Transitive sets and Graham's radius conjecture

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Abstract

We prove that every nonempty finite Euclidean Ramsey set of circumradius ρ is Ramsey on sufficiently high-dimensional spheres of every prescribed radius $R > \rho$. We include and use an elementary proof that all finite transitive Euclidean sets are Ramsey. We also use the solvable dense block theorem derived from the Kneser–shift theorem [10] and adapt the necessity argument from OpenAI's characterization of Ramsey sets [8].

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Author's note. ChatGPT produced for me the first draft of a proof that all finite transitive Euclidean sets are Ramsey on 4 September 2026. I announced the result in the arXiv version of [9] submitted on 20 September, before OpenAI's classification preprint dated 23 September [8]. I feel quite bad for not being able to produce a proper write-up during such a long time, sorry everyone!

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1 Introduction

A finite Euclidean set P is *Ramsey* if for every r some Euclidean space contains a monochromatic congruent copy of P in every r-coloring. A finite set is *transitive* if its isometry group acts transitively on its points. Leader, Russell and Walters [7] conjectured that a finite set is Ramsey exactly when it embeds in a finite transitive set. Kříž [5] proved Ramseyness for configurations with a solvable transitive isometry group. OpenAI [8] established the implication from subtransitivity in its classification of finite Ramsey configurations, and showed that the converse fails using the cyclic kites of [6]. Section 3 gives a separate proof of the implication from subtransitivity using the letter-splitting argument of the earlier draft [11]. It also proves the fixed-block-size statement for every template, as in the conjecture of Ivan, Leader and Walters [3].

Every nonempty finite Ramsey set is spherical, as we also show in Appendix A. We measure its circumradius in its affine span. Call a set *nearcircumsphere-Ramsey* if every finite coloring of a sufficiently high-dimensional sphere of each prescribed radius greater than its circumradius has a monochromatic congruent copy. Graham studied Euclidean Ramsey problems on spheres [1], and Reiher [12] described the radius conjecture for finite spherical sets. That formulation includes configurations that are already non-Ramsey in Euclidean space, such as the cyclic heptagon in [9]. Theorem 1.1 proves the radius conclusion with the Ramsey hypothesis. The Kneser–shift dense block theorem of [10] gave this conclusion for solvable transitive sets. We use it in Section 3 to handle arbitrary finite group actions, and in the direct kite proof in Section 4.

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Theorem 1.1. *Let P be a nonempty finite Ramsey set of circumradius ρ. For every r and every $R > \rho$, there is an N such that every r-coloring of the sphere of radius R in $\mathbb{R}^N$ contains a monochromatic congruent copy of P.*

The proof is in Section 2. It uses the orbit lemma from Section 3 and the table lemma proved in Appendix A.

Theorem 1.2. *Every finite transitive Euclidean set, and hence each of its subsets, is Ramsey.*

The proof is in Section 3.1.

2 Words on a sphere

We use tables of real numbers as points in Euclidean space. For a table of length m put $\|x\|^2 = m^{-1}\sum_{j=1}^{m} x_j^2$. This simply scales the usual coordinates by $1/\sqrt{m}$. In particular repeating an entire table any number of times preserves all distances.

Lemma 2.1. *Let $P = \{p_1, \dots, p_s\}$ be Ramsey, with circumradius 1. For every $\delta > 0$ there are a finite list $b = (b_t)_{t\in T}$ and lists $v_1, \dots, v_s$ on a common finite index set L, with the following properties.*

1. *Each list has sum zero. In the normalized Euclidean norm, $\|b\|^2 = 1 + \delta$, $\|v_i\|^2 = \delta$, and $\|v_i - v_j\|^2 = \delta\|p_i - p_j\|^2$.*
2. *For each i, fill a $T \times L$ table with entries $b_t + v_i(l)$. Its entries can be rearranged into a table f_i so that*

$$\|f_i - f_j\|^2 = (1 + \delta)\|p_i - p_j\|^2 \quad \text{for all } i, j.$$

The rearrangement may depend on i. This is the only place in the proof that asks which finite configurations are Ramsey. Its proof is in Appendix A.

For color control we use Lemma 3.5, proved in Section 3. Given a finite orthogonal group acting on a finite set X on a sphere, we may embed X isometrically in a slightly larger prescribed sphere so that each orbit has one color. In particular, once an entire orbit of tables is in X, we may make the allowed rearrangements without losing the color.

Proof of Theorem 1.1. The singleton case is immediate. Rescale to $\rho = 1$. Choose a Hales–Jewett length [2] n for s letters and r colors. Take $M > n$ large enough that $1 + (n + 1)/M < R^2$, and use Lemma 2.1 with $\delta = 1/M$.

For each word $w \in [s]^n$ make the following table on $T \times L^n$:

$$B(w)(t, l_1, \dots, l_n) = b_t + \sum_{k=1}^{n} v_{w_k}(l_k).$$

All n letters share the same entry b_t. When we square and sum, every cross term vanishes: its two factors depend on different indices and each has sum zero. Thus $\|B(w)\|^2 = 1+(n+1)/M$ for every word. Form a longer table $V(w)$ by concatenating M identical copies of $B(w)$. Its norm is the same.

Allow arbitrary permutations of entries within each of the M copies, independently. Let X consist of all $V(w)$ and all their images under this finite permutation group. The orbit lemma embeds X into the prescribed colored sphere so that the color of each table survives these permutations. Give each word w the color of $V(w)$ in this embedding. Hales–Jewett

gives a monochromatic line. Write $w(i)$ for its word with variable letter i, and q for the number of variable positions.

Choose just one of those positions, say j. Inside $B(w(i))$, hold all indices other than t, l_j fixed. The resulting slice is the small table $b_t + v_i(l_j)$ plus a constant. Apply the permutation from Lemma 2.1 in every such slice. The result is

$$f_i(t, l_j) + \sum_{k \neq j} v_{w(i)_k}(l_k).$$

Make this switch in the first $M - q$ copies of $B(w(i))$. It is an allowed permutation, so the resulting long tables still have the same color.

For roles i, i', put $D = \|p_i - p_{i'}\|^2$. In an untouched copy the q variable positions contribute $q\delta D$ to the squared distance. In a switched copy the distinguished position contributes $(1 + \delta)D$ and the other $q - 1$ positions still contribute δD each. Cross terms again vanish by summing a zero-sum difference over its own index. Thus switching one of the M copies gains exactly D/M in the squared distance of the long tables. Their final squared distance is

$$q\delta D + \frac{M - q}{M} D = D.$$

This holds for every pair of roles and proves the theorem. □

Consequently the Ramsey, sphere-Ramsey and nearcircumsphere-Ramsey properties coincide for finite Euclidean sets. Indeed, the reverse implications follow by restricting a coloring to a sphere.

3 Dense fillings from solvable permutation groups

A permutation of positions acts on a word by permuting its coordinates. We shall use the following elementary observation repeatedly. Suppose two lists of words, all of the same length L, can be matched by a permutation of the lists in K, and each matched pair can be matched internally by an element of $H \leq S_L$. Then their concatenations are in the same orbit of the wreath product $H^n \rtimes K$, where n is the length of the lists. This group is solvable whenever H and K are solvable. A coordinatewise change of alphabet commutes with every permutation of positions.

For a finite group G acting on a finite set A, a *filling* is a word $F(x)$ whose entries are fixed elements of A or expressions hx with $h \in G$. We call the latter positions active, including when two expressions happen to agree as functions. The activity is the fraction of active positions. Say that a filling is *certified* on specified orbits of G if a fixed solvable group of position permutations puts $F(x)$ and $F(x')$ in the same orbit whenever x, x' belong to the same specified orbit. Different input orbits can give different output orbits.

Lemma 3.1. *Every finite group G has a positive-activity filling for its left regular action certified on G.*

Proof. We give the deterministic version of the splitting argument in [11], which builds on the product and orbit method of Kříž [5]. First, a regular-action filling for a group gives certified fillings on every finite alphabet on which the group acts. Indeed, to treat the orbit of a_0, replace a fixed letter c by ca_0 and each active formula hg by hx. On that orbit the result is the coordinatewise image of the regular filling under $g \mapsto ga_0$. To combine a filling F certified on some earlier orbits and a filling D certified on a new orbit, replace each letter of $D(x)$ by its F-word. On the new orbit a permutation matches the outer letters exactly.

On an earlier orbit corresponding outer letters are either equal fixed letters or belong to that same earlier orbit, so their inner words match independently. The wreath product certifies the result, whose activity is the product of the two activities. Iteration treats every orbit.

An ordinary substitution in variables $z_1, \dots, z_k$ uses fixed letters and occurrences of those variables. In the following induction every variable occurs the same positive number of times. For a template w, write $\mathcal{R}(w)$ for its rearrangements. We say a substitution is certified on $\mathcal{R}(w)$ if its values there belong to one orbit of a solvable group of position permutations.

We first prove the lemma for S_m by induction. The group S_1 is immediate. Suppose $m \geq 2$ and the regular filling is available for S_{m-1}. Put $k = m - 1$. On the alphabet of k-letter words over any finite alphabet A, use the certified S_k-filling obtained in the first paragraph, with S_k permuting positions in a letter. Expand each letter to its k ordinary entries. The result repeats each of k variables equally often, since an active letter contains every variable once. Inputs with the same content give expanded words matched by the solvable group permuting the k-letter cells. This construction also works for a pair of k-letter words. Apply the same substitution to each word, and allow exchange of the two cells. The two-word substitution, denoted by $D(u, v)$, is certified whenever the unordered pair of contents is the same, and each of the $2k$ variables occurs equally often. Its certifying group is a wreath product with S_2.

For example, the splitting step for 112 sends $112, 121, 211$ respectively to

$$(12 \mid 11), \qquad (21 \mid 11), \qquad (11 \mid 21).$$

Rearranging inside a half and exchanging the halves matches all three. The extra fixed 1 makes this possible. Proposition 3.4 below explains how to recover almost all of the positions spent on such fixed letters.

We construct certified ordinary substitutions for all the length-m templates

$$0^m, \quad 0^{m-1}1, \quad \dots, \quad 0\,1 \cdots (m-1).$$

For the constant template take the identity substitution. In passing to the next template w, let a, b be the two letters identified by the map η onto the previous alphabet. Delete one occurrence of each from w, leaving a fixed word v of length $m - 2$. For $1 \leq j \leq m$ set

$$L_j(z) = (z_1 \cdots z_{j-1} z_{j+1} \cdots z_m,\ z_j v).$$

For $u \in \mathcal{R}(w)$, the unordered pair of contents of $L_j(u)$ depends only on $\eta(u_j)$. In particular, when $u_j = a$ or b those contents are the contents of av and bv. Choose a representative pair for each possible value of $\eta(u_j)$. In the certified substitution for $\mathcal{R}(\eta(w))$, replace a fixed symbol by its representative pair and a variable z_j by $L_j(z)$. Apply D to each resulting pair. The certifying permutation of the outer positions matches the η-symbols, so matched pairs have identical unordered contents and the inner words match as well. Each variable position of the outer substitution gives one occurrence of every original variable before applying D. Thus uniform positive multiplicities are preserved.

We have obtained an ordinary substitution B certified on all permutations of $1, \dots, m$. Let $H \leq S_m$ be the stabilizer of m, and set $s_j = (j\, m)$, taking s_m to be the identity. By induction and the first paragraph, the left H-action on S_m has a filling E certified on every orbit. In B replace a fixed j by s_j and a variable z_j by $s_j \pi$, and apply E to each slot. The H-orbit of a permutation γ is determined by $\gamma^{-1}(m)$. The list of orbit indices of the slots is therefore exactly

$$B(\pi^{-1}(1), \dots, \pi^{-1}(m)).$$

These lists are matched by the certifying group of B, and matched E-words are matched internally. The resulting word is a certified regular S_m-filling, with positive activity because both substitutions have active positions. This closes the induction.

Finally, list a general finite group as $g_1, \dots, g_m$. Use the certified ordinary substitution on all permutations of m distinct letters just proved, relabeling its alphabet by the g_j. On inputs $(g_1 g, \dots, g_m g)$ its values are in one solvable orbit and its active entries are precisely of the form $g_j g$. This gives the desired filling. □

3.1 Fixed block sizes and transitive sets

A b-uniform block substitution on words of length m repeats each of the m input positions in exactly b output positions and fixes all other positions. The output blocks need not be intervals. The proof of Lemma 3.1 gives a single such substitution for each template whose images lie in a single orbit of a finite solvable group of coordinate permutations.

Theorem 3.2. *For each m there is a block size b_m such that for every finite alphabet A, every $w \in A^m$ and every r, some n makes every r-coloring of A^n contain a monochromatic b_m-uniform block copy of $\mathcal{R}(w)$.*

Proof. Take the certified substitution B for the distinct-letter template $12\cdots m$ in the proof of Lemma 3.1, with block size b and solvable coordinate group K. Let Y be the K-orbit of $B(12\cdots m)$. By the uniform Hales–Jewett theorem for finite solvable groups of Kanellopoulos and Karamanlis [4, Theorem 1.5], applied to the transitive action on Y, there is a total number h of active positions, independent of r, such that every r-coloring of a sufficiently long power of Y is constant on the image of a K-filling with h active positions. A coloring of words over $[m]$ induces a coloring of this power by concatenation. Composing the filling with B and expanding the words over Y repeats every original input position hb times, since an active position applies a coordinate permutation of B and a fixed position supplies a fixed word. This proves the claim for $12\cdots m$, with $b_m = hb$. For an arbitrary w, pull the coloring back under the letter map $j \mapsto w_j$ and then apply the same map to the fixed symbols of the resulting substitution. □

Corollary 3.3. *Every finite group has the uniform Hales–Jewett property. For any template w, the conclusion of Theorem 3.2 also holds in a layer of words containing equally many copies of every letter of the alphabet.*

Proof. For a group $G = \{g_1, \dots, g_m\}$, the words $(g_1 g, \dots, g_m g)$, for $g \in G$, are rearrangements of $(g_1, \dots, g_m)$. Their images under the substitution in Theorem 3.2 give a monochromatic G-filling in which each formula $g_j g$ occurs exactly b_m times. For the balanced layer, take an n from Theorem 3.2 and append to each n-letter word exactly the letters needed to reach n copies of each alphabet symbol, in a fixed order. Pull the coloring of the balanced layer back by this map. All words of the resulting monochromatic block copy have the same content, so their appended suffix is a common fixed word. □

Proof of Theorem 1.2. The empty set is immediate. Let $P = Gx_0$, where G is the isometry group of P in its affine span. It acts orthogonally after placing the centroid at the origin in its affine span. By Lemma 3.1, there is a filling $F(g)$ of length N, with $b > 0$ active positions, whose values lie in one orbit of a finite solvable permutation group K. Evaluate each letter at x_0 and scale all coordinates by $1/\sqrt{b}$. The resulting point depends only on $x = gx_0$, and each active position contributes $\|x - y\|^2/b$ to the squared distance between the images of x, y. Thus the map is an isometry on P. Its image lies inside one finite K-orbit, a solvable transitive Euclidean configuration. The Ramsey theorem of Kříž [5] applies to this orbit, and the property passes to subsets. □

3.2 Fillings with arbitrarily few fixed positions

Proposition 3.4. *For every finite group action and every $\delta > 0$ there is a filling of activity exceeding $1 - \delta$ certified on every orbit of the action.*

Proof. For the regular action choose the filling $F(g)$ from Lemma 3.1, with activity $q > 0$ and certifying group H. Write $y = F(e)$. In any certified regular filling with activity d, expand each active entry $a = hg$ to the word ya, multiplying every coordinate on the right. All entries of this block are active formulas. Expand each fixed letter a to the word $F(ga^{-1})a$. Its active entries are of the form $h'g$, and its fixed entries are fixed. For each g this latter block is in the H-orbit of ya, since right multiplication by a commutes with coordinate permutations. Thus each expanded block can be matched to the canonical word ya determined only by its actual old letter a. The wreath product certifies the expanded filling, and the new activity is $d+(1-d)q$. The inactive fraction tends to zero under iteration. For an arbitrary action use the one-orbit transfer and composition from the opening of the proof of Lemma 3.1. The number of orbits is finite, and the product of their chosen activities can be taken greater than $1 - \delta$. □

Lemma 3.5. *Let X be a finite subset of an origin-centered sphere of radius $s > 0$ and let a finite orthogonal group G preserve X. For every $R > s$ and number of colors r, some R-sphere has the following property. Every r-coloring admits a single isometric embedding of X whose color is constant on each G-orbit.*

Proof. Choose the certified filling from Proposition 3.4 of length N with b active positions, so that $s\sqrt{N/b} < R$. The map $x \mapsto b^{-1/2}F(x)$ is isometric, since every active position applies an orthogonal transformation and the other positions are fixed. The entire orbit of its image under the solvable certifying group of position permutations lies on the sphere of radius $s\sqrt{N/b}$. Apply the solvable dense block theorem of [10] to this orbit union, with one variable. More explicitly, put $s' = s\sqrt{N/b}$ and take loss small enough that the theorem's output length n and active size k satisfy $n(s')^2/k \le R^2$. Embed the word space on that orbit union into the R-sphere by scaling each of the n coordinates by $1/\sqrt{k}$ and appending the common scalar $\sqrt{R^2 - n(s')^2/k}$. The resulting orbit-insensitive block map is an isometry on its alphabet, by the same distance calculation. Its composition with $b^{-1/2}F$ gives the assertion. □

Corollary 3.6. *Every nonempty finite transitive Euclidean set of circumradius ρ embeds isometrically into a finite solvable transitive set of circumradius at most R for every $R > \rho$.*

Proof. For $\rho > 0$, use the construction in Theorem 1.2 with the filling from Proposition 3.4 chosen so that $\rho\sqrt{N/b} < R$. The orbit lies on the sphere of radius $\rho\sqrt{N/b}$, and its circumradius is at most this quantity. A singleton is already solvable transitive. □

4 A direct construction for the kite

For $0 < a < 1$, set $b = \sqrt{1 - a^2}$. We use the solvable orbitwise lemma of [10] directly to show that the six points $(\pm 1, 0)$, $(\pm a, \pm b)$ with independent signs are nearcircumsphere-Ramsey.

Fix r and write $t = a^2$. Choose $r+1$ consecutive integers k so large that $c_k = t^{k+1}/(1-t) < 1$. Sort the numbers $0, 1$ and all the c_k. Let the gaps between consecutive numbers be columns, with widths d_l. Make an array of cells (j, l), taking levels $j = 0, \dots, 1 + \max k$. Give cell (j, l) weight $(1 - t)t^j d_l$. Take S_k to consist of all cells in levels $1, \dots, k$ and the initial columns of level zero up to c_k. The total weight of S_k is $t(1 - t^k) + (1 - t)c_k = t$. Define a unit vector x_k by taking the square root of the cell weight divided by $\sqrt{t}$ on S_k, and zero elsewhere. Let T shift the finite stack of levels cyclically, preserving the interval labels. For $h < k$ the shifted support of x_h lies in S_k and the ratio between consecutive cell weights is t. Thus

$$x_k = aTx_h + be_{hk}$$

for a unit vector e_{hk} disjointly supported from Tx_h.

The coordinate sign changes and T generate a finite solvable group. Apply the solvable orbitwise lemma to the union of its orbits through the $r + 1$ vectors. Two orbits, indexed by $h < k$, have the same color in the resulting isometric copy. The first contains $\pm Tx_h$, and the second contains all four vectors $\pm aTx_h \pm be_{hk}$. This proves the claim, including the transcendental parameters in the Leader–Russell–Walters family [6]. The value $a = 0$ gives a square.

A Construction of the two tables

We prove Lemma 2.1, beginning with the signed list needed in the construction. We give the separation argument explicitly. It follows the modular coloring in [8, Proposition 2.4], applied directly to histograms of affine forms.

A finite signed list from the Ramsey property. Take the affine span of P as $\mathbb{R}^d$. Let V be the rational vector space with one formal basis vector $[t]$ for each real number t. In $V^s \oplus \mathrm{Sym}_d(\mathbb{R})$, considered over $\mathbb{Q}$, take the rational span of all vectors

$$([b + u^\top p_1], \dots, [b + u^\top p_s], uu^\top) \qquad (b \in \mathbb{R},\ u \in \mathbb{R}^d).$$

We claim that this span contains $(0, \dots, 0, I_d)$. Otherwise extend a basis of the span to obtain a rational linear functional vanishing on it and taking value 1 at this last vector. Its coordinate functionals give functions $h_i : \mathbb{R} \to \mathbb{R}$ and a rational linear functional L on symmetric matrices such that

$$\sum_i h_i(b + u^\top p_i) = -L(uu^\top), \qquad L(I_d) = 1.$$

Putting $u = 0$ gives $\sum_i h_i(b) = 0$ for every b. In any ambient dimension put $H_i(x) = \sum_j h_i(x_j)$. Any ordered congruent copy has points $x_i = t + Qp_i$ for a linear isometry Q (the differences have the same Gram matrix by polarization). Summing the displayed identity over the rows of Q, using $Q^\top Q = I_d$, gives $\sum_i H_i(x_i) = -1$. On the other hand $\sum_i H_i(x_1) = 0$.

Divide $[0, 2)$ into $2s + 1$ equal half-open intervals. Color x by the interval containing $H_i(x)$ modulo 2 for each i. In a monochromatic copy every difference $H_i(x_i) - H_i(x_1)$ is within $1/s$ (strictly) of an even integer. Their sum is -1, an impossibility. The number of colors depends only on s, contradicting the Ramsey property.

Writing the target as a finite rational combination of the displayed generators gives a finite rational signed list of rows (b, u). At each p_i its evaluation histogram is zero, while its slope second moment is exactly I_d. In particular its mass is zero. Evaluating the full second moment at $(1, p_i)$ gives an equation $\|p_i\|^2 + 2c \cdot p_i + c_0 = 0$ common to all i. Completing the square proves sphericity. For two or more points we can now center and rescale to circumradius one, and apply the same span argument to this Ramsey configuration. In these coordinates the affine span condition gives $c = 0$ and $c_0 = -1$, so the full signed second moment is $J = \mathrm{diag}(-1, I_d)$.

Use separate rational bases for the intercepts and for the slope vectors in the signed list. Rescale the bases to clear denominators. Thus rows are Dz for integer addresses $z = (x, y) \in \mathbb{Z}^{k_c} \times \mathbb{Z}^{k_u}$, with the two blocks giving the intercept and slope separately. Write $k = k_c + k_u$. The signed list becomes a rational function $\nu(z)$ of finite support and mass zero. Its first moment is zero: its physical mean is zero by equality of histograms and affine spanning, and the basis columns are rationally independent. Put $C = \sum_z \nu(z) z z^\top$, so that $DCD^\top = J$. A translate of ν also has zero histogram at every p_i, because translating the addresses adds the same scalar to every evaluation there.

Two walks. We choose two finite symmetric rational step distributions P_0, Q_0 on $\mathbb{Z}^k$. The intercept and slope addresses of a P_0-step will be independent. Their second moment matrices will be $S, S + C$, where $DSD^\top$ is as close as desired to $\mathrm{diag}(1 + \delta/2, (\delta/2) I_d)$.

The matrix D has full row rank since J is invertible. With $D^+ = D^\top (DD^\top)^{-1}$, lift the indicated target matrix V to $D^+ V (D^+)^\top + L \Pi_{\ker D}$. It is block diagonal in our two address blocks. This lift and its sum with C are positive definite for sufficiently large L: their restrictions to $\mathrm{im}\, D^\top$ are positive definite since $V, V + J > 0$, and a large L absorbs the cross terms by completing the square. Take a sufficiently close rational block diagonal approximation S. Every rational positive definite matrix is a sum of positive rational multiples of $vv^\top$ with integer v, by symmetric Gaussian elimination over the rationals and clearing denominators. Realize each term by jumps $\pm Hv$ of sufficiently small equal rational probability, taking H large and putting the unused probability at zero. Apply this separately to the two blocks of S and take their product to get P_0. Apply it to $S + C$ to get Q_0.

Write $*$ for addition of independent steps (convolution), and powers for repeated convolution. To compare n steps of the two walks, replace them one at a time. Thus, with

$$W_n = \sum_{j=0}^{n-1} Q_0^{*j} * P_0^{*(n-1-j)}, \quad Q_0^{*n} - P_0^{*n} = (Q_0 - P_0) * W_n.$$

The signed step $E = \nu - (Q_0 - P_0)$ has zero moments through degree two. Replacing $Q_0 - P_0$ by ν makes the right side a weighted sum of such translates. We show that after a little averaging, the resulting error can be covered at negligible cost by padding the two address lists separately.

Counting the error. Let $m = \lfloor \sqrt{n} / \log n \rfloor$. In each of the k address coordinates add four independent uniform integers from $[-m, m]$, independently across coordinates. Denote this product distribution by K. Put

$$H_0 = P_0^{*n} * K, \qquad \mu = \nu * W_n * K.$$

The table H_0 is a product in the two address blocks. The exact identity

$$H_0 + \mu = Q_0^{*n} * K + E * W_n * K$$

reduces possible negative entries to the last term.

The first pointwise estimate is

$$\|E * W_n * K\|_\infty = O(nm^{-k-3}).$$

To see this, form the finite Laurent polynomial $\sum_z E(z)X^z$. Its value and all first and second derivatives vanish at $(1,\dots,1)$ by the zero moments. After multiplication by a monomial to clear negative powers, expand in the variables $X_j - 1$. Every term contains at least three difference factors. Thus $E * K$ is a fixed linear combination of translates of third differences of K. In a coordinate differentiated $h \le 3$ times, apply one difference to each of h uniform factors. Each differenced uniform factor has sum of absolute values $2/(2m+1)$, and one remaining uniform factor has maximum $1/(2m+1)$. Hence the maximum in that coordinate is $O(m^{-1-h})$. Multiplication across coordinates gives $O(m^{-k-3})$. Finally W_n has mass n.

Second, outside the box $[-\sqrt{n}\log n, \sqrt{n}\log n]^k$ the error is $O(n\exp(-c\log^2 n))$ for a fixed $c > 0$. Each summand of W_n is a sum of independent, centered, bounded steps. For one coordinate whose step magnitude is at most B, symmetry gives $\mathbb{E}e^{tX} = \mathbb{E}\cosh(tX) \le e^{t^2B^2/2}$; multiplying and optimizing the exponential Markov bound gives a tail at u of at most $2e^{-u^2/(2nB^2)}$. The support shifts from E and K are respectively $O(1)$ and $O(m)$, which gives the estimate after summing over coordinates and the n terms. All distributions are supported in a larger box $[-Bn - O(m), Bn + O(m)]^k$, enlarging B if necessary.

Retaining a rectangle. Write $H_0 = A_0 \otimes B_0$ for the two probability lists. Round the central and larger boxes outwards to integer cubes. On each cube I dominate the corresponding pointwise error bound by $e_I/|I|$, choosing rational $e_I > 0$ within a factor two of the bound times $|I|$. For the central cube, $e_I = O(n^{-1/2}(\log n)^{2k+3})$. For the larger cube, using the tail estimate for its part outside the central cube, e_I is smaller than any fixed inverse power of n. Let U_I, V_I be the uniform probabilities on the two factors of I. Choose rational s_I between $\sqrt{e_I}$ and $2\sqrt{e_I}$. Replace H_0 by the product

$$H = \left(A_0 + \sum_I s_I U_I\right) \otimes \left(B_0 + \sum_I s_I V_I\right).$$

Expanding shows that $H - H_0$ dominates $e_I U_I \otimes V_I$ on each cube. Consequently $H + \mu \ge 0$ everywhere. The two lists $H, H + \mu$ have identical evaluation histograms and equal rational mass $h = 1 + o(1)$.

On the scale $z/\sqrt{n}$, the second moments of H converge to S. Indeed, those of H_0 are $S + O(m^2/n)$. Each central padding marginal adds $O(s_I \log^2 n) = o(1)$ to the rescaled second moments, and the outer marginal adds $O(s_I n) = o(1)$. All marginal masses tend to one and their means vanish by symmetry, so the product has the asserted limit. Meanwhile μ has zero mass and first moment and second moment exactly nC: every translate of ν has these same moments as ν, and $W_n * K$ has mass n.

Finishing the lists. Normalize $H, H + \mu$ by h and use physical rows $\sqrt{h/n}\,Dz$. Their second moment difference is exactly J. Their means vanish. In the first list the intercept and slope are independent, and their full covariance approaches $DSD^\top$. Choose S close enough to the target V and then n large enough that the first intercept variance is below $1 + \delta$ and its slope covariance is strictly below δI_d. Add the same independent centered random affine form to both lists to make up the two deficits exactly. Use an independent uniform sign times the square root of the scalar deficit, and a vector of independent uniform signs multiplied by a square root of the slope deficit. This preserves both the evaluation histograms and the product structure of the first list.

Expand rational probabilities to uniform lists by repetitions. The first list now has rows indexed by the product of its intercept and slope lists. Repeat the intercept list if necessary to make the size of the rectangle divisible by all denominators of the second list. Place that entire list of affine forms on the same rectangle in any fixed order, and evaluate both arrays at p_i. The matching histograms give the required rearrangements. Writing $v_i(l) = u_l^\top p_i$, the small slope covariance δI_d gives $\|v_i\|^2 = \delta$ and $\|v_i - v_j\|^2 = \delta\|p_i - p_j\|^2$. The large slope covariance $(1+\delta)I_d$ gives the asserted distances for the rearranged tables.

Statement on the use of artificial intelligence

ChatGPT developed the proofs in this manuscript and wrote both precursor drafts and the merged text. I prompted ChatGPT to investigate the problems. In case of the transitive result, I first told it to prove it for S_5, then for S_6, then when it got stuck for S_7, I asked it to try a general argument, and then it solved the whole thing. For the ncs-Ramsey result, I recommended it to use my ncs-Ramsey result, because prior to this it was stuck. I asked for revisions to make the exposition more readable. My review of the proofs is incomplete. The argument builds on the earlier results credited in the text.

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