An explicit bound below one half for a quartic Liouville correlation

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Abstract

Let λ be the Liouville function. We prove that every limit point of the normalized harmonic averages of $\lambda(n)\lambda(n+1)\lambda(3n+1)\lambda(3n+2)$ has absolute value at most $196567331/402653184 < 1/2$. The estimate holds on every sequence of windows whose harmonic mass tends to infinity, and on each fixed arithmetic progression. The same bound applies to four equally spaced Liouville values with any fixed step and origin class. The proof retains the prescribed arithmetic law in a strongly stationary construction, reduces its four-term averages to real centred profiles, and combines a Fourier estimate for the vertical contribution with an exact integer certificate for the horizontal form. We deduce that each extracted harmonic law contains at least twenty-six sign words of length five and forty-four of length six. For sixteen specified five-letter words, we give explicit quantitative refinements of positivity obtainable from earlier correlation results. We also obtain an explicit entropy deficit forced by a nonzero quartic bias. A separate transfer gives the same numerical bound for ordinary averages outside a set of cutoffs of logarithmic density zero. For the displayed quartic, the dyadic identity and an ordinary two-point estimate give a bound strictly below $2/3$ on all ordinary prefixes. A complementary argument bounds the signed additive-parallelogram energy of the Möbius pair $\mu(n)\mu(n+1)$ by $5/9$ of its unsigned energy. No existence or vanishing of the quartic limit is assumed.

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Contents

1 Introduction **3**

2 An elementary identity and the information it loses **7**

3 Keeping the arithmetic progression in the ergodic reduction **9**

4 The horizontal form and its vertical error **11**

5 A finite certificate for the circle inequality **13**

6 Five consecutive signs **18**

7 Six-letter words and an exact finite certificate **21**

8 Twenty-eight words at common difference five **23**

9 The entropy cost of a quartic bias **26**

10 Ordinary averages **32**

11 Cancellation in additive parallelograms of the Möbius pair **35**

12 What the estimate leaves open **39**

Artificial-intelligence assistance and verification **39**

A Sources and reproducibility **39**

B Local inequalities and the meaning of the certificate tables **40**

C A fixed-side parallelogram estimate in a residue class **51**

D The three-copy affine classification **54**

E Multiplicative examples for the dyadic identity **56**

1 Introduction

The Liouville function $\lambda(n) = (-1)^{\Omega(n)}$ records the parity of the number of prime factors of n, counted with multiplicity. Its values are conjectured to behave, in each fixed additive configuration, like independent fair signs. For four signs, the first question beyond the known logarithmic binary and odd-order cancellations is whether their product has mean zero. This article proves an explicit restriction on that mean without determining it.

Our principal arithmetic observable is

$$C(n) = \lambda(n)\lambda(n+1)\lambda(3n+1)\lambda(3n+2). \tag{1.1}$$

Complete multiplicativity gives

$$C(n) = \lambda(3n)\lambda(3n+1)\lambda(3n+2)\lambda(3n+3).$$

Thus C is a consecutive four-point product whose origin is constrained to be a multiple of three. This origin condition matters. A bound for the average over all origins does not by itself give the same bound on a prescribed residue class.

For $I = (U, V]$, $1 \le U < V$, and a fixed class $a \pmod q$, define

$$H_{I;a,q} = \sum_{\substack{U<n\le V\\ n\equiv a\ (q)}} \frac{1}{n}, \qquad \mathbb{E}^{\log}_{I;a,q} f = \frac{1}{H_{I;a,q}} \sum_{\substack{U<n\le V\\ n\equiv a\ (q)}} \frac{f(n)}{n}. \tag{1.2}$$

The denominator is positive for all sufficiently large windows in the regimes below, and $H_{I;a,q} = q^{-1}\log(V/U) + O_q(1)$. We omit the residue subscripts when $q = 1$. All limits over windows are sequential: the assertions hold along every sequence $I_j = (U_j, V_j]$ with $U_j \ge 1$ and $\log(V_j/U_j) \to \infty$. Residues, moduli, shifts and finite collections of observables are fixed before this limit.

Theorem 1.1 (The quartic bound)**.** *Set*

$$B = \frac{196567331}{402653184} = 0.4881802474459012\ldots < \frac{1}{2}. \tag{1.3}$$

For every fixed $q \ge 1$ and $a \in \mathbb{Z}$, and every sequence of windows of diverging harmonic mass,

$$\limsup_{j\to\infty} \left|\mathbb{E}^{\log}_{I_j;a,q} C\right| \le B. \tag{1.4}$$

More generally, for every fixed $h \ge 1$,

$$\limsup_{j\to\infty} \left|\mathbb{E}^{\log}_{I_j;a,q} \prod_{i=0}^{3} \lambda(n+ih)\right| \le B. \tag{1.5}$$

The theorem bounds all subsequential limits. It does not assert that the averages converge. The constant is independent of the fixed parameters, but the proof supplies no rate uniform when these parameters grow with the window. The more general statement in (1.5) is proved by the same four-profile estimate; it is not inferred by identifying a conditioned average with an unconditioned one.

Corollary 1.2 (Both signs and four-letter words)**.** *For $\varepsilon \in \{-1, 1\}$,*

$$\liminf_j \mathbb{E}^{\log}_{I_j;a,q} \mathbf{1}_{\{C=\varepsilon\}} \ge \frac{1-B}{2} > 0.2559098762.$$

For every fixed $h \geq 1$ and each $s \in \{-1,1\}^4$,

$$\liminf_j \mathbb{E}^{\log}_{I_j;a,q} \mathbf{1}_{\{\lambda(n+ih)=s_i,\ 0\leq i\leq 3\}} \geq \frac{1-B}{16} > 0.0319887345.$$

The latter conclusion also holds for the canonical four values in (1.1)*, with their prescribed ordering.*

Proof. For a sign-valued random variable, the mass of sign ε is $(1+\varepsilon\mathbb{E}C)/2$. For the four-letter assertion, expand the indicator as $2^{-4}\prod_i(1+s_i\lambda(n+ih))$. All proper nonempty moments vanish: the binary moments by Tao's theorem [26], and the odd moments by Tao–Teräväinen [30, Corollary 1.8 and Appendix A]. These inputs apply after the fixed progression substitution. The only remaining nonconstant moment is the quartic, bounded by Theorem 1.1. The four affine forms in (1.1) are pairwise nonproportional, so the same proper-moment argument applies to their signs. □

1.1 Consequences for words and entropy

A *harmonic law of Liouville* is a subsequential limit of all finite blocks $(\lambda(n+j))_{j\in\mathbb{Z}}$ under $\mathbb{E}^{\log}_{n\in I}$ on such windows. It is a shift-invariant probability measure on $\{-1,1\}^{\mathbb{Z}}$. For a word $w \in \{-1,1\}^k$, let $\nu[w]$ be its cylinder probability. At a fixed step d, replace the coordinates by $\lambda(n+dj)$. Denote the mth largest word probability, counting individual words rather than symmetry classes, by $p_{k,d,(m)}$.

Theorem 1.3 (Short sign words)**.** *Every harmonic law of Liouville satisfies the following bounds:*

Configuration	*Number of words*	*Lower bound for each selected word*
Five consecutive signs	26	$\frac{4759261}{6442450944}$
Six consecutive signs	44	$\frac{8707}{2064851523584}$
Five signs at step five	28	$\frac{1}{891000}$

Equivalently, these are lower bounds for $p_{5,1,(26)}$, $p_{6,1,(44)}$ and $p_{5,5,(28)}$. The selected words may depend on the law. Consequently at least that many words have positive upper natural density in the corresponding configurations.

For the sixteen specified five-letter words with $w_0 \neq w_4$, the following quantitative refinement holds. Each word has positive lower natural density. On every harmonic law, its probability is at least $1/96$ if also $w_1 \neq w_3$, and at least $2167/1246272$ otherwise.

The distinction between selected and specified words is essential. The theorem does not provide one fixed list of twenty-six or forty-four words with a common lower logarithmic density. The sixteen-word assertion identifies its words independently of the extraction. Its qualitative positivity follows from earlier correlation results; the explicit harmonic minorants are quantified below. The twenty-six-word count is analytic. The forty-four-word and step-five counts use finite rational witnesses. The step-five witnesses already force twenty-eight words with the earlier quartic bound $1/2$; the new B improves their explicit minorant. The validation criteria and these dependencies are given in the corresponding sections.

For entropy, retain a single extraction on which the required joint laws and the quartic means exist. Let ν denote the unconditioned sign process on the tripled windows $3I_j$, and let

$$h_{\mathrm{bit}}(\nu) = \lim_{k\to\infty} \frac{1}{k} H_{\mathrm{bit}}(X_0,\dots,X_{k-1}) \in [0,1]$$

be its entropy rate, in bits per sign. The limit exists by subadditivity.

Theorem 1.4 (Entropy required by a quartic bias)**.** *For every fixed residue class on the same extraction, its quartic limit $c_{a,q}$ satisfies*

$$\frac{439}{250}|c_{a,q}| \le 1 - h_{\mathrm{bit}}(\nu). \tag{1.6}$$

In particular, $h_{\mathrm{bit}}(\nu) > 311/750$ implies $|c_{a,q}| < 1/3$ for every fixed class retained in that extraction.

This is a necessary entropy deficit for a nonzero bias. It does not establish a positive entropy lower bound for Liouville. Section 10 also transfers Theorem 1.1 to ordinary averages outside a set of cutoffs of logarithmic density zero. The argument gives no bound of this strength on the exceptional scales. For the special quartic $C(n)$ on all positive integers n, the dyadic identity and the ordinary binary theorem give a strict saving below $2/3$ at all cutoffs (Theorem 10.5). For four equally spaced signs with unrestricted origins, Proposition 10.6 gives the bound $1/2$ at all cutoffs.

1.2 Earlier results and the contribution here

Tao and Teräväinen proved the bound $1/2$ for the unconditioned consecutive logarithmic four-point correlation [30, Proposition 7.1]. They credit the translate–product argument to Will Sawin and independently Kaisa Matomäki. A related argument appears in Klurman–Mangerel [13, Lemma 5.3], where the relevant threshold for a unimodular multiplicative function is $1/\sqrt{2}$. Tao and Teräväinen's Corollary 7.2 [30] already gives positive lower natural density for all sixteen words of length four. We therefore use the numerical improvement in Corollary 1.2, not the existence of these words, as the new conclusion at length four.

At length five, Tao–Teräväinen [28, Theorem 1.14 and its proof, pp. 51–53] obtain twenty-four words with positive upper natural density and twelve specified words with positive lower natural density. The other four words with opposite endpoints also follow from the earlier moment identities, the bound $1/2$, and the run-exit mechanism of [17, Corollary 2.8]; the deduction is included in Proposition 6.2. Thus the assertion for those sixteen words is a quantitative refinement of positivity obtainable from earlier results. Its contribution here is the explicit harmonic minorants, including the weighted run estimate. The twenty-six-word conclusion has a different dependency: the boundary value $1/2$ in the earlier constraints is ruled out by the new quartic estimate. The selected twenty-six words may depend on the extracted law.

Qualitative cancellation for the exact polynomial $n(n+1)(3n+1)(3n+2)$ predates the present estimate. Teräväinen proves that both signs occur with positive lower natural density for a polynomial with positive leading coefficient that splits over $\mathbb{Q}$ and is not a constant times a square [32, Corollary 2.1]. His Theorem 2.6 gives a positive configuration-dependent saving in ordinary averages, also after a fixed progression substitution. This is a conclusion at every ordinary cutoff; the explicit constant here has a different strength and is proved first for harmonic windows. Its transfer to almost all ordinary scales uses the isotopy theorem of [29, Theorem 1.19(i)].

Teräväinen–Walker [33, Theorem 1.8] also give a qualitative saving from the trivial bound for correlations of real 1-bounded multiplicative functions, with at least one nonpretentious factor, along pairwise nonproportional affine forms on convex Bohr sets. Taking the Bohr set to be all integers covers the four forms in (1.1). Their statement concerns logarithmic averages up to a cutoff and gives a configuration-dependent saving; it does not supply the explicit constant below $1/2$ or, by its statement alone, the terminal-window assertion used here.

Recent developments in the binary problem include logarithmic power savings of Pilatte [21] and quantitative correlation results of Tao–Teräväinen [31]. Klurman–Mangerel–Teräväinen [14] prove ordinary binary cancellation along a set of cutoffs of full upper logarithmic density; this is distinct from the exceptional-set formulation used in Section 10. The ordinary two-point estimate of [20, Theorem 1.1] supplies the binary input for the all-cutoff results in that section. The proof of the harmonic bound B uses Tao's logarithmic two-point theorem [26].

The elementary bound in Section 2 concerns the same arithmetic observable C as Theorem 1.1. Multiplying its values at $n, 2n, 2n+1$ leaves just two Liouville factors. Binary cancellation then gives $|c| \le 2/3$ for each unconditioned harmonic limit c of C. The sign laws at equality explain where this argument loses information. They also give a way to read the stronger bound: on every fixed origin class, both possible majorities in the dyadic triple have positive mass bounded explicitly in terms of B. The published $1/2$ comparison above concerns unrestricted origins. Theorem 1.1 gives B for both unrestricted and prescribed origins and for every fixed common difference. Its proof uses the geometry of four real centred profiles, their arithmetic transfer and the certificate described below.

The relation to the author's public manuscript on arithmetic clusters is through its deterministic interface [23, Section 10]. There, indicator expansions identify signed packets which must be estimated for a deterministic sequence. The present article controls one fixed even-order observable. The Poisson results for random completely multiplicative sources do not supply this deterministic estimate, and the quartic estimate alone does not provide the growing families of packet bounds required for a Poisson law.

Word complexity provides a separate comparison. If $s(k)$ denotes the number of length-k words attained by Liouville, Frantzikinakis–Host [5] proved superlinear growth, McNamara [19] obtained a quadratic lower bound, and Matomäki–Radziwiłł–Tao–Teräväinen–Ziegler [18, Theorem 1.9] proved $s(k) \gg_A k^A$ for every $A \ge 1$. These occurrence counts do not give a positive entropy lower bound for each harmonic limit law, or the entropy hypothesis in Theorem 1.4.

1.3 How the estimate is obtained

There are two distinct reductions. First, the arithmetic law on a fixed progression is realized on every affine line of a strongly stationary sign process. Pairwise cancellation centres the four observables on almost every ergodic component. The components are totally ergodic, so the characteristic factor of the four-term average is an inverse limit of two-step nilsystems.

Second, on a connected nilsystem the zero vertical frequency gives the horizontal form

$$\mathcal{Q}(f_0, f_1, f_2, f_3) = \int_{\mathbb{T}^2} f_0(2u - v) f_1(u) f_2(v) f_3(2v - u)\, du\, dv, \qquad P = \prod_i \|f_i\|_2,$$

with four real centred functions bounded by one. The nonzero vertical frequencies contribute at most $(1 - \sqrt{P})^2/3$. The coefficient $1/3$ comes from a fourth Fourier moment estimate based on the connected-group Riesz–Sobolev inequality of Christ and Iliopoulou [3].

Two related sharp comparisons concern more restricted problems. For one centred real observable F bounded by one in a totally ergodic system, $\|F\|_{U^3}^8 \le 1/3$. For four semicircle signs pulled back by nontrivial characters of a compact connected abelian group, the horizontal four-term form has absolute value at most $1/3$ [24, Theorems 2.1 and 3.1]. Here the four profiles can be distinct real functions, and the vertical contribution must also be controlled.

The finite computation proves the single inequality

$$|\mathcal{Q}(f_0, f_1, f_2, f_3)| + \frac{1}{3}(1 - \sqrt{P})^2 \le B. \tag{1.7}$$

It alternately projects two profiles onto their Fourier frequencies divisible by three. Their losses of norm, the signed patterns on nine translations, and the norms of the other pair constrain the whole sequence of projections. Centring forces the remaining norms to tend to zero. The integer recurrence covers every allowed sequence with outward bounds; it does not sample profiles or infer a universal inequality from a numerical search. Equation (1.7), integrated over the ergodic components, is the step that turns the finite certificate into Theorem 1.1.

The proofs are arranged in that order. The twenty-six- and forty-four-word conclusions use the strict improvement below $1/2$. The quantitative sixteen-word argument uses earlier moment identities, and the step-five witnesses are presented first with the earlier bound and then with B. Entropy uses additional estimates for the same horizontal profiles. The final application to Möbius pairs uses a connected-fibre energy estimate of its own, which is distinguished from a numerical substitution into (1.7).

The appendices give the computational record and the local inequalities, complete the fixed-side parallelogram estimate, and classify the three-copy affine identities underlying the dyadic reduction. Appendix E examines two classes of multiplicative examples to determine which features of an endpoint law survive under additional arithmetic assumptions.

2 An elementary identity and the information it loses

Put $Q(T) = T(T+1)(3T+1)(3T+2)$, so that $C(n) = \lambda(Q(n))$. Direct multiplication gives

$$\begin{aligned} Q(n)Q(2n)Q(2n+1) = 16\big[n(n+1)(2n+1)(3n+1)(3n+2)\big]^2 \\ \cdot (6n+1)(6n+5). \end{aligned} \tag{2.1}$$

The square has Liouville value one. Hence

$$C(n)C(2n)C(2n+1) = \lambda(6n+1)\lambda(6n+5) =: R(n). \tag{2.2}$$

The mean of the right-hand side is zero on every admissible harmonic window by the logarithmic two-point theorem [26].

For three signs x, y, z, the elementary identity

$$x + y + z = 2 + xyz - 4\mathbf{1}_{\{\text{at least two of } x,y,z \text{ are negative}\}} \tag{2.3}$$

identifies the slack in $x + y + z \le 2 + xyz$, one of the parity inequalities on the sign vectors (x, y, z, xyz); see Jeroslow [10] for the general hypercube setting. For every bounded sequence f and every such window,

$$\mathbb{E}^{\log}_{n\in I}\{f(2n) + f(2n+1)\} = 2\mathbb{E}^{\log}_{n\in I} f(n) + o(1).$$

Indeed, separate the even and odd integers in $2I$; replacing the odd weight $1/(2n+1)$ by $1/(2n)$ has bounded total error. The symmetric difference of I and $2I$ has bounded harmonic mass. Both errors vanish after division by $\log(V/U)$.

Let $D_-(n)$ indicate that at least two of $C(n), C(2n), C(2n+1)$ are negative. Averaging (2.3) and using (2.2) yields

$$\mathbb{E}^{\log}_I C = \frac{2}{3} + \frac{1}{3}\mathbb{E}^{\log}_I R - \frac{4}{3}\mathbb{E}^{\log}_I D_- + o(1). \tag{2.4}$$

Writing $D_+ = 1 - D_-$ gives the lower-endpoint identity

$$\mathbb{E}^{\log}_I C = -\frac{2}{3} + \frac{1}{3}\mathbb{E}^{\log}_I R + \frac{4}{3}\mathbb{E}^{\log}_I D_+ + o(1).$$

Thus $|\mathbb{E}_I^{\log} C| \le 2/3 + o(1)$ and either sign of C has harmonic mass at least $1/6 - o(1)$. An earlier three-copy argument for a different quartic appears in Tao and Teräväinen [28, Lemma 7.3], where the copies are translates. A dyadic argument yielding $1/6$ for adjacent Liouville sign patterns appears in Matomäki et al. [17, Theorem 2.7].

Proposition 2.1 (Oriented laws at the two endpoints)**.** *Let (X_0, X_L, X_R) be three signs satisfying*

$$\mathbb{E}(X_0 X_L X_R) = 0, \qquad \mathbb{E}(X_L + X_R) = 2\mathbb{E}X_0.$$

Write $m = \mathbb{E}X_0$. For $\sigma = \pm 1$, the probability that at least two coordinates equal $-\sigma$ is $(2 - 3\sigma m)/4$. Consequently $|m| \le 2/3$, and $m = 2\sigma/3$ precisely when the law of $(\sigma X_0, \sigma X_L, \sigma X_R)$ assigns the following masses:

state	$+++$	$-++$	$+-+$	$++-$
mass	$1/2$	$1/6$	θ	$1/3 - \theta$

$0 \le \theta \le 1/3.$

All other states then have mass zero.

Proof. Apply (2.3) to $(\sigma X_0, \sigma X_L, \sigma X_R)$. At equality only $+++$ and the three single-minus states occur. Write their masses as p, q_0, q_L, q_R in the order shown. The product condition gives $p = q_0 + q_L + q_R = 1/2$, and the marginal condition gives $q_L + q_R = 2q_0$. Conversely every row of the table satisfies both conditions and has the indicated root mean. □

Every subsequential harmonic law of $(C(n), C(2n), C(2n+1))$ on unrestricted windows satisfies the two hypotheses. They determine only the sum of the child means. Averaging a law with its image under exchange of the children selects $\theta = 1/6$ at equality.

The symmetrized endpoint law extends to a process on the entire binary tree. Use a bifurcating Markov chain in the sense of Guyon [7], with negative root probability $1/6$. A positive parent has children $++, -+, +-$ with probabilities $3/5, 1/5, 1/5$; a negative parent has children $++$. Child pairs are generated independently at distinct vertices conditional on the parents. Each child has negative probability $1/6$, so every descendant subtree has the same law. Its local triple law is

$$\frac{1}{2}\delta_{+++} + \frac{1}{6}(\delta_{-++} + \delta_{+-+} + \delta_{++-}).$$

Its triple product is centred and each coordinate has mean $2/3$. Reversing every sign gives the lower endpoint. Hence the bound is sharp among descendant-stationary tree processes with centred triple product. Appendix E also realizes the upper endpoint by mixing two completely multiplicative sources, and computes the residual mean for sources supported on a fixed finite set of sign changes at primes. For Liouville itself, the arithmetic structure used in Theorem 1.1 imposes a positive majority defect on both sides.

There is also a useful seven-coordinate formulation. For signs $Z_0, \dots, Z_6$, define

$$T_0 = Z_0 Z_2 Z_4 Z_6, \qquad T_1 = Z_0 Z_1 Z_2 Z_3, \qquad T_2 = Z_3 Z_4 Z_5 Z_6.$$

Their product is $Z_1 Z_5$. Whenever the pair has zero mean and all three quartics have mean at most B, (2.3) gives

$$\mathbb{P}\{\text{at least two of } T_0, T_1, T_2 \text{ are negative}\} \ge \frac{2 - 3B}{4} = \frac{71868125}{536870912}. \tag{2.5}$$

The corresponding lower-majority event has the same bound if each mean is at least $-B$. In the affine-line lift of Lemma 3.1 for the seven offsets $0, \dots, 6$, the three quartics are covered separately by the geometric estimate below: their offsets are arithmetic progressions with steps one or two. The preservation of the arithmetic laws on all affine lines is part of this assertion. Formula (2.5) is the resulting bound for the diagonal defect in that class. It is obtained from the three quartic estimates, rather than by inverting the earlier inequality involving that defect.

Corollary 2.2 (Majority defect on a fixed origin class)**.** *For each fixed $q \geq 1$, $a \in \mathbb{Z}$, and either choice of sign,*

$$\liminf_{\log(V/U)\to\infty} \mathbb{E}^{\log}_{I;a,q} D_{\pm} \;\geq\; \frac{2-3B}{4}, \qquad I=(U,V].$$

Proof. Under $n \equiv a \pmod q$, each of the three means of $C(n), C(2n), C(2n+1)$ is bounded in absolute value by $B+o(1)$. For a child $j=0,1$, set $m=2n+j$. The image has $m \equiv 2a+j \pmod{2q}$ on $(2U+j, 2V+j]$. Replacing $1/n$ by $2/m$ changes the numerator and denominator by $O(1)$, since the difference of the weights is $O(n^{-2})$. The harmonic mass on the fixed class diverges, so Theorem 1.1 applies to the reindexed mean. The residual R has mean $o(1)$ by the two-point theorem on this class: after writing $n=qk+a$ with $0 \leq a < q$, the two forms are $6qk+6a+1$ and $6qk+6a+5$, with determinant $24q$. The possible term $k=0$ and the change from $1/(qk+a)$ to $1/(qk)$ for $k \geq 1$ contribute $O_q(1)$ before normalization. Now average (2.3) for the triple and for its negative. The argument uses a bound on each of the three means separately. □

3 Keeping the arithmetic progression in the ergodic reduction

The geometric estimate used below is uniform in four real functions, but its application to Liouville requires more than global mean zero. The functions must remain centred on the ergodic components on which the estimate is applied. We give a construction which retains the prescribed residue class throughout and obtains this centring from the two-point theorem.

For an interval $I=(U,V]$ write

$$\mathbb{E}^{\log}_{n\in I} a(n) = \frac{\sum_{U<n\leq V} a(n)/n}{\sum_{U<n\leq V} 1/n}.$$

All intervals in this section have $U \geq 1$ and $\log(V/U) \to \infty$. A fixed translation of the summation variable changes a bounded logarithmic average by $o(1)$. Replacing $1/n$ by $Q/(Qn+r)$ has the same effect when Q, r are fixed and positive arguments are retained. These assertions follow by summing the difference of the weights and discarding finitely many initial terms. Consequently a logarithmic law restricted to $m \equiv r \pmod Q$ can be written as a law of the values $\lambda(Qn+r+i)$ with the usual weight $1/n$.

Lemma 3.1 (An arithmetic law on every affine line)**.** *Fix $Q \geq 1$, $r \in \mathbb{Z}$, and a finite set $U \subset \mathbb{Z}$ of distinct offsets. Let ρ be an extracted logarithmic law of $(\lambda(Qn+r+i))_{i\in U}$. There is a probability law of signs*

$$(X_i(t))_{i\in U,\ t\in\mathbb{Z}}$$

with the following properties:

1. *its cylinder laws are invariant under $(i,t) \mapsto (i, dt+bi+c)$ for every $d \geq 1$ and $b, c \in \mathbb{Z}$;*
2. *for all $s, j \in \mathbb{Z}$, the vector $(X_i(si+j))_{i\in U}$ has law ρ;*
3. *distinct coordinates are orthogonal: $\mathbb{E}X_i(t)X_{i'}(t') = 0$ whenever $(i,t) \neq (i',t')$.*

Proof. Pass to a common subsequence on which every fixed translated correlation used below exists. This is possible because the collection is countable. For $S \subset U$, put

$$f_S(D) = \lim_j \mathbb{E}^{\log}_{n\in I_j} \prod_{i\in S} \lambda(Qn + D(r+i)), \qquad D \geq 1.$$

Tao–Teräväinen's structural theorem with affine coefficients [30, Theorem A.1(i)] says that each f_S is a uniform limit of periodic functions. Hence, for every $\varepsilon > 0$, there is a common period T, divisible by Q and larger than the diameter of U, such that

$$|f_S(D) - f_S(1)| \le 2\varepsilon \quad (D \ge 1,\ D \equiv 1 \pmod T,\ S \subset U). \tag{3.1}$$

Only uniform periodic approximation is used here; no uniformity in a growing modulus is asserted.

For fixed positive integers v, w, extract the logarithmic law of

$$X_i^{v,w}(t) = \lambda(Qn + r + i + T(wi + vt)). \tag{3.2}$$

It is stationary in t, because Tv is divisible by Q. Any two distinct coordinates in (3.2) are evaluations at distinct affine forms: different colours remain distinct modulo T, and for the same colour the coefficient Tv is nonzero. Their correlation vanishes by the logarithmic two-point theorem [26, Theorem 1.3].

Let $p_1, p_2, \dots$ be the primes and define

$$V_N = \left\{ \prod_{k \le N} p_k^{e_k} : 0 \le e_k < N \right\}, \qquad H_N = N^2 \max V_N.$$

Average the laws (3.2) uniformly over $v \in V_N$ and $H_N < w \le 2H_N$. For fixed $d \ge 1$, the uniform distributions on V_N and dV_N differ in total variation by $o_d(1)$: after all prime factors of d have appeared, only a fixed number of boundary layers of the exponent boxes are lost. The shear $t \mapsto t + bi$ replaces w by $w + bv$ and changes its uniform distribution by at most $|b| \max V_N / H_N = |b|/N^2$. Time translation already preserves each law. A subsequential limit as $N \to \infty$ therefore has all the invariances in (i).

On a fixed line $t = si + j$, the argument in (3.2) is

$$Qn + r + Di + Tvj, \qquad D = 1 + T(w + sv).$$

For large N, $D > 0$ and $D \equiv 1 \pmod T$. The difference $Tvj - (D-1)r$ is divisible by Q. A fixed translation of n therefore identifies each moment on this line with $f_S(D)$, up to an error which vanishes in the arithmetic limit. Equation (3.1) places it within 2ε of the corresponding moment of ρ. Here the arithmetic limits were taken first, with v, w fixed. Finally let $\varepsilon \downarrow 0$ and take another limit of cylinder laws. Moments determine laws on a finite sign alphabet, so (ii) follows. Orthogonality and all the invariances are closed cylinder conditions and survive both limiting operations. □

The marginal process $t \mapsto (X_i(t))_{i \in U}$ is strongly stationary in the usual sense: all its finite-dimensional laws are unchanged when every time is multiplied by the same positive integer. The additional shear invariance in Lemma 3.1 records why the required arithmetic law is retained.

Proposition 3.2 (Reduction to totally ergodic systems)**.** *Let $h \ge 1$ and let a be any extracted logarithmic mean of*

$$\prod_{i=0}^{3} \lambda(Qn + r + ih),$$

where Q, r, h are fixed. Suppose that a constant B bounds

$$\left| \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \int F_0 \, (F_1 \circ S^k) \, (F_2 \circ S^{2k})(F_3 \circ S^{3k}) \, d\mu \right| \le B \tag{3.3}$$

for every totally ergodic probability-preserving system and every four real F_i satisfying $|F_i| \le 1$ and $\int F_i \, d\mu = 0$. Then $|a| \le B$.

Proof. Apply Lemma 3.1 with offsets $0, h, 2h, 3h$. Write S for the time shift and $F_i = X_{ih}(0)$. Property (ii) gives $\int \prod_i F_i \circ S^{ihk}\, d\mu = a$ for every positive integer k. By [4, Theorem 6.6], almost every ergodic component of a strongly stationary system is totally ergodic. The global pair orthogonality yields

$$\left\| \frac{1}{N} \sum_{k=1}^{N} F_i \circ S^k \right\|_2^2 = \frac{1}{N}.$$

The mean ergodic theorem implies $\mathbb{E}(F_i \mid \mathcal{I}_S) = 0$. Thus F_i has mean zero on almost every ergodic component, simultaneously for the four colours. This is the componentwise centring needed in (3.3). On each such component S^h is totally ergodic. The multiple averages converge there by the Host–Kra theorem. Integrating their limits, using bounded convergence and the identity before the decomposition, gives a as an average of quantities bounded in modulus by B. □

In particular this proposition applies both to an unrestricted origin and to any fixed residue class of that origin. The original quartic is included because complete multiplicativity gives the exact identity

$$\lambda(n)\lambda(n+1)\lambda(3n+1)\lambda(3n+2) = \prod_{i=0}^{3} \lambda(3n+i). \tag{3.4}$$

If $n \equiv a \pmod q$, the origin on the right lies in $3a \pmod{3q}$. This identity does not equate its mean with the mean over all origins. Proposition 3.2 supplies the same bound for both means separately.

4 The horizontal form and its vertical error

The four-term average in (3.3) is controlled by the factor Z_2. We use the characteristic-factor theorems of Host–Kra and Ziegler [9, 37] and Ziegler's nilsystem progression formula [36]; the exact forms needed are also stated as Theorems 6.2 and 6.3 in [4]. We make the ensuing real-variable estimate explicit, since its dependence on the horizontal norms is part of the computer certificate.

Lemma 4.1 (A fourth Fourier moment on a connected group)**.** *Let K be a compact connected abelian group with normalized Haar measure. For a real measurable $f : K \to [-1, 1]$ of mean m,*

$$\sum_{\chi \neq 0} |\widehat{f}(\chi)|^4 \leq (1 - |m|)^3 \left(\frac{1}{3} + |m| \right) \leq \frac{1}{3}(1 - m^2)^2. \tag{4.1}$$

Proof. The assertion is immediate for the trivial group. Otherwise Haar measure is atomless. On the weak-star compact convex set of real $[-1, 1]$-valued functions of mean m, the sum on the left is convex and weak-star lower semicontinuous: its finite subsums have these properties. The extreme points are sign functions. Indeed, a function which stays a positive distance from both endpoints on a set of positive measure admits two opposite nonzero mean-zero perturbations on that set. The closed convex sublevel set of any proposed bound contains the whole set once it contains these extreme points, by Krein–Milman.

For a sign function, change its overall sign if necessary and write $f = 2\mathbf{1}_A - 1$ with $\alpha = |A| = (1 - |m|)/2 \leq 1/2$. Let $g = \mathbf{1}_A * \mathbf{1}_{-A}$. The Riesz–Sobolev inequality on compact connected abelian groups [3, Theorem 1.1] bounds $\int_C g$ for every prescribed $|C|$ by the corresponding quantity for centred arcs on the circle. Since

$$\int (g - u)_+ = \sup_C \left(\int_C g - u|C| \right),$$

integration in $u \geq 0$ also bounds $\int g^2$ by the value for arcs. Their autocorrelation is the triangular function $(\alpha - \|t\|)_+$, whose squared integral is $2\alpha^3/3$. Parseval consequently gives

$$\sum_{\chi \neq 0} |\widehat{f}(\chi)|^4 = 16(\|g\|_2^2 - \alpha^4) \leq 16(2\alpha^3/3 - \alpha^4).$$

This is the first expression in (4.1). For $r = |m|$, the second inequality follows from $(1+r)^2/3 - (1-r)(1/3+r) = 4r^2/3$. □

For four real centred functions on K, define

$$\begin{aligned} \mathcal{Q}_K(f_0, f_1, f_2, f_3) &= \int_{K^2} f_0(2u - v) f_1(u) f_2(v) f_3(2v - u)\, du\, dv, \\ P &= \prod_{i=0}^{3} \|f_i\|_2. \end{aligned} \tag{4.2}$$

Proposition 4.2 (Reduction with the norm-dependent error)**.** *Suppose that, for every compact connected abelian K and every real centred $f_i : K \to [-1, 1]$,*

$$|\mathcal{Q}_K(f_0, f_1, f_2, f_3)| + \frac{1}{3}(1 - \sqrt{P})^2 \leq B. \tag{4.3}$$

Then (3.3) *holds with the same B.*

Proof. Project the four observables to Z_2, then approximate by their conditional expectations on finite two-step nilsystem factors. Conditional expectation preserves reality, the bound by one, and mean zero. A telescoping expansion of the four factors bounds the error in the integral by the sum of the four L^1 approximation errors. The nilsystem factors of a totally ergodic system remain totally ergodic.

Consider one such nilsystem $Y = N/\Gamma$, with translation by a, in the standard presentation $N = \langle N^0, a \rangle$. The compact vertical group V induced by the commutators is connected. In step two it is generated by the connected images $[N^0, N^0]$ and $[a, N^0]$ containing the identity; their closure is connected as well. The horizontal quotient is a compact connected abelian rotation: a nontrivial finite component quotient would contradict total ergodicity.

Let ν be the integrated four-term progression measure. It is the corresponding coordinate marginal of Haar measure on the Hall–Petresco nilmanifold [5, Proposition 4.7]. Its polynomial description [5, Appendix B.1, equation (45)] makes it invariant under all vertical quadratic jets

$$(v_0, v_0 v_1, v_0 v_1^2 v_2, v_0 v_1^3 v_2^3), \qquad v_0, v_1, v_2 \in V.$$

To justify the Fourier expansion for measurable observables, choose a positive real trigonometric approximate identity K_ℓ on V and set

$$F_{i,\ell}(y) = \int_V K_\ell(v) F_i(vy)\, dv.$$

These functions have finite vertical Fourier support, remain real and bounded by one, and have the same vertical means f_i as F_i. They converge to F_i in $L^1(Y)$. Since every coordinate marginal of ν is Haar measure on Y, telescoping gives

$$\left| \int \prod_{i=0}^{3} F_i(y_i)\, d\nu - \int \prod_{i=0}^{3} F_{i,\ell}(y_i)\, d\nu \right| \leq \sum_{i=0}^{3} \|F_i - F_{i,\ell}\|_{L^1(Y)} \longrightarrow 0. \tag{4.4}$$

For the smoothed functions, the jet invariances show that the only possible vertical frequency tuples are

$$(-\chi, 3\chi, -3\chi, \chi).$$

This follows by solving the three character equations for the powers of v_0, v_1, v_2. The dual of V is torsion-free, so multiplication by three is injective. The zero tuple contributes the horizontal expression (4.2), where f_i is the vertical mean of F_i. The change of variables is $u = x + y$, $v = x + 2y$ in the horizontal progression $(x, x + y, x + 2y, x + 3y)$.

For each fixed horizontal progression, apply Hölder to the finite sum of the nonzero vertical modes and then use Lemma 4.1 in each fibre. Choices of fibre sections introduce phases of modulus one and do not affect the bound. The resulting error is at most

$$\frac{1}{3}\prod_{i=0}^{3}(1 - f_i(x_i)^2)^{1/2}.$$

The horizontal progression measure has product Haar marginals on every pair of distinct sites. Indeed, multiplication by a nonzero integer is surjective on a connected compact abelian group. Cauchy–Schwarz, with the sites grouped into two pairs, therefore bounds the integrated error by

$$\frac{1}{3}\prod_{i=0}^{3}(1 - \|f_i\|_2^2)^{1/2}.$$

For $v_i = \|f_i\|_2^2$, Hölder on a two-point space gives

$$\left(\prod_i v_i\right)^{1/4} + \left(\prod_i (1 - v_i)\right)^{1/4} \leq 1.$$

Since $(\prod_i v_i)^{1/4} = \sqrt{P}$, the error is at most $(1 - \sqrt{P})^2/3$. This bound is independent of ℓ, because positive vertical smoothing leaves every f_i unchanged. Passing to the limit in (4.4) proves (3.3) at every finite nilsystem stage by (4.3), and then at the inverse limit by the initial L^1 approximation. □

Lemma 4.3 (It suffices to use the circle)**.** *If* (4.3) *holds for* $K = \mathbb{T}$*, it holds for every compact connected abelian group.*

Proof. Approximate the four profiles in L^2 by real centred trigonometric polynomials, still bounded by one, using positive approximate identities. Their finitely many characters generate a finitely generated torsion-free abelian group, so the four polynomials factor through a torus $\mathbb{T}^d$. Choose $v \in \mathbb{Z}^d$ outside the finitely many hyperplanes arising from nonzero frequencies in the means, in the squared norms, and in the four-term expansion. Substitution $z = vt$ then preserves all four means, all four norms, and the four-term integral exactly. It also preserves the pointwise bound by one. Apply the circle inequality and let the approximation errors tend to zero. Continuity follows from Cauchy–Schwarz and the product Haar marginals of the two pairs of sites. □

The remaining task is thus the single concrete inequality (4.3) on $\mathbb{T}$. Its right-hand side is established by the finite certificate in the next section. Combining that inequality with Propositions 4.2 and 3.2 proves the logarithmic bound, without assuming that the original correlation has a limit.

5 A finite certificate for the circle inequality

We now prove the circle estimate required by Proposition 4.2. The exact constant is

$$B_* = \frac{196567331}{402653184}.$$

The proof combines analytic bounds for a single ternary projection with a finite interval calculation. The intervals cover all real parameters; sampling a collection of test functions is not used. The accompanying source package contains the integer tables and the programs which verify them. Appendix B gives the analytic derivations and the exact mathematical specification of every local input. We describe the mathematical objects checked by those programs and the invariant of the final recurrence.

5.1 Ternary projection and successive losses

It is convenient in this section to write

$$Q = \int_{\mathbb{T}^2} A(u)B(v)E(2u-v)O(2v-u)\,du\,dv.$$

The four profiles are real, centred and bounded by one. Define

$$T_3F(t) = \frac{1}{3}\sum_{a=0}^{2} F(t+a/3), \qquad v_F = \|F\|_2^2, \qquad q_F = \frac{\|T_3F\|_2^2}{v_F}.$$

A zero norm makes $Q = 0$ and can be handled separately. At a given step the two active profiles are A, B. Put

$$a = \|A\|_2\|B\|_2, \quad b = \|E\|_2\|O\|_2, \quad P = ab, \quad t = \sqrt{q_Aq_B}, \quad x = \sqrt{(1-q_A)(1-q_B)}.$$

Then $t + x \le 1$. Replacing the two active profiles by their projections gives a new form Q'. Since $T_3A(u) = A'(3u)$ and similarly for B, the change of variables $(e, o) = (2u - v, 2v - u)$ gives $3u = 2e + o$, $3v = e + 2o$. After two harmless reflections, Q' has the same form, with the previous passive pair active, and

$$(a, b, P) \longmapsto (b, at, Pt). \tag{5.1}$$

Iteration defines $Q_j, a_j, b_j, P_j, t_j, x_j$. Every two steps each original profile has undergone one more projection to frequencies divisible by three, followed by division of those indices. For centred trigonometric polynomials the descent terminates. Approximation and Parseval give $P_j \to 0$ for the original measurable profiles.

The two geometric identities used to retain the losses are

$$|Q_j - Q_{j+1}| \le \frac{P_jx_j}{\sqrt{2}}\sqrt{1 - t_{j+1}^2 - x_{j+1}^2/2}, \tag{5.2}$$

and

$$v_F(1-3q_F) \le \frac{2}{3}, \qquad U_F := \int F(s)F(s+1/3)\,ds = \frac{v_F(3q_F-1)}{2} \ge -\frac{1}{3}. \tag{5.3}$$

For (5.2), condition on the cosets of $\mathbb{T}[3]$. The two active mean-zero triples have three relative correlations. Their mean is zero and their squared mean is half the product of their variances, because cyclic shift acts as a rotation through $2\pi/3$ on the real two-dimensional mean-zero subspace. Centre the passive product on the same three relative positions and apply Cauchy–Schwarz. Its exact squared norm is $b^2 - (b')^2 - e_Ee_O/2$, where $e_F = v_F - \|T_3F\|_2^2$. This is (5.2). For (5.3), the sum of the three pairwise products of numbers in $[-1, 1]$ is at least -1, as is checked at the vertices of the cube. Integration and the identity for the squared sum of the triple give both assertions.

5.2 Information retained by the local bounds

There are three nonzero Fourier channels at a projection step. If the two active residues are $1, 1$ modulo three, the passive residue pairs are $(-1,-1),(1,0),(0,1)$. The conjugate active pair supplies their conjugates. The passive products are orthogonal in $L^2(\mathbb{T}^2)$, since the passive coordinates are independent Haar variables. Subdividing the two active classes modulo nine records more of this orthogonality.

Let $p=(p_0,p_1,p_2)$ and $r=(r_0,r_1,r_2)$ be their normalized energy distributions on the three subclasses. For $d\in\mathbb{Z}/3\mathbb{Z}$ set $w_d=\sum_i p_i r_{i+d}$ and $m_d=\max_i p_i r_{i+d}$. The channel coefficients are bounded by

$$C_0=\sqrt{w_0},\qquad C_d=\begin{cases}\sqrt{w_d}, & 2m_d\le w_d,\\ (\sqrt{m_d}+\sqrt{w_d-m_d})/\sqrt{2}, & 2m_d>w_d\end{cases}\quad(d=\pm1).$$

To see this, apply Bessel to the three orthogonal active products within one channel. In a nonzero channel the real centred passive profile gives the additional individual bound $1/\sqrt{2}$; the resulting optimization is the support function of the positive unit ball with each coordinate at most $1/\sqrt{2}$. The elementary Fourier lemma producing this additional bound, including its dyadic frequency partition, is proved in Appendix B.1.

The normalized projection loss is bounded by the norm of

$$M(p,r)=\begin{pmatrix}C_0/2 & C_1/\sqrt{2}\\ C_{-1}/\sqrt{2} & 0\end{pmatrix}.$$

The interval proof on the two probability simplexes gives

$$\|M(p,r)\|\le\frac{59}{100},\qquad |Q_j-Q_{j+1}|\le\frac{59}{100}(P_j-P_{j+1}),\qquad |Q_j|\le\frac{59}{100}P_j. \tag{5.4}$$

For clarity, the matrix inequality has a rational certificate. With $W_d\ge C_d^2$ and $f=W_0+2(W_1+W_{-1})$, it is enough to check

$$f\le 8h^2,\qquad 4h^4-h^2f+W_1W_{-1}\ge 0,\qquad h=59/100.$$

These conditions place both squared singular values below h^2. Parametrize each simplex by $(s,(1-s)t,(1-s)(1-t))$ and subdivide the four-dimensional cube. Each w_d is separately affine, so its extrema on a box occur at vertices. The expression defining C_d as a support function is increasing in each weight, which gives directed bounds for the remaining terms. The verified subdivision has 18 528 168 terminal boxes. The associated directional bounds and six further applications of the one-step recurrence give the table `hom59.inc`; the data and their verifiers are supplied.

Fourier bounds are combined with finite identities on nine translates. For each of four words of length nine, let $m_i\in\{0,1,2,3\}$ count its nonconstant triples at positions differing by three. Let I be the sum of the 81 products in the four-form pattern on $(\mathbb{Z}/9\mathbb{Z})^2$, and let I' be the analogous sum after projecting the two active words. The enumerator records the exact maxima of $|I-kI'/10|$ for $k=0,\dots,9$ and for every tuple (m_1,m_2,m_3,m_4). Integer linear majorants in

$$1,\quad U_1+U_2,\quad U_3+U_4,\quad U_1U_2,\quad U_3U_4,\quad (U_1+U_2)(U_3+U_4)$$

are checked against these maxima on all 4^4 tuples. Here the finite substitution is $U_i=1-4m_i/9$.

For the passage to real profiles, round each value independently to a sign with that expectation and average over Haar. Multilinearity gives $\mathbb{E}I=81Q$, $\mathbb{E}I'=81Q'$. Moreover

$\mathbb{E}m_i = 9(1-U_i)/4$, and statistics at distinct sites are pairwise independent under their Haar coordinates. Thus each checked polynomial majorant G gives the continuous inequality

$$|Q| \le G(U_1, U_2, U_3, U_4) + (k/10)|Q'|.$$

The maximum of a separately affine G on a box lies at a vertex. This explains why the discrete identities constrain the original real profiles rather than only sign-valued test functions.

Two further tables retain the same variances in the sign and Fourier bounds. They contain 32 340 energy bounds and 1 187 certified rectangles in (a, b, t), respectively. A rectangle can improve a grid cell only when it contains that cell entirely; the latter rectangles improve 451 cells of a 40^3 table. In every transition that meets several cells, the program uses the largest of their upper bounds. The detailed inequalities, including the three-valued rounding used when the passive norms are below one, appear in Appendix B.5. They preserve the relations (5.3) before maximizing. Dropping those relations would lose the gain below one half.

5.3 The invariant of the finite recurrence

The recurrence stores upper bounds for

$$J_j = |Q_j| - \gamma P_j, \qquad \gamma = 1/32.$$

The product P_j is divided into N equal intervals. Each pair norm a_j, b_j has a separate label among H intervals. A transition retains the shared passive label and follows the relation (5.1). There are also L intervals for the loss x_j, with endpoints $1-(1-i/L)^2$. All upper and lower arithmetic bounds are rational; square roots and divisions are rounded in the required direction with integer operations.

The proof of the recurrence is induction on the product interval. Strictly smaller product intervals have already been bounded. For a cell $I_q = [q/N, (q+1)/N]$ above the starting index, let j_0 be an entry rank and m the last rank in that cell, so that $P_{m+1} < q/N$. Such a last rank exists since $q > 0$ and $P_j \to 0$. If three successive products P_j, P_{j+1}, P_{j+2} lie in I_q, then $t_{j+1} \ge q/(q+1)$ and (5.2) gives

$$|Q_j - Q_{j+1}| \le \sqrt{\frac{2q+1}{2(q+1)^2}}\,(P_j - P_{j+1}).$$

Above the starting index this coefficient is at most γ, so $J_j \le J_{j+1}$ for $j_0 \le j \le m-2$. This does not cover the transition from $m-1$ to m. The last two transitions must both be paid. More precisely, if $E_j \ge |Q_j - Q_{j+1}|$ is the certified cost used for transition j, then, whenever $j_0 \le m-1$,

$$\begin{aligned} J_{j_0} \le J_{m-1} \le J_{m+1} + E_{m-1} - \gamma(P_{m-1} - P_m) \\ + E_m - \gamma(P_m - P_{m+1}). \end{aligned} \tag{5.5}$$

For a visit consisting of the single rank $j_0 = m$, the corresponding bound is simply $J_m \le J_{m+1} + E_m - \gamma(P_m - P_{m+1})$.

Both implementations, `rec_xyz.cpp` and `rec_disque.cpp`, follow precisely this order. The passes with `r<q` first store the exiting candidates, including their exit costs. The pass with `r==q` reads those candidates through `future` and adds the preceding transition cost. Only then does the closure using `last`, `forward` and `reverse` transfer the resulting bound to earlier ranks in the same cell. Single-rank visits remain covered by the already stored exit candidates. The starting-index test also ensures that every same-cell transition has $x_j \le 1/(q+1)$ below the first positive loss-grid endpoint, which justifies performing this closure in loss bin zero.

During the internal steps the two pair norms may exchange roles, but each physical pair only loses mass. Between two states in I_q, the value of either physical pair is at least its earlier value times $q/(q+1)$. The closure therefore retains both orientations and every pair of labels compatible with these inequalities; it does not replace the two paid transitions by a free exit. Below the starting index the recurrence uses (5.4).

The threshold used to discard impossible states is also justified by an inequality on the entire trajectory. The cumulative projection bound is

$$|Q_0-Q_j| \le D(a_j,b_j,P_j), \qquad D(a,b,P)=\min\{4(1-\sqrt{P})^2,(1-b)(3-2a-b),(1-a)^2+(1-b)^2\}.$$

Its proof expands the cumulative ternary projections and uses the two-residual cancellations proved in Appendix B.4. Since $P_j \le P_0$, a trajectory with $|Q_0|+(1-\sqrt{P_0})^2/3 \ge g$ must satisfy at every step

$$|Q_j| \ge g - D(a_j,b_j,P_j) - (1-\sqrt{P_j})^2/3.$$

Only a state violating this necessary condition can be removed. The reported upper bound is the maximum of g and the upper bound for all surviving initial states. Thus g is a computational threshold, not an assumption on the correlation.

The retained candidates are pruned only by componentwise domination within a fixed passive-weight label: a candidate is removed when another has at least its admissible mass, transition coefficient and value. The Python reference keeps the whole candidate list and uses arbitrary precision integers. This gives a separate implementation of the same induction on small grids.

The source also retains an optional earlier stopping bound $b(P)=5/8-\sqrt{135/512}(1-P)$. It is inactive in the present calculation: $b(P)-(59/100)P$ is affine and is at least $7/200$ on $[0,1]$. The directed rounding errors at the stated integer scale are smaller than this margin. Consequently the proof of the present bound does not require that earlier numerical certificate.

Proposition 5.1 (Certified circle estimate)**.** *For four real centred $f_i : \mathbb{T} \to [-1,1]$, with $P=\prod_i \|f_i\|_2$,*

$$|\mathcal{Q}_{\mathbb{T}}(f_0,f_1,f_2,f_3)| + \frac{1}{3}(1-\sqrt{P})^2 \le \frac{196567331}{402653184}.$$

Proof. Use the local bounds and recurrence above with

$$(N,L,H,\gamma)=(3072,1,256,1/32), \qquad g_0=488/1000.$$

The integer scale is $T=805306368$ and the actual threshold is $g=\lfloor Tg_0\rfloor/T$. The exact maximum of the surviving initial states, after restoring γP_0 and the vertical term with outward rounding, is $393134662/T$, reached in product cell 3071. It exceeds g. The recurrence invariant therefore bounds every admissible trajectory by $393134662/805306368 = 196567331/402653184$. All entries in this calculation are integer comparisons over covering intervals. Passing from trigonometric polynomials to the measurable profiles follows by the continuity already used in Lemma 4.3. □

The verifier distinguishes three tasks: enumeration of the finite word maxima, certification of the continuous local tables, and evaluation of the final recurrence. Repeating the last task alone presupposes the checked input tables. Appendix A identifies the reference outputs and the commands that reproduce each stage.

6 Five consecutive signs

The five-letter calculation of Tao–Teräväinen [28, Section 7] is the starting point. The strict quartic bound below one half forces at least five of the eight opposite pairs with equal endpoints to have positive mass, yielding twenty-six words in total. The qualitative positivity of the other sixteen words already follows from earlier correlation results and a run-exit argument. We first make that deduction and its quantitative refinement explicit.

Fix a logarithmic limit law of $X_j = \lambda(n+j)$ on windows whose harmonic mass tends to infinity. All moments below belong to this same law. Put

$$a = \mathbb{E}X_0X_1X_2X_3, \qquad b = \mathbb{E}X_0X_1X_2X_4, \qquad d = \mathbb{E}X_0X_1X_3X_4.$$

The number a averages over all origins. It need not equal the correlation c conditioned on origins divisible by three. For $s \in \{-1,1\}^5$, let $P_s = \mathbb{P}((X_0,\dots,X_4) = s)$.

Lemma 6.1 (The five-letter distribution)**.** *Writing $t = s_0s_1s_2s_3s_4$, one has*

$$32P_s = 1 + t\big((s_0+s_4)a + (s_1+s_3)b + s_2d\big), \qquad |d| \le \frac{2}{3}. \tag{6.1}$$

In particular, the law is invariant under global sign reversal and under reflection of the five sites. The earlier bound $|a| \le 1/2$ also holds. All these statements remain valid for $(X_{hj})_j$ at any fixed positive integer common difference h, with parameters a_h, b_h, d_h.

Proof. The formula (6.1) is [28, (7.4)], with its third parameter renamed d. We recall its derivation to specify the inputs used below. Expand the cylinder indicator as $2^{-5}\prod_{i=0}^{4}(1+s_iX_i)$. Tao's two-point theorem and Tao–Teräväinen's odd-order theorem annihilate all nonconstant terms except those of order four. Stationarity identifies the first and last consecutive quadruples. Reflection invariance of the even correlations, obtained from the isotypy theorem, identifies the two quadruples of type b; see [26, 30] and [28, Lemma 7.1]. Vanishing of all odd moments also gives global sign symmetry of the full process. This gives (6.1). The following estimate makes quantitative the three-translates mechanism of [28, Lemma 7.3]. For $Y_j = X_jX_{j+1}X_{j+3}X_{j+4}$ one has $Y_0Y_1Y_2 = X_0X_6$, whose mean is zero. If σ is a preferred sign of Y_j, then

$$\frac{1}{2} = \mathbb{P}(Y_0Y_1Y_2 \ne \sigma) \le \sum_{j=0}^{2}\mathbb{P}(Y_j \ne \sigma) = \frac{3(1-|d|)}{2}.$$

Thus $|d| \le 2/3$. Finally, put $Q_j = X_jX_{j+1}X_{j+2}X_{j+3}$. Since $Q_0Q_1 = X_0X_4$ has mean zero,

$$2|a| = |\mathbb{E}(Q_0+Q_1)| \le \mathbb{E}|Q_0+Q_1| = 1 + \mathbb{E}(Q_0Q_1) = 1.$$

This is the earlier one-half bound; compare [30, Proposition 7.1]. Replacing every shift by its multiple by h proves the assertions at a fixed common difference. Throughout, the origins are unrestricted; no reflection identity on a single prescribed origin class is needed. □

The proof of [28, Theorem 1.14] already gives positive lower natural density for twelve of the following words: the eight with $s_0 \ne s_4$ and $s_1 \ne s_3$, and the four obtained from $++++-$ by sign reversal and reflection. The other four are the orbit of $+-+--$ under these symmetries. Their qualitative positivity follows from the same earlier inputs, as the elementary run estimate below shows.

Proposition 6.2 (Quantitative bounds for sixteen specified words)**.** *Every word with $s_0 \ne s_4$ has positive mass. More precisely,*

$$P_s \ge \begin{cases} 1/96, & s_1 \ne s_3, \\ 2167/1246272, & s_1 = s_3. \end{cases}$$

These bounds hold in every logarithmic limit law on the stated windows. Each of these sixteen specified words consequently has positive lower natural density. The same assertions hold at every fixed positive integer common difference.

Proof. In the first case $ts_2 = s_0s_4s_1s_3 = 1$, and (6.1) gives $P_s = (1+d)/32 \geq 1/96$. The remaining eight words are the exits from a constant or alternating four-letter run, together with their reflections and opposites. It is enough to treat the two words $++++-$ and $+-+--$.

For the alternating case replace X_j by $(-1)^j X_j$. Global sign symmetry makes the resulting process stationary, and all distinct pairs remain orthogonal. We may therefore consider a stationary, sign-symmetric process Y with orthogonal distinct pairs and write

$$p = \mathbb{P}(Y_0 = \cdots = Y_3 = 1, Y_4 = -1), \qquad \rho = \mathbb{P}(Y_0 = \cdots = Y_3 = 1) = \frac{1+a}{16}.$$

The four-letter product is unchanged by the alternating substitution.

First, the old one-half bound alone gives a useful uniform threshold. Let $A_L = \{Y_0 = \cdots = Y_{L-1}\}$, with both signs counted. Then $\mathbb{P}(A_4) = 2\rho = (1+a)/8 \geq 1/16$. Pair orthogonality gives $\mathbb{E}(\sum_{j=0}^{L-1} Y_j)^2 = L$, so that $\mathbb{P}(A_L) \leq 1/L$. If a run of length four exits before position L, its first exit is one of $++++-$ and $----+$ at one of $L-4$ possible starts. By stationarity and sign symmetry, for every $L > 4$,

$$\frac{1}{16} \leq \mathbb{P}(A_4) \leq \frac{1}{L} + 2(L-4)p.$$

Choosing $L = 30$ yields

$$p \geq \frac{1/16 - 1/30}{2(30-4)} = \frac{7}{12480} > \frac{1}{2000}. \tag{6.2}$$

The run-propagation method appears in [17, Corollary 2.8]; (6.2) records the explicit constant obtained by the present calculation. Reflection and the alternating substitution cover all eight run exits. Together with the first case, this proves from the earlier inputs alone that

$$P_s > \frac{1}{2000} \qquad (s_0 \neq s_4). \tag{6.3}$$

The argument applies unchanged to $(X_{hj})_j$ for every fixed h.

We now sharpen (6.2) by weighting the runs. For the appropriate choice of sign, (6.1) gives $32p = 1 - d \pm 2b$. Positivity of the eight equal-endpoint numerators gives

$$d \leq 1 - 2|a|, \qquad 2|b| \leq 1 + d - 2|a|, \qquad 32p \geq 6|a| - 2.$$

In particular $a \geq -(16p+1)/3$ and $2\rho \geq 1/12 - 2p/3$.

Take positive weights $w_1, \ldots, w_k$ of sum one and set

$$C_w = \sum_i w_i^2, \qquad D_w = 2\sum_{i,j} |i-j| w_i w_j, \qquad S = \sum_i w_i Y_i.$$

Pair orthogonality gives $\mathbb{E}S^2 = C_w$. Let π_r be the intensity of starts of maximal constant runs of length r, with both signs counted. Infinite runs have zero mass, since $\mathbb{P}(|\sum_{j<m} Y_j| = m) \leq 1/m$. Counting exits and the possible starts of a four-letter subword inside a run yields

$$\sum_{r \geq 4} \pi_r = 2p, \qquad \sum_{r \geq 4} (r-3)\pi_r = 2\rho.$$

For $\phi(u) = (2u-1)_+^2$ define

$$G(r) = \sum_{s\in\mathbb{Z}} \phi\left(\sum_{i=s}^{s+r-1} w_i\right), \qquad w_i = 0 \quad (i \notin \{1,\dots,k\}).$$

A run carrying weight $u > 1/2$ forces $S^2 \ge (2u-1)^2$. There can be at most one such run in a configuration, hence $C_w \ge \sum_r \pi_r G(r)$. Convexity of ϕ bounds $G(r+1) - G(r)$ by one: the increase obtained by appending a weight is no larger than the corresponding increase after all earlier weights have been included, and these latter increases telescope. For $r \ge k$, a direct count with $F_j = \sum_{i\le j} w_i$ gives

$$G(r) = r - k + 1 + \sum_{j=1}^{k-1} (2F_j - 1)^2 = r - D_w.$$

Here $D_w = 4\sum_{j=1}^{k-1} F_j(1-F_j)$. The increment bound therefore gives $G(r) \ge r - D_w$ for every r. Keeping only $r \ge 4$ and using the preceding identities, we obtain

$$C_w \ge 2\rho - 2p(D_w - 3) \ge \frac{1}{12} - 2p\left(D_w - \frac{8}{3}\right).$$

For $D_w > 8/3$ it follows that

$$p \ge \frac{1 - 12C_w}{24D_w - 64}. \tag{6.4}$$

Use the twenty-five weights

$$w_{i+13} = \frac{169 - i^2}{2925} \quad (-12 \le i \le 12).$$

They satisfy $C_w = 677/14625$ and $D_w = 64928/4875$. Substitution into (6.4) gives $p \ge 2167/1246272$. Sign reversal and reflection cover all eight exits.

For completeness, the implication for lower natural density uses all admissible windows, rather than only logarithmic prefixes. If a specified word had prefix frequencies $\epsilon_j \to 0$ at N_j, choose $\omega_j \to \infty$ so slowly that $\omega_j\epsilon_j \to 0$ and $N_j/\omega_j \to \infty$. Its normalized harmonic mass in $(N_j/\omega_j, N_j]$ would then be at most $\omega_j\epsilon_j/(\log\omega_j + o(1))$, contradicting the uniform positive lower bound in every extracted law. □

Theorem 6.3 (Twenty-six five-letter words)**.** *Let $p_{5,(j)}$ denote the j-th largest five-letter probability. For $B = 196567331/402653184$,*

$$p_{5,(26)} \ge \frac{1-2B}{32} = \frac{4759261}{6442450944} > 0.0007387345. \tag{6.5}$$

Consequently at least twenty-six five-letter words occur with positive upper natural density. The additional ten words need not be the same in different logarithmic limit laws.

Proof. This is where the strict improvement $|a| \le B < 1/2$ is used in the five-letter count. We insert it into the five-letter law of [28] to extend the earlier count. Put $v_i = ts_i$. Opposite words have the same v, and $\prod_i v_i = 1$. For equal endpoints put $x = v_0 = v_4$. For each $x = \pm 1$, the four values of $32P_s$ on opposite pairs are

$$1 + d + 2ax + 2b, \quad 1 + d + 2ax - 2b, \quad 1 - d + 2ax, \quad 1 - d + 2ax. \tag{6.6}$$

Let $u = 1 - 2B$. For the stated B, one has $0 < u < 1/6$, and $u/32$ is smaller than both bounds in Proposition 6.2. If $4|a| \ge u$, every value in the group $x = \operatorname{sgn}(a)$ exceeds its nonnegative

counterpart by $4|a|$, and hence is at least u. The average of the four values in the other group is $1-2|a| \geq 1-2B = u$. At least one more value is therefore at least u. If $4|a| < u$, in each group the two repeated values and the larger of the first two are at least $1-|d|-2|a| > 1/3-u/2 > u$. Thus at least five opposite pairs with equal endpoints meet the threshold. Together with the sixteen specified words this proves (6.5).

Choose a convergent subsequence of logarithmic prefix laws and retain the twenty-six words given by that limit. Each has upper logarithmic density at least the stated threshold. Summation by parts bounds upper logarithmic density by upper natural density, proving the last assertion. □

7 Six-letter words and an exact finite certificate

The five-letter law alone leaves correlations of order six undetermined. For six-letter words we use stationarity as a flow condition and retain the exact divisions of Liouville on finite residue classes. The finite calculation has rational witnesses and an exhaustive integer verifier.

Theorem 7.1 (Forty-four six-letter words)**.** *Let $p_{6,(44)}$ be the forty-fourth largest six-letter probability and put $u = (1+2a)/4$. Every logarithmic Liouville limit law satisfies*

$$p_{6,(44)} \geq \min\left\{\frac{8707}{2064851523584}, \frac{u^2}{4096}\right\}. \tag{7.1}$$

In particular the bound $|a| \leq B$ in this paper gives

$$p_{6,(44)} \geq \frac{8707}{2064851523584} > \frac{1}{237148447}.$$

At least forty-four words therefore have positive upper natural density.

Proof. Write $Q_i = X_iX_{i+1}X_{i+2}X_{i+3}$. Since $Q_0Q_1 = X_0X_4$ has mean zero, $u = \mathbb{P}(Q_0 = Q_1 = 1) = (1+2a)/4$ and $0 \leq u \leq 1/2$. Let $\eta = p_{6,(44)}$. Global sign reversal and reflection partition the sixty-four words into orbits of size two or four. The words with mass greater than η are at most forty-two in number. Complete their union of orbits, if necessary, to a mask M containing forty or forty-two words. Its complementary mass is at most 32η.

Regard a six-letter word as an edge from its five-letter prefix to its five-letter suffix. Stationarity makes its probability a circulation on this thirty-two-vertex graph. Decompose the circulation into simple directed cycles, each of length at most thirty-two. Every cycle using an edge outside the recurrent part of M must also use an edge outside M. The mass outside that recurrent part is at most 1024η.

The earlier inputs already give at least twenty-four vertices of mass at least $1/2000$, including all sixteen specified opposite-endpoint vertices by (6.3). To see the other eight without using B, put $\theta = 32/2000$ in (6.6). If $4|a| \geq \theta$, the four entries in the group $x = \operatorname{sgn}(a)$ are at least θ, because their counterparts are nonnegative. If $4|a| < \theta$, at least three entries in each group are at least $1-|d|-2|a| > 1/3-\theta/2 > \theta$. In either case four opposite pairs with equal endpoints meet the required threshold. If the recurrent part omits a specified vertex or has fewer than twenty-four vertices, then $\eta \geq 1/(2000 \cdot 1024)$, which is stronger than the stated result. Otherwise it lies inside a maximal admissible recurrent mask F. Write e for its complementary mass; still $e \leq 1024\eta$.

The exact enumeration and its witnesses have the following counts:

unions of forty or forty-two words	110880
distinct admissible recurrent masks	20257
maximal admissible masks	9247
excluded by stationary scores	8973
excluded by division scores	272
remaining masks	2.

We describe the two kinds of witness so that the logical content of the computation is explicit.

A stationary score on an odd-length block of N signs is an integer linear combination of reflected pair products, differences between prefix and suffix cylinder indicators, the mean of the two end quartic products, and indicators of the specified five-letter orbits at the centre. Its expectation has a rational lower bound

$$L = -\frac{|c_4|}{2} + \sum_o c_o|o|\rho_o, \qquad c_o \geq 0, \quad \rho_o \in \{1/96, 1/2000\}.$$

Here the quartic contribution uses only $|a| \leq 1/2$, and the cylinder contributions use the bounds established from the earlier inputs above. The verifier enumerates every path permitted by F and obtains an integer maximum $h < L$. If A is a bound for the absolute value of the score on any word, a forbidden six-letter subword occurs with probability at most $(N-5)e$. Thus

$$\eta \geq \frac{L-h}{1024(N-5)(A-h)}. \tag{7.2}$$

The stationary witnesses use $N = 7, 9, 11, 13$, with respectively $7074, 1218, 192, 489$ masks.

For a division witness put $D = \lfloor (N-1)/5 \rfloor$ and $q = \operatorname{lcm}(1, \ldots, D)$, and condition the origin on one class $r \pmod q$. Whenever $1 \leq d \leq D$, $j + 5d < N$ and $r + j \equiv 0 \pmod d$, exact multiplicativity identifies the six signs at $j, j+d, \ldots, j+5d$ with $\lambda(d)$ times a consecutive word on a class modulo q/d. Since F is sign-symmetric, the probability of failure of that reading is at most $(q/d)e$. Fixed dilation or translation of a window changes its unnormalized harmonic mass by $O_{d,j}(1)$, so this comparison holds in the same extracted law. Let

$$K = \sum_{\substack{1 \leq d \leq D,\ 0 \leq j < N-5d \\ d \mid r+j}} \frac{q}{d}.$$

The joint failure probability is at most Ke. A score $f = \sum_{i<j} c_{ij} X_i X_j$ has conditional expectation zero by the two-point theorem on distinct affine forms. The verifier checks $f \leq h < 0$ on every accepted word and uses $A = \sum |c_{ij}|$ on the rest. This gives

$$\eta \geq \frac{-h}{1024K(A-h)}. \tag{7.3}$$

If there are no accepted words, the bound is $1/(1024K)$. The 272 division witnesses use block lengths at most thirty-one. Taking the exact minimum in (7.2) and (7.3), including the elementary vertex bound, gives $8707/2064851523584$.

It remains to treat the two masks not excluded by these witnesses. Let D_5 be the eight five-letter words for which both adjacent quartics are positive. In binary notation, with a minus sign coded by one, these words have codes

$$0, 6, 10, 12, 19, 21, 25, 31.$$

Let F_0 be the forty six-letter words whose prefix and suffix both avoid D_5. The two remaining masks are precisely

$$F_{\mathrm{c}} = F_0 \cup \{000000, 111111\}, \qquad F_{\mathrm{a}} = F_0 \cup \{010101, 101010\}.$$

In the first case, a prefix in D_5 must be constant if the first edge is allowed, and its only allowed continuation stays constant. In the second case the same assertion holds with alternating signs. For either mask, pair orthogonality bounds the probability of a run of the prescribed type and length N by $1/N$. A first departure is detected by one of the consecutive six-letter windows. Hence, for $N \geq 6$,

$$u \leq \frac{1}{N} + (N-4)e.$$

Choose $N = \max(6, \lceil 2/u \rceil)$ when $u > 0$. Since $u \leq 1/2$, one has $1/N \leq u/2$ and $N - 4 \leq 2/u$. Therefore $e \geq u^2/4$ and $\eta \geq u^2/4096$. This proves (7.1). The new quartic bound is needed to exclude $a = -1/2$ in the two remaining masks and to make the resulting minimum uniformly positive. Indeed $u \geq (1-2B)/4$, and direct rational comparison gives

$$\frac{(1-2B)^2}{65536} = \frac{22650565266121}{2656331146614175432704} > \frac{8707}{2064851523584}.$$

The passage to upper natural density is the same as in Theorem 6.3. □

Reproducibility of the finite step. The data consist of the two integer witness files `cert13.jsonl` and `dilate.jsonl` in the six-letter certificate directory. Running `python3 verifier.py 42` re-enumerates all orbit masks, computes their strongly connected components, checks every score on every admissible word, and reconstructs the two exceptional masks. The verification uses only Python's standard library and exact integer or rational arithmetic. The discovery of the witness coefficients is not a step of the proof.

8 Twenty-eight words at common difference five

The comparison between two fixed common differences supplies a further consequence. It uses exact divisions by two, three and five, rather than an identification of the correlation at all common differences. The finite witnesses already give twenty-eight words using the earlier quartic bound $1/2$. We separate this consequence from the improvement of its probability threshold obtained with B.

For $h \geq 1$ put

$$a_h = \mathbb{E}(X_0 X_h X_{2h} X_{3h}), \qquad \eta_h = p_{5,h,(28)},$$

where the latter is the twenty-eighth largest probability at common difference h. Each h is fixed before taking the logarithmic limit.

Theorem 8.1 (Twenty-eight words from the earlier quartic bound)**.** *For every fixed positive integer d coprime to six,*

$$\eta_{5d} \geq \frac{1}{2800000}.$$

In particular at least twenty-eight five-letter words at common difference five have positive upper natural density. For any five prescribed words at that common difference, the union of their occurrence sets has positive lower natural density.

Proof. We first state precisely the finite inequalities used in the argument. For $\sigma, \tau \in \{-1, 1\}$ define a sign-symmetric family of six five-letter words by

$$F_{\sigma,\tau} = \left\{ s : \ 12 + (s_0 s_1 s_2 s_3 s_4)\big(5\sigma(s_0 + s_4) + 2\tau(s_1 + s_3) + 2s_2\big) = 0 \right\}.$$

For $\sigma = -1$, representatives with initial sign + are

$$F_{-,+}: +--++,\ +-+-+,\ ++--+; \qquad F_{-,-}: +++++,\ +--++,\ ++--+.$$

The positive families have representatives $+-+++, ++-++, +++-+$ and $+-++ +, +++-+, +---+$, respectively.

If $\eta_h < k_0 = 1/2000$, at least one of these families satisfies

$$\mathbb{P}_h(F_{\sigma,\tau}) \le 6\eta_h. \tag{8.1}$$

Indeed all sixteen opposite-endpoint words have mass greater than k_0 by (6.3), whose proof at every fixed common difference uses only the earlier correlation results. If $\eta_h < k_0$, at least three of the eight remaining opposite pairs are small. Use the numerators (6.6) for the process $(X_{hj})_j$. If $4|a_h| \le 32\eta_h$, at least three numerators in each group are $\ge 1/3 - 16\eta_h > 32\eta_h$, a contradiction. Otherwise every small pair lies in the group with $a_h x = -|a_h|$. Three small entries must include the repeated entry and one of the two others, which gives one of the four displayed families.

The integer certificates supply, for either negative family,

$$a_{5h} - a_h \ge L_\Delta - T_\Delta \mathbb{P}_h(F)/6, \tag{8.2}$$

$$-a_h \ge L_a - T_a \mathbb{P}_h(F)/6. \tag{8.3}$$

Their exact coefficients are

	$F_{-,+}$	$F_{-,-}$
L_Δ	876965201/12000000000	1102289587/12000000000
T_Δ	1534424290237/600000000	21879529009333/6000000000
L_a	864219901/2000000000	1313720921/3000000000
T_a	1081358738797/800000000	7137984547037/2400000000.

For the two positive families the certificates give $0 \ge L_+ - T_+ \mathbb{P}_h(F)/6$, with

$$(L_+, T_+) = \left(\frac{27982891}{4000000000}, \frac{686068371347}{3000000000}\right), \quad \left(\frac{32040557}{12000000000}, \frac{2814334371461}{12000000000}\right).$$

Both ratios L_+/T_+ exceed $1/90000$.

Here is the common verification of these inequalities. On N sites ($N = 17$ or 19), use the variables $x_{r,[w]} = \mathbb{P}(J = r \pmod 6, (X_0, \dots, X_{N-1}) \in \{w, -w\})$. The equalities are normalization, stationarity, vanishing conditional pair correlations, and the exact divisions

$$\begin{aligned} &\mathbb{P}\big(J = er \pmod 6, (X_0, X_e, \dots, X_{(l-1)e}) \in [v]\big) \\ &\quad = \frac{1}{e} \sum_{u \equiv r \ (6/e)} \mathbb{P}\big(J = u \pmod 6, (X_0, \dots, X_{l-1}) \in [v]\big), \end{aligned} \tag{8.4}$$

where $e = 2, 3$ and $l = 1 + \lfloor (N-1)/e \rfloor$. Reflection may be imposed by symmetrizing the joint law of the signs and the origin; the targets and the families F are preserved. In a column (r, w), let $k(r, w)$ count the occurrences of F at common differences $e = 1, 2, 3$ and starts i such that $e \mid r + i$. Its mean is

$$\sum_{r,w} k(r,w) x_{r,w} = \mathbb{P}_h(F) \sum_{e=1}^{3} \frac{N - 4e}{e}.$$

For integer multipliers Y, integer scale S, constraint matrix A and target c, the verifier computes $z(r, w) = 6Sc(r, w) - (6A_{r,w})^{\mathsf{T}} Y$. Let $\bar{z}$ be the average with the reflected column, $\delta = \min_{k=0} \bar{z}$, and

$$H = \max\left(0, \max_{k>0} \frac{\delta - \bar{z}}{6Sk}\right).$$

The inequality $\bar{z} \geq \delta - 6SHk$, integrated against x, gives the stated bound $L - T\mathbb{P}_h(F)/6$, with $L = (Y_{\mathrm{mass}} + \delta/6)/S$ and $T = 6H\sum_{e=1}^{3}(N - 4e)/e$. Every column, including those containing forbidden words, is tested.

These laws and calculations apply at a fixed h coprime to six. One uses the process X_{hj} and recodes the finite origin as $J' = h^{-1}J \pmod 6$. Under a shift by h, this origin advances by one, and division by two or three gives (8.4) for the recoded process. Pair orthogonality on each class follows from Tao's theorem on affine forms, while the odd moments and reflection identities follow from the structure theorem. The elementary proof in Lemma 6.1 supplies $|a_h| \leq 1/2$ at each such fixed common difference. The new four-profile bound is not used in these finite inequalities or their applicability.

Apply (8.1) at $h = 5d$. If the family is positive, the preceding ratios already give $\eta_{5d} > 1/90000$. For a negative family, exact division on origins divisible by five gives

$$\mathbb{P}_d(F) \leq 5\mathbb{P}_{5d}(F) \leq 30\eta_{5d}.$$

Adding (8.2) at d to (8.3) at $5d$ yields

$$-a_d \geq L_\Delta + L_a - (5T_\Delta + T_a)\eta_{5d}.$$

Consequently any upper bound $-a_d \leq M$ gives, in this branch,

$$\eta_{5d} \geq \frac{L_\Delta + L_a - M}{5T_\Delta + T_a}. \tag{8.5}$$

For the two negative families, respectively,

$$L_\Delta + L_a = \frac{6062284607}{12000000000}, \qquad L_\Delta + L_a = \frac{2119057757}{4000000000}.$$

Both exceed $1/2$. With the earlier bound $M = 1/2$, the two right-hand sides of (8.5) are

$$\frac{62284607}{169662810105655}, \qquad \frac{357173271}{254485212828515},$$

both exceeding $1/2800000$. The positive-family branch and the alternative $\eta_{5d} \geq k_0$ give stronger bounds. This proves the displayed threshold without B.

Upper natural density follows from a logarithmic prefix extraction. Any set of five prescribed words meets the twenty-eight words provided by every limit law. Its union therefore has mass at least $1/2800000$ in every such law. The terminal-window argument in Proposition 6.2 proves the asserted positive lower natural density of the union. It does not select the same twenty-eight words in all limit laws. □

Corollary 8.2 (Improved threshold)**.** *With $B = 196567331/402653184$, for every fixed positive integer d coprime to six one has*

$$\eta_{5d} \geq \frac{1}{891000}.$$

Proof. The four-profile estimate of this paper gives $-a_d \leq B$. Taking $M = B$ in (8.5) gives, for the two negative families,

$$\frac{13377315644977}{11119021923084206080}, \qquad \frac{32703139128881}{16677942907929559040},$$

both greater than $1/891000$. The positive-family bound $1/90000$ and the alternative $\eta_{5d} \geq 1/2000$ are stronger as well. □

Files for the common-difference calculation. The six witnesses are the files beginning `pas_bornee_17`, `pas_bornee_19`, and `pas_farkas_positive_bornee_19` in the step-five certificate directory. The command `python3 certifier_pas_fixe.py -stabilite` checks the column inequalities and their penalties; explicit filenames may be supplied to restrict the run to these six witnesses.

9 The entropy cost of a quartic bias

The entropy in this section belongs to the same extracted law as the correlation. It is a Shannon entropy rate in bits per symbol, not the logarithm of the number of allowed finite words. Let $F = X_0$ in the joint system with the profinite origin and let $H = \mathbb{E}(F \mid Z_2)$, where Z_2 is the Host–Kra factor of order two. Write $h_{\mathrm{bit}}(\mu)$ for the entropy rate of the sign process. The profinite extension has entropy zero and does not change this rate.

Theorem 9.1. *For the same logarithmic extraction and every fixed progression in the four-profile transfer,*

$$\frac{439}{250}|c_{a,q}| \le 1 - h_{\mathrm{bit}}(\mu), \qquad |c_{a,q}| \le \frac{131}{250}\int |H|^2\, d\mu.$$

Thus $h_{\mathrm{bit}}(\mu) > 311/750$ implies $|c_{a,q}| < 1/3$. No positive lower bound for the actual entropy rate is asserted here.

The scalar functions. Let

$$h_{\mathrm{b}}(p) = -p\log_2 p - (1-p)\log_2(1-p), \qquad d(t) = 1 - h_{\mathrm{b}}((1+t)/2), \qquad D(v) = d(\sqrt{v}).$$

The convergent positive-coefficient series

$$D(v) = \frac{1}{\log 2}\sum_{k\ge 1}\frac{v^k}{2k(2k-1)} \quad (0 \le v \le 1) \tag{9.1}$$

shows that D is increasing, convex, and superadditive where the sum of its arguments is at most one.

A horizontal fibre. For the four real centred horizontal profiles f_i, bounded by one, put $P = \prod_i \|f_i\|_2$ and $e = \frac{1}{4}\sum_i \|f_i\|_2^2$. Thus $P \le e^2$. Let Λ be the horizontal four-linear form, and let ν be the average of the four distributions of $|f_i|$. The homogeneous certificate gives $|\Lambda| \le KP$, where $K = 294643/500000$. The absolute certificate gives

$$|\Lambda| + \frac{(1-\sqrt{P})^2}{3} \le B.$$

In addition, the intermediate maxima G_j/s of the recurrence bound the same left-hand side whenever the initial rank is at most j. For $e \le b$, define the upper bound

$$M(b) = \min\left\{Kb^2,\ B - \frac{(1-b)^2}{3},\ \frac{G_{\lfloor Nb^2\rfloor}}{s} - \frac{(1-b)^2}{3}\right\}, \tag{9.2}$$

using the last entry only when that rank is in the computed prefix. The rank convention includes profiles on a cell boundary.

We now explain how the certificate bounds entropy, including the truncation needed between the low-energy and high-energy ranges. For $0 < r < 1$, truncate a centred f to $[-r, r]$ and

restore its zero mean by reducing the values of the sign in excess. This gives a centred g, $|g| \le r$, such that

$$\|f-g\|_1 \le 2\int (|f|-r)_+ .$$

Replace the four profiles successively and average over the twenty-four orders. If j earlier profiles have been replaced, the other factors give at most r^j in the variation of the form. Multilinearity yields

$$|\Lambda| \le Br^4 + C_r \int (t-r)_+ \, d\nu(t), \qquad C_r = 2(1+r+r^2+r^3). \tag{9.3}$$

Use the rational constants

$$r=\frac{1549}{1600}, \quad L=\frac{249}{1600}, \quad m=\frac{10239}{8000}, \quad d_0=-\frac{325852}{10^6}, \quad \beta=\frac{439}{250}, \quad \alpha=\frac{131}{250}.$$

The scalar verification proves

$$d(t) \ge d_0 + mt^2 + LC_r(t-r)_+ \qquad (0 \le t \le 1). \tag{9.4}$$

On $[0,r]$ it uses a tangent to the convex function D at $(928504027/10^9)^2$. On $[r,1]$, the function $d(t)-mt^2$ is convex, and a tangent at $988501482/10^9$ supplies the second inequality. All logarithms in these tests are bounded by the positive series for $2\,\mathrm{artanh}(x)$ with an explicit remainder.

For $e \le 0.875$, (9.1) shows that $D(e)/e^2$ decreases and that $D(0.875) > \beta K(0.875)^2$. For $e \ge 0.925$, it gives $D(e) \ge D(0.925) > \beta B$. Between these values, the verifier checks on every interval $[a,b]$ of length 10^{-6} that

$$d_0 + ma - LBr^4 \ge (\beta - L) M(b). \tag{9.5}$$

Integrating (9.4) and using (9.3) gives

$$\int d \, d\nu \ge d_0 + me + L(|\Lambda| - Br^4) \ge \beta |\Lambda|.$$

The same rational mesh, extended to 0.935, verifies $M(b) \le \alpha a$. The inequality $K(0.875) < \alpha$ covers the range below it, and $B < 0.935\alpha$ covers the range above it. We have proved on every horizontal fibre

$$\beta|\Lambda| \le \int d \, d\nu, \qquad |\Lambda| \le \alpha e. \tag{9.6}$$

The vertical contribution. Only an elementary bound is needed for the constants in the theorem. On a connected central fibre, reality pairs the nonzero Fourier coefficients at χ and $-\chi$. If its variance is v, then

$$\sum_{\chi \ne 0} |\widehat{f}(\chi)|^4 \le \frac{v^2}{2}.$$

Indeed the paired squared magnitudes have total v; concentrating them in one pair maximizes their sum of squares. Write G_i for the four functions before vertical projection, f_i for their fibre means, and $v_i(x) = \int |G_i|^2 - |f_i(x)|^2$ for their fibre variances. Hölder in the four coupled frequencies and Cauchy–Schwarz on the pairwise independent base coordinates bound the vertical form by

$$\frac{1}{2}\prod_i \Big(\int v_i\Big)^{1/2} \le \frac{a^2}{2}, \qquad a = \frac{1}{4}\sum_i \int v_i \in [0,1].$$

Conditional Jensen and superadditivity of D give, on each fibre, $\int d(G_i) - d(f_i) \geq D(v_i)$. Jensen once more therefore bounds the average of these four entropy deficits from below by $D(a)$. The first three terms of (9.1) suffice:

$$\frac{D(a)}{a^2} \geq \frac{1/(2a) + 1/12 + a/30}{\log 2} \geq \frac{37}{60 \log 2} > \frac{37}{42} > \frac{439}{500} = \frac{\beta}{2}.$$

Here the middle expression decreases for $0 < a \leq 1$, and $\log 2 < 7/10$. Thus the entropy cost covers β times the vertical form. Also $a^2/2 \leq a/2 < \alpha a$. This argument does not need a separate absolute bound of one third for the vertical part.

9.1 The original law, its clock, and its structured projection

For the entropy assertion we work with the original arithmetic law, not with the auxiliary law of Lemma 3.1. The profinite coordinate and the division maps belong to the established arithmetic Furstenberg framework of Tao [27, Proposition 1] and Sawin [25, Definition 1.1]. We give the projection and conditioning argument required here, including the joint independence assertion for the sign and its structured projections.

Along the chosen logarithmic extraction, record both $(\lambda(n+j))_{j\in\mathbb{Z}}$ and $n \in \widehat{\mathbb{Z}}$. Denote the resulting system by (Ω, μ, T), its clock by z, and its sign at zero by F. Thus $z \circ T = z + 1$, $F \in \{-1, 1\}$, and the clock has Haar law. We take a common extraction whenever an additional countable collection of correlations is needed. Put

$$H = \mathbb{E}_\mu(F \mid Z_2(T)).$$

Lemma 9.2 (Powers, restrictions, and factor maps)**.** *For an invertible probability-preserving transformation, possibly nonergodic, and integers $d \geq 1$, $s \geq 1$,*

$$Z_s(T^d) = Z_s(T).$$

The conditional expectations onto these factors commute with restriction to an invariant set of positive measure and with pullback by a factor map.

Proof. Write $\| \cdot \|_{k,T}$ for the Gowers–Host–Kra seminorm of order k. The comparison of seminorms under powers [6, Lemma 2.1], applied with all k directions equal to T, gives for $k \geq 2$

$$\|f\|_{k,T} \leq \|f\|_{k,T^d} \leq d^{k/2^k} \|f\|_{k,T}.$$

The two seminorms therefore have the same null space. The characterization $\mathbb{E}(f \mid Z_s(T)) = 0$ if and only if $\|f\|_{s+1,T} = 0$ proves the first assertion. This argument does not assert equality of $\mathcal{I}_T$ and $\mathcal{I}_{T^d}$.

The cube measures, and hence their null spaces, decompose over invariant sets; this proves the restriction assertion. If π is a factor map, the iterative-average formula for the seminorms gives $\|f \circ \pi\|_k = \|f\|_k$. Moreover the pullbacks of the cube dual functions are the cube dual functions of the pullbacks. Since the dual functions generate Z_s, the pullback of Z_s is contained in the corresponding factor upstairs. Decompose f into its Z_s projection and a function of zero seminorm. The two preceding observations show that the pullback of this decomposition is precisely its orthogonal decomposition upstairs. This proves the factor-map assertion. □

Lemma 9.3 (Division and independence of one site from the clock)**.** *The joint law of (F, H) is independent of z and invariant under $(F, H) \mapsto (-F, -H)$. The same assertion holds with any finite list of the projections $\mathbb{E}(F \mid Z_s(T))$, $s \geq 1$.*

Proof. For $d \geq 1$, let $Y_d = \{z \in d\widehat{\mathbb{Z}}\}$. The normalized restriction $\mu_d = d\mathbf{1}_{Y_d}\mu$ is T^d-invariant, and

$$\pi_d((x_j)_{j\in\mathbb{Z}}, z) = ((\lambda(d)x_{dj})_{j\in\mathbb{Z}}, z/d)$$

is a factor map from (Y_d, μ_d, T^d) to (Ω, μ, T). Indeed this follows on every arithmetic cylinder from complete multiplicativity and the substitution $n = dm$. Replacing the original window by its dilation by $1/d$ changes a bounded harmonic sum by $O_d(1)$ before normalization. The normalized difference thus tends to zero. The finitely many nonpositive arguments of any fixed cylinder also have negligible normalized weight.

Since $F \circ \pi_d = \lambda(d)F$ on Y_d, Lemma 9.2 gives

$$H \circ \pi_d = \lambda(d)H \quad \text{on } Y_d. \tag{9.7}$$

It gives the same identity for every higher structured projection.

Here is a direct argument which turns these division identities into independence. Let φ be a bounded function of the listed one-site variables, of parity $\sigma \in \{1, -1\}$ under simultaneous sign reversal, and set $M(z) = \mathbb{E}(\varphi \mid z)$. Disintegration of (9.7) implies, for every prime p,

$$M(pu) = \sigma M(u)$$

for Haar-almost every u. For a real function g modulo an integer W, the functions

$$e_p(z) = \frac{p\mathbf{1}_{p|z} - 1}{\sqrt{p-1}} \qquad (p \nmid W)$$

form an orthonormal family. Bessel's inequality applied to Mg gives

$$\sum_{p\nmid W} \frac{|\sigma \int M(u)g(pu)\,du - \int M(u)g(u)\,du|^2}{p-1} < \infty.$$

For each invertible residue class of p modulo W, the numerator is constant. The divergence of the sum of reciprocals of primes in that class forces this numerator to vanish.

If $\sigma = -1$, the class $p \equiv 1 \pmod W$ gives $\int Mg = 0$ for every periodic g, hence $M = 0$. If $\sigma = 1$, the conditional expectation of M modulo W is invariant under multiplication by every unit modulo W. It depends only on $\gcd(z, W)$. The division identity for a general d, applied to this even φ, also gives

$$\int_{d|z} M(z)\,dz = \frac{1}{d}\int M(z)\,dz.$$

For all $d \mid W$ these equations determine, by inversion on the divisor lattice, the integral on each class of a fixed greatest common divisor. They are the Haar masses multiplied by $\int M$. The conditional expectation modulo W is therefore constant. Letting W run through the factorials proves $M = \int M$. Decomposing a general φ into its even and odd parts proves both assertions of the lemma. □

Lemma 9.4 (Centring on all finite cyclic components)**.** *For every $d \geq 1$,*

$$\mathbb{E}(F \mid \mathcal{I}_{T^d}) = \mathbb{E}(H \mid \mathcal{I}_{T^d}) = 0.$$

Consequently every finite nilsystem approximation of H is centred on each of its connected components, as are its fixed translates on the corresponding components.

Proof. The logarithmic two-point theorem gives $\int F\,(F \circ T^{dj})\,d\mu = 0$ for every nonzero j. Thus

$$\left\| \frac{1}{N}\sum_{j=1}^{N} F \circ T^{dj} \right\|_2^2 = \frac{1}{N}.$$

The mean ergodic theorem proves the first equality. By Lemma 9.2, $\mathcal{I}_{T^d} \subset Z_1(T^d) = Z_1(T) \subset Z_2(T)$; the tower property proves the second. These equalities hold on almost every ergodic component, simultaneously for the countably many d. In an ergodic nilsystem with q connected components, each component is invariant under T^q. Its preimage in the original system therefore has zero integral of F, and hence of every relevant conditional expectation of F. Translation permutes the components and preserves this centring. □

Lemma 9.5 (Characteristic-factor transfer inside the original law)**.** *Let $c_{Q,r}$ be the extracted logarithmic correlation of $C(n)$ on $n \equiv r \pmod{Q}$, with $0 \le r < Q$. Set $b = 3Q$ and $s = 3r$. For every $\varepsilon > 0$ there is a positive multiple W of b such that*

$$|c_{Q,r} - A_W(H)| \le \varepsilon,$$

where

$$A_W(H) = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} b \int_{z\equiv s \ (b)} \prod_{i=0}^{3} H \circ T^{i(1+Wm)} \, d\mu.$$

Proof. The identity $C(n) = \prod_{i=0}^{3} \lambda(3n+i)$ gives

$$f(D) = \lim_j \mathbb{E}^{\log}_{n\in I_j} \prod_{i=0}^{3} \lambda(bn + D(s+i)), \qquad f(1) = c_{Q,r}.$$

The affine-coefficient theorem of Tao–Teräväinen [30, Theorem A.1(i)] makes f a uniform limit of periodic functions. Choose a common approximate period W, enlarged to a multiple of b, so that $|f(1+Wm) - f(1)| \le \varepsilon$ for all $m \ge 1$. A fixed change of the summation variable gives

$$f(1+Wm) = b \int_{z\equiv s \ (b)} \prod_{i=0}^{3} F \circ T^{i(1+Wm)} \, d\mu.$$

The four-term characteristic-factor theorem applies to these Cesàro averages for T^W. Its factor is $Z_2(T^W) = Z_2(T)$ by Lemma 9.2. The clock mask belongs to Z_1 and is T^W-invariant. It therefore does not obstruct replacement of the three moving functions by their Z_2 projections. To replace the zeroth function too, write the resulting error with the mask times $F - H$ and translate the whole integral by T^{-Wm}. That function then occupies the moving position $-m$, orthogonal to $Z_2(T^W)$; the same characteristic-factor theorem eliminates the error. Fixed translations T^i are absorbed in the four observables. Cesàro averaging now proves the assertion. □

Proposition 9.6 (Transfer of the fibre costs)**.** *Suppose the horizontal and vertical estimates proved above give, for four real centred functions G_i on any totally ergodic two-step nilsystem, with horizontal projections g_i,*

$$\beta|b_h| \le \frac{1}{4}\sum_i \int d(g_i), \qquad \beta|b_v| \le \frac{1}{4}\sum_i \int \big(d(G_i) - d(g_i)\big),$$
$$|b_h| \le \frac{\alpha}{4}\sum_i \int g_i^2, \qquad |b_v| \le \frac{1}{8}\sum_i \int (G_i^2 - g_i^2),$$

where $d(t) = 1 - h_{\mathrm{b}}((1+t)/2)$ and $\alpha \ge 1/2$. Then, in the original logarithmic law,

$$\beta|c_{Q,r}| \le \int d(H)\, d\mu, \qquad |c_{Q,r}| \le \alpha \int H^2 \, d\mu.$$

Proof. Adding the two costs and using the triangle inequality gives on every such connected nilsystem

$$\beta|b_h + b_v| \le \frac{1}{4}\sum_i \int d(G_i), \qquad |b_h + b_v| \le \frac{\alpha}{4}\sum_i \int G_i^2. \tag{9.8}$$

We explain carefully why these estimates also bound $A_W(H)$.

First decompose the original system into ergodic components. On each component approximate H in L^1 by its conditional expectation on a finite two-step nilsystem factor. Include the finite clock modulo b in this factor; adjoining a finite cyclic rotation preserves the class of nilsystem factors. Denote the approximating observable by G. All errors in a product of four bounded functions are at most the sum of the four L^1 errors, so it is enough to prove an estimate uniform over these finite factors and then pass to the inverse limit.

Suppose the finite factor has q connected components. Its translation cycles them, and its qth power is totally ergodic on each component. Split the Cesàro variable m into its classes u modulo q and write

$$1 + Wm = j_0 + Wq\ell, \qquad j_0 = 1 + Wu.$$

On a chosen connected component take the four functions $G_i = G \circ T^{ij_0}$ and the transformation T^{qW}. They are centred by Lemma 9.4, so (9.8) applies. This decomposition is performed separately on every finite factor; it does not require a single W to be divisible by all of their component counts.

Average the bounds over the initial components and over u. Because W is divisible by b, translation by ij_0 sends the clock class s to $s + i$. For the given ergodic component ω the result is

$$\beta|A_W(G;\mu_\omega)| \le \frac{b}{4}\sum_{i=0}^{3} \int_{z\equiv s+i \ (b)} d(G)\, d\mu_\omega,$$

and the analogous bound with β, d replaced by $1, \alpha t^2$. Conditional Jensen, using the inclusion of the finite clock in the factor, bounds this right-hand side by the same expression with H. Pass to the inverse limit on the ergodic component, then integrate over ω. The uniform boundedness permits dominated convergence. Lemma 9.3 finally gives

$$\beta|A_W(H)| \le \int d(H)\, d\mu, \qquad |A_W(H)| \le \alpha \int H^2\, d\mu.$$

Use Lemma 9.5 and let ε tend to zero. □

The factor Z_2 has entropy zero, by its inverse-limit nilsystem description on the ergodic components and the entropy decomposition formula. It contains the clock. The sign partition together with the clock generates the original system, so entropy addition and the one-symbol conditional-entropy bound give

$$h_{\mathrm{bit}}(\mu) \le \int h_{\mathrm{b}}((1 + H)/2)\, d\mu = 1 - \int d(H)\, d\mu.$$

Adjoining the zero-entropy clock does not change the entropy rate of the sign process. Taking $\beta = 439/250$ and $\alpha = 131/250$ in Proposition 9.6 therefore proves the two entropy and energy assertions for the original arithmetic law.

The numerical threshold in Theorem 9.1 follows from $1 - (439/250)/3 = 311/750$.

Reproduction of the scalar inequalities. The script `certifier_constantes.py` checks the tangents, series remainders and all 60000 closed intervals in exact rational arithmetic. It accepts either archived rank profile, of size 1536 or 3072; both yield the same two coefficients. The minimum truncation margin in the latter check is greater than $38942/10^8$. This scalar check presupposes the recurrence bounds for its input profile; regeneration of that profile is part of the geometric certificate, as specified in the reproducibility appendix.

10 Ordinary averages

The harmonic bound B transfers to ordinary prefixes outside an exceptional set. For the special quartic C a separate dyadic argument gives a bound strictly below $2/3$ on all prefixes. We give both arguments.

10.1 The bound B at almost all scales

An exceptional set $E \subset \mathbb{N}$ has logarithmic density zero if $\sum_{n \le X,\, n \in E} n^{-1} = o(\log X)$.

Lemma 10.1 (A transfer under fixed-ratio isotopy)**.** *Let $a\colon \mathbb{N} \to \mathbb{C}$ satisfy $|a(n)| \le 1$, and put*

$$A(x) = \frac{1}{x} \sum_{n \le x} a(n) \qquad (x \ge 1).$$

Suppose that $E \subset \mathbb{N}$ and that, for every fixed $r \in \mathbb{Q}_{>0}$,

$$A(N/r) - A(N) \longrightarrow 0 \quad \text{as } N \to \infty,\ N \notin E. \tag{10.1}$$

Suppose also that $B \ge 0$ and that every sequence of windows $1 \le U_j < V_j$ with $\log(V_j/U_j) \to \infty$ satisfies

$$\limsup_{j \to \infty} \left| \frac{\sum_{U_j < n \le V_j} a(n)/n}{\sum_{U_j < n \le V_j} 1/n} \right| \le B. \tag{10.2}$$

Then

$$\limsup_{N \to \infty,\, N \notin E} |A(N)| \le B.$$

In particular, if E has logarithmic density zero, the same bound holds at almost all ordinary scales in that precise sense.

Proof. For $1 \le u \le v$, boundedness of a gives

$$|A(v) - A(u)| \le 2\frac{v-u}{v} + \frac{1}{v} \le 2\log(v/u) + \frac{1}{v}.$$

Consequently, for every fixed $R > 1$, a finite rational net in $[1, R]$ and (10.1) imply

$$\sup_{1 \le r \le R} |A(N/r) - A(N)| \longrightarrow 0 \quad (N \to \infty,\ N \notin E). \tag{10.3}$$

Indeed, the displayed bound makes these functions equicontinuous in $\log r$, up to an error $O_R(N^{-1})$, and the isotopy holds at every point of the finite net.

If the conclusion fails, choose $\varepsilon > 0$ and an increasing sequence $N_j \notin E$ with $|A(N_j)| \ge B + \varepsilon$. By (10.3) and a diagonal choice, there are $R_j \to \infty$, with $R_j \le \sqrt{N_j}$, for which

$$\delta_j := \sup_{1 \le r \le R_j} |A(N_j/r) - A(N_j)| \longrightarrow 0.$$

Set $U_j = N_j/R_j$ and $V_j = N_j$. Partial summation gives the exact identity

$$\sum_{U_j < n \le V_j} \frac{a(n)}{n} = A(V_j) - A(U_j) + \int_{U_j}^{V_j} \frac{A(t)}{t}\, dt.$$

The integral is $A(N_j)\log R_j + O(\delta_j \log R_j)$; the boundary term is bounded by 2. Since

$$\sum_{U_j < n \le V_j} \frac{1}{n} = \log R_j + O(U_j^{-1}),$$

the normalized harmonic average on this window equals $A(N_j) + o(1)$. This contradicts (10.2). □

Proposition 10.2 (Isotopy for even affine Liouville products)**.** *Fix an even positive integer k, positive integers $a_1, \dots, a_k$, and integers $b_1, \dots, b_k$. Extend the finitely many terms for which an argument is nonpositive arbitrarily, and let*

$$a(n) = \prod_{i=1}^{k} \lambda(a_i n + b_i), \qquad A(x) = \frac{1}{x} \sum_{n \le x} a(n).$$

There is a set $E \subset \mathbb{N}$ of logarithmic density zero such that (10.1) *holds for every fixed $r \in \mathbb{Q}_{>0}$. The forms, and hence the exceptional set, are fixed before taking the limit.*

Proof. Put

$$L = \operatorname{lcm}(a_1, \dots, a_k), \qquad c_i = L/a_i, \qquad H_i = c_i b_i, \qquad \kappa = \prod_{i=1}^{k} \lambda(c_i).$$

Complete multiplicativity gives

$$a(n) = \kappa \prod_{i=1}^{k} \lambda(Ln + H_i)$$

apart from the irrelevant initial terms. For each Dirichlet character χ modulo L, define

$$A_\chi(y) = \frac{1}{y} \sum_{m \le y} \chi(m+1) \prod_{i=1}^{k} \lambda(m + H_i).$$

Character orthogonality gives, including nonunits,

$$\mathbf{1}_{L \mid m} = \frac{1}{\varphi(L)} \sum_{\chi \bmod L} \chi(m+1).$$

Thus for real $x \to \infty$,

$$A(x) = \frac{\kappa L}{\varphi(L)} \sum_{\chi \bmod L} A_\chi(Lx) + O_{\boldsymbol{a},\boldsymbol{b}}(x^{-1}). \tag{10.4}$$

The $k+1$ multiplicative functions in A_χ are k copies of λ and one copy of χ; their product is exactly χ because k is even. Tao–Teräväinen's archimedean isotopy theorem [29, Theorem 1.19(i)] therefore applies with twist parameter $t = 0$. For every such χ, it supplies a set $E_\chi \subset \mathbb{N}$ of logarithmic density zero such that

$$A_\chi(M/r) - A_\chi(M) \longrightarrow 0 \quad (M \to \infty,\ M \notin E_\chi)$$

for every fixed positive rational r. That theorem permits coincident shifts, so the extra shift 1 causes no restriction.

Let $E = \{N \in \mathbb{N} : LN \in \bigcup_\chi E_\chi\}$. This set has logarithmic density zero, since for each χ

$$\sum_{\substack{N \le X \\ LN \in E_\chi}} \frac{1}{N} \le L \sum_{\substack{M \le LX \\ M \in E_\chi}} \frac{1}{M} = o(\log X).$$

The finite decomposition (10.4) now proves (10.1). □

Corollary 10.3 (Ordinary averages on each fixed progression)**.** *Fix $q \geq 1$ and $a \in \mathbb{Z}$. Set*

$$C(n) = \lambda(n)\lambda(n+1)\lambda(3n+1)\lambda(3n+2).$$

With B as in Theorem 1.1, there is an exceptional set $E_{a,q} \subset \mathbb{N}$ of logarithmic density zero such that

$$\limsup_{N\to\infty,\, N\notin E_{a,q}} \left| \frac{1}{N} \sum_{n\leq N} C(qn+a) \right| \leq B.$$

The same conclusion holds for a fixed even affine Liouville product in place of C, provided its harmonic averages on all such windows satisfy the analogous bound B.

Proof. Apply Proposition 10.2 to the four forms $qn+a$, $qn+a+1$, $3qn+3a+1$, and $3qn+3a+2$, and then apply Lemma 10.1. The harmonic hypothesis is supplied by Theorem 1.1, with the change of normalization described below. □

Remark 10.4 (Conditional progression normalization)*.* If the harmonic input is stated as a conditional average over $m \equiv a \pmod q$, the substitution $m = qn + a$ gives the hypothesis of Corollary 10.3. For fixed a, q, replacing $(qn + a)^{-1}$ by $(qn)^{-1}$ changes the relevant weighted sums by $O_{a,q}(1)$, and the same is true of the denominators and fixed endpoint adjustments. Division by the diverging harmonic mass makes these errors vanish. The ordinary conclusion is equivalently expressible using the number of terms of that progression up to a physical cutoff; the change of scale is fixed and affine and preserves logarithmic density-zero exceptional sets. The parameters are fixed throughout. This argument gives no bound of this strength on the exceptional scales.

10.2 A strict saving below two thirds on all prefixes

Theorem 10.5. *There is a constant $\kappa > 0$ such that the quartic C in* (1.1) *satisfies*

$$\limsup_{X\to\infty} \left| \frac{1}{X} \sum_{n\leq X} C(n) \right| \leq \frac{2}{3} - \kappa.$$

The sum runs over all positive integers $n \leq X$. The argument gives no numerical value for κ.

Proof. Write $S(X) = \sum_{n\leq X} C(n)$ and $R(n) = \lambda(6n+1)\lambda(6n+5)$. The ordinary two-point theorem [20, Theorem 1.1], applied to the two fixed forms of determinant 24, gives

$$\sum_{n\leq x} R(n) = O(x(\log x)^{-\eta}) \qquad (x \geq 3) \tag{10.5}$$

for some $\eta > 0$. Fix a sign s. Let $H_s(n)$ indicate that the majority of $C(n), C(2n), C(2n+1)$ equals $-s$. Take $J_X = (X/4, X/2]$ as the set of parents. Their children $2n, 2n+1$ cover $(X/2, X]$ up to $O(1)$ endpoints. With $|J_X|$ denoting its integer count, the cocycle (2.2) and the majority identity (2.3) give

$$s\{S(X) - S(X/4)\} = 2|J_X| + s \sum_{n\in J_X} R(n) - 4 \sum_{n\in J_X} H_s(n) + O(1). \tag{10.6}$$

If $L_s = \limsup_{X\to\infty} sS(X)/X$, division by X yields $L_s \leq \frac{1}{2} + \frac{1}{4}L_s$, hence $L_s \leq 2/3$. In fact, summing (10.6) at the scales $X/4^k$ gives $|S(X)|/X \leq 2/3 + O((\log X)^{-\eta})$. Above $\sqrt{X}$ the errors from (10.5) sum geometrically, and the total length of the smaller shells is $O(\sqrt{X})$.

Suppose $sS(X_j)/X_j \to 2/3$. Equality in the limiting upper estimate in (10.6) forces both

$$\frac{sS(X_j/4)}{X_j/4} \longrightarrow \frac{2}{3}, \qquad \sum_{n \in J_{X_j}} H_s(n) = o(X_j).$$

Induction gives the same two conclusions with X_j replaced by $X_j/4^k$ for every fixed k. Fix such a k, put $x = X_j/4^k$, and let U, V be the normalized harmonic means of sC on $(x/4, x/2]$ and $(x/2, x]$ respectively. Each interval has harmonic mass $\log 2 + o(1)$. Summing the pointwise majority identity on the first interval with weight $1/n$ gives

$$U + 2V = 2 + o(1).$$

Indeed the weighted defects have mass $o(1)$ since $1/n \le 4/x$ there. The residual term is $o(1)$ by partial summation in (10.5). For each pair of children the parent weight differs from twice the corresponding child weight by $O(n^{-2})$, and the endpoint discrepancies have weight $O(x^{-1})$. Since $V \le 1$, the harmonic mean over the two intervals combined is at least $1/2 - o(1)$.

For each fixed positive integer K, sum this conclusion over $k = 0, \dots, K-1$ to obtain the same lower bound on $(X_j/4^K, X_j]$ as $j \to \infty$. Choose $j = j(K)$ so far along the sequence that the mean is at least $1/2 - 1/K$ and $X_j/4^K \ge K$. These windows have harmonic mass tending to infinity, whereas Theorem 1.1 bounds the limiting upper mean of sC by $B < 1/2$. This contradiction excludes $L_s = 2/3$ for each of the two signs. Taking $\kappa = 2/3 - \max(L_+, L_-)$, where $L_\pm = L_{\pm 1}$, proves the claim. □

Proposition 10.6 (Four equally spaced signs on every prefix)**.** *Let $Y_h(m) = \prod_{i=0}^{3} \lambda(m + ih)$ for fixed $h \ge 1$. For $q \ge 1$ dividing h and every fixed residue $a \bmod q$,*

$$\limsup_{X \to \infty} \left| \frac{\sum_{1 \le m \le X,\, m \equiv a \ (q)} Y_h(m)}{\#\{1 \le m \le X : m \equiv a \pmod q\}} \right| \le \frac{1}{2}.$$

In particular this holds for unrestricted origins ($q = 1$).

Proof. The identity $Y_h(m)Y_h(m+h) = \lambda(m)\lambda(m+4h)$ leaves a binary product. Its ordinary mean on the class tends to zero by [20, Theorem 1.1] after writing $m = qn + a$. Translation by h preserves the class and changes only $O_h(1)$ terms at the endpoints. Thus twice the absolute mean of Y_h is at most the mean of $|Y_h(m) + Y_h(m+h)|$, up to $o(1)$. For signs u, v one has $|u + v| = 1 + uv$, which gives the bound. The quartic $C(n) = Y_1(3n)$ samples a class that translation by 1 does not preserve. □

11 Cancellation in additive parallelograms of the Möbius pair

Put

$$A(n) = \lambda(n)\lambda(n+1), \qquad F(n) = \mu(n)\mu(n+1), \qquad P(n) = |F(n)|.$$

Thus $F = AP$. For a real sequence f with $|f| \le 1$, define

$$\mathcal{E}_{H,I}(f) = \frac{1}{H^3} \sum_{\substack{1 \le j_1, j_2, j_3, j_4 \le H \\ j_1 + j_2 = j_3 + j_4}} \mathbb{E}^{\log}_{x \in I} \prod_{i=1}^{4} f(x + j_i). \tag{11.1}$$

The quadruples are ordered. The logarithmic window limit below always precedes the limit in the block length.

Theorem 11.1. *Let*

$$D_0 = \frac{2}{3} \prod_{p \text{ prime}} \left(1 - \frac{8}{p^2} + \frac{24}{p^4} - \frac{26}{p^6}\right) > 0.$$

Then

$$\lim_{H\to\infty} \lim_{\log(V/U)\to\infty} \mathcal{E}_{H,(U,V]}(P) = D_0, \tag{11.2}$$

$$\limsup_{H\to\infty} \limsup_{\log(V/U)\to\infty} \mathcal{E}_{H,(U,V]}(F) \le \frac{5}{9} D_0. \tag{11.3}$$

Consequently the lower limiting proportion of negative products, among the quadruples in (11.1) *with nonzero product, is at least* $2/9$*. The same statement holds after excluding repeated indices.*

The argument uses the following fixed-side estimate, proved in Appendix C. For each extracted logarithmic law and each fixed residue class of the origin,

$$\limsup_{k\to\infty} \left|\mathbb{E}\big(A(0)A(k) \mid z \equiv b \pmod{Q}\big)\right| \le \frac{1}{3}. \tag{11.4}$$

The threshold may depend on the law, Q, and b. Uniformity in these parameters is not needed.

11.1 A conditional second moment on the profinite factor

The profinite model of the squarefree mask and its Euler-product correlations are classical. We use the framework of Cellarosi–Sinai [2, Main Theorem and Section 3, equation (15)], where the correlation formula is attributed to Mirsky, and spell out its specialization to the adjacent-pair mask. The fixed-side estimate and the factor $5/9$ require the separate analytic argument below.

Record the translates of λ, A, F, P in a joint compact shift space, and adjoin their profinite coordinate $z \in \widehat{\mathbb{Z}}$, with the shift acting by $z \mapsto z + 1$. Every logarithmic limit gives this coordinate Haar measure. Moreover,

$$P(z) = \prod_p \mathbf{1}_{z \not\equiv 0,-1 \pmod{p^2}}, \qquad F = AP(z) \tag{11.5}$$

almost surely. To justify this identity, truncate the product at $p \le R$. Counting multiples of p^2 up to X gives a tail bound $2X \sum_{p>R} p^{-2} + O(\sqrt{X})$; partial summation therefore bounds the limiting logarithmic error by $2\sum_{p>R} p^{-2}$. In particular, (11.5) also survives weak limits of logarithmic laws.

Let $\nu = \int \nu_t \, dt$ be the ergodic decomposition of such a stationary law. Each ν_t projects to Haar measure on $\widehat{\mathbb{Z}}$, because translation by 1 on this group is uniquely ergodic. Set

$$m_t(z) = \mathbb{E}_{\nu_t}(A \mid z), \qquad m_{t,Q}(b) = \mathbb{E}_{\nu_t}(A \mid z \equiv b \pmod{Q}).$$

On the class $b \pmod{Q}$, translation by Q preserves the conditional measure. Jensen's inequality, followed by integration in t, gives

$$\int |m_{t,Q}(b)|^2 \, dt \le \frac{1}{J^2} \sum_{i,j=1}^{J} \mathbb{E}_\nu(A(iQ)A(jQ) \mid z \equiv b \pmod{Q}).$$

By (11.4), the terms with $|i - j|$ sufficiently large are at most $1/3 + \varepsilon$. The remaining terms contribute $O_{\varepsilon,Q,b,\nu}(J^{-1})$. First let $J \to \infty$, then let $\varepsilon \downarrow 0$. Finally take Q through the factorials. Martingale convergence in $L^2(dt\,dz)$, followed by a subsequence in $L^1(dz)$ for the squared norms, yields

$$\int |m_t(z)|^2 \, dt \le \frac{1}{3} \quad \text{for Haar-almost every } z. \tag{11.6}$$

11.2 Fourth moments on connected fibres

For a compact abelian group K and a real bounded function v, write

$$\mathcal{S}_K(v) = \sum_{\chi \in \widehat{K}} |\widehat{v}(\chi)|^4 = \int_{K^3} v(x_1)v(x_2)v(x_3)v(x_1+x_2-x_3)\,dx_1dx_2dx_3.$$

Adding the zero-frequency term to Lemma 4.1 shows that, on a connected group and for $m = \int v$, $|v| \le 1$,

$$\mathcal{S}_K(v) \le \frac{1 - 6m^2 + 8|m|^3}{3} \le \frac{1+2m^2}{3}. \tag{11.7}$$

For the trivial group the same inequalities follow directly from $\mathcal{S}_K(v) = m^4$.

The Kronecker factor K_t of an ergodic component ν_t is a compact abelian group rotation and maps onto $\widehat{\mathbb{Z}}$. Its kernel N_t is connected. Indeed, its character group is $\Gamma_t/(\mathbb{Q}/\mathbb{Z})$, where $\Gamma_t \subset \mathbb{R}/\mathbb{Z}$ is the eigenvalue group, and this quotient is torsion-free. Put $a_t = \mathbb{E}_{\nu_t}(A \mid K_t)$. On the fibre above z, this is a real function bounded by 1 with mean $m_t(z)$.

Fix z_1, z_2, z_3 and put $z_4 = z_1 + z_2 - z_3$. The parallelogram integral over the corresponding four fibres is, after choosing representatives, an integral on N_t^3, possibly with a translation in its last factor. Fourier expansion and Hölder bound its absolute value by

$$\prod_{i=1}^{4} \mathcal{S}_{N_t}(a_t|_{z_i})^{1/4} \le \prod_{i=1}^{4} \left(\frac{1+2m_t(z_i)^2}{3}\right)^{1/4}.$$

Hölder in t and (11.6) bound the integral of this expression by $5/9$, for almost every triple (z_1, z_2, z_3). Multiplication by the nonnegative weight $\prod_i P(z_i)$ and integration give

$$\int \mathcal{S}_{K_t}(\mathbb{E}_{\nu_t}(F \mid K_t))\,dt \le \frac{5}{9}\mathcal{S}_{\widehat{\mathbb{Z}}}(P). \tag{11.8}$$

Only the positivity of the profinite weight was used in this step.

11.3 A uniform comparison for block energies

For a stationary law η, let $\mathcal{E}_H(\eta; f)$ denote the expression in (11.1) with expectation under η. Write $\mathcal{A}_L v = L^{-1}\sum_{j=1}^{L} v \circ T^j$ and $g_h = f(f \circ T^h)$. Stationarity gives the exact identity

$$\mathcal{E}_H(\eta; f) = H^{-3} \sum_{|h|<H} (H - |h|)^2 \|\mathcal{A}_{H-|h|} g_h\|^2_{L^2(\eta)}.$$

For $\|v\|_2 \le 1$, comparison of $\mathcal{A}_L v$ with $\mathcal{A}_L\mathcal{A}_K v$ gives $\|\mathcal{A}_L v\|_2^2 \le \|\mathcal{A}_K v\|_2^2 + O(K/L)$. Use this inequality once, interchange the two differences in the parallelogram, and then use it again. The elementary identity

$$(H-s)_+^2 = 2\sum_{R=1}^{H-1}(R-s)_+ + (H-s)_+ \qquad (s \in \mathbb{Z}_{\ge 0})$$

shows that, after the first comparison, the coefficient of $(K-|k|)/K^2$ is

$$\frac{2}{H^3}\sum_{R=1}^{H-1} R^2 \|\mathcal{A}_R g_k\|_2^2 + \frac{1}{H}\|\mathcal{A}_H g_k\|_2^2.$$

The weights in this expression sum to $a_H = (2 + H^{-2})/3$. The second comparison costs $O(M/H)$. Thus, uniformly in the stationary law,

$$\mathcal{E}_H(\eta; f) \le a_H \sum_{|k|<K} \frac{K - |k|}{K^2} \|\mathcal{A}_M g_k\|^2_{L^2(\eta)} + O\left(\frac{K + M}{H}\right), \qquad a_H = \frac{2 + H^{-2}}{3}. \tag{11.9}$$

For each H, choose a logarithmic law ν_H realizing the inner upper limit in (11.3); an arbitrarily small error would also suffice. Along a subsequence realizing the outer upper limit, let ν_H converge weakly to ν. A diagonal selection of windows shows that ν is again a logarithmic law. Keep K, M fixed in (11.9) and pass to this limit. The mean ergodic theorem then allows $M \to \infty$. Finally the Fejér average in k and the spectral theorem give

$$\limsup_{H\to\infty} \mathcal{E}_H(\nu_H; F) \le \frac{2}{3} \int \mathcal{S}_{K_t}(\mathbb{E}_{\nu_t}(F \mid K_t))\, dt. \tag{11.10}$$

For completeness, the intermediate integrand is $|\langle F, F \circ T^k \rangle_{\nu_t}|^2$. Its Fejér average tends to the sum of the squares of the atomic masses of the spectral measure of F; on an ergodic component these are exactly the fourth powers of its Kronecker Fourier coefficients. Bounded convergence justifies the integration over t. This argument is the reason that a bound for each fixed law suffices even though the law ν_H can vary with H.

Combining (11.8) and (11.10) proves the signed estimate once the positive constant is evaluated.

11.4 The positive constant and the sign proportion

For a periodic mask, equidistribution of the three free indices in $1 \le j_1, j_2, j_3, j_1 + j_2 - j_3 \le H$ gives $\lim_H \mathcal{E}_H(P) = \frac{2}{3}\mathcal{S}(P)$. The factor $2/3$ is the volume of this region after scaling by H. Periodic approximation using (11.5) extends this identity to P. The Chinese remainder theorem factorizes $\mathcal{S}(P)$ into its prime-square factors, in the same way as the classical squarefree correlations [2, Section 3]. For $q = p^2$, the local mask is $u_q = \mathbf{1}_{\mathbb{Z}/q\mathbb{Z}\setminus\{0,-1\}}$. Its zero Fourier coefficient is $1 - 2/q$, and its nonzero coefficients have modulus $|1 + e^{2\pi i r/q}|/q$. Since $q \ge 4$,

$$\mathcal{S}(u_q) = \left(1 - \frac{2}{q}\right)^4 + \frac{6q - 16}{q^4} = 1 - \frac{8}{q} + \frac{24}{q^2} - \frac{26}{q^3}.$$

These factors are positive and are $1 + O(p^{-2})$, so their product converges to a positive number. This proves (11.2) and the expression for D_0.

On a nonzero quadruple the signed product equals 1 or -1. Its negative proportion is therefore exactly

$$\frac{1}{2}\left(1 - \frac{\mathcal{E}_{H,I}(F)}{\mathcal{E}_{H,I}(P)}\right).$$

Equations (11.2)– (11.3) give the lower bound $2/9$ in the stated order of limits. Quadruples with a repeated index number $O(H^2)$, so deleting them has no effect on the limiting result.

Remark 11.2. The same proof applies to $F = AW(z)$ and its unsigned weight $W(z)$, where $0 \le W \le 1$ is approximable by periodic masks in logarithmic mean. The positive constant becomes $\frac{2}{3}\mathcal{S}_{\widehat{\mathbb{Z}}}(W)$ and the relative upper bound remains $5/9$. In particular it applies to any fixed pair of square parts with positive support. This is a statement about averaged block parallelograms; it does not estimate a fixed multiplicative dilation correlation such as $\mathbb{E}^{\log} A(n)A(3n + 1)$.

12 What the estimate leaves open

The improvement below one half excludes an endpoint that obstructs the five-letter count. It does not determine a quartic limit, and the proof does not supply cancellation tending to zero. A further improvement of the finite circle certificate would strengthen the numerical bounds derived from it, but would not by itself establish the conjectural independence of Liouville signs. The connected horizontal form retains nonzero correlations, even for centred real profiles; additional arithmetic information is needed to exclude such structured contributions.

There is also a direct limitation of the particular expression certified in (1.7). Taking $f_0 = f_1 = f_2 = f_3 = 0$ gives $P = 0$ and

$$|\mathcal{Q}(0,0,0,0)| + \frac{1}{3}(1-\sqrt{0})^2 = \frac{1}{3}.$$

Consequently no universal upper bound for this same expression can be smaller than $1/3$. Improving only its numerical certificate cannot prove vanishing of the arithmetic quartic. This is a limitation of the stated profile estimate; it is not a positive lower bound for a Liouville correlation. Additional arithmetic information or a more discriminating vertical estimate would be needed to pass this obstruction. For a complementary analysis of the sign patterns forced by arithmetic dynamical axioms, see Sawin [25].

The entropy inequality identifies one possible form of that information. A sufficiently large entropy rate of the same extracted arithmetic law would force a smaller quartic bias. The estimate proved here places a cost on a given bias; it does not establish that the arithmetic law has the entropy needed to rule the bias out. The distinction also explains why a finite table of word frequencies cannot be substituted for a statement about an entropy rate.

Artificial-intelligence assistance and verification

GPT-5.6 Sol (OpenAI) assisted with the exploration, writing, exact computations and formal development of the dyadic results. GPT-6 Astra Pro (OpenAI) assisted with the quartic estimate, mathematical reviews, checks against primary references, English exposition, computational verification and preparation of the source package. OpenAI Codex assisted with the exposition and verification of the dyadic arguments and the all-prefix estimates, and with preparation of the source archive. The author is responsible for the statements and their presentation.

The finite calculations have the specific scope described in Appendix A. They check the stated integer and rational inequalities and the indicated implementations. The arithmetic transfer and the limiting arguments are proved in the text.

A Sources and reproducibility

The accompanying archive `quartic-liouville-sources.zip` contains the manuscript sources, the finite certificates, their checking programs and the reference outputs used for exact comparison. The archive is available from the Zenodo record and is also embedded in the PDF distributed there. The file `README.md` gives the build and verification commands. File digests identify the supplied inputs and outputs.

The package-level driver `reproduce_all.py` provides a single entry point for the active stages. Its `full` level regenerates the upstream finite inequalities, runs the final recurrence, and checks the word and entropy certificates in dependency order. The map `ACTIVE_INPUTS_EN.md` identifies the inputs of each stage and the optional smaller entropy profile. A `quick` level

checks a smaller set of interfaces and exact identities. The driver works in a separate copy and records commands, input digests, exit statuses and comparisons without changing the distributed certificate tree.

The roles of the computations are distinct. The circle bound uses finite word enumeration, directed interval certificates for continuous local inequalities, and the final recurrence over product and pair-norm intervals. The word counts use rational dual witnesses against every allowed finite column. The entropy calculation checks scalar inequalities over covering intervals, using the geometric recurrence profile as an input. The numerical value returned by one stage is used only after its input contract has been specified in the corresponding proof.

For the elementary dyadic results, `verification/cocycle_models.py` checks the vertices of the endpoint laws, the modified-character recurrences and the local factors at the primes up to 43. The factorization for an arbitrary finite set of primes is proved in Appendix E.

Checking the computations. With $N = 3072$, $L = 1$, $H = 256$ and $\gamma = 1/32$, the recurrence returns $393134662/805306368$. The maximum occurs in product cell 3071. The driver compares the independent arbitrary-precision Python, vector C++ and mapped C++ computations on a smaller grid before evaluating the full recurrence. To regenerate the inputs to that recurrence, it checks the ten nine-translate maximum tables, the 32340 energy bounds, the signed linear majorants and the coverage of all 1187 continuous rectangles. It also recomputes the homogeneous $59/100$ matrix estimate, its directional bounds and the six-step envelope.

For the six-letter conclusion the checker enumerates 110880 orbit masks. The stationary and division witnesses cover the 9247 maximal admissible recurrent masks except for the two cases handled analytically in Section 7. For the step-five result it tests six witnesses on every allowed column, including the columns outside each preferred support. The rational word-threshold checker then compares the resulting coefficients with the inequalities in the text.

The entropy calculation has two stages. The first regenerates the 3072-cell rank profile through rank 2650. The second takes its 2651 cumulative bounds and checks the scalar inequalities on 60000 covering intervals. Each stage has its own reference output. The sharper homogeneous coefficient $294643/500000$ used in the entropy calculation is checked by both a C++ interval program and an exact-rational Python implementation.

B Local inequalities and the meaning of the certificate tables

This appendix supplies the analytic derivations and finite mathematical specifications needed to check the local input to the recurrence. The data files are part of the certificate: a row in a file is an integer inequality with the meaning specified below, and its checker proves that inequality by exhaustion or by covering a compact parameter domain.

Throughout the appendix,

$$Q = \int_{\mathbb{T}^2} F_1(u)F_2(v)F_e(2u-v)F_o(2v-u)\,du\,dv, \qquad F_i : \mathbb{T} \longrightarrow [-1,1], \quad \int F_i = 0.$$

Write $P_3F = \frac{1}{3}\sum_{j=0}^{2} F(\cdot + j/3)$, $v_i = \|F_i\|_2^2$, $q_i = \|P_3F_i\|_2^2/v_i$, and $U_i = v_i(3q_i-1)/2$. Zero energies are treated by continuity, or by the immediate identity $Q = 0$. Set

$$A = \sqrt{v_1v_2}, \quad B = \sqrt{v_ev_o}, \quad P = AB, \quad t = \sqrt{q_1q_2}, \quad x = \sqrt{(1-q_1)(1-q_2)}.$$

Primes indicate the next half-step. Thus the active and passive pair norms become B and At, respectively. Every pair of distinct site coordinates has product Haar distribution: its integer

matrix has nonzero determinant and hence defines a surjective homomorphism of tori. This pairwise independence, including independently randomized values at each site, will be used repeatedly. Independence of three sites is never used.

B.1 A real Fourier factor and the channel bounds

For four pairwise independent integer linear forms in two variables, if one profile is real and centred and the other three are complex, then

$$\left| \int \prod_{i=1}^{4} G_i(L_i) \right| \le 2^{-1/2} \prod_{i=1}^{4} \|G_i\|_2. \tag{B.1}$$

Here is the frequency argument, including the point where realness enters. Let l be the frequency of the real profile and choose one of the other frequencies as k. Permutation of the three other indices and integral dilations of the frequency sequences reduce the resonance equations to

$$i = ak + cl, \qquad j = bk + dl, \qquad a \text{ even}, \quad b, c, d \text{ odd}.$$

Indeed, Cramer's rule expresses the coefficients as quotients of determinants. The three products of complementary determinants satisfy the Plücker relation. Their nonzero terms have two equal least 2-adic valuations and a strictly larger third valuation. Choose the permutation with $v_2(ad) > v_2(bc)$; powers of two applied to the four indices make the valuations of b, c, d zero and that of a positive. Odd denominators can then be cleared. The transported Fourier sequences are completed by zero, which preserves their norms and the symmetry at the real site. Enlarging the resulting resonance lattice only increases the sum of coefficient moduli.

Write $h_l = |\widehat{G}_4(l)|$, so $h_0 = 0$ and $h_l = h_{-l}$. On the layer $v_2(l) = s$, put $m = 2^{s+1}$ and fix $k \equiv r \pmod m$. Cauchy–Schwarz applied to the groups (h_l, A_i) and (C_k, B_j) gives at most

$$2^{-1/2} \|h_{v_2=s}\|_2 \|A\|_{ar+c2^s,m} \|C\|_{r,m} \|B\|_{br+d2^s,m}.$$

The subscript denotes the norm on a residue class. The factor $2^{-1/2}$ follows because the two classes $l \equiv \pm 2^s \pmod{2m}$ carry equal energies and give complementary classes of i inside $ar + c2^s \pmod m$. Since b is odd, the classes of j are permuted as r varies. After Cauchy–Schwarz in r, each pair (i, k) occurs only in the layer $v_2(i - ak) = s$. Summing these disjoint layers and applying Cauchy–Schwarz once more proves (B.1). Approximation proves the assertion in L^2. For the present four forms one may use directly $i = 2k - l$, $j = 2l - k$ when the real profile has frequency l.

Let $F_{i,r}$ be the projection to frequencies $r \pmod 3$. The difference $Q - Q'$ is twice the real part of the sum with active residues $1, 1$ and passive pairs $(1, 0), (0, 1), (-1, -1)$. Their passive products are orthogonal in $L^2(\mathbb{T}^2)$. If their normalized inner products with the active product are z_{10}, z_{01}, z_{--}, then

$$z_{10}^2 + z_{01}^2 + z_{--}^2 \le 1, \qquad z_{10}, z_{01} \le 1/\sqrt{2}.$$

For $a = \min(q_e, q_o)$ and $b = \max(q_e, q_o)$ the three weights are

$$\sqrt{(1-a)b/2}, \quad \sqrt{a(1-b)/2}, \quad \sqrt{(1-a)(1-b)}/2.$$

Consequently $|Q - Q'| \le PxK(a, b)$, where

$$R = (1-a)b/2, \quad S = (1-b)(1+a)/4, \qquad K(a,b) = \begin{cases} \sqrt{R+S}, & R \le S, \\ (\sqrt{R} + \sqrt{S})/\sqrt{2}, & R > S. \end{cases} \tag{B.2}$$

This is the maximum of $\sqrt{R}z+\sqrt{S}\sqrt{1-z^2}$ for $0 \le z \le 1/\sqrt{2}$. In the second case $K^2 \le (3-a)/8$. In the first case $K^2 = (1+a+(1-3a)b)/4$: for $a \ge 1/3$ use $b \ge a$, and for $a < 1/3$ use $b \le (1+a)/(3-a)$. These give $K^2 \le 3/8$ in both cases. Summing the three channels separately also gives

$$K \le \frac{x' + \sqrt{1-(t'-x')^2}}{2}. \tag{B.3}$$

Both are used along with the geometric coefficient $\sqrt{(1-t'^2-x'^2/2)/2}$ and the finer modulo-nine support bound in the main text.

For completeness, the directional table has the following exact meaning. For nonnegative weights $w = (w_0, w_1, w_2)$ put

$$S(w) = \sup_{p,r \in \Delta_2} \sum_{d=0,1,-1} w_d C_d(p,r),$$

with C_d as in Section 5. Each of the 91 rows (i,j,k,h) of `directionswater.txt` asserts $S(i,j,k) \le 12h/1000$, where $i+j+k=12$; equivalently, $S(i/12, j/12, k/12) \le h/1000$. The verifier uses its recorded integer normalization directly. Convexity and positive homogeneity of S extend the upper values by barycentric interpolation over each triangle of this simplex. The symmetry exchanging the last two weights is checked in the data. The function still majorizes the original geometric coefficient divided by $\sqrt{2}$: take the first primitive probability vector to be a point mass and optimize the second. The resulting support is $\sqrt{w_0^2 + (w_1^2+w_2^2)/2} \ge \|w\|_2/\sqrt{2}$. This observation justifies the rarity test below even after the finer channel bound has replaced the geometric bound.

The six-step table `hom59.inc` is also a finite support calculation. Cover the ordered passive fractions $0 \le a \le b \le 1$ by the 400 by 400 grid. For each box retain an upper channel coefficient k and an initial upper tail coefficient

$$v \le \min\{(59/100)(t+x), \sqrt{t^2 + x^2/2}\}.$$

Given all retained pairs (k,v), one improvement replaces a current tail coefficient by the maximum of $xk+tv$ over these pairs and over the current box subject to $t+x \le 1$. The upper convex hull of the pairs suffices. Six repetitions, each taking the minimum with the previous value, give the table. The final function is the upper envelope of $(1-t)k+tv$. Its values at $t = i/1000$ are rounded upward, and convexity permits linear interpolation between these values. The integer verifier uses scale 10^6, retains respectively $57, 99, 98, 97, 97, 97$ hull vertices, and checks all 1001 recorded entries.

The sharper homogeneous coefficient used for entropy. The same channel problem admits the separately certified bound

$$|Q| \le K_* P, \qquad K_* = \frac{294643}{500000}. \tag{B.4}$$

It does not replace an input to the recurrence producing B_*; it is an additional homogeneous estimate. We specify its certificate because it is used in the entropy application. Put $w_d = C_d^2$ and $g = K_*^2$. For $d \ne 0$, $h_i = p_i r_{i+d}$, any index i, and any $0 < a \le 1$, weighted Cauchy–Schwarz gives the global affine majorant

$$w_d \le \frac{1+a}{2} h_i + \frac{1+1/a}{2} \sum_{j \ne i} h_j.$$

Indeed, the second Cauchy–Schwarz factor is at most $2[a+(1-a)z_i^2]/(1+a) \le 1$. The index and the slope may be selected at the centre of a box because the resulting inequality holds on the entire box. One may also use $w_d \le \sum_i h_i$.

Since $w_1, w_{-1} \le 1/2$ and $\sum_d w_d \le 1$, the trace of MM^* is at most $1/2 < 2g$. Thus the norm bound is equivalent to

$$gw_0 + 2g(w_1 + w_{-1}) - w_1 w_{-1} \le 4g^2.$$

If $j \le w_1$ and $z \le w_{-1}$ are lower bounds on a box, then $w_1 w_{-1} \ge j w_{-1} + z w_1 - jz$. It therefore suffices to upper bound

$$gw_0 + (2g - z)w_1 + (2g - j)w_{-1} + jz.$$

All three coefficients are positive. Substitute the affine majorants just proved and parametrize each probability vector by $(s, (1-s)t, (1-s)(1-t))$. The result is separately affine in the four parameters and can be bounded at their sixteen exact rational corners. The same simultaneous permutation symmetry as above permits $p_0 \ge p_1 \ge p_2$.

The checker `matrice_tangentes.cpp` uses scale 2^{28}, rounds upper coordinates outward and the channel lower bounds downward, and proves the assertion on 46 220 terminal boxes. The independent `matrice_tangentes_exacte.py` evaluates the sixteen rational corners exactly, uses scale 2^{29} for lower bounds, and terminates on 50 213 boxes. Both reach depth 52. They are run with the exact arguments `589286 1000000`; their normalized constants agree with (B.4). In either case the resulting norm bound gives $|Q - Q'| \le K_* P x \le K_*(P - P')$. Telescoping the alternating ternary projections and then using L^2 approximation proves (B.4) for all four real centred profiles.

B.2 Physical constraints and local projection costs

For three numbers in $[-1, 1]$ the variance is at most $8/9$ and the sum of pair products is at least -1. Both assertions follow by separate convexity or affinity and examination of the eight vertices. Thus

$$-\frac{1}{3} \le U_i \le 1, \qquad v_i(1 - 3q_i) \le \frac{2}{3}, \qquad v_i(1 - q_i) \le \frac{8}{9}. \tag{B.5}$$

In particular $Ax \le 8/9$. The two fractions q_1, q_2 are the roots of $X^2 - (1 + t^2 - x^2)X + t^2$, and

$$v_i \le m(q_i), \qquad m(q) = \begin{cases} 2/[3(1-3q)], & q < 1/9, \\ 1, & q \ge 1/9. \end{cases}$$

Therefore $A^2 \le m(q_1)m(q_2)$. Also

$$81A^2t^2 \ge 3A^2 - 2, \qquad t \le 1/3 \implies A^2(3x^2 + 6t^2 - 2) \le 4/9. \tag{B.6}$$

For the first inequality only $A^2 > 2/3$ needs consideration. Then $v_1, v_2 > 2/3$ and $(v_1 - 2/3)(v_2 - 2/3) \ge A^2/3 - 2/9$, because $v_1 + v_2 \le 1 + A^2$. Use $3v_i q_i \ge v_i - 2/3$. For the second, $(1 - 3q_1)(1 - 3q_2) = 3x^2 + 6t^2 - 2$; either this product is nonpositive or both factors are nonnegative and (B.5) applies separately.

Put $E = Q - Q'$. Independently round the six active values on two third-order fibres to signs of their given expectations. The centred convolution vanishes unless both triples are mixed. If both are mixed, its three values are, up to sign and permutation, $(8, -4, -4)/9$, whose mean absolute value is $16/27$. The probability that the triple at site i is mixed is $a_i = 3(1 - U_i)/4$. Pairwise independence gives

$$|E| \le \frac{(1 - U_1)(1 - U_2)}{3}. \tag{B.7}$$

The centred convolution of two arbitrary real centred triples is $kl\cos(\theta + 2\pi j/3)$, where k, l are their standard deviations. The mean absolute value of these three cosines is at most $2/3$.

Hence $|E| \le 2Ax/3$. The following additional bounds, all used by the recurrence, follow from (B.7):

$$|E| \le \frac{3}{4}(1-At)^2, \qquad |E| \le \frac{(5+A^2-6At)^2}{48}, \tag{B.8}$$

$$|E| \le \frac{(1-A)^2}{3} + \frac{(1-A^2)(1-t^2+x^2)}{2} + \frac{3}{4}A^2x^2, \tag{B.9}$$

$$3t^2 \ge 1 \quad \Longrightarrow \quad |E| \le \frac{(2-A\sqrt{3x^2+6t^2-2})^2}{12}. \tag{B.10}$$

Indeed $U_1 + U_2 \ge 3At - (1+A^2)/2 \ge 3At - 1$, and the arithmetic– geometric mean inequality bounds $(1-U_1)(1-U_2)$. For (B.9), write $1-U_i = 1-v_i + \frac{3}{2}v_i(1-q_i)$, expand the product, and use $(1-v_1)(1-v_2) \le (1-A)^2$, $v_i \ge A^2$, and $(1-q_1)+(1-q_2) = 1-t^2+x^2$. If $3t^2 \ge 1$, both $q_i \ge 1/3$ and both $U_i \ge 0$; use $U_1+U_2 \ge 2\sqrt{U_1U_2}$ and $4U_1U_2 = A^2(3x^2+6t^2-2)$ to obtain (B.10).

The loss also requires at least one mixed exterior triple. Consequently

$$|E| \le \frac{16}{27}\min(a_1,a_2)(a_e+a_o). \tag{B.11}$$

This follows by the union bound and independence between each active site and each exterior site. If an upper channel coefficient is k, then $U_eU_o \ge B^2(1-6k^2)$, by the final observation in Section B.1. When this exceeds $1/9$, both exterior U's are positive and $a_e + a_o \le \frac{3}{2}(1 - B\sqrt{1-6k^2})$. Otherwise this refinement is omitted. Also $\min(a_1,a_2) \le 3/4 - 3(U_1+U_2)/8$. These are precisely the data needed by the monotone rarity bound.

Two immediate stopping estimates are

$$|Q| \le P\sqrt{t^2+x^2/2}, \qquad |Q| \le \frac{1+U_1U_2}{2} - \frac{1-B}{3}. \tag{B.12}$$

The first is Cauchy–Schwarz after projection onto the three-element kernel. For the second, for three signs h_0,h_1,h_2, $|(h_0+h_1+h_2)/3| = (1+\sum_{i<j} h_ih_j/3)/2$ and this absolute value is at least $1/3$. Apply this to the paired active products and then integrate. Discarding the passive weight loses at least $(1-\mathbb{E}|F_eF_o|)/3 \ge (1-B)/3$. Independent sign rounding and Jensen give the same inequality for real profiles.

B.3 Finite words, orbit statistics, and the phase inequalities

All word indices in this subsection lie in $\mathbb{Z}/9\mathbb{Z}$. For four sign words define

$$I = \sum_{i,j} A_iB_jE_{2i-j}O_{2j-i}, \qquad L = |I|.$$

Each word has three triples w_s, w_{s+3}, w_{s+6}; call a triple constant or mixed according as its signs agree or do not agree. Let U,V,W be the counts of constant triples at the two active sites and at exterior site e, and K the count of mixed triples at o. For a sign word its label is the least binary code among all cyclic rotations and global sign changes. Let G be the event that the active labels form one of $(15,85),(51,15),(85,51)$. Choose the exterior orientation so the mixity at e is at least that at o. Let H count the nine pairs of active triples whose minority positions align with that of the corresponding mixed triple at e: in the representatives $i,j,z = (2i-j) \bmod 3$ the condition is

$$m_e(z) = 2m_1(i) - m_2(j) + (2i-j-z)/3 \pmod 3.$$

Write J for the event $(3-W,K)\in\{(3,0),(0,3)\}$. The following finite inequalities are input to the continuous proof:

$$9L \le 729 - 162(6-U-V) + 68(3-U)(3-V), \tag{B.13}$$

$$L \le 37 + 4\mathbf{1}_J + 5UV, \tag{B.14}$$

$$9L \le 9(37 + 4\mathbf{1}_G + 5UV) + 4WK, \tag{B.15}$$

$$L - 27 - 2H \le -2K + 4K\mathbf{1}_{G^c} + 4UV + 4(U+V)(W+K) + 10WK - 6(U+V+W). \tag{B.16}$$

There are also three further integer inequalities, with $S = U + V$:

$$3L \le 135 - 36S - 24W - 12K + 18UV + 14SW + 15SK + 12WK - 12\mathbf{1}_G, \tag{B.17}$$

$$4L \le 168 - 42S - 20W - 10K + 24UV + 14SW + 17SK + 10WK - 4\mathbf{1}_G, \tag{B.18}$$

$$9L \le 369 - 90S - 90W - 36K + 54UV + 46SW + 36SK + 72WK + 18K\mathbf{1}_{G^c}. \tag{B.19}$$

These statements have a direct exhaustive proof: enumerate the 30 rotation-and-sign representatives for each active word, 256 exterior words at e modulo global sign, and all 512 words at o. Compute the displayed sums and counts and compare the integers. The independent active rotations induce exterior rotations and preserve the assertions, so these 117 964 800 configurations cover all sign words. The programs `verifier_demi.cpp` and `verifier_trois_plans.cpp` implement these tests. In particular, averaging (B.13) gives

$$36|Q| \le 17 + U_1 + U_2 + 17U_1U_2. \tag{B.20}$$

To specify the signed comparison, for $s,t\in\{0,1,2\}$ put $w = (2t-s) \bmod 3$ and define b_{st} by the multilinear identity

$$-8b_{st} = \sum_{p,q,k=0}^{2} (3\mathbf{1}_{p+q+k=0\ (3)} - 1) A_{s+3p} B_{t+3q} E_{2s-t+3k}.$$

Set $V_9 = \sum_{s,t} b_{st} \sum_{n=0}^{2} O_{w+3n}$. On three mixed triples this is the product of their majorities times $3\mathbf{1}_{\text{alignment}} - 1$; if a triple is constant it is zero. The finite signed inequalities are

$$|45I + 18V_9| \le 981 - 180S - 180W + 132K + 204UV + 146SW + 138SK + 246WK, \tag{B.21}$$

$$|45I + 18V_9| \le 981 - 162S - 54W + 132K + 204UV + 109SW + 120SK + 120WK, \tag{B.22}$$

$$15L \le 615 - 90S - 79W - 11K + 87UV + 33SW + 11SK + 30WK. \tag{B.23}$$

The same enumeration, now preserving the joint signs of I, V_9, proves these statements. More generally the files `sig_k.txt`, $0 \le k \le 5$, record the exact maxima of $|5I + kV_9|$ for each of the 4^4 tuples of mixity counts. Every row $(k, d, c_0, c_1, \dots, c_7)$ of `signalplanes_petit.txt` asserts

$$\frac{|I + (k/5)V_9|}{81} \le \frac{c_0 + c_1\alpha + c_2 r + c_3 y + c_4 pq + c_5 \alpha r + c_6 \alpha y + c_7 ry}{d}, \tag{B.24}$$

where $p = U/3$, $q = V/3$, $r = W/3$, $y = K/3$ and $\alpha = (p+q)/2$. The plane checker verifies these inequalities against all 256 maxima with rational arithmetic.

Here is the continuous meaning and proof of the phase bound used with these planes. After independent sign rounding and Haar averaging, now p, q, r denote the three probabilities of a constant triple and y the probability that the small exterior triple is mixed. Thus

$$p = (1+3U_1)/4, \quad q = (1+3U_2)/4, \quad U_e = (4r-1)/3, \quad U_o = (3-4y)/3, \quad r + y \le 1.$$

Put $\alpha = (p+q)/2$ and

$$D = (v_e - U_e)(v_o + 2U_o)/4.$$

The exact Fourier identity on a triple, with $\zeta = e^{2\pi i/3}$, is $M\mathbf{1}_{\mathrm{mixed}}\zeta^{jk} = -\frac{1}{2}\sum_n s_n\zeta^{jn}$ for $j = 1, 2$. After expectation this phase profile has squared norm $\frac{3}{4}(v - U)$; the fourth profile is the real centred projection P_3F_o of squared norm $(v_o + 2U_o)/3$. Applying (B.1), and bounding the two active phase norms by $\sqrt{1-p}, \sqrt{1-q}$, proves

$$|\mathbb{E}V_9| \le 54(1-\alpha)\sqrt{D/2}. \tag{B.25}$$

In particular every checked plane gives a bound for $|Q|$ by its expected polynomial plus $(2k/15)(1-\alpha)\sqrt{D/2}$. All polynomial monomials involve at most two sites, so pairwise independence suffices. Positive coefficients of pq may use $pq \le \alpha^2$.

We give the remaining inequalities used to enclose the continuous parameters, so that the local phase certificate is specified independently of its implementation. For an alleged value $|Q| \ge h$ with $0 \le \alpha \le 1/4$, (B.13)–(B.15) imply

$$\begin{aligned} 81h &\le 41 - 28\alpha + 68\alpha^2, \\ 81h &\le 41 - 4(r + y - 2ry) + 45\alpha^2, \\ g &= \frac{81h - 37 - 45\alpha^2 - 4ry}{4} \le \mathbb{P}(G). \end{aligned}$$

Define

$$\sigma = \max\{0, \sqrt{(2g-1)_+}, \sqrt{(2g - (1-\alpha)^2)_+} - \alpha\}.$$

If two independent label distributions have matching probability $g > 1/2$, their dominant labels must match, and their dominant masses b, c satisfy $g \le bc + (1-b)(1-c)$; hence $b + c \ge 1 + \sqrt{2g-1}$. Restricting to the three admissible labels, whose total masses are at most $1-p, 1-q$, gives the stronger alternative $b + c \ge 1 - \alpha + \sqrt{2g - (1-\alpha)^2}$ whenever it is useful. Independence of the exterior mixity count with each individual active label and the union bound now give $\mathbb{E}[K\mathbf{1}_{G^c}] \le 3y(1-\sigma)$. With $\rho = \mathbb{E}H/9$, (B.16) yields

$$\begin{aligned} \rho &\ge \max\left\{0, \frac{81h - 27 + Z}{18}\right\}, \\ Z &= 18(2\alpha + r) - 36\alpha^2 - 72\alpha(r+y) - 90ry + (12\sigma - 6)y. \end{aligned}$$

The Riesz–Sobolev inequality also gives

$$\rho \le (1-r)(1-\alpha) - (1-r)^2/4. \tag{B.26}$$

To see the normalization, the three minority selector densities have masses $(1-p)/3, (1-q)/3, (1-r)/3$ and are bounded by one. Rearrangement replaces them by arcs. Balancing the first two lengths at fixed sum increases their convolution against the third arc. Integration of the resulting triangle, and multiplication by the factor nine in the alignment count, gives (B.26). The third length is at most the sum of the balanced first two because $\alpha \le 1/4$.

For the unsigned phase residual one has

$$|Q| \le \frac{5}{9}(1-\alpha)\sqrt{D/2} + \min(R_0, R_1, R_2), \tag{B.27}$$

where

$$\begin{aligned}R_0 &= \frac{1}{18}\big[1-\rho+2\alpha+r+3(\alpha^2+2\alpha r)+5\min(\alpha^2,\alpha r)\\ &\qquad +8\alpha y+8ry+4\min(\alpha^2,\alpha y)\big]+\frac{11}{27}(1-\max(\alpha,r))y,\\ R_1 &= \frac{1}{54}\big[17-11(2\alpha+r)+8y+51\alpha^2+46\alpha r\\ &\qquad -8\alpha y+16ry+(8/3)(1-r)y-17\rho\big],\\ R_2 &= \frac{1}{54}\big[3+3(2\alpha+r)+20y+37\alpha^2+18\alpha r\\ &\qquad -8\alpha y+4ry-3\rho\big].\end{aligned}$$

These residual bounds involve only a ternary finite check. For clarity, if I_1, I_2, I_e are the three indicators of constance, m is exterior mixity, L_0 is authentic minority alignment, S_0 is the event of equal minority positions at two mixed exteriors, M is the product of the four majorities, and T the average over nine ternary lifts, form

$$R_* = |54T + 5M(3-2m)(3L_0-1)(1-I_1)(1-I_2)(1-I_e)|.$$

With $Z_* = (1, I_1, I_2, I_e, m, I_1I_2, I_1I_e, I_1m, I_2I_e, I_2m, I_em, S_0, -L_0)$, the two coefficient vectors

$$(17,-11,-11,-11,8,51,23,-4,23,-4,16,8,17), \qquad (3,3,3,3,22,37,9,-4,9,-4,2,-6,3)$$

majorize R_* by their scalar products with Z_*. Check them on the 4^4 triples with positive majority; changing majorities does not affect either absolute value. Averaging gives R_1, R_2, since $\mathbb{E}S_0 = (1-r)y/3$. For R_0, when all three large triples are mixed the local residual after adding the phase is at most $(1-L_0)/18 + 11m/27$. Outside that event, with N constant large triples and $m=0$, the values are $1/9, 1/3, 1$ for $N = 1, 2, 3$. For $m = 1$ their increase is at most $2m(I_1+I_2)/9 + 4mI_e/9 + 2mI_1I_2/9$. The resulting expectation is bounded using pairwise products and $\mathbb{P}(E_1 \cap E_2 \cap E_3) \le \min_{i<j} \mathbb{P}(E_i \cap E_j)$, which gives exactly R_0 above. These small finite assertions are checked by `verifier_fibres.py` and `verifier_phase_ponderee.py`.

Finally the three inequalities (B.17)– (B.19) and the signed inequalities (B.21)–(B.23) are averaged in the same way. The former use $\mathbb{P}(G) \ge g$ and $\mathbb{E}[K\mathbf{1}_{G^c}] \le 3y(1-\sigma)$; the latter use (B.25). The admissible energies obey

$$\max(U_e, -2U_e) \le v_e \le 1, \quad \max(U_o, -2U_o) \le v_o \le 1, \quad v_e v_o \le B^2.$$

Besides direct interval evaluation of D, the check uses

$$4D \le B^2 + 2U_o v_e - U_e v_o - 2U_e U_o, \qquad D \le (1-r)(1-8y/9)(2+B^2)/3.$$

For the second, relative to sign rounding the two squared phase norms are reduced by factors at most $(1+3v_e)/4$ and $(2+v_o)/3$. At fixed $v_e v_o$ their product is maximized at an endpoint, giving $(2+v_e v_o)/3$.

The programs `newphase.py` and `phase_locale.py` bisect the longest side of $[0, \alpha_{\max}] \times [0,1]^2$, discarding a box only when an outward bound contradicts one of the inequalities just specified. Square roots in σ are rounded down on $2^{-24}\mathbb{Z}$; a positive square-root upper test is squared only after verifying the positivity of the other side. The certified local values are the following

exact rationals:

B at most	α at most	$\lvert Q\rvert$ at most
1	1/4	467/1000
9/10	1/4	465/1000
82/100	1/4	463/1000
4/5	1/4	461/1000
85/100	1/16	462/1000
91/100	1/32	465/1000
86/100	1/32	463/1000
82/100	1/32	462/1000
9/10	1/20	465/1000

In the first four rows the application is to $U_1, U_2 \le 0$, which implies $p, q \le 1/4$ and hence $\alpha \le 1/4$. A bound $\alpha \le 1/8$ also implies $p, q \le 1/4$ separately, and permits the same application. The smaller values of α in the last five rows are retained explicitly.

The remaining sign branches require no new table. If $u = U_+ \ge 0$ and $-w = U_- \le 0$, then

$$w \ge \frac{A^2}{2}\left(1 - \frac{9t^2}{1+2u}\right)_+, \qquad 36|Q| \le 17 + u - (1+17u)w. \tag{B.28}$$

Indeed $q_+ \ge (1+2u)/3$, $q_- \le 3t^2/(1+2u)$ and $v_- \ge A^2$. The code maximizes these inequalities over 16 covering intervals for u. It also uses the resulting bounds 189/400 for $t \le 1/3$ and 471/1000 for $t \le 3/10$. For the first, if $|Q| \ge h = 189/400$, the homogeneous bound $|Q| \le \sqrt{3/8}\prod_i \|F_i\|_2$ gives $v_- \ge \beta = 8h^2/3$. With $d = 36h - 17 = 1/100$, (B.28) would imply $(17\beta - 2)u^2 + (\beta + 2d - 1)u + d \le 0$; its leading coefficient is positive and its discriminant negative. For the second, put $h = 471/1000$ and $d = 36h - 17 = -11/250$. The same inequalities imply

$$9t^2 \ge (1+2u)\left(1 - \frac{3(u-d)}{4h^2(1+17u)}\right).$$

The right side is increasing for $0 \le u \le 1$: the derivative of $(1+2u)(u-d)/(1+17u)$ is decreasing and equals 17/50 at zero. Its value at zero is greater than 81/100. For the older opposite-sign cuts at $h = 487/1000$ or 489/1000, put $d = 36h - 17 > 0$ and use the same displayed lower bound for $9t^2$. Now $u \ge d$. Its right side is increasing on $[d, 1]$ and has minimum $1 + 2d$, respectively 2.064 or 2.208. Both exceed $9(11/25)^2$, proving those cuts whenever $t \le 11/25$. When both active U's are positive, put $z = 3t - 1$. If $17A^2 \ge 2$, (B.20) gives $36|Q| \le 17 + z + 17A^2z^2/4$; otherwise replace A^2 by one. This follows by eliminating $q_1 + q_2 \ge 2t$, whose coefficient is nonpositive. Without a sign hypothesis the analogous bound is $36|Q| \le 17 + 3t - A + 17A^2(3t-1)^2/4$ whenever $17A^2 \ge 2$. The expression is convex in t, so its maximum on an interval is at an endpoint. These facts also imply the older 489/1000 cut for $t \le 17/40$; in the opposite-sign branch a value at least 489/1000 would already force $t > 11/25$.

B.4 Projected weights and cumulative losses

Let $p = (1+3U_1)/4$, $q = (1+3U_2)/4$, and $\theta = (1+3U_eU_o)/4$. The next-stage stopping estimate is

$$27|Q'| \le (1+2p)(1+2q) + 2\theta + 4\min(\theta, p+q-pq) + 12\min(\theta, pq). \tag{B.29}$$

After independent sign rounding, the modulus of a projected triple is one when it is constant and 1/3 otherwise. The product G of two such moduli has the decomposition $G = 1/9 + (2/9)\mathbf{1}_{N\ge1} + (2/3)\mathbf{1}_{N=2}$, where $\mathbb{P}(N \ge 1) = p + q - pq$ and $\mathbb{P}(N = 2) = pq$. The mean of the three paired

active sign products at the next stage has modulus $H = 1/3 + (2/3)\mathbf{1}_S$, with $\mathbb{P}(S) = \theta$. The projected weight is constant on the relevant kernel, so $|Q'| \le \mathbb{E}GH$. Bounding the intersections by the minima of their probabilities proves (B.29), without assuming independence between G and S.

If $\eta = |Q - Q'|/(Px)$ (set $\eta = 0$ when $Px = 0$), the geometric loss bound gives $U_e U_o \le B^2(1 - 3\eta^2)$. Partitioning $0 \le \eta \le 59/100$ into intervals of length $1/64$ therefore bounds simultaneously the jump and θ in (B.29). The lower endpoint bounds θ and the upper endpoint bounds the jump. The stored eight passive weight labels are upper roundings of θ to $1/4, 3/10, 1/3, 3/8, 9/20, 11/20, 7/10, 1$.

We also prove the cumulative estimate used in threshold pruning. In the original coordinates, the profiles after j steps are $G_i = E_{d_i} F_i$, where E_d projects to frequencies divisible by d, and the four d_i are all 3^k, or are 3^{k+1} at the original active pair and 3^k at the other pair. Put $R_i = F_i - G_i$. A resonance has $n_1 = -2n_e + n_o$, $n_2 = n_e - 2n_o$. These equations show that a term with exactly one residual frequency is impossible, and a term with exactly two residual frequencies can only use the current passive pair. For a crossed pair the determinant is prime to three; inversion first forces divisibility by 3^k, and the residual congruence $n_1 \equiv n_2 \pmod{3^{k+1}}$ excludes the remaining unequal-level case.

Let $a_i = 1 - \|G_i\|_2^2$. Conditional convexity gives $\|R_i\|_1 \le a_i$. A double telescoping expansion of the product, ordered with the active site of largest defect first, has no terms with one residual and no surviving two-residual term containing that first site. More explicitly, when $a_1 \ge a_2$, the support cancellations reduce the difference to

$$Q_0 - Q_j = \int (F_1 R_2 R_e F_o + F_1 R_2 G_e R_o + F_1 G_2 R_e R_o),$$

with every factor evaluated at its corresponding site. The other factors have modulus at most one. Pairwise independence thus gives

$$|Q_0 - Q_j| \le \min(a_1, a_2)(a_e + a_o) + a_e a_o \le (1 - B)(3 - 2A - B).$$

Here A, B are the products of current active and passive norms. The left expression is at most $(\sum a_i)^2/4$, while $\sum a_i \le 4(1 - \sqrt{AB})$. For a second estimate, draw independent signs r_i, s_i of conditional means F_i, G_i at each site and put $d_i = \mathbf{1}_{r_i \ne s_i}$. Then $\mathbb{E}d_i = a_i/2$ and these indicators are pairwise independent. With $T = \prod s_i$, the support cancellations give $\mathbb{E}[T d_i] = 0$ and $\mathbb{E}[T d_i d_l] = 0$ for crossed pairs. Writing $n = d_1 + d_2$, $m = d_e + d_o$, we obtain

$$Q_0 - Q_j = \mathbb{E}T\{(-1)^{n+m} - 1 + 2(n + m) - 4nm\}.$$

For $n, m \in \{0, 1, 2\}$ the absolute value of the braces is $4(\mathbf{1}_{n=2} + \mathbf{1}_{m=2})$. Hence $|Q_0 - Q_j| \le a_1 a_2 + a_e a_o \le (1 - A)^2 + (1 - B)^2$. Combining the three estimates proves the function $D(A, B, AB)$ used in Section 5.

The refinement retaining the previous active norm is equally explicit. At the previous step the pair norms were (D, A) with $B \le D \le 1$, and its compression ratio was B/D. Add the previous cumulative bound

$$\min\{(1 - A)(3 - 2D - A), (1 - A)^2 + (1 - D)^2\}$$

to the minimum of the four jump bounds

$$\frac{59}{100}A(D - B), \quad \frac{3}{4}(1 - B)^2, \quad \frac{(5 + D^2 - 6B)^2}{48}, \quad \frac{(1 - D)^2}{3} + (1 - D^2)(1 - B/D) + \frac{3}{4}(D - B)^2.$$

These follow from (B.8), (B.9) and $x \le 1 - t$. The recurrence maximizes this sum over 32 covering intervals for D. The zero case follows by continuity.

B.5 Three-valued rounding and the coupled continuous tables

For each exterior value $f \in [-1,1]$, choose a random $X \in \{-1,0,1\}$ with

$$\mathbb{P}(X=0 \mid f) = \tau(1-|f|), \qquad \mathbb{P}(X=\pm 1 \mid f) = [1-\tau(1-|f|) \pm f]/2.$$

Its expectation is f. Every integrated zero frequency in $[0, 1-\|F\|_2]$ is realizable, because $\|F\|_1 \le \|F\|_2$. Use independent such roundings at the exterior sites and sign roundings at the active sites. The polynomial definitions of I, V_9 are multilinear, so their expectations remain $81Q, \mathbb{E}V_9$. For an exterior word set

$$r(w) = \frac{1}{12}\sum_{i=0}^{2}(1 + w_i w_{i+3} + w_i w_{i+6} + w_{i+3}w_{i+6}), \quad s(w) = \#\{i : w_i = 0\}/9.$$

Their expectations are $(1+3U_F)/4$ and the chosen zero frequency.

The 55 rows in `twosig_tous_petit.txt` are majorants of $|I + (k/5)V_9|/81$, $k = 0,1,2$, in the monomials

$$1, \alpha, r, d, s, z, J, \alpha r, \alpha d, \alpha s, \alpha z, rd, rz, sd, sz,$$

where $J = pq$ and $\alpha = (p+q)/2$ are active constance statistics, (r,s) and (d,z) are the two exterior statistics. The attached integer rows specify all coefficients and denominators. To verify a row, take the maximum for each of the 36 864 admissible statistic tuples and test the row there. The enumerators fix the 30 active representatives as above, enumerate the 9842 ternary exterior words modulo global sign, and maximize linearly over the final word. For each prescribed statistic of one of its triples, take the maximum over its 27 possible values, and then assemble the three triples. This is exhaustive because the objective is linear at the fourth site. The algebraic definition of V_9 is used also when zero entries occur.

For a passive norm product at most B, choose $z_0 \in [B,1]$ with $\|F_e\|_2 \le z_0$ and $\|F_o\|_2 \le B/z_0$. Set the two zero frequencies to $1-z_0, 1-B/z_0$. Such a choice always exists; all choices are covered. The continuous checker imposes

$$-v_F/2 \le U_F \le v_F, \quad v_e \le z_0^2, \quad v_o \le (B/z_0)^2, \quad \max(0, 2\alpha - 1) \le J \le \alpha^2.$$

It bounds the phase by (B.25). The remaining polynomials are separately affine. A positive coefficient of J uses $J \le \alpha^2$, giving a convex quadratic in α; a negative coefficient uses its piecewise affine lower bound, whose only breakpoint is $1/2$. Thus vertices and this breakpoint suffice. Subdivision in z_0, α, r, d, with outward rational bounds, proves the 200 nontrivial entries of `maillage.txt`. They are the table `phys.inc`; uncovered entries use the trivial bound one.

The 32 340 entries of `affines.inc` are bounds for

$$A \in [a/100, (a+1)/100], \quad t \in [b/100, (b+1)/100], \quad B \le B_i, \quad 67 \le a \le 99, \quad 0 \le b \le 69,$$

with the fourteen passive ceilings B_i listed in `phys.inc`. For each cell, order $q_1 \le q_2$, divide $q_2 \in [t_-, 1]$ into 80 intervals, and use $q_1 = t^2/q_2$. Divide the admissible interval of v_1 into twelve intervals and use $v_2 = A^2/v_1$. Impose $v_i \le m(q_i)$ and $U_i = v_i(3q_i - 1)/2$ throughout. On every resulting box take the minimum of: (B.20), the second bound in (B.12), the applicable entries of `phys.inc`, the first bound in (B.12), $|Q| \le (59/100)Pt + (1-U_1)(1-U_2)/3$, and the applicable nonpositive phase bound above. The maximum over all boxes is the recorded cell value. `verifier_energies.py` performs these operations independently with arbitrary precision integers, at scale 10^6, and reconstructs every entry.

The final coupled table uses all four q_i simultaneously. For $k = 0, \dots, 9$, let I' be the sum I after projecting its two active words to their triples. Then $9I'$ is integral. The file

`J_0_k.txt` records, at every mixity tuple, the maximum of $|90I - 9kI'|$. Dividing by 7290 gives the maximum of $|I - (k/10)I'|/81$. Rows in `plans_quatre.inc` and `etage_quatre.inc` are integer linear majorants in

$$1, \quad U_1 + U_2, \quad U_e + U_o, \quad U_1U_2, \quad U_eU_o, \quad (U_1 + U_2)(U_e + U_o).$$

Their denominators and, for the latter file, the coefficient k are part of each row. At a word with m mixed triples, substitute $U = 1 - 4m/9$. Testing every row on all 4^4 tuples proves the finite inequalities. Independent sign rounding then proves

$$|Q| \le G(U_1, U_2, U_e, U_o) + (k/10)|Q'|$$

for the original real profiles. Only pairwise independence is needed for these monomials.

Each row $(A_-, A_+, B_-, B_+, t_-, t_+, h)$ of `regions4_vraies.txt`, in units 1/1000, asserts $|Q| < h$ on that closed rectangle. The verifier starts with all four fractions $q_i \in [0, 1]$, orders the fractions within each pair in the symmetric bounds, and imposes $q_1q_2 \in [t_-^2, t_+^2]$ and $v_1v_2 \in [A_-^2, A_+^2]$, $v_ev_o \in [B_-^2, B_+^2]$. The constraints $v_i \le m(q_i)$ enclose each U_i. It discards a box if these constraints are incompatible, or if either a checked signed plane or the following two-step bound is below h:

$$|Q| \le \min\{PxK(q_e, q_o), (1 - U_1)(1 - U_2)/3\} + Pt \min\{H(t'), \sqrt{t'^2 + x'^2/2}, (59/100)(t' + x')\}.$$

Here K is the minimum of the geometric, separate-channel, Bessel, and modulo-nine support bounds already proved. The longest fraction interval is bisected until every box has been discarded. Strict comparisons prove the stated closed-rectangle bound. All 1187 rows are verified in this way. A 1/40 grid cell is improved only when it lies entirely in a certified rectangle; taking the minimum of all containing rectangles produces `table_xyz.inc`, with 451 nontrivial cells. A transition touching several cells uses their largest upper value.

B.6 Redundant legacy inequalities and a closed dependency list

The source keeps earlier upper bounds for continuity of the recorded calculation. They introduce no additional mathematical dependency. On $U_1, U_2 \le 0$, the checked bounds 467/1000, 465/1000, 463/1000, 461/1000 dominate the earlier phase bounds and the earlier 1/2 bound with a correction of at most 1/1000. For opposite signs, the weighted estimate in (B.12) and (B.28) supply the older branch cuts directly. In particular the legacy correction to 1/2 has size at most $(1 - B^2)/90 \le (1 - B)/45$, whereas the weighted estimate saves $(1 - B)/3$. For positive signs the order-nine calculation after (B.28) supplies them. The optional historical 5/8 stopping bound is dominated everywhere by $(59/100)P$, as proved in Section 5.

Thus a self-contained mathematical supplement consists of the main recurrence specification, this appendix, the exact data files, and the following finite verifiers: the modulo-nine matrix and directional checkers with `h59.py`; the word enumerators `verifier_demi.cpp`, `verifier_trois_plans.cpp`, `verifier_signal_neuf.cpp`, `signal9.cpp`, `tritwo.cpp`, `tritwosig.cpp`, and `joint9.cpp`; the plane and local phase checkers; `boites_creuses.cpp` with `revalide.py`; `verifier_energies.py`; `accouple_resid.cpp` with `valide_49.py`; and the final integer recurrence with its Python reference implementation. The verification driver supplies separate stages for exact regeneration of word maxima, verification of plane coefficients, coverage of continuous domains, and evaluation of the recurrence.

C A fixed-side parallelogram estimate in a residue class

This appendix supplies the additional input for Theorem 11.1. Only one side of the parallelogram tends to infinity. No assertion about a varying modulus or a varying logarithmic law is

needed.

Lemma C.1. *Let ν be a logarithmic limit law of Liouville, jointly with its profinite coordinate z, along windows of divergent logarithmic mass. For every fixed $q \geq 1$ and $b \pmod q$,*

$$\limsup_{k\to\infty} \left| \int f\,(f\circ T)\,(f\circ T^k)\,(f\circ T^{k+1})\,d\nu(\,\cdot \mid z \equiv b \pmod q) \right| \leq \frac{1}{3},$$

where f is the Liouville coordinate. The same conclusion holds as $k \to -\infty$.

Proof. We retain only the finite coordinate $z \pmod q$. Denote the integral in the statement by $R_b(k)$.

Arithmetic transfer with the finite coordinate retained. Fix $k \geq 2$ and put $(b_0, b_1, b_2, b_3) = (0, 1, k, k+1)$. Choose d with $b + d \equiv 1 \pmod q$. After refining the extraction, all the following countably many means exist. For a Dirichlet character $\chi \pmod q$, extended by zero on nonunits, set

$$F_\chi(u) = \lim_I \mathbb{E}^{\log}_{n\in I} \chi(n+ud) \prod_{i=0}^{3} \lambda(n+ub_i).$$

The pointwise product of the five multiplicative functions here is χ. By [30, Theorem 1.2(i),(iii)], F_χ has uniform periodic approximants that are χ-isotypic. Consequently,

$$F_\chi(p) - \chi(p)F_\chi(1) \longrightarrow 0 \qquad (p \to \infty,\ p \text{ prime}).$$

For primes $p \equiv 1 \pmod q$, character orthogonality and $n + pd \equiv n + d \pmod q$ identify the corresponding residue-class moment with $R_b(k)$ in the limit.

Let $\nu_{\mathcal{P}}$ be the arithmetic-progression measure with prime steps associated with the joint base (X, ν, T):

$$\int \prod_j f_j(x_j)\,d\nu_{\mathcal{P}} = \lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\leq N} \int \prod_j f_j(T^{pj}x)\,d\nu(x).$$

Existence and the required factor properties are given in [5, Theorem 4.3 and Section 4.5]. If z_j is the finite coordinate of x_j, the probability measure

$$d\eta = q\varphi(q)\,\mathbf{1}_{z_0=b,\ z_1-z_0=1 \pmod q}\,d\nu_{\mathcal{P}} \tag{C.1}$$

satisfies

$$R_b(k) = \int f(x_0)f(x_1)f(x_k)f(x_{k+1})\,d\eta. \tag{C.2}$$

The normalization follows from the uniform origin coordinate and the equidistribution of primes over the invertible residue classes. This step uses a fixed k only. Having obtained (C.2) for every k, all subsequent approximations take place in one fixed progression system.

Centring and finite nilpotent factors. The logarithmic two-point theorem [26, Theorem 1.3] gives $\int f(f\circ T^h)\,d\nu = 0$ for every $h \neq 0$. Hence, for every $e \geq 1$,

$$\left\| \frac{1}{J} \sum_{j=1}^{J} f\circ T^{ej} \right\|_2^2 = \frac{1}{J}, \qquad \mathbb{E}(f \mid \mathcal{I}_{T^e}) = 0.$$

By the mean ergodic theorem on the ergodic components and bounded convergence, these centring identities hold on almost every component of the base, simultaneously for all e.

The progression construction commutes with this decomposition. On each ergodic component the density in (C.1) still has integral 1. The characteristic-factor assertion in [5, Theorem 4.3 and equation (29)] allows each copy of f to be replaced exactly by $h = \mathbb{E}(f \mid Z_\infty)$ on this component. For the two finite-coordinate indicators, expand the density into products of indicators at ranks 0 and 1 and use that this finite factor is contained in Z_∞. At the rank 0, the same projection follows by integration, after the other observables have been projected. All gaps can be filled with the constant observable 1 when applying the theorem.

Approximate h in L^1 by its conditional expectation g on a finite nilsystem factor containing the finite coordinate. The observable remains real and bounded by 1. On every connected component of this nilsystem it has mean zero: the components form a finite cycle, and the corresponding power of the transformation preserves each component. Replacing all four observables costs at most $4q\varphi(q)\|h - g\|_1$, independently of k, since every coordinate of the unconditioned progression law has the original marginal.

Removal of the central frequencies. Consider such an ergodic nilsystem G/Γ. We use the standard presentation in which G is generated by its identity component and the translating element. Then the lower central groups G_j are connected for $j \geq 2$. The integer-step progression law is Haar on the Hall–Petresco nilmanifold [5, Section 4.2.1 and Proposition 4.7]. If the nilsystem has D connected components, its prime-step law is the integer-step law restricted to invertible increments of the component label, with density $D/\varphi(D)$; this is [5, Theorem 4.4, equations (24)–(26)]. We have $q \mid D$. The additional restriction in (C.1) concerns only these finite labels.

If the nilpotency class is $s \geq 2$, the last central group defines a connected torus $V = G_s/(G_s \cap \Gamma)$. In the progression law the three transformations

$$x_j \longmapsto v_0 v_1^j v_2^{\binom{j}{2}} x_j, \qquad v_0, v_1, v_2 \in V,$$

preserve Haar measure. This follows directly from the Hall–Petresco formula [5, Appendix B.1, equation (45)]. They also preserve all finite labels, and therefore preserve the conditioned law.

Convolve g with a positive Fejér kernel on V. This produces a finite vertical Fourier expansion, remains real and bounded by 1, and changes the four-observable cylinder by an arbitrarily small error uniformly in k. For a contribution with vertical frequencies $m_0, m_1, m_2, m_3 \in \widehat{V}$, the three invariances force

$$\begin{aligned} m_0 + m_1 + m_2 + m_3 &= 0, \\ m_1 + k m_2 + (k+1) m_3 &= 0, \\ \binom{k}{2} m_2 + \binom{k+1}{2} m_3 &= 0. \end{aligned}$$

The frequencies range over one fixed finite set. For all sufficiently large k, the second equation forces $m_2 + m_3 = 0$ and $m_1 + m_3 = 0$. The first then gives $m_0 = m_3$, and the third becomes $k m_3 = 0$. Since $\widehat{V}$ is torsion-free, all four frequencies are zero. Thus the truncated cylinder equals the cylinder of the vertical average of g, evaluated on the quotient nilsystem. Iterating this argument reduces the desired large-k bound to the abelian factor. The number of reductions is finite after the nilsystem factor has been fixed, and each approximation error is uniform in k.

The abelian calculation. The abelian factor is a rotation on a compact abelian Lie group $K \simeq B \times \mathbb{Z}/D\mathbb{Z}$, with B a torus. The conditioned progression law chooses an origin label $j \equiv b \pmod q$ and an increment label $r \in (\mathbb{Z}/D\mathbb{Z})^\times$ with $r \equiv 1 \pmod q$. Conditional on

these labels, its connected coordinates are $x + iy$, where x, y are independent Haar points of B. This follows from [5, Proposition 4.7 and Theorem 4.4, especially equation (25)].

Let g_j denote the observable on component j and let $c_j(\alpha)$ be its Fourier coefficients on B. A positive Fejér approximation gives one finite Fourier support for all g_j, while preserving their zero means. When k is sufficiently large, integration over x, y forces the four horizontal frequencies to have the form $(\alpha, -\alpha, -\alpha, \alpha)$. The resulting expression is

$$\sum_{\alpha} c_j(\alpha)\overline{c_{j+r}(\alpha)c_{j+kr}(\alpha)}c_{j+(k+1)r}(\alpha).$$

Its absolute value is, by Hölder, at most

$$\prod_{i\in\{0,1,k,k+1\}} \left(\sum_{\alpha} |c_{j+ir}(\alpha)|^4\right)^{1/4} \leq \frac{1}{3}.$$

The last inequality is the centred case of Lemma 4.1, whose input is the connected-group rearrangement inequality of Christ–Iliopoulou [3, Theorem 1.1]. Averaging over the permitted labels preserves the bound. Letting all approximation errors tend to zero proves the upper limit $1/3$ on each base ergodic component. Reverse Fatou, applied to the bounded absolute values in (C.2), gives the same upper limit on the original base. The identical finite-mode argument applies as $k \to -\infty$ (or one can translate the origin and use the finitely many residue classes). This proves the lemma. □

D The three-copy affine classification

Which triples of nonconstant affine copies of $Q(t) = t(t+1)(3t+1)(3t+2)$ have a product whose squarefree part has degree two? There are seven rational classes. Among integral affine arguments, only the dyadic identity (2.1) occurs, up to the symmetries specified below. The classification follows from a finite enumeration of overlaps among the root sets and is independent of the analytic estimates in the main text.

There is earlier work on the underlying polynomial-block problem. Yıldız and Gürel [35] study products of blocks of consecutive integers, and Yıldız's thesis [34, Sections 3.1.1–3.1.3; Table B.1] gives polynomial search procedures and relevant examples. For $b_4(x) = x(x+1)(x+2)(x+3)$, the relation

$$b_4(3t) = 9Q(t) \tag{D.1}$$

puts those examples in the same rational affine setting. We identify their two classes explicitly after the proof of Theorem D.1.

The earlier block problem requires disjoint blocks. Overlaps contribute to the affine classification here. For example, the dyadic triple includes the two blocks beginning at $6M$ and $6M + 3$, which share the entry $6M + 3$. For pure dilations of $\{1, \ldots, n\}$, Pilz [22, Corollary 2] proved that the symmetric difference of any nonempty finite family of distinct positive integer dilates has at least n elements when $n \leq 6$, which includes $n = 4$. See also Holdridge and Pach [8, Lemma 3.3]. Ke and Meyer [11] proved the same lower bound for every n when there are at most three dilates. The affine arguments below also allow independent translations.

For a nonzero polynomial $P = c\prod_{\pi} \pi^{e_\pi} \in \mathbb{Q}[T]$, where the π are distinct monic irreducible polynomials, write

$$\mathrm{sqk}(P) = \prod_{\pi} \pi^{e_\pi \bmod 2}.$$

Thus nonzero constants are discarded. The relevant intrinsic symmetry is

$$Q(-t-1) = Q(t). \tag{D.2}$$

Two triples of nonconstant rational affine forms are called equivalent if one is obtained from the other by a common invertible rational affine change of the variable, a permutation of the three forms, and independent reflections $L_i \mapsto -L_i - 1$.

Theorem D.1 (Three-copy affine classification)**.** *Let $L_1, L_2, L_3 \in \mathbb{Q}[T]$ be nonconstant affine forms. The condition*

$$\deg \operatorname{sqk}\big(Q(L_1)Q(L_2)Q(L_3)\big) = 2 \tag{D.3}$$

defines exactly seven equivalence classes. Representatives are listed in Table 1.

If $L_1, L_2, L_3 \in \mathbb{Z}[T]$, condition (D.3) *holds if and only if, after permutation and independent reflections, there is a nonconstant $M \in \mathbb{Z}[T]$ such that*

$$(L_1, L_2, L_3) = (M, 2M, 2M + 1). \tag{D.4}$$

Table 1: The seven rational affine classes. The last column counts unordered triples of root sets containing the fixed set $\{0, 1, 2, 3\}$ in the finite enumeration used in the proof.

Class	Representative	Normalized triples
R_1	$(t/4,\ t,\ t + 1/3)$	4
R_2	$(t/3 - 1/9,\ t/2 - 1/6,\ t)$	6
R_3	$(t/4 - 1/3,\ t,\ t + 1/3)$	2
R_4	$(t/3 - 2/9,\ t/2 - 1/3,\ t)$	6
R_5	$(t/2,\ t,\ t + 1)$	3
R_6	$(t/2,\ t,\ t + 1/3)$	6
R_7	$(3t/4,\ t,\ t + 1/3)$	2

Proof. The four roots of Q form a rational affine copy of $S = \{0, 1, 2, 3\}$. Let S_i be the root set of $Q(L_i)$. All roots of each individual factor are simple, so (D.3) is equivalent to

$$|S_1 \triangle S_2 \triangle S_3| = 2. \tag{D.5}$$

The sets S_i are pairwise distinct: if two were equal, the symmetric difference would be the remaining four-point set. Moreover, each root set determines its affine argument up to (D.2). It therefore suffices to classify triples of four-term rational arithmetic progressions, up to a common affine map and permutation.

We first reduce exhaustiveness to a finite computation. Let u_j be the number of points lying in exactly j of the three sets. Then

$$u_1 + u_3 = 2, \qquad u_1 + 2u_2 + 3u_3 = 12, \qquad \sum_{i<j} |S_i \cap S_j| = 5 + 2u_3 \geq 5.$$

Some pair thus meets in at least two points. Normalize one member of that pair to S. Every possible second set $B = u + vS$ is obtained by choosing distinct $i, j \in S$ and distinct $p, q \in S$ and setting

$$v = \frac{p - q}{i - j}, \qquad u = p - vi. \tag{D.6}$$

The 144 choices give 54 distinct affine maps and 27 distinct second sets. The candidate $B = S$ cannot lead to (D.5).

For $B \neq S$ put $D = S \triangle B$. Since B meets S in two or three points, $|D|$ is four or two. A third four-point progression C satisfies $|D \triangle C| = 2$ precisely when

$$|D \cap C| = \frac{|D| + 2}{2}.$$

In particular, C contains at least two points of D. Consequently, all candidates for C are obtained from (D.6), now with distinct $p, q \in D$ and distinct $i, j \in S$. Filtering these candidates by the symmetric-difference condition gives 40 ordered occurrences (B, C). Sorting the three root sets and removing duplicates gives 29 unordered triples. These steps impose no bound on the coefficients of the original affine forms.

For completeness, the orbit comparison has a similarly finite description. For each triple, map each of its three root sets to S in each of the two possible orientations. Sort the points in each image and then sort the three image sets; take the lexicographically smallest result among these six normalizations. This is a complete orbit invariant, since the only affine automorphisms of S are $x \mapsto x$ and $x \mapsto 3 - x$. Exact rational evaluation gives seven distinct invariants, with multiplicities shown in Table 1. The representatives in that table satisfy (D.5) and give these seven distinct invariants. The accompanying program `aff3_verify.py` performs the stated finite operations using rational arithmetic and checks an explicit affine map and permutation from each of the 29 triples to its representative; these witnesses are recorded in `aff3_certificate.json`. This proves the rational classification by an exhaustive finite computation.

Suppose now that the original arguments are integral. After the permitted reflections and permutation, the rational classification writes them as one of the displayed representatives with $t = AT + B$ a nonconstant rational affine form. Each representative contains the argument t itself, so $A, B \in \mathbb{Z}$. Classes R_1, R_3, R_6, R_7 would require both t and $t + 1/3$ to be integral. Class R_2 would require $(3B - 1)/9 \in \mathbb{Z}$, and class R_4 would require $(3B - 2)/9 \in \mathbb{Z}$; both are impossible modulo three. Only R_5 remains. Its first argument $t/2$ is integral, so $t = 2M$ for a nonconstant $M \in \mathbb{Z}[T]$, giving (D.4).

Conversely, direct polynomial multiplication gives

$$Q(M)Q(2M)Q(2M+1) = 16\big[M(M+1)(2M+1)(3M+1)(3M+2)\big]^2 \times (6M+1)(6M+5). \tag{D.7}$$

For nonconstant affine M, the two residual linear factors are distinct, proving (D.3). □

Remark D.2 (Earlier examples and the integral restriction). The six entries with $(k, r) = (4, 3)$ in Yıldız [34, Table B.1], after division of each block argument by three as in (D.1), belong respectively to

$$R_2, \quad R_4, \quad R_2, \quad R_2, \quad R_4, \quad R_2.$$

Their explicit affine equivalences are included in `aff3_yildiz_comparison.json`. Those block arguments do not become three integral arguments of Q under this division. They are therefore consistent with the unique integral class R_5. The thesis searches depend on coefficient bounds and serve to produce examples; the overlap reduction above is what makes the present seven-class enumeration exhaustive without such bounds. The integral conclusion concerns exactly three nonconstant affine copies and a residual degree of exactly two. It makes no classification claim for more copies or for nonlinear substitutions.

Remark D.3 (Exact verification). The accompanying program uses Python's standard library and rational arithmetic. It checks the finite enumeration, the seven-class partition, all 29 affine witnesses, the integral obstructions, and identity (D.7).

E Multiplicative examples for the dyadic identity

For a completely multiplicative sign function f, write

$$C_f(n) = f(n)f(n+1)f(3n+1)f(3n+2), \qquad R_f(n) = f(6n+1)f(6n+5).$$

Identity (2.1) gives $C_f(n)C_f(2n)C_f(2n+1) = R_f(n)$. In the proof of the elementary bound, the arithmetic input was the vanishing mean of this residual pair. Two examples show how the source of that cancellation matters.

E.1 A mixture of modified characters

Fix $\varepsilon \in \{-1, 1\}$ and define ψ_ε on primes by

$$\psi_\varepsilon(3) = \varepsilon, \qquad \psi_\varepsilon(p) = \left(\frac{p}{3}\right) \quad (p \neq 3),$$

extending completely multiplicatively. These are the classical modified Legendre symbols $f_3^{\pm}$; see Lehmer and Lehmer [16] and Klurman et al. [14, §2.2]. The choice ψ_+ is the function λ_3 of Borwein et al. [1].

Proposition E.1 (A centred residual after mixing)**.** *For every $n \geq 1$, $R_{\psi_\varepsilon}(n) = -1$. For every finite interval J of positive integers,*

$$\sum_{n \in J} (C_{\psi_\varepsilon}(n) - 1/3) = O(1)$$

with an absolute constant. Choose a single function F for all n, with equal probabilities on the constant function 1 and on ψ_ε. Then $\mathbb{E}_F R_F(n) = 0$ at every n, and for $I = (U, V]$ of positive harmonic mass $H_I = \sum_{n \in I} 1/n$,

$$\mathbb{E}_F \mathbb{E}_I^{\log} C_F = \frac{2}{3} + O((UH_I)^{-1}).$$

Proof. On the two unit classes modulo three, $\psi = \psi_\varepsilon$ has values $1, -1$. Hence $R_\psi(n) = -1$ and $C_\psi(n) = -\psi(n)\psi(n+1)$. For $m \geq 1$ the recurrence $\psi(3m) = \varepsilon\psi(m)$ gives the three values

$$C_\psi(3m) = -\varepsilon\psi(m), \qquad C_\psi(3m+1) = 1, \qquad C_\psi(3m+2) = \varepsilon\psi(m+1).$$

The sum over a complete block differs from 1 by $\varepsilon(\psi(m+1) - \psi(m))$. These differences telescope. The incomplete blocks at the ends of J, including possible terms at $n = 1, 2$, have bounded total contribution. Partial summation against $1/n$ now gives $\mathbb{E}_I^{\log} C_\psi = 1/3 + O((UH_I)^{-1})$. The other source has $C_1 = R_1 = 1$, which gives the result. □

The support constraint of Proposition 2.1 is already visible at each n. One of $n, 2n, 2n+1$ is 1 mod 3, where C_ψ equals 1. Since the triple product is -1, exactly one of the three signs is negative. The constant source contributes $+++$. Conditional on the global choice of source the residual is identically 1 or identically -1; its mean is zero after averaging over the choice. The upper endpoint is attained under this averaging.

The resulting three-coordinate law is the symmetric point $\theta = 1/6$. Indeed, $\psi(m)$ is determined by $m \bmod 3^k$ when $3^k \nmid m$, since this residue gives $v_3(m)$ and the unit part modulo three. Hence $C_\psi(m) = -\psi(m)\psi(m+1)$ agrees with a periodic function of modulus 3^k off the two classes $0, -1$. Each map $n \mapsto 2n + j$ permutes all residue classes modulo 3^k. Equidistribution on fixed classes gives the same harmonic mean for the periodic function at n and at $2n + j$; choosing it sign-valued on the exceptional classes incurs an error $O(3^{-k})$ in the limiting means. Letting $k \to \infty$ shows that both child means, like the root mean, are $1/3$ for ψ and $2/3$ after mixing. Thus each single-minus state has limiting mass $1/6$. This identifies the local triple law; it uses no independence between subtrees.

E.2 Signs changed at finitely many primes

For a finite set of primes S, let $f_S(m) = (-1)^{\sum_{p\in S} v_p(m)}$, the function λ_S in the notation of Krishnamoorthy [15]. These signs admit products of local means. For general results on such products see Klurman [12]; the local factor for distinct shifts also appears in Krishnamoorthy [15, Theorem 1].

Proposition E.2 (Comparison for a fixed set of primes)**.** *Let c_S, r_S denote the natural means of C_{f_S}, R_{f_S}, respectively. They exist and agree with the limits of the unrestricted harmonic averages on every sequence of windows of diverging harmonic mass. They are given by $c_S = \prod_{p\in S} c_p$ and $r_S = \prod_{p\in S} r_p$, with*

	$p=2$	$p=3$	$p\geq 5$
c_p	$-1/3$	0	$(p-7)/(p+1)$
r_p	1	1	$(p-3)/(p+1)$.

Moreover $|c_S| \leq 3r_S^2$, with equality precisely for $S = \{5\}$.

Proof. First truncate the valuations of the individual affine factors at K. For each p the truncated product is periodic modulo p^K, so the Chinese remainder theorem factors the mean over S. Truncation can change a value only when p^K divides one of the factors. Its solution set has density $O(p^{-K})$ for each form here: a unit slope gives one class, and a prime dividing the slope does not divide the constant term. For fixed S the error therefore tends to zero as $K \to \infty$. This proves factorization for natural means. The same proof applies on the stated harmonic windows, since each fixed residue class has its usual limiting weight there.

For $p \geq 5$, the four roots of the factors defining C_f are distinct modulo p, as are the two roots for R_f. For an affine form of unit slope the density of odd valuation is $\sum_{j\geq 0}(p^{-(2j+1)} - p^{-(2j+2)}) = 1/(p+1)$. With distinct roots these events are disjoint. The mean sign for d such forms is $1 - 2d/(p+1)$, giving the last column.

For $p = 2$, both residual factors are odd. Among the four quartic factors exactly two are even. On each class $n = 4k + s$, one has valuation one and the other is four times a linear form whose slope is odd. For $s = 0, 1, 2, 3$, the latter form is respectively $k, 3k+1, 3k+2, k+1$. Its sign has mean $1 - 2/(2+1) = 1/3$, which gives $c_2 = -1/3$. For $p = 3$ the residual factors and the last two quartic factors are units. The conditional quartic means on $n = 0, 1, 2 \bmod 3$ are $-1/2, 1, -1/2$: in either nonconstant case the divisible factor is three times a form of unit slope. Thus $c_3 = 0$.

All r_p are positive. If 3 or 7 belongs to S then $c_S = 0$. Otherwise the factors above give

$$\frac{|c_S|}{r_S^2} = 3^{\mathbf{1}_{\{5\in S\}} - \mathbf{1}_{\{2\in S\}}} \prod_{\substack{p\in S\\ p\geq 11}} \left(1 - \frac{16}{(p-3)^2}\right).$$

Every factor in the product lies strictly between zero and one. The ratio equals three exactly for $S = \{5\}$. □

Each fixed finite set has $r_S > 0$. Along any sequence of finite sets for which these limiting residual means tend to zero, the associated limiting quartic means also tend to zero. The estimate concerns means already taken with S fixed. The truncation argument gives no uniform estimate when S varies with the observation window.

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