of the cocycle, characteristic polynomial of R	essential
Lemma B.1	ranks over $\mathbb{F}_2$ for $m = 3, 5, 7$	check
Prop. 6.2, step (i)	the centralizers act on the ground states by scalars, for $g = Z_0$ and $g = X_0Z_0$	essential
App. B.3, case $m = 3$	378 coordinates and 420 reciprocity equations; the solution space has dimension 14	essential
Prop. 6.5, step (1)	1792 equations over $\mathbb{F}_2$ in 128 unknowns, of rank 64, solved by the trivial correction	essential
Theorem 6.4, Step 6	the symmetries of the four-copy construction fix a subspace of W_6 of dimension at most 76,430,266	essential
Proposition 7.1	the congruence $\phi \equiv 1 \bmod 331$ from the Hecke algebra action	essential
Table 8	orders of the groups of Lie type	essential

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