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THE SHAPES OF MAXIMAL DEGREE 147 WITH STARTING CORNER (6, 15) IN THE PLANE JACOBIAN PROBLEM

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Abstract. In the list of Guccione, Guccione, Horruitiner and Valqui of the possible shapes of a counterexample to the plane Jacobian conjecture with degrees at most 150, the families F_7 and F_8 have starting corner $(6, 15)$ and final corners $(7/3, 4)$ and $(8/3, 5)$; for degrees at most 150 they give the standard (m, n)-pairs with $(m, n) = (2, 7)$ or $(7, 2)$ (degrees 42 and 147) and $(m, n) = (3, 7)$ or $(7, 3)$ (degrees 63 and 147). We show that no such pair exists, the pair being normalised as in Corollary 5.21 of Guccione–Guccione–Valqui. The left side of the Newton polygon is controlled by a lattice argument applied to translates of the pair, after which the pair lies in an explicit polynomial ring with a weight coming from the final corner. The top part of the system has three solutions for F_8, conjugate over a cubic field, and one solution for F_7, over $\mathbb{Q}(\sqrt{-15})$. Over these fields the remaining equations are solved by linear algebra layer by layer, and the compatibility conditions, polynomials in six unknowns, generate the unit ideal; an explicit certificate $1 = \sum c_i f_i$ is computed and checked exactly. The computation is reproduced by an ancillary script (SymPy and python-flint). All statements are proved relative to the cited results. Nothing in this note has been checked by a third party.

1. Statement

Throughout, K is an algebraically closed field of characteristic zero, $L = K[x, y]$ and $[P, Q] = P_x Q_y - P_y Q_x$. We use the notation of [1]: for a weight (ρ, σ), $v_{\rho,\sigma}$ is the weighted degree, $\ell_{\rho,\sigma}$ the leading form, $\mathrm{st}_{\rho,\sigma}$ and $\mathrm{en}_{\rho,\sigma}$ its end points; $L^{(l)} = K[x^{\pm 1/l}, y]$. *Standard (m, n)-pairs*, their *starting triple* and their *regular corners* are defined in [1, Definitions 4.3, 6.2, 5.5]. Put $t := xy^3$.

Standing data 1.1. (m, n) is one of $(2, 7)$, $(7, 2)$, $(3, 7)$, $(7, 3)$; (P, Q) is a standard (m, n)-pair in L with $[P, Q] = J \in K^\times$, $\deg P = 21m$, $\deg Q = 21n$, starting triple $\big((6, 15), (1, 0), (3, -1)\big)$, and $(-1, 1) < \mathrm{Succ}_P(1, 0) < (-1, 0)$ as in [1, Corollary 5.21(4)]. Thus:

(D1) the support of P lies in $\{(i, j) : 3i - j \le 3m\}$, that of Q in $\{3i - j \le 3n\}$, and

$$\ell_{3,-1}(P) = \lambda_P \big(x\, h(t)\big)^m, \qquad \ell_{3,-1}(Q) = \lambda_Q \big(x\, h(t)\big)^n, \qquad \lambda_P, \lambda_Q \in K^\times,$$

with $h \in K[t]$ monic of degree 5, $h(0) \neq 0$ [1, Proposition 6.1, Theorem 7.6(2), Remark 3.1].

Moreover h has a root c of multiplicity $k \in \{4, 5\}$: for $k = 5$ (family F_8), $\{m, n\} = \{3, 7\}$ and $h = (t - c)^5$; for $k = 4$ (family F_7), $\{m, n\} = \{2, 7\}$ and $h = (t - c)^4 (t - c')$ with $c' \neq c$.

Theorem 1.2. *There is no pair (P, Q) as in Standing data 1.1.*

Remark 1.3. A root of h of multiplicity k satisfies inequality (5.9) of [1, Proposition 5.18], which here reads $km \ge \frac{m}{l} \frac{a\rho + bl\sigma}{\rho + \sigma} = \frac{3m}{2}$ (with $l = 1$, $(a, b) = (6, 15)$, $(\rho, \sigma) = (3, -1)$), exactly when $k \ge 2$, and then generates the regular corner $(1, 0) + k(1/3, 1)$ of an (m, n)-pair in $L^{(3)}$ (Lemma 3.1 below). The last condition of the standing data is the case of the corners $(7/3, 4)$ and $(8/3, 5)$: these are the families F_7 and F_8 of [2, Section 5], listed in [2, Section 6] with $(m, n) = (2, 7)$ and $(3, 7)$ (maximal degree 147), the cases $(7, 2)$ and $(7, 3)$ being obtained by swapping m and n as indicated there. Apart from the preprint recalled below, we found no exclusion of these shapes in the literature we searched; their novelty is not otherwise verified.

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The preprint [3] of the four authors discards all cases with $\max\{\deg P, \deg Q\} \leq 150$ except $(75, 125)$, using a formula for the intersection number of a Jacobian pair; in the later version of that work [4] only an inequality is obtained, and the authors state that the infinite families cannot be discarded in this way. They treat in [5] the shapes of degree below 125 and obtain the bound 108.

Replacing $P(x, y), Q(x, y)$ by $P(cx, y), Q(cx, y)$ multiplies J by c, keeps the standing data and replaces $h(t)$ by $c^{-5}h(ct)$; *we assume from now on that* $c = 1$.

Lemma 1.4. *Let F be the element of* [1, Theorem 2.6] *for P and the weight $(3, -1)$. Then $F = xy\, f(t)$ with $f \in K[t]$ of degree 3, so that $\mathrm{en}_{3,-1}(F) = (4, 10) = \frac{2}{3}(6, 15)$. For F_7, c' is a root of $2u^2 - 5u + 5$, and $H := 1/(1 - c')$ is a root of $2H^2 + H + 2$.*

Proof. $F \in L$ is $(3, -1)$-homogeneous with $v_{3,-1}(F) = 3 - 1 = 2$ [1, Theorem 2.6(1)], so $F = xy\, f(t)$ with $f \in K[t]$, and $[F, \ell_{3,-1}(P)] = \ell_{3,-1}(P)$. By [1, Theorem 2.6(3)], $\mathrm{en}(F) = (1 + d, 1 + 3d)$, $d = \deg f$, is either $(1, 1)$ or proportional to $\mathrm{en}(P) = m(6, 15)$; the second case gives $d = 3$; in the first, $F = \mu xy$ and $[xy, x^i y^j] = (j - i)x^i y^j$, while the monomials $x^{m+r}y^{3r}$ of $\ell(P)$ have the non-constant values $2r - m$ of $j - i$, which is impossible. Solving $[xy\, f(t), \lambda_P(xh(t))^m] = \lambda_P(xh(t))^m$ for f of degree 3 (ancillary script) gives a unique f for F_8, and, for F_7, a solution exactly when $c' \in \{0, 1\}$ or $2c'^2 - 5c' + 5 = 0$; here $c' \neq 0$ as $h(0) \neq 0$ and $c' \neq 1 = c$. □

2. The left side

Lemma 2.1. *Let $R = x^2y^5 + \gamma y^2$ with $\gamma \in K^\times$. There are no $i \in \mathbb{N}_0$ and no $(-3, 2)$-homogeneous $G \in L$ with $[G, R] = R^i$.*

Proof. The substitution $x \mapsto \kappa x$, $\kappa^2 = \gamma$, multiplies the bracket by κ and R by γ; so we may assume $\gamma = 1$, at the cost of a constant factor, which is absorbed in G. Put $z := x^{-2}y^{-3}$ and $r := 1 + z$, so that $R = x^2y^5r(z)$. A $(-3, 2)$-homogeneous G with $[G, R] = R^i$ has the value fixed by i and can be written $G = x^{2i-1}y^{5i-4}f(z)$ with $f = \sum_k f_k z^k$ a finite sum; its monomials $x^{2i-1-2k}y^{5i-4-3k}$ have integral exponents, so $k \in \mathbb{Z}$. Since $[x^A y^B, x^C y^D] = (AD - BC)x^{A+C-1}y^{B+D-1}$, applied term by term with $A = a - 2k$, $B = b - 3k$, $C = 2 - 2k'$, $D = 5 - 3k'$, one has

$$[x^a y^b f(z),\, x^2y^5r(z)] = x^{a+1}y^{b+4}\big((5a - 2b)fr + (2b - 3a)zfr' - 4zf'r\big),$$

so the equation reads $E := 3fr + (4i - 5)zfr' - 4zf'r = r^i$. If k_0 is the smallest index with $f_{k_0} \neq 0$, the coefficient of z^{k_0} in E is $(3 - 4k_0)f_{k_0}$, and that of r^i is 0 for $k_0 < 0$; hence $k_0 \geq 0$ and $f \in K[z]$. If $d = \deg f$, the coefficient of z^{d+1} in E is $(4i - 2 - 4d)f_d \neq 0$; so $d + 1 \leq i$, that is, $d \leq i - 1$ and $i \geq 1$. On the other hand, if $f = r^j g$ with $0 \leq j < i$, then $E = r^j\big(3rg + (4i - 5 - 4j)zg - 4zrg'\big)$, and evaluating $E/r^j = r^{i-j}$ at $z = -1$ gives $(4i - 5 - 4j)g(-1) = 0$, hence $g(-1) = 0$ since $4(i - j) - 5$ is odd. By induction r^i divides f, which contradicts $0 \leq \deg f \leq i - 1$. □

Lemma 2.2. *Put $v := y^{-1}$ and $\tau := xy^2 - y^{-1}$, so that $L \subset K[v^{\pm 1}, \tau]$ ($x = \tau v^2 + v^3$). There are $\alpha, \beta \in K$ such that, with $g(T) := (T - \alpha)(T - \beta)$,*

$$Y_1 := \tau - \alpha, \qquad Y_2 := g(\tau)/v,$$

both P and Q belong to the subring $K[v, Y_1, Y_2]$ of $K[v^{\pm 1}, \tau]$.

Proof. Step 1: the tool. Apply the flip ψ_1 of [1, Section 6] (as used in [1, Section 8]), so that the weight $(3, -1)$ becomes $(\rho_0, \sigma_0) = (-1, 3)$, with $\mathrm{st}_{\rho_0,\sigma_0} = m(15, 6)$ and $\mathrm{en}_{\rho_0,\sigma_0} = m(0, 1)$ for the flipped P, and $\frac{1}{m}\mathrm{st}(P) = \frac{1}{n}\mathrm{st}(Q) = (15, 6)$. Let F be the element of [1, Theorem 2.6] for (ρ_0, σ_0); then $v_{-1,3}(F) = 2$. If $\mathrm{st}(F) = (1, 1)$ then $\mathrm{en}(F) \neq (1, 1)$, since F is not a monomial [1, Remark 2.5], so $\mathrm{en}(F) \sim \mathrm{en}(P) \sim (0, 1)$ and $\mathrm{en}(F) = (0, 2/3)$, which is impossible; hence $\mathrm{st}(F) \sim (15, 6)$, so $\mathrm{st}(F) = (10, 4) = \frac{2}{3}(15, 6)$ (as in Lemma 1.4) and $q = 3$ in [1, Corollary 7.4]. That corollary applies to the flipped pair (its hypotheses hold with $(a/l, b) = (15, 6)$, and $\rho_0 + \sigma_0 > 0$), and to every pair obtained from it by a translation of the second variable by

a Laurent monomial in the first which leaves ℓ_{ρ_0,σ_0} unchanged: such a pair lies in $L^{(1)}$, the degrees $v_{1,1}$, $v_{0,1}$ of the flipped pair are unchanged, and ℓ_{ρ_0,σ_0} is unchanged, so that the argument above (with $(0, 2/3)$ excluded in $L^{(1)}$) gives $\mathrm{st}(F) = (10, 4)$ and $q = 3$ for any element F of [1, Theorem 2.6] attached to the translated pair. Back in the original variables, it yields: *for the pair $(\tilde P, \tilde Q)$ obtained by a substitution $x \mapsto x + \zeta y^{-2}$, $\zeta \in K$ (which leaves $\ell_{3,-1}$ unchanged, as $v_{3,-1}(y^{-2}) = 2 < 3$), and for every weight $(-\theta, 1)$, $\theta > 0$, at which $\tilde P$ has an edge and such that the value of $\tilde P$ is positive at all weights from $(3, -1)$ to $(-\theta, 1)$, the edge form is a power R^{3m} with R a $(-\theta, 1)$-homogeneous element of $K[x, y^{\pm 1}]$.* The polynomial $\tilde P$ lies in $K[x, y^{\pm 1}]$, and so does R.

Step 2: the first edge. The point $(6m, 15m) = \mathrm{en}_{3,-1}(P)$ is the starting point of the edge of P at $\mathrm{Succ}_P(1, 0)$: an edge with a direction strictly between $(3, -1)$ and $(1, 1)$ starting at $(6m, 15m)$ would end at a point with $v_{1,1} > 21m = \deg P$ [1, Remark 1.8], and by the condition on $\mathrm{Succ}_P(1, 0)$ there is no direction of P in $](1, 0), (-1, 1)]$; then [1, Proposition 3.7] applies. Consider the weights $(-\theta, 1)$ with $\theta < 5/2$; the value of $(6m, 15m)$ there is $3m(5 - 2\theta) > 0$. As long as $(6m, 15m)$ is the starting point of an edge of weight $(-\theta, 1)$ with $\theta < 5/2$, it is leading at all weights from $(3, -1)$ to $(-\theta, 1)$, where the value of P is therefore positive, and by Step 1 (with $\zeta = 0$) the edge form is R^{3m}, where R starts at $(2, 5)$ and can only contain the monomials x^2y^5, $xy^{5-\theta}$, $y^{5-2\theta}$, with integral exponents; and $\theta > 1$ by the condition on $\mathrm{Succ}_P(1, 0)$. So $\theta \in \{3/2, 2\}$. For $\theta = 3/2$ one gets $R = \mu(x^2y^5 + \gamma y^2)$, $\gamma \neq 0$, a polynomial which is not a proper power. By [1, Lemma 2.2], applied with the weight $(-3, 2)$, at which $v(P) = 12m \neq 0$, there is $G_0 \in K[P, Q] \subset L$ with $C := [\ell(G_0), \ell(P)] \neq 0$ and $[C, \ell(P)] = 0$; by [1, Proposition 2.1(2b)] (with $v(P) > 0$, the case $C, \ell(P) \in K[x^{\pm 1}]$ being excluded as $\ell(P)$ involves y), C and $\ell(P) = \lambda R^{3m}$ are proportional to non-negative powers of a common element of L (the case where the two values have opposite signs being the excluded one), which is a power of R, because $R = y^2(x^2y^3 + \gamma)$ with $x^2y^3 + \gamma$ irreducible and $\gcd(2, 1) = 1$; so $C = \mu' R^e$; then $[\ell(G_0), R] = \mu'' R^{e-3m+1}$, and multiplying $\ell(G_0)$ by a power of R and by a constant gives $G \in L$, $(-3, 2)$-homogeneous, with $[G, R] = R^i$, $i \geq 0$, against Lemma 2.1. Hence either no edge of weight $(-\theta, 1)$ with $\theta < 5/2$ starts at $(6m, 15m)$, and we put $\alpha = \beta = 0$, or the first one has $\theta = 2$ and its form is $\lambda_P\big(y\, g(xy^2)\big)^{3m}$ with $g(T) = (T - \alpha)(T - \beta)$ monic of degree 2, not of the form T^2.

Step 3: a criterion. Write $f \in K[v^{\pm 1}, \tau]$ as $\sum_e v^e \varphi_e(\tau)$. Since $K[v, Y_1, Y_2] = \sum_{d \geq 0} (g(\tau)/v)^d K[v, \tau]$, $f \in K[v, Y_1, Y_2]$ if and only if g^d divides φ_{-d} for every $d > 0$, that is, if and only if for every root ζ of g, of multiplicity e_ζ, the monomial valuation μ_ζ of $K[v^{\pm 1}, \tau - \zeta]$ with $\mu_\zeta(v) = e_\zeta$, $\mu_\zeta(\tau - \zeta) = 1$ satisfies $\mu_\zeta(f) \geq 0$. Let $\tilde P, \tilde Q$ be obtained by $x \mapsto x + \zeta y^{-2}$, and $\tilde x := x - \zeta y^{-2}$, so that $\tau - \zeta = \tilde x y^2 - y^{-1}$ and $P(x, y) = \tilde P(\tilde x, y)$. If $e_\zeta = 2$, let ν be the monomial valuation of $K[\tilde x, y^{\pm 1}]$ with $\nu(\tilde x) = 5$, $\nu(y) = -2$; then $\nu(v) = 2$, and $\tau - \zeta$ has ν-value 1 with initial form $\tilde x y^2$. If $e_\zeta = 1$, let $\nu(\tilde x) = 3$, $\nu(y) = -1$; then $\nu(v) = 1$, and $\tau - \zeta$ has ν-value 1 with initial form $y^{-1}(\tilde x y^3 - 1)$. In both cases the initial forms of v and $\tau - \zeta$ are algebraically independent, so ν and μ_ζ agree on $K[v^{\pm 1}, \tau]$. Hence $\mu_\zeta(P) \geq 0$ if and only if every monomial $\tilde x^i y^j$ of $\tilde P$ has $5i - 2j \geq 0$ (if $e_\zeta = 2$), respectively $3i - j \geq 0$ (if $e_\zeta = 1$); likewise for Q.

Step 4: a double root. Let $e_\zeta = 2$ (this covers $\alpha = \beta = 0$). The substitution is $(-2, 1)$-homogeneous, so $\ell_{-2,1}(\tilde P) = \lambda_P \tilde x^{6m} y^{15m}$, with $(6m, 15m)$ leading from $(3, -1)$ to $(-2, 1)$. Suppose that an edge of $\tilde P$ of weight $(-\theta, 1)$ with $2 < \theta < 5/2$ starts at $(6m, 15m)$, and take the first one. The value of $(6m, 15m)$ is positive at all weights from $(3, -1)$ to $(-\theta, 1)$, so by Step 1 the edge form is R^{3m}, with R starting at $(2, 5)$ and other monomials among $\tilde x y^{5-\theta}$, $y^{5-2\theta}$, with integral exponents: $\theta \in \mathbb{Z}$ or $2\theta \in \mathbb{Z}$, which is impossible for $2 < \theta < 5/2$. So $(6m, 15m)$ is leading for all weights $(-\theta, 1)$, $2 \leq \theta < 5/2$ [1, Proposition 3.7], and by continuity for $\theta = 5/2$: every monomial of $\tilde P$ has $j - \frac{5}{2} i \leq 0$.

Step 5: a simple root. Let $e_\zeta = 1$, and let ζ' be the other root. Then $\ell_{-2,1}(\tilde P) = \lambda_P y^{3m} (\tilde x y^2)^{3m} (\tilde x y^2 + \zeta - \zeta')^{3m}$, an edge from $(6m, 15m)$ to $(3m, 9m)$. Suppose that an edge of $\tilde P$ of weight $(-\theta, 1)$ with $2 < \theta < 3$ starts at $(3m, 9m)$, and take the first one. The values at all weights from $(3, -1)$ to $(-2, 1)$ are positive (the point $(6m, 15m)$), and from $(-2, 1)$ to

$(-\theta,1)$ the value of $(3m,9m)$ is $3m(3-\theta)>0$; by Step 1 the edge form is R^{3m}, with R starting at $(1,3)$ and other monomials among $y^{3-\theta}$: $\theta\in\mathbb{Z}$, which is impossible. As in Step 4, every monomial of $\tilde P$ has $j-3i\le 0$.

Step 6: Q. The point $(6n,15n)$ lies in the support of $\tilde Q$, since $\ell_{3,-1}(\tilde Q)=\ell_{3,-1}(Q)$, and $v_{3,-1}(\tilde Q)=3n$. At a weight $(-\theta,1)$ with $1<\theta<5/2$ the values of $\tilde P$ and $\tilde Q$ are therefore positive, and since $\rho+\sigma=1-\theta<0$ while $[\tilde P,\tilde Q]=J$ has value 0, the leading forms commute [1, Proposition 1.13]; by [1, Proposition 2.1(2a)] some powers of them are proportional. In Step 4, for $2\le\theta<5/2$, $\ell(\tilde P)$ is the monomial at $(6m,15m)$, so $\ell(\tilde Q)$ is a monomial at a point $c(6,15)$ of the support of $\tilde Q$; comparing its value with that of $(6n,15n)$ gives $c\ge n$, and $v_{3,-1}(\tilde Q)=3n$ gives $c\le n$. So $j-\theta i\le 3n(5-2\theta)$ on the support of $\tilde Q$, and letting θ tend to $5/2$ gives $5i-2j\ge 0$. In Step 5, at $(-2,1)$, $\ell(\tilde P)=\lambda_P R_0^{3m}$ with $R_0:=\tilde x y^3(\tilde x y^2+\zeta-\zeta')$, which is not a proper power; so $\ell(\tilde Q)=\mu R_0^N$, whose point of largest $\tilde x$-degree $N(2,5)$ has $(-2,1)$-value $N\ge 3n$ (the value of $(6n,15n)$) and $(3,-1)$-value $N\le 3n$; hence $N=3n$, and $(3n,9n)=3n(1,3)$ lies in the support of $\tilde Q$. For $2<\theta<3$, $\ell(\tilde P)$ is the monomial at $(3m,9m)$, so $\ell(\tilde Q)$ is a monomial at a point $c(1,3)$ with $c\ge 3n$ (compare with $(3n,9n)$) and $c\le v_{-2,1}(\tilde Q)=3n$; so $j-\theta i\le 3n(3-\theta)$ on the support of $\tilde Q$, and letting θ tend to 3 gives $3i-j\ge 0$.

By Step 3, $\mu_\zeta(P)\ge 0$ and $\mu_\zeta(Q)\ge 0$ for every root ζ of g, and the lemma follows. □

3. The right side

Write $X:=x^{1/3}$, so that $L^{(3)}=K[X^{\pm1},y]$. Put $W(X^i\eta^j):=7i-11j$ for F_8 and $W(X^i\eta^j):=7i-12j$ for F_7.

Lemma 3.1. *There is $\omega\in K$ with $\omega^3=1$ such that, with*

$$Y:=\eta+\omega X^{-1},\qquad P_1(X,\eta):=P\big(X^3,Y\big),\qquad Q_1(X,\eta):=Q\big(X^3,Y\big),$$

every monomial of P_1 has $W\le m$ and every monomial of Q_1 has $W\le n$.

Proof. In $\tilde t:=x^{1/3}y$ one has $\ell_{3,-1}(P)=\lambda_P x^m\,h(\tilde t^3)^m$, so every root ω of $\tilde t^3=1$ has multiplicity $km\ge\frac{3m}{2}$ (Remark 1.3); moreover $[\ell_{3,-1}(P),\ell_{3,-1}(Q)]=0$ by (D1), and $h(\tilde t^3)^m$ has at least three distinct roots, so that the polynomial attached to $\ell_{3,-1}(P)$ has more than one factor. By [1, Proposition 5.18] the automorphism $\varphi(y)=y+\omega x^{-1/3}$ gives an (m,n)-pair $(\varphi P,\varphi Q)$ in $L^{(3)}$ with a regular corner at $A_1:=(1,0)+k(1/3,1)=\big((3+k)/3,k\big)$, of direction $(\rho',\sigma')<(3,-1)$. We follow this corner by the method of the proof of [1, Proposition 5.19]: Case III steps keep the point A_1 and the ring $L^{(3)}$, since $\rho'\mid 3$ [1, Proposition 5.17], and lower the direction, and after finitely many steps Case I or Case II occurs. Case II is impossible at every step, because [1, Proposition 5.14(4)] requires $\gcd(3+k,k)>1$, while $\gcd(8,5)=\gcd(7,4)=1$. In Case I, $v_{\rho,\sigma}(P_1)+v_{\rho,\sigma}(Q_1)=\rho+\sigma$ [1, Proposition 1.13], while $\operatorname{en}(P_1)=mA_1=\frac{m}{n}\operatorname{en}(Q_1)$ [1, Corollary 5.7(2)]. Hence $(m+n)\big(\frac{3+k}{3}\rho+k\sigma\big)=\rho+\sigma$, which gives $(\rho,\sigma)=(21,-11)$, $v(A_1)=1$ for F_8 ($m+n=10$, $k=5$), and $(\rho,\sigma)=(7,-4)$, $v(A_1)=1/3$ for F_7 ($m+n=9$, $k=4$). A Case III direction would lie strictly between this direction, which is determined by the point A_1, and $(3,-1)$; no direction (ρ',σ') with $\rho'\in\{1,3\}$ does (ancillary script). So Case I occurs for $(\varphi P,\varphi Q)$ itself, and $v_{\rho,\sigma}(P_1)=m\,v(A_1)$, $v_{\rho,\sigma}(Q_1)=n\,v(A_1)$. Finally $v_{21,-11}(x^{i/3}y^j)=7i-11j$ and $3v_{7,-4}(x^{i/3}y^j)=7i-12j$. □

4. Common coordinates

Let α,β be as in Lemma 2.2, $\delta:=\alpha-\beta$, and put

$$s:=xy^3-1-\alpha y,\qquad v=y^{-1},\qquad \text{so that}\quad x=(s+1)v^3+\alpha v^2,\quad Y_1=sv,\quad Y_2=s(sv+\delta),$$

and $Y_1^2+\delta Y_1=vY_2$. Every element of $K[v,Y_1,Y_2]$ is a unique combination of the elements $v^iY_2^jY_1^\varepsilon$ ($i,j\ge 0$, $\varepsilon\in\{0,1\}$); we index such an element by its *key* $(b,a):=(2j+\varepsilon,\,i+j+\varepsilon)$,

so that the keys are the pairs with $0 \le b \le 2a$, and we write it $e_{b,a}$. In $K[s,v]$,

$$e_{b,a} = \sum_{r=0}^{\lfloor b/2 \rfloor} \binom{\lfloor b/2 \rfloor}{r} \delta^r \, s^{b-r} v^{a-r}, \tag{1}$$

with leading term $s^b v^a$ (uniqueness follows). Let $W(b,a) := 7a-4b$ for F_8 and $W(b,a) := 7a-5b$ for F_7, and $w_0 := W(1,1)$ (3, resp. 2); the terms $r \ge 1$ of (1) have $W(b-r, a-r) = W(b,a) - r w_0$.

Lemma 4.1. *Write $P = \sum p_{b,a} e_{b,a}$, $Q = \sum q_{b,a} e_{b,a}$ (Lemma 2.2). Then:*

(a) $p_{b,a} \ne 0$ implies $a \le 3m$ and $W(b,a) \le m$; $q_{b,a} \ne 0$ implies $a \le 3n$ and $W(b,a) \le n$;
(b) $\sum_b p_{b,3m} s^b = \lambda_P (s+1)^m h(s+1)^m$ and $\sum_b q_{b,3n} s^b = \lambda_Q (s+1)^n h(s+1)^n$;
(c) $P_s Q_v - P_v Q_s = -J\, v$.

Proof. (c) follows from $dx \wedge dy = -v\, ds \wedge dv$ (ancillary script).

(a), (b) for the weight $(3,-1)$: v and $\tau = xy^2 - y^{-1} = v(t-1)$ are $(3,-1)$-homogeneous of value 1, the leading forms of Y_1 and Y_2 are τ and $\tau^2 v^{-1}$, so the leading form of $e_{b,a}$ is $\tau^b v^{a-b} = (t-1)^b v^a$, of value a; these are linearly independent for distinct keys. So $v_{3,-1}(P)$ is the largest a with some $p_{b,a} \ne 0$, which is $3m$ by (D1), and $\ell_{3,-1}(P) = v^{3m} \sum_b p_{b,3m} (t-1)^b$. On the other hand $\ell_{3,-1}(P) = \lambda_P \big(x\, h(t)\big)^m = \lambda_P v^{3m} \big(t\, h(t)\big)^m$, as $x = tv^3$; comparing the coefficients of $(t-1)^b$ gives (b), where the variable s of (b) stands for $t-1$.

(a) for W: let $\mathcal{A}$ be the set of formal sums $\sum c_{ij} X^i \eta^j$ ($i \in \mathbb{Z}$, $j \ge 0$) such that, for every r, only finitely many terms have $W \ge r$; it is a ring containing $K[X^{\pm 1}, \eta]$, on which the largest weight W of the terms is additive. The assignment $v \mapsto Y^{-1}$, $s \mapsto X^3 Y^3 - 1 - \alpha Y$ defines a ring morphism $K[s, v^{\pm 1}] \to \mathcal{A}$ which sends P to P_1 and Q to Q_1. With $u := \omega^{-1} \eta X$, $Y = \omega X^{-1}(1+u)$ is invertible, $v \mapsto \omega^{-1} X (1+u)^{-1}$ has leading term $\omega^{-1} X$ of weight 7, and $s \mapsto (1+u)^3 - 1 - \alpha\omega X^{-1}(1+u)$ has leading term $3u = 3\omega^{-1} X \eta$, of weight -4 for F_8 and -5 for F_7 (ancillary script). So the image of $e_{b,a}$ has leading term a non-zero multiple of $X^{a+b}\eta^b$, of weight $W(b,a)$, and these monomials are distinct for distinct keys. Lemma 3.1 gives (a). □

5. The finite system

Let $(U,V) := (P,Q)$ if $m < n$ and $(U,V) := (Q,P)$ if $m > n$; we write $(m_0, n_0) := (\min, \max)$, which is $(3,7)$ for F_8 and $(2,7)$ for F_7. By Lemmas 2.2–4.1 the coefficients $u_{b,a}, \bar{u}_{b,a}$ of U, V in the basis $(e_{b,a})$ and $\delta \in K$ satisfy:

(S1) $u_{b,a} \ne 0$ only if $0 \le b \le 2a \le 6m_0$ and $W(b,a) \le m_0$; $\bar{u}_{b,a} \ne 0$ only if $0 \le b \le 2a \le 42$ and $W(b,a) \le 7$;
(S2) $\sum_b u_{b,3m_0} s^b = \lambda_U (s+1)^{m_0} h(s+1)^{m_0}$, $\sum_b \bar{u}_{b,21} s^b = \lambda_V (s+1)^7 h(s+1)^7$, $\lambda_U \lambda_V \ne 0$;
(S3) $U_s V_v - U_v V_s = -J'\, v$ with $J' \in K^\times$.

For $m > n$ the bracket changes sign, which is absorbed in J'. Let $w := s^7 v^4$ for F_8 and $w := s^7 v^5$ for F_7. The keys of weight m_0 in (S1) are $(1,1)$, $(8,5)$, $(15,9)$ for F_8 and $(1,1)$, $(8,6)$ for F_7, and those of weight 7 are $(7l, 1+4l)$, $0 \le l \le 5$, for F_8 and $(7l, 1+5l)$, $0 \le l \le 4$, for F_7; so the parts of U and V of top weight are, in $K[s,v]$, $sv\, f(w)$ and $v\, g(w)$ with $\deg f \le 2$, $\deg g \le 5$ for F_8 and $\deg f \le 1$, $\deg g \le 4$ for F_7, and the part of weight $m_0 + 7 - w_0$ of (S3) is

$$\{sv\, f(w),\, v\, g(w)\}_{s,v} = -J'\, v. \tag{2}$$

Let $E_8 := \mathbb{Q}(a)$ with $162a^3 - 162a^2 + 45a - 4 = 0$ (an irreducible polynomial) and $E_7 := \mathbb{Q}(H)$ with $2H^2 + H + 2 = 0$.

Lemma 5.1. *If (S1)–(S3) has a solution, then it has one with $u_{0,0} = \bar{u}_{0,0} = 0$, with $\bar{u}_{k,k} = 0$ for $1 \le k \le 3 - \epsilon$, where $\epsilon = 1$ for F_8 and $\epsilon = 0$ for F_7, and with:*

- *for F_8: $\lambda_U = \lambda_V = 1$, $f = a + w + w^2$,*

$$g = w^5 + \tfrac{7}{3} w^4 + \tfrac{21a+14}{9} w^3 + \tfrac{252a+14}{81} w^2 + \tfrac{378a^2+126a-7}{243} w + \tfrac{378a^2-84a+7}{729}, \qquad J' = -\tfrac{882a^2-294a+28}{2187},$$

where a is the image of $a \in E_8$ under a field embedding $E_8 \to K$;

- *for F_7:* $\lambda_U(1-c')^2 = \lambda_V(1-c')^7 = 1$, $H = 1/(1-c')$ *(a root of* $2H^2+H+2$ *by Lemma 1.4),* $f = 1+w$,

$$g = w^4 + \tfrac{7}{2}w^3 + \tfrac{35}{8}w^2 + \tfrac{35}{16}w + \tfrac{35}{128}, \qquad J' = -\tfrac{35}{128}.$$

In particular, by (S2), $\sum_b u_{b,9}s^b = (s+1)^3 s^{15}$ *and* $\sum_b \bar{u}_{b,21}s^b = (s+1)^7 s^{35}$ *for* F_8*, and* $\sum_b u_{b,6}s^b = \big((1+s)(1+Hs)\big)^2 s^8$ *and* $\sum_b \bar{u}_{b,21}s^b = \big((1+s)(1+Hs)\big)^7 s^{28}$ *for* F_7*.*

Proof. The system is preserved by: adding constants to U and V; multiplying U, V by non-zero constants (which changes λ_U, λ_V, J'); the substitution $v \mapsto \kappa v$, $\kappa \in K^\times$, followed by $\delta \mapsto \delta/\kappa$, under which $e_{b,a}$ becomes $\kappa^a e_{b,a}$ by (1), so that (S1) is kept, λ_U, λ_V are multiplied by κ^{3m_0}, κ^{21} and J' by κ^2; and the replacement of V by $V - \gamma U^k$, $k \geq 1$, discussed below. We first add constants to get $u_{0,0} = \bar{u}_{0,0} = 0$ and use the multiplications to get the stated values of λ_U, λ_V; the later steps keep $u_{0,0} = 0$, and the constant of V is removed again at the end, which changes nothing else. Then the coefficient of w^2, resp. w, in f (key $(15,9)$, resp. $(8,6)$) and the leading coefficient of g (key $(35,21)$, resp. $(28,21)$) equal 1 by (S2), and the substitution $v \mapsto \kappa v$ followed by the normalisation of λ_U, λ_V replaces, for F_8, $f = a_0 + a_1 w + w^2$ by $a_0\kappa^{-8} + a_1\kappa^{-4}w + w^2$, and for F_7 it replaces $f = a_0 + w$ by $a_0\kappa^{-5} + w$. Equation (2) with $J' \neq 0$ has the following solutions (ancillary script): for F_8, none with $a_1 = 0$, and, for $a_1 = 1$, exactly three, with a_0 a root of $162a^3 - 162a^2 + 45a - 4$ and g, J' as stated; for F_7, none with $a_0 = 0$, and, for $a_0 = 1$, exactly the one stated. So, after a suitable κ, f, g and J' are as stated.

The keys of U other than $(0,0)$ have $b \geq 1$, since $W(0,a) = 7a > m_0$ for $a \geq 1$. Hence the part of top weight of U^k is $(sv\,f(w))^k$, which contains $s^k v^k$ with the coefficient $f(0)^k \neq 0$; the key (k,k), of weight $kw_0 = km_0$, is therefore present in U^k with a non-zero coefficient, while all keys of U^k have weight at most km_0 and $a \leq 3m_0 k$. For $k \leq 3 - \epsilon$ we have $km_0 \leq 6 < 7$ and $3m_0 k \leq 18 < 21$, so $V - \gamma U^k$ satisfies (S1)–(S3) with the same top parts and the same J'. Subtracting successively suitable multiples of $U^{3-\epsilon}, \dots, U^2, U$ (the key (k',k') does not occur in U^k for $k' > k$, as its weight $k'm_0$ exceeds km_0) gives $\bar{u}_{k,k} = 0$ for $1 \leq k \leq 3 - \epsilon$. □

Proposition 5.2. *For every field embedding* $E_8 \to K$*, resp.* $E_7 \to K$*, the system* (S1)–(S3) *with the normalisations of Lemma 5.1 has no solution.*

Proof (exact computation). It suffices to work over $E := E_8$, resp. E_7, with δ as an indeterminate, and to show that the polynomials obtained below generate the unit ideal of $E[\delta, z_0, \dots, z_4]$: applying the embedding to a relation $1 = \sum h_i f_i$ gives the statement for it. Write the coefficient of weight $7 + m_0 - w_0 - k$ of (S3), $k = 1, 2, \dots$, in the monomials $s^b v^a$ of $K[s,v]$, the coefficients $u_{b,a}$, $\bar{u}_{b,a}$ being taken from the top weight downwards and expanded by (1). In the k-th coefficient the unknown coefficients of U of weight $m_0 - k$ and of V of weight $7 - k$ occur linearly, multiplied by the known top parts, and all other terms involve coefficients of higher weight, already expressed. Reducing each of these linear systems to echelon form over E (with δ and the previous free unknowns as parameters) expresses the pivot unknowns through the others; the free unknowns are new parameters z_j, and the rows without pivot give *conditions*, polynomials in δ and the z_j with coefficients in E, which vanish on every solution. Carrying this out (ancillary script):

- for F_8, over the coefficients of weights $6, 5, \dots, -11$ of (S3): five free unknowns, the coefficients $\bar{u}_{32,19}$, $\bar{u}_{34,20}$, $\bar{u}_{31,18}$, $\bar{u}_{33,19}$, $\bar{u}_{35,20}$, and 16 conditions; there are polynomials h_i with coefficients in E_8 such that $1 = \sum h_i f_i$, where f_i runs over 10 of these conditions and $\deg(h_i f_i) \leq 11$;
- for F_7, over the coefficients of weights $6, 5, \dots, -23$: five free unknowns, the coefficients $\bar{u}_{27,20}$, $\bar{u}_{26,19}$, $\bar{u}_{25,18}$, $\bar{u}_{28,20}$, $\bar{u}_{24,17}$, and 17 conditions; there are h_i with coefficients in E_7 with $1 = \sum h_i f_i$, where f_i runs over 8 of these conditions and $\deg(h_i f_i) \leq 14$.

The multipliers h_i are obtained by solving a square linear system over $\mathbb{Q}$ (of size 3480, resp. 8200, with coefficients of up to 3907, resp. 466, digits), and the identity $\sum h_i f_i = 1$ is then checked by an independent exact expansion in $E[\delta, z_0, \dots, z_4]$. □

As a check of the procedure, the script applies the same elimination, for F_8 and $\delta = 2$, to a pair (U, V) with the normalised top parts and randomly chosen lower coefficients, with the right-hand side of (S3) replaced by its bracket: all the conditions then vanish at the chosen values, and many of them no longer do so after the right-hand side is modified.

Proof of Theorem 1.2. A pair as in the standing data gives a solution of (S1)–(S3) by Lemmas 1.4–4.1, hence one as in Lemma 5.1, contradicting Proposition 5.2. □

6. Scope

- **Proved here, relative to the cited results:** the reduction of Theorem 1.2 to Proposition 5.2 (Lemmas 1.4–5.1). The inputs are (D1), which rests on [1, Proposition 6.1, Theorem 7.6(2), Remark 3.1], and [1, Lemma 2.2, Theorem 2.6, Remarks 1.8, 2.5, Corollaries 5.7, 7.4, Propositions 1.13, 2.1, 3.7, 5.14, 5.17, 5.18, 6.1] and the method of the proof of [1, Proposition 5.19], as indicated; the substitutions $x \mapsto x + \zeta y^{-2}$ and $y \mapsto y + \omega x^{-1/3}$ have Jacobian 1. The condition on $\mathrm{Succ}_P(1, 0)$ in the standing data is the one of [1, Corollary 5.21(4)], which is established there for the standard pairs attached to a minimal counterexample; the theorem is stated under this condition.
- **Verified by exact computation:** Proposition 5.2; the solutions of (2) in Lemma 5.1; the element F in Lemma 1.4; the arithmetic of Lemma 3.1; the expansions in Lemma 4.1 (ancillary file `verify_F7F8.py` in the folder `anc`, with its expected output; SymPy and python-flint). The result depends on the correctness of these computations.
- **Not claimed:** a statement on the plane Jacobian conjecture itself; on the other shapes of maximal degree 147 in [2]; a new bound on the degree of a counterexample. The completeness of the list of [2] is not re-examined.
- [1, Corollary 7.4], on which Lemma 2.2 rests, is stated in [1] with the indication that its proof mimics that of Corollary 7.2, using [1, Proposition 7.3], whose proof mimics that of Proposition 7.1; it is applied here to Laurent translates, as in [1, Section 8] and [5]. The references [2] and [5] are cited in their arXiv versions.
- The argument has been prepared by the author with AI assistance (Claude, Anthropic). Nothing in this note has been checked by a third party; it has not been validated by professionals.

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