SUBADDITIVITY OF THE SMOOTH NUMBER COUNTING FUNCTION FOR EVERY SMOOTHNESS BOUND

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Abstract. Let $\Psi(x,y)$ be the number of positive integers up to x with no prime factor exceeding y. Hildebrand proved in 1985 that $\Psi(x+z,y) \le \Psi(x,y)+\Psi(z,y)$ for all $x,z \ge y$, provided that y is sufficiently large, and Granville asked whether the inequality holds for all $x,z > y \ge 2$. We give an elementary proof that it does. The inequality also holds for all nonnegative integers x and z. The proof follows Hildebrand's method, with his estimates replaced by exact statements. We are not aware of an earlier proof for all y, but our literature search was not exhaustive; see Section 1.2.

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1. Introduction

For real $x \ge 0$ and $y \ge 2$ let $\Psi(x,y)$ denote the number of integers m with $1 \le m \le x$ all of whose prime factors are at most y. Hildebrand [5, Theorem 4] proved that there is a constant y_1, which he does not specify, such that

$$\Psi(x+z,y) \;\le\; \Psi(x,y)+\Psi(z,y) \tag{1}$$

for all $y \ge y_1$ and all $x,z \ge y$. In other words, once y is large, an interval $(x,x+z]$ with $x,z \ge y$ never contains more y-smooth integers than the initial interval $[1,z]$ of the same length. In his survey of smooth numbers, Granville [3, Section 1.4, p. 274] quotes this bound for sufficiently large y and asks whether it holds for all $x,z > y \ge 2$. This note answers the question in the affirmative.

Theorem 1. *Let $y \ge 2$ be real. For all integers $a,b \ge 0$,*

$$\Psi(a+b,y) \;\le\; \Psi(a,y)+\Psi(b,y).$$

Equivalently, any n consecutive positive integers contain at most $\Psi(n,y)$ integers free of prime factors exceeding y.

Theorem 2. *Let $y \ge 2$ be real. For all real $x,z \ge y$,*

$$\Psi(x+z,y) \;\le\; \Psi(x,y)+\Psi(z,y).$$

If x and z are integers, the inequality is strict.

For real arguments a lower bound on x and z cannot be removed altogether: $\Psi(3,3) = 3 > 2 = 2\,\Psi(3/2,3)$. Without any lower bound, Theorem 1 still gives $\Psi(x+z,y) \le \Psi(x,y)+\Psi(z,y)+1$ for all real $x,z \ge 0$ and $y \ge 2$, because $\lfloor x+z \rfloor \le \lfloor x \rfloor + \lfloor z \rfloor + 1$.

The proof of Theorem 1 uses nothing beyond the arithmetic–geometric mean inequality and the trivial bound $\pi(p) \le p/2$ for primes $p \ge 11$. Theorem 2 uses in addition the theorem of Sylvester and Schur [6, 1]: if $N \ge 2k \ge 2$, then $\binom{N}{k}$ has a prime factor exceeding k.

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1.1. **Relation to Hildebrand's proof.** The method is Hildebrand's. His starting point is the identity [5, (3)]

$$\Psi(x,y)\log x-\int_1^x \frac{\Psi(t,y)}{t}\,dt \;=\sum_{\substack{r^k\le x\\ r\le y}}\Psi\Big(\frac{x}{r^k},y\Big)\log r,$$

where r runs over primes, which he applies at $x+z$ and at x and then iterates. The left side is $\sum\log s$ over the y-smooth $s\le x$, and in that form the identity is Lemma 3 below. Applied to an interval, it produces terms caused by rounding x/r^k and z/r^k to integers. In the proof of [5, Theorem 4] these terms are bounded by $\pi(y)\log(2x)$, the range $y\le z\le 4y$ is treated separately by means of [5, Theorem 1 and Lemma 2], and in both ranges the argument is closed by estimates that are valid only when y is sufficiently large; this is where the restriction on y enters. Hildebrand also remarks, after the statement of his Theorem 4, that the condition $x,z\ge y$ could be dropped for positive integers x and z, which is the form of Theorem 1.

What is new here is bookkeeping. We keep the identity as an equality for integer arguments (equation (3)), account for the rounding terms exactly (Lemma 6), treat the range $x+z\le p^2$ by an exact formula, and close the argument with an elementary inequality that holds for every prime (Lemma 7).

1.2. **Status of the question.** We are not aware of a previous proof of (1) for all $y\ge 2$, and we make no stronger claim than that. The evidence is as follows. Recent papers continue to quote (1) with Hildebrand's restriction: Granville and Lamzouri [4, Lemma 3.2] state it for $y\ge y_0$ with an unspecified constant y_0, Gorodetsky [2, proof of Lemma 3.5] for $y\ge C$, and Wright [7, (5)] for a sufficiently large smoothness bound. A November 2026 search of citations to [5] and [3], including full-text searches where available, found no earlier proof or report that Granville's question had been settled. That search did not have full-text coverage of every citing paper, so a proof may exist that it missed. We would be grateful for a reference if so.

2. Notation and Hildebrand's identity

Throughout, $y\ge 2$ is fixed and p is the largest prime not exceeding y. Let S be the set of positive integers all of whose prime factors are at most p (so $1\in S$), and write

$$\Psi(t)=\Psi(t,y)=\#\{s\in S: s\le t\},\qquad T(t)=\sum_{\substack{s\in S\\ s\le t}}\log s\qquad (t\ge 0).$$

The letter r always denotes a prime, and $\pi(p)$ is the number of primes $r\le p$. For integers $a,b\ge 0$ put

$$E(a,b)=\Psi(a)+\Psi(b)-\Psi(a+b).$$

Theorem 1 states that $E(a,b)\ge 0$.

Lemma 3 (Hildebrand's identity)**.** *For every $t\ge 0$,*

$$T(t)=\sum_{r\le p}\sum_{k\ge 1}\Psi\Big(\frac{t}{r^k}\Big)\log r.$$

Proof. For $s\in S$ we have $\log s=\sum\log r$, the sum being over the pairs (r,k) with $k\ge 1$ and $r^k\mid s$. The elements $s\le t$ of S divisible by r^k are the numbers $r^k s'$ with $s'\in S$ and $s'\le t/r^k$. Summing over $s\le t$ gives the claim. □

3. Proof of Theorem 1

We prove $E(a,b)\geq 0$ by strong induction on $a+b$. Write $x=\max(a,b)$ and $n=\min(a,b)$. If $n=0$ then $E(x,0)=\Psi(0)=0$. Let $n\geq 1$ and suppose that $E(a',b')\geq 0$ whenever $a'+b'<x+n$. Let W be the set of integers in $(x,x+n]$ and $W_S=W\cap S$, so that

$$E(x,n)=\Psi(n)-\#W_S.$$

Case 1: $n\leq p$**.** Every positive integer up to p lies in S, so $\Psi(n)=n=\#W\geq\#W_S$.

Case 2: $x+n\leq p^2$**.** A positive integer $m\leq p^2$ outside S has exactly one prime factor $r>p$, to the first power, since two such factors would give $m\geq(p+1)^2$. Hence $t-\Psi(t)=\sum_{r>p}\lfloor t/r\rfloor$ for every integer $0\leq t\leq p^2$, and therefore

$$E(x,n)=\sum_{r>p}\left(\left\lfloor\frac{x+n}{r}\right\rfloor-\left\lfloor\frac{x}{r}\right\rfloor-\left\lfloor\frac{n}{r}\right\rfloor\right)\ \geq\ 0. \tag{2}$$

Case 3: $p<n\leq x$ **and** $x+n>p^2$**.** This case is settled by the following proposition, which gives slightly more than is needed.

Proposition 4. *Let $p<n\leq x$ be integers with $x+n>p^2$, and suppose that $E(a',b')\geq 0$ for all integers $a',b'\geq 0$ with $a'+b'<x+n$. Then $E(x,n)\geq 1$.*

The rest of this section proves Proposition 4, assuming Lemma 7, which is proved in Section 4. Note that $x\geq n\geq 3$, so $\log x>0$.

3.1. **The identity.** For a prime power $q=r^k$ with $r\leq p$ and $k\geq 1$, put

$$\delta(q)=\left\lfloor\frac{x+n}{q}\right\rfloor-\left\lfloor\frac{x}{q}\right\rfloor-\left\lfloor\frac{n}{q}\right\rfloor\in\{0,1\},\qquad \chi(q)=\begin{cases}1 & \text{if } \delta(q)=1 \text{ and } \lfloor(x+n)/q\rfloor\in S,\\ 0 & \text{otherwise.}\end{cases}$$

Since $\Psi(c+1)-\Psi(c)$ is 1 or 0 according as $c+1\in S$ or not,

$$\Psi\left(\left\lfloor\frac{x+n}{q}\right\rfloor\right)-\Psi\left(\left\lfloor\frac{x}{q}\right\rfloor\right)-\Psi\left(\left\lfloor\frac{n}{q}\right\rfloor\right)=\chi(q)-E\left(\left\lfloor\frac{x}{q}\right\rfloor,\left\lfloor\frac{n}{q}\right\rfloor\right).$$

Apply Lemma 3 at $t=x+n$, x and n and subtract. With

$$X=\sum_{q=r^k}\chi(q)\log r,\qquad \Sigma=\sum_{q=r^k}E\left(\left\lfloor\frac{x}{q}\right\rfloor,\left\lfloor\frac{n}{q}\right\rfloor\right)\log r$$

(both sums are finite, as the terms vanish for $q>x+n$), this gives $T(x+n)-T(x)-T(n)=X-\Sigma$. On the other hand

$$T(x+n)-T(x)=\sum_{m\in W_S}\log m=\#W_S\cdot\log x+A,\qquad A=\sum_{m\in W_S}\log\frac{m}{x}\ \geq 0.$$

Substituting $\#W_S=\Psi(n)-E(x,n)$ we obtain the exact identity

$$E(x,n)\log x=\Psi(n)\log x-T(n)+A-X+\Sigma. \tag{3}$$

Every pair occurring in Σ has $\lfloor x/q\rfloor+\lfloor n/q\rfloor\leq(x+n)/2<x+n$, so $\Sigma\geq 0$ by the induction hypothesis.

Example 5. Let $p=3$, $x=5$, $n=4$. Then $W_S=\{6,8,9\}$, $\Psi(4)=4$ and $E(5,4)=1$. Here $A=\log\frac{432}{125}$, $T(4)=\log 24$, the prime powers with $\chi(q)=1$ are 8, 3 and 9, so $X=\log 18$, and $\Sigma=0$. Identity (3) reads $\log 5=\log\left(\frac{625}{24}\cdot\frac{432}{125}\cdot\frac{1}{18}\right)$.

3.2. **The rounding terms.**

Lemma 6. *With the notation above, $A - X \geq -\pi(p)\log x$.*

Proof. For each prime $r \leq p$ let $\ell_r \geq 0$ be the integer with $r^{\ell_r} \leq n < r^{\ell_r+1}$, and let $L = \sum_{r\leq p} \ell_r \log r$. Split $X = X_{\leq} + X_{>}$ according to whether $q \leq n$ or $q > n$.

There are exactly ℓ_r powers of r not exceeding n, and $\chi \leq 1$, so $X_{\leq} \leq L$.

Let $q = r^k > n$. Then $\lfloor n/q \rfloor = 0$ and $\delta(q)$ is the number of multiples of q in W, which is at most one because W consists of $n < q$ consecutive integers. If W contains a multiple m of q, then $m/q = \lfloor (x+n)/q \rfloor$, and $m/q \in S$ if and only if $m \in S$. Thus $\chi(q) = 1$ exactly when some $m \in W_S$ is divisible by q, and so

$$X_{>} = \sum_{m\in W_S} \sum_{\substack{r^k \mid m \\ r^k > n}} \log r = \sum_{m\in W_S} \log \frac{m}{\tau(m)}, \qquad \tau(m) = \prod_{r\mid m} r^{\min(v_r(m),\, \ell_r)},$$

where $v_r(m)$ is the exponent of r in m; indeed the number of k with $r^k \mid m$ and $r^k > n$ is $\max(v_r(m) - \ell_r, 0)$. Consequently

$$A - X_{>} = \sum_{m\in W_S} \log \frac{\tau(m)}{x}.$$

If $\tau(m) = m$, the term is $\log(m/x) > 0$. Otherwise there is a prime r with $v_r(m) \geq \ell_r + 1$; choose one and call it $\rho(m)$. Since $r^{\ell_r+1} > n$, at most one element of W is divisible by r^{ℓ_r+1}, so ρ is injective on the set of $m \in W_S$ with $\tau(m) \neq m$. For such m and $r = \rho(m)$ we have $\tau(m) \geq r^{\ell_r}$, so the term is at least $-\log(x/r^{\ell_r})$, and $\log(x/r^{\ell_r}) \geq 0$ because $r^{\ell_r} \leq n \leq x$. Hence

$$A - X_{>} \;\geq\; -\sum_{r\leq p} \log \frac{x}{r^{\ell_r}} = -\pi(p)\log x + L,$$

and $A - X = (A - X_{>}) - X_{\leq} \geq -\pi(p)\log x$. □

3.3. **The closing inequality.** From (3), Lemma 6 and $\Sigma \geq 0$,

$$E(x,n)\log x \;\geq\; \big(\Psi(n) - \pi(p)\big)\log x - T(n). \tag{4}$$

Lemma 7. *Let p be a prime and $n > p$ an integer, and let $x_0 = \max(n,\, p^2 - n + 1)$. Then for every real $x \geq x_0$,*

$$\big(\Psi(n) - \pi(p)\big)\log x \;>\; T(n).$$

Under the hypotheses of Proposition 4 we have $x \geq n$ and $x \geq p^2 + 1 - n$, that is, $x \geq x_0$. So (4) and Lemma 7 give $E(x,n)\log x > 0$, hence $E(x,n) \geq 1$. This proves Proposition 4, and with it Theorem 1. The window form of the theorem is the case $a = x$, $b = n$ for the integers $x+1, \ldots, x+n$. □

4. Proof of Lemma 7

Let C be the set of elements of S that are at most n and not prime; in particular $1 \in C$. Since $p < n$, the primes in $S \cap [1,n]$ are exactly the primes $r \leq p$, so $\#C = \Psi(n) - \pi(p) \geq 1$ and $T(n) = \sum_{r\leq p} \log r + \sum_{s\in C} \log s$. The left side of the lemma is increasing in x, so it suffices to take $x = x_0$, and then the claim is equivalent to

$$\prod_{s\in C} \frac{x_0}{s} \;>\; \prod_{r\leq p} r. \tag{5}$$

Every factor on the left is at least 1, because $s \leq n \leq x_0$. So we may discard factors.

Let D be the set of non-primes in $[1,p]$. Every positive integer up to p lies in S and is less than n, so $D \subseteq C$; moreover $\#D = p - \pi(p)$ and $\prod_{s\in D} s \cdot \prod_{r\le p} r = p!$. Keeping only the factors with $s \in D$, we see that (5) follows from

$$x_0^{\,p-\pi(p)} \;>\; p!. \tag{6}$$

Since x_0 is at least the average of n and $p^2 - n + 1$,

$$x_0 \;\ge\; \frac{p^2+1}{2} \;>\; \left(\frac{p+1}{2}\right)^2, \tag{7}$$

the second inequality being $(p-1)^2 > 0$.

The case $p \ge 11$. The primes up to p are 2 and some of the $(p-1)/2$ odd numbers in $[3,p]$, one of which is 9. Hence $\pi(p) \le (p-1)/2$ and $p - \pi(p) > p/2$. By (7) and the arithmetic–geometric mean inequality $p! \le \left(\frac{1+2+\cdots+p}{p}\right)^p$,

$$x_0^{\,p-\pi(p)} \;\ge\; x_0^{\,p/2} \;>\; \left(\frac{p+1}{2}\right)^p \;\ge\; p!,$$

which is (6).

The cases $p = 2, 5, 7$. As x_0 is an integer, (7) gives $x_0 \ge 3$, 13, 25 respectively, and (6) holds because $3 > 2!$, $13^2 = 169 > 5! = 120$ and $25^3 = 15625 > 7! = 5040$.

The case $p = 3$. Here (7) gives $x_0 \ge 5$, and (6) can fail (for $n = 5$ it reads $5 > 6$). But $n \ge 4$, so C contains both 1 and 4, and the left side of (5) is at least $x_0 \cdot \frac{x_0}{4} \ge \frac{25}{4} > 6 = 2 \cdot 3$. □

5. Real arguments: proof of Theorem 2

Proposition 8. *If $a \ge b \ge p$ are integers, then $E(a,b) \ge 1$.*

Proof. We use the Sylvester–Schur theorem in two equivalent forms: for $N \ge 2k \ge 2$ the binomial coefficient $\binom{N}{k}$ has a prime factor exceeding k; and the product of k consecutive integers, each greater than k, has a prime factor exceeding k.

If $b = p$, then $E(a,p) = p - \#\big(S \cap (a, a+p]\big)$. The integers $a+1, \dots, a+p$ all exceed p, so one of them has a prime factor exceeding p and is not in S. Hence $E(a,p) \ge 1$.

If $b > p$ and $a + b \le p^2$, then $E(a,b)$ is given by (2). Since $a + b \ge 2b$, there is a prime $r > b$ dividing $\binom{a+b}{b}$. By Legendre's formula the exponent of r in $\binom{a+b}{b}$ is $\sum_{k\ge1}\big(\lfloor (a+b)/r^k \rfloor - \lfloor a/r^k \rfloor - \lfloor b/r^k \rfloor\big)$, and the terms with $k \ge 2$ vanish because $r^2 > p^2 \ge a+b$. So the term $k = 1$ equals 1, and as $r > p$ it is one of the terms of (2). Hence $E(a,b) \ge 1$.

If $b > p$ and $a+b > p^2$, Proposition 4 applies, its hypothesis being supplied by Theorem 1. □

Proof of Theorem 2. Let $x, z \ge y$ be real, and put $a = \lfloor x \rfloor$, $b = \lfloor z \rfloor$. Then $a, b \ge p$, since $p \le y$ is an integer. By symmetry and Proposition 8, $\Psi(a+b) \le \Psi(a) + \Psi(b) - 1$, which is the asserted strict inequality when x and z are integers. In general $\lfloor x + z \rfloor \le a + b + 1$, so

$$\Psi(x+z) \;\le\; \Psi(a+b+1) \;\le\; \Psi(a+b)+1 \;\le\; \Psi(a)+\Psi(b) = \Psi(x)+\Psi(z).$$ □

6. Remarks

Remark 9. The proof of Theorem 1 uses no information about the distribution of primes: neither the prime number theorem nor Chebyshev-type bounds nor Bertrand's postulate. The Sylvester–Schur theorem enters only through Proposition 8, in the two boundary ranges $b = p$ and $a + b \le p^2$. We do not know whether it can be avoided there.

Remark 10. No computation is used in the proofs. As an independent check, the conclusions of Theorem 1 and Proposition 8 were confirmed by direct enumeration for all primes $p \le 61$ and all $0 \le b \le a \le 4000$, identity (3) and Lemma 6 on a range of triples (p, n, x), and Lemma 7 in exact integer arithmetic for all primes $p < n \le 700$. No counterexample was found.

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