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THERE IS NO SYMMETRIC CONFIGURATION 59_8

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Abstract. A symmetric configuration v_k is an incidence structure with v points and v lines, k points on every line and k lines through every point, in which two points lie on at most one common line. For $v = k^2-k+3$ (deficiency two), the leave graph joins two distinct points exactly when they are not collinear. It is 2-regular, as is the dual leave graph joining distinct lines that do not meet. We show that the incidence matrix intertwines the adjacency matrices of these two graphs, and derive from this a short list of exact necessary conditions on the cycle lengths: integral block moments, a periodicity condition and cross-moment identities. For $v = 59$, $k = 8$ these conditions exclude every cycle structure except a single 59-cycle, which is ruled out by an elementary argument modulo 353 that uses only the integrality of the incidence matrix. Hence no configuration 59_8 exists. No cyclicity or automorphism is assumed. The 59_8 case is listed as undecided in the 2014 tables of Davydov, Faina, Giulietti, Marcugini and Pambianco. All arguments are elementary. The finite part is a small exhaustive computation, and the complete proof, including the finite part, has been formalized and checked in the Lean 4 proof assistant.

1. Introduction

A *symmetric configuration* v_k consists of v points and v lines such that every line contains k points, every point is on k lines, and two distinct points are on at most one common line. Equivalently, it is a $v \times v$ matrix A with entries in $\{0, 1\}$, all row and column sums equal to k, and no all-ones 2×2 submatrix. Counting the points collinear with a fixed point gives $v \geq k^2 - k + 1$, with equality exactly for projective planes of order $k - 1$; the integer $d = v - (k^2 - k + 1)$ is the *deficiency*. The existence problem for v_k has a long history (see Gropp [2] and the tables of Davydov et al. [1]); for each k it is open only for finitely many v, and the smallest deficiencies are the hardest cases.

For deficiency one, $v = k^2-k+2$, Philbrook and Peet [3] exclude several parameter sets, among them 58_8. For deficiency two, Philbrook and Peet [3] proved that the "non-intersection" structure decomposes into cycles whose number must be odd (apart from the Pappus configuration 9_3). The tables of [1] distinguish two questions: Table 6.1 already excludes a *cyclic* 59_8 configuration, whereas Table 7.1 leaves the unrestricted 59_8 existence question undecided. Our theorem settles the latter.

This is not the first unrestricted deficiency-two nonexistence result: Kaski and Östergård [4] proved nonexistence of 33_6 by exhaustive search. The Levi graph of a hypothetical 59_8 configuration would be an 8-regular bipartite graph on 118 vertices, of girth six, with excess four over the bipartite Moore bound $2(1 + 7 + 7^2) = 114$. Filipovski [5] studied the cycle structure and spectra of bipartite graphs of excess four; his stated exclusions do not cover degree eight and girth six. Tactical block decompositions of symmetric configurations had also been studied by Funk, Labbate and Napolitano [6]. Finally, the determinant and local quadratic-form techniques applied by Francetić, Herke and Horsley [7] to *symmetric pair coverings* are a relevant methodological precursor; the covering excess and configuration leave obey different Gram identities.

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Here the parameter-specific advance is the combination of exact block-incidence restrictions with the elementary integral obstruction at 353, without any automorphism assumption.

Theorem 1. *There is no symmetric configuration* 59_8*. Equivalently, there is no* 59×59 *matrix with entries in* $\{0,1\}$*, exactly eight ones in each row and column, and no all-ones* 2×2 *submatrix.*

The proof does not assume any automorphism. Section 2 develops necessary conditions valid for every deficiency-two configuration; Section 3 applies them to 59_8. The finite verification is described in Section 4 and the formalization in Section 5.

2. Necessary conditions at deficiency two

Throughout, A is the incidence matrix of a symmetric configuration v_k with $k \geq 4$ and $v = k^2 - k + 3$; rows index lines and columns index points (the roles are symmetric). Let I and J be the identity and all-ones matrices.

2.1. **The leave graphs and the Gram identities.** Two distinct rows share at most one column. A fixed row has k ones, and each of its columns contains $k-1$ further ones, so the row meets exactly $k(k-1)$ other rows, each in exactly one column. Since there are $v-1 = k(k-1)+2$ other rows, exactly two rows are disjoint from it. Let R be the adjacency matrix of the graph on rows in which two rows are adjacent when they are disjoint, and define C on columns in the same way. Both graphs are 2-regular and simple, hence disjoint unions of cycles of lengths at least 3. By construction

$$AA^\top = (k-1)I + J - R, \qquad A^\top A = (k-1)I + J - C. \tag{1}$$

For the adjacency matrix X of a 2-regular graph define integer matrices $P_n(X)$ by $P_0 = 2I$, $P_1 = X$ and $P_{n+1} = XP_n - P_{n-1}$. If the graph is a single cycle with vertex set $\mathbb{Z}/\ell\mathbb{Z}$ and S is the cyclic shift, then $P_n(X) = S^n + S^{-n}$: this holds for $n = 0, 1$ and is preserved by the recurrence, since $(S + S^{-1})(S^n + S^{-n}) = S^{n+1} + S^{-(n+1)} + S^{n-1} + S^{-(n-1)}$. Hence on a disjoint union of cycles every row of $P_n(X)$ sums to 2, and $P_n(X)_{xx}$ equals 2 if the length of the cycle through x divides n and 0 otherwise.

Lemma 2. *$RA = AC$, and hence $P_n(R)A = AP_n(C)$ for all $n \geq 0$. The row and column leave graphs have the same multiset $L = (\ell_1, \ldots, \ell_c)$ of cycle lengths.*

Proof. Using $JA = AJ = kJ$, compute $(AA^\top)A = (k-1)A + kJ - RA$ and $A(A^\top A) = (k-1)A + kJ - AC$. The two left sides are equal, so $RA = AC$, and induction on the recurrence gives $P_n(R)A = AP_n(C)$. Therefore $\operatorname{tr}(P_n(R)AA^\top) = \operatorname{tr}(AP_n(C)A^\top) = \operatorname{tr}(P_n(C)A^\top A)$. Put $t_n = \operatorname{tr} P_n(R)$ and $t'_n = \operatorname{tr} P_n(C)$. Using (1), $RP_n = P_{n+1} + P_{n-1}$ and $\mathbf{1}^\top P_n \mathbf{1} = 2v$, this identity becomes $(k-1)t_n - t_{n+1} - t_{n-1} = (k-1)t'_n - t'_{n+1} - t'_{n-1}$ for $n \geq 1$. Since $t_0 = t'_0 = 2v$ and $t_1 = t'_1 = 0$, we get $t_n = t'_n$ for all n. Finally $t_n = 2\sum_{\ell \mid n} \ell\, c_\ell$, where c_ℓ is the number of row cycles of length ℓ, and similarly for t'_n; strong induction on n gives $c_\ell = c'_\ell$ for every ℓ. Neither invertibility of A nor any spectral theory is used. □

Order rows and columns so that each cycle of R and C occupies a consecutive block of indices, in cyclic order around the cycle, and the i-th row cycle and the i-th column cycle both have length ℓ_i. Write $A = (A_{ij})$ in the corresponding block form, so A_{ij} is $\ell_i \times \ell_j$, and let Z_ℓ denote the adjacency matrix of the ℓ-cycle. Lemma 2 becomes

$$Z_{\ell_i} A_{ij} = A_{ij} Z_{\ell_j} \qquad \text{for all } i, j. \tag{2}$$

Lemma 3 (constant block sums)**.** *Each block A_{ij} has constant row sum u_{ij} and constant column sum v_{ij}, with $\ell_i u_{ij} = \ell_j v_{ij}$ and $0 \leq u_{ij}, v_{ij} \leq k$. In particular v_{ij} is a multiple of $\ell_i / \gcd(\ell_i, \ell_j)$.*

Proof. Applying (2) to the all-ones vector $\mathbf{1}$ gives $Z_{\ell_i}x = 2x$ for $x = A_{ij}\mathbf{1}$, so each entry of x is the average of its two neighbours on the cycle. At a maximal entry both neighbours are maximal, and the cycle is connected, so x is constant. The column statement follows by transposition, and counting the ones of A_{ij} in two ways gives $\ell_i u_{ij} = \ell_j v_{ij}$. □

Lemma 4 (moment identities)**.** *For every row cycle i,*

$$\sum_j \ell_j v_{ij} = k\,\ell_i, \tag{3}$$

$$\sum_j \ell_j v_{ij}^2 = \ell_i(\ell_i + k - 3), \tag{4}$$

and for distinct row cycles $i \neq h$,

$$\sum_j \ell_j v_{ij} v_{hj} = \ell_i \ell_h. \tag{5}$$

Proof. (3) counts the ones in the rows of cycle i. For (4), sum $|r \cap r'|$ over ordered pairs of rows r, r' in cycle i. Each row meets itself in k columns, is disjoint from its two neighbours on the cycle, and meets the remaining $\ell_i - 3$ rows of the cycle once, giving $\ell_i(k + \ell_i - 3)$ in total. Counting instead by columns, a column of column cycle j lies in exactly v_{ij} rows of cycle i, giving $\sum_j \ell_j v_{ij}^2$. For (5), rows in different cycles are never disjoint, so every one of the $\ell_i \ell_h$ pairs meets in exactly one column, and the same double count applies. □

Lemma 5 (periodicity)**.** *Let $a = \ell_i$, $b = \ell_j$ and $g = \gcd(a, b)$, and index the rows of A_{ij} by $\mathbb{Z}/a\mathbb{Z}$. Then $A_{ij}[r + g, s] = A_{ij}[r, s]$ for all r, s.*

Proof. Write $X = A_{ij}$. Restricting Lemma 2 to the block gives $P_b(Z_a)X = XP_b(Z_b) = X(S^b + S^{-b}) = 2X$, that is, $X[r + b, s] + X[r - b, s] = 2X[r, s]$. Since the entries are 0 or 1, this forces $X[r + b, s] = X[r, s]$. The multiples of b in $\mathbb{Z}/a\mathbb{Z}$ are exactly the multiples of g, so $X[r + g, s] = X[r, s]$. □

Corollary 6 (periodicity condition)**.** *For every row cycle i and every $0 < \delta < \ell_i$,*

$$Q_i(\delta) = \sum_{j:\ \gcd(\ell_i, \ell_j) \mid \delta} u_{ij} \ \leq\ 1.$$

Proof. By Lemma 5, the distinct rows r and $r + \delta$ of cycle i have identical support inside every block A_{ij} with $\gcd(\ell_i, \ell_j) \mid \delta$, so they share at least $Q_i(\delta)$ columns. Two distinct rows share at most one column. □

Remark 7 (parity). The number c of cycles of R is odd. We record this for comparison with [3]. The proof of Theorem 1 does not use it.

Proof. The matrix $(k - 1)I + J - R$ has eigenvalue $(k - 1) + v - 2$ on $\mathbf{1}$; eigenvalue $k - 3$ on the $(c - 1)$-dimensional space of vectors constant on each cycle and orthogonal to $\mathbf{1}$; and eigenvalues $k - 1 - 2\cos(2\pi t/\ell)$, $1 \leq t \leq \ell - 1$, for each cycle of length ℓ. For $k = 8$ the last product over one cycle equals $\prod_{t=1}^{\ell-1}(7 - 2\cos(2\pi t/\ell)) = (L_{4\ell} - 2)/5 = F_{2\ell}^2$, where F_n and L_n are the Fibonacci and Lucas numbers, since $7 = \varphi^4 + \varphi^{-4}$. Hence $\det(AA^\top) = 64 \cdot 5^{c-1} \prod_i F_{2\ell_i}^2$. As $\det(AA^\top) = \det(A)^2$ is a square, c is odd. (For general k the same conclusion holds; it is Theorem 6.1 of [3]. We need only $k = 8$.) □

3. The case 59_8

Now let $v = 59$ and $k = 8$. Lemma 2 gives a common cycle multiset L, a partition of 59 into parts ≥ 3. For each such L and each cycle length $a \in L$, call a vector $(v_{ij})_j$ *admissible* if it satisfies Lemma 3, (3), (4) and Corollary 6 (with $u_{ij} = \ell_j v_{ij}/a$).

Proposition 8. *There are* 29,269 *partitions of* 59 *into parts* ≥ 3. *Exactly fifteen of them admit, for every cycle length present, a vector satisfying Lemma 3,* (3) *and* (4)*:*

$$59,\quad 3{+}21{+}35,\quad 10{+}14{+}35,\quad 3{+}24{+}32,\quad 5{+}24{+}30,$$

$$3{+}5{+}6{+}15{+}30,\quad 3{+}8{+}12^2{+}24,\quad 3{+}6^2{+}8{+}12{+}24,$$

$$3{+}7{+}14^2{+}21,\quad 3{+}7^3{+}14{+}21,\quad 3{+}7^5{+}21,\quad 3^3{+}5{+}15^3,\quad 3^3{+}5^2{+}10{+}15^2,\quad 3^3{+}5^4{+}15^2,\quad 3^8{+}5^7.$$

Imposing Corollary 6 leaves only 59, $3^3{+}5{+}15^3$ *and* $3^8{+}5^7$.

This is a finite computation, described in Section 4. For each excluded profile among the fifteen, a cycle length a and shift δ for which *every* moment solution gives $Q(\delta) \geq 2$ are:

profile	(a, δ)
$3^3{+}5^4{+}15^2$	$(15, 5)$
$3{+}5{+}6{+}15{+}30$	$(30, 15)$
$3{+}7^5{+}21,\ 3{+}7{+}14^2{+}21,\ 3{+}21{+}35$	$(21, 7)$
$3{+}8{+}12^2{+}24,\ 3{+}6^2{+}8{+}12{+}24$	$(8, 4)$
$3{+}7^3{+}14{+}21$	$(14, 7)$
$3{+}24{+}32$	$(24, 8)$
$5{+}24{+}30$	$(24, 6)$
$10{+}14{+}35,\ 3^3{+}5^2{+}10{+}15^2$	$(10, 5)$

A worked periodicity certificate. Consider $L = 3 + 21 + 35$ and a row cycle of length $a = 21$. The first two moments admit exactly one column-sum vector $(v_{i1}, v_{i2}, v_{i3}) = (7, 2, 3)$, in column-cycle order $(3, 21, 35)$. Indeed, $3{\cdot}7{+}21{\cdot}2{+}35{\cdot}3 = 168 = 8a$, and $3{\cdot}7^2{+}21{\cdot}2^2{+}35{\cdot}3^2 = 546 = a(a{+}5)$. The corresponding block row sums are $(1, 2, 5)$. Since $\gcd(21, 35) = 7$ divides $\delta = 7$, while $\gcd(21, 3) = 3$ and $\gcd(21, 21) = 21$ do not, the two rows at distance 7 share at least $Q_i(7) = 5$ columns. This contradicts the bound $Q_i(7) \leq 1$ and excludes the entire profile.

Lemma 9. *The profiles* $3^8{+}5^7$ *and* $3^3{+}5{+}15^3$ *violate the cross-moment identity* (5).

Proof. For $3^8{+}5^7$, consider a row 3-cycle. In a column 5-cycle its block column sum is divisible by 3, so any positive value contributes at least $5 \cdot 3^2 = 45$ to the second moment, exceeding $3(3 + 5) = 24$. It must therefore vanish. Similarly, for a row 5-cycle, any positive block column sum on a column 3-cycle is at least 5 and contributes at least $3 \cdot 5^2 = 75 > 5(5 + 5) = 50$. The supports of the two types of row cycles are disjoint, so their cross moment is $0 \neq 3 \cdot 5$.

For $3^3{+}5{+}15^3$, the moments and block divisibility give the following complete support description. A row 3-cycle has column sums 1 on each of the three column 3-cycles, 0 on the column 5-cycle, and 1 on exactly one of the three column 15-cycles. The row 5-cycle has column sum 2 on the column 5-cycle, 0 on the column 3-cycles and 1 on exactly two column 15-cycles. For any two row 3-cycles, their cross moment already receives $3(1 \cdot 1)$ from each of the three column 3-cycles, for a total of 9, which is the required value $3 \cdot 3$. Hence these row cycles must choose distinct column 15-cycles. All three column 15-cycles are thus chosen once; one is omitted by the row 5-cycle. The row 3-cycle using the omitted column has cross moment 0 with that row 5-cycle, contrary to the required value $3 \cdot 5 = 15$. □

Lemma 10. *The profile* $L = (59)$ *is impossible.*

Proof. After relabelling the rows, $AA^\top = G := 7I + J - Z_{59}$ with A an integer matrix. Let $p = 353$. Write $S = \sum_{r=0}^{58} x_r$ for a vector x indexed cyclically. The equation $Gx = 0 \bmod p$ has coordinate sum $64S = 0 \bmod p$, hence $S = 0 \bmod p$, and reduces to the recurrence $x_{r+1} = 7x_r - x_{r-1} \bmod p$. It follows that the kernel has dimension at most two. Define integer vectors k_1, k_2 by the least nonnegative residues of this recurrence, starting with $(x_0, x_1) = (1, 0)$ and $(0, 1)$ respectively. In each case the recurrence returns to its initial seeds after 59 steps, and $\sum_r x_r = 0 \bmod 353$; these are finite integer checks. Thus $Gk_1 = Gk_2 = 0 \bmod p$, and since the two seed pairs are independent, the rank is exactly 57. The integers $k_i^\top G k_j$ are divisible by p, and the residues $b_{ij} = k_i^\top G k_j / p \bmod p$ give

$$(b_{ij}) = \begin{pmatrix} 69 & 288 \\ 288 & 69 \end{pmatrix}, \qquad b_{12}^2 - b_{11}b_{22} \equiv 170 \pmod{353},$$

and 170 is not a square modulo 353 (since $170^{176} \equiv -1 \bmod 353$). Hence the binary form $b(c) = b_{11}c_1^2 + 2b_{12}c_1c_2 + b_{22}c_2^2$ is anisotropic over $\mathbb{F}_{353}$.

Since $\det(A)^2 = \det G \equiv 0 \pmod p$, the reduction of A is singular, so there is an integer vector $n \not\equiv 0 \pmod p$ with $A^\top n \equiv 0 \pmod p$. Then $Gn = A(A^\top n) \equiv 0$, so $n = c_1k_1 + c_2k_2 + pw$ for some integers c_1, c_2, not both divisible by p, and an integer vector w. Every coordinate of $A^\top n$ is divisible by p, so $n^\top Gn = |A^\top n|^2 \equiv 0 \pmod{p^2}$. On the other hand $Gk_i \equiv 0 \pmod p$ gives $n^\top Gn \equiv (c_1k_1 + c_2k_2)^\top G(c_1k_1 + c_2k_2) = p\, b(c) \pmod{p^2}$. Hence $b(c) \equiv 0 \pmod p$, contradicting anisotropy. □

Remark 11. Equivalently, the 353-adic Hasse invariant of G is -1, while that of $MM^\top$ is $+1$ for every rational invertible matrix M. The argument above needs only the integrality of A.

Proof of Theorem 1. By Lemma 2 a configuration 59_8 determines a common cycle profile L. Proposition 8 leaves three profiles; Lemma 9 excludes two and Lemma 10 the third. □

4. The computation

Proposition 8 and the modular computation in Lemma 10 use exact integer arithmetic only: no floating point, solvers or randomness is used for the proof. Lemma 9 is now entirely analytic. For each partition and each cycle length a in it, the program enumerates the multisets of block column sums for each distinct column-cycle length, subject to the divisibility in Lemma 3 and the bounds (3)–(4), and then tests Corollary 6 for every shift. The executable source specifies an exhaustive, deterministic search; run time depends on the Python interpreter and machine.

The proof package supplies two algorithmically distinct implementations of the row-vector feasibility tests: one uses memoised aggregate states for the remaining first and second moments, and the other enumerates block-sum multisets directly. They share the partition generator, so they are not presented as independent verifications of partition generation. The integrated fail-closed replay verifies all 29,269 partitions, the 15 parity-free moment candidates, all 12 periodicity witnesses, the two mixed-case conclusions and the anisotropic kernel at 353. It passes under ordinary and optimized Python (`python -O`); regression tests deliberately corrupt stored certificates. The Lean proof in Section 5 instead verifies the required profile exclusions and the unrestricted nonexistence theorem; it does not assert each displayed intermediate count. As a nonessential cross-check of Lemma 10, the 353-adic Hasse invariant of G was computed by exact rational diagonalisation and equals -1. Randomly generated rational Gram matrices provided additional software tests; those randomized tests play no role in the proof. The equality $\det G = 64F_{118}^2$ is checked exactly. The Python verifier, regression tests, saved certificates, and SHA-256 manifest are available at `https://github.com/jjtripodo-lgtm/symconf59-checker`. The version used here is commit `71db581317719397e01fa182cdce91358a444b0b`.

5. Formal verification

The complete proof has been formalized in Lean 4 (version 4.31.0-rc1) using the Mathlib library (revision `0ed37c92`). The final theorem is

```
theorem no_unrestricted_incidence59 :
¬∃ A : Matrix (Fin 59) (Fin 59) ℤ, Incidence59 A
```

where `Incidence59 A` states exactly the hypotheses of Theorem 1: every entry is 0 or 1, every row and every column sums to 8, and two distinct rows have at most one common 1. Nothing else is assumed. The column version of the intersection condition, the 2-regularity of both leave graphs, their decomposition into cycles and a cyclic ordering of each cycle are proved. The formal proof follows the elementary route of this paper: Lemma 2 by the trace recurrence, Lemmas 3 and 5 by the averaging and P_n arguments, Lemma 4 by double counting, and Lemma 10 by the modular argument. The formal finite argument uses a complete partition list and an exact profile-key certificate: weaker local tests reduce the list to five candidates, two more are rejected by explicit shifted tests, and the remaining profiles are eliminated in the original incidence model. This certifies the needed exclusion, not every intermediate enumeration tally in Proposition 8. The formal finite checks run in Lean's kernel without `native_decide`. The theorem depends only on Lean's three standard axioms (`propext`, `Classical.choice` and `Quot.sound`). The development comprises about 100 source files. The public Lean project is in the `lean/` directory of the repository and commit given above. The terminal declaration is in `Symconf59/Unrestricted59.lean`.

Remark 12. The same method (intertwining, block moments, periodicity, cross moments and local invariants) applies to every deficiency-two parameter set. Applications to 75_9, 93_{10} and 113_{11}, where additional structure arguments are needed, will be treated separately.

Provenance. The manuscript text was primarily AI-generated. AI tools assisted with drafting, proof development and review, checker implementation, and Lean formalization. The author reports that no part has been checked and understood by a human. The Python replay and Lean kernel verification provide computational checks.

References

[1] A. A. Davydov, G. Faina, M. Giulietti, S. Marcugini and F. Pambianco, *Tables of parameters of symmetric configurations v_k*, arXiv:1312.3837v2 (2014).

[2] H. Gropp, Configurations and their realization, *Discrete Math.* **174** (1997), 137–151.

[3] J. Philbrook and B. Peet, On the impossibility of certain $(n^2+n+k)_{n+1}$ configurations, *Involve* **19** (2026), 297–310; arXiv:2404.17514.

[4] P. Kaski and P. R. J. Östergård, There exists no symmetric configuration with 33 points and line size 6, *Australas. J. Combin.* **38** (2007), 273–277.

[5] S. Filipovski, On bipartite cages of excess 4, *Electron. J. Combin.* **24** (2017), Paper P1.40.

[6] M. Funk, D. Labbate and V. Napolitano, Tactical (de-)compositions of symmetric configurations, *Discrete Math.* **309** (2009), 741–747.

[7] N. Francetić, S. Herke and D. Horsley, More nonexistence results for symmetric pair coverings, *Linear Algebra Appl.* **487** (2015), 43–73.