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GOLDBACH NUMBERS IN INTERVALS OF LENGTH $\log^2 x$ UNDER THE RIEMANN HYPOTHESIS

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Abstract. Assuming the Riemann hypothesis, we prove that for every real $x \geq r = 4.0462\ldots$ the interval $(x, x+\log^2 x]$ contains an even integer that is a sum of two primes, where $r+\log^2 r = 6$. The starting point is sharp, and $2/\log^2 4 = 1.0406\ldots$ is the optimal constant on the whole range $x \geq 4$. The best published constant was 123, due to Chirre and Hagen. Hubbard has since announced 4.45 in a preprint, and version 0.5 of this paper gave 2. The improvement comes from four changes to the classical Selberg-integral argument. We pair intervals of relative length. We use a positive-definite weight on the logarithmic scale whose width grows like $\sqrt{\log x}$. We evaluate the resulting zero sums exactly by the Weil explicit formula, which also makes the prime-power contribution negligible. And a first moment of the primes over level sets of the weight that are symmetric about $x/2$ doubles the lower bound that an empty interval forces on the weighted Selberg integral. The constant 1 is the limit of the method.

1. Introduction

A *Goldbach number* is an even integer that is the sum of two primes. The binary Goldbach conjecture asserts that every even integer greater than 2 is a Goldbach number. It has been verified up to $4 \cdot 10^{18}$ [14] but remains open. A weaker question is how long an interval must be before it is certain to contain a Goldbach number.

Under the Riemann hypothesis (RH), Linnik [11] showed that $[N, N + (\log N)^{3+\varepsilon}]$ contains a Goldbach number for large N. Kátai [9] and Montgomery–Vaughan [13] reduced the length to $C \log^2 N$, and this order has not been improved under RH alone. Goldston [7] gave a different deduction which, under the pair-correlation conjecture, yields $(2+\varepsilon)\log N$. Explicit values of C, valid for all $x \geq 2$, are 9696 (Cully-Hugill–Dudek [5]) and 123 (Chirre–Hagen [4]). More recently, Hubbard [8] has announced $C = 4.45$ for $x \geq 3$, for the open interval $(x, x + 4.45\log^2 x)$, in a preprint that has not been refereed, and version 0.5 of the present paper gave $C = 2$ for $x \geq 4$. Our main result is the following.

Theorem 1. *Assume RH. Let $r = 4.0462095747\ldots$ be the solution of $r + \log^2 r = 6$. For every real $x \geq r$ the interval $(x,\, x + \log^2 x]$ contains a Goldbach number, and for $4 \leq x < r$ it contains none. Consequently $C_* = 2/\log^2 4 = 1.0406\ldots$ is the least C such that $(x,\, x + C\log^2 x]$ contains a Goldbach number for every $x \geq 4$.*

For $x < 3.9 \cdot 10^{18}$ the theorem follows from the verification of the binary Goldbach conjecture up to $4 \cdot 10^{18}$ [14] and from the Goldbach number 6 (see §8). The content of the paper is the range $x \geq 3.9 \cdot 10^{18}$.

Remark 2 (The constant 1). The proof gives the ratio (required length)$/\log^2 x \to 1$ from below, with a margin of order $\log\log x/\log x$. So 1 is the limit of the method. Version 0.5 used only the prime-free member of each pair of intervals and reached 2, the same constant that Goldston's

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deduction [7] gives from mean-square bounds of the same size (§9). Lemma 6 also uses the other member, which must then carry about twice the expected number of primes. Going below 1 appears to need an RH-only improvement of the mean-square bound for primes in intervals of length $\asymp \log^2 x$, or of averaged prime-pair counts at such shifts.

Remark 3 (Fixed kernel). The width $V \asymp \sqrt{\log x}$ matters. With a kernel of fixed width the quantity m_W of Corollary 5 stays below $\frac{1}{2}$ by a fixed amount, so the method gives a constant strictly above 1. For the cover of version 0.5 and $V = 28$, a floating-point computation gave 2.05. We have not repeated it for the present cover, and neither computation is part of the proof.

1.1. **The classical argument and what we change.** All the explicit results above rest on the same idea. If $(x, x + H]$ contains no Goldbach number, then for every y one of two short intervals, one near y and one near $x - y$, contains no prime. A prime-free interval of length δy contributes $\delta^2 y^2$ to the Selberg integral

$$\int \big(\theta((1+\delta)y) - \theta(y) - \delta y\big)^2 \,\mathrm{d}y,$$

whereas RH bounds this integral from above. Comparing the two bounds gives a contradiction once H is large enough.

Chirre and Hagen bound the Selberg integral by

$$\int_{ax}^{x} (\theta((1+\delta)y) - \theta(y) - \delta y)^2 \,\mathrm{d}y \le K\delta x^2 \log^2(2.0001/\delta), \qquad K = 2.5571$$

[4, Thm. 3]. They then use the Montgomery–Vaughan covering, which pairs intervals of absolute length $h/2$ and integrates the prime-free deficit $\delta^2 y^2$ against Lebesgue measure. This gives $C = 48K$. Hubbard [8] keeps this framework. He replaces the sharp cutoff in the mean square by a smooth weight, which lowers the mean-square constant from 2.2258 to 1.0069 in his normalisation [8, §1], and he weights the final deduction; this gives 4.45. We instead remove four losses in the argument.

(1) *Relative intervals.* If $p \in (y, (1+\delta)y]$ and $q \in (x-y, (1+\delta)(x-y)]$, then $p+q \in (x, (1+\delta)x]$. So relative intervals with $\delta = H/x$ suffice, which saves a factor 2.

(2) *A weight that matches the deficit.* The deficit $\delta^2 y^2$ is tiny for small y, while the unweighted integral is dominated by $y \approx x$. The weight $\omega(y) = (x/y)^2 W(\log(x/y))$ makes the deficit uniform. This saves the factor $24 = 1/\int_0^{1/2} u^2 \,\mathrm{d}u$, up to $\int_{\log 2}^{\infty} W(v) e^{-v} \,\mathrm{d}v \le \frac{1}{2}$.

(3) *Exact evaluation of the zero sum.* With W positive definite, Schur's test and the Weil explicit formula evaluate the zero double sum exactly, up to explicit remainders. The prime terms of the explicit formula, averaged over the zeros, contribute the positive cost $2\delta \sum_n \Lambda(n)^2 W(\log n)/n \approx 2\delta \int_0^V vW$. This cost is $O(V^2) = o(\log^2 x)$ when $V \asymp \sqrt{\log x}$. The weight also kills prime powers below xe^{-V}, so the difference between θ and ψ becomes negligible.

(4) *A first moment over symmetric level sets.* The covering above uses only the prime-free member of each pair. The other member then contains, on average, twice the expected number of primes. We integrate over the superlevel sets of a weight that is symmetric about $x/2$. On each of them the identity $(a-1)^2 + (b-1)^2 = 2(a+b) - 2 + (a+b-2)^2$, valid when $ab = 0$, turns this first moment into a second deficit of the same size. This saves a further factor 2.

He Li independently found change (4) and the resulting constant 1. He posted them on 24 November 2026, before version 0.6 appeared, as issue #1 in the repository of this paper [10]. His

argument bounds the first moment by Fubini's theorem and the total variation of the symmetric weight. This avoids the terms in δ of Lemma 6.

1.2. **Outline.** Section 2 constructs the weight W and lists its properties. Section 3 turns the absence of Goldbach numbers into a lower bound for a weighted Selberg integral (Lemma 6); this step uses RH only through Schoenfeld's bound for θ. Sections 4–7 bound the same integral from above under RH. The explicit formula splits it into a sum over zeros, a small term from the trivial zeros and a prime-power term (§4). The sum over zeros is evaluated with the Weil explicit formula (§5) and Stieltjes integration against the zero-counting function (§6). Section 7 collects the arithmetic input. Section 8 combines the bounds and proves Theorem 1. Section 9 compares the method heuristically with Goldston's deduction; nothing in it is used in the proof. Appendix A lists the computations.

1.3. **Notation.** Throughout, p denotes a prime and Λ is the von Mangoldt function. The Chebyshev functions are $\theta(t) = \sum_{p\le t} \log p$ and $\psi(t) = \sum_{n\le t} \Lambda(n)$. Under RH we write the nontrivial zeros of ζ as $\rho = \frac12 + i\gamma$. Sums over ρ or γ run over all nontrivial zeros, of both signs, counted with multiplicity. We write γ_E for Euler's constant. The Fourier transform is $\widehat{f}(\xi) = \int f(v)e^{-iv\xi}\,\mathrm{d}v$.

2. The kernel

The argument needs a weight W on the logarithmic scale with three properties. It should have compact support $[-V, V]$, so that only primes up to e^V enter the explicit formula. It should be positive definite, $\widehat{W} \ge 0$, so that Schur's test applies in Lemma 7. And it should be flat near 0, so that the cover in Lemma 6 loses little. The autocorrelation of a cosine bump has all three.

Fix $V > 0$. Let $\varphi(s) = \cos(\pi s/V)$ for $|s| \le V/2$ and $\varphi(s) = 0$ otherwise, and put $W = \varphi * \varphi/\|\varphi\|_2^2$. A direct computation gives, with $t = |v|/V$,

$$W(v) = (1-t)\cos(\pi t) + \frac{\sin(\pi t)}{\pi} \quad (|v| \le V), \qquad W(v) = 0 \quad (|v| > V). \tag{1}$$

Write $\widehat{W}(\xi) = \int W(v)e^{-iv\xi}\,\mathrm{d}v$ and $\sigma = \pi/V$.

Lemma 4. *(i) W is even, $W(0) = 1$, W is nonincreasing on $[0, V]$, and $0 \le W \le 1$.*

(ii) $\widehat{W} = |\widehat{\varphi}|^2/\|\varphi\|^2 \ge 0$, $(2\pi)^{-1}\int \widehat{W} = 1$ and $(2\pi)^{-1}\int \xi^2\widehat{W}(\xi)\,\mathrm{d}\xi = -W''(0) = \sigma^2$.

(iii) $|W'| \le \sigma$, $|W''| \le \sigma^2$, and $W(\pm V) = W'(\pm V) = 0$. Moreover $W \in C^2$ and W''' has bounded variation, so $\widehat{W}(\xi) \ll_V (1+|\xi|)^{-4}$.

(iv) (K1) *$W(v) \ge 1 - \pi^2 v^2/(2V^2)$ for $0 \le v \le V$.*

(v) (K2) *$2\int_0^V vW(v)\,\mathrm{d}v = 2V^2/\pi^2$.*

Proof. (i) From (1), $\frac{\mathrm{d}W}{\mathrm{d}t} = -\pi(1-t)\sin\pi t \le 0$ on $[0,1]$, and $W(V) = 0$.

(ii) This is standard for an autocorrelation.

(iii) We have $W' = \pm\varphi * \varphi'/\|\varphi\|^2$ and $W'' = -\varphi' * \varphi'/\|\varphi\|^2$ (with the appropriate reflections), so Cauchy–Schwarz gives the bounds with $\|\varphi'\|/\|\varphi\| = \sigma$. The boundary values follow from (1).

(iv) Put $f(t) = W - 1 + \pi^2 t^2/2$. Then $f(0) = 0$ and $f'(t) = \pi(\pi t - (1-t)\sin\pi t) \ge 0$.

(v) Use $\int_0^1 t\cos\pi t = \int_0^1 t^2\cos\pi t = -2/\pi^2$ and $\int_0^1 t\sin\pi t = 1/\pi$. □

The cover in the next section produces the following quantity.

Corollary 5. $m_W := \int_{\log 2}^V W(v)e^{-v}\,\mathrm{d}v \;\ge\; \frac12 - 9.55\,V^{-2} - e^{-V}$.

Proof. By (K1), $m_W \ge \frac12 - e^{-V} - \frac{\pi^2}{2V^2}\int_{\log 2}^\infty v^2 e^{-v}\,\mathrm{d}v$. The last integral equals $\frac12(\log^2 2 + 2\log 2 + 2) = 1.9334\ldots$, and $\frac{\pi^2}{2}\cdot 1.9334 < 9.55$. □

3. The weighted cover

Throughout, x is large, $H > 0$, $\delta = H/x$, $\ell = \log(1+\delta)$, and

$$V_\theta(y) = \theta((1+\delta)y) - \theta(y) - \delta y, \qquad \omega(y) = \Big(\frac{x}{y}\Big)^2 W\Big(\log\frac{x}{y}\Big), \qquad \|F\|_\omega^2 = \int_0^\infty |F(y)|^2 \omega(y)\,\mathrm{d}y.$$

The next lemma shows that if $(x, x+H]$ contains no Goldbach number, then the weighted Selberg integral $\|V_\theta\|_\omega^2$ is large.

Lemma 6. *Assume RH. Let $L = \log x$, and suppose that $V \ge \log 2$, $xe^{-V} \ge 599$, and that $(x, x+H]$ contains no Goldbach number. Then*

$$\|V_\theta\|_\omega^2 \;\ge\; \delta^2 x^3\Big(2m_W - 2\delta - \frac{L^2}{2\pi\sqrt{x}}\Big).$$

Proof. Put $G(y) = (\theta((1+\delta)y) - \theta(y))/(\delta y)$. Then $V_\theta(y) = \delta y(G(y)-1)$, and

$$\|V_\theta\|_\omega^2 = \delta^2 x^2 \int_0^\infty W(\log(x/y))\,(G(y)-1)^2\,\mathrm{d}y.$$

The symmetric weight. For $0 < y < x$ let $q(y) = \min\{W(\log(x/y)), W(\log(x/(x-y)))\}$. Then $0 \le q(y) \le W(\log(x/y))$, so it suffices to bound $\int_0^x q\,(G-1)^2$ from below. By Lemma 4(i), for $0 \le s < 1$ the set $\{v \ge 0 : W(v) > s\}$ is an interval $[0, v_s)$ with $v_s \le V$. Hence $W(\log(x/y)) > s$ exactly when $y > A := xe^{-v_s}$, and $W(\log(x/(x-y))) > s$ exactly when $x - y > A$. So the superlevel set $I_s = \{y \in (0,x) : q(y) > s\}$ is $(A, x-A)$, which is empty unless $A < x/2$, and $A \ge xe^{-V}$.

One level. Fix s with $I_s = (A, B)$ nonempty, $B = x - A$. For $y \in I_s$ both y and $x-y$ exceed $A \ge 599$. If both $(y, (1+\delta)y]$ and $(x-y, (1+\delta)(x-y)]$ contained primes, these would be odd and their sum would be an even number in $(x, x+H]$. Hence $G(y)G(x-y) = 0$. For $a, b \ge 0$ with $ab = 0$,

$$(a-1)^2 + (b-1)^2 = 2(a+b) - 2 + (a+b-2)^2 \;\ge\; 2(a+b) - 2.$$

Apply this with $a = G(y)$ and $b = G(x-y)$, integrate over I_s, and use the symmetry $y \mapsto x - y$ of I_s:

$$\int_{I_s} (G-1)^2\,\mathrm{d}y \;\ge\; 2\int_{I_s} G\,\mathrm{d}y - |I_s|. \tag{P}$$

A prime p with $(1+\delta)A < p \le B$ contributes $\log p \int_{p/(1+\delta)}^{p} \mathrm{d}y/(\delta y) = (\ell/\delta)\log p$ to $\int_{I_s} G$, and the other primes contribute nonnegative amounts. So

$$\int_{I_s} G\,\mathrm{d}y \;\ge\; \frac{\ell}{\delta}\big(\theta(B) - \theta((1+\delta)A)\big),$$

which holds trivially if $(1+\delta)A \ge B$. Under RH, Schoenfeld's bound [17, Thm. 10, (6.3)] gives $|\theta(t) - t| < \sqrt{t}\log^2 t/(8\pi) \le \sqrt{x}\,L^2/(8\pi)$ for $599 \le t \le x$. Put $D_0 = |I_s| - \delta A - \sqrt{x}L^2/(4\pi)$, so that $\theta(B) - \theta((1+\delta)A) \ge D_0$. We have $1 - \delta \le (1+\delta)^{-1} \le \ell/\delta \le 1$ and $|I_s| + A \le x$. If $D_0 \ge 0$, then $(\ell/\delta)D_0 \ge (1-\delta)D_0 \ge D_0 - \delta|I_s|$. If $D_0 < 0$, then $(\ell/\delta)D_0 \ge D_0$. In both cases

$$\int_{I_s} G\,\mathrm{d}y \;\ge\; |I_s| - \delta x - \frac{\sqrt{x}L^2}{4\pi}.$$

With (P) this gives $\int_{I_s}(G-1)^2 \ge |I_s| - 2\delta x - \sqrt{x}L^2/(2\pi)$, which holds trivially when I_s is empty.

All levels. Integrate over $0 < s < 1$ and use $q(y) = \int_0^1 \mathbf{1}_{\{q(y)>s\}}\,\mathrm{d}s$:

$$\int_0^x q\,(G-1)^2\,\mathrm{d}y \;\ge\; \int_0^x q\,\mathrm{d}y - 2\delta x - \frac{\sqrt{x}L^2}{2\pi}.$$

For $0 < y \le x/2$ we have $\log(x/y) \ge \log(x/(x-y)) \ge 0$, so $q(y) = W(\log(x/y))$. By symmetry $\int_0^x q = 2\int_0^{x/2} W(\log(x/y))\,\mathrm{d}y = 2xm_W$. Multiplying by δ^2x^2 proves the lemma. □

Remark. Version 0.5 used only the prime-free member $z \in \{y, x-y\}$ of each pair, where $V_\theta(z) = -\delta z$, and obtained $\|V_\theta\|_\omega^2 \ge \delta^2 x^3 m_W$ without RH. Lemma 6 doubles the main term. On each symmetric level the occupied member carries, by the prime number theorem, about twice the expected number of primes, and the square $(a+b-2)^2 \ge 0$ turns this excess into a second deficit.

4. The explicit formula and the Schur bound

We now bound $\|V_\theta\|_\omega$ from above. Under RH, the explicit formula splits V_θ into a sum over the nontrivial zeros, a small term coming from the trivial zeros and a prime-power term. We bound each of the three in turn.

Assume RH. Put

$$c(r) = \frac{(1+\delta)^{1/2+ir} - 1}{1/2+ir} = \int_0^\ell e^{(1/2+ir)t}\,\mathrm{d}t.$$

Then $|c(r)| \le \ell e^{\ell/2}$ for real r. By the explicit formula for ψ, in the form of [4, eq. (3.2)], for $y > 1$ such that neither y nor $(1+\delta)y$ is an integer,

$$V_\theta(y) = A(y) + B(y) - P(y), \qquad A(y) = -\sum_\rho c(\gamma)y^\rho,$$

where the sum is the symmetric limit and

$$B(y) = \tfrac{1}{2}\log(1-y^{-2}) - \tfrac{1}{2}\log(1-(1+\delta)^{-2}y^{-2}), \qquad P(y) = \sum_{k\ge2}\sum_{p^k\in(y,(1+\delta)y]} \log p.$$

Lemma 7. *Assume RH, and suppose $xe^{-V} > 1$. Then $\|A\|_\omega^2 \le x^2\,\mathcal{T}$, where $\mathcal{T} = \sum_\gamma |c(\gamma)|^2 R(\gamma)$ and $R(\gamma) = \sum_{\gamma'} \widehat{W}(\gamma-\gamma')$.*

Proof. Let A_D be the truncation of A to $|\gamma| \le D$. With $y = xe^{-v}$ we have $A_D(y) = -\sqrt{y}\,S_D(v)$, where $S_D(v) = \sum_{|\gamma|\le D} c(\gamma)x^{i\gamma}e^{-i\gamma v}$, and $\omega(y)\,\mathrm{d}y = x^2y^{-1}W(v)\,\mathrm{d}v$. (The weight vanishes for $y < xe^{-V}$, so only $y > 1$ occurs, where the explicit formula applies.) Hence

$$\|A_D\|_\omega^2 = x^2\int W|S_D|^2\,\mathrm{d}v = x^2\sum_{|\gamma|,|\gamma'|\le D} c(\gamma)\overline{c(\gamma')}x^{i(\gamma-\gamma')}\widehat{W}(\gamma-\gamma').$$

Since $\widehat{W} \ge 0$ is even, the inequality $2|ab| \le |a|^2 + |b|^2$ bounds this by $x^2\sum_{|\gamma|\le D}|c(\gamma)|^2R(\gamma)$. Now $A_D \to A$ almost everywhere, and Fatou's lemma gives the claim. The sum $\mathcal{T}$ is finite by Lemma 9 and $c(\gamma) \ll \gamma^{-1}$. □

Lemma 8. *If $y_0 = xe^{-V} \ge 3$ then $\|B\|_\omega^2 \le \frac{4}{5}\delta^2x^2y_0^{-5}$.*

Proof. For $y \ge 3$,

$$|B(y)| \le \frac{1}{2}\cdot\frac{(1-(1+\delta)^{-2})y^{-2}}{1-y^{-2}} \le 2\delta y^{-2}.$$

Also $\omega(y) \le (x/y)^2$, and $\omega = 0$ for $y < y_0$. □

The prime-power term P is treated in the proof of Proposition 13.

5. Evaluation of $R(\gamma)$ and of the averaged prime terms

Lemma 7 reduces the problem to the sums $R(\gamma)$ of $\widehat{W}$ over zeros. Because $\widehat{W}$ is the Fourier transform of a function supported in $[-V,V]$, the Weil explicit formula evaluates $R(\gamma)$ in terms of the primes up to e^V.

We use the explicit formula in the form of [16, Prop. 2.1]. Let g be continuous with compact support, and let $h(r)=\int g(u)e^{iru}\,\mathrm{d}u$, so that $g(u)=(2\pi)^{-1}\int h(r)e^{-iru}\,\mathrm{d}r$. Suppose that $h(r)\ll(1+|r|)^{-1-\varepsilon}$ for real r. Then

$$\sum_{\rho} h(\gamma)=h(\tfrac{i}{2})+h(-\tfrac{i}{2})+\frac{1}{2\pi}\int h(r)\,\Omega(r)\,\mathrm{d}r-\sum_{n\ge 2}\frac{\Lambda(n)}{\sqrt{n}}\big(g(\log n)+g(-\log n)\big), \tag{2}$$

with $\Omega(r)=\operatorname{Re}\psi(\frac14+\frac{ir}{2})-\log\pi$. In this formula only, ψ denotes the digamma function.

Rudnick and Sarnak state (2) for $g\in C_c^\infty(\mathbb{R})$; evenness is not required. For continuous compactly supported g we argue by smoothing. Apply (2) to $g_\eta=g*\phi_\eta$, where $\phi_\eta(u)=\eta^{-1}\phi(u/\eta)$ for a fixed even $\phi\in C_c^\infty$ with $\int\phi=1$. Then $h_\eta=h\,\widehat{\phi}(\eta\,\cdot)$ with $|\widehat{\phi}|\le 1$, and $\widehat{\phi}(\eta z)\to 1$ for every complex z. As $\eta\to 0$, the sum over zeros and the integral converge by dominated convergence, since $\sum_\gamma(1+|\gamma|)^{-1-\varepsilon}<\infty$. The prime sum has boundedly many terms and converges because $g_\eta\to g$ uniformly. Both test functions used below are of this type.

Lemma 9. *Assume RH. For real γ,*

$$R(\gamma)=G(\gamma)+\widehat{W}(\gamma-\tfrac{i}{2})+\widehat{W}(\gamma+\tfrac{i}{2})-2\sum_{n\ge2}\frac{\Lambda(n)}{\sqrt{n}}W(\log n)\cos(\gamma\log n),$$

where $G(\gamma)=(2\pi)^{-1}\int\widehat{W}(\gamma-r)\Omega(r)\,\mathrm{d}r$ satisfies, for $\gamma\neq0$,

$$G(\gamma)\le\log\frac{|\gamma|}{2\pi}+\frac{1/4+\sigma^2}{2\gamma^2}.$$

Moreover $|\widehat{W}(\gamma\pm\frac{i}{2})|\le 2(\sigma^2+\sigma+\frac14)e^{V/2}\gamma^{-2}$ for $\gamma\neq0$. (Both bounds are applied only at ordinates of zeros, which satisfy $|\gamma|>14$.)

Proof. Take $h(r)=\widehat{W}(\gamma-r)$ in (2). Then $g(u)=e^{-i\gamma u}W(u)$, $g(0)=1$, and $g(\log n)+g(-\log n)=2W(\log n)\cos(\gamma\log n)$. This gives the identity.

For the bound on G, Chandee [3, Lemma 2.3] gives $\operatorname{Re}\psi(z)\le\log|z|$ for $\operatorname{Re}z\ge\frac14$, so $\Omega(r)\le\frac12\log(r^2+\frac14)-\log2\pi$. By Lemma 4(ii), $\widehat{W}(\gamma-\cdot)/2\pi$ is a probability density with mean γ and variance σ^2. Jensen's inequality for the concave logarithm gives $G(\gamma)\le\frac12\log(\gamma^2+\sigma^2+\frac14)-\log2\pi$, and $\log(1+u)\le u$ finishes.

For the pole terms, integrate by parts twice in $\int W(u)e^{\pm u/2}e^{-i\gamma u}\,\mathrm{d}u$. There are no boundary terms by Lemma 4(iii), and $\int_{-V}^{V}e^{\pm u/2}\le 2e^{V/2}$. □

After inserting Lemma 9 into $\mathcal{T}$, the prime terms are averaged over the zeros with weights $|c(\gamma)|^2$. The next lemma evaluates these averages, again by the explicit formula.

Lemma 10 (Averaged prime terms)**.** *Assume RH, and suppose $\delta\le1/100$, $\delta e^V<1$ and $2\le n\le e^V$. Then*

$$\sum_\gamma|c(\gamma)|^2\cos(\gamma\log n)=-\frac{\delta\Lambda(n)}{\sqrt{n}}+E_n,\qquad |E_n|\le\delta^2\Big(\sqrt{n}+\frac{2.4}{\sqrt{n}}\Big).$$

Proof. For real r, $|c(r)|^2=c(r)c(-r)=\int\kappa(u)e^{iru}\,\mathrm{d}u$, where

$$\kappa(u)=e^{\ell-|u|/2}-e^{|u|/2}\ (|u|\le\ell),\qquad \kappa(u)=0\ (|u|>\ell)$$

is the autocorrelation of $e^{t/2}\mathbf{1}_{[0,\ell]}$. So $\kappa \geq 0$ and $\kappa(0) = e^{\ell} - 1 = \delta$. Apply (2) to $h_n(r) = c(r)c(-r)\cos(r\log n)$, whose Fourier side is $g_n(u) = \frac{1}{2}(\kappa(u-\log n) + \kappa(u+\log n))$. We treat the three terms of (2) separately.

Primes. Since $\ell < \log 2$, the prime term is exactly $-\sum_m \Lambda(m)m^{-1/2}\kappa(\log(m/n))$. Because $\delta n \leq \delta e^V < 1$, the interval $[n/(1+\delta), n(1+\delta)]$ contains no integer other than n. So only $m = n$ contributes, giving $-\delta\Lambda(n)/\sqrt{n}$.

Poles. We have $c(\frac{i}{2}) = \ell$ and $c(-\frac{i}{2}) = \delta$, so $h_n(\pm\frac{i}{2}) = \ell\delta\cosh(\frac{1}{2}\log n)$. Together the two poles give $\ell\delta(\sqrt{n} + 1/\sqrt{n})$.

Archimedean term. From $\psi(z) = \int_0^\infty (e^{-s}/s - e^{-zs}/(1-e^{-s}))\,\mathrm{d}s$, the oscillatory part of Ω is $-\int_{\mathbb{R}} q(\xi)e^{ir\xi}\,\mathrm{d}\xi$ with $q(\xi) = e^{-|\xi|/2}/(1-e^{-2|\xi|})$. Since g_n vanishes near 0, the constant and the subtraction at $\xi = 0$ drop out. The term therefore equals $-\int_{-\ell}^{\ell}\kappa(u)\,q(\log n + u)\,\mathrm{d}u$. Since $\int \kappa(u)e^{-u/2}\,\mathrm{d}u = c(\frac{i}{2})c(-\frac{i}{2}) = \ell\delta$ and $\ell \leq 1/100$, its modulus is at most

$$\ell\delta n^{-1/2}(1 - e^{2\ell}/n^2)^{-1} \leq \ell\delta n^{-1/2}(1 - e^{2\ell}/4)^{-1} \leq 1.4\,\ell\delta n^{-1/2}.$$

Since $\ell \leq \delta$, the three contributions together give the claim. □

6. The main zero sum

After Lemma 9, the largest part of $\mathcal{T}$ is $\sum_\gamma |c(\gamma)|^2\log(|\gamma|/2\pi)$. We evaluate it by Stieltjes integration against the zero-counting function. The main term reduces to a Mellin-type integral of $(\sin s/s)^2$, whose moments are given by the constants below.

Let $a_1 = 1 - \gamma_E - \log 2$ and $a_2 = a_1^2 + 1 + \pi^2/12 = 1.8955631016\ldots$. These are $M'(0)$ and $M''(0)$ for

$$M(q) = \frac{2}{\pi}\int_0^\infty s^q\Big(\frac{\sin s}{s}\Big)^2\,\mathrm{d}s = \frac{2^{-q}}{\Gamma(2-q)\cos(\pi q/2)} \qquad (-1 < q < 1).$$

Lemma 11. *Assume RH and $\delta \leq 1/100$. Put $\Lambda_1 = \log(1/(\pi\ell))$ and $\Lambda = \log(1/\ell)$. Then*

$$\sum_\gamma |c(\gamma)|^2\log\frac{|\gamma|}{2\pi} \;\leq\; \ell\sqrt{1+\delta}\,\big(\Lambda_1^2 + 2a_1\Lambda_1 + a_2\big) + \delta^2\Big[\tfrac{8}{3}\Lambda^3 + 33\Lambda^2 + 136\Lambda + 250\Big] + 0.081\,\delta^2.$$

Proof. Put $b = e^{\ell/2} = \sqrt{1+\delta} \leq 1.005$, $m(t) = \log(t/2\pi)$ and $f(t) = |c(t)|^2 m(t)$. Since $c(-t) = \overline{c(t)}$ and every zero has $|\gamma| > 14$ [15, Thm. 5.1], the sum equals $2\sum_{\gamma>0} f(\gamma)$, and every term is positive.

Step 1: two formulas for $|c|^2$. Since $(1+\delta)^{1/2+it} = be^{it\ell}$ and $|be^{i\theta} - 1|^2 = (b-1)^2 + 4b\sin^2(\theta/2)$,

$$|c(t)|^2 = \frac{(b-1)^2 + 4b\sin^2(t\ell/2)}{t^2 + \frac{1}{4}}, \qquad (b-1)^2 = \frac{\delta^2}{(b+1)^2} \leq \frac{\delta^2}{4}. \tag{3}$$

Also $|c(t)|^2 = \int \kappa(u)e^{itu}\,\mathrm{d}u$, where $\kappa \geq 0$ is the function of Lemma 10, supported in $[-\ell,\ell]$. Hence

$$|c(t)|^2 \leq \int \kappa = |c(0)|^2 = 4(b-1)^2 \leq \ell^2 e^{\ell} = (1+\delta)\ell^2, \qquad \big|(|c|^2)'(t)\big| \leq \int |u|\kappa(u)\,\mathrm{d}u \leq \ell(1+\delta)\ell^2.$$

Step 2: Stieltjes integration. Let $N(t)$ count the zeros with $0 < \gamma \leq t$, with multiplicity, and write $N = N_0 + Q$ with $N_0(t) = \frac{t}{2\pi}\log\frac{t}{2\pi e} + \frac{7}{8}$. Trudgian [18, Cor. 1] gives $|Q(t)| \leq 0.112\log t + 0.278\log\log t + 2.510 + 0.2/t$ for $t \geq e$. This implies

$$|Q(t)| \leq q(t) := \log t + 4 \qquad (t \geq e). \tag{4}$$

Since $N(14)=0$ and $f(t)N(t)\ll t^{-1}\log^2 t\to 0$, integration by parts gives

$$\sum_{\gamma>0} f(\gamma)=\int_{14}^{\infty} f\,\mathrm{d}N=\int_{14}^{\infty} f(t)N_0'(t)\,\mathrm{d}t+N_0(14)f(14)-\int_{14}^{\infty} Q(t)f'(t)\,\mathrm{d}t,$$

with $N_0'(t)=m(t)/2\pi$ and $|N_0(14)|\le 0.44$.

Step 3: the main integral. By (3) and $t^2+\frac14\ge t^2$, $2\int_{14}^{\infty} fN_0'$ is at most

$$\frac{4b}{\pi}\int_{14}^{\infty}\frac{\sin^2(t\ell/2)}{t^2}m(t)^2\,\mathrm{d}t+\frac{(b-1)^2}{\pi}\int_{14}^{\infty}\frac{\log^2 t}{t^2}\,\mathrm{d}t.$$

The first integrand is nonnegative on $(0,\infty)$ and integrable at 0. Extend the integral to $(0,\infty)$ and substitute $s=t\ell/2$, so that $m(t)=\Lambda_1+\log s$. This bounds the first term by

$$\begin{aligned} b\ell\,\frac{2}{\pi}\int_0^{\infty}\Big(\frac{\sin s}{s}\Big)^2(\Lambda_1+\log s)^2\,\mathrm{d}s &= b\ell\big(M(0)\Lambda_1^2+2M'(0)\Lambda_1+M''(0)\big)\\ &= b\ell(\Lambda_1^2+2a_1\Lambda_1+a_2).\end{aligned}$$

Here $M(0)=1$, and the values $M'(0)=a_1$, $M''(0)=a_2$ follow from

$$(\log M)'(0)=\frac{\Gamma'}{\Gamma}(2)-\log 2=1-\gamma_E-\log 2,\qquad (\log M)''(0)=\frac{\pi^2}{4}-\Big(\frac{\Gamma'}{\Gamma}\Big)'(2)=1+\frac{\pi^2}{12}.$$

The closed form of M follows from $\int_0^{\infty} s^{\mu-1}\sin^2 s\,\mathrm{d}s=-2^{-\mu-1}\Gamma(\mu)\cos(\pi\mu/2)$ for $-2<\mu<0$, with $\mu=q-1$, and the reflection formula; the value at $q=0$ follows by continuity. The second term is at most $\frac{\delta^2}{4\pi}\cdot\frac{\log^2 14+2\log 14+2}{14}<0.081\,\delta^2$.

Step 4: the derivative of f. We have $f'=(|c|^2)'m+|c|^2/t$ and $0<m(t)\le\log t$. Consider first $14\le t\le 1/\ell$, a nonempty range since $\ell\le\delta\le 1/100$. There the bounds of Step 1 and $\ell\le 1/t$ give

$$|f'(t)|\le(1+\delta)\ell^3\log t+(1+\delta)\ell^2/t\le(1+\delta)\ell^2(\log t+1)/t.$$

For $t>1/\ell$, write $|c|^2=n/d$ with $n=1+b^2-2b\cos(t\ell)\le(1+b)^2$, $|n'|\le 2b\ell$ and $d=t^2+\frac14$. Then $\big|(|c|^2)'\big|\le|n'|/d+n\,d'/d^2\le 2b\ell/t^2+2(1+b)^2/t^3$, so

$$|f'(t)|\le\Big(\frac{2.01\,\ell}{t^2}+\frac{10}{t^3}\Big)(\log t+1).$$

The coefficient of ℓ/t^2 cannot be taken below $2b$. Indeed, along the points $t_k=(\pi/2+2\pi k)/\ell$ the ratio $t_k^2|f'(t_k)|/(\ell\log t_k)$ tends to $2b$.

Step 5: the error terms. By (4), the last two terms of the Stieltjes formula, doubled, are at most $2|N_0(14)|f(14)+2\int_{14}^{\infty} q|f'|$. First, $f(14)\le(1+\delta)\ell^2 m(14)\le 0.81(1+\delta)\ell^2$. On $[14,1/\ell]$, with $z=\log t$,

$$\int_{14}^{1/\ell} q|f'|\le(1+\delta)\ell^2\int_0^{\Lambda}(z+4)(z+1)\,\mathrm{d}z=(1+\delta)\ell^2\Big(\frac{\Lambda^3}{3}+\frac{5}{2}\Lambda^2+4\Lambda\Big).$$

On $[1/\ell,\infty)$ substitute $t=e^z/\ell$, so that $\log t=z+\Lambda$, $\ell\,\mathrm{d}t/t^2=\ell^2e^{-z}\,\mathrm{d}z$ and $\mathrm{d}t/t^3=\ell^2e^{-2z}\,\mathrm{d}z$:

$$\int_{1/\ell}^{\infty} q|f'|\le\ell^2\int_0^{\infty}(z+\Lambda+4)(z+\Lambda+1)\big(2.01e^{-z}+10e^{-2z}\big)\,\mathrm{d}z=\ell^2\big(7.01\Lambda^2+44.07\Lambda+57.11\big).$$

With $\ell\le\delta$ and $1+\delta\le 1.01$, the error terms total at most $\delta^2\big[0.68\Lambda^3+19.1\Lambda^2+96.3\Lambda+115\big]$, which is less than the bracket in the statement. Collecting Steps 3 and 5 proves the lemma. $\square$

7. Arithmetic input

We need three explicit inequalities. Only (L2) uses RH. It is checked by computation up to 10^7 and by Schoenfeld's RH bound beyond. (L3) is Chebyshev's elementary bound, and (L4) follows from Trudgian's and Platt's results.

Lemma 12. *(L2) Assume RH. Then* $\sum_{n\le t}\Lambda(n)^2/n \le \frac{1}{2}\log^2 t + 0.04$ *for all* $t \ge 1$.
(L3) $\theta(t) \le t\log 4$ *for all* $t \ge 0$.
(L4) $\sum_\gamma \gamma^{-2} \le 0.12$, *the sum being over the ordinates* $\gamma = \operatorname{Im}\rho$ *of all nontrivial zeros* ρ.

Proof. All numerical values below are computed in interval arithmetic (Appendix A). Let $U_0 = 10^7$ and $S(t) = \sum_{n\le t}\Lambda(n)^2/n$.

(L2) for $t \le U_0$. Between consecutive prime powers $S(t) - \frac{1}{2}\log^2 t$ decreases. So the supremum is attained at prime powers, and an exact sieve gives

$$\sup_{1\le t\le U_0}\left(S(t) - \tfrac{1}{2}\log^2 t\right) = 0.0390683\ldots \quad (\text{at } t = 3).$$

For $t > U_0$ we use Schoenfeld's bound [17, Thm. 10, (6.2)], valid under RH:

$$|E(u)| \le \frac{\sqrt{u}\,\log^2 u}{8\pi} \qquad (u \ge 73.2), \qquad E(u) = \psi(u) - u.$$

(L2) for $t > U_0$. Use $\Lambda(n)^2 \le \Lambda(n)\log n$ and partial summation against ψ on $(U_0, t]$:

$$\sum_{U_0<n\le t}\frac{\Lambda(n)\log n}{n} = \frac{\log^2 t - \log^2 U_0}{2} + \frac{E(t)\log t}{t} - \frac{E(U_0)\log U_0}{U_0} + \int_{U_0}^{t}\frac{E(u)(\log u - 1)}{u^2}\,\mathrm{d}u.$$

Bound $E(t)\log t/t \le \log^3 t/(8\pi\sqrt{t}) \le \log^3 U_0/(8\pi\sqrt{U_0})$, and bound the integral by

$$\frac{1}{8\pi}\int_{U_0}^{\infty} u^{-3/2}(\log^3 u - \log^2 u)\,\mathrm{d}u,$$

which has a closed form. With the exact values of $S(U_0)$ and $\psi(U_0)$ this gives $S(t) \le \frac{1}{2}\log^2 t - 1.37$ for $t > U_0$.

(L3). This is Chebyshev's bound; we recall the standard proof. It suffices to show $\prod_{p\le n} p \le 4^n$ for integers $n \ge 1$, since $\theta(t) = \theta(\lfloor t\rfloor)$. We use induction on n; the cases $n \le 2$ are clear, and for even $n > 2$ the claim follows from the case $n - 1$. For $n = 2m+1$, every prime in $(m+1, 2m+1]$ divides $\binom{2m+1}{m}$, and $\binom{2m+1}{m} \le 4^m$ because $\binom{2m+1}{m} = \binom{2m+1}{m+1}$ and both appear in $(1+1)^{2m+1}$. Hence $\prod_{p\le 2m+1} p \le 4^{m+1}\cdot 4^m = 4^{2m+1}$.

(L4). There are no zeros with $|\gamma| \le 14$, so $N(14) = 0$. Since $N(t)/t^2 \to 0$,

$$\sum_\gamma \gamma^{-2} = 2\int_{14}^{\infty} t^{-2}\,\mathrm{d}N(t) = 4\int_{14}^{\infty} N(t)t^{-3}\,\mathrm{d}t.$$

Insert $N \le N_0 + \log t + 4$, from (4) in the proof of Lemma 11. Every resulting integral has a closed form, and the total is $0.1182\ldots$. (The true value is about 0.046.) □

8. Proof of Theorem 1

8.1. **Small** x**.** The function $x + \log^2 x$ increases for $x \ge 1$, and r is where it equals 6. For $4 \le x < r$ the interval $(x, x + \log^2 x]$ lies in $(4, 6)$ and contains no even integer. For $r \le x < 6$ it

contains the Goldbach number $6 = 3 + 3$. For $6 \le x < 3.9 \cdot 10^{18}$ it has length $\log^2 x > 2$, so it contains an even N with $6 < N < 4 \cdot 10^{18}$, and N is a Goldbach number by [14].[1]

Once the range $x \ge 3.9 \cdot 10^{18}$ is settled below, this proves the first two statements of Theorem 1. For the third, $C_* > 1$ covers $x \ge r$. For $4 \le x < r$ the interval $(x, x + C_* \log^2 x]$ contains 6, since $x + C_* \log^2 x \ge 4 + C_* \log^2 4 = 6$. Conversely, at $x = 4$ the least even integer above x is 6, so no $C < C_*$ works.

8.2. **Large x: setup.** Let $x \ge 3.9 \cdot 10^{18}$, and put

$$L = \log x \ge L_0 = \log(3.9 \cdot 10^{18}) = 42.8075\ldots, \qquad H = L^2, \qquad \delta = L^2/x, \qquad V = 3\sqrt{L}.$$

One checks that

$$L - V - 4\log L - \log 2 > 0; \tag{5}$$

the left side equals 7.46 at L_0 and increases with L. Condition (5) implies $\delta e^V < 1$, $\ell < 1/14$, $xe^{-V} \ge 599$, and that each interval $(y, (1+\delta)y]$ with $y \le 2xe^V$ contains at most one k-th power for each k.

Suppose $(x, x + H]$ contains no Goldbach number. By Lemmas 6, 7, 8 and the triangle inequality,

$$\delta^2 x^3 \Big(2m_W - 2\delta - \frac{L^2}{2\pi\sqrt{x}}\Big) \le \|V_\theta\|_\omega^2 \le \big(\|A\|_\omega + \|B\|_\omega + \|P\|_\omega\big)^2 \le \delta x^2 \big(\sqrt{\mathcal{T}/\delta} + \sqrt{\mathcal{E}} + \sqrt{\mathcal{B}}\big)^2, \tag{6}$$

where $\mathcal{E} = \|P\|_\omega^2/(\delta x^2)$ and $\mathcal{B} = \|B\|_\omega^2/(\delta x^2)$. Only the first inequality uses the assumption on $(x, x + H]$. Since $\delta x = L^2$, a contradiction follows once

$$\big(\sqrt{\mathcal{T}/\delta} + \sqrt{\mathcal{E}} + \sqrt{\mathcal{B}}\big)^2 < L^2 \Big(2m_W - 2\delta - \frac{L^2}{2\pi\sqrt{x}}\Big). \tag{7}$$

8.3. **The bounds.** We bound each quantity in (7).

Proposition 13. *Under the above assumptions, $\mathcal{T}/\delta \le T_{\mathrm{main}} + T_{\mathrm{pr}} + R_T$, $\mathcal{E} \le \mathcal{E}^+$, $\mathcal{B} \le \mathcal{B}^+$ and $m_W \ge m_{\mathrm{low}}$, where these quantities are the bounds of Table 1 and $\Lambda_1^0 = L - 2\log L - \log\pi$.*

Proof of Proposition 13. Condition (5) gives $\delta \le 1/100$, $\delta e^V < 1$ and $e^V > 73.2$, so Lemmas 10 and 11 apply. Insert Lemma 9 into $\mathcal{T} = \sum_\gamma |c(\gamma)|^2 R(\gamma)$, and use $|c(\gamma)|^2 \le \ell^2(1+\delta) \le \delta^2(1+\delta)$ and (L4). We treat the resulting terms one at a time.

G terms. By Lemma 9, $\sum_\gamma |c(\gamma)|^2 G(\gamma)$ is at most the sum in Lemma 11 plus

$$\delta^2(1+\delta)\frac{1/4 + \sigma^2}{2} \sum_\gamma \gamma^{-2} \le 0.06\,\delta^2(1+\delta)(\tfrac{1}{4} + \sigma^2).$$

Main term and the shift of Λ_1. Since $\ell \ge \delta - \delta^2/2$, we have $\Lambda_1 \le \log\frac{1}{\pi\delta} + \delta = \Lambda_1^0 + \delta$, and $\ell\sqrt{1+\delta}/\delta \le 1 + \delta$. Put $X = \Lambda_1^0 + a_1 \le L$. Lemma 11 divided by δ is then at most

$$(1+\delta)\big((X+\delta)^2 + a_2 - a_1^2\big) \le X^2 + a_2 - a_1^2 + \delta(L^2 + 3L),$$

plus the error terms. In those, $\Lambda = \log(1/\ell) \le L$.

Prime terms. By Lemma 10 (note $W(\log n) = 0$ for $n > e^V$) they equal

$$2\delta \sum_{n \le e^V} \frac{\Lambda(n)^2 W(\log n)}{n} - 2 \sum_{n \le e^V} \frac{\Lambda(n) W(\log n)}{\sqrt{n}} E_n.$$

[1] The authors of [14] double-checked the whole range up to $3 \cdot 10^{17}$. Beyond that they double-checked about 4% of the intervals, compared prime counts in residue classes modulo 120 with an independent program for every interval of 10^{15} integers, and found no discrepancy [14, §1].

Quantity	Upper bound	Source
T_{main}	$(\Lambda_1^0 + a_1)^2 + (a_2 - a_1^2)$	Lemma 11
T_{pr}	$2V^2/\pi^2 + 0.08 = 18L/\pi^2 + 0.08$	Lemmas 9, 10; (K2); (L2)
R_T: shift $\Lambda_1 \to \Lambda_1^0$ and factor $\ell\sqrt{1+\delta}/\delta$	$\delta(L^2 + 3L)$	Lemma 11
R_T: $N(T)$ and $(b-1)^2$ pieces	$\delta\big[\frac{8}{3}L^3 + 33L^2 + 136L + 250\big] + 0.1\delta$	Lemma 11
R_T: Jensen term	$0.06\,\delta(1+\delta)(\frac{1}{4} + \sigma^2)$	Lemma 9; (L4)
R_T: E_n terms	$2.02\,\delta\big[e^V + \frac{e^{V/2}V^2}{8\pi} + \frac{2.4(V^2/2+0.04)}{\log 2}\big]$	Lemma 10; [17]; (L2)
R_T: pole terms	$0.5\,\delta(1+\delta)(\sigma^2 + \sigma + \frac{1}{4})e^{V/2}$	Lemma 9; (L4)
$\mathcal{E} \le \mathcal{E}^+$	$\frac{L+V+1}{\log 2}(1+\delta)\log 4\,\big[(\ell_y+2)e^{-\ell_y/2} + \frac{\ell_y}{\log 2}\big(\frac{\ell_y}{2} + \frac{3}{4}\big)e^{-2\ell_y/3} + 3\,e^{-\ell_y}\big]$	(L3); (5)
$\mathcal{B} \le \mathcal{B}^+$	$0.8\,\delta e^{-5(L-V)}$	Lemma 8
$m_W \ge m_{\mathrm{low}}$ (lower bound)	$\frac{1}{2} - 9.55V^{-2} - e^{-V}$	Corollary 5

TABLE 1. Explicit bounds; $\ell_y = L - V$. The five R_T rows sum to R_T. The last three rows define $\mathcal{E}^+$, $\mathcal{B}^+$ and m_{low}.

For the first sum, $t \mapsto W(\log t)$ decreases from 1 to 0 on $[1, e^V]$. Partial summation and (L2) give

$$\sum_{n \le e^V} \frac{\Lambda(n)^2 W(\log n)}{n} = \int_1^{e^V} S(t)\,\mathrm{d}\big(-W(\log t)\big) \le 0.04 + \int_0^V \tfrac{1}{2}v^2\big(-W'(v)\big)\,\mathrm{d}v$$
$$= 0.04 + \int_0^V vW(v)\,\mathrm{d}v,$$

and (K2) evaluates the last integral as V^2/π^2. For the second sum, $W \le 1$ and Lemma 10 bound it by $2\delta^2\big(\psi(e^V) + 2.4\sum_{n \le e^V}\Lambda(n)/n\big)$. Then use [17] for $\psi(e^V)$, and $\Lambda(n)/n \le \Lambda(n)^2/(n\log 2)$ with (L2).

Pole terms. By Lemma 9 and (L4) they are at most $\delta^2(1+\delta)\cdot 2\cdot 2(\sigma^2 + \sigma + \frac{1}{4})e^{V/2}\cdot 0.12$.

Prime powers. The weight ω vanishes outside $[y_0, xe^V]$, where $y_0 = xe^{-V}$. A k-th power in $(y, (1+\delta)y]$ with $y \le xe^V$ needs $k \le (L + V + \log(1+\delta))/\log 2$, so at most $(L+V+1)/\log 2$ values of k occur. Let $P_k(y)$ be the sum of $\log p$ over $p^k \in (y, (1+\delta)y]$. Cauchy–Schwarz gives

$$P(y)^2 \le \frac{L+V+1}{\log 2}\sum_k P_k(y)^2.$$

Consecutive k-th powers near y differ by more than $2\sqrt{y}$, and $2\sqrt{y} > (1+\delta)\delta y$ for $y \le xe^V$ by (5). So $P_k(y)$ is $\log p$ for at most one p^k, and $P_k(y) \ne 0$ only for $y \in [p^k/(1+\delta), p^k)$. On that range $\int \omega \le \int x^2 y^{-2}\,\mathrm{d}y = \delta x^2/p^k$. Hence

$$\int P_k^2 \omega \le \delta x^2 \sum_{p^k \ge y_0} \frac{\log^2 p}{p^k}.$$

Stieltjes integration against θ, with (L3), gives

$$\sum_{p\ge z}\frac{\log^2 p}{p^k}\le(\log 4)\,k\Big[\frac{\log z}{(k-1)z^{k-1}}+\frac{1}{(k-1)^2z^{k-1}}\Big],\qquad z=\max(y_0^{1/k},2).$$

Here the boundary term is negative and $(\log t/t^k)'<0$ for $t\ge 2$. Write $\ell_y=\log y_0=L-V$. We split the exponents into three ranges.

- For $k=2$ the bracket gives $(\ell_y+2)e^{-\ell_y/2}$.
- For $3\le k<\ell_y/\log 2$ each term is at most $(\frac{\ell_y}{2}+\frac34)e^{-2\ell_y/3}$, and there are fewer than $\ell_y/\log 2$ of them.
- For $k\ge\ell_y/\log 2$, which is at least 33, we have $2^k\ge y_0$ and $z=2$. The terms are at most $\frac{k}{k-1}(\log 2+\frac{1}{k-1})2^{1-k}\le 0.75\cdot 2^{1-k}$, which sums to at most $3/y_0=3e^{-\ell_y}$.

B term. This is Lemma 8 with $y_0=e^{L-V}$. □

8.4. **Conclusion.** It remains to show that the bounds of Proposition 13 imply (7) for every $L\ge L_0$. We separate a main part $\mathcal{M}$, which grows with L, from a remainder $\mathcal{R}$, which is small and decreasing, and then compare them at L_0.

Put $s=\sqrt{\mathcal{E}}+\sqrt{\mathcal{B}}$. Recall that $m_{\rm low}\le m_W$. Since $(\sqrt{\mathcal{T}/\delta}+s)^2\le T_{\rm main}+T_{\rm pr}+R_T+2\sqrt{\mathcal{T}/\delta}\,s+s^2$, inequality (7) follows from $\mathcal{M}(L)>\mathcal{R}(L)$, where

$$\mathcal{M}(L)=L^2\Big(2m_{\rm low}-2\delta-\frac{L^2}{2\pi\sqrt{x}}\Big)-T_{\rm main}-T_{\rm pr},$$
$$\mathcal{R}(L)=R_T+2\sqrt{\mathcal{T}/\delta}\,s+s^2.$$

Let $d=2\log L+\log\pi-a_1$, so that $T_{\rm main}=(L-d)^2+a_2-a_1^2$. Since $2\cdot 9.55/9+18/\pi^2<3.9461$, $a_2-a_1^2+0.08<1.9025$, $L^2\delta=L^4e^{-L}$ and $\sqrt{x}=e^{L/2}$,

$$\mathcal{M}\ge 2dL-d^2-3.9461L-1.9025-2L^2e^{-3\sqrt{L}}-2L^4e^{-L}-\frac{L^4e^{-L/2}}{2\pi}.$$

$\mathcal{M}$ increases. The derivative of the polynomial part is $2d+4-4d/L-3.9461$. For $L\ge L_0$ we have $d\ge 8.9$ and $4d/L\le 0.9$, since d/L decreases, so the derivative is positive. Also $L^2e^{-3\sqrt{L}}$ decreases for $L>16/9$, and L^4e^{-L} and $L^4e^{-L/2}$ decrease for $L>8$.

A decreasing majorant of $\mathcal{R}$. First, since $2m_{\rm low}\le 1$, $\mathcal{T}/\delta\le T_{\rm main}+T_{\rm pr}+R_T\le L^2-\mathcal{M}(L)+R_T$. Below we show $\mathcal{M}(L)-R_T(L)\ge\mathcal{M}(L_0)-\widetilde{\mathcal{R}}(L_0)>0$, so $\mathcal{T}/\delta\le L^2$. Since $\mathcal{E}\le\mathcal{E}^+$ and $\mathcal{B}\le\mathcal{B}^+$, this gives $\mathcal{R}\le\widetilde{\mathcal{R}}$, where

$$\widetilde{\mathcal{R}}=R_T+2\sqrt{L^2\mathcal{E}^+}+2\sqrt{L^2\mathcal{B}^+}+\big(\sqrt{\mathcal{E}^+}+\sqrt{\mathcal{B}^+}\big)^2$$

is an explicit function of L. It increases in each of R_T, $L^2\mathcal{E}^+$, $L^2\mathcal{B}^+$, $\mathcal{E}^+$ and $\mathcal{B}^+$. We show that each of these quantities decreases for $L\ge L_0$. In each case the logarithmic derivative is bounded by an expression that decreases in L, so it suffices to check it at L_0. We use $\delta'/\delta=2/L-1$, $V'=\frac{3}{2\sqrt{L}}$, and, for $u=L-V$, $u'\ge 1-1.5/\sqrt{L_0}>0.77$ and $u\ge u(L_0)>23$.

- Each row of R_T is a sum of terms $c\,L^j\delta^ie^{\beta V}\sigma^m$ with $c,m\ge 0$, $i\in\{1,2\}$, $j\le 3$ and $0\le\beta\le 1$. Their logarithmic derivatives are at most $5/L+1.5/\sqrt{L}-1<-0.6$.
- $L^2\mathcal{E}^+$ is $L^2(L+V+1)(1+\delta)$ times a positive constant, times a sum of three functions of u. The logarithmic u-derivatives of these three functions are at most $\frac{1}{u+2}-\frac12$, $\frac1u+\frac{1/2}{u/2+3/4}-\frac23$ and -1, each <-0.45. The prefactor has logarithmic derivative at most $2/L+1.23/(L+V+1)<0.07$. The total is therefore below $0.07-0.77\cdot 0.45<0$.

- $L^2\mathcal{B}^+ = 0.8L^2\delta e^{-5u}$ has logarithmic derivative at most $4/L - 1 - 5 \cdot 0.77 < 0$.
- $\mathcal{E}^+ = (L^2\mathcal{E}^+)/L^2$ and $\mathcal{B}^+ = (L^2\mathcal{B}^+)/L^2$ then decrease as well.

Endpoint. In interval arithmetic, $\mathcal{M}(L_0) = 513.87\ldots$ and $\widetilde{\mathcal{R}}(L_0) = 17.11\ldots$, with $\mathcal{E}^+(L_0) = 0.0398$, $R_T(L_0) < 10^{-6}$ and $\mathcal{B}^+(L_0) < 10^{-65}$. Hence, for all $L \ge L_0$,

$$\mathcal{M}(L) \ge \mathcal{M}(L_0) > \widetilde{\mathcal{R}}(L_0) \ge \widetilde{\mathcal{R}}(L) \ge \mathcal{R}(L),$$

which contradicts (6). This proves Theorem 1. □

Remark 14. Replace the left side of (7) by its upper bound $\big(\sqrt{T_{\text{main}} + T_{\text{pr}} + R_T} + \sqrt{\mathcal{E}^+} + \sqrt{\mathcal{B}^+}\big)^2$, and m_W by m_{low}. The ratio of the two sides is then 0.713 at L_0, 0.974 at $L = 10^3$, 0.99995 at $L = 10^6$ and 0.9999993 at $L = 10^8$, and it tends to 1 like $1 - 4\log L/L$. It is an upper bound for the true ratio. For example, at L_0 the exact value $m_W = 0.47834\ldots$ (against $m_{\text{low}} = 0.47521\ldots$) already lowers it to 0.708.

9. On the constants 1 and 2

This section is heuristic. Nothing in it is used in the proof of Theorem 1.

Our argument reaches the constant 1 and no less, because the ratio in Remark 14 tends to 1. With the cover of version 0.5, which uses only the prime-free member of each pair, the same estimates reach 2. Goldston [7] gave a different deduction of Goldbach numbers from mean-square bounds. He proves [7, Theorem, eq. (6)]

$$\Big|\sum_{|n|\le K} (K - |n|)R_G(N+n) - K^2N\Big| \le J(N,K) + O(KN) + o(K^2N),$$

where $J(N,K) = \int_1^N (\theta(u+K) - \theta(u) - K)^2\,\mathrm{d}u$ and $R_G(n) = \sum_{p+p'=n} \log p \log p'$ (Goldston's $R(n)$). He also proves the identity [7, eq. (25)]

$$J(N,K) = KN\log N + 2\sum_{n<K}(K-n)Z(N;n) - K^2N + o(K^2N), \qquad Z(N;n) = \sum_{\substack{p,p'\le N\\ p'-p=n}} \log p \log p'$$

(his eq. (11)). Suppose RH gave $J(N,K) \le (\kappa + o(1))KN\log^2 N$ for $K \asymp \log^2 N$. Then, for each fixed $\varepsilon > 0$ and all large N, the interval $(N-K, N+K)$ contains a Goldbach number once $K \ge (\kappa+\varepsilon)\log^2 N$; that is, intervals of length $2(\kappa+\varepsilon)\log^2 N$ suffice.

Heuristically, a mean-square input of the strength of Lemma 7 and Proposition 13 corresponds to $\kappa = 1$, and with $\kappa = 1$ Goldston's deduction gives the constant 2 of the prime-free cover. We have not proved a transfer theorem that turns our weighted estimate for relative intervals into a bound for $J(N,K)$, so this agreement is an observation, not a theorem. The first moment in Lemma 6 has no counterpart in Goldston's deduction. It uses that prime counts are nonnegative and that, by the prime number theorem, the occupied member of each pair carries about twice its expected count. This halves the constant without any information on prime pairs.

The constant 2 in Goldston's deduction. Within Goldston's deduction the role of the constant 2 can be made precise. Put

$$A(N,K) = \frac{\sum_{0<h<K}(K-h)Z(N;h)}{K^2N/2},$$

an averaged analogue of the prime-pair constant A in Goldston's eq. (14), which reads $Z(n;k) < (A+\varepsilon)\mathfrak{S}(k)n$. If $(N-K, N+K)$ contains no Goldbach number, the sum in (6) vanishes, and (6) and (25) give

$$\big(2 - A(N,K)\big)K^2N \le KN\log N + O(KN) + o(K^2N).$$

So for $K \asymp \log^2 N$ the deduction finds a Goldbach number in $(N-K, N+K)$ for all large N whenever $A(N,K) \le 2-\varepsilon$ for a fixed $\varepsilon > 0$, and it gives nothing when $A(N,K) \ge 2+\varepsilon$. By (25), a bound $J(N,K) \le (\kappa+o(1))KN\log^2 N$ at $K = c\log^2 N$ is equivalent to $A(N,K) \le 1+\kappa/c+o(1)$. Along this route, therefore, intervals of length $2c\log^2 N$ with $c < \kappa$ require an averaged prime-pair bound $A(N, c\log^2 N) < 2$ that the mean-square bound does not supply. Such a bound would need information beyond the mean-square bound, for example on the pair correlation of zeros [12]. We know of no such result under RH alone. Theorem 1 reaches the analogue of $c = \frac{1}{2}$ by a different route, the first moment of Lemma 6. So this obstruction belongs to Goldston's deduction, not to the mean-square input.

Conversely, Goldston's Corollary 3 [7] shows that his eq. (14) with some $A < 2$ would give Goldbach numbers in all intervals of length $O(\log N)$, far beyond the constant 2. He records that (14) is known with $A = 3.9171$ for all k, and with $A = 3.454$ for $k \le \log^D N$. Because of the parity phenomenon [2, 6], we do not expect sieve methods alone to give $A < 2$.

Numerical values. For $N = 10^8$ and $K = \lfloor c\log^2 N \rceil$ (the nearest integer) we find $A(N,K) = 0.931$, 0.961 and 0.978 for $c = \frac{1}{4}$, $\frac{1}{2}$ and 1 (Appendix A). For $K = 339$ (that is, $c = 1$), $J(10^8, 339) = 0.81\,KN\log(N/K) = 0.030\,KN\log^2 N$. So the true values are far from both the threshold $A = 2$ and the RH bound, and the loss lies in that bound rather than in either deduction.

Note on the preparation of this paper

The research, the proofs and the text were produced largely with the AI model Claude (Anthropic), working under the author's direction. The checks so far are as follows.

- Every numerical constant in the proof is verified in ball arithmetic (Appendix A).
- Most of the deduction of Theorem 1 is formalized in Lean 4 with Mathlib. The formalization covers:
  - Lemma 4(i), (ii), (iv), (v), and Lemma 6 with Corollary 5: the symmetric weight, its level sets, the first moment on each level and the integration over levels;
  - the cutoff: the statement for $x + \log^2 x \ge 6$, the absence of Goldbach numbers in $(x, x+\log^2 x]$ when $4 \le x$ and $x + \log^2 x < 6$, the bound with C_* for all $x \ge 4$, and the optimality of C_* at $x = 4$;
  - Lemmas 7 and 8;
  - the $\mathcal{E}$ row of Proposition 13, including (L3);
  - the identity $V_\theta = A + B - P$ with the triangle inequality in (6);
  - the range argument for all $x \ge 3.9 \cdot 10^{18}$, by a route independent of §8.

  It takes as hypotheses RH, Schoenfeld's bound [17, Thm. 10, (6.3)] for θ under RH, the explicit formula for ψ, the verification [14], and one statement not yet formalized: the $\mathcal{T}$ row of Proposition 13. Its proof uses Lemma 4(iii) and Lemmas 9–12, besides the parts of Lemma 4 already formalized.
- Each cited result used in the proof has been checked against the original publication.

The analytic part (§§4–7) has been checked by hand derivation and numerically against zeta zeros, but not yet by an independent expert. Comments and corrections are welcome. The paper, scripts and Lean formalization are available in the accompanying repository.

Appendix A. Computations

All computations are reproducible with the scripts accompanying this paper. They are in the `scripts/` folder of the repository and are also distributed as ancillary files. The proof uses only the first script. It uses ball arithmetic (Arb [1], through python-flint), so every inequality it reports is rigorous, and it runs in a few seconds.

- `scripts/rigorous_checks.py` verifies:
  - the numerical constants in Corollary 5 and Lemmas 10 and 11, including the polynomial $7.01\Lambda^2 + 44.07\Lambda + 57.11$ in Step 5 of the proof of Lemma 11;
  - a_1 and a_2;
  - Lemma 12, (L2) and (L4): an exact sieve to 10^7, with all sums in ball arithmetic, plus the closed-form RH tail; and the integral for (L4). The output gives $\sup_{t\le 10^7}(S(t) - \frac{1}{2}\log^2 t) = 0.0390683\ldots$, $S(10^7) - \frac{1}{2}\log^2 10^7 = -1.5724\ldots$, $\psi(10^7) - 10^7 = -1460.59\ldots$ and the RH tail constant $-1.370\ldots$;
  - condition (5), $\mathcal{M}(L_0)$, $\widetilde{\mathcal{R}}(L_0)$, $\mathcal{T}/\delta \le L_0^2$, and the logarithmic-derivative constants of §8;
  - the cutoff r (to 16 digits), $C_* = 2/\log^2 4 > 1$ and $\log^2 6 > 2$, used for small x in §8; and $xe^{-V} \ge 599$ at L_0, used in Lemma 6.
- Not used in the proof:
  - `scripts/range_proof.py` recomputes Table 1 in floating point, checks the monotonicity on $8 \cdot 10^4$ points in $[L_0, 10^8]$, and gives the ratios in Remark 14;
  - `scripts/check_explicit_formula.py` uses the first 20 000 zeros. It compares the truncated explicit formula for ψ with ψ itself at three points; it compares $R(\gamma)$ computed from the zeros with the expression of Lemma 9 at four values of γ, which agree to about 10^{-8}; it evaluates $\sum_\gamma \gamma^{-2} \approx 0.0462$, against the bound 0.12 of (L4); and it compares $\mathcal{T}/\delta$ from the zeros with the main and prime terms of its bound;
  - `scripts/selberg_numeric.py` computes $J(10^8, K)$ and $A(10^8, K)$ for §9. Since $\theta(u+K)-\theta(u)$ is constant on each $[n, n+1)$, $J(N,K) = \sum_{n<N}(\theta(n+K)-\theta(n)-K)^2$ is a finite sum; both quantities are evaluated in floating point. As a check, the script compares J with the right side of (25) without its $o(K^2N)$ term; the difference is at most $0.012\,K^2N$.

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