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TWO SHAPES OF MAXIMAL DEGREE 147 IN THE PLANE JACOBIAN PROBLEM: EXCLUSION OF THE DEGREE PAIRS $(147, 98)$ AND $(98, 147)$ WITH CORNERS $(7, 42)$ AND $(13/7, 6)$

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Abstract. In the list of Guccione, Guccione, Horruitiner and Valqui of the possible shapes of a counterexample to the plane Jacobian conjecture with degrees at most 150, two shapes have starting corner $(7, 42)$ and final corner $(13/7, 6)$: the standard $(3, 2)$-pairs of degrees $(147, 98)$ and the standard $(2, 3)$-pairs of degrees $(98, 147)$. We show that no such pair exists, the pair being normalised as in Corollary 5.21 of Guccione–Guccione–Valqui; no assumption is made on the polynomial attached to the starting corner, of which an arithmetic argument on regular corners shows that it has a single root. The two sides of the Newton polygon are described by the results of Guccione–Guccione–Valqui, and in suitable polynomial coordinates the pair is confined to an explicit finite system of polynomial equations. Its leading part is a pair of polynomials of degrees 18 and 12 in one variable with $u_3^2 - w_2^3$ of degree 7; up to the symmetries of the system there are exactly six such pairs, conjugate over a number field of degree 6. Over that field the remaining equations are solved by three steps of linear algebra, and the 46 compatibility conditions, polynomials in six unknowns, generate the unit ideal. The computation is exact and is reproduced by an ancillary script (SymPy, python-flint, PARI, and Singular for the Gröbner bases). All statements are proved relative to the cited results. Nothing in this note has been checked by a third party.

1. Statement

Throughout, K is an algebraically closed field of characteristic zero, $L = K[x, y]$, $[P, Q] = P_x Q_y - P_y Q_x$ and (m, n) is $(3, 2)$ or $(2, 3)$. We use the notation of [1]: for a weight (ρ, σ), $v_{\rho,\sigma}$ is the weighted degree, $\ell_{\rho,\sigma}$ the leading form, $\mathrm{st}_{\rho,\sigma}$ and $\mathrm{en}_{\rho,\sigma}$ its end points; $L^{(l)} = K[x^{\pm 1/l}, y]$. *Standard (m, n)-pairs*, their *starting triple* and their *regular corners* are defined in [1, Definitions 4.3, 6.2, 5.5]. Put $t := xy^7$.

Standing data 1.1. (P, Q) is a standard (m, n)-pair in L with $[P, Q] = J \in K^\times$, $\deg P = 49m$, $\deg Q = 49n$, starting triple $\big((7, 42), (1, 0), (7, -1)\big)$, and $(-1, 1) < \mathrm{Succ}_P(1, 0) < (-1, 0)$ as in [1, Corollary 5.21(4)]. Thus:

(D1) the support of P lies in $\{(i, j) : 7i - j \le 7m\}$, that of Q in $\{7i - j \le 7n\}$, and

$$\ell_{7,-1}(P) = \lambda_P \big(x\, h(t)\big)^m, \qquad \ell_{7,-1}(Q) = \lambda_Q \big(x\, h(t)\big)^n, \qquad \lambda_P, \lambda_Q \in K^\times,$$

with $h \in K[t]$ monic of degree 6, $h(0) \neq 0$ [1, Proposition 6.1, Theorem 7.6(2), Remark 3.1].

Theorem 1.2. *There is no pair (P, Q) as in Standing data 1.1, neither for $(m, n) = (3, 2)$ nor for $(m, n) = (2, 3)$.*

Lemma 1.3. *In the situation of Standing data 1.1, $h = (t - c)^6$ with $c \in K^\times$.*

Proof. In $L^{(7)}$, with $\tilde{t} := x^{1/7} y$, one has $\ell_{7,-1}(P) = \lambda_P x^m\, h(\tilde{t}^7)^m$, and $h(\tilde{t}^7)$ has 42 roots counted with multiplicity, at least 7 of them distinct, so $\ell_{7,-1}(P)$ has more than one factor; moreover $[\ell_{7,-1}(P), \ell_{7,-1}(Q)] = 0$ by (D1) and $v_{1,-1}(\mathrm{st}_{7,-1}(P)) = v_{1,-1}(m, 0) = m > 0$. A root of h of multiplicity k gives roots of multiplicity km of $h(\tilde{t}^7)^m$. By [1, Propositions 5.16, 5.18] some root satisfies inequality (5.9) of [1, Proposition 5.18], $km \ge \frac{m}{l} \frac{a\rho + bl\sigma}{\rho + \sigma} = \frac{7m}{6}$ (with $l = 1$,

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$(a,b)=(7,42)$, $(\rho,\sigma)=(7,-1)$), so $k\geq 2$; conversely every root of multiplicity $k\geq 2$ satisfies it, since $km\geq 2m>\frac{7m}{6}$. For each such root [1, Proposition 5.18] gives an (m,n)-pair in $L^{(7)}$ with a regular corner at $A:=(1,0)+k(1/7,1)=\big((7+k)/7,k\big)$, of direction smaller than $(7,-1)$. We follow it as in the proof of [1, Proposition 5.19]: in Case III, [1, Proposition 5.17] gives $\rho'\mid 7$, and the translation $y\mapsto y+\lambda' x^{\sigma'/\rho'}$ produces an (m,n)-pair in $L^{(7)}$ with a regular corner at the same point A and a smaller direction; after finitely many steps Case I or Case II occurs, at the point A and in $L^{(7)}$. Assume $2\leq k\leq 5$. Case II is excluded by [1, Proposition 5.14(4)], which requires $\gcd(7+k,k)=\gcd(7,k)>1$. In Case I, for the pair (P',Q') and the direction (ρ'',σ'') reached, $v_{\rho'',\sigma''}(P')+v_{\rho'',\sigma''}(Q')=\rho''+\sigma''$ [1, Proposition 1.13] and $\operatorname{en}(P')=mA=\frac{m}{n}\operatorname{en}(Q')$ [1, Corollary 5.7(2)], so $5\big(\frac{7+k}{7}\rho''+k\sigma''\big)=\rho''+\sigma''$, that is $(\rho'',\sigma'')=\big(7(5k-1),-(5k+28)\big)$, which is primitive for $k=2,3,4,5$ (the pairs are $(63,-38)$, $(98,-43)$, $(133,-48)$, $(168,-53)$), with $\rho''+\sigma''=5(6k-7)$, $v(P')=m(6k-7)$ and $v(Q')=n(6k-7)$. By [1, Proposition 5.13], either (a) $\frac{1}{m}\operatorname{st}(P')=(s/7,0)$ with $s\in\mathbb{Z}$, whose value $(5k-1)s$ would equal $6k-7$, which is impossible since $6k-7$ is not a multiple of $5k-1$ for $2\leq k\leq 5$; or (b) the starting point of P' or of Q' is $(k'/7,0)$ with $k'\in\mathbb{N}$, of value $(5k-1)k'$, which would equal $2(6k-7)$ or $3(6k-7)$, that is $9k'\in\{10,15\}$, $14k'\in\{22,33\}$, $19k'\in\{34,51\}$ or $24k'\in\{46,69\}$, which has no solution. Hence every root of h of multiplicity at least 2 has multiplicity 6, and since some root has multiplicity at least 2 and $\deg h=6$, $h=(t-c)^6$; $c\neq 0$ as $h(0)\neq 0$. □

Remark 1.4. The line through $(7,42)$ and $(13/7,6)$ has direction $(7,-1)$ and meets the horizontal axis at $(1,0)$, and $(13/7,6)=(1,0)+6(1/7,1)$; by Lemma 1.3 and Lemma 3.1 below, every pair as in the standing data has the corners $(7,42)$ and $(13/7,6)$, and these are the two shapes of [2, Section 6] with these corners. Theorem 1.2 thus needs no assumption on h beyond (D1). Apart from the preprint recalled below, we found no exclusion of these two shapes in the literature we searched; their novelty is not otherwise verified. The preprint [3] of the four authors discards all cases with $\max\{\deg P,\deg Q\}\leq 150$ except $(75,125)$, using a formula for the intersection number of a Jacobian pair; in the later version of that work [4] only an inequality is obtained, and the authors state that the infinite families cannot be discarded in this way. They treat in [5] the shapes of degree below 125 and obtain the bound 108.

2. The left side

Lemma 2.1. *There are $\gamma_2,\dots,\gamma_5\in K$ such that*

$$\hat P(z,v):=P\Big(zv^6+\sum_{s=2}^{5}\gamma_s v^s,\ v^{-1}\Big),\qquad \hat Q(z,v):=Q\Big(zv^6+\sum_{s=2}^{5}\gamma_s v^s,\ v^{-1}\Big)$$

are polynomials in z,v. Then $\hat P$ has total degree $7m$ with homogeneous part of degree $7m$ equal to $\lambda_P\big(z(z-cv)^6\big)^m$, $\hat Q$ has total degree $7n$ with homogeneous part of degree $7n$ equal to $\lambda_Q\big(z(z-cv)^6\big)^n$, and

$$\hat P_z\hat Q_v-\hat P_v\hat Q_z=-J\,v^4.$$

Proof. The argument is the one of [5, Section 4, Case $(7,21)$]; we give it in detail. By Lemma 1.3, $h=(t-c)^6$. Apply the flip ψ_1 of [1, Section 6] (as used in [1, Section 8]), so that the weight $(7,-1)$ becomes $(\rho_0,\sigma_0)=(-1,7)$, with $\operatorname{st}_{\rho_0,\sigma_0}=m(42,7)$ and $\operatorname{en}_{\rho_0,\sigma_0}=m(0,1)$ for the flipped P, and $\frac{1}{m}\operatorname{st}(P)=\frac{1}{n}\operatorname{st}(Q)=(42,7)$. Let F be the element of [1, Theorem 2.6] for (ρ_0,σ_0); then $v_{-1,7}(F)=6$. If $\operatorname{st}(F)=(1,1)$ then $\operatorname{en}(F)\neq(1,1)$, since F is not a monomial [1, Remark 2.5], so $\operatorname{en}(F)\sim\operatorname{en}(P)\sim(0,1)$ and $\operatorname{en}(F)=(0,6/7)$, which is impossible; hence $\operatorname{st}(F)\sim(42,7)$, so $\operatorname{st}(F)=(36,6)=\frac{6}{7}(42,7)$ and $q=7$ in [1, Corollary 7.4]. That corollary applies to the flipped pair (its hypotheses hold with $(a/l,b)=(42,7)$, and $\rho_0+\sigma_0>0$), and to every pair obtained from it by a translation of the second variable by a Laurent monomial in the first which leaves ℓ_{ρ_0,σ_0} unchanged: such a pair lies in $L^{(1)}$, the degrees $v_{1,1}$, $v_{0,1}$ of the flipped pair are unchanged, and so is F (the element G_0 of [1, Theorem 2.6] can be kept, and $(0,6/7)$ is excluded in $L^{(1)}$ as well). It yields: for every weight (ρ,σ) between $(1,-6)$ and (ρ_0,σ_0) at

which the pair has an edge, and such that the value of P is positive at all weights from (ρ,σ) to (ρ_0,σ_0), the edge form of P is a power R^{7m}.

In the original variables, consider the weights $(-s,1)$ with $s<6$. The point $(7m,42m)=\mathrm{en}_{7,-1}(P)$ is the starting point of the edge of P at $\mathrm{Succ}_P(1,0)$: an edge with a direction strictly between $(7,-1)$ and $(1,1)$ starting at $(7m,42m)$ would end at a point with $v_{1,1}>49m=\deg P$ [1, Remark 1.8], and by the condition on $\mathrm{Succ}_P(1,0)$ there is no direction of P in $](1,0),(-1,1)]$; then [1, Proposition 3.7] applies. As long as $(7m,42m)$ is the starting point of an edge of weight $(-s,1)$ with $s<6$, the value of P there is $7m(6-s)>0$, the point $(7m,42m)$ is leading at all weights from $(7,-1)$ to $(-s,1)$, where the value of P is therefore positive as well, and the edge form is a power R^{7m} with R a $(-s,1)$-homogeneous element of $K[x,y^{\pm1}]$ whose only possible monomials are xy^6 and y^{6-s}; so the form is $(xy^6-\gamma y^{6-s})^{7m}$ up to a constant, $s\in\mathbb{Z}$, and $s\geq 2$ by the condition on $\mathrm{Succ}_P(1,0)$, that is $s\in\{2,3,4,5\}$. The translation $x\mapsto x+\gamma y^{-s}$ removes this edge and leaves unchanged the leading forms at the weights between $(7,-1)$ and $(-s,1)$, while the $(-s,1)$-form becomes the monomial $x^{7m}y^{42m}$ ([1, Proposition 3.9], applied after the flip); so the next edge starting at $(7m,42m)$, if any, has a larger s. After at most four such translations ($s=2,3,4,5$ in this order; we write $\gamma_2,\dots,\gamma_5$ for the constants, which may be 0) no edge of weight $(-s,1)$, $s<6$, starts at $(7m,42m)$. Since the weight $(-6,1)$ lies within a half-turn of $(7,-1)$, the point $(7m,42m)$ is then still leading for $(-6,1)$ [1, Remark 2.10, Proposition 3.7]; hence every monomial x^iy^j of the translated P satisfies $j\leq 6i$, that is, $x^iy^j=z^iv^{6i-j}$ is a monomial in $z=xy^6$, $v=y^{-1}$.

A translation $x\mapsto x+\gamma y^{-s}$, $2\leq s\leq 5$, replaces x^iy^j by monomials $x^{i-r}y^{j-sr}$ of $(7,-1)$-weight $7i-j-(7-s)r$, so (D1) holds for the translated pair with the same leading forms. Since $x^iy^j=z^iv^{6i-j}$ has $7i-j=i+(6i-j)$, the bound $7i-j\leq 7m$ is the bound on the total degree in z,v, and for $x=zv^6$, $y=v^{-1}$ one has $t=zv^{-1}$ and $x(t-c)^6=z(z-cv)^6$, which gives the homogeneous parts of top degree (that $\hat{Q}$ is a polynomial is shown below). The Jacobian follows from $dx\wedge dy=-v^4\,dz\wedge dv$.

For Q: for the pair obtained after the translations and $5\leq s<6$, the values of P and Q are positive, because $(7m,42m)$ and $(7n,42n)$ are in the supports; the leading forms commute by [1, Proposition 1.13] (the sum of the values exceeds $1-s$), so $\ell_{-s,1}(Q)$ is proportional to a power of the monomial $\ell_{-s,1}(P)$ [1, Proposition 2.1(2)], hence a monomial at a point $(i_0,6i_0)$ of the support of the translated Q; by (D1), which holds for the translated pair as shown above, every point $(i,6i)$ of this support has $7i-6i\leq 7n$, that is $i\leq 7n$, and $(7n,42n)$ belongs to it, so $i_0=7n$ by maximality of the value $i(6-s)$ on such points, and $j-si\leq 7n(6-s)$ on the support of the translated Q. Letting s tend to 6 gives $j\leq 6i$ there. □

3. The right side

Write $X:=x^{1/7}$, so that $L^{(7)}=K[X^{\pm1},y]$.

Lemma 3.1. *There is $\lambda\in K^\times$ with $\lambda^7=c$ such that, with*

$$Y:=\eta+\lambda X^{-1},\qquad P_1(X,\eta):=P\big(X^7,Y\big),\qquad Q_1(X,\eta):=Q\big(X^7,Y\big),$$

and with the weight $W(X^i\eta^j):=i-2j$, every monomial of P_1 has $W\leq m$ and every monomial of Q_1 has $W\leq n$.

Proof. In $\tilde{t}:=x^{1/7}y$ one has $\ell_{7,-1}(P)=\lambda_P x^m(\tilde{t}^7-c)^{6m}$, so every root λ of $\tilde{t}^7=c$ has multiplicity $6m>\frac{m}{l}\frac{a\rho+bl\sigma}{\rho+\sigma}=\frac{7m}{6}$ (with $l=1$, $(a,b)=(7,42)$, $(\rho,\sigma)=(7,-1)$). By [1, Proposition 5.18] the automorphism $\varphi(y)=y+\lambda x^{-1/7}$ gives an (m,n)-pair $(\varphi P,\varphi Q)$ in $L^{(7)}$ with a regular corner at $A_1:=(1,0)+6(1/7,1)=(13/7,6)$, of direction $(\rho',\sigma')<(7,-1)$. We follow this corner as in the proofs of [1, Proposition 5.19] and of Lemma 1.3: Case III steps keep the point A_1 and the ring $L^{(7)}$, since $\rho'\mid 7$ [1, Proposition 5.17], and lower the direction, and after finitely many steps Case I or Case II occurs. Case II is impossible at every step, because [1, Proposition 5.14(4)] requires $\gcd(13,6)>1$, a condition on the point A_1 only. In Case I, $v_{\rho,\sigma}(P_1)+v_{\rho,\sigma}(Q_1)=\rho+\sigma$ [1, Proposition 1.13], while $\mathrm{en}(P_1)=mA_1=\frac{m}{n}\,\mathrm{en}(Q_1)$ [1, Corollary 5.7(2)]. Hence $m(\frac{13}{7}\rho+6\sigma)=\frac{m}{5}(\rho+\sigma)$, that is $(\rho,\sigma)=(7,-2)$, $v(P_1)=m$, $v(Q_1)=n$. A

Case III direction would lie strictly between the direction $(7,-2)$ of the final Case I corner, which is determined by the point A_1, and $(7,-1)$; no direction (ρ',σ') with $\rho' \in \{1,7\}$ and $\sigma' \in \mathbb{Z}$ does, since $-2 < 7\sigma'/\rho' < -1$ has no solution. So Case I occurs for $(\varphi P, \varphi Q)$ itself, with $(\rho,\sigma) = (7,-2)$, and $v_{7,-2}(x^{i/7}y^j) = i - 2j$. □

4. Common coordinates

Let $\mathcal{A}$ be the set of formal sums $\sum c_{ij}X^i\eta^j$ $(i \in \mathbb{Z},\, j \geq 0)$ such that, for every r, only finitely many terms have $W = i - 2j \geq r$. It is a ring containing $K[X^{\pm 1}, \eta]$; for $f \neq 0$ let $W(f)$ be the largest weight of its terms and $\ell(f)$ the sum of the terms of that weight. Then $\ell(fg) = \ell(f)\ell(g)$ and $W(fg) = W(f) + W(g)$, and $1 + \varepsilon$ is invertible in $\mathcal{A}$ when $W(\varepsilon) < 0$.

Lemma 4.1. *Let $\gamma_2, \dots, \gamma_5$ and λ be as in Lemmas 2.1 and 3.1, and put $\zeta := z - cv$. Then $K[z,v] = K[\zeta,v]$. As polynomials in ζ, v, $\hat{P}$ has degree at most m in v and total degree $7m$ with homogeneous part of degree $7m$ equal to $\lambda_P(\zeta + cv)^m\zeta^{6m}$; $\hat{Q}$ has degree at most n in v and total degree $7n$ with homogeneous part of degree $7n$ equal to $\lambda_Q(\zeta + cv)^n\zeta^{6n}$; and*

$$\hat{P}_\zeta \hat{Q}_v - \hat{P}_v \hat{Q}_\zeta = -J\, v^4.$$

Proof. Since ζ is homogeneous of degree 1 in z, v, the change of coordinates preserves total degrees and homogeneous parts, which gives the statements on the total degree by Lemma 2.1, with $z(z - cv)^6 = (\zeta + cv)\zeta^6$. Since $d\zeta \wedge dv = dz \wedge dv$, the bracket is unchanged. It remains to see that $\hat{P}$, written as $\sum p_{ab} z^a \zeta^b$, has $a \leq m$ (this is equivalent to $\deg_v \leq m$ in the coordinates ζ, v, as $z = \zeta + cv$), and likewise for $\hat{Q}$.

In $\mathcal{A}$ the element Y is $\lambda X^{-1}(1+u)$ with $u := \lambda^{-1}\eta X$, $W(u) = -1$; it is invertible. The assignment $v \mapsto Y^{-1}$, $z \mapsto X^7Y^6 - \sum_{s=2}^{5} \gamma_s Y^{6-s}$ defines a ring morphism $K[z,v] \to \mathcal{A}$ which sends $\hat{P}$ to P_1 and $\hat{Q}$ to Q_1, because $P(x,y) = \hat{P}\big((x - \sum \gamma_s y^{-s})y^6, y^{-1}\big)$ in $K[x, y^{\pm 1}]$. Since $X^7Y^6 = \lambda^6 X(1+u)^6$, $cY^{-1} = \lambda^6 X(1+u)^{-1}$ and $W(Y^{6-s}) = s - 6 < 0$, the image of z is $\lambda^6 X +$ (terms of $W \leq 0$) and the image of ζ is

$$\lambda^6 X\big((1+u)^6 - (1+u)^{-1}\big) - \sum_{s=2}^{5} \gamma_s Y^{6-s} = 7\lambda^5 \eta X^2 + (\text{terms of } W \leq -1).$$

Hence the image of $z^a\zeta^b$ has leading term a non-zero multiple of $\eta^b X^{a+2b}$, of weight a, and these monomials are distinct for distinct (a,b). Therefore, for a non-zero polynomial, the value of W on the image of $\sum c_{ab} z^a \zeta^b$ is $\max\{a : c_{ab} \neq 0\}$, and Lemma 3.1 gives $a \leq m$ for $\hat{P}$ and $a \leq n$ for $\hat{Q}$. □

5. The finite system

Let $(U,V) := (\hat{P}, \hat{Q})$ if $(m,n) = (3,2)$ and $(U,V) := (\hat{Q}, \hat{P})$ if $(m,n) = (2,3)$. Writing x for ζ, replacing v by v/c and dividing U, V by their leading constants, a pair as in the standing data yields in both cases a solution of the following system.

- (S1) $U = u_3v^3 + u_2v^2 + u_1v + u_0$ and $V = w_2v^2 + w_1v + w_0$ with $u_i, w_i \in K[x]$, $\deg u_i \leq 21 - i$, $\deg w_i \leq 14 - i$, and the homogeneous parts of degree 21 of U and 14 of V are $(x+v)^3x^{18}$ and $(x+v)^2x^{12}$.
- (S2) $U_xV_v - U_vV_x = -J'\, v^4$ with $J' \in K^\times$.

For $(m,n) = (2,3)$ the bracket changes sign, which is absorbed in J'. In particular u_3 and w_2 are monic of degrees 18 and 12, and the coefficients of x^{19}, x^{20}, x^{21} in u_2, u_1, u_0 are $3, 3, 1$, those

of x^{13}, x^{14} in w_1, w_0 are $2, 1$. The coefficients of $v^4, v^3, v^2, v, 1$ in (S2) are

$$2u_3'w_2 - 3u_3w_2' = -J', \tag{E4}$$

$$u_3'w_1 + 2u_2'w_2 - 3u_3w_1' - 2u_2w_2' = 0, \tag{E3}$$

$$u_2'w_1 + 2u_1'w_2 - 3u_3w_0' - 2u_2w_1' - u_1w_2' = 0, \tag{E2}$$

$$u_1'w_1 + 2u_0'w_2 - 2u_2w_0' - u_1w_1' = 0, \tag{E1}$$

$$u_0'w_1 - u_1w_0' = 0. \tag{E0}$$

Let $E := \mathbb{Q}(a)$ with $a^6 - 2a^5 - 8a^4 + 10a^3 + 12a^2 - 48a + 80 = 0$ (an irreducible polynomial), a number field of degree 6. The ancillary file `leading_pair_147.txt` gives $w_2^* = x^{12} + \sum_{i=0}^{10} c_i x^i$ with $c_i \in E$; let u_3^* be the polynomial part of $(w_2^*)^{3/2}$, that is, the monic polynomial of degree 18 with $\deg\big((u_3^*)^2 - (w_2^*)^3\big) \le 17$.

Lemma 5.1. *(a) One has* $\deg\big((u_3^*)^2 - (w_2^*)^3\big) = 7$, $2u_3^{*\prime}w_2^* - 3u_3^*w_2^{*\prime}$ *is a non-zero constant* $-J^*$, $c_{10} \neq 0$, *and* c_8/c_{10}^2 *generates* E *(the representative is normalised by* $c_9 = c_{10}$, *which is not used).*

(b) If (S1)–(S2) *has a solution, then there are a field embedding* $\sigma\colon E \to K$ *and a solution with* $u_3 = \sigma(u_3^*)$, $w_2 = \sigma(w_2^*)$, $J' = \sigma(J^*)$, $u_0(0) = w_0(0) = 0$, *and no term of degree* 14 *in* u_0.

Proof. (a) is an exact computation in $E[x]$ (ancillary script; the minimal polynomial of c_8/c_{10}^2 over $\mathbb{Q}$ is irreducible of degree 6).

(b) The system is preserved by: adding constants to U and V; replacing U by $U - \alpha V$ (the homogeneous part of degree 21 of U is unchanged, as $\deg V = 14$); the translation $x \mapsto x + \tau$; and the scaling $U \mapsto \nu^{-21}U(\nu x, \nu v)$, $V \mapsto \nu^{-14}V(\nu x, \nu v)$, $\nu \in K^\times$, which preserves (S1) and replaces J' by $\nu^{-29}J'$ and the coefficient of x^i in w_2 by ν^{i-12} times itself. Using the translation, then $U \mapsto U - \alpha V$, then the addition of constants, we may assume that w_2 has no term of degree 11, that u_0 has no term of degree 14 (the coefficient of x^{14} in w_0 is 1) and that $u_0(0) = w_0(0) = 0$; the scalings preserve these conditions.

Put $D := u_3^2 - w_2^3$ and $W := 2u_3'w_2 - 3u_3w_2'$, so that $(u_3^2/w_2^3)' = u_3W/w_2^4$. Since u_3^2 and w_2^3 are monic of degree 36, D/w_2^3 has a zero at infinity. If $W = -J'$ is constant, the order at infinity of $(D/w_2^3)' = -J'u_3/w_2^4$ is 30, so D/w_2^3 has order 29 there: $\deg D = 7$ and $J' = 29d_7$, where d_7 is the leading coefficient of D. Conversely, if $\deg D \le 7$, then u_3W/w_2^4 has order at least 30 at infinity, so W is constant, equal to $-29d_7$. Moreover u_3 is determined by w_2, as the polynomial part of $w_2^{3/2}$. Write $w_2 = x^{12} + \sum_{i=0}^{10} b_i x^i$. The leading pairs (u_3, w_2) of solutions of (S1)–(S2) normalised as above therefore correspond to the points $(b_0, \dots, b_{10})$ at which the coefficients of $x^8, \dots, x^{17}$ of D vanish and $d_7 \neq 0$. There are at most 174 such points with $d_7 = 1$. Indeed, $W = -J'$ vanishes at no point of K, while it vanishes at a multiple root of w_2 or of u_3 and at a common root of w_2 and u_3; so w_2 and u_3 have simple roots and no common root, and (w_2^3, u_3^2) is a Davenport–Zannier pair with passport $\big((3^{12}), (2^{18})\big)$ in the sense of [6, Definition 1.2]: here $n = 36$, $p = 12$, $q = 18$, $\deg(w_2^3 - u_3^2) = 7 = (n+1) - (p+q)$, and the conditions of [6, Assumption 1.1] hold ($\gcd(3,2) = 1$ and $p + q \le n + 1$). The eleven equations considered here (the vanishing of the coefficients of $x^8, \dots, x^{17}$ of D, and $d_7 = 1$) are defined over $\mathbb{Q}$, so it suffices to count their solutions over $\mathbb{C}$: a zero-dimensional system defined over $\mathbb{Q}$ has its solutions in $\overline{\mathbb{Q}}$, and the same number of them in K. By [6, Proposition 1.4] (established in [7]; a consequence of Riemann's existence theorem, see [9, 8]), the classes of such pairs under $(\tilde P, \tilde Q) = (\theta\, P(\alpha x + \beta), \theta\, Q(\alpha x + \beta))$, $\theta\alpha \neq 0$, are in bijection with the weighted bicolored plane trees of this passport. In such a tree every edge has weight 1 or 2 (an edge of weight 3 would end at a white vertex of degree, that is total weight, at least 3), and since the 29 edges have total weight 36, exactly 7 edges have weight 2; each of them joins a white leaf to a black vertex whose other edge has weight 1, and the other black vertices carry three edges of weight 1. Deleting the 7 white leaves and the white vertices with two edges of weight 1, which only subdivide edges, gives a bijection with the plane trees having 7 leaves and 5 vertices of degree 3, that is, with the triangulations of a heptagon up to rotation; there are $42/7 = 6$ of them,

since no triangulation of a heptagon is invariant under a non-trivial rotation. Finally, in a class the normalisations (w_2 monic without term of degree 11, $d_7 = 1$) leave at most 29 pairs: d_7 is multiplied by α^{-29} when the pair is made monic again, and then β is determined. As a computational confirmation, the 11 equations with $d_7 = 1$ define an ideal of dimension 0 and degree 174 modulo three primes (Singular [10], ancillary script).

On the other hand, by (a), for each of the 6 embeddings σ and each of the 29 values of ν with $\nu^{29} = \sigma(J^*)/29$, the scaling with parameter ν of $(\sigma(u_3^*), \sigma(w_2^*))$ is such a point with $d_7 = 1$; for it $b_{10} = \nu^{-2}\sigma(c_{10})$ and $b_8/b_{10}^2 = \sigma(c_8/c_{10}^2)$. These 174 points are distinct: distinct embeddings take distinct values on the generator c_8/c_{10}^2 of E, and for a fixed σ the conditions $\nu_1^{-2} = \nu_2^{-2}$ and $\nu_1^{29} = \nu_2^{29}$ give $\nu_1 = \nu_2$. Hence they are all the points with $d_7 = 1$. A normalised solution of (S1)–(S2) has $d_7 = J'/29 \neq 0$; after a scaling it has $d_7 = 1$, so its leading pair is one of these points, and applying the inverse of the scaling which produced that point from $\sigma(u_3^*, w_2^*)$ gives the solution of the statement. □

Proposition 5.2. *For every embedding $\sigma\colon E \to K$, the system* (S1)–(S2) *with $u_3 = \sigma(u_3^*)$, $w_2 = \sigma(w_2^*)$, $J' = \sigma(J^*)$, $u_0(0) = w_0(0) = 0$ and no term of degree* 14 *in u_0 has no solution.*

Proof (exact computation). It suffices to treat the system with coefficients in E and to show that the polynomials obtained below generate the unit ideal over E: applying σ to a relation $1 = \sum h_i f_i$ gives the statement for σ. The unknowns are the coefficients of u_2, u_1, u_0, w_1, w_0 below their prescribed leading terms: $19 + 20 + 19 + 13 + 13 = 84$ unknowns. Each of (E3), (E2), (E1) is, coefficient by coefficient, a system of linear equations with matrix over E: (E3) in the coefficients of u_2, w_1; then (E2) in those of u_1, w_0; then (E1) in those of u_0; the right-hand sides depend on the solutions of the previous steps. Reducing each matrix to echelon form expresses the pivot unknowns through the others, and the rows without pivot give *conditions*, polynomials with coefficients in E in the unknowns which remain free, which vanish on every solution. Carrying this out (ancillary script): (E3), in 32 unknowns, has rank 29, leaves (for the pivot choice of the script) the coefficients of x^{12} in u_2 and of x^{11}, x^{12} in w_1 free and gives no condition; (E2), in 33 unknowns, has rank 30, leaves the coefficients of x^{11}, x^{12}, x^{13} in w_0 free and gives no condition; (E1) determines u_0 and gives 13 conditions; finally the coefficients of (E0) give 33 conditions. These 46 polynomials, of degrees 3 and 4 in the six free unknowns, generate the unit ideal (Gröbner basis over E, Singular); the 13 conditions coming from (E1) already do, and in fact 1 is an E-linear combination of these 13 cubic polynomials (comparison of ranks of their coefficient matrix over E, with and without the constant vector; ancillary script). □

As an independent confirmation, the script builds the full normalised system (84 unknowns, 126 equations) by direct expansion modulo the prime $p = 536853127$, at which the defining polynomial of E has six roots, and finds it inconsistent at each of them. As a check of the procedure, it applies the same elimination to a pair (U, V) with $u_3 = u_3^*$, $w_2 = w_2^*$ and randomly chosen lower coefficients, with the right-hand side of (S2) replaced by its bracket: the conditions then vanish at the chosen values, and no longer all do so after the right-hand side is modified.

Proof of Theorem 1.2. A pair as in the standing data gives a solution of (S1)–(S2) by Lemmas 1.3–4.1, hence one as in Lemma 5.1(b), contradicting Proposition 5.2. □

6. Scope

- **Proved here, relative to the cited results:** the reduction of Theorem 1.2 to Proposition 5.2 (Lemmas 1.3–5.1). The inputs are (D1), which rests on [1, Proposition 6.1, Theorem 7.6(2), Remark 3.1], and [1, Theorem 2.6, Remarks 1.8, 2.5, 2.10, Corollaries 5.7, 7.4, Propositions 1.13, 2.1, 3.7, 3.9, 5.13, 5.14, 5.16, 5.17, 5.18, 5.19] and [6, Proposition 1.4], as indicated. The condition on $\mathrm{Succ}_P(1, 0)$ in the standing data is the one of [1, Corollary 5.21(4)], which is established there for the standard pairs attached to a minimal counterexample; the theorem is stated under this condition.

- **Verified by exact computation:** Proposition 5.2; Lemma 5.1(a); the count of leading pairs in its proof modulo three primes (the bound 174 itself is derived from [6], as indicated); the expansions in Lemma 4.1 and the Jacobian in Lemma 2.1 (ancillary files `verify_147.py` and `leading_pair_147.txt` in the folder `anc`; SymPy, python-flint, PARI, and Singular for the Gröbner bases, for which the script writes the input files). The result depends on the correctness of these computations.
- **Not claimed:** a statement on the plane Jacobian conjecture itself; on the other shapes of maximal degree 147 in [2]; a new bound on the degree of a counterexample. The completeness of the list of [2] is not re-examined.
- [1, Corollary 7.4], on which Lemma 2.1 rests, is stated in [1] with the indication that its proof mimics that of Corollary 7.2, using [1, Proposition 7.3], whose proof mimics that of Proposition 7.1; it is applied here to Laurent translates, as in [1, Section 8] and [5]. The references [2] and [5] are cited in their arXiv versions.
- The argument has been prepared by the author with AI assistance (Claude, Anthropic). Nothing in this note has been checked by a third party; it has not been validated by professionals.

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