hexagon:2611.00892v1

Sharp inequalities and stability for real profiles on arithmetic progressions

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Abstract

Real bounded observables obey extremal laws that differ from those of complex polynomial phases. We prove that a centred real observable on a totally ergodic system satisfies $\|F\|_{U^3}^8 \leq 1/3$. Equality holds precisely for signs of semicircles on a Kronecker circle factor, and near equality forces proximity in L^1 to this class, uniformly over all systems. For four such signs at consecutive times we classify the maximising frequencies and phases, and obtain a gap to $1/4$ when no relation with coefficients in $\{-1, 1\}$ is present. We then determine the exact range of four-term averages with prescribed horizontal projections and realise every value in one totally ergodic extension with an enlarged Kronecker factor. For seven real profiles on a common quadratic Haar progression, we determine both the lower convex boundary at two fixed anchor means and the boundary attainable by a single tuple when those means vanish. Their separation has optimal coefficient $(4 - \sqrt{2})/3$. Finally, an explicit continuous family of four-profile configurations retains correlation $1/3$ while its interior profiles cease to be semicircle signs. The proofs combine connected-group rearrangement, an autocorrelation tail inequality, compactness of ordered Fourier coefficients, and exact finite lattice checks.

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1 Introduction

The sign of a semicircle is a simple real function with substantial additive structure. Put

$$S(t) = \operatorname{sgn}\cos(2\pi t), \qquad t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}.$$

Its mean is zero, its values are ± 1 almost everywhere, and its autocorrelation is the triangular function

$$R_S(t) = \int_{\mathbb{T}} S(x)S(x+t)\,dx = 1 - 4\|t\|_{\mathbb{T}}. \tag{1.1}$$

In particular, $|R_S(t)|$ is uniformly distributed on $[0,1]$ when t is Haar distributed. This elementary profile accounts for several sharp constants below. The main issue is to determine which extremal problems force it, and which allow other profiles with the same value.

We work with real measurable functions bounded in absolute value by one. On an invertible probability-preserving system (X, μ, T), write $T^n F = F \circ T^n$ and

$$\Lambda(F_0, F_1, F_2, F_3) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X \prod_{i=0}^{3} T^{in} F_i \, d\mu. \tag{1.2}$$

The limit exists by the characteristic-factor theory of Host and Kra [8]. A system is *totally ergodic* if T^r is ergodic for every nonzero integer r. Its Kronecker factor is a rotation on a compact connected abelian group. This connectedness is essential: a nonconstant sign character on a finite group can have mean zero and every Gowers norm equal to one.

1.1 One observable, four profiles, and seven profiles

Our first result concerns a single observable. For a centred real F on a totally ergodic standard system,

$$\|F\|_{U^3}^8 \le \frac{1}{3}. \tag{1.3}$$

The equality and stability statements in Section 2 identify precisely the signs $S(\chi + b)$, where χ is a nonzero character of the Kronecker factor. Stability is uniform in the following sense: for every $\varepsilon > 0$ there is an $\eta > 0$, independent of the system, such that a deficit at most η in (1.3) implies L^1 distance less than ε from one of these signs. We do not assign a numerical value or a power law to this modulus.

The proof first controls the distribution of an autocorrelation by its second moment. Near equality then forces the observable onto its Kronecker factor. A Fourier argument reduces exact extremisers to a circle; a bounded-variation argument makes their jumps equally spaced. To obtain a uniform statement when the systems vary, we compactify the ordered Fourier data and prove that no L^2 mass is lost.

Four possibly different observables require a separate analysis. If each is a semicircle sign, Theorem 3.1 proves that their average at times $0, 1, 2, 3$ has absolute value at most $1/3$. Equality requires both an exact relation between the four character directions and an alignment of their phases. Without a relation with coefficients ± 1, the absolute value is at most $1/4$. The phase deficit is quadratic, with explicit uniform bounds.

The same classification does not hold for four arbitrary centred profiles. Section 7 constructs functions g_a, $0 \le a \le 1/12$, for which

$$\int_{\mathbb{T}^2} g(x)g_a(x+h)g_b(x+2h)g(x+3h)\,dx\,dh = \frac{1}{3} - 32(a-b)^2, \tag{1.4}$$

where $g = 2\mathbf{1}_{[0,1/2)} - 1$. Every point on the diagonal $a = b$ gives $1/3$, although g_a is not a semicircle sign for $0 < a < 1/12$. Thus the rigidity of (1.3) is a statement about one observable,

and the rigidity of Theorem 3.1 is a statement within a specified four-profile class. Neither identifies all four-profile configurations with average $1/3$.

Between these questions lies an exact optimisation at fixed Kronecker projections. Section 4 gives upper and lower envelopes U_0 and $U_{1/2}$ for the central fibres of a four-term progression. Theorem 4.3 constructs one totally ergodic extension in which every value between the integrated envelopes occurs. Consequently the envelopes retain all the freedom available in the central variable, while preserving the prescribed horizontal functions. The construction enlarges the full Kronecker factor from K to $K \times \mathbb{T}$; the prescribed functions remain $m_i(x)$, independent of the added coordinate.

Our seven-profile problem makes a different distinction visible. For seven numbers $z_i \in [-1, 1]$, set

$$D_\sigma(z) = \frac{2 - \sigma(z_0z_2z_4z_6 + z_0z_1z_2z_3 + z_3z_4z_5z_6 - z_1z_5)}{4}, \qquad \sigma \in \{-1, 1\}. \tag{1.5}$$

We evaluate seven functions on the *same* quadratic Haar progression $Q_i = z + it + \binom{i}{2}v$ in a compact connected abelian group. If μ_i are their means, then $D = D_\sigma(\mu)$ is the defect of the means and $d = \mathbb{E}D_\sigma(F_i(Q_i))$ is the actual central defect. The word *anchor* will refer only to positions 1 and 5.

When both anchor means vanish, the lower convex boundary of the pairs (D, d) is

$$L(D) = \max\left\{\frac{1}{4},\, 2D - \frac{3}{4}\right\}, \qquad \frac{1}{4} \le D \le \frac{3}{4}. \tag{1.6}$$

Let $F_{\mathrm{nc}}(D)$ denote the minimum over a single seven-tuple, with no mixing of tuples. We compute this function exactly and prove

$$F_{\mathrm{nc}}(D) \ge L(D) + \frac{4 - \sqrt{2}}{3} \operatorname{dist}\left(D, \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}\right), \tag{1.7}$$

with the best possible coefficient. For example,

$$L(2/3) = \frac{7}{12}, \qquad F_{\mathrm{nc}}(2/3) = \frac{3}{4} - \frac{\sqrt{6}}{27}.$$

The first number can be attained by mixing two tuples with different values of D; the second is attained by one tuple with $D = 2/3$. Knowing only $\mathbb{E}D$ allows the first construction. The stronger bound therefore requires that the relevant constraints hold together within each fibre.

1.2 Relation to earlier work

The Gowers–Host–Kra seminorms and their nilsystem factors are supplied by Host–Kra theory [8]; we use the progression measure in the explicit Hall–Petresco form recorded by Frantzikinakis and Host [7, Proposition 4.7 and Appendix B]. These structural results are inputs to the paper. We verify below how their central fibres enter our real optimisation problem.

Eisner and Tao [6, Theorems 1.3 and 1.8] study large Gowers–Host–Kra seminorms in the complex class, including near equality at the upper endpoint and the appearance of quadratic phases above a specified U^3 threshold. Their conclusions do not state (1.3), its real centred equality classification, or the seven-profile boundaries considered here. Our proof uses the real constraint through the exact fixed-mean second-order inequality and through the subadditivity of $1 - |R|$.

For varying compact groups, we use the Fourier compactness method developed by Szegedy [12], which records ordered coefficients together with the relations among their characters. The additional step here is to derive the absence of L^2 mass loss from near extremality. This

permits passage to the U^3 cube integral and yields stability uniformly over the underlying systems.

Christ and Iliopoulou [3, Theorems 1.5 and 1.6] prove the relaxed Riesz–Sobolev inequality and its quantitative stability on compact connected abelian groups. We use both results with their hypotheses intact. The direct inequality has no strict-admissibility restriction; the stability theorem does. Our stability applications have masses in a compact subset of the strictly admissible region. The four-function rearrangement, the compatibility of three quartic maxima, and the optimisation at fixed anchor means are the explicit steps that connect those inputs to our conclusions. The sumset inequality used in the autocorrelation argument is Kneser's [9]; it is also stated in [3, (1.2)].

The cosine-sign agreement problem studied by Dou et al. [5] and Zang et al. [13] provides a nearby extremal question in one variable. The translation-and-doubling argument in [13, Proposition 3.9] supplies an earlier lower bound $-1/3$ in zero-phase odd-frequency cases. We give its precise transfer to a subclass of our progression model in Section 3. The full conclusion of Theorem 3.1 also treats arbitrary phases and directions, both signs of the correlation, and the equality and gap classifications.

For the Fourier analysis of four semicircle signs we use the Genocchi–Hermite formula for divided differences [4]. The compact quadratic models considered here are also related to the continuous model of complexity-two configurations developed by Candela and Szegedy [2]. Their approximation theorem concerns configurations on large prime cyclic groups. Our realisation theorem instead prescribes four horizontal functions and gives their attainable interval in one explicit dynamical extension.

In the Boolean application, the Walsh sum is the classical noise stability at parameter $-1/5$ [10, Section 2.4, Theorem 2.49]. The finite torsion certificate provides the geometric inequality to which that spectral identity is applied.

These results are intended to be reusable analytic inputs for questions about multiplicative correlations. We state their interface with such applications in Section 8. No arithmetic distribution of the horizontal profiles is assumed or established here.

1.3 Conventions and organisation

Table 1 records the constraints that distinguish the optimisation problems. In particular, the value $1/3$ has different roles for a single observable and for a quartet of profiles.

All compact groups below are abelian and their Haar measure has total mass one. They are metrizable whenever a standard dynamical system or a sequential compactness argument is involved. Functions and identities are understood up to null sets. We use $\widehat{f}(\chi) = \int f(x)e^{-2\pi i\chi(x)}\,dx$ for additively written characters $\chi : K \to \mathbb{T}$. A nonzero character of a connected compact group maps onto $\mathbb{T}$, and multiplication by every nonzero integer preserves Haar measure.

The proof for one observable occupies Section 2. Section 3 treats four semicircle signs. Section 4 determines the central envelopes and constructs their realisations. Sections 5 and 6 treat the convex and single-tuple seven-profile boundaries. Section 7 gives the continuous family (1.4). The last sections record the geometric interfaces and the precisely restricted Boolean application. The finite computations used in the semicircle classification are specified in Appendix A.

Objects	Quantity	Constraints and conclusion
One observable F	$\|F\|_{U^3}^8$	Real, centred, $\|F\| \le 1$, totally ergodic; sharp bound $1/3$ with equality and stability.
Four semicircle signs	$\|\Lambda\|$	Arbitrary nonzero character directions and phases; bound $1/3$ and exact classification.
Four real profiles	Range of Λ	Prescribed horizontal projections; exact interval realised with full Kronecker factor $K \times \mathbb{T}$.
Seven profiles	Pairs (D, d)	One common quadratic Haar jet; convex boundary at fixed anchor means, single-tuple boundary for zero anchors.
General centred quartets	Values of Λ	An explicit nonrigid family attains $1/3$; no global optimum for this unrestricted class is claimed.
Boolean middle profile	Horizontal defect	Characters independent modulo five and two centred anchors; restricted margins from a finite certificate.

Table 1: The optimisation domains used in the paper. All real profiles are bounded in absolute value by one.

2 The sharp real U^3 inequality

The example of a half-circle has a useful feature: its correlation with a translate is a triangular wave. We first show that this wave also controls the upper tails of the autocorrelation of every centred real profile. Combining this fact with a sharp U^2 estimate gives the U^3 bound. Equality requires two further steps: the observable must lie on the Kronecker factor, and its sign changes must be equally spaced on one circle.

Throughout this section, a system is an invertible probability-preserving transformation T of a standard probability space (X, μ). The system is *totally ergodic* if T^q is ergodic for every nonzero integer q. We use the Host–Kra normalisation

$$\|F\|_{U^1(T)} = \left| \int F \, d\mu \right|, \qquad \|F\|_{U^{s+1}(T)}^{2^{s+1}} = \lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \|F\, \overline{T^h F}\|_{U^s(T)}^{2^s}. \tag{2.1}$$

Here $T^h F = F \circ T^h$. These limits and the associated cube measures are those of [8]. All observables used below are real, so conjugation can be omitted.

Write $\pi : X \to K$ for the Kronecker factor. Thus K is a compact metrizable abelian group, its measure is normalised Haar measure, and $\pi(Tx) = \pi(x) + \theta$. Total ergodicity makes K connected: a torsion character would give a nonconstant root-of-unity eigenfunction. Conversely, the rotation by θ and each of its nonzero integer multiples are ergodic. A character is written additively as $\chi : K \to \mathbb{T}$, and its complex representative is $e(\chi(x)) = \exp(2\pi i \chi(x))$.

Theorem 2.1 (Sharp real bound and equality)**.** *Let T be totally ergodic and let $F : X \to [-1, 1]$ be real with $\int F \, d\mu = 0$. Then*

$$\|F\|_{U^3(T)}^8 \le \frac{1}{3}. \tag{2.2}$$

Equality holds if and only if, up to a null set,

$$F(x) = \operatorname{sgn} \cos\bigl(2\pi(\chi(\pi(x)) + b)\bigr)$$

for a nonzero character χ of K and $b \in \mathbb{T}$. The values assigned at the two boundary points are immaterial.

The exponent eight in (2.2) is part of the statement. The corresponding bound for the seminorm itself is $3^{-1/8}$. We prove the estimate first, then the equality statement, and finally a stability theorem whose constants do not depend on the system.

2.1 An autocorrelation inequality from connectedness

Lemma 2.2 (Upper tails of a real autocorrelation)**.** *Let K be a compact connected abelian group and let $f : K \to [-1, 1]$ be real with $\int_K f = 0$. Put*

$$R(t) = \int_K f(x)f(x+t)\,dx, \qquad Y = |R(t)| \quad (t \text{ Haar distributed}), \qquad e = \int_K |R(t)|^2\,dt.$$

Then, for every $0 \le a \le 1$,

$$\mathbb{E}(Y - a)_+^2 \le (1 - a)^3 e. \tag{2.3}$$

In particular,

$$\mathbb{E}Y^3 \le \frac{3}{4}e. \tag{2.4}$$

If $f(t) = \operatorname{sgn}\cos(2\pi t)$ on $\mathbb{T}$, equality holds in (2.3) for every a.

Proof. The pointwise inequality

$$1 - uw \le (1 - uv) + (1 - vw) \qquad (-1 \le u, v, w \le 1)$$

follows from $v(u + w) \le |u + w| \le 1 + uw$. Apply it to three translates of f, choosing the signs of the middle and last translates to make the first two averaged products nonnegative. Integration gives

$$|R(s + t)| \ge |R(s)| + |R(t)| - 1.$$

Consequently

$$d(s + t) \le d(s) + d(t), \qquad d(t) = 1 - |R(t)|.$$

The function R is continuous, $R(0) = \|f\|_2^2$, and $\int_K R = 0$. Unless $f = 0$, connectedness therefore implies that R vanishes somewhere. For $0 < v < 1$, the open set

$$B_v = \{t : d(t) < v\}$$

has measure $b(v) < 1$: a neighbourhood of a zero of R lies outside B_v. Subadditivity gives $B_s + B_t \subseteq B_{s+t}$. The connected-group form of Kneser's inequality [9], applied to nonempty open sets, yields

$$b(s + t) \ge b(s) + b(t) \qquad (s, t > 0,\ s + t < 1).$$

Indeed, the alternative lower bound 1 is excluded by $b(s + t) < 1$. If one of the sets is empty, the same inequality follows from monotonicity.

Set $B(v) = \int_0^v b(u)\,du$. Integrating $b(v) \ge b(u)+b(v-u)$ over $0 < u < v$ gives $vb(v) \ge 2B(v)$. Thus $B(v)/v^2$ is nondecreasing on $(0, 1)$. For $0 < s \le 1$, integration against the positive weight $v^2\,dv$ now gives

$$\int_0^s B(v)\,dv \le s^3 \int_0^1 B(v)\,dv. \tag{2.5}$$

For clarity, dividing each side by its respective integral of v^2 compares the weighted average of the increasing function $B(v)/v^2$ on $(0, s)$ with that on $(0, 1)$.

Layer integration gives

$$\mathbb{E}(Y - a)_+^2 = 2\int_0^{1-a} B(v)\,dv, \qquad \mathbb{E}Y^2 = 2\int_0^1 B(v)\,dv.$$

This proves (2.3), including the endpoints by continuity. Integration with respect to $a \in [0, 1]$ gives (2.4). For the half-circle, $R(t) = 1 - 4\|t\|_{\mathbb{T}}$, and $|R(t)|$ is uniform on $[0, 1]$. □

Both centring and connectedness enter the proof at the assertion that $b(v) < 1$. The use of real profiles enters earlier, in the elementary inequality for u, v, w.

2.2 The U^2 problem at a prescribed mean

We need the bound at U^2, together with its equality and near-equality information. The rearrangement theorem is an external input; the reduction from the fourth Fourier moment to that theorem is given here.

Lemma 2.3 (A mean-sensitive U^2 bound)**.** *Let T be totally ergodic, let $G : X \to [-1, 1]$ be real, and write $m = \int G\, d\mu$. Then*

$$\|G\|_{U^2(T)}^4 \le H(m), \qquad H(m) = \frac{1}{3} - 2m^2 + \frac{8}{3}|m|^3. \tag{2.6}$$

For $-1 < m < 1$, equality holds precisely for signs of arcs lifted by a nonzero character of the Kronecker factor, with the arc length chosen to give mean m. In addition, uniformly over all these systems,

$$m_j \longrightarrow 0, \quad \|G_j\|_{U^2(T_j)}^4 \longrightarrow \frac{1}{3} \quad \Longrightarrow \quad \inf_{\chi \neq 0,\ b \in \mathbb{T}} \|G_j - \operatorname{sgn}\cos(2\pi(\chi \circ \pi_j + b))\|_1 \longrightarrow 0.$$

Proof. Let $g_0 = \mathbb{E}(G \mid K)$, $g = (1 + g_0)/2$, and $\alpha = (1 + m)/2$. The Fourier formula for U^2 is

$$\|G\|_{U^2(T)}^4 = \sum_{\xi \in \widehat{K}} |\widehat{g}_0(\xi)|^4 = 16\|g * g\|_2^2 - 16\alpha^4 + m^4.$$

Changing G to $-G$ allows us to assume $0 \le \alpha \le 1/2$. Let $I_\alpha \subseteq \mathbb{T}$ be the centred arc of length α and $T_\alpha = \mathbf{1}_{I_\alpha} * \mathbf{1}_{I_\alpha}$. The relaxed rearrangement inequality [3, Theorem 1.5] states that for $0 \le g_1, g_2, g_3 \le 1$ on a compact connected group, the convolution integral is at most that of centred circle arcs with the same three masses. Applied to $g, g, \mathbf{1}_E$, it gives

$$\int_E g * g \le \int_{I_{|E|}} T_\alpha. \tag{2.7}$$

If K is trivial, $g_0 = m$ and (2.6) is immediate; it cannot be an equality for $|m| < 1$. We may therefore use (2.7) on a nontrivial connected group.

For every $t \ge 0$, take $E = \{g * g > t\}$ in (2.7). Optimising the right side over its arc length gives

$$\int_K (g * g - t)_+ \le \int_{\mathbb{T}} (T_\alpha - t)_+.$$

Integration in t yields $\|g * g\|_2^2 \le \|T_\alpha\|_2^2$. Hence the largest fourth Fourier moment is attained by the sign of an arc. For this sign, put $u = |m|$. Its autocorrelation decreases linearly from 1 to $2u - 1$ on an interval of length $(1 - u)/2$, is constant on an interval of length u, and increases symmetrically. The integral of its square is

$$2\int_0^{(1-u)/2} (1 - 4t)^2\, dt + u(2u - 1)^2 = \frac{1}{3} - 2u^2 + \frac{8}{3}u^3,$$

which proves (2.6).

We record why the same reduction carries the equality information. Define

$$d_\alpha(t) = \int (T_\alpha - t)_+ - \int (g * g - t)_+.$$

These nonnegative functions are uniformly Lipschitz, and

$$\int_0^\infty d_\alpha(t)\,dt = \frac{1}{2}(\|T_\alpha\|_2^2 - \|g*g\|_2^2).$$

Thus exact equality forces $d_\alpha(t) = 0$ for every t. In a near-extremising sequence with $\alpha \to 1/2$, it forces $d_\alpha(\alpha/2) \to 0$. Put $E = \{g * g > \alpha/2\}$ and $b = |E|$. The function

$$J_\alpha(b) = \int_{I_b} T_\alpha - \frac{\alpha b}{2}$$

has its unique maximum at $b = \alpha$: for $0 \le b \le 2\alpha$, it is $\alpha b/2 - b^2/4$, and for $2\alpha \le b \le 1$, it is $\alpha^2 - \alpha b/2$. Moreover,

$$d_\alpha(\alpha/2) = (J_\alpha(\alpha) - J_\alpha(b)) + \left(\int_{I_b} T_\alpha - \int_E g*g\right),$$

and both summands are nonnegative. It follows that $b = \alpha$ at equality, while $b - \alpha \to 0$ in the near-extremising sequence, and the rearrangement deficit for $(g, g, \mathbf{1}_E)$ tends to zero.

The masses (α, α, α) are strictly admissible for each fixed $0 < \alpha \le 1/2$. When $\alpha \to 1/2$, the actual masses (α, α, b) remain in one uniformly strictly admissible, bounded range. The stability theorem [3, Theorem 1.6] therefore implies that g is an indicator of a character-preimage arc at equality, and is at $o(1)$ distance in L^1 from such an indicator in the sequence. In the latter case, changing the arc length to $1/2$ costs $2|\alpha - 1/2| = |m|$ for the sign profile.

Finally, if S is any K-measurable sign and $|G| \le 1$, then

$$\|G - S\|_1 = 1 - \int GS\,d\mu = 1 - \int g_0 S\,d\mu = \|g_0 - S\|_1. \tag{2.8}$$

This lifts the equality and stability conclusions from the Kronecker projection to G. □

2.3 Proof of the numerical bound

Let $f = \mathbb{E}(F \mid K)$, $R(t) = \int_K f(x)f(x+t)\,dx$, and $e = \int_K |R|^2$. For $m(h) = \int_X F\,T^h F\,d\mu$, the spectral theorem gives

$$m(h) = R(h\theta) + c(h), \qquad \lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} |c(h)|^2 = 0. \tag{2.9}$$

The second assertion is Wiener's lemma for the continuous part of the spectral measure. Since R is continuous and the rotation is ergodic, any continuous function of $m(h)$ has the same Cesàro limit as its Haar average with $m(h)$ replaced by $R(t)$. The same assertion holds along rh for each nonzero fixed integer r, by total ergodicity.

Apply Lemma 2.3 to $G_h = F\,T^h F$ and use (2.1). We obtain

$$\|F\|_{U^3(T)}^8 \le \int_K H(R(t))\,dt = \frac{1}{3} - 2e + \frac{8}{3}\int_K |R(t)|^3\,dt \le \frac{1}{3}, \tag{2.10}$$

where the final inequality is (2.4). For a half-circle on a circle rotation, $|R|$ is uniform on $[0, 1]$. For every translation t, the derivative $f(x)f(x+t)$ is a sign of an arc lifted under the doubling map. It is therefore an equality case of Lemma 2.3. Both inequalities in (2.10) are equalities, proving sharpness.

2.4 Why the horizontal energy cannot vanish

The next elementary obstruction is the reason that an almost extremiser cannot hide outside its Kronecker factor.

Lemma 2.4 (The steps $r, 2r, 4r$)**.** *Let $F : X \to [-1,1]$ be real, write $m(r) = \int F\,T^r F\,d\mu$, and let S be a half-circle sign on the Kronecker factor of a totally ergodic system. Then*

$$19\|F\,T^r F - S\|_1 \geq 1 - 4|m(2r)| - |m(4r)|. \tag{2.11}$$

If F is sign-valued, the coefficient 19 can be replaced by 12.

Proof. Set $G = F\,T^r F$, $d = \|G - S\|_1$, and $a = 1 - \int F^2\,d\mu$. Since $\int |G| \leq \int F^2$, we have $a \leq d$. Let ϑ be the rotation angle of the character of S under T^r, and put $R_S(\vartheta) = 1 - 4\|\vartheta\|_{\mathbb{T}}$. Expanding $G\,T^r G$, and replacing the one intervening square of F by 1, gives

$$|R_S(\vartheta) - m(2r)| \leq 2d + a.$$

Likewise, the product of four successive copies of G contains three intervening squares, so

$$\left| m(4r) - \int \prod_{j=0}^{3} T^{jr} S\,d\mu \right| \leq 4d + 3a.$$

For a half-circle, $\|S + T^{2r}S\|_1 = 2|R_S(\vartheta)|$. Comparing the pairs of indices $(0,2)$ and $(1,3)$ consequently yields

$$1 - \int \prod_{j=0}^{3} T^{jr} S\,d\mu \leq 4|R_S(\vartheta)|.$$

Combining the three estimates gives

$$1 - |m(4r)| \leq 4|m(2r)| + 12d + 7a.$$

Use $a \leq d$; when F is a sign, use $a = 0$. □

Lemma 2.5 (Concentration on Kronecker near equality)**.** *For every sequence of totally ergodic systems and centred real $F_j : X_j \to [-1,1]$ such that $\|F_j\|_{U^3(T_j)}^8 \to 1/3$, put $f_j = \mathbb{E}(F_j \mid K_j)$, $R_j(t) = \int f_j(x) f_j(x+t)\,dx$, and $e_j = \int |R_j|^2$. Then*

$$\liminf_j e_j > 0, \qquad \|f_j\|_2 \longrightarrow 1, \qquad \|F_j - f_j\|_2 \longrightarrow 0. \tag{2.12}$$

More precisely, along any subsequence for which $e_j \to e_* > 0$, every weak limit of the laws of $|R_j|$ is

$$(1 - 3e_*)\delta_0 + 3e_*\,\mathrm{Unif}[0,1]. \tag{2.13}$$

Proof. Suppose first that a subsequence satisfies $e_j \to 0$. By (2.9) and total ergodicity, the Cesàro means of $|m_j(h)|^2, |m_j(2h)|^2, |m_j(4h)|^2$ all tend to e_j. Choose a sufficiently long finite average in each system so that these three means, and the recursion in (2.1), have their stated limits up to $o(1)$. Outside a set of relative size $o(1)$, the three correlations are $o(1)$. On that set of good h, Lemma 2.3 bounds $\|F_j T_j^h F_j\|_{U^2}^4$ by $1/3 + o(1)$; everywhere it is at most 1. Its full average tends to $1/3$. There is therefore a good h_j for which $\|F_j T_j^{h_j} F_j\|_{U^2}^4 \to 1/3$. Lemma 2.3 places this derivative at $o(1)$ distance from a half-circle sign. This contradicts (2.11), because $m_j(2h_j), m_j(4h_j) \to 0$. The argument also proves that some universal positive lower bound for e holds in a sufficiently small neighbourhood of equality.

Let $Y_j = |R_j|$, and pass to a subsequence on which $e_j \to e_* > 0$ and the laws of Y_j converge. Define the nonnegative deficit

$$D_j(a) = e_j(1-a)^3 - \mathbb{E}(Y_j - a)_+^2.$$

By (2.10),

$$0 \le 8\int_0^1 D_j(a)\,da \le \frac{1}{3} - \|F_j\|_{U^3(T_j)}^8 \longrightarrow 0.$$

The limiting deficit is continuous and nonnegative, so it vanishes identically. If Y_* has the limiting law, then

$$\mathbb{E}(Y_* - a)_+^2 = e_*(1-a)^3 \qquad (0 \le a \le 1).$$

Twice differentiating on $(0,1)$, in the distributional sense, gives $\mathbb{P}(Y_* > a) = 3e_*(1-a)$, which is (2.13). In particular the limiting support reaches 1.

Cauchy–Schwarz gives $Y_j \le \|f_j\|_2^2 \le 1$. Since $e_* > 0$, the preceding limiting law forces $\|f_j\|_2 \to 1$. Finally, orthogonal projection gives

$$\|F_j - f_j\|_2^2 = \|F_j\|_2^2 - \|f_j\|_2^2 \le 1 - \|f_j\|_2^2 \longrightarrow 0.$$

Every subsequence has the same conclusion, proving (2.12). □

2.5 Classification of the exact extremisers

Assume equality in (2.2). Apply Lemma 2.5 to the constant sequence. It follows that $F = f$ is Kronecker measurable and $|f| = 1$ almost everywhere. Its cube seminorm is thus the compact-group cube integral

$$\int_{K^4} \prod_{\omega\in\{0,1\}^3} f(x + \omega_1 h_1 + \omega_2 h_2 + \omega_3 h_3)\,dx\,dh_1\,dh_2\,dh_3.$$

Put $g_t(x) = f(x)f(x+t)$. The nonnegative function

$$H(R(t)) - \|g_t\|_{U^2(K)}^4$$

is continuous in t, by continuity of translations in L^1, and its integral is zero. Hence it vanishes for every t. Whenever $|R(t)| < 1$, Lemma 2.3 says that g_t is a sign of an arc lifted by a nonzero character.

We first reduce to a single circle. The set $\{|R| < 1\}$ is a nonempty open set. Almost every $t \in K$ generates a dense cyclic subgroup: the dual is countable, and each nonzero character has a Haar-null kernel. Choose such a t with $|R(t)| < 1$, and let $\chi \neq 0$ be a character through which g_t factors. For each $k \in \ker\chi$, the sign

$$x \longmapsto f(x+k)f(x)$$

is invariant under translation by t, and is therefore almost everywhere constant. Every character in the Fourier support of f consequently has the same restriction to $\ker\chi$. If γ belongs to that support, the entire support lies in $\gamma + \mathbb{Z}\chi$. Reality makes $-\gamma$ belong to the support too, so $2\gamma = m\chi$ for an integer m. The subgroup generated by γ, χ is cyclic. Indeed, for even m, torsion-freeness gives $\gamma = (m/2)\chi$; for odd m, $\gamma - ((m-1)/2)\chi$ is a generator. Thus f factors through a nonzero character $K \to \mathbb{T}$. Its correlation distribution and cube integral are unchanged by this passage to the quotient.

It remains to classify the resulting sign f on $\mathbb{T}$. The set

$$Z = \{t : |R(t)| = 1\}$$

is the closed subgroup of translations preserving f up to sign. It is finite: otherwise it would be the entire circle, contrary to (2.13). Write $m_0 = |Z|$ and $e = \int |R|^2 > 0$. That law gives

$$|\{1 - |R| < v\}| = 3ev \qquad (0 < v < 1).$$

As $v \downarrow 0$, this set lies in arbitrarily small neighbourhoods of Z, and is invariant under Z. Its portion near zero has measure $3ev/m_0$. We can therefore choose $t_v \to 0$ in that portion with $|t_v| \geq 3ev/(4m_0)$. Since $R(t_v) > 0$ for small v,

$$\int_{\mathbb{T}} |f(x + t_v) - f(x)|\, dx = 1 - R(t_v) < v \leq \frac{4m_0}{3e}|t_v|.$$

This subsequential difference-quotient bound already implies bounded variation. Testing it against a smooth function bounds the distributional derivative of f as a functional on continuous functions with the uniform norm, so that derivative is a finite measure.

A sign of bounded variation on a circle is constant, up to null sets, on finitely many intervals. Let $2m$ be its number of jumps. For positive t shorter than every gap between jumps and with $2mt < 1/2$, the set

$$\{x : f(x)f(x + t) = -1\}$$

is a disjoint union of $2m$ arcs of length t, each ending at one of those jumps. Equality in Lemma 2.3 says that this same set is the inverse image of a single arc under a circle character. Its component arcs therefore have equally spaced endpoints. All jumps of f are equally spaced, and their signs alternate. This is exactly a half-circle sign lifted by a circle character. Together with the sharpness calculation, this completes the proof of Theorem 2.1.

2.6 Uniform stability when the system varies

Theorem 2.6 (Uniform stability)**.** *For every $\varepsilon > 0$ there is $\eta > 0$ such that, for every totally ergodic standard system and every real centred $F : X \to [-1, 1]$,*

$$\|F\|_{U^3(T)}^8 \geq \frac{1}{3} - \eta \quad \Longrightarrow \quad \inf_{\chi \neq 0,\ b \in \mathbb{T}} \|F - \operatorname{sgn}\cos(2\pi(\chi \circ \pi + b))\|_{L^1(\mu)} < \varepsilon.$$

The character is taken from the Kronecker factor of the given system. The modulus $\eta(\varepsilon)$ is independent of that system; no explicit numerical modulus is asserted.

We isolate the compactness argument needed in the proof. Its construction belongs to the ordered-Fourier approach of Szegedy [12, Lemma 2.2, Proposition 2.2 and Theorem 2, arXiv version], in which coefficients are retained together with their character relations. An additional equality of L^2 norms will allow the U^3 cube integral to pass to the limit. Near extremality will provide that equality in the application.

Lemma 2.7 (Fourier compactness on connected groups)**.** *Let K_j be compact connected metrizable abelian groups with normalised Haar measures, and let $f_j : K_j \to [-1, 1]$ be real and centred. After passage to a subsequence, there are continuous homomorphisms $\Phi_j : K_j \to \mathbb{T}^{\mathbb{N}}$, a compact connected subgroup $K_* \subseteq \mathbb{T}^{\mathbb{N}}$ with Haar measure ν_*, a real centred function $f_* : K_* \to [-1, 1]$, and a number ℓ with the following properties.*

The probability measures $\nu_j = (\Phi_j)_*(dx)$ and the real signed measures $(\Phi_j)_*(f_j\, dx)$ satisfy

$$\nu_j \implies \nu_*, \qquad (\Phi_j)_*(f_j\, dx) \implies f_*\nu_*, \qquad \|f_j\|_2^2 \longrightarrow \ell \geq \|f_*\|_2^2, \tag{2.14}$$

where the measure convergences are weak on $\mathbb{T}^{\mathbb{N}}$. Writing

$$R_j(t) = \int_{K_j} f_j(x)f_j(x + t)\, dx, \qquad R_*(t) = \int_{K_*} f_*(x)f_*(x + t)\, d\nu_*(x),$$

the Haar-distributed random variables R_j converge in law to R_*.

For every real continuous cylinder function $P : \mathbb{T}^{\mathbb{N}} \to [-1, 1]$,

$$\lim_j \|f_j - P \circ \Phi_j\|_2^2 = \ell - \|f_*\|_2^2 + \|f_* - P\|_{L^2(\nu_*)}^2. \tag{2.15}$$

In particular, if $\ell = \|f_*\|_2^2$, then

$$\|f_j\|_{U^3(K_j)}^8 \longrightarrow \|f_*\|_{U^3(K_*)}^8. \tag{2.16}$$

Finally, if $f_* = \operatorname{sgn}\cos(2\pi(\chi_* + b))$ for a nonzero character χ_* of K_*, then there are nonzero characters χ_j of K_j, for all sufficiently large j, such that

$$\|f_j - \operatorname{sgn}\cos(2\pi(\chi_j + b))\|_1 \longrightarrow 0. \tag{2.17}$$

The compactness assertion permits the limit group to be trivial.

Proof. For each j, list the nonzero Fourier coefficients of f_j in nonincreasing order of magnitude, with corresponding characters $\chi_{j1}, \chi_{j2}, \ldots$. Pad a finite list by zero characters and zero coefficients. Since f_j is centred, its coefficient at the zero character is zero. Set

$$\Phi_j(x) = (\chi_{j1}(x), \chi_{j2}(x), \ldots).$$

After passage to a subsequence, the measures ν_j, the signed measures $(\Phi_j)_*(f_j\,dx)$, all listed coefficients, and the numbers $\|f_j\|_2^2$ converge. For each finitely supported integer vector n, the Fourier coefficient $\widehat{\nu_j}(n) \in \{0, 1\}$ is also eventually constant. There are countably many such vectors, so all these choices can be made in one diagonal extraction.

Let

$$L = \{n : \widehat{\nu_j}(n) = 1 \text{ eventually}\}, \qquad K_* = \{z \in \mathbb{T}^{\mathbb{N}} : n \cdot z = 0 \text{ for every } n \in L\}.$$

The limiting Fourier coefficients identify the limiting probability measure as Haar measure on K_*. The relation subgroup L is saturated: if $kn \in L$ for a nonzero integer k, connectedness of every K_j makes its dual torsion-free, and hence $n \in L$. Thus K_* is connected. The inequalities $-\nu_j \le (\Phi_j)_*(f_j\,dx) \le \nu_j$ pass to the limit against nonnegative continuous functions. The limiting signed measure therefore has a real density $f_* \in [-1, 1]$, and its integral is zero.

Write $a_{jr} = \widehat{f_j}(\chi_{jr})$. Parseval and the ordering give

$$|a_{jr}| \le r^{-1/2}, \qquad \left\| R_j - \sum_{r \le N} |a_{jr}|^2 e(\chi_{jr}) \right\|_2^2 = \sum_{r > N} |a_{jr}|^4 \le \frac{1}{N+1}.$$

For each N, the finite sums converge in law. We verify that the limiting Fourier series is exactly that of R_*. Every character of K_* is the restriction of a finitely supported integer monomial n on $\mathbb{T}^{\mathbb{N}}$. If its coefficient in f_* is nonzero, the corresponding coefficients of f_j are bounded away from zero, and their ranks are bounded. One rank r occurs infinitely often. The relation $n \cdot \Phi_j = \chi_{jr}$ then holds infinitely often, so eventual stabilisation makes it hold for all sufficiently large j. Conversely, each nonzero limiting listed coefficient is the coefficient of f_* at the corresponding coordinate character. Two such ranks remain distinct, since their equality would be an eventual relation already excluded on the original groups. This identifies all nonzero coefficients of f_*. The uniform L^2 tail bound now proves convergence in law of the full autocorrelations.

Let $\ell = \lim_j \|f_j\|_2^2$. Expanding the square, the two weak measure convergences give

$$\lim_j \|f_j - P \circ \Phi_j\|_2^2 = \ell - 2\int_{K_*} f_* P\,d\nu_* + \int_{K_*} P^2\,d\nu_*.$$

Continuous real cylinder functions bounded by one approximate f_* in $L^2(\nu_*)$. Nonnegativity of the left side therefore implies $\ell \geq \|f_*\|_2^2$, and proves (2.15).

Suppose now that there is no loss of mass: $\ell = \|f_*\|_2^2$. For fixed P, the cube integrals of $P \circ \Phi_j$ converge to that of P, because $\nu_j^{\otimes 4} \Longrightarrow \nu_*^{\otimes 4}$ and the addition maps are continuous. Telescoping eight factors in $[-1,1]$ bounds the change of a cube integral by eight times the L^1 error. Equation (2.15), followed by approximation of f_*, thus proves (2.16). The equality of the norms is an additional hypothesis at this step.

For the last assertion, express χ_* as a finite monomial n. Since it is nonzero on K_*, the character $\chi_j = n \cdot \Phi_j$ is nonzero for all sufficiently large j. All these characters, including χ_*, send Haar measure to Lebesgue measure on the circle. Approximate $\operatorname{sgn}\cos(2\pi(t+b))$ in circle L^1 by continuous functions bounded by one. The approximation error is the same on every group, while weak convergence of the signed measures handles each continuous approximant. Hence

$$\int_{K_j} f_j(x) \operatorname{sgn}\cos(2\pi(\chi_j(x)+b))\,dx \longrightarrow \int_{K_*} f_*(x)^2\,d\nu_*(x) = 1.$$

For a sign S and $f \in [-1,1]$, $\|f - S\|_1 = 1 - \int fS$, giving (2.17). □

Proof of Theorem 2.6. Suppose that a sequence approaches $1/3$ while staying a fixed positive L^1 distance from every permitted half-circle. Put $f_j = \mathbb{E}(F_j \mid K_j)$. Lemma 2.5 gives $\|F_j - f_j\|_2 \to 0$, $\|f_j\|_2 \to 1$, and $\liminf_j e_j > 0$, where $e_j = \int |R_j|^2$. After extraction, let $e_j \to e_* > 0$. The cube measures have the original measure as each coordinate marginal, so telescoping gives

$$\left| \|F_j\|_{U^3(T_j)}^8 - \|f_j\|_{U^3(K_j)}^8 \right| \leq 8\|F_j - f_j\|_1.$$

Thus the compact-group cube integrals of f_j also approach $1/3$.

Apply Lemma 2.7. Its limit satisfies $\mathcal{L}(|R_*|) = (1-3e_*)\delta_0 + 3e_* \operatorname{Unif}[0,1]$, by Lemma 2.5 and convergence of the autocorrelation laws. This law reaches arbitrarily close to one. Since $|R_*(t)| \leq \|f_*\|_2^2 \leq 1$, it follows that $\|f_*\|_2^2 = 1$. The limiting value ℓ is also one, so the no-loss hypothesis in (2.16) holds and $\|f_*\|_{U^3(K_*)}^8 = 1/3$. In particular, K_* is nontrivial: a centred function on a trivial group cannot have L^2 norm one.

A compact connected metrizable abelian group admits a totally ergodic rotation. Choose an element on which every nonzero character has an irrational value, outside a countable union of Haar-null sets. Theorem 2.1 applied to such a rotation on K_* makes f_* a half-circle sign of a nonzero character. Equation (2.17) gives corresponding half-circle signs approximating f_j in L^1. Together with $\|F_j - f_j\|_1 \to 0$, this contradicts the assumed positive distance and proves the theorem. □

3 Four semicircle signs: frequencies and phases

For four different semicircle signs, large U^3 norms do not by themselves determine the progression average. Their character directions must also fit together. We describe this compatibility exactly.

Let K be a compact connected abelian group, let $\chi_0, \dots, \chi_3 \in \widehat{K} \setminus \{0\}$, and put

$$u_i = (\chi_i, i\chi_i) \in \widehat{K^2}.$$

For a phase vector $\mathbf{b} = (b_0, b_1, b_2, b_3) \in \mathbb{T}^4$, write

$$L(\mathbf{b}) = \int_{K^2} \prod_{i=0}^{3} S(\chi_i(x+ih) + b_i)\,dx\,dh. \tag{3.1}$$

We call $\epsilon \in \{-1,1\}^4$ a *unit relation* if $\sum_i \epsilon_i u_i = 0$. This term refers to the four coefficients, not to a multiplicative structure on the characters.

Theorem 3.1 (Classification for four semicircle signs)**.** *For (3.1), $|L(\mathbf{b})| \le 1/3$. If no unit relation exists, then $|L(\mathbf{b})| \le 1/4$. If a unit relation exists, it is unique up to an overall sign. Choose one and set $\beta = \sum_i \epsilon_i b_i$ in $\mathbb{T}$. For $\sigma \in \{-1,1\}$, define*

$$d_\sigma(\beta) = \begin{cases} \|\beta\|_{\mathbb{T}}, & \sigma = 1, \\ \|\beta - 1/2\|_{\mathbb{T}}, & \sigma = -1. \end{cases}$$

Then

$$\frac{8}{3} d_\sigma(\beta)^2 \le \frac{1}{3} - \sigma L(\mathbf{b}) \le 8 d_\sigma(\beta)^2. \tag{3.2}$$

Consequently $|L| = 1/3$ exactly when a unit relation exists and $\beta \in \{0, 1/2\}$; the two values give $L = 1/3$ and $L = -1/3$, respectively. In particular, $|L| \ge 1/3 - \eta > 1/4$ forces an exact unit relation and distance at most $\sqrt{3\eta/8}$ from the corresponding extremal phase.

For instance, if all four characters are the same, then $(1,-1,-1,1)$ is a unit relation. Zero phases give $L = 1/3$. The signed phase controls the deficit without determining the whole integral. With these same four characters on the circle, the phase vectors $(1/8, 0, 0, 0)$ and $(1/8, 1/8, -1/8, 0)$ both give $\beta = 1/8$. Their integrals are $1/4$ and $5/24$, respectively, by the phase formula (3.10) below. The theorem allows different characters and makes no bound on their frequencies. The gap $1/4$ is used as a uniform separation; we do not claim that it is the best constant in the class without unit relations.

Comparison with cosine-sign agreement. Dou et al. [5] and Zang et al. [13] study integer-frequency cosine signs with one common circle parameter. Pair correlations in that model need not vanish. Here any two directions u_i, u_j are independent, as verified below, so the probability that the four signs in (3.1) agree is $(1 + L(\mathbf{b}))/8$. The two optimisation problems therefore retain different lower-order correlations.

The translation-and-doubling argument of [13, Proposition 3.9] does give part of the lower bound in a specific subclass. We record its transfer explicitly. Put $G = K^2$, $F_{\mathbf{b}}(z) = \prod_i S(u_i(z) + b_i)$, and suppose there is $q \in G$ such that $u_i(q) \in \{1/4, 3/4\}$ for every i. The scalar identity $S(t)S(t+c) = S(2t+c)$ for $c \in \{1/4, 3/4\}$ gives, almost everywhere,

$$F_{\mathbf{b}}(z) F_{\mathbf{b}}(z+q) = F_{2\mathbf{b}}(2z+q).$$

Translation and doubling preserve Haar measure on G, and hence

$$0 \le \int_G (1 \pm F_{\mathbf{b}}(z))(1 \pm F_{\mathbf{b}}(z+q))\, dz = 1 \pm 2L(\mathbf{b}) + L(2\mathbf{b}).$$

At $\mathbf{b} = \mathbf{0}$ this gives $L(\mathbf{0}) \ge -1/3$. It also applies when the phases are absorbed by a common translation, $b_i = u_i(z_0)$. For example, if $K = \mathbb{T}$ and $\chi_i(x) = w_i x$ with every w_i odd, take $q = (1/4, 0)$.

The condition on q is not automatic. For weights $(4,1,2,1)$, the relation $(1,-8,2,0)$ among the u_i has odd coefficient sum, which contradicts $2u_i(q) = 1/2$ for every i. Moreover, at arbitrary phases the displayed inequalities couple $L(\mathbf{b})$ to $L(2\mathbf{b})$ and do not give the two-sided bound or its equality classification. This compact-group transfer is a deduction from the argument of Zang et al.; the phase and frequency analysis below establishes the full theorem.

3.1 The relation lattice and absolute convergence

Let

$$\mathcal{R} = \left\{ m \in \mathbb{Z}^4 : \sum_{i=0}^{3} m_i u_i = 0 \right\}.$$

Any two u_i, u_j are linearly independent over $\mathbb{Q}$: subtracting i times the first component of a relation from the second gives $(j-i)m_j\chi_j = 0$, and the dual of a connected group is torsion-free. Thus $\operatorname{rank}\mathcal{R} \le 2$, and the coordinates on any two sites determine a relation uniquely.

The nonzero Fourier coefficients of S are

$$\widehat{S}(m) = \frac{2\sin(\pi m/2)}{\pi m}, \qquad m \text{ odd}. \tag{3.3}$$

It follows that

$$L(\mathbf{b}) = \sum_{\substack{m\in\mathcal{R}\\ m_i \text{ odd}}} \prod_{i=0}^{3} \widehat{S}(m_i)\, e^{2\pi i \sum_i m_i b_i}. \tag{3.4}$$

This series is absolutely convergent. For a partition into two pairs, Cauchy–Schwarz bounds the sum of its absolute values by the product of two square roots of sums of squared pair coefficients. Each sum is at most one by Parseval and the uniqueness of completion. One may therefore first use the Fejér approximants of S and then pass to (3.4).

If the lattice has rank zero, the sum vanishes. At rank one, either it has no relation with all coordinates odd or these relations are the odd multiples of a primitive odd vector R. Since $\sum_{k \text{ odd}} k^{-4} = \pi^4/48$, with both signs counted,

$$|L| \le \frac{1}{3\prod_i |R_i|}. \tag{3.5}$$

This is at most $1/9$ unless R is a unit relation.

At rank two all χ_i are commensurable. Indeed, if two of them were independent, writing a third u_k as a rational combination of the corresponding two directions and subtracting the coordinate equations would force both coefficients to vanish. The finitely generated subgroup generated by the four characters is therefore a free abelian group of rank one. We may write $\chi_i = w_i\chi$, with w_i positive integers after replacing (χ_i, b_i) by $(-\chi_i, -b_i)$ where necessary. The relation equations become

$$\sum_i w_i m_i = 0, \qquad \sum_i i w_i m_i = 0. \tag{3.6}$$

3.2 A unit relation and a periodised simplex density

Orient each (u_i, b_i) by the chosen unit relation, so that the sum of the oriented phases is β and $\mathbf{1} = (1,1,1,1) \in \mathcal{R}$. At rank two, a basis of the integer relation lattice is $\mathbf{1}, v$, where $v_0 = 0$ and v_1, v_2, v_3 have greatest common divisor one. To see this, subtract the first coordinate times $\mathbf{1}$ from any relation. The remaining rank-one kernel is saturated and hence has a primitive generator. The four v_i are distinct; otherwise subtracting a multiple of $\mathbf{1}$ would give a nonzero relation supported on at most two sites. The odd relations are precisely

$$m_i = k + 2jv_i, \qquad k \text{ odd}, \quad j \in \mathbb{Z}.$$

Write

$$u = \sum_i b_i \pmod 1, \quad a = 2\sum_i v_i b_i + \frac{1}{2}\sum_i v_i \pmod 1.$$

Changing all phase signs if necessary, take $0 \le u \le 1/2$ and set $w = 2u$. The contribution from $j = 0$ is

$$M(w) = \frac{16}{\pi^4} \sum_{k \text{ odd}} \frac{e^{\pi i k w}}{k^4} = \frac{1}{3} - 2w^2 + \frac{4}{3} w^3, \qquad 0 \le w \le 1. \tag{3.7}$$

Let $B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$ on $[0,1]$, extended periodically. For $0 < w < 1$, partial fractions in k give

$$L = M(w) + \frac{4}{3}[v_0, v_1, v_2, v_3]\big(B_3(a - w\,\cdot)\big). \tag{3.8}$$

Here $[v_0, v_1, v_2, v_3] f = \sum_i f(v_i) / \prod_{\ell \ne i} (v_i - v_\ell)$. For completeness, the identities used in the calculation are

$$\prod_i (k + 2j v_i)^{-1} = -\frac{1}{(2j)^3} \sum_i \frac{1}{\prod_{\ell \ne i} (v_i - v_\ell)(k + 2j v_i)},$$

$$\sum_{k \text{ odd}} \frac{e^{2\pi i k u}}{k} = \frac{i\pi}{2} \quad (0 < u < 1/2), \qquad \sum_{j \ne 0} \frac{e^{2\pi i j x}}{j^3} = \frac{4\pi^3 i}{3} B_3(x).$$

The first series is taken symmetrically. The product of the four sine factors in (3.3) contributes $(-1)^{j \sum_i v_i}$, already included in a. Absolute convergence of (3.4) and convergence of the resulting j^{-3} series justify the grouping.

Let Θ be uniform on the simplex $\{\Theta_i \ge 0 : \sum_i \Theta_i = 1\}$, and let b_v be the density of $V = \sum_i \Theta_i v_i$. This density is continuous, compactly supported, and unimodal. Its values are proportional to areas of parallel sections of a tetrahedron; Brunn–Minkowski makes the square root of the section area concave. The four distinct knots ensure continuity and zero boundary values. The Genocchi–Hermite formula [4, Section 9] reads

$$[v_0, v_1, v_2, v_3] f = \frac{1}{6} \mathbb{E} f'''(V).$$

Apply it to $f(t) = e^{2\pi i k t}$ for $k \in \mathbb{Z} \setminus \{0\}$. The divided difference is zero because the knots are integers. Thus the periodisation of b_v has all nonconstant Fourier coefficients zero, and by continuity

$$\sum_{n \in \mathbb{Z}} b_v(t - n) = 1 \qquad (t \in \mathbb{R}). \tag{3.9}$$

The third distributional derivative of periodic B_3 is $6 - 6 \sum_{n \in \mathbb{Z}} \delta_n$. Mollifying, applying Genocchi–Hermite, and using the continuity of b_v converts (3.8) into

$$L = \frac{1}{3} - 2w^2 + \frac{4}{3} w^3 D_w(a), \qquad D_w(a) = \frac{1}{w} \sum_{n \in \mathbb{Z}} b_v((a - n)/w). \tag{3.10}$$

For $0 < w \le 1$,

$$0 \le D_w(a) \le \frac{1}{w}. \tag{3.11}$$

To prove the upper bound, bracket a mode t of b_v by two consecutive points $A \le t \le B$ of the grid of spacing $1/w$. The intervals $[A, t]$ and $[t - 1, B - 1]$ intersect. For s in their intersection, each point of $s + \mathbb{Z}$ lies between the corresponding point of the coarser grid and the mode, on its respective side. Unimodality bounds the coarse-grid sum by $\sum_n b_v(s + n) = 1$.

Equations (3.10)–(3.11) yield

$$\frac{2}{3} w^2 \le \frac{1}{3} - L \le 2w^2.$$

Continuity of translations in L^1 includes the endpoints $w = 0, 1$. At rank one the same bounds follow directly from (3.7). Replacing one phase by its translate by $1/2$ changes the sign of L. This proves (3.2) whenever a unit relation exists.

3.3 The gap when there is no unit relation

The rank-zero and rank-one cases were settled above, so the relation lattice now has rank two. We give the reduction to four small finite lattice problems explicitly. The calculations left by the reduction use rational arithmetic only and are specified in Appendix A.

For distinct sites i, j with complement k, ℓ, put $U_{k\ell} = \mathbb{Z}u_k + \mathbb{Z}u_\ell$ and work in the group generated by the directions modulo $2U_{k\ell}$. Set

$$\delta = [u_k + u_\ell], \qquad \alpha = [u_i], \qquad \beta = [u_j] - \delta.$$

Since $2\delta = 0$, two odd integers m, n extend to an odd relation exactly when $m\alpha + n\beta = 0$. Give odd integers the probability law

$$p(m) = \frac{4}{\pi^2 m^2},$$

and let P_{ij} be the probability of this event for independent m, n. Cauchy–Schwarz in (3.4) gives

$$|L|^2 \le P_{ij} P_{k\ell}. \tag{3.12}$$

If $\langle\alpha\rangle \neq \langle\beta\rangle$, one coordinate must be divisible by an integer $d > 1$, or must vanish if the relevant quotient order is infinite. Even d gives zero probability; odd d gives probability $d^{-2} \le 1/9$. If the cyclic subgroups coincide and $\beta = \pm\alpha$, an odd relation exists whose two coordinates on the chosen pair have absolute value one. Otherwise the common order N is finite and $\beta = q\alpha$ for a unit $q \not\equiv \pm 1 \pmod N$. With $Q = N$ for even N and $Q = 2N$ for odd N, the masses on odd residue classes are

$$p_Q(r) = \frac{4}{Q^2 \sin^2(\pi r/Q)}. \tag{3.13}$$

For $Q \ge 12$, the largest mass outside ± 1 is at most $1/18$. The two entries at ± 1 and all remaining entries together then give $P_{ij} \le 1/9$. The exceptional possibilities are $P_{ij} = 13/125$ at $Q = 10$ and $P_{ij} = 1/8$ at $Q = 8$. Thus a pair without a relation unit on that pair always has $P_{ij} \le 1/8$.

If all six pairs admitted a relation unit on that pair, Appendix A would force the existence of a global unit relation. Hence, in the absence of a global unit relation, at least one pair fails the test. Choose such a pair. Suppose, towards a contradiction, that $|L| > 1/4$. Then

$$P_{ij} > \frac{1}{16}, \qquad P_{k\ell} > \frac{1}{2}. \tag{3.14}$$

A projection of the odd-integer law onto a cyclic element of order greater than two has every atom of mass at most $1/2$, by (3.13); infinite order is covered directly. Conditioning on either coordinate shows that $P_{k\ell} > 1/2$ forces both relevant orders to be at most two. Odd multiples are then fixed, so $P_{k\ell} = 1$. Comparing two compatible pairs differing by two in one coordinate gives $U_{k\ell} \subset U_{ij}$, since the ambient group is torsion-free.

If $\langle\alpha\rangle \neq \langle\beta\rangle$, the divisibility order in the preceding argument must be three, by (3.14). Conditionally on $3 \mid m$, the law of $m/3$ is again p. The compatibility probability for $3\alpha, \beta$ is therefore $9P_{ij} > 1/2$; these two elements have order at most two and coincide. The quotient $U_{ij}/2U_{k\ell}$ is generated by an element of order three or six and an element of order at most two. It surjects onto $U_{ij}/2U_{ij} \simeq (\mathbb{Z}/2\mathbb{Z})^2$, and its order is divisible by three. It consequently has order twelve, giving

$$[U_{ij} : U_{k\ell}] = 3. \tag{3.15}$$

The argument is unchanged if the divisibility condition is on n.

If the cyclic subgroups coincide, the quotient is generated by α of order N and δ of order at most two. Surjectivity onto $(\mathbb{Z}/2\mathbb{Z})^2$ forces N even, $\delta \notin \langle \alpha \rangle$, and quotient order $2N$. Thus $[U_{ij} : U_{k\ell}] = N/2$. For $8 \le N \le 22$ even, the exact finite Fourier identity

$$P_{ij} = \frac{1}{N} \sum_{v=0}^{N-1} (1 - 4\|v/N\|_{\mathbb{T}}) \, (1 - 4\|qv/N\|_{\mathbb{T}}) \tag{3.16}$$

leaves only $N = 8, 10, 14$ when $P_{ij} > 1/16$.

For even $N \ge 24$, the distances of q and q^{-1} from zero are at least three, and one is at least five: if both were three, N would divide $9 - 1$ or $9 + 1$. Writing $p_j = p_N(j)$, separate the terms whose input or output is ± 1 and use Cauchy–Schwarz on the rest. This gives

$$P_{ij} \le 2p_1(p_3 + p_5) + 2p_3^2 + p_5(1 - 2p_1 - 2p_3) < \frac{1}{16}.$$

The last strict inequality follows from the rational bounds

$$p_1 < \frac{409}{1000}, \quad p_3 < \frac{19}{400}, \quad p_5 < \frac{19}{1000}, \quad 1 - 2p_1 - 2p_3 < \frac{51}{500}.$$

For the first three, $N \sin(\pi j/N)$ increases with $N \ge 2j$ and $\sin x \ge x - x^3/6$ suffices at $N = 24$; the last follows from $p_j \ge 4/(\pi j)^2$. The rational estimates $314159/100000 < \pi < 22/7$ make all four bounds explicit.

Only the indices $3, 4, 5, 7$ remain. The exhaustive rational calculation in Appendix A gives, in every configuration without a unit relation,

$$\min_{\{i,j\} \sqcup \{k,\ell\} = \{0,1,2,3\}} P_{ij} P_{k\ell} < \frac{1}{16}.$$

This contradicts (3.12) and $|L| > 1/4$. Finally, two different unit relations not equal up to sign would have a sum or difference supported on at most two sites. Pairwise independence excludes this. The theorem is proved.

Corollary 3.2 (A consequence of individual near extremality)**.** *In a sequence of totally ergodic systems, let $F_0, \dots, F_3$ be centred real functions bounded by one. If all four values $\|F_i\|_{U^3}^8$ tend to $1/3$, then*

$$\limsup |\Lambda(F_0, F_1, F_2, F_3)| \le \frac{1}{3}.$$

More generally, writing $\mathcal{S}$ for the semicircle signs of the Kronecker factor,

$$\sum_{i=0}^{3} \inf_{S_i \in \mathcal{S}} \|F_i - S_i\|_1 \ge \big(|\Lambda(F_0, F_1, F_2, F_3)| - 1/3\big)_+. \tag{3.17}$$

Proof. Replacing the four profiles costs at most the sum of their L^1 distances, by telescoping and invariance of μ. For Kronecker-measurable signs the average is (3.1). Theorem 3.1 gives (3.17), and the uniform stability theorem in Section 2 gives the first assertion. □

4 Exact central envelopes and their realisation

The Kronecker projections of four profiles retain their horizontal dependence but forget the variation inside a nilpotent fibre. We determine the largest and smallest possible contributions of that variation, and then realise the result without allowing one colour to read another colour's position.

4.1 Four real functions on a parallelogram

We record a four-function consequence of the relaxed Riesz–Sobolev inequality of Christ and Iliopoulou [3, Theorem 1.5], in a form adapted to real profiles with prescribed means.

For $m \in [-1,1]$, let

$$A_m(t) = 2\mathbf{1}_{\{\|t\|_{\mathbb{T}} \le (1+m)/4\}} - 1.$$

It has mean m, and, for $n \neq 0$, Fourier coefficient

$$a_n(m) = \frac{2\sin(\pi n(1+m)/2)}{\pi n}. \tag{4.1}$$

For $\delta \in \{0, 1/2\}$ and $\mathbf{m} = (m_0, m_1, m_2, m_3)$ define

$$U_\delta(\mathbf{m}) = \prod_{i=0}^{3} m_i + \sum_{n\neq 0} e^{2\pi i n\delta} \prod_{i=0}^{3} a_n(m_i). \tag{4.2}$$

The series converges absolutely and uniformly on $[-1,1]^4$.

Lemma 4.1 (The real four-function rearrangement)**.** *Let C be a compact connected abelian group and let $f_i : C \to [-1,1]$ have means m_i. Then*

$$U_{1/2}(\mathbf{m}) \le \int_{C^3} f_0(s) f_1(t) f_2(w) f_3(s-t+w)\, ds\, dt\, dw \le U_0(\mathbf{m}). \tag{4.3}$$

Both functions U_δ are 1-Lipschitz for the ℓ^1 distance on the four means.

Proof. Write $p_i = (1+f_i)/2$ and $\alpha_i = \int p_i$. Let I_{α_i} be the centred circle interval of length α_i. The positive four-function integral is

$$\int_C (p_0 * p_2)(z)(p_1 * p_3)(z)\, dz.$$

For every measurable set E, the relaxed rearrangement theorem [3, Theorem 1.5] bounds $\int_E p_0 * p_2$ by the integral of $\mathbf{1}_{I_{\alpha_0}} * \mathbf{1}_{I_{\alpha_2}}$ over a centred interval of measure $|E|$. The same statement holds for $p_1 * p_3$. Layer-cake integration first rearranges the latter convolution and then uses that the former interval convolution is symmetric decreasing. It follows that

$$\int_C (p_0 * p_2)(p_1 * p_3) \le \int_{\mathbb{T}} (\mathbf{1}_{I_{\alpha_0}} * \mathbf{1}_{I_{\alpha_2}})(\mathbf{1}_{I_{\alpha_1}} * \mathbf{1}_{I_{\alpha_3}}).$$

This use of the direct inequality needs no strict admissibility of the four masses. Every proper subset of the four parallelogram coordinates is Haar independent. When $\prod_i (2p_i - 1)$ is expanded, all terms of degree less than four are therefore fixed by the means. Only the positive quartic term is changed by the rearrangement. The resulting upper bound is the integral of the four A_{m_i}, which is U_0 by Fourier expansion.

Apply the upper bound after replacing f_0 by $-f_0$. The identity $a_n(-m) = (-1)^{n+1} a_n(m)$ gives $-U_0(-m_0, m_1, m_2, m_3) = U_{1/2}(\mathbf{m})$. On the trivial group the upper bound follows instead from the nonnegative covariance of two symmetric decreasing interval convolutions; the lower bound follows by the same sign change.

Finally, the centred intervals are nested and $\|A_m - A_{m'}\|_1 = |m - m'|$. Use the integral representation with the first argument translated by δ and telescope the four factors. Each argument is Haar distributed, giving the stated Lipschitz constant. □

The envelope has a finite formula useful for exact calculations. Set $\alpha_i = (1+m_i)/2$, $A = \sum_i \alpha_i$, and

$$T_4(\boldsymbol{\alpha}) = \frac{1}{6}\sum_{q=-2}^{2} \sum_{J\subset\{0,1,2,3\}} (-1)^{|J|} \left(q + \frac{A}{2} - \sum_{j\in J}\alpha_j\right)_+^3. \tag{4.4}$$

Then

$$U_0(\mathbf{m}) = \prod_i m_i + 16\left(T_4(\boldsymbol{\alpha}) - \prod_i \alpha_i\right). \tag{4.5}$$

Indeed the convolution of four centred interval indicators on the line is the truncated-power expression inside (4.4). Its support lies in $[-2,2]$; periodisation gives its value at zero on the circle. Thus U_0 is piecewise cubic and can be evaluated exactly at rational means.

4.2 Why these fibres occur in a totally ergodic system

We spell out the structural input because the means and the three central parameters must belong to the same progression measure.

Lemma 4.2 (Central disintegration of a four-term average)**.** *Let (X,μ,T) be a totally ergodic standard invertible system, let K be its Kronecker factor, and let $m_i = \mathbb{E}(F_i \mid K)$ for four real functions $|F_i| \le 1$. Then*

$$\int_{K^2} U_{1/2}\big((m_i(x+ih))_{i=0}^3\big)\,dx\,dh \le \Lambda(F_0,F_1,F_2,F_3) \le \int_{K^2} U_0\big((m_i(x+ih))_{i=0}^3\big)\,dx\,dh. \tag{4.6}$$

Proof. First suppose that $X = G/\Gamma$ is a totally ergodic two-step nilsystem, with translation τ. We use the usual presentation $G = \langle G^0, \tau\rangle$. The group G need not be connected. Its commutator group G_2 is connected, and $K = G/(G_2\Gamma)$ is the Kronecker factor [1, Theorem 4.1(4),(6)]. Put $C = G_2/(G_2\cap\Gamma)$, a compact connected abelian group.

The averaged progression measure is Haar measure on the Hall–Petresco nilmanifold [7, Proposition 4.7 and Appendix B]. In step two its configurations are represented by

$$(a_0 a_1^i a_2^{\binom{i}{2}})_{i=0}^3, \qquad a_0,a_1 \in G, \quad a_2 \in G_2.$$

Projection to the abelian quotient gives the Haar pair $(x,h)\in K^2$. Over a fixed pair, the central configurations form a torsor under C^3, whose action on the four coordinates is

$$(z,t,v) \longmapsto \big(z + it + \tbinom{i}{2}v\big)_{i=0}^3. \tag{4.7}$$

Here is the fibre identification. Two configurations with the same horizontal pair have their first two parameters equal modulo $G_2\Gamma$. Choose $\gamma_0,\gamma_1\in\Gamma$ and right-multiply one configuration by $(\gamma_0\gamma_1^i)_{i=0}^3$ so that coordinates $0,1$ agree modulo G_2. Their degree-one images in G/G_2 then agree at every coordinate. The coordinatewise quotient lies in G_2. Centrality then makes this quotient a quadratic sequence of the form (4.7). Its lattice stabiliser is $(G_2\cap\Gamma)^3$: coordinates $i=0,1,2$ give a triangular integer matrix with diagonal entries one. Invariance of the conditional measure makes it Haar on this torsor. After choosing an origin, z,t,v are independent Haar variables. The choices of origins in the four fibres may depend on (x,h); this only translates the four local functions and preserves their means.

On a connected group, any three of the four expressions in (4.7) are Haar independent. The corresponding integer evaluation matrix has determinant 1 or 3, hence is surjective. The fourth coordinate obeys $Q_3 = Q_0 - 3Q_1 + 3Q_2$. For $j=1,2$, put $\bar f_j(u) = \mathbb{E}(f_j(q) \mid 3q = u)$.

Since multiplication by three preserves Haar measure, conditioning gives the parallelogram integral

$$\int_{C^3} f_0(s)\bar{f}_1(t)\bar{f}_2(w)f_3(s-t+w)\,ds\,dt\,dw.$$

This conditioning preserves the bound one and the means $m_i(x+ih)$. Lemma 4.1 gives the asserted bounds in each fibre, and integration proves (4.6).

For a general system, the factor Z_2 is characteristic for the three moving functions in (1.2) [8, Lemma 4.3 and Theorem 12.1]. After those functions are projected, their product is Z_2-measurable, so F_0 may also be projected. The factor Z_2 is an inverse limit of two-step nilsystems [8, Theorem 10.1], and all its factors inherit total ergodicity. Approximate the four projected functions by their conditional expectations on an increasing sequence of these factors.

The Kronecker factors K_j of these stages generate K. Indeed, an eigenfunction has a nonzero projection to some sufficiently large stage; that projection is an eigenfunction with the same eigenvalue, and ergodicity makes the eigenspace one-dimensional. Thus $m_{i,j} = \mathbb{E}(F_i \mid K_j) \to m_i$ in L^1. The progression averages converge by telescoping four bounded factors, and the integrals of U_δ converge by the Lipschitz bound in Lemma 4.1. This proves the result. □

4.3 One extension realises the entire interval

Theorem 4.3 (Prescribed horizontal functions)**.** *Let $x \mapsto x+a$ be a totally ergodic rotation of a compact connected metrizable abelian group K, and let $m_i : K \to [-1,1]$ be measurable. There is a totally ergodic extension on $K \times \mathbb{T}^2$ whose full Kronecker factor is the rotation on $K \times \mathbb{T}$ such that every value in the closed interval in (4.6) is realised by binary profiles $F_0, \dots, F_3$ on this one extension, with*

$$\mathbb{E}(F_i \mid K \times \mathbb{T})(x,y) = m_i(x), \qquad 0 \le i \le 3.$$

Thus the prescribed functions are pulled back from K and do not depend on the added horizontal coordinate. The two integrated bounds are optimal over the class of extensions allowing this enlargement and preserving these prescribed projections.

Proof. The dual of K is countable. Choose $\alpha \in \mathbb{T}$ so that the rotation by (a,α) on $K \times \mathbb{T}$ is totally ergodic. This excludes only countably many values of α. On $K \times \mathbb{T}^2$ define

$$T(x,y,z) = (x+a, y+\alpha, z+y), \qquad F_i(x,y,z) = A_{m_i(x)}(d_i z + b_i), \quad d = (1,3,3,1). \tag{4.8}$$

The Kronecker factor of T is precisely (x,y). To verify this, expand an eigenfunction in Fourier characters of $K \times \mathbb{T}^2$. A nonzero vertical frequency would force coefficients of the same absolute value along an infinite arithmetic progression of frequencies in y, contradicting square summability. The same argument applies to every power T^r. All remaining eigenfunctions belong to the rotation by (a,α); every power of this rotation is ergodic. Hence T is totally ergodic and the conditional means of the F_i on its Kronecker factor are $m_i(x)$.

At time in, the central coordinate is

$$z + i\left(ny + \binom{n}{2}\alpha\right) + \binom{i}{2}n^2\alpha.$$

For each nonzero n, integration in y makes the middle parameter uniform and independent of x, z. The pair $(na, n^2\alpha)$ is equidistributed on $K \times \mathbb{T}$: on any nontrivial character, either the quadratic coefficient is irrational, or the remaining linear character of the totally ergodic rotation is irrational. Weyl's criterion applies. Thus the average has the same three independent central Haar parameters as (4.7), independent of the horizontal pair (x,h).

For the four weighted coordinates d_iQ_i, the character relations are exactly the integer multiples of $(1,-1,1,-1)$. Taking $b_0=\delta$ and $b_1=b_2=b_3=0$ therefore gives

$$\int_{K^2} U_\delta((m_i(x+ih))_{i=0}^3)\,dx\,dh$$

at $\delta=0$ and $\delta=1/2$. Dependence on δ is continuous in L^1, so every value between these endpoints occurs on the same system. Approximation of the bounded measurable profiles by continuous functions in L^1 justifies the equidistribution step: all single-coordinate marginals are Haar, and the telescoping error is bounded by the sum of the four L^1 errors. □

Remark 4.4 (Reduction of the free optimisation to a circle). The supremum of $\int_{K^2} U_0((m_i(x+ih))_i)$ over centred bounded real m_i and connected compact metrizable groups can be approximated using $K=\mathbb{T}$. To see this, approximate the four functions by bounded centred trigonometric polynomials on a finite-dimensional torus $\mathbb{T}^d$. Positive convolution kernels preserve both constraints. Pull them back by $t\mapsto(t,Nt,\dots,N^{d-1}t)$. For each fixed Fourier support, the means are preserved when N is sufficiently large; the pushforwards of one and of two independent circle variables tend to Haar by character orthogonality. Continuity and the Lipschitz bound for U_0 finish the argument. This reduces the free optimisation problem but does not compute its value.

4.4 The exact envelope at a fixed horizontal amplitude

Suppose $m_i=rS_i$, $0\le r\le 1$, where the S_i are arbitrary binary horizontal profiles, and put $L=\int_{K^2}\prod_i S_i(x+ih)$. Let $v=\min(r,1-r)$ and set

$$\begin{aligned} E(r)&=\frac{2}{3}v^3-v^4, & Q(r)&=\frac{1}{3}(1-r)^3(1+3r),\\ B(r)&=r^4+E(r), & O(r)&=Q(r)-E(r). \end{aligned} \tag{4.9}$$

Changing r to $-r$ preserves $a_n(r)$ for odd n and reverses its sign for even n. The sum of the fourth powers of the even nonzero coefficients is $E(r)$: these coefficients are those of a circle interval of length r, and its nonconstant convolution energy is $2v^3/3-v^4$. The sum over all nonzero coefficients is $Q(r)$, by the fixed-mean U^2 formula of Section 2. The attainable interval is therefore exactly

$$[\,B(r)L-O(r),\ B(r)L+O(r)\,]. \tag{4.10}$$

Corollary 4.5 (Semicircle horizontal projections)**.** *If every S_i is a semicircle sign of a nonzero character, the maximum possible absolute value, over all characters, phases and central fluctuations, is*

$$M(r)=O(r)+\frac{B(r)}{3}=\begin{cases}\dfrac{1}{3}-2r^2+\dfrac{20}{9}r^3, & 0\le r\le 1/2,\\[2mm] \dfrac{5}{9}-\dfrac{4}{3}r+\dfrac{2}{3}r^2+\dfrac{4}{9}r^3, & 1/2\le r\le 1.\end{cases} \tag{4.11}$$

It is attained. In particular, $M(0)=M(1)=1/3$, $M(1/2)=1/9$, and its minimum is

$$M\left(\frac{\sqrt{5}-1}{2}\right)=\frac{12-5\sqrt{5}}{9}.$$

Proof. Theorem 3.1 gives $|L|\le 1/3$; taking the same semicircle sign with zero phase at all sites gives equality. Theorem 4.3 then attains the upper endpoint of (4.10) for this very choice of horizontal functions. The formulas follow by substitution and differentiation. □

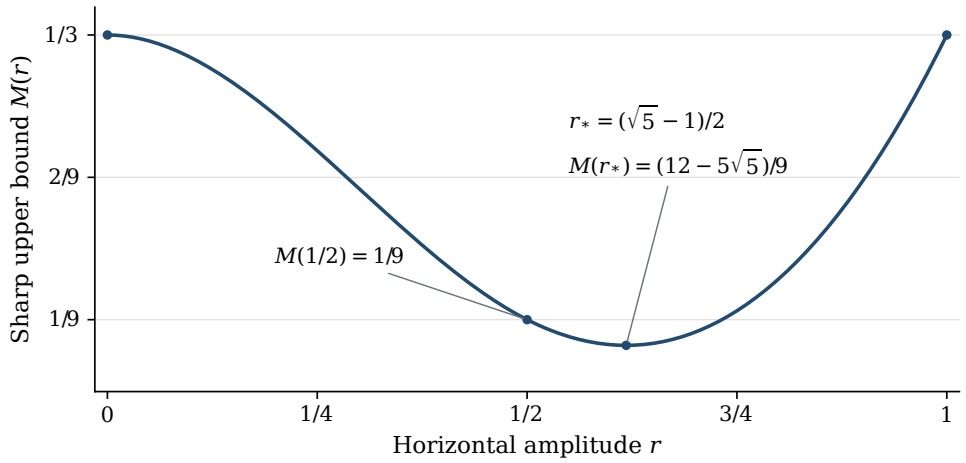

Figure 1: The exact four-profile envelope when every Kronecker projection is a semicircle sign with common amplitude r. Central freedom is included in the optimisation. The curve is the polynomial formula (4.11); it is not a simulation.

This result is robust at the level of the horizontal means alone. For arbitrary $m_i = \mathbb{E}(F_i \mid K)$, the Lipschitz bound yields

$$|\Lambda(F_0, F_1, F_2, F_3)| \le M(r) + \sum_{i=0}^{3} \inf_{S_i \in \mathcal{S}} \|m_i - rS_i\|_{L^1(K)} \qquad (0 < r \le 1), \tag{4.12}$$

whenever the semicircle class $\mathcal{S}$ is nonempty. At $r = 0$ the same argument gives $|\Lambda| \le 1/3 + \sum_i \|m_i\|_1$, without a nontriviality assumption on K. The error terms pay for arbitrary central fluctuations, because those fluctuations have already been optimised in (4.6).

5 Seven profiles on one quadratic Haar jet

A four-function rearrangement is not automatically compatible with a second rearrangement sharing two of its functions. The configuration considered here has a useful exception: a single choice of characters makes three quartic rearrangements sharp at once. This permits an exact comparison between the means of seven profiles and the correlations of those same profiles. The comparison distinguishes one septuple from a probabilistic mixture of septuples.

Let Z be a compact connected abelian group, with normalized Haar measure. For independent Haar variables t_0, t_1, t_2, put

$$Q_i = t_0 + it_1 + \binom{i}{2} t_2, \qquad 0 \le i \le 6. \tag{5.1}$$

For real measurable functions $F_i : Z \to [-1, 1]$, write $\mu_i = \int_Z F_i$. Given $\sigma \in \{-1, 1\}$, define

$$D_\sigma(z_0, \dots, z_6) = \frac{2 - \sigma(z_0 z_2 z_4 z_6 + z_0 z_1 z_2 z_3 + z_3 z_4 z_5 z_6 - z_1 z_5)}{4} \tag{5.2}$$

and

$$D = D_\sigma(\mu_0, \dots, \mu_6), \qquad d = \mathbb{E} D_\sigma(F_0(Q_0), \dots, F_6(Q_6)), \qquad a = \mu_1, \quad b = \mu_5. \tag{5.3}$$

We call positions 1 and 5 the anchors. The two numbers a, b always refer to means in this very same Haar jet.

The function D_σ takes values in $[0,1]$. Indeed, at a vertex of the sign cube, if A, B, C are its three quartic products, its pair product is ABC, and $A+B+C-ABC$ is either -2 or 2. Multiaffine interpolation gives the assertion on the entire cube. Changing the signs of F_0 and F_5 reverses σ, with b replaced by $-b$.

5.1 Three compatible rearrangements

For $x \in [-1,1]$, let

$$h_x = 2\mathbf{1}_{I_{(1+x)/2}} - 1,$$

where I_s is the centered interval of length s in $\mathbb{T}$. The four-function rearrangement of Lemma 4.1 gives

$$\int_{Z^3} f_0(s) f_1(t) f_2(w) f_3(s-t+w) \le \varphi(x_0, x_1, x_2, x_3),$$

where $x_i = \int f_i$ and

$$\varphi(x_0, x_1, x_2, x_3) = \int_{\mathbb{T}} (h_{x_0} * h_{x_2})(z)(h_{x_1} * h_{x_3})(z)\, dz. \tag{5.4}$$

The right-hand side is symmetric in its four arguments. This symmetry also follows from its Fourier expression and the evenness of h_x.

Here is why this rearrangement applies simultaneously. On any four-term progression of indices, the first three phases in (5.1) are independent Haar variables, and the fourth satisfies $Q_{r+3q} = Q_r - 3Q_{r+q} + 3Q_{r+2q}$. The relevant integer matrix has nonzero determinant q^3; it is surjective on Z^3 because multiplication by a nonzero integer is surjective on a connected compact abelian group. Conditioning the two middle functions under multiplication by 3 turns the correlation into the displayed parallelogram form, without changing their means or their bounds.

For the progressions $(0,2,4,6)$, $(0,1,2,3)$ and $(3,4,5,6)$, the multipliers

$$(k_0, \dots, k_6) = (1,3,3,1,3,3,1) \tag{5.5}$$

have the order $(1,3,3,1)$ in each progression. Therefore the seven functions $F_i(z) = h_{\mu_i}(k_i \chi(z))$, with any nonzero continuous character $\chi : Z \to \mathbb{T}$, attain all three upper bounds simultaneously. The character is surjective by connectedness. Furthermore, Q_1, Q_5 are independent Haar variables, so their pair term is exactly ab.

Consequently, for $\sigma = 1$, the exact lower envelope at seven prescribed means is

$$4d \ge 2 + ab - \varphi(\mu_0, \mu_2, \mu_4, \mu_6) - \varphi(\mu_0, a, \mu_2, \mu_3) - \varphi(\mu_3, \mu_4, b, \mu_6). \tag{5.6}$$

On every nontrivial Z, equality is attained for every seven-tuple of means. The inequality remains valid on the trivial group, where $d = D$.

5.2 The convex boundary at two prescribed anchor means

Set

$$\begin{aligned}
g(t) &= \frac{1}{3} - \frac{t^2}{2} + \frac{t^3}{6} \quad (0 \le t \le 1), & S &= \frac{1}{3} + g(|a|) + g(|b|), \\
\ell &= \frac{1 + \sigma ab - |a + \sigma b|}{4}, & r &= \frac{3 + \sigma ab + |a - \sigma b|}{4}, \\
\Delta &= \frac{2 + \sigma ab}{4}, & d_0 &= \Delta - \frac{S}{4}, \\
u &= 1 - \frac{S}{1 + |a + \sigma b|}, & v &= 1 + \frac{S}{1 + |a - \sigma b|}.
\end{aligned} \tag{5.7}$$

All denominators are positive, and $0 \le u \le 1 \le v \le 2$.

Theorem 5.1 (The convex boundary)**.** *For the quantities in* (5.3)*,*

$$\ell \le D \le r, \qquad d \ge L_{a,b,\sigma}(D) := \max\{\ell + u(D-\ell),\ r + v(D-r)\}. \tag{5.8}$$

The two segments of $L_{a,b,\sigma}$ *join*

$$(\ell,\ell), \qquad (\Delta, d_0), \qquad (r,r).$$

If Z *is nontrivial, each of these three points is attained by one septuple with the prescribed anchor means. Thus* $L_{a,b,\sigma}$ *is the lower boundary of the convex hull of the realizable pairs* (D,d) *at those means. Its segments are attained by mixtures of two septuples.*

We prove the theorem by reducing the seven means to nine configurations. For uniform $t \in [-1,1]$, put

$$U_{x,y}(t) = \max\{-1 + |x+y|,\ \min(t, 1-|x-y|)\}. \tag{5.9}$$

The intersection formula for two intervals gives

$$(h_x * h_y)(z) = U_{x,y}(1 - 4\|z\|_{\mathbb{T}}), \qquad \mathbb{E}U_{x,y} = xy.$$

The random variable $1 - 4\|z\|_{\mathbb{T}}$ is uniform on $[-1,1]$. Every interval of clipping levels $[L,R] \subset [-1,1]$ arises in this way: take $x = (2+L-R)/2$ and $y = (L+R)/2$.

Lemma 5.2 (Clipping against a decreasing function)**.** *If* H *is decreasing and integrable on* $[-1,1]$*, and* $C_{L,R}(t) = \max(L, \min(t,R))$*, then*

$$\mathbb{E}(C_{L,R}H) \ge \min\{\mathbb{E}(tH),\ \mathbb{E}H,\ -\mathbb{E}H\}.$$

Proof. At fixed R, the derivative with respect to L is $\frac{1}{2}\int_{-1}^{L} H$. The average of H over this prefix is decreasing in L, so its sign can change only from positive to negative. The minimum is therefore at $L = -1$ or $L = R$. On the diagonal $L = R$, the expression is affine in this common level and is minimized by one of the constants $-1, 1$. On $L = -1$, the derivative with respect to R is $\frac{1}{2}\int_{R}^{1} H$. Its normalized suffix average is again decreasing, so the endpoints $R = -1, 1$ suffice. Absolute continuity justifies the same argument when H has discontinuities or zero intervals. □

For the proof of Theorem 5.1, it is enough to take $\sigma = 1$. Let $c = \mu_3$ and

$$U = U_{\mu_0,\mu_2}, \quad W = U_{\mu_4,\mu_6}, \quad V_a = U_{a,c}, \quad V_b = U_{b,c}, \quad p = \mathbb{E}U, \quad q = \mathbb{E}W.$$

If C denotes the right-hand side of (5.6) and $A = 4D$, then

$$C = 2 + ab - \mathbb{E}(UW + UV_a + WV_b), \qquad A = 2 + ab - pq - acp - bcq. \tag{5.10}$$

For any real slope k, the kernels of $C - kA$ as a function of U and then of W are

$$-W - V_a + k(q + ac), \qquad -U - V_b + k(p + bc),$$

both decreasing. Lemma 5.2 thus reduces a lower bound on $C - kA$ to the nine choices $U, W \in \{-1, t, 1\}$. No joint convexity assumption is involved.

One mixed case uses the following elementary estimate. Direct integration of (5.9) gives

$$J(b,c) := \mathbb{E}[tU_{b,c}(t)] = g(M) - \frac{1-M}{2}m^2, \qquad M = \max(|b|,|c|), \quad m = \min(|b|,|c|). \tag{5.11}$$

In particular $0 \le J(b,c) \le \min(g(|b|), g(|c|))$. For $A_0, B_0, C_0 \in [0,1]$,

$$J(B_0, C_0) \le \left(\frac{1}{3} + g(A_0) + g(B_0)\right)\left(1 - \frac{A_0 C_0}{1 + |A_0 - B_0|}\right). \tag{5.12}$$

To verify this, use $A_0/(1+|A_0-B_0|) \le 1/(2-B_0)$ and discard $g(A_0)$. If $C_0 \le B_0$, the resulting right-hand side is at least $g(B_0)$, since its difference at $C_0 = B_0$ is

$$\frac{2(1-B_0)/3 - B_0 g(B_0)}{2-B_0} \ge 0.$$

Here $g(B_0) \le (1-B_0)/2$. If $C_0 \ge B_0$ and $B_0 < 1$, write $C_0 = B_0 + (1-B_0)s$. The relevant difference equals

$$\begin{aligned}&\left(\frac{1}{3} + g(B_0)\right)(2 - B_0 - C_0) - (2-B_0)g(C_0)\\ &\quad = \frac{(2-B_0)(1-B_0)}{6}\left[(1-s)(2-B_0^2) + sB_0(5-3B_0) + (1-B_0)^2 s^2(3-s)\right] \ge 0.\end{aligned}$$

The remaining endpoint is immediate.

Proof of Theorem 5.1. The extrema of D in the five means other than a, b occur at signs, by multiaffinity. At those signs, the extrema of $pq + c(ap+bq)$ are $1+|a+b|$ and $-1-|a-b|$. This gives $\ell \le D \le r$. Taking those five profiles constant realizes both endpoints, with $d = D$.

To prove each of the two affine bounds in (5.8), apply the preceding nine-configuration reduction at its slope. If U, W are constant signs, $C = A$, and each line lies below the diagonal on $[\ell, r]$. If $U = W = t$, then $D = \Delta$ and

$$C = 2 + ab - \frac{1}{3} - J(a,c) - J(b,c) \ge 2 + ab - S = 4d_0.$$

Equality holds at $c = 0$. Finally, if $U = p \in \{-1, 1\}$ and $W = t$, then

$$D = \Delta - \frac{acp}{4}, \qquad \frac{C}{4} = D - \frac{J(b,c)}{4}.$$

The required inequalities are

$$J(b,c) \le S\left(1 - \frac{acp}{1+|a+b|}\right), \qquad J(b,c) \le S\left(1 + \frac{acp}{1+|a-b|}\right).$$

Both follow from (5.12), because their right-hand sides are at least $S(1-|a||c|/(1+||a|-|b||))$. Interchanging a, b handles the other mixed choices. The sign reversal described after (5.3) gives both orientations.

The central point is attained by taking every mean other than a, b equal to zero and using the compatible interval profiles of (5.5). Finally, any convex minorant of all realizable pairs lies below the two chords between the three attained points. Since (5.8) is itself valid and convex, it is the exact convex boundary. □

5.3 One polarized anchor and a weighted stability bound

Corollary 5.3. *For every septuple in (5.3),*

$$d \ge K_\sigma(a,b) D, \qquad K_\sigma(a,b) = 1 - \frac{\frac{1}{3} + g(|a|) + g(|b|)}{2 + \sigma ab}. \tag{5.13}$$

For nontrivial Z, this factor is optimal for every fixed a, b. Moreover,

$$K_\sigma(a,b) \ge \frac{1}{2} + \frac{a^2+b^2}{12}. \tag{5.14}$$

The coefficient $1/12$ is optimal. If $|a| = 1$ or $|b| = 1$, then $d \ge 5D/8$, and $5/8$ is the best constant uniform over that sector.

Proof. The ratio $K = d_0/\Delta$ lies between the two slopes u, v, so the two segments in Theorem 5.1 lie above KD. The central witness gives equality. For (5.14), put $A_0 = |a|$, $B_0 = |b|$, $s = A_0^2 + B_0^2$ and $\rho = \sigma ab$. Multiplication of the proposed difference by $12(2+\rho)$ gives

$$P = 4s - 2(A_0^3 + B_0^3) + \rho(6 - s).$$

The smaller of the two signs occurs at $\rho = -A_0 B_0$, where

$$P = 2(2 - A_0 - B_0)(A_0 - B_0)^2 + A_0 B_0\big[(1 - A_0)^2 + (1 - B_0)^2\big] \geq 0.$$

Equality at $A_0 = B_0 = 1$, $\rho = -1$ forces the coefficient $1/12$.

If one anchor has modulus one, the factor depends only on ρ:

$$K = \begin{cases} (\rho^2 + \rho + 4)/6, & -1 \leq \rho \leq 0, \\ (8 + 6\rho + 3\rho^2 - \rho^3)/(6(2+\rho)), & 0 \leq \rho \leq 1. \end{cases}$$

The first branch is $5/8 + (2\rho + 1)^2/24$, while the second is at least $2/3$. The central witness with $a = 1$, $b = -\sigma/2$ has $D = 3/8$, $d = 15/64$, proving sharpness. □

A mean of modulus one makes its profile a constant sign almost everywhere. It is a substantially stronger assumption than merely requiring the profile to take its values in $\{-1, 1\}$.

6 The exact boundary for a single septuple

When $a = b = 0$, Theorem 5.1 becomes

$$\frac{1}{4} \leq D \leq \frac{3}{4}, \qquad d \geq L(D) := \max\left\{\frac{1}{4},\ 2D - \frac{3}{4}\right\}. \tag{6.1}$$

The line segment above $D = 1/2$ is obtainable by mixing different septuples. Retaining one septuple gives a strictly larger answer except at the three vertices.

Fix a nontrivial compact connected abelian group Z. Let $F_{\mathrm{nc}}(D)$ be the infimum of d over all seven profiles on this group with $\mu_1 = \mu_5 = 0$ and with the specified scalar defect D.

Theorem 6.1 (The boundary before mixing)**.** *For $1/4 \leq D \leq 1/2$, set $z = (2 - 4D)^{1/4}$. Then*

$$F_{\mathrm{nc}}(D) = \min\left\{\frac{1}{4} + z^2 - z^3,\ \frac{5}{12} - \frac{z^6}{6}\right\}. \tag{6.2}$$

For $1/2 \leq D \leq 3/4$, set $z = (4D - 2)^{1/4}$ and

$$B(z) = \begin{cases} \frac{1}{4} + z^2 - \frac{2}{3}z^3, & 0 \leq z \leq \frac{1}{2}, \\ \frac{1}{12} + z - z^2 + \frac{2}{3}z^3, & \frac{1}{2} \leq z \leq 1. \end{cases}$$

Then

$$F_{\mathrm{nc}}(D) = \min\left\{B(z),\ \frac{5}{12} + \frac{z^4}{2} - \frac{z^6}{6}\right\}. \tag{6.3}$$

Each minimum is attained by one septuple on every such Z. The answer is the same for both orientations σ.

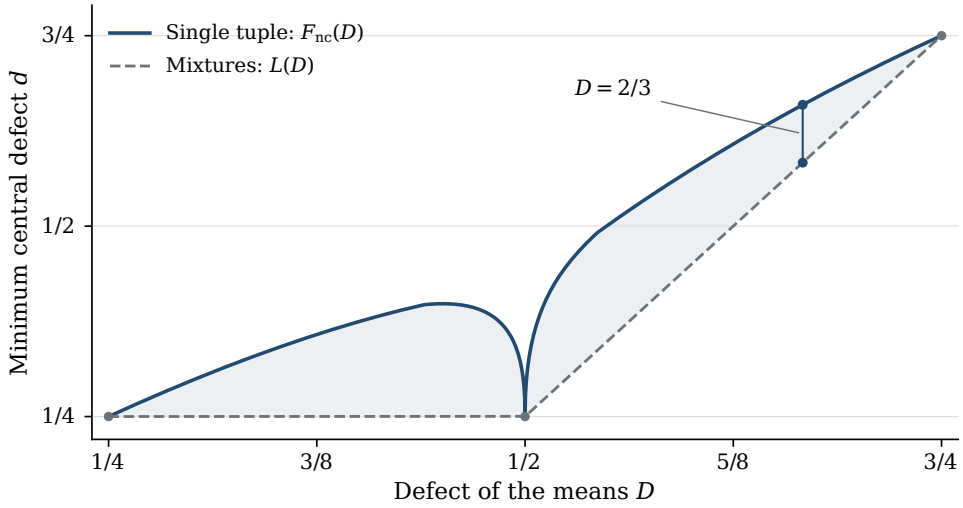

Figure 2: The boundary for one seven-tuple and its lower convex envelope, with both anchor means zero. The shaded region is the loss allowed by mixing tuples with different values of D. At $D = 2/3$, the endpoints are $7/12$ and $3/4 - \sqrt{6}/27$. Both curves are evaluated from the exact formulas.

6.1 Widening the intervals without changing their means

Take $\sigma = 1$ and retain U, W, p, q from (5.10). With $V = U_{0,\mu_3}$, the exact normal form is

$$4d \geq 2 - \mathbb{E}[UW + (U + W)V], \qquad 4D = 2 - pq. \tag{6.4}$$

The function V is odd, with $0 \leq V(t) \leq t$ for $0 \leq t \leq 1$. Since $U + W$ is increasing,

$$\mathbb{E}[(U + W)V] = \frac{1}{2}\int_0^1 (U(t) + W(t) - U(-t) - W(-t))V(t)\,dt \leq \mathbb{E}[(U + W)t].$$

Thus choosing $\mu_3 = 0$ can only improve the objective, and leaves D unchanged.

For $p \in [-1, 1]$, define

$$U_p(t) = \begin{cases} \max(t, 2\sqrt{p} - 1), & p \geq 0, \\ \min(t, 1 - 2\sqrt{-p}), & p \leq 0. \end{cases} \tag{6.5}$$

The two definitions agree at $p = 0$, and $\mathbb{E}U_p = p$. They are realized by the mean pairs $(\sqrt{p}, \sqrt{p})$ and $(\sqrt{-p}, -\sqrt{-p})$, respectively.

Lemma 6.2 (Widening at a fixed mean)**.** *If $U = U_{x,y}$, $xy = p$, and K is increasing, then $\mathbb{E}(U_pK) \geq \mathbb{E}(UK)$.*

Proof. The clipping interval of U_p contains that of U: for $p \geq 0$ use $|x + y| \geq 2\sqrt{p}$, and for $p \leq 0$ use $|x - y| \geq 2\sqrt{-p}$. If $[L, R]$ is the interval for U, then $U_p - U$ is nonpositive to the left of L, zero on $[L, R]$, and nonnegative to its right. Its mean is zero. For any $t_* \in [L, R]$,

$$\mathbb{E}[(U_p - U)K] = \mathbb{E}[(U_p - U)(K - K(t_*))] \geq 0.$$

This also covers a constant clipping interval. □

Apply Lemma 6.2 first to U with kernel $W + t$, and then to W with kernel $U_p + t$. Both means are separately preserved. Hence Theorem 6.1 is equivalent to maximizing

$$M(p, q) = \mathbb{E}[U_pU_q + (U_p + U_q)t], \qquad pq = 2 - 4D. \tag{6.6}$$

The exact attainment in (6.4) ensures that this is an equivalent optimization, rather than a relaxation.

6.2 Only two families can maximize the product problem

First suppose $k = 2 - 4D \geq 0$. The identity $U_{-p}(t) = -U_p(-t)$ allows us to take $p = x^2$, $q = y^2$, $0 \leq y \leq x \leq 1$. Put $r = \sqrt{k}$, so $xy = r$. Integration over the clipping intervals gives

$$M(x^2, y^2) = \frac{4x^3 - 6x^2 + 6xy^2 + 2y^3 - 6y^2 + 3}{3}.$$

For $0 < r < 1$ and $\sqrt{r} \leq x \leq 1$,

$$\frac{d}{dx} M(x^2, r^2/x^2) = \frac{2(x^2 - r)}{x^4} P_r(x), \qquad P_r(x) = r^2 + 2x(x-1)(r + x^2).$$

Writing $s = r/x^2 \in [0, 1]$ gives

$$P_r(x) = x^3\{x(s^2 + 2s + 2) - 2(s+1)\}, \qquad \frac{2(s+1)}{s^2 + 2s + 2} \geq \frac{4}{5}.$$

Thus $P_r(x) < 0$ for $x < 4/5$, and for $x \geq 4/5$,

$$P_r'(x) = 2x^2(4x - 3) + 2r(2x - 1) > 0.$$

The objective can therefore decrease and then increase, but has no interior strict maximum. Its maximum is at $x = y = \sqrt{r}$ or at $x = 1, y = r$. At $r = 0$, the objective is $1 - 2x^2 + 4x^3/3$, decreasing on $[0, 1]$; at $r = 1$ there is only one point. With $z = k^{1/4}$ the two candidate values are

$$M(z^2, z^2) = 1 - 4z^2 + 4z^3, \qquad M(1, z^4) = \frac{1}{3} + \frac{2}{3} z^6.$$

The identity $d = (2 - M)/4$ proves (6.2).

Now suppose $k \leq 0$. Write $p = x^2$, $q = -y^2$, $r = \sqrt{-k}$ and $w = x + y$. The exact range is $2\sqrt{r} \leq w \leq 1 + r$. Direct integration gives

$$M = \begin{cases} 1 - 2w^2 + \frac{4}{3} w^3 + 4r(1 - w), & w \leq 1, \\ \frac{5}{3} - 2w + \frac{2}{3} w^3 + 4r(1 - w), & w \geq 1. \end{cases}$$

The first branch has derivative $4(w^2 - w - r) \leq 0$; the second has second derivative $4w > 0$. Their common value at $w = 1$ is $1/3$. Consequently the maximum is again at an endpoint: either $x = y = \sqrt{r}$ or $\{x, y\} = \{1, r\}$. Substituting $z = (-k)^{1/4}$ gives (6.3), including $r = 0, 1$.

For explicit witnesses, take $\mu_1 = \mu_3 = \mu_5 = 0$ and use the following means for $(\mu_0, \mu_2, \mu_4, \mu_6)$:

	balanced pair products	one constant pair
$D \leq 1/2$	(z, z, z, z)	$(1, 1, z^2, z^2)$
$D \geq 1/2$	$(z, z, z, -z)$	$(1, 1, z^2, -z^2)$.

Choose the branch with smaller defect and use $F_i = h_{\mu_i} \circ (k_i \chi)$ with (5.5). These witnesses attain the exact seven-mean envelope simultaneously. This finishes the proof of Theorem 6.1.

For example, at $D = 2/3$, $z^4 = 2/3$. The second branch of (6.3) is smaller because the difference is

$$\frac{(z-1)^2}{6}(z^4 + 2z^3 + 2z - 2) > 0.$$

Thus

$$F_{\mathrm{nc}}(2/3) = \frac{3}{4} - \frac{\sqrt{6}}{27} = 0.659278\ldots, \qquad L(2/3) = \frac{7}{12}. \tag{6.7}$$

The latter value comes from weights $1/3, 2/3$ on the two vertices $(1/2, 1/4)$ and $(3/4, 3/4)$. It cannot be obtained by one septuple.

6.3 An optimal separation from the convex boundary

Theorem 6.3. *Let $V = \{1/4, 1/2, 3/4\}$ and $\eta_* = (4 - \sqrt{2})/3$. Then*

$$F_{\mathrm{nc}}(D) \geq L(D) + \eta_* \operatorname{dist}(D, V) \qquad (1/4 \leq D \leq 3/4). \tag{6.8}$$

The coefficient is optimal; apart from the three zero-distance vertices, equality holds at $D = 3/8$ and $D = 5/8$.

Proof. Put $z_0 = 2^{-1/4}$. In either parametrization of Theorem 6.1,

$$\operatorname{dist}(D, V) = \tfrac{1}{4} \min(z^4, 1 - z^4).$$

For the constant-pair branch the gap above L is $(1 - z^6)/6$. Its ratio to this distance is

$$\frac{2}{3}(z^{-4} - z^2) \quad (0 < z \leq z_0), \qquad \frac{2}{3}\frac{1 + z^2 + z^4}{1 + z^2} \quad (z_0 \leq z < 1).$$

The first decreases and the second increases. Their common minimum is η_*.

For the balanced branch on $D \leq 1/2$, the gap is $z^2(1 - z)$, and the ratio decreases up to z_0 and increases afterwards. Indeed, its two expressions are $4(1-z)/z^2$ and $4z^2/(1+z+z^2+z^3)$; the derivative of the second has the sign of $2 + z - z^3 > 0$. Its minimum is $\gamma_+ = 4\sqrt{2} - 4\,2^{1/4} > \eta_*$.

For the balanced branch on $D \geq 1/2$ and $z \leq 1/2$, the ratio is $4z^{-2} - \frac{8}{3}z^{-1} - 2$, decreasing to $26/3$. For $z \geq 1/2$ the gap is

$$\frac{1 - z}{6}(3z^3 - z^2 + 5z - 1).$$

The derivatives of its ratio, below and above z_0, are respectively

$$-\frac{4(2z^3 - 6z^2 + 9z - 2)}{3z^5}, \qquad \frac{4(2z^4 - 2z^3 + 3z^2 + 3)}{3(z + 1)^2(z^2 + 1)^2}.$$

They are negative and positive on the indicated intervals, so the minimum γ_- is again at z_0. To compare both balanced minima with η_*, put $t = 2^{1/4} > 19/16$. One obtains

$$3(\gamma_+ - \eta_*) = 13t^2 - 12t - 4 > 0, \qquad 3(\gamma_- - \eta_*) = 12t^3 - 11t^2 + 8t - 14 > 0.$$

Both polynomials increase for $t \geq 19/16$, and their values there are $21/256$ and $85/1024$. All branches therefore satisfy (6.8). At $z = z_0$ the constant-pair branch is strictly smaller than the balanced branch, and attains equality. Its two values of D are $3/8, 5/8$. □

Corollary 6.4 (Integration over actual common fibres)**.** *Suppose a probability space carries a measurable family of the septuples above, with both anchor means zero in almost every fibre. Write $m = \mathbb{E}D$ and $\delta = \mathbb{E}d$. Then*

$$\delta \geq L(m) + \eta_* \mathbb{E} \operatorname{dist}(D, V) + \mathbb{E}L(D) - L(m).$$

In particular, $\delta \leq L(m) + \varepsilon$ implies $\mathbb{E} \operatorname{dist}(D, V) \leq \varepsilon/\eta_*$.

Proof. Integrate (6.8) and use convexity of L. More explicitly,

$$\mathbb{E}L(D) - L(m) = 2\left(\mathbb{E}(D - \tfrac{1}{2})_+ - (m - \tfrac{1}{2})_+\right) \geq 0.$$

□

No independence assumption is needed in this corollary. Its hypothesis is instead a common family of actual jets, with both anchors centered inside each jet. Global centering alone does not imply it. If only m is prescribed, mixing the three vertices still makes $L(m)$ optimal; the extra gain records the distribution of local defects.

7 A continuous family beyond semicircle rigidity

The four-profile classification in Section 3 assumes that each profile is already a semicircle sign. We now exhibit a deformation that preserves the value $1/3$ and leaves that class. The construction takes place on a circle rotation, so it does not use central nilpotent fluctuations.

Let $g = 2\mathbf{1}_{[0,1/2)} - 1$ on $\mathbb{T}$. For a measurable $E \subset [-1/12, 1/12]$, let g_E be obtained from g by changing its sign on

$$B_E = (1/4 + E) \cup (3/4 + E).$$

The two copies are read modulo one. The identity $g_E(x + 1/2) = -g_E(x)$ shows that g_E is centred. Write

$$\ell(E) = |E|, \qquad M(E) = \int_E |s|\, ds, \qquad Q(f_0, f_1, f_2, f_3) = \int_{\mathbb{T}^2} \prod_{i=0}^{3} f_i(x + ih)\, dx\, dh.$$

Theorem 7.1 (Exact deformation identity)**.** *For measurable $E_1, E_2 \subset [-1/12, 1/12]$,*

$$Q(g, g_{E_1}, g_{E_2}, g) = \frac{1}{3} - 32\big(M(E_1) + M(E_2)\big) + 16\ell(E_1)\ell(E_2). \tag{7.1}$$

In particular, for $g_a = g_{[-a,a]}$ and $0 \le a, b \le 1/12$,

$$Q(g, g_a, g_b, g) = \frac{1}{3} - 32(a - b)^2. \tag{7.2}$$

For $0 < a < 1/12$, g_a is not of the form $S(kx + \theta)$ with $k \in \mathbb{Z} \setminus \{0\}$ and $\theta \in \mathbb{T}$.

Proof. Use the independent middle coordinates $u = x + h$ and $v = x + 2h$; the endpoint coordinates are $2u - v$ and $2v - u$. The coefficient of a change in the first middle profile is

$$H(u) = \int_{\mathbb{T}} g(2u - v)g(v)g(2v - u)\, dv.$$

For $u = 1/4 + s$, $0 < s < 1/12$, the successive breakpoints of the integrand in v are

$$0,\ 2s,\ u/2,\ u/2 + 1/4,\ 1/2,\ 2u,\ u/2 + 1/2,\ u/2 + 3/4,\ 1.$$

They occur in this order; the signs alternate, starting with $+1$. The alternating sum of the interval lengths is $8s$. Reflection about $1/4$ and translation by $1/2$ consequently give

$$g(u)H(u) = 8|s| \quad \text{for } u \in \{1/4 + s, 3/4 + s\}, \quad |s| \le 1/12.$$

Changing the first middle sign therefore contributes $-32M(E_1)$; interchanging u, v gives $-32M(E_2)$ for the second change.

In the cross term put $u = c_1 + s$ and $v = c_2 + t$, where $c_1, c_2 \in \{1/4, 3/4\}$. The endpoint centres are respectively c_2, c_1 modulo one. Their deviations, $2s - t$ and $2t - s$, have absolute value at most $1/4$. Hence the product of the four original signs is $+1$ almost everywhere on $B_{E_1} \times B_{E_2}$. The cross term is $4|B_{E_1}||B_{E_2}| = 16\ell(E_1)\ell(E_2)$. The starting value is $Q(g, g, g, g) = 1/3$, by Theorem 3.1. This proves (7.1).

For $E = [-a, a]$, $\ell(E) = 2a$ and $M(E) = a^2$, giving (7.2). The positive set of g_a has three components of lengths $1/4 - a$, $1/4 - a$, and $2a$. These are not all equal when $0 < a < 1/12$. Every preimage of a semicircle under a nonzero circle character has components of equal length. This excludes the claimed representation. At the endpoints, $g_0 = g$ and $g_{1/12} = g(3\,\cdot)$. □

Proposition 7.2 (Stability inside the deformation sector)**.** *Let* $\ell_i = |E_i|$ *and* $I_i = [-\ell_i/2, \ell_i/2]$. *Then*

$$\frac{1}{3} - Q(g, g_{E_1}, g_{E_2}, g) \geq 8(\ell_1 - \ell_2)^2 + 4\sum_{i=1}^{2} |E_i \triangle I_i|^2. \tag{7.3}$$

Equality $Q = 1/3$ *in this sector holds precisely when the two* E_i *are centred intervals of the same length, up to null sets.*

Proof. If $e = |E \setminus I| = |I \setminus E|$, moving mass e from inside the centred interval I to outside it increases $\int |s|$ by at least $e^2/2$. Indeed the least expensive move removes layers of width $e/2$ just inside each endpoint and adds layers of the same width just outside it. Thus

$$M(E) \geq \frac{\ell(E)^2}{4} + \frac{|E \triangle I|^2}{8}.$$

Substitution in (7.1) gives (7.3). Its equality consequence and the converse follow from (7.2). □

The construction has two uses. It supplies a continuous path of four-profile configurations with average 1/3, from frequency one to frequency three in the middle sites. It also specifies a limitation: the maximum over all centred real four-tuples is a different optimisation problem, and the value 1/3 by itself does not force the four profiles into the semicircle class.

8 Geometric consequences and the arithmetic interface

The preceding theorems keep separate three kinds of information: individual seminorms, horizontal projections, and constraints that hold in a common fibre. This section gives two consequences that preserve those distinctions. They also indicate which additional information is needed in an arithmetic application.

8.1 Simultaneous envelopes for seven semicircle projections

Corollary 8.1. *Consider seven real profiles on a totally ergodic system whose Kronecker projections have the form* rS_i, *with a common* $0 \leq r \leq 1$ *and with every* S_i *a semicircle sign. For either orientation* σ,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X D_\sigma((T^{in}F_i)_{i=0}^{6})\, d\mu \geq \frac{2 - 3M(r)}{4}. \tag{8.1}$$

The constant is attained in the class of seven freely chosen colours. At $r = 0$ *and* $r = 1$ *it is* $1/4$, *and at* $r = 1/2$ *it is* $5/12$.

Proof. The three quartic terms in D_σ are four-term progressions, of steps two, one and one. Apply Corollary 4.5 to each. Total ergodicity is preserved on replacing T by T^2; its Kronecker factor is the same, since eigenvalues do not merge under squaring. The pair term has average $(\int F_1)(\int F_5) = 0$ by ergodicity of T^4. This proves the bound.

For attainment, take one irrational circle rotation and the same semicircle sign S at all seven horizontal positions. Use the construction (4.8) with seven profiles, means $rS(x)$, and central multipliers $(1, 3, 3, 1, 3, 3, 1)$. The three horizontal quartics all equal $1/3$: the first follows by the Haar-preserving substitution $h \mapsto 2h$, and the last by translating x. The multipliers occur in the order $(1, 3, 3, 1)$ in all three quartics, so zero central phases attain their upper envelopes simultaneously. Reversing colours 0 and 5 gives the other orientation. The numerical values follow from (4.11). □

The word *freely* matters here. The construction chooses seven observables. It does not assert that an observable specified in advance by an arithmetic sequence has these projections.

8.2 Adjacent triples cannot saturate together

For $0 < s < 1/2$, the relaxed rearrangement inequality gives $3s^2/4$ as the maximum of a triple convolution of three functions in $[0,1]$ with mean s. Two overlapping three-term progressions cannot both attain this maximum. The strict separation is uniform when s ranges over any compact subinterval of $(0, 1/2)$.

Proposition 8.2 (Incompatible equalities on two adjacent triples)**.** *Let J be a compact subinterval of $(0,1/2)$. There is $\eta_J > 0$ such that, for every compact connected abelian group K, every $s \in J$, and every four measurable functions $g_i : K \to [0,1]$ of mean s,*

$$\frac{1}{2}\int_{K^2} (g_0(x)g_1(x+h)g_2(x+2h) + g_1(x+h)g_2(x+2h)g_3(x+3h))\,dx\,dh \le \frac{3}{4}s^2 - \eta_J. \quad (8.2)$$

The constant depends only on J, not on the group or the four profiles.

Proof. On the trivial group both triples equal s^3, which is separated uniformly from $3s^2/4$ for $s \in J$. We may therefore suppose that K is nontrivial.

Let $L_2 g$ denote the density of the pushforward of $g(x)\,dx$ under $x \mapsto 2x$. Connectedness makes this map Haar preserving; thus $0 \le L_2 g \le 1$ and its mean is s. Writing $w = x + h$, the two triples are

$$T_{012} = \langle g_0 * g_2, L_2 g_1 \rangle, \qquad T_{123} = \langle g_1 * g_3, L_2 g_2 \rangle.$$

Each is at most the corresponding circle-interval value $\int_{-s/2}^{s/2} (s - |u|)\,du = 3s^2/4$.

We first record the uniform Fourier separation used to compare their near-extremisers. For $0 < x < \pi$ and every integer $k \ge 2$,

$$\frac{|\sin(kx)|}{\sin x} = \left| \sum_{j=0}^{k-1} e^{i(k-1-2j)x} \right| < k.$$

The inequality is strict because the successive unit phases have quotient $e^{-2ix} \ne 1$. Consequently

$$\Delta_J := \inf_{\substack{s \in J \\ k \ge 2}} \left\{ \frac{\sin(\pi s)}{\pi} - \frac{|\sin(k\pi s)|}{\pi k} \right\} > 0. \quad (8.3)$$

Indeed, $\sin(\pi s)$ has a positive minimum on J, and the second term is at most $1/(\pi k)$. All sufficiently large k are therefore uniformly separated; for the finitely many remaining k, compactness and the strict inequality give a positive gap. Moreover,

$$\left| \frac{d}{ds} \frac{\sin(k\pi s)}{\pi k} \right| = |\cos(k\pi s)| \le 1,$$

uniformly in every nonzero integer k. Thus the gap survives changes $s + o(1)$ in the interval lengths, even if the harmonic order varies.

If (8.2) failed, a sequence would make both triple deficits tend to zero. The three equal masses $s \in J$ remain uniformly strictly admissible and bounded in the sense of [3, Theorem 1.6]. For the first triple, that theorem makes $g_0, g_2, L_2 g_1$ $o(1)$-close in L^1 to interval preimages under a common nonzero character χ, with lengths $s + o(1)$. For a binary target $\mathbf{1}_B$,

$$\|g_1 - \mathbf{1}_B(2\,\cdot)\|_1 = \|L_2 g_1 - \mathbf{1}_B\|_1.$$

It follows that the interval approximants for g_1, g_2 have characters $2\chi, \chi$, respectively. The second triple provides approximants for these same profiles with characters $\psi, 2\psi$.

Two interval preimages of lengths $s + o(1)$, $s \in J$, cannot be $o(1)$-close in L^1 unless their characters agree up to sign. To see this, let the first character be γ and the second ζ. At frequency γ, the first indicator has Fourier coefficient of modulus $\sin(\pi s)/\pi + o(1)$. The second coefficient is zero unless $\gamma = k\zeta$ for some integer k. If $|k| \geq 2$, its modulus is

$$\frac{|\sin(k\pi s)|}{\pi |k|} + o(1),$$

where the error is uniform in k by the derivative bound above. The positive gap (8.3), or the positive fundamental coefficient in the zero case, contradicts L^1 proximity. Hence $\gamma = \pm\zeta$.

Applied to the two shared profiles, this gives, for every sufficiently large member of the sequence,

$$2\chi = \pm\psi, \qquad \chi = \pm 2\psi.$$

These relations force $3\chi = 0$ or $5\chi = 0$. The dual of a connected compact abelian group is torsion-free, and $\chi \neq 0$, so both possibilities are excluded. This proves the uniform positive deficit. □

8.3 What an application must preserve

Suppose an arithmetic correlation is represented, after fixing its finite congruence data, by an average of totally ergodic components. The inequalities above can be applied component by component provided the component profiles retain their stated bounds and centring. For example, Lemma 4.2 is an inequality for the four actual horizontal means. It does not allow these means to be replaced by an unrelated quartet having the same separate distributions.

Likewise, the single-tuple boundary of Section 6 requires both anchors to be centred in each relevant central fibre. If a measurable family of such fibres is given, its stronger term is the average of $\operatorname{dist}(D, \{1/4, 1/2, 3/4\})$, as in Corollary 6.4. Replacing the data by $\mathbb{E}D$ alone recovers the convex boundary and can erase this term. The arithmetic representation and its centring conditions must therefore be proved where the geometric result is applied.

For the Liouville observable $\lambda(n) = (-1)^{\Omega(n)}$, the expression

$$\lambda(n)\lambda(n+1)\lambda(3n+1)\lambda(3n+2) = \prod_{i=0}^{3} \lambda(3n+i)$$

is indeed a four-term sign product. The equality follows from complete multiplicativity and $\lambda(3)^2 = 1$. It still singles out origins divisible by three. For this Liouville correlation, Pouly [11] proves the bound $196567331/402653184$ in absolute value on each fixed arithmetic progression along windows of diverging harmonic mass. In that arithmetic setting D_+ and D_- indicate the two majority events for three observed sign products. Their means are the corresponding probabilities. Here D is a polynomial of the seven means, and d is its expectation on the seven profiles in one quadratic Haar jet. To apply the bound $d \geq 1/4$ to the arithmetic defects, one would have to identify those profiles on a common jet and prove that both anchors are centred in each relevant fibre. Global centring allows conditional means of opposite signs to cancel. The realisation in Theorem 4.3 constructs an extension with an enlarged Kronecker factor and does not identify it with the specified arithmetic law.

Proposition 8.2 provides another reusable input: any reduction that produces the same four positive profiles in two adjacent triples inherits its strict deficit. The simultaneous presence of the shared profiles is the essential information; two separate maximisation problems would each still have value $3s^2/4$.

9 A finite torsion certificate and Boolean profiles

The bounds in this section concern a horizontal progression, rather than the three-parameter jet of Section 5. Let K be a compact connected abelian group and let $h_i : K \to [-1, 1]$, $0 \le i \le 6$, be real measurable functions, with $\int h_1 = \int h_5 = 0$. Define

$$d_{\mathrm{hor}} = \int_{K^2} D_+(h_i(x + it))\, dx\, dt, \qquad C_h(t) = \int_K h(x)h(x + t)\, dx.$$

The notation $D_+(h_i(x+it))$ denotes the seven arguments in their order $i = 0, \dots, 6$. We retain the same distinguished color 3 throughout this section.

9.1 The finite inequality and its transfer

Lemma 9.1 (Five-point certificate)**.** *Let $f_i : \mathbb{F}_5 \to \{-1, 1\}$ be seven sign tables. Let N be the number of pairs $(x, t) \in \mathbb{F}_5^2$ for which $D_+(f_i(x + it)) = 1$, and let n be the number for which $f_1(x + t)f_5(x) = -1$. Put $S = \sum_{j\in\mathbb{F}_5} f_3(j)$. Then*

$$6N + n + S^2 \ge 21. \tag{9.1}$$

Exact finite verification. An affine change of the five inputs, and simultaneous reversal of colors $1, 3, 5$, preserve the four monomials in D_+. The affine group of $\mathbb{F}_5$ is transitive on points and on unordered pairs. It therefore suffices to fix f_3 in three configurations, distinguished by its number of minority signs. The exact minima are

minority signs in f_3	0	1	2
S^2	25	9	1
$\min(6N + n)$	0	12	20.

For completeness, the following specifies the entire finite search. Encode a sign table by a five-bit integer, a set bit meaning -1. Pull it back in color i to the 25-bit word whose bit (x, t) is the bit at $x + it$ of that integer. For each pair of tables form the two families of words

$$P = f_0(x)f_2(x + 2t), \qquad Q = f_4(x + 4t)f_6(x + t),$$

where multiplication of signs means bitwise exclusive-or. Each family has exactly 512 distinct words: the two nonparallel coordinate forms are independent, and two pairs of sign tables give the same product exactly when both signs are reversed. Each word thus represents two of the 32^2 pairs.

For each of the 32^2 choices of f_1, f_5 and each of the 512^2 choices of P, Q, set, in bits,

$$B = P \oplus f_1 \oplus f_3, \qquad E = Q \oplus f_5 \oplus f_3, \qquad R = f_1 \oplus f_5.$$

The word of defects is

$$(B \,\&\, E) \mid ((\neg R) \,\&\, (B \mid E)),$$

with complementation restricted to the 25 bits. Its population count is N, and the population count of R is n. Minimization gives the three entries in the table. All operations are on integers, and all 2^{30} choices of the six free sign tables are covered, with their known multiplicity four. The accompanying code contains this verification and a second enumeration using sign gauges and the majority formula for the three quartic bits. The two enumerations give the same minimum at every possible n; their minimizing witnesses are also checked directly on the 25 points. The displayed table proves (9.1). □

The finite statement includes uncentered tables. Centering enters only after integration on the connected group.

Proposition 9.2. *If $\tau \in K$ has order exactly five, then*

$$d_{\mathrm{hor}} \geq \frac{7}{300} - \frac{C_{h_3}(\tau) + C_{h_3}(2\tau)}{15}. \tag{9.2}$$

Proof. For sign tables, write $d_5 = N/25$, $r_5 = 1 - 2n/25$ and $m_3 = S/5$. Lemma 9.1 is equivalent to

$$d_5 \geq \frac{17}{300} - \frac{m_3^2}{6} + \frac{r_5}{12}.$$

For real tables in $[-1, 1]$, replace the 35 entries by independent signs of their respective means. Multiaffinity preserves the expectations of d_5, r_5, whereas the squared central mean acquires the correction $25^{-1} \sum_j (1 - f_3(j)^2)$. Therefore

$$d_5 \geq \frac{17}{300} - \frac{m_3^2}{6} - \frac{1}{150} \sum_{j \in \mathbb{F}_5} (1 - f_3(j)^2) + \frac{r_5}{12}.$$

Apply this to $f_i(j) = h_i(x + it + j\tau)$ and integrate over x, t. Haar invariance makes the averaged d_5 equal to d_{hor}. The averaged pair term vanishes because $(x, t) \mapsto (x+t, x+5t)$ is surjective on K^2, its determinant being 4, and both anchors have mean zero. With $M_\tau h = 5^{-1} \sum_{j=0}^{4} h(\cdot + j\tau)$, the result is

$$d_{\mathrm{hor}} \geq \frac{7}{300} + \frac{\|h_3\|_2^2}{30} - \frac{\|M_\tau h_3\|_2^2}{6}.$$

The operator M_τ is an orthogonal projection, and

$$\|M_\tau h\|_2^2 = \frac{1}{5}(\|h\|_2^2 + 2C_h(\tau) + 2C_h(2\tau)).$$

Substitution gives (9.2). □

9.2 A dimension-independent class of profiles

Let $\Gamma = \widehat{K}$, let $r \geq 1$, and take characters $\chi_1, \ldots, \chi_r$ whose classes in $\Gamma/5\Gamma$ are linearly independent over $\mathbb{F}_5$. Equivalently,

$$(\chi_1, \ldots, \chi_r) : K[5] \longrightarrow \mathbb{T}[5]^r \quad \text{is surjective.} \tag{9.3}$$

These characters are also independent over $\mathbb{Z}$. Otherwise, removing the largest common power of five from an integer relation would give a nonzero relation modulo five, since Γ is torsion-free. Thus $(\chi_1, \ldots, \chi_r)$ pushes Haar measure onto Haar measure on $\mathbb{T}^r$.

Let $s(t) = 1$ on $[0, 1/2)$ and $s(t) = -1$ on $[1/2, 1)$, periodically extended. For a function $B : \{-1, 1\}^r \to [-1, 1]$ and fixed phases θ_j, set

$$H(x) = B(s(\chi_1(x) + \theta_1), \ldots, s(\chi_r(x) + \theta_r)), \qquad \mu = \mathbb{E}B, \quad v = \mathbb{E}B^2,$$

where the expectations are over independent uniform signs.

Theorem 9.3 (Boolean profile margins)**.** *If $h_3 = H$, then*

$$d_{\mathrm{hor}} \geq \max\left\{0, \frac{7}{300} - \frac{2v}{375} - \frac{16\mu^2}{125}\right\} \geq \max\left\{0, \frac{9 - 64\mu^2}{500}\right\}. \tag{9.4}$$

In particular, $\mathbb{E}B = 0$ implies $d_{\mathrm{hor}} \geq 9/500$. If $B(-z) = -B(z)$, then $d_{\mathrm{hor}} \geq 7/300$. All six other profiles are arbitrary subject to the two anchor centerings. The constants do not depend on r or on the complexity of B.

Proof. The four nonzero fifth-torsion autocorrelations of s are

$$C_s(1/5) = \frac{1}{5}, \quad C_s(2/5) = -\frac{3}{5}, \quad C_s(3/5) = -\frac{3}{5}, \quad C_s(4/5) = \frac{1}{5}.$$

Their average is $-1/5$, and multiplication by two permutes them. Choose $u_1, \dots, u_r$ independently and uniformly in $\mathbb{F}_5^*$. By (9.3), choose one element $\tau_u \in K[5]$ with $\chi_j(\tau_u) = u_j/5$ for each of these finitely many vectors. Every such element has order five.

Expand B in the Walsh basis,

$$B(z) = \sum_{S \subseteq [r]} b_S \prod_{j \in S} z_j, \qquad b_\varnothing = \mu, \qquad \sum_S b_S^2 = v.$$

In the product $H(x)H(x + \tau_u)$, different Walsh supports have zero integral because one coordinate occurs once and its half-circle sign is centered. Independence of the character coordinates gives

$$\mathbb{E}_u C_H(\tau_u) = \mathbb{E}_u C_H(2\tau_u) = \sum_S b_S^2 (-1/5)^{|S|}.$$

The right-hand side is the noise stability $\mathrm{Stab}_{-1/5}(B)$: it is $\mathbb{E}[B(\epsilon)B(\epsilon')]$ when the pairs $(\epsilon_j, \epsilon'_j)$ are independent across j, each sign is uniform, and $\mathbb{E}(\epsilon_j \epsilon'_j) = -1/5$. This Walsh representation is the standard spectral identity [10, Section 2.4, Theorem 2.49]. Averaging (9.2) over these translations yields

$$d_{\mathrm{hor}} \geq \frac{7}{300} - \frac{2}{15} \sum_S b_S^2 (-1/5)^{|S|}. \tag{9.5}$$

For every positive integer k, $(-1/5)^k \leq 1/25$. The sum is therefore at most $\mu^2 + (v - \mu^2)/25$, giving (9.4) since $v \leq 1$. If B is odd, only odd degrees remain, the sum is nonpositive, and the stronger margin follows. □

The bound $1/25$ on the centered Walsh subspace is attained by $B(z) = z_1 z_2$ when $r \geq 2$. Thus $9/500$ is optimal for the particular averaged certificate (9.5) using only centering and $|B| \leq 1$. This does not assert that $9/500$ is the minimum of the full horizontal defect.

The torsion assumption is substantive. The character $x \mapsto 5x$ on $\mathbb{T}$ is nontrivial but vanishes on $\mathbb{T}[5]$; the characters x and $2x$ on $\mathbb{T}$ do not provide independent coordinates. The five-adic solenoid is connected and has no nonzero element of order five. None of these cases satisfies (9.3). The theorem does apply to coordinate characters on tori of arbitrary dimension, or to integer character matrices of full row rank modulo five.

9.3 Perturbations and the role of the hypothesis

For fixed six other profiles,

$$|d_{\mathrm{hor}}(h_3) - d_{\mathrm{hor}}(H)| \leq \tfrac{1}{2} \|h_3 - H\|_1,$$

since color 3 appears in two quartics with coefficient $1/4$. Three applications of Cauchy–Schwarz to each four-term progression give also

$$|d_{\mathrm{hor}}(h_3) - d_{\mathrm{hor}}(H)| \leq \tfrac{1}{2} \|h_3 - H\|_{U^3(K)}.$$

Here $U^3(K)$ is the Haar cube norm on the compact abelian group. Thus distance at most η in either norm from a centered profile of Theorem 9.3 gives $d_{\mathrm{hor}} \geq \max(0, 9/500 - \eta/2)$. This is a conditional perturbation statement: a general structured profile supplied by a uniformity decomposition need not admit these independent fifth-torsion characters or this Boolean representation.

AI assistance and verification

The author used ChatGPT to explore the problems and GPT-6 Astra Pro for mathematical checking, literature searches, English exposition and preparation of the source files and exact verification programs. The author takes responsibility for the article. The public source archive includes the programs, their inputs and instructions for reproducing the finite calculations. The continuous results are proved in the text.

A The finite lattice checks in the semicircle theorem

This appendix specifies the finite part of Theorem 3.1. All variables below are integers or exact rational numbers. The program `verify_arcs.py`, distributed in the public source archive, implements these enumerations. The analytic reduction to the listed finite domains is part of Section 3; the program verifies those domains, not arbitrary frequency quadruples.

A.1 Six pairs that each admit unit coordinates

Suppose, for a contradiction, that all six pairs admit a relation unit on that pair but that no global unit relation exists. This last assumption is used when selecting a nonunit coordinate below.

Choose indices i, j such that $w_i \le w_j$ are the two smallest positive weights. Let R be an odd relation with $R_i = 1$ and $R_j = \pm 1$. For the two other sites s, t,

$$|R_s| w_s \le \frac{|t-i| w_i + |t-j| w_j}{|t-s|} \le 5 w_j,$$

so $|R_s| \le 5$. Since no global unit relation exists, some k satisfies $|R_k| \ge 3$, whence $w_k \le 5w_j/3$. A relation V that is unit on i, k has its other coordinates bounded by

$$|V_s| \le \frac{3w_i + 3w_k}{w_j} \le 8.$$

These coordinates are odd, so their absolute values are at most seven. The two relations are independent, because their i coordinates have absolute value one while their k coordinates have different absolute values.

Enumerate the twelve ordered choices (i, j), the two choices of R_j, and the remaining coordinates of R in $\{\pm 1, \pm 3, \pm 5\}$. Choose a site k with $|R_k| \ge 3$, and enumerate $V_i = 1$, $V_k = \pm 1$, with its other two coordinates in $\{\pm 1, \pm 3, \pm 5, \pm 7\}$. The four rows

$$R, \quad (iR_i)_{i=0}^{3}, \quad V, \quad (iV_i)_{i=0}^{3}$$

have rank at least three. Otherwise $V_i/R_i = A + Bi$ and iV_i/R_i would both be affine in i at four distinct values, forcing $B = 0$ and dependence of R, V. Signed 3×3 minors give the possible weights up to scale. Retain only positive weights satisfying all four equations and $w_i \le w_j \le \min(w_k, w_\ell)$. After primitive normalisation and reversal, the list consists solely of $(1, 1, 1, 1)$, which has the unit relation $(1, -1, -1, 1)$. This contradicts the assumption and proves the required implication.

A.2 Enumerating fixed lattice indices

Suppose $U_{k\ell} \subset U_{ij}$ and $[U_{ij} : U_{k\ell}] = I$. Write

$$u_k = a u_i + c u_j, \qquad u_\ell = b u_i + d u_j, \qquad D = ad - bc = \pm I.$$

The four slopes $0, 1, 2, 3$ force

$$ad = D\frac{(j-k)(\ell-i)}{(j-i)(\ell-k)}, \qquad bc = ad - D. \tag{A.1}$$

Both products are nonzero. If the first expression is not integral, discard the case. Otherwise enumerate every signed divisor pair of each product. With $w_i = 1$, the weights must be

$$w_k = \frac{a(j-i)}{j-k}, \qquad w_\ell = \frac{b(j-i)}{j-\ell}, \qquad w_j = \frac{w_k(k-i)}{(j-i)c}. \tag{A.2}$$

Check also $w_\ell(\ell-i) = d(j-i)w_j$, positivity, and primitive normalisation. These operations are reversible, so the enumeration is exhaustive.

For a pair of odd modes m, n at i, j, completion is possible precisely when both numbers

$$\frac{(\ell-i)mw_i + (\ell-j)nw_j}{(\ell-k)w_k}, \qquad \frac{(i-k)mw_i + (j-k)nw_j}{(\ell-k)w_\ell}$$

are odd integers; an irrelevant common minus sign does not affect this test. Subtracting only the definitely incompatible modes $-19 \le m, n \le 19$ gives the rigorous upper bound

$$P_{ij} \le U_{ij}^{(19)} := 1 - 16(7/22)^4 \sum_{\substack{m,n \text{ odd},\ |m|,|n|\le 19 \\ (m,n) \text{ incompatible}}} \frac{1}{m^2 n^2}, \tag{A.3}$$

because $\pi < 22/7$. The omitted modes cost nothing in this upper bound: they remain in the initial total mass one.

Index I	Primitive weights	Without unit relation	Certified product bound
3	22	20	49/1000
4	32	32	3/50
5	24	24	47/1000
7	24	24	4/125

The first two numerical columns identify weights up to reversal. Each entry in the last column bounds the worst configuration in its row and is strictly less than $1/16$. The same program enumerates the units in (3.16) for $N = 8, 10, \ldots, 22$; the only orders allowing probability greater than $1/16$ are $8, 10, 14$.

A.3 Reproduction and the scope of the computation

Run `python3 verification/verify_arcs.py` from the source directory. This uses only the Python standard library. The archived output records the exact rational maxima, not just decimal approximations. The companion `verify_identities.py` performs exact polygonal integrations for the phase formula and independent rational checks of the central envelope and the deformation family. The five-point enumeration used in Section 9 is documented separately with its two integer implementations.

The source archive provides the six exhaustive enumeration tables and the commands for replaying their witnesses or recomputing the tables. These checks certify the stated finite calculations; the connected-group arguments, passage to characteristic factors, and limiting arguments remain the written mathematical proofs given in the body of the article.

References

[1] Vitaly Bergelson, Bernard Host, and Bryna Kra. Multiple recurrence and nilsequences. *Inventiones Mathematicae*, 160(2):261–303, 2005. doi: 10.1007/s00222-004-0428-6. URL https://people.math.osu.edu/bergelson.1/BHK.pdf. With an appendix by Imre Ruzsa. See Theorem 4.1(4), (6), and Theorem 5.4.

[2] Pablo Candela and Balázs Szegedy. A continuous model for systems of complexity 2 on simple abelian groups. *Journal d'Analyse Mathématique*, 135(2):437–471, 2018. doi: 10.1007/s11854-018-0037-8. URL https://arxiv.org/abs/1509.04485v2. See Theorem 1.5, Example 1.6, and Theorem 2.1 of arXiv:1509.04485v2.

[3] Michael Christ and Marina Iliopoulou. Inequalities of Riesz–Sobolev type for compact connected Abelian groups. *American Journal of Mathematics*, 144(5):1367–1435, 2022. doi: 10.1353/ajm.2022.0032. URL https://arxiv.org/abs/1808.08368v2. Theorems 1.5 and 1.6 are cited in arXiv:1808.08368v2 (17 August 2019).

[4] Carl de Boor. Divided differences. *Surveys in Approximation Theory*, 1:46–69, 2005. URL https://pages.cs.wisc.edu/~deboor/sat/papers/2/2.pdf. See Section 9, equation (52), p. 64, for the Genocchi–Hermite formula.

[5] Shilin Dou, Ansel Goh, Kevin Liu, Madeline Legate, and Gavin Pettigrew. Cosine sign correlation. *Journal of Fourier Analysis and Applications*, 30:Article 13, 2024. doi: 10.1007/s00041-024-10067-1. URL https://arxiv.org/abs/2212.02496v1. Published 8 February 2024; preprint first submitted 5 December 2022. See the unnumbered main theorem in Section 1.2.

[6] Tanja Eisner and Terence Tao. Large values of the Gowers–Host–Kra seminorms. *Journal d'Analyse Mathématique*, 117(1):133–186, 2012. doi: 10.1007/s11854-012-0018-2. URL https://arxiv.org/abs/1012.3509v2. Theorem numbers 1.3 and 1.8 refer to arXiv:1012.3509v2 (26 June 2011).

[7] Nikos Frantzikinakis and Bernard Host. The logarithmic Sarnak conjecture for ergodic weights. *Annals of Mathematics*, 187(3):869–931, 2018. doi: 10.4007/annals.2018.187.3.6. URL https://annals.math.princeton.edu/wp-content/uploads/annals-v187-n3-p06-p.pdf. See Section 4.2.1, Proposition 4.7, and Appendix B.1–B.2, especially equation (45) and Lemma B.2.

[8] Bernard Host and Bryna Kra. Nonconventional ergodic averages and nilmanifolds. *Annals of Mathematics*, 161(1):397–488, 2005. doi: 10.4007/annals.2005.161.397. URL https://annals.math.princeton.edu/wp-content/uploads/annals-v161-n1-p08.pdf. See Lemma 4.3, Theorem 10.1, and Theorem 12.1.

[9] Martin Kneser. Summenmengen in lokalkompakten abelschen Gruppen. *Mathematische Zeitschrift*, 66:88–110, 1956. doi: 10.1007/BF01186598. URL https://link.springer.com/article/10.1007/BF01186598.

[10] Ryan O'Donnell. *Analysis of Boolean Functions*. Cambridge University Press, 2014. doi: 10.1017/CBO9781139814782. URL https://arxiv.org/abs/2105.10386v1. References to Section 2.4 and Theorem 2.49 use the corrected May 2021 arXiv edition.

[11] Brice Pouly. An explicit bound below one half for a quartic Liouville correlation. Zenodo, 2026. URL https://doi.org/10.5281/zenodo.24443184.

[12] Balázs Szegedy. Limits of functions on groups. *Transactions of the American Mathematical Society*, 370(11):8135–8153, 2018. doi: 10.1090/tran/7432. URL https://arxiv.org/abs/1502.07861v1. See Lemma 2.2, Proposition 2.2, and Theorems 2–4 of the arXiv version, first submitted 27 February 2015.

[13] Shengyang Zang, Yingyu Zhao, Yunhao Zhao, and Chen Zhou. Extremal cosine sign correlations. arXiv preprint arXiv:2610.33190v1, 2026. URL https://arxiv.org/abs/2610.33190v1. Submitted 23 October 2026. See Proposition 3.9, p. 12.