The entropy photon-number inequality without an energy assumption

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November 21, 2026

Abstract

We deduce the entropy photon-number inequality for two independent bosonic inputs of finite von Neumann entropy from OpenAI's finite-energy theorem. The result holds for any finite number of modes, even when both inputs have infinite mean photon number. Applying the same photon-number filter at both ports of the beam splitter gives an operator bound on the retained marginal and hence an upper bound on its entropy. At transmissivities strictly between zero and one, an input of infinite entropy forces the output entropy to be infinite. We also obtain the sharp minimum output entropy of a thermal attenuator at prescribed input entropy, without an energy constraint.

2020 Mathematics Subject Classification: 81P17 (primary), 81P47, 81V73 (secondary).

Keywords: entropy photon-number inequality, bosonic Gaussian channels, von Neumann entropy.

hexagon:2611.00882v1

1 Introduction

Mix two independent thermal beams of mean photon numbers a and b per mode. The retained beam is thermal with mean photon number $\eta a + (1-\eta)b$ at transmissivity η. Guha, Erkmen, and Shapiro conjectured that this pair minimizes the output entropy among all independent inputs with the same respective entropies [1]. OpenAI proved the conjecture for any finite number of modes when both inputs have finite mean photon number [2, Theorem 1.1]. The energy hypothesis already excludes a simple one-mode state. In the photon-number basis, give k photons probability

$$p_k = \frac{6}{\pi^2 (k+1)^2}, \qquad k = 0, 1, \dots .$$

Its entropy is finite since $\sum_{k\ge 1} \log(k+1)/(k+1)^2$ converges. Its mean photon number is infinite since kp_k is asymptotic to $6/(\pi^2 k)$.

Write $S(\rho) = -\operatorname{Tr}(\rho \log \rho) \in [0, \infty]$ for the von Neumann entropy, using natural logarithms and $0 \log 0 = 0$. The entropy of a one-mode thermal state of mean photon number x is

$$g(x) = (1+x)\log(1+x) - x \log x, \qquad x \ge 0.$$

The function g increases continuously from zero to infinity. Thus $g^{-1}(S(\rho)/n)$ is the mean photon number per mode of the product of n identical thermal states having the same entropy as ρ.

Theorem 1.1. *Let A and B be states of n bosonic modes each, with $1 \le n < \infty$ and $S(A), S(B) < \infty$. Mix the product state $A \otimes B$ at a beam splitter of transmissivity $0 \le \eta \le 1$. If a_j, b_j are the input annihilation operators, let C be the state of the modes $c_j = \sqrt{\eta}\, a_j + \sqrt{1-\eta}\, b_j$, $1 \le j \le n$. Then*

$$S(C) \;\ge\; n\, g\!\left(\eta\, g^{-1}\!\left(\frac{S(A)}{n}\right) + (1-\eta)\, g^{-1}\!\left(\frac{S(B)}{n}\right)\right). \tag{1.1}$$

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The modes within either input may be entangled. An infinite value of $S(C)$ satisfies (1.1) in the extended reals. These are the hypotheses of Problem 418 in ProofAtlas [3]. For the Lean proof of Theorem 1.1, see [4].

The difficulty in deducing Theorem 1.1 by approximation is at the output port. If $C_r \to C$ in trace norm, lower semicontinuity gives $S(C) \le \liminf S(C_r)$. To pass a lower bound on $S(C_r)$ to $S(C)$ one needs an upper bound on the approximating entropies. We obtain it by filtering both input ports by r^N, where N counts photons. Since the beam splitter conserves total photon number, the same product filter acts on the two outputs. The filter on the discarded port can then be removed at the cost of an operator inequality on the retained port.

2 An entropy estimate for filtered states

For a positive trace-class operator T of trace at most one, put

$$\Phi(T) = \operatorname{Tr} f(T), \qquad f(t) = t(1 - \log t) \quad (0 < t \le 1), \qquad f(0) = 0.$$

The added trace term makes f increasing on the whole state spectrum. The following estimate is an elementary form of the spectral domination used in Simon's convergence theorems for entropy [5, Appendix, Theorems A.2–A.4].

Lemma 2.1. *Let ρ and σ be states on a separable Hilbert space, let D be a contraction, and suppose $q\sigma \le D\rho D^*$ for some $0 < q \le 1$. Then*

$$q(1 + S(\sigma) - \log q) \le 1 + S(\rho). \tag{2.1}$$

Proof. Write $\lambda_j(T)$ for the eigenvalues of a positive compact operator, in decreasing order with multiplicity, padded by zeros in the finite-rank case. The min–max principle gives $\lambda_j(q\sigma) \le \lambda_j(D\rho D^*)$ for each j. Taking $K = D\rho^{1/2}$, we have $KK^* = D\rho D^*$ and $K^*K = \rho^{1/2}D^*D\rho^{1/2} \le \rho$. The nonzero eigenvalues of KK^* and K^*K agree, so a second application of min–max gives $\lambda_j(q\sigma) \le \lambda_j(\rho)$. Applying f and summing the nonnegative terms yields $\Phi(q\sigma) \le \Phi(\rho)$. Since $\Phi(q\sigma) = q(1 + S(\sigma) - \log q)$ and $\Phi(\rho) = 1 + S(\rho)$, this proves (2.1), including the case of an infinite trace of $f(\rho)$. □

In particular, suppose $\sigma_r \to \rho$ in trace norm and $q_r\sigma_r \le D_r\rho D_r^*$ with contractions D_r and $0 < q_r \le 1$ tending to one. If $S(\rho) < \infty$, (2.1) gives $\limsup S(\sigma_r) \le S(\rho)$. Lower semicontinuity gives $S(\rho) \le \liminf S(\sigma_r)$. Indeed, trace-norm convergence implies convergence of each ordered eigenvalue by min–max, and Fatou's lemma applies to the nonnegative terms $-\lambda_j(\sigma_r)\log\lambda_j(\sigma_r)$. The same lower bound suffices when $S(\rho) = \infty$. Thus in either case

$$S(\sigma_r) \longrightarrow S(\rho) \quad \text{in } [0, \infty]. \tag{2.2}$$

3 Photon-number damping at the two ports

On the n-mode Fock space $\mathcal{F}_n$, let P_k be the projection onto the subspace of total photon number k, so $N = \sum_{k\ge 0} kP_k$. For $0 < r < 1$ and a state ρ define

$$D_r = r^N, \qquad p_r(\rho) = \operatorname{Tr}(D_r\rho D_r), \qquad \rho_r = \frac{D_r\rho D_r}{p_r(\rho)}. \tag{3.1}$$

Writing $w_k = \mathrm{Tr}(P_k\rho)$, we have $p_r(\rho) = \sum_{k\geq 0} r^{2k} w_k > 0$ and $p_r(\rho) \uparrow 1$ as $r \uparrow 1$. Each ρ_r has finite mean photon number, since

$$\mathrm{Tr}(N\rho_r) = \frac{\sum_{k\geq 0} k r^{2k} w_k}{p_r(\rho)} \leq \frac{\sup_{k\geq 0} k r^{2k}}{p_r(\rho)} < \infty.$$

For trace-norm convergence, factor $\rho = \rho^{1/2}\rho^{1/2}$ and use the Hilbert–Schmidt inequality to obtain

$$\|D_r\rho D_r - \rho\|_1 \leq 2\|(I - D_r)\rho^{1/2}\|_2, \qquad \|(I - D_r)\rho^{1/2}\|_2^2 = \sum_{k\geq 0}(1 - r^k)^2 w_k \longrightarrow 0.$$

Hence $\rho_r \to \rho$ in trace norm. Taking $q_r = p_r(\rho)$ in (2.2) gives $S(\rho_r) \to S(\rho)$.

Let U_η be the beam-splitter unitary on $\mathcal{F}_n \otimes \mathcal{F}_n$, let $\omega = U_\eta(A \otimes B)U_\eta^*$, and take the first output port as the retained one, so $C = \mathrm{Tr}_2\,\omega$. Write C_r for the output from the independently filtered inputs A_r, B_r, and set $q_r = p_r(A)p_r(B)$. In general C_r differs from the state obtained by filtering C alone.

Proposition 3.1. *For every $0 < r < 1$,*

$$0 \leq q_r C_r \leq D_r C D_r. \tag{3.2}$$

Consequently $S(C_r) \to S(C)$ in $[0, \infty]$ as $r \uparrow 1$.

Proof. The unitary U_η preserves each eigenspace of $N \otimes I + I \otimes N$. On the eigenspace of total number m, $D_r \otimes D_r$ is multiplication by r^m. It therefore commutes with U_η, and

$$q_r C_r = \mathrm{Tr}_2\big[(D_r \otimes D_r)\omega(D_r \otimes D_r)\big].$$

Put $E_r = (I - D_r^2)^{1/2}$. For any bounded test operator H on the first port, the pairing of $D_r C D_r - q_r C_r$ with H is $\mathrm{Tr}[\omega(D_r H D_r \otimes E_r^2)]$. Thus

$$D_r C D_r - q_r C_r = \mathrm{Tr}_2\big[(D_r \otimes E_r)\omega(D_r \otimes E_r)\big] \geq 0.$$

By Lemma 2.1, (3.2) gives the upper bound $q_r(1 + S(C_r) - \log q_r) \leq 1 + S(C)$. The input products converge in trace norm because $A_r \to A$ and $B_r \to B$ in that norm. Unitary conjugation and partial trace give $C_r \to C$ in trace norm, so (2.2) applies with $q_r \to 1$. □

Proof of Theorem 1.1. Both filtered inputs have finite mean photon number. Their joint output has finite total mean photon number by conservation, so each marginal has finite entropy by the Gibbs bound [2, Lemma 2.2]. The finite-energy inequality [2, Theorem 1.1] therefore gives

$$S(C_r) \geq n\, g\!\left(\eta\, g^{-1}\!\left(\frac{S(A_r)}{n}\right) + (1 - \eta)\, g^{-1}\!\left(\frac{S(B_r)}{n}\right)\right). \tag{3.3}$$

By (3.1) and (2.2) the input entropies converge to $S(A)$ and $S(B)$. Letting $r \uparrow 1$ and using the continuity of g and g^{-1} on the right and Proposition 3.1 on the left proves (1.1). □

Corollary 3.2. *Suppose A and B are independent and $0 < \eta < 1$. If $S(A) = \infty$ or $S(B) = \infty$, then $S(C) = \infty$.*

Proof. The filtered inputs have finite energy, so (3.3) applies for every $0 < r < 1$. If $S(A) = \infty$, then $S(A_r) \to \infty$. Since $\eta > 0$ and $g^{-1}(s) \to \infty$ as $s \to \infty$, the right side of (3.3) tends to infinity regardless of $S(B_r)$. Proposition 3.1 gives the claim. If $S(B) = \infty$ use $1 - \eta > 0$. At $\eta = 0$ or 1 the retained port carries just one input, so the restriction on η is necessary. □

4 The minimum for a thermal attenuator

Let τ_x denote the one-mode thermal state with mean photon number $x \geq 0$. Its photon counts have the geometric distribution of mean x, with τ_0 the vacuum. Let $\mathcal{E}_{\eta,b}$ be the one-mode channel obtained by mixing the input with an independent copy of τ_b and retaining the first output. For one mode, De Palma, Trevisan, and Giovannetti proved the constrained minimum-output-entropy theorem for phase-covariant Gaussian channels [6]. Multimode results include the entanglement-breaking case [7] and the heating range [8]. OpenAI derived the following minimum over finite-energy competitors [2, Corollary 1.2].

Corollary 4.1. *For $1 \leq n < \infty$, $a, b \geq 0$, and $0 \leq \eta \leq 1$,*

$$\min_{\substack{\rho \text{ on } \mathcal{F}_n \\ S(\rho) = ng(a)}} S\Big(\mathcal{E}_{\eta,b}^{\otimes n}(\rho)\Big) = n\, g(\eta a + (1-\eta)b).$$

The product thermal input $\tau_a^{\otimes n}$ attains the minimum.

Proof. The environment is $\tau_b^{\otimes n}$ and has entropy $ng(b)$. Theorem 1.1 gives the stated lower bound for every input of entropy $ng(a)$. For attainment, the thermal mixing identity [2, Proposition 2.3] gives $\mathcal{E}_{\eta,b}(\tau_a) = \tau_{\eta a + (1-\eta)b}$, including the vacuum cases. Taking the n-fold product gives equality. □

AI Assistance Disclosure

OpenAI's GPT-6.2 Astra was used in an autoresearch loop and reported this discovery to the author.

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