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Three isolated primes in the ternary Goldbach problem

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Abstract

Every sufficiently large odd integer N is the sum of three distinct primes, each at distance at least $c \log N \, (\log\log N)^{1/325565}$ from every other prime. One of the three has an isolation radius of order $\log N \, \log_2 N \, \log_4 N / \log_3 N$. This answers the question of Maier and Rassias with the smaller growing function allowed in their remark. For all but $O(X^{1-\beta})$ even integers in $[X, 2X]$, there are at least cX^{ν} binary representations with both primes isolated at the smaller radius, for absolute $\beta, \nu > 0$. The proof keeps the centres of a two-interval covering coprime to its modulus and applies classical Goldbach theorems in arithmetic progressions.

Maier and Rassias ask whether every sufficiently large odd integer is a sum of three isolated primes, allowing a smaller growing function than in their two-isolated-prime theorem [10, remark after Theorem 1.3]. We obtain one prime at the full radius and two at a common superlogarithmic radius. The binary argument also gives a power-sized family of isolated representations for almost all even totals. Write $\log_j$ for the jth iterated natural logarithm and set

$$F(N) = \log N \, \frac{\log_2 N \log_4 N}{\log_3 N}, \qquad B(N) = \log N \, (\log_2 N)^{1/325565}.$$

The full radius $F(N)$ is the order obtained by Ford, Green, Konyagin, Maynard and Tao for large gaps between primes [4]. Their covering bound (1.2) supplies the full-radius class in Appendix A. Maier and Rassias use the same order in Definition 1.2 of [10]. A prime is isolated to radius h here if every integer at nonzero distance at most h from it is composite.

Theorem 1. *There are positive constants c, β, ν with the following unconditional assertions for sufficiently large integers.*

1. *Every odd N is a sum of three distinct primes, one isolated to radius $cF(N)$ and the other two to radius $cB(N)$.*
2. *Outside $O(X^{1-\beta})$ even integers $m \in [X, 2X]$, there are $\gg X^{\nu}$ ordered representations $m = p + p'$ with both ordinary primes isolated to radius $cB(m)$.*
3. *Every odd N is a sum of three distinct primes, two isolated to radius $cF(N)$ and the third to radius $c \log N$.*

Every prime in these statements may be taken larger than N^{ν} or m^{ν} after decreasing ν. The moduli which certify compositeness of neighbours use prime factors $O((\log N)^2)$.

Assertion 1 answers the three-prime question in the Remark after Maier–Rassias Theorem 1.3 with $g(n) \asymp (\log_2 n)^{1/325565}$ in their Definition 1.1. In assertion 3 the two larger radii have the order in their Definition 1.2. Three summands at the full radius of their Definition 1.2 remain open.

The smaller radius B comes from Gabdullin and Radomskii [5]. They express each large integer as a sum of two integers at distance at least $(\log N)(\log_2 N)^{C(1/2)-o(1)}$ from every prime, with $C(1/2) >$

1/325565. Their covered intervals include the two centres. Radomskii has extended the covering to composite polynomial values and to integers outside the sums of two squares [11, 12]. Appendix A keeps both centres coprime to the covering modulus, allowing them to be primes.

At the scale $c \log p$, an upper-bound sieve for prime pairs and Gallagher's mean of the singular series [6] show that the primes with no other prime within $c \log p$ have lower relative density $1 - O(c)$ in each fixed reduced progression. The implied constant is absolute: for a fixed progression one averages the conditioned pair singular series over the shifts; at primes dividing the modulus the allowed shifts have density $1 - 1/\ell$, cancelling their factor $(1 - 1/\ell)^{-1}$. For small c the density theorems of Shao [13] and Alsetri–Shao [1] give three such summands for every sufficiently large odd integer and two for almost all even integers. Here the radii F and B exceed any fixed multiple of $\log N$ as N tends to infinity.

Published progression inputs

M.-C. Liu and T. Zhan, *The Goldbach problem with primes in arithmetic progressions* [9], give absolute constants $a, d > 0$ for the following statements. Every sufficiently large odd N is a sum of primes in any three prescribed reduced classes modulo $W \le N^a$ whose sum is N mod W. Y. Wang [14, p. 218] gives these quantifiers explicitly. For binary Goldbach at a fixed $W \le X^a$ and two reduced classes u, v, the number of even $m \le X$, $m \equiv u + v \pmod W$, missing the prescribed representation is $\ll X^{1-d}/\varphi(W)$. This is Bauer–Wang, (1.2), with attribution to M.-C. Liu and T. Zhan. The earlier analytic theorem is a direct prior result for simultaneity of the binary classes at a smaller power. Indeed summing over $W \le (2X)^\omega$ and its reduced pairs costs at most $\sum_{W \le (2X)^\omega} \varphi(W) \ll X^{2\omega}$. Thus for $\omega < \min(a, d/5)$ there is one omitted set of $O(X^{1-d/2})$ even $m \le 2X$ outside which every admissible pair of classes at every one of those moduli has a prime pair. For the first argument one can alternatively use the three-prime progression result of Zhang and Wang [15], as recorded after Theorem 0 in Kong, Song, Ma and Wang [7]. J. Liu and T. Zhan [8] give the earlier almost-all-moduli result used in Section 3 of Maier and Rassias.

Keeping the primes large and distinct. Let $R \asymp X^{2\nu}$ be an auxiliary prime much larger than every factor of the covering modulus W. Keep $WR \le X^\omega$ (or N^a in the ternary application). Among the R possible first residues modulo R in a compatible pair of classes, $\gg R$ give both residues in $[R/10, R)$ and unequal. At most two short intervals and one residue are disallowed. For three slots there are $\gg R^2$ compatible triples with all three residues in that range and distinct, by the same elementary union bound on ordered pairs. If one prime was chosen in advance, exclude its residue too. Each solution in the lifted classes has every newly chosen prime $\ge R/10$; different first residues give distinct representations. The auxiliary prime serves only to select the sizes and separate solutions.

Representations

For the binary assertion take $x = \lambda \log X$ with λ a sufficiently small fixed constant. Apply item 1 of the punctured-cover theorem (Theorem 2) to m, getting a squarefree W of logarithm $O(x)$ and a class u such that $u, m - u$ are units and both punctured intervals of radius $\gg B(m)$ are covered. The absolute implied constant in $\log W$ is independent of λ. Choose λ, then ν, so $WR \le X^\omega$. Apply the binary progression input on the common good set to the lifted classes. Every neighbouring value exceeds all its possible covering divisors, hence is composite. This proves assertion 2 with $\beta = d/2$, including the multiplicity.

For assertion 3 take $x = \lambda \log N$ in item 2 of the same covering theorem. The residues $u, u, N - 2u$ are admissible and have the claimed radii. Apply the ternary progression theorem with the auxiliary

prime. This proves assertion 3, including distinctness.

The statement with two full radii and an unrestricted third prime follows immediately from assertion 3, which also gives logarithmic isolation of the third prime. Maier–Rassias Theorem 1.3 assumes the generalized Riemann hypothesis for Dirichlet L-functions. The announced proof of this hypothesis by Alpöge [16], if confirmed, would also give the two-isolated-prime statement directly from their theorem.

For assertion 1 write $T = N/12$, $I = [3T, 4T]$ and use the common binary good set at $X = N/2$. Then $m = N - p$ lies in $[2N/3, 3N/4] \subset [X, 2X]$ for every $p \in I$. Use the class $u = b$ of Theorem 2, item 2, with the excluded primes specified below. Only the covered interval about u is used. Fix a sufficiently small exponent e from Banks–Freiberg–Maynard, Lemma 4.1 and Theorem 4.2. Apply their Theorem 4.2 at both $3T$ and $4T$. Let $\mathcal{Z}$ consist of the entries different from one among $Z_{(3T)^{2e}}$ and $Z_{(4T)^{2e}}$ from their (4.8). By their (4.7) each is a prime $\gg_e \log_2 N$, as required by our cover. Choose λ so small that $W < T^e$ and $W < N^{d/8}$. Its largest prime divisor is $O((\log N)^2)$. Subtracting the estimates for the single $q = W$ term of their (4.10) at $4T$ and $3T$, with base modulus $q_0 = W$ and the prime indicator, gives

$$\#\{p \in I \cap \mathbb{P} : p \equiv u \ (W)\} \gg \frac{N}{\varphi(W) \log N}.$$

Rounding T to an integer affects the interval endpoints by $O(1)$. All these primes are isolated to radius $\gg F(N)$. The even remainders $m = N - p$ are distinct and $m \asymp N$. Only $O(N^{1-d/2})$ fail the common binary progression statement. There are therefore many choices of p for which the preceding binary construction applies with the smaller covered radius at each of the other two primes. Avoid $p \bmod R$ there to ensure distinctness. This proves the theorem.

A A covering with two protected centres

Write $P(x) = \prod_{p \le x} p$ and use iterated natural logarithms. A residue $b \bmod W$ has a covered punctured interval of radius h if $(b, W) = 1$ and $(b + j, W) > 1$ whenever $0 < |j| \le h$. Every prime larger than $P^+(W) + h$ in that class then has only composite neighbours in the interval. We use two published covers whose initial centres need not be units.

Theorem 2. *Fix $K \ge 1$ and $c_Z > 0$. For sufficiently large x, fix an integer N with $e^{x/K} \le N \le e^{Kx}$ and let $\mathcal{Z}$ be a set of at most two specified primes, each at least $c_Z \log x$. For each of the following assertions take its data first, including k in item 3. For those data there exist a squarefree modulus W and a class $b \bmod W$ satisfying that assertion, with $(W, Z) = 1$ for each $Z \in \mathcal{Z}$, $\log W = O_{c_Z}(x)$ and $P^+(W) \le 3x^2$. The choices of W and b may depend on the item, N, $\mathcal{Z}$, and, in item 3, k.*

1. *If N is even, both $b, N - b$ have covered punctured intervals of common radius $c_1 x (\log x)^{1/325565}$.*
2. *Take $s = 2$ for odd N or $s = 1$ for even N. Then b has a covered punctured interval of radius $c_2 x \log x \log_3 x / \log_2 x$, and $N - sb$ has one of radius $c_3 x$.*
3. *Given any even k up to $c_4 x \log x \log_3 x / \log_2 x$, both $b, b + k$ are units and every $b + j$ with $0 < j < k$ has a common factor with W. One can also cover the two exterior intervals of length $c_4 x \log x \log_3 x / \log_2 x$ at the endpoints.*

The constant in the logarithmic modulus bound is independent of K. The construction and bound also work in item 1 for $\log N = x^{1+o(1)}$.

The two inputs are Gabdullin, (2.2) and the completion after (2.1), also in Gabdullin–Radomskii, and Ford–Green–Konyagin–Maynard–Tao, (1.2). The first constructs, for each prescribed even N, a class $b_0 \bmod P(Cx)$ with every integer within distance $y = \lfloor x(\log x)^{1/325565} \rfloor$ of b_0 and $N - b_0$ covered by a prime $\le Cx$. The covering statement is uniform in the integer N. In passing from their union of surviving offsets to two lists one may simply use twice as many primes in the final interval $(x/2, 3x/2]$, so one can take $C = 3/2$. The second input gives one such covered interval about some b_0, with $y \asymp x \log x \log_3 x / \log_2 x$. Translate the interval to put its centre at zero in offset coordinates. In each use $y \ll x \log x$.

Excluded primes. Delete the primes of $\mathcal{Z}$ that occur in the initial cover. In each initial interval a newly exposed offset belongs to a covering class modulo at least one deleted prime Z and has escaped all retained primes. Count these offsets for each Z by a union bound, including offsets originally covered by both deleted primes. For $Z \le \sqrt{y}$, apply the elementary linear upper sieve on that progression to a small power of y/Z, omitting all of $\mathcal{Z}$. All other small primes exclude one class, with remainder $O(1)$ per squarefree modulus in the progression parameter. Omitting at most two primes $\ge c_Z \log x$ changes the sieve product by a bounded factor. This gives $O(y/(Z \log x)) = O(x/\log x)$ offsets. For $Z > \sqrt{y}$, the progression contains $O(\sqrt{y} + 1)$ offsets. Summing over the at most two deleted primes and over the initial intervals costs $O(x/\log x)$, with zero cost for empty $\mathcal{Z}$.

Choosing the centres. Move b_0 to $b = b_0 + t$ with $|t| \le y/4$. The desired second unit is $N - b$, $N - sb$, or $b + k$ respectively. In the last case take $k \le y/8$. Put $z = (\log x)^{10}$. These two affine forms of t have at most two roots at each prime, with fewer roots than the prime everywhere, by the parity assumptions. At 2 the second form in item 2 is odd for all odd b. The dimension-two lower sieve shows that $\gg y/(\log z)^2$ shifts avoid all those roots below z. Indeed the local product is at least a constant times $(\log z)^{-2}$; the level $y^{1/3}$ has total CRT rounding $O(y^{1/3+o(1)})$ and its logarithmic ratio to z tends to infinity. The constants here are absolute.

For each surviving t, delete all the remaining primes $l \le Cx$ which divide a protected value. Assign expense $y/l + 1$ to each deleted prime, sufficient up to an absolute constant for both lists of neighbours. On summing over *all* shifts, the total expense at primes $l > z$ is

$$\ll \sum_{z < l \le Cx} (y/l + 1)^2 \ll y^2/z + y \log\log(3x) + x. \tag{1}$$

The leading coefficients are units at each such l. Dividing by the number of sifted shifts gives $o(x/\log x)$ for some admissible t. For this choice retain all other congruences with their original translated residues. In case 1 both covered intervals still contain the radius $y/2$ about the new centres. In case 2 this holds for the first, and in case 3 every desired point lies inside the original interval if c_4 is small.

In case 2 we also need the shorter interval about $N - sb$. Before deleting any primes its possible survivors on this list are $O(x/\log x)$ by the ordinary upper sieve on an interval of length $\asymp x$ at a small power of x, one excluded residue at every prime there. Deleting the primes charged in (1) costs at most the same expense. For each $Z \in \mathcal{Z}$ use the preceding one-residue upper on this list as well. Consequently in all three cases $O(x/\log x)$ desired positions still need a divisor, after omitting the protected positions themselves.

Restoring the punctured intervals. Use a separate fresh prime $r \in (x^2, 2x^2)$ for each exposed position and prescribe b to make its value zero modulo r. Each unwanted value is affine with leading coefficient ± 1 or -2 in b. Avoid the prime divisors of its two determinants with the protected forms. In cases 1 and 2 the determinants are nonzero and of size $\ll N + y$, including $N + O(y)$ and a nonzero multiple of the offset. In case 3 omit the offsets of both endpoints; the relevant differences are then

nonzero and $O(y)$. At most $O(1+\log(2N)/\log x)$ primes in the pool are forbidden for any one position, besides those already used and the at most two primes in $\mathcal{Z}$. The prime number theorem provides $\gg x^2/\log x$ choices. CRT now keeps every protected value a unit, covers all requested offsets, and uses total logarithmic modulus $O(x) + (\log(2x^2))O(x/\log x) = O(x)$. This also verifies the asserted relaxed size condition on N.

Additional units. The same argument protects any fixed admissible set of translates in one initial gap. The lower sieve then has the corresponding fixed dimension, (1) gains only the number of roots, and the last step omits the finitely many pairwise coincidences. In particular the method for case 2 also gives a single covered prime class without reference to N. All estimates concern ordinary integer divisibility.

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