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Proof of a conjectural formula of Guillera for $1/\pi^2$

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Abstract

We prove the identity

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{8})_n(\frac{3}{8})_n(\frac{5}{8})_n(\frac{7}{8})_n}{n!^5} \frac{1920n^2+304n+15}{7^{4n}} = \frac{56\sqrt{7}}{\pi^2},$$

which Guillera discovered experimentally in 2003. Up to a constant factor, the generating function of the coefficients is a period of a one-parameter family of fourfolds, namely double covers of $\mathbb{C}\times(\mathbb{C}^*)^3$. For the parameter value corresponding to 7^{-4} we construct an algebraic surface on the fourfold. It is swept out by lifts of rational curves totally tangent to the branch divisor and is birational to the product of an elliptic curve and the projective line. Using the inhomogeneous Picard–Fuchs equation of the associated normal function, we show that on a suitable isotypic component of the cohomology the class of the 4-form whose period is a constant multiple of the sum of the series equals $56\sqrt{-7}\,(2\pi\mathrm{i})^2$ times the class of the surface. The formula follows by computing the intersection number of the surface with the cycle over which the period is taken. One step, the evaluation of a residue sum in the function field of the elliptic curve, is done by computer in exact arithmetic.

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1 Introduction

For a complex number a and an integer $n \geq 0$ let $(a)_n = a(a+1)\cdots(a+n-1)$ denote the Pochhammer symbol. In 1914 Ramanujan [1] recorded seventeen series for $1/\pi$ of the form $\sum_n c_n\,(\alpha+\beta n)\,z_0^n$, in which c_n is a product of three Pochhammer symbols divided by $n!^3$ and z_0 is algebraic; see [2] for a survey of their proofs and of the many generalisations found since. Guillera discovered analogous series for $1/\pi^2$, in which five Pochhammer symbols occur and the linear polynomial is replaced by a quadratic one. He proved several of them by the Wilf–Zeilberger method [3], for instance

$$\sum_{n=0}^{\infty}(-1)^n\frac{(\frac12)_n^5}{n!^5}\,\frac{820n^2+180n+13}{2^{10n}}=\frac{128}{\pi^2}.$$

Further proofs, generalisations and interpretations can be found in [4–7].

In 2003 Guillera found the following identity by an integer relation search [8, (2-5)].

Theorem 1.1

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac18)_n(\frac38)_n(\frac58)_n(\frac78)_n}{n!^5}\,\frac{1920n^2+304n+15}{7^{4n}}=\frac{56\sqrt{7}}{\pi^2}.$$

The identity has so far been conjectural. It is entry #15 of the list of Guillera's formulas for $1/\pi^2$ in [9, Table 2.10] and appears there again as the conjectural identity [9, (6.3)]. The same paper conjectures that the underlying hypergeometric motive is related to the Asai L-function of a Hilbert modular form [9, Conjecture 5.1]. Au observes that it does not succumb to the method of Wilf–Zeilberger seeds, which has settled many other conjectures of this type [7, §3.4]. In this paper we prove Theorem 1.1. In our proof the value $56\sqrt{7}/\pi^2$ arises because a certain period equals $(2\pi\mathrm{i})^2$ times an algebraic number times the intersection number of an algebraic surface with the cycle of integration.

Algebraic cycles and normal functions have appeared before in the study of hypergeometric identities. For example, Kerr [10] uses hypergeometric normal functions to explain "upside-down" Ramanujan-type identities for $\zeta(3)$ and $\zeta(5)$ due to Guillera and to Au.

1.1 Outline of the proof

Write

$$a_n=\frac{(\frac12)_n(\frac18)_n(\frac38)_n(\frac58)_n(\frac78)_n}{n!^5},\qquad F(z)=\sum_{n\geq0}a_nz^n,\qquad P(n)=1920n^2+304n+15,$$

and $\theta = z\,\mathrm{d}/\mathrm{d}z$. The left-hand side of Theorem 1.1 is $(P(\theta)F)(7^{-4})$. The proof has four steps.

Step 1: a period. For $t\in\mathbb{C}$ let X_t be the fourfold $Y^2=(1-u^4)\big(1-tu(x_1+x_2+x_3+x_4)\big)$, $x_1x_2x_3x_4=1$, with the 4-form $\Omega_t = u\,\mathrm{d}u\wedge\omega_0/Y$, where $\omega_0 =$

$\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3/(x_1x_2x_3)$. For $|t| < 1/4$ there is a compact 4-cycle γ on X_t with $\int_\gamma \Omega_t = \pi(2\pi\mathrm{i})^3 F(256t^4)$ (Lemma 2.3). Since $256t_0^4 = 7^{-4}$ for $t_0 = 1/28$, the series in Theorem 1.1 equals $(\pi(2\pi\mathrm{i})^3)^{-1} \int_\gamma P(\theta)\Omega_{t_0}$, where $\theta = \frac{t}{4}\partial_t$ corresponds to $z\,\mathrm{d}/\mathrm{d}z$ under $z = 256t^4$.

Step 2: an algebraic surface. The fourfold X_t is fibred over the u-line. Its fibres are double covers of the torus $\{x_1x_2x_3x_4 = 1\}$, branched along the surfaces $x_1 + x_2 + x_3 + x_4 = 1/(tu)$. We construct rational curves in the torus that are totally tangent to these branch surfaces, one for each point of the elliptic curve $E\colon h^2 = L(2L^2 - 3L + 2)$. The preimage of each curve in the double cover splits into two components, and one of them sweeps out an algebraic surface $S \subset X_{t_0}$ (Section 4). The construction rests on an identity between binary octic forms (Lemma 4.1). Geometrically, it says that certain plane sections of an octic threefold split into two quartic curves (Remark 4.3).

Step 3: the class of S. The difference of the two components defines a normal function over the u-line. By a theorem of Griffiths [11], the normal function satisfies an inhomogeneous Picard–Fuchs equation (see also [12–14]). In our situation the inhomogeneous term can be computed from local data at the points of tangency. We carry this out in Section 5 and translate the result into cohomology in Section 6. On a suitable isotypic component this gives $(2\pi\mathrm{i})^2\ \mathrm{PD}(S) = [P(\theta)\Omega_{t_0}]/(56\sqrt{-7})$; see Theorem 6.4 for the precise statement.

Step 4: an intersection number. The surface S meets γ transversally in exactly two points, both with local intersection number $+1$ (Proposition 4.6). Pairing the identity of Step 3 with γ gives $\int_\gamma P(\theta)\Omega_{t_0} = 2 \cdot 56\sqrt{-7}\,(2\pi\mathrm{i})^2$, which by Step 1 implies Theorem 1.1 (Section 7).

The evaluation of the residue sum in Proposition 5.6 is a finite but lengthy computation in the function field of E. We carried it out in exact arithmetic with SymPy and checked it by numerical contour integration (Appendix A). All other computations in the paper can be checked by hand.

1.2 Conventions

Complex manifolds, their complex submanifolds and algebraic cycles carry their canonical orientations. Let A and B be oriented submanifolds, or piecewise smooth chains, of complementary dimensions in an oriented manifold, meeting transversally at a point p. The local intersection number at p is $+1$ if an oriented basis of T_pA followed by an oriented basis of T_pB is an oriented basis of the ambient tangent space, and -1 otherwise; if A and B meet transversally in finitely many points, $A \cdot B$ is the sum of the local intersection numbers. For a closed, not necessarily compact, oriented cycle Z of real codimension k we write $\mathrm{PD}(Z)$ for its Poincaré dual in $H^k(-, \mathbb{Q})$, normalised by $\int_c \mathrm{PD}(Z) = Z \cdot c$ for every compact k-cycle c meeting Z transversally; equivalently $\int \alpha \wedge \mathrm{PD}(Z) = \int_Z \alpha$ for closed compactly supported forms α [15]. Cohomology with complex coefficients is identified with de Rham cohomology. Finally, $\sqrt{-7} := \mathrm{i}\sqrt{7}$.

2 The fourfold and the period

2.1 The fourfold

Let $\mathbb{T} = \{x \in (\mathbb{C}^*)^4 : x_1x_2x_3x_4 = 1\}$; we use x_1, x_2, x_3 as coordinates and $x_4 = (x_1x_2x_3)^{-1}$. Put $\Phi(x) = x_1 + x_2 + x_3 + x_4$ and $\omega_0 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3/(x_1x_2x_3)$, and let $\mathbb{T}_{\mathbb{R}} = \{|x_1| = |x_2| = |x_3| = 1\}$, oriented by $\mathrm{d}\arg x_1 \wedge \mathrm{d}\arg x_2 \wedge \mathrm{d}\arg x_3$, so that $\int_{\mathbb{T}_{\mathbb{R}}} \omega_0 = (2\pi\mathrm{i})^3$.

For $t \in \mathbb{C}$ let $X_t \subset \mathbb{C} \times \mathbb{T} \times \mathbb{C}$ be the fourfold defined by $Y^2 = f$, where $f = (1-u^4)\big(1-tu\Phi(x)\big)$, and let $\mathrm{pr}\colon X_t \to \mathbb{C}$ be the projection $(u, x, Y) \mapsto u$. The singular points of X_t are the points with $Y = 0$ and $\mathrm{d}f = 0$. Where $u^4 = 1 \neq tu\Phi$ we have $\partial_u f = -4u^3(1 - tu\Phi) \neq 0$, and where $tu\Phi = 1 \neq u^4$ we have $\partial_u f = -t\Phi\,(1 - u^4) \neq 0$, since $t\Phi = 1/u$ there. As $\mathrm{d}f = 0$ where both factors vanish, we obtain $\operatorname{Sing} X_t = \{u^4 = 1,\ tu\Phi(x) = 1,\ Y = 0\}$.

Let $X_t^\circ = X_t \setminus \operatorname{Sing} X_t$ and $X_t^\flat = \{(u, x, Y) \in X_t : tu\Phi(x) \neq 1\}$, an open subset of X_t°. Put $\Omega_t = u\,\mathrm{d}u \wedge \omega_0/Y$. We write $\theta = \frac{t}{4}\,\partial_t$ for the derivation with respect to the parameter, acting on forms with u and x fixed; this is compatible with $\theta = z\,\mathrm{d}/\mathrm{d}z$ for $z = 256t^4$.

Lemma 2.1 *Ω_t is a holomorphic 4-form on X_t°. For every $j \geq 0$, $(1-tu\Phi)^j\,\theta^j\Omega_t$ is Ω_t times a polynomial in $tu\Phi$ with rational coefficients; in particular $\theta^j\Omega_t$ is holomorphic on $X_t^\flat$.*

Proof Away from $\{Y = 0\}$ there is nothing to prove. Near a point of X_t° with $Y = 0$ we have $\mathrm{d}f \neq 0$, and $2Y\,\mathrm{d}Y = \partial_u f\,\mathrm{d}u + \sum_{j\leq 3} \partial_{x_j} f\,\mathrm{d}x_j$. If $\partial_u f \neq 0$, then (Y, x_1, x_2, x_3) are local coordinates and $\Omega_t = 2u\,\mathrm{d}Y \wedge \omega_0/\partial_u f$ is regular. Otherwise some $\partial_{x_j} f$ is nonzero, say $j = 3$; then (u, x_1, x_2, Y) are local coordinates and $\Omega_t = 2u\,\mathrm{d}u \wedge \frac{\mathrm{d}x_1}{x_1} \wedge \frac{\mathrm{d}x_2}{x_2} \wedge \mathrm{d}Y/(x_3\partial_{x_3} f)$ is regular. The second assertion follows by induction on j, since θ acts on functions of $w = tu\Phi$ as $\frac{w}{4}\frac{\mathrm{d}}{\mathrm{d}w}$ and $\frac{w}{4}\frac{\mathrm{d}}{\mathrm{d}w}(1-w)^{-1/2} = \frac{w}{8}(1-w)^{-3/2}$. □

2.2 The period

Since $0 < (a)_n/n! \leq 1$ for $0 < a \leq 1$, we have $0 < a_n \leq 1$. Hence F converges for $|z| < 1$, and the left-hand side of Theorem 1.1 equals $(P(\theta)F)(7^{-4})$.

Lemma 2.2 *Let $|t| < 1/4$. On $[0,1]\times\mathbb{T}_{\mathbb{R}}$ let $Y = \sqrt{1-u^4}\,\sqrt{1-tu\Phi(x)}$ with principal square roots, and orient $[0,1] \times \mathbb{T}_{\mathbb{R}}$ by $\mathrm{d}u$ followed by the orientation of $\mathbb{T}_{\mathbb{R}}$. Then $\int_{[0,1]\times\mathbb{T}_{\mathbb{R}}} \Omega_t = \frac{\pi}{4}(2\pi\mathrm{i})^3 F(256t^4)$.*

Proof For $0 \leq u \leq 1$ and $x \in \mathbb{T}_{\mathbb{R}}$ we have $|tu\Phi(x)| \leq 4|t| < 1$, so the binomial series $(1-tu\Phi)^{-1/2} = \sum_{m\geq 0} \frac{(1/2)_m}{m!}(tu\Phi)^m$ converges uniformly on $[0,1]\times\mathbb{T}_{\mathbb{R}}$. The constant term of the Laurent polynomial Φ^m on $\mathbb{T}$ vanishes unless $m = 4n$, in which case it equals $(4n)!/n!^4$,

because a monomial $x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}$ is constant on $\mathbb{T}$ only if $k_1=k_2=k_3=k_4$. Therefore

$$\int_{\mathbb{T}_\mathbb{R}} \frac{\omega_0}{\sqrt{1-tu\Phi}} = (2\pi\mathrm{i})^3 \sum_{n\ge 0} \frac{(\frac12)_{4n}}{n!^4}(tu)^{4n}. \tag{2.1}$$

Substituting $w=u^2$ and then $w=\sin\varphi$ gives $\int_0^1 u^{4n+1}(1-u^4)^{-1/2}\,\mathrm{d}u = \frac12\int_0^{\pi/2}\sin^{2n}\varphi\,\mathrm{d}\varphi = \frac{\pi}{4}\frac{(1/2)_n}{n!}$. Together with Gauss's multiplication formula $(\frac12)_{4n} = 4^{4n}(\frac18)_n(\frac38)_n(\frac58)_n(\frac78)_n$ this proves the lemma. □

2.3 Symmetries

The symmetric group $\mathfrak{S}_4$ acts on X_t by permuting $x_1,\dots,x_4$. Let $\iota(u,x,Y) = (u,x,-Y)$ be the covering involution, and let $\rho(u,x,Y) = (\mathrm{i}u,-\mathrm{i}x,Y)$, which preserves X_t because $(\mathrm{i}u)^4 = u^4$, $(-\mathrm{i})^4=1$ and $(\mathrm{i}u)\Phi(-\mathrm{i}x) = u\Phi(x)$. The automorphisms ι and ρ commute with each other and with $\mathfrak{S}_4$; all of them preserve X_t° and $X_t^\flat$, and they generate a group $G\cong\mathfrak{S}_4\times\mathbb{Z}/2\times\mathbb{Z}/4$. A transposition of two coordinates changes the sign of ω_0 (for instance, exchanging x_3 and x_4 replaces $\mathrm{d}x_3/x_3$ by $\mathrm{d}x_4/x_4 = -\sum_{j\le 3}\mathrm{d}x_j/x_j$), and $\rho^*(u\,\mathrm{d}u) = -u\,\mathrm{d}u$. Therefore

$$\sigma^*\Omega_t = \operatorname{sgn}(\sigma)\,\Omega_t\ (\sigma\in\mathfrak{S}_4),\qquad \iota^*\Omega_t = -\Omega_t,\qquad \rho^*\Omega_t = -\Omega_t, \tag{2.2}$$

and the same holds for $\theta^j\Omega_t$, since these automorphisms do not involve t.

Let χ_0 be the character of $\mathfrak{S}_4\times\langle\iota\rangle$ given by $\chi_0(\sigma\iota^a) = \operatorname{sgn}(\sigma)(-1)^a$, and $e_{\chi_0} = \frac{1}{48}\sum_{g\in\mathfrak{S}_4\times\langle\iota\rangle}\chi_0(g)\,g$ the corresponding idempotent. For a representation V of G we write $V^{\chi_0} = e_{\chi_0}V$. We use this for cohomology, with g acting by pull-back, and for homology, with g acting by push-forward. Since χ_0 takes the values ±1, in both cases V^{χ_0} is the subspace on which every $g\in\mathfrak{S}_4\times\langle\iota\rangle$ acts by $\chi_0(g)$. As $\int_{g_*c}\alpha = \int_c g^*\alpha$, we have $\int_{e_{\chi_0}c}\alpha = \int_c e_{\chi_0}\alpha$. By (2.2), the classes of $\theta^j\Omega_t$ lie in $H^4(X_t^\flat,\mathbb{C})^{\chi_0}$, and ρ^* acts on them by -1.

2.4 The cycle γ

Fix t with $|t|<1/4$. For $a\in\{\pm1,\pm\mathrm{i}\}$ let $\Lambda_a^\pm$ be the images of

$$[0,1]\times\mathbb{T}_\mathbb{R}\to X_t,\qquad (\varrho,x)\mapsto\Big(\varrho a,\ x,\ \pm\sqrt{1-\varrho^4}\,\sqrt{1-ta\varrho\Phi(x)}\Big),$$

with principal square roots, oriented as $[0,1]\times\mathbb{T}_\mathbb{R}$. Put $\Lambda_a = \Lambda_a^+ - \Lambda_a^-$ and $\gamma = \Lambda_1 - \Lambda_\mathrm{i}$. Thus Λ_a runs over $\mathbb{T}_\mathbb{R}$ from $u=0$ to $u=a$ on the sheet with $Y=1$ at $u=0$, and back on the other sheet. In terms of these parametrisations, $\sigma\in\mathfrak{S}_4$ maps $\Lambda_a^\pm$ onto itself by $(\varrho,x)\mapsto(\varrho,\sigma(x))$, which multiplies the orientation by $\operatorname{sgn}(\sigma)$. The involution ι maps $\Lambda_a^\pm$ onto $\Lambda_a^\mp$ by the identity, and ρ maps $\Lambda_a^\pm$ onto $\Lambda_{\mathrm{i}a}^\pm$ by $(\varrho,x)\mapsto(\varrho,-\mathrm{i}x)$, which preserves the orientation.

Lemma 2.3 *(a) γ is a compact 4-cycle contained in $X_t^\flat$.*
(b) $\int_\gamma\theta^j\Omega_t = \pi(2\pi\mathrm{i})^3\,(\theta^jF)(256t^4)$ for all $j\ge0$. In particular, for $t=t_0=1/28$ the left-hand side of Theorem 1.1 equals $(\pi(2\pi\mathrm{i})^3)^{-1}\int_\gamma P(\theta)\Omega_{t_0}$.

(c) $\sigma_*\gamma = \mathrm{sgn}(\sigma)\gamma$ *for* $\sigma \in \mathfrak{S}_4$, $\iota_*\gamma = -\gamma$, *and* $\gamma + \rho_*\gamma + \rho_*^2\gamma + \rho_*^3\gamma = 0$.

Proof (a) At $\varrho = 1$ the two sheets meet, since $Y = 0$ there, so $\partial\Lambda_a^+$ and $\partial\Lambda_a^-$ have the same component over $u = a$. Therefore $\partial\Lambda_a = \mathbb{T}_\mathbb{R} \times \{(u, Y) = (0, -1)\} - \mathbb{T}_\mathbb{R} \times \{(u, Y) = (0, 1)\}$, which does not depend on a, and $\partial\gamma = 0$. On γ we have $|tu\Phi| < 1$, so $\gamma \subset X_t^\flat$.

(b) By these relations and (2.2), $\int_{\Lambda_a^-} \theta^j\Omega_t = -\int_{\Lambda_a^+} \theta^j\Omega_t$ and $\int_{\Lambda_\mathrm{i}} \theta^j\Omega_t = -\int_{\Lambda_1} \theta^j\Omega_t$, so $\int_\gamma \theta^j\Omega_t = 4\int_{\Lambda_1^+} \theta^j\Omega_t$. On Λ_1^+ the integrand of $\theta^j\Omega_t$ is $(1-tu\Phi)^{-j}$ times a polynomial in $tu\Phi$ times that of Ω_t (Lemma 2.1). The factor $1-tu\Phi$ is bounded away from 0 locally uniformly in $|t| < 1/4$, and the only singularity, $(1-u^4)^{-1/2}$, is integrable. We may therefore differentiate under the integral sign and obtain $\int_{\Lambda_1^+} \theta^j\Omega_t = \theta^j \int_{\Lambda_1^+} \Omega_t$. Now (b) follows from Lemma 2.2, and the last assertion from $256t_0^4 = 7^{-4}$.

(c) This follows from the same relations, since $\rho_*^k\gamma = \Lambda_{\mathrm{i}^k} - \Lambda_{\mathrm{i}^{k+1}}$ for all k. □

3 The fibres

3.1 The threefolds Z_s

Let $\mathcal{P} = \{s \in \mathbb{C}^* : 256s^4 \neq 1\}$. For $s \in \mathcal{P}$ let $Z_s = \{(x, v) \in \mathbb{T} \times \mathbb{C} : v^2 = 1 - s\Phi(x)\}$, $R_s = \{v = 0\} \subset Z_s$, $U_s = Z_s \setminus R_s$ and $\eta_s = \omega_0/v$. The critical points of Φ on $\mathbb{T}$ are the points $x_1 = x_2 = x_3 = x_4 = \zeta$ with $\zeta^4 = 1$, and the critical values are 4ζ; since $4\zeta s \neq 1$, the surface $\{\Phi = 1/s\}$ is smooth. Thus Z_s is a smooth affine threefold, the double cover of $\mathbb{T}$ branched along the surface $\{\Phi = 1/s\}$, with ramification divisor R_s, and, by the argument of Lemma 2.1, η_s is a holomorphic 3-form on Z_s. The group $\mathfrak{S}_4$ and the involution $\iota(x, v) = (x, -v)$ act on Z_s, with $\sigma^*\eta_s = \mathrm{sgn}(\sigma)\eta_s$ and $\iota^*\eta_s = -\eta_s$. For a space with an action of ι we write $H^\bullet(-)^-$ and $H_\bullet(-)^-$ for the ι-anti-invariant parts of its cohomology and homology.

The fibres of X_t are of this form. Let $u \in \mathbb{C}$ with $u^4 \neq 1$ and $tu \in \mathcal{P}$, put $s = tu$, and let δ be a square root of $1 - u^4$. Then

$$Z_s \xrightarrow{\sim} \mathrm{pr}^{-1}(u), \qquad (x, v) \mapsto (u, x, \delta v), \tag{3.1}$$

and this isomorphism maps U_s onto $\mathrm{pr}^{-1}(u) \cap X_t^\flat$. Under it ω_0/Y corresponds to η_s/δ, so over $\{u : u^4 \neq 1,\ tu \in \mathcal{P}\}$

$$\Omega_t = \frac{\mathrm{d}u}{u} \wedge \frac{u^2}{\delta}\,\eta_{tu}, \tag{3.2}$$

where $\eta_{tu}/\delta = \omega_0/Y$ does not depend on the choice of δ.

3.2 Cohomology of the fibres

Let $s \in \mathcal{P}$ and $\psi = 1/s$. Let $\widetilde{Z} \subset \mathbb{P}^4$ be the octic threefold $z_0^8 + z_1^8 + z_2^8 + z_3^8 + z_4^8 = \psi\,(z_1z_2z_3z_4)^2$. The group $G_0 = \{(\zeta_0, \dots, \zeta_4) \in \mu_8^5 : \zeta_1\zeta_2\zeta_3\zeta_4 = \zeta_0^4\}/\mu_8$, with μ_8 embedded diagonally, has order 512 and acts on $\widetilde{Z}$ by $z_i \mapsto \zeta_i z_i$ $(0 \le i \le 4)$. Let

$B_i = \widetilde{Z} \cap \{z_i = 0\}$ for $0 \le i \le 4$. Fix a square root of ψ. The map

$$\widetilde{Z} \setminus (B_1 \cup \cdots \cup B_4) \to Z_s, \qquad x_i = \frac{z_i^8}{(z_1 z_2 z_3 z_4)^2}, \quad v = \frac{z_0^4}{\sqrt{\psi}\, z_1 z_2 z_3 z_4},$$

is G_0-invariant, finite and surjective, of degree $512 = 128 \cdot 4$: the map $(z_1 : \cdots : z_4) \mapsto (x_1, x_2, x_3)$ between tori has degree $\det \left(\begin{smallmatrix} 6 & -2 & -2 \\ -2 & 6 & -2 \\ -2 & -2 & 6 \end{smallmatrix} \right) = 128$, and $z_0 \mapsto v$ has degree 4. Since the degree equals $|G_0|$ and Z_s is normal, the map identifies Z_s with the quotient of $\widetilde{Z} \setminus (B_1 \cup \cdots \cup B_4)$ by G_0, and U_s with the quotient of $\widetilde{Z} \setminus (B_0 \cup \cdots \cup B_4)$. We put $\overline{Z}_s = \widetilde{Z}/G_0$, $\overline{R}_s = B_0/G_0$ and $\overline{U}_s = \overline{Z}_s \setminus \overline{R}_s$. Thus $\overline{Z}_s$ is a projective variety containing Z_s as a dense open subset, $\overline{R}_s$ is the closure of R_s, and $U_s \subset \overline{U}_s$. The automorphism $\tilde{\iota}\colon z_0 \mapsto e^{\pi \mathrm{i}/4} z_0$ of $\widetilde{Z}$ commutes with G_0 and induces an automorphism of $\overline{Z}_s$ extending ι, again denoted by ι. Being quotients of smooth varieties by a finite group, $\overline{Z}_s$ and $\overline{U}_s$ are rational homology manifolds: Poincaré duality with rational coefficients holds for them, and their complex cohomology can be computed with G_0-invariant differential forms on $\widetilde{Z}$ and on $\widetilde{Z} \setminus B_0$.

Proposition 3.1 *Let $s \in \mathcal{P}$.*

(a) $H^j(Z_s, \mathbb{Q})^- = 0$ for $j \ne 3$, and $\dim H^3(Z_s, \mathbb{Q})^- = 4$.
(b) The restriction maps $H^j(Z_s, \mathbb{Q})^- \to H^j(U_s, \mathbb{Q})^-$ and $H^j(\overline{U}_s, \mathbb{Q})^- \to H^j(U_s, \mathbb{Q})^-$ are isomorphisms for all j.
(c) The families $\{Z_s\}$, $\{U_s\}$ and $\{\overline{U}_s\}$, $s \in \mathcal{P}$, are locally trivial.

Proof We first show that $\widetilde{Z}$ is smooth and that $B_0 + \cdots + B_4$ is a simple normal crossing divisor. At a singular point of $\widetilde{Z}$ we would have $z_0 = 0$ and $4z_i^8 = \psi\,(z_1 z_2 z_3 z_4)^2$ for $i = 1, \ldots, 4$ (if some z_i with $i \ge 1$ vanished, the vanishing of the partial derivatives would force all coordinates to vanish); multiplying these four equations gives $\psi^4 = 256$, which is excluded. So $\widetilde{Z}$ is smooth, and the same computation shows that B_0 is smooth. For $\emptyset \ne I \subset \{1, \ldots, 4\}$ the intersection $B_I = \bigcap_{i \in I} B_i$ is the Fermat variety $z_0^8 + \sum_{j \in \{1,\ldots,4\} \setminus I} z_j^8 = 0$, because the right-hand side of the equation of $\widetilde{Z}$ vanishes there, and $B_I \cap B_0$ is the Fermat variety $\sum_{j \in \{1,\ldots,4\} \setminus I} z_j^8 = 0$ in $\{z_0 = 0\}$. At points of B_I the differential of the equation of $\widetilde{Z}$ is $8 \sum_{j \in \{0,\ldots,4\} \setminus I} z_j^7 \, \mathrm{d}z_j$; it is linearly independent of the $\mathrm{d}z_i$ with $i \in I$, and also of $\mathrm{d}z_0$ at points of B_0. Therefore $B_0 + \cdots + B_4$ is a simple normal crossing divisor whose strata are smooth; apart from $\widetilde{Z}$ and B_0, they do not depend on ψ.

As ψ varies, the pairs $(\widetilde{Z}, B_0 + \cdots + B_4)$ form a family over $\{\psi \in \mathbb{C}^* : \psi^4 \ne 256\}$ of smooth projective varieties with simple normal crossing divisors, all of whose strata form smooth proper families. By Ehresmann's theorem for such pairs, a special case of Thom's first isotopy lemma [16], this family is locally trivial, and by using vector fields tangent to all strata and invariant under G_0 and $\tilde{\iota}$ (obtained by averaging) the trivialisations can be chosen equivariant. Passing to the quotients, we obtain (c).

Since G_0 is finite, the rational cohomology of Z_s, U_s and $\overline{U}_s$ is the G_0-invariant part of the cohomology of the complements in $\widetilde{Z}$ of $B_1 \cup \cdots \cup B_4$, of $B_0 \cup \cdots \cup B_4$ and of B_0, respectively, with ι acting through $\tilde{\iota}$. On B_i $(1 \le i \le 4)$ the automorphism $\tilde{\iota}$ agrees with the element of G_0 given by $\zeta_i = -1$, $\zeta_0 = e^{\pi \mathrm{i}/4}$ and $\zeta_j = 1$ for the other j; hence $\tilde{\iota}$ acts trivially

on the G_0-invariant cohomology of every G_0-stable locally closed subset of B_i. The same holds for B_0, which $\tilde\iota$ fixes pointwise. Now remove the divisors one at a time, at each step the part of B_i not yet removed, and use the Gysin sequences. These are compatible with G_0 and $\tilde\iota$, which act holomorphically and so preserve the orientations of normal bundles. Since taking the part on which G_0 acts trivially and $\tilde\iota$ acts by -1 is exact, all restriction maps between the cohomology of $\widetilde Z$ and that of the complements above induce isomorphisms on these parts. This proves (b), and it shows that $H^j(Z_s,\mathbb{Q})^- \cong H^j(\widetilde Z,\mathbb{Q})^{G_0,\,\tilde\iota=-1}$.

By the Lefschetz hyperplane theorem and Poincaré duality [17], for $j\neq 3$ the group $H^j(\widetilde Z,\mathbb{Q})$ is zero or spanned by a power of the hyperplane class, which is invariant under G_0 and $\tilde\iota$, as these are induced by linear automorphisms of $\mathbb{P}^4$. Therefore $H^j(Z_s,\mathbb{Q})^- = 0$ for $j\neq 3$.

To compute the dimension, we use Euler characteristics. For fixed (x_1,x_2) the equation $\Phi=\psi$ reads $x_3^2-(\psi-x_1-x_2)x_3+(x_1x_2)^{-1}=0$, whose roots are nonzero. So the projection to (x_1,x_2) exhibits the surface $\{\Phi=\psi\}\cong R_s$ as a double cover of $(\mathbb{C}^*)^2$ branched along the curve $x_1x_2(\psi-x_1-x_2)^2=4$. On this curve $w=(\psi-x_1-x_2)/2$ is nonzero, $x_1+x_2=\psi-2w$ and $x_1x_2=w^{-2}$; so the curve is a double cover of $\mathbb{C}^*_w$ branched at the roots of $w^2(\psi-2w)^2-4$. These roots are nonzero, and they are distinct because the discriminant of this quartic is $4096(\psi^4-256)\neq 0$. Thus the curve has Euler characteristic -4, so that $\chi(R_s)=2\chi\big((\mathbb{C}^*)^2\big)+4=4$ and $\chi(Z_s)=2\chi(\mathbb{T})-\chi(R_s)=-4$. Since $Z_s/\iota\cong\mathbb{T}$, the ι-invariant part of $H^\bullet(Z_s,\mathbb{Q})$ is $H^\bullet(\mathbb{T},\mathbb{Q})$, whose Euler characteristic is 0. Therefore $\sum_j(-1)^j\dim H^j(Z_s,\mathbb{Q})^- = \chi(Z_s) = -4$, and since only $j=3$ contributes, $\dim H^3(Z_s,\mathbb{Q})^- = 4$. □

3.3 The Picard–Fuchs equation

On the family $\{U_s\}_{s\in\mathcal{P}}$ we differentiate forms with respect to s keeping x fixed, and write $\vartheta = s\,\partial_s$. Let

$$\mathcal{L} = \vartheta^4 - 256s^4\Big(\vartheta+\frac12\Big)\Big(\vartheta+\frac32\Big)\Big(\vartheta+\frac52\Big)\Big(\vartheta+\frac72\Big).$$

In terms of $\lambda = 256s^4$ and $D_\lambda = \lambda\,\mathrm{d}/\mathrm{d}\lambda = \frac14\vartheta$, this is 256 times the hypergeometric operator $D_\lambda^4 - \lambda\prod_\alpha(D_\lambda+\alpha)$, where α runs over $\{\frac18,\frac38,\frac58,\frac78\}$.

Put $q_i = sx_i$ for $1\le i\le 4$, $p_k=\sum_{i=1}^4 q_i^k$ (so that $p_1 = 1-v^2$ on Z_s), $g = \frac13(16+232p_1+1492p_1^2)-480p_2$ and

$$N_i = g\Big(q_i-\frac{p_1}{4}\Big) - (64+976p_1)\Big(q_i^2-\frac{p_2}{4}\Big) + 960\Big(q_i^3-\frac{p_3}{4}\Big) \qquad (1\le i\le 4),$$

and let $\beta_s = V\lrcorner\omega_0$ be the contraction of ω_0 with the vector field $V=\sum_{i=1}^4 \frac{N_i}{8v^7}\,x_i\frac{\partial}{\partial x_i}$. Since $\sum_i N_i = 0$, V is tangent to $\mathbb{T}$, and in the coordinates x_1,x_2,x_3

$$\beta_s = \frac{1}{8v^7}\Big(N_1\frac{\mathrm{d}x_2}{x_2}\wedge\frac{\mathrm{d}x_3}{x_3} - N_2\frac{\mathrm{d}x_1}{x_1}\wedge\frac{\mathrm{d}x_3}{x_3} + N_3\frac{\mathrm{d}x_1}{x_1}\wedge\frac{\mathrm{d}x_2}{x_2}\Big).$$

Proposition 3.2 *On U_s we have $\mathcal{L}\eta_s = \mathrm{d}\beta_s$. The form β_s satisfies $\sigma^*\beta_s = \mathrm{sgn}(\sigma)\beta_s$ for $\sigma\in\mathfrak{S}_4$ and $\iota^*\beta_s = -\beta_s$. The forms $\vartheta^j\eta_s$ $(j\ge 0)$ and β_s extend to holomorphic forms on $\overline{U}_s$, and $\mathcal{L}\eta_s = \mathrm{d}\beta_s$ holds on $\overline{U}_s$.*

Proof Let $f(q) = (1 - q_1 - q_2 - q_3 - q_4)^{-1/2}$, regarded as a function of four independent variables, and $\vartheta_i = q_i\partial/\partial q_i$. On U_s, with $q_i = sx_i$, the operator ϑ acts on functions of q as $\vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4$, and for $j \le 3$ the vector field $x_j\partial/\partial x_j$ acts as $\vartheta_j - \vartheta_4$. Expanding $f = \sum_m \frac{(1/2)_m}{m!} p_1^m$ gives $\prod_{k=1}^4(\sum_i \vartheta_i + k - \frac{1}{2})f = \frac{105}{16}(1-p_1)^{-9/2}$, while $\vartheta_1\vartheta_2\vartheta_3\vartheta_4 f = q_1q_2q_3q_4\,\partial_1\partial_2\partial_3\partial_4 f = \frac{105}{16}\,q_1q_2q_3q_4(1-p_1)^{-9/2}$. Since $q_1q_2q_3q_4 = s^4$ on U_s, it follows that $\mathcal{L}(1/v) = \big[\big((\sum_i \vartheta_i)^4 - 256\vartheta_1\vartheta_2\vartheta_3\vartheta_4\big)f\big]_{q=sx}$.

For indeterminates $y_1, \dots, y_4$ let $\bar{y} = \frac{1}{4}\sum y_i$, $d_i = y_i - \bar{y}$, $M_k = \sum_i d_i^k$, and

$$W_i = (128\bar{y}^2 - 32M_2)\,d_i - \frac{256}{3}\,\bar{y}\Big(d_i^2 - \frac{M_2}{4}\Big) + 64\Big(d_i^3 - \frac{M_3}{4}\Big).$$

Then $\sum_i W_i = 0$ and

$$\Big(\sum_i y_i\Big)^4 - 256\,y_1y_2y_3y_4 = \sum_{i=1}^4 y_iW_i = \sum_{i=1}^3 (y_i - y_4)W_i. \tag{3.3}$$

Since $\sum_i d_i = 0 = \sum_i W_i$, the middle term of (3.3) equals $\sum_i d_iW_i = 128\bar{y}^2M_2 - 32M_2^2 - \frac{256}{3}\bar{y}M_3 + 64M_4$. On the other hand, the elementary symmetric functions of the d_i are 0, $-\frac{M_2}{2}$, $\frac{M_3}{3}$ and $\frac{M_2^2}{8} - \frac{M_4}{4}$, so $256\bar{y}^4 - 256\prod_i(\bar{y} + d_i)$ equals the same expression.

Since f depends on q only through p_1, we have $\vartheta_i f = q_i f'$, $\vartheta_j\vartheta_i f = \delta_{ij}q_i f' + q_iq_j f''$ and

$$\vartheta_k\vartheta_j\vartheta_i f = \delta_{ij}\delta_{ik}q_i f' + (\delta_{ij}q_iq_k + \delta_{ik}q_iq_j + \delta_{jk}q_iq_j)f'' + q_iq_jq_k f''',$$

where f', f'', f''' are the derivatives with respect to p_1, namely $\frac{1}{2}(1-p_1)^{-3/2}$, $\frac{3}{4}(1-p_1)^{-5/2}$ and $\frac{15}{8}(1-p_1)^{-7/2}$. Inserting these into the cubic polynomial W_i and collecting terms, one finds $W_i(\vartheta_1, \dots, \vartheta_4)f = N_i/(8(1-p_1)^{7/2})$, which becomes $N_i/(8v^7)$ at $q = sx$. By (3.3), $\mathcal{L}(1/v) = \sum_{j=1}^3 x_j\partial_{x_j}\big(N_j/(8v^7)\big)$, and this equals the coefficient of ω_0 in $\mathrm{d}\beta_s$. Hence $\mathcal{L}\eta_s = \mathrm{d}\beta_s$.

The permutation σ permutes the functions $N_i/(8v^7)$ and the vector fields $x_i\partial/\partial x_i$ in the same way, so $\sigma_*V = V$ and $\sigma^*\beta_s = \mathrm{sgn}(\sigma)\beta_s$. The involution ι changes the sign of v^{-7} and fixes the N_i, so $\iota^*\beta_s = -\beta_s$.

The pull-backs of $\vartheta^j\eta_s$ and β_s to $\widetilde{Z}\setminus B_0$ are holomorphic. By symmetry it suffices to check this in the chart $z_4 = 1$. There z_1, z_2, z_3 are local coordinates on $\widetilde{Z}\setminus B_0$, since the derivative of the equation of $\widetilde{Z}$ with respect to z_0 does not vanish. As $\mathrm{d}x_i/x_i = 6\,\mathrm{d}z_i/z_i - 2\sum_{j\ne i}\mathrm{d}z_j/z_j$ $(i, j \le 3)$, we have $\omega_0 = 128\,\mathrm{d}z_1\wedge\mathrm{d}z_2\wedge\mathrm{d}z_3/(z_1z_2z_3)$ and $\eta_s = 128\sqrt{\psi}\,\mathrm{d}z_1\wedge\mathrm{d}z_2\wedge\mathrm{d}z_3/z_0^4$. By induction on j, as in Lemma 2.1, $\vartheta^j\eta_s/\eta_s$ is $(1 - s\Phi)^{-j} = v^{-2j}$ times a polynomial of degree at most j in $s\Phi = 1 - v^2$, hence a polynomial in $v^{-2} = \psi(z_1z_2z_3)^2/z_0^8$. Finally, N_i is a polynomial of degree at most 3 in the $q_j = sz_j^8/(z_1z_2z_3)^2$ (where $z_4 = 1$), so N_i/v^7 is a polynomial in z_1, z_2, z_3 divisible by $z_1z_2z_3$, divided by z_0^{28}. This cancels the denominators of the forms $\frac{\mathrm{d}z_k}{z_k}\wedge\frac{\mathrm{d}z_l}{z_l}$, of which β_s is a linear combination. □

3.4 Monodromy

By Proposition 3.1(c), the groups $\mathbb{V}_s = H^3(U_s, \mathbb{Q})^-$ form a local system $\mathbb{V}$ on $\mathcal{P}$. For $0 \le j \le 3$ let $E_j(s) = [\vartheta^j\eta_s] \in H^3(U_s, \mathbb{C})^-$.

Lemma 3.3*(a) For $0 < |s| < 1/4$ let $c_{\mathbb{T}}(s)$ be the lift $x \mapsto \bigl(x, \sqrt{1 - s\Phi(x)}\bigr)$ of $\mathbb{T}_{\mathbb{R}}$ to U_s (principal square root), oriented as $\mathbb{T}_{\mathbb{R}}$. Then*

$$\int_{c_{\mathbb{T}}(s)} \eta_s = (2\pi\mathrm{i})^3 \sum_{n \geq 0} b_n \lambda^n, \qquad b_n = \frac{(\frac{1}{8})_n (\frac{3}{8})_n (\frac{5}{8})_n (\frac{7}{8})_n}{n!^4}.$$

(b) For every $s \in \mathcal{P}$ the classes $E_0(s), \dots, E_3(s)$ form a basis of $H^3(U_s, \mathbb{C})^-$, and $\mathcal{L}E_0 = 0$.
(c) $\mathfrak{S}_4$ acts on $H^3(U_s, \mathbb{Q})^-$ by the sign character.
(d) The local system $\mathbb{V}$ is the pull-back, under $s \mapsto \lambda = 256 s^4$, of a local system on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ whose monodromy representation is irreducible.

Proof (a) follows from (2.1), with tu replaced by s, and Gauss's multiplication formula.

The isomorphisms $Z_s \to Z_{\mathrm{i}s}$, $(x, v) \mapsto (-\mathrm{i}x, v)$, map η_s to $\eta_{\mathrm{i}s}$ and commute with ι and with ϑ. Therefore the family $\{U_s\}$ descends to the λ-line, and so do $\mathbb{V}$ and the classes E_j. For a flat family of cycles c the period $\int_c \eta_s$ is annihilated by $\mathcal{L}$ (Proposition 3.2). The span of these periods is stable under monodromy, and it is nonzero by (a). Since none of $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ is an integer, the hypergeometric local system is irreducible by the criterion of Beukers and Heckman [18], so the periods span the full four-dimensional solution space.

(b) $\mathcal{L}E_0 = [\mathrm{d}\beta_s] = 0$. Suppose $\sum_j c_j E_j(s_0) = 0$ with constants c_j. Then $\sum_j c_j (\vartheta^j y)(s_0) = 0$ for every period y, hence for every solution y of $\mathcal{L}y = 0$. Since the Wronskian of $\mathcal{L}$ does not vanish at $s_0 \in \mathcal{P}$, all c_j vanish. As $\dim H^3(U_s)^- = 4$ (Proposition 3.1), the E_j form a basis.

(c) Since σ commutes with ϑ, each E_j transforms by sgn, and the E_j span $H^3(U_s, \mathbb{C})^-$ by (b).

(d) By (b) and duality, the period map $c \mapsto (\int_c E_j)_j$ is an isomorphism from $H_3(U_s, \mathbb{C})^-$ onto the solution space of the hypergeometric equation, compatible with monodromy. Hence the descended local system is dual to the hypergeometric local system, which is irreducible, as shown above. □

3.5 Normalisation of the intersection form

Let

$$\mathsf{B}(\lambda) = \begin{pmatrix} 0 & -\frac{11}{2}\lambda & -4\lambda & 1-\lambda \\ \frac{11}{2}\lambda & 0 & -(1-\lambda) & 0 \\ 4\lambda & 1-\lambda & 0 & 0 \\ -(1-\lambda) & 0 & 0 & 0 \end{pmatrix}, \qquad \kappa = 128\,(2\pi\mathrm{i})^3.$$

For $c \in H_3(U_s, \mathbb{Q})^-$ write $\boldsymbol{\varpi}_c = \bigl(\int_c E_0, \dots, \int_c E_3\bigr)^{\mathsf{T}}$ and $\mathbf{E} = (E_0, E_1, E_2, E_3)^{\mathsf{T}}$.

Proposition 3.4 *For $s \in \mathcal{P}$ and $c, c' \in H_3(U_s, \mathbb{Q})^-$,*

$$\boldsymbol{\varpi}_c^{\mathsf{T}}\, \mathsf{B}(\lambda)\, \boldsymbol{\varpi}_{c'} = -\kappa\,(c \cdot c'). \tag{3.4}$$

In particular the intersection form on $H_3(U_s, \mathbb{Q})^-$ is nondegenerate. The class $\mathrm{PD}(c) \in H^3(U_s, \mathbb{C})^-$, characterised by $\int_{c'} \mathrm{PD}(c) = c \cdot c'$ for all c', is

$$\mathrm{PD}(c) = -\kappa^{-1}\, \boldsymbol{\varpi}_c^{\mathsf{T}}\, \mathsf{B}(\lambda)\, \mathbf{E}. \tag{3.5}$$

Proof Step 1: flatness. Since $\prod_{k=1}^{4}(\vartheta+k-\frac12)=\vartheta^4+8\vartheta^3+\frac{43}{2}\vartheta^2+22\vartheta+\frac{105}{16}$, the relation $\mathcal{L}E_0=0$ reads $\vartheta\mathbf{E}=\mathsf{M}\mathbf{E}$, where M is the companion matrix with rows $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ and $\frac{\lambda}{1-\lambda}\big(\frac{105}{16},22,\frac{43}{2},8\big)$. A direct computation, using $\vartheta\lambda=4\lambda$, shows

$$\vartheta\mathsf{B}+\mathsf{M}^{\mathsf{T}}\mathsf{B}+\mathsf{B}\mathsf{M}=0. \tag{3.6}$$

For flat families of cycles we have $\vartheta\boldsymbol{\varpi}_c=\mathsf{M}\boldsymbol{\varpi}_c$, so $\boldsymbol{\varpi}_c^{\mathsf{T}}\mathsf{B}\boldsymbol{\varpi}_{c'}$ is locally constant.

Step 2: proportionality. The isomorphisms $Z_s\cong Z_{\mathrm{is}}$ from the proof of Lemma 3.3 are holomorphic and preserve intersection numbers, and B depends only on λ. So both sides of (3.4) are alternating bilinear forms on $H_3(U_s,\mathbb{C})^-$ that are invariant under the monodromy of the descended local system. The left-hand side is nondegenerate, because $\det\mathsf{B}=(1-\lambda)^4$ and the period map is an isomorphism. By irreducibility (Lemma 3.3(d)) and Schur's lemma, the intersection form is a constant multiple of it. We determine the constant near $\lambda=1$, for real s.

Step 3: two cycles near $\lambda=1$. Let $0<s<1/4$, $a=1-4s$ and $\varepsilon=1-\lambda=4a+O(a^2)$. Let $c_{\mathbb{T}}$ be the cycle of Lemma 3.3(a); its period is $\varpi_{\mathbb{T}}=\int_{c_{\mathbb{T}}}\eta_s=(2\pi\mathrm{i})^3\sum_n b_n\lambda^n$. The cycle $c_{\mathbb{T}}$ is not ι-anti-invariant, but since $\iota^*\eta_s=-\eta_s$ and ι preserves intersection numbers, its projection $\frac12(c_{\mathbb{T}}-\iota_*c_{\mathbb{T}})$ has the same periods as $c_{\mathbb{T}}$ and the same intersection number with every ι-anti-invariant cycle. We may therefore work with $c_{\mathbb{T}}$ itself.

At $\lambda=1$ the fibre acquires an ordinary double point at $x_1=x_2=x_3=x_4=1$, $v=0$. Write $x_j=e^{\xi_j}$ for $j\le3$. Then $\Phi(e^\xi)=4+\frac12\xi^{\mathsf{T}}\mathsf{H}\xi+O(|\xi|^3)$ with $\mathsf{H}=I+\mathbf{1}\mathbf{1}^{\mathsf{T}}$ and $\det\mathsf{H}=4$. By Taylor's formula with integral remainder, $\frac14(\Phi(e^\xi)-4)=\xi^{\mathsf{T}}\mathsf{K}(\xi)\xi$, where $\mathsf{K}(\xi)$ is real analytic, symmetric, and positive definite for small real ξ, with $\mathsf{K}(0)=\mathsf{H}/8$. Consequently $y=\mathsf{K}(\xi)^{1/2}\xi$ are real-analytic coordinates near $\xi=0$ with $\frac14(\Phi(e^\xi)-4)=|y|^2$ and $\mathrm{d}\xi_1\wedge\mathrm{d}\xi_2\wedge\mathrm{d}\xi_3=J(y)\,\mathrm{d}y_1\wedge\mathrm{d}y_2\wedge\mathrm{d}y_3$, where $J(0)=\det(\mathsf{H}/8)^{-1/2}=8\sqrt2$. For real ξ the equation of the fibre becomes $v^2=a-(1-a)|y|^2$. The vanishing cycle c_{van} consists of the two sheets $v=\pm\sqrt{a-(1-a)|y|^2}$ over the ball $|y|^2\le a/(1-a)$ in real ξ-space, the upper sheet oriented by $\mathrm{d}\xi_1\wedge\mathrm{d}\xi_2\wedge\mathrm{d}\xi_3$ and the lower one by the opposite orientation. It is an embedded 3-sphere with $\iota_*c_{\mathrm{van}}=-c_{\mathrm{van}}$, and the two sheets contribute equally to its period, so that

$$\varpi_{\mathrm{van}}=\int_{c_{\mathrm{van}}}\eta_s=2\int_{|y|^2\le\frac{a}{1-a}}\frac{J(y)\,\mathrm{d}^3y}{\sqrt{a-(1-a)|y|^2}}=16\sqrt2\,\pi^2a+O(a^2)=4\sqrt2\,\pi^2\varepsilon+O(\varepsilon^2).$$

Here we substituted $y=\sqrt{a/(1-a)}\,w$ and used that the linear term of J integrates to zero and that $\int_{|w|\le1}(1-|w|^2)^{-1/2}\,\mathrm{d}^3w=\pi^2$. The sphere c_{van} lies in Z_s but meets R_s. By Proposition 3.1(b) and duality, its class is the image of a unique class in $H_3(U_s,\mathbb{Q})^-$, again denoted by c_{van}, which has the same intersection number with $c_{\mathbb{T}}\subset U_s$ and, as η_s is holomorphic on Z_s, the same η_s-period.

The cycles meet only at $\xi=0$ on the upper sheet: $c_{\mathbb{T}}$ is contained in $\{\mathrm{Re}\,\xi=0\}$ near this point, and c_{van} in $\{\mathrm{Im}\,\xi=0\}$. There $(\partial_{\mathrm{Re}\,\xi_1},\partial_{\mathrm{Re}\,\xi_2},\partial_{\mathrm{Re}\,\xi_3})$ and $(\partial_{\mathrm{Im}\,\xi_1},\partial_{\mathrm{Im}\,\xi_2},\partial_{\mathrm{Im}\,\xi_3})$ are oriented bases of the tangent spaces of c_{van} and $c_{\mathbb{T}}$. The first followed by the second differs from the complex orientation $(\mathrm{Re}\,\xi_1,\mathrm{Im}\,\xi_1,\mathrm{Re}\,\xi_2,\mathrm{Im}\,\xi_2,\mathrm{Re}\,\xi_3,\mathrm{Im}\,\xi_3)$ by a permutation with three inversions, so $c_{\mathrm{van}}\cdot c_{\mathbb{T}}=-1$.

Step 4: expansions at $\lambda=1$. The local exponents of $\mathcal{L}$ at $\lambda=1$ are $0,1,1,2$. The Frobenius method shows that near $\varepsilon=0$ the solutions of $\mathcal{L}y=0$ are spanned by three power series with leading terms 1, ε, ε^2 and one solution of the form $f\log\varepsilon+\tilde f$, where f and $\tilde f$ are holomorphic at 0. Here f is itself a solution, determined up to a constant factor by the requirement that such an $\tilde f$ exists; normalised by $f=\varepsilon+O(\varepsilon^2)$, it is $f=\varepsilon+\frac{43}{64}\varepsilon^2+O(\varepsilon^3)$, as one finds by inserting $f\log\varepsilon+\tilde f$ into $\mathcal{L}y=0$ and comparing the coefficients of $\varepsilon^k\log\varepsilon$ and ε^k for $k\le2$. The local monodromy T_1 maps $f\log\varepsilon+\tilde f$ to itself plus $2\pi\mathrm{i}f$. On the other hand, by

the Picard–Lefschetz formula [19, §3.2], applied to the family $\{Z_s\}$, and Proposition 3.1(b), $(T_1-1)c$ is a multiple of c_{van} for every cycle c, so T_1-1 maps every period into $\mathbb{C}\,\varpi_{\mathrm{van}}$. The formula applies here: the total space $\{v^2=1-s\Phi(x)\}$ is smooth, s has a nondegenerate critical point at the node, and near $s=1/4$ the family is topologically trivial away from the node, because the singular points of $\widetilde{Z}$ for $\psi=4$ satisfy $z_1z_2z_3z_4\neq 0$ and the strata of $B_1+\cdots+B_4$ do not depend on ψ. Hence ϖ_{van} is proportional to f, and by the computation above $\varpi_{\mathrm{van}}=\alpha_{\mathrm{van}}f$ with $\alpha_{\mathrm{van}}=4\sqrt{2}\,\pi^2$. By Stirling's formula and the reflection formula, $b_n=(1+O(n^{-1}))\,n^{-2}/\bigl(\Gamma(\frac18)\Gamma(\frac38)\Gamma(\frac58)\Gamma(\frac78)\bigr)=(1+O(n^{-1}))/(2\sqrt{2}\,\pi^2n^2)$. Comparing with $\mathrm{Li}_2(1-\varepsilon)=\frac{\pi^2}{6}+\varepsilon\log\varepsilon-\varepsilon+O(\varepsilon^2\log\varepsilon)$, and noting that a power series with coefficients $O(n^{-3})$ has a continuous derivative up to $\lambda=1$, we find that the coefficient of $\varepsilon\log\varepsilon$ in $\varpi_{\mathbb{T}}$ is $\alpha_{\mathbb{T}}=(2\pi\mathrm{i})^3/(2\sqrt{2}\,\pi^2)=-2\sqrt{2}\,\pi\mathrm{i}$. Thus $\varpi_{\mathbb{T}}=\alpha_{\mathbb{T}}f\log\varepsilon+g_{\mathbb{T}}(\varepsilon)$ with $g_{\mathbb{T}}$ holomorphic.

Step 5: the constant. Put $y=\varpi_{\mathrm{van}}$ and $z=\varpi_{\mathbb{T}}$. In terms of ε we have $\vartheta=-4(1-\varepsilon)\,\mathrm{d}/\mathrm{d}\varepsilon$, and as $\varepsilon\to 0$

$$\begin{gathered}
y=\alpha_{\mathrm{van}}\varepsilon+O(\varepsilon^2),\qquad \vartheta y=-4\alpha_{\mathrm{van}}+O(\varepsilon),\\
\vartheta^2y=\tfrac{11}{2}\alpha_{\mathrm{van}}+O(\varepsilon),\qquad \vartheta^3y=O(1),\\
z=z_*+O(\varepsilon\log\varepsilon),\qquad \vartheta z=-4\alpha_{\mathbb{T}}\log\varepsilon+O(1),\\
\vartheta^2z=\frac{16\alpha_{\mathbb{T}}}{\varepsilon}+O(\log\varepsilon),\qquad \vartheta^3z=\frac{64\alpha_{\mathbb{T}}}{\varepsilon^2}+O\Bigl(\frac{1}{\varepsilon}\Bigr);
\end{gathered}$$

here z_* is the value of $\varpi_{\mathbb{T}}$ at $\lambda=1$, and the value $\vartheta^2y(0)=\frac{11}{2}\alpha_{\mathrm{van}}$ uses the coefficient $\frac{43}{64}$. In the expansion

$$\boldsymbol{\varpi}_{c_{\mathrm{van}}}^{\mathsf{T}}\,\mathsf{B}\,\boldsymbol{\varpi}_{c_{\mathbb{T}}}=y\Bigl(-\tfrac{11}{2}\lambda\vartheta z-4\lambda\vartheta^2z+\varepsilon\vartheta^3z\Bigr)+\vartheta y\Bigl(\tfrac{11}{2}\lambda z-\varepsilon\vartheta^2z\Bigr)+\vartheta^2y\bigl(4\lambda z+\varepsilon\vartheta z\bigr)-\varepsilon z\,\vartheta^3y$$

the terms $\frac{11}{2}\lambda\,\vartheta y\,z$ and $4\lambda\,\vartheta^2y\,z$ tend to $-22\alpha_{\mathrm{van}}z_*$ and $22\alpha_{\mathrm{van}}z_*$, the terms $-4\lambda y\,\vartheta^2z$, $\varepsilon y\,\vartheta^3z$ and $-\varepsilon\,\vartheta y\,\vartheta^2z$ tend to $-64\alpha_{\mathrm{van}}\alpha_{\mathbb{T}}$, $64\alpha_{\mathrm{van}}\alpha_{\mathbb{T}}$ and $64\alpha_{\mathrm{van}}\alpha_{\mathbb{T}}$, and the remaining three terms tend to 0. Since the left-hand side is constant, it equals $64\,\alpha_{\mathrm{van}}\alpha_{\mathbb{T}}=-1024\pi^3\mathrm{i}=\kappa$. Together with $c_{\mathrm{van}}\cdot c_{\mathbb{T}}=-1$ this proves (3.4).

Step 6: Poincaré duals. Nondegeneracy follows from (3.4). For $c'\in H_3(U_s,\mathbb{Q})^-$ the right-hand side of (3.5) has period $-\kappa^{-1}\boldsymbol{\varpi}_c^{\mathsf{T}}\mathsf{B}\boldsymbol{\varpi}_{c'}=c\cdot c'$ over c'. Since $H^3(U_s,\mathbb{C})^-$ is dual to $H_3(U_s,\mathbb{C})^-$, this proves (3.5). □

By Proposition 3.1(b), the inclusion induces an isomorphism $H_3(U_s,\mathbb{Q})^-\to H_3(\overline{U}_s,\mathbb{Q})^-$, compatible with intersection numbers and with the periods of the forms E_j, which are defined on $\overline{U}_s$ by Proposition 3.2. Therefore (3.4) and (3.5) also hold for $c,c'\in H_3(\overline{U}_s,\mathbb{Q})^-$, with $\mathrm{PD}(c)\in H^3(\overline{U}_s,\mathbb{C})^-$.

4 The surface

4.1 A family of totally tangent curves

Let E be the elliptic curve with affine equation $h^2=LT$, where $T=2L^2-3L+2$. On E we use the functions $D=L^2-L+1$, $A=D-h$, $u=\mathrm{i}h/L$ and $k=7(L-h)^2/(L^2Dh^2)$, which satisfy

$$\begin{gathered}
(L-h)^2=2LA,\qquad L+L^{-1}=\frac{3-u^2}{2},\qquad 1-u^4=-\frac{4(L-1)^2D}{L^2},\\
A^2=L^4+1-2Dh.
\end{gathered}\tag{4.1}$$

The first follows from $h^2 = LT$, the second and third from $u^2 = -h^2/L^2 = -T/L$ and $1 - u^4 = (L^2 - T^2)/L^2$, and the fourth from $D^2 + h^2 = L^4 + 1$. For a variable r put

$$P_0 = r(r+1)(r+L), \qquad Q = \frac{r^4}{L^2} + \frac{2(L+1)r^3}{Lh} + \Big(\frac{7}{h} - \frac{1}{L}\Big)r^2 + \frac{2(L+1)r}{h} + 1,$$

and $w_1 = A^2/(2L^4Dh)$, $w_2 = A^2/(2Dh)$, $w_3 = -1/(2Dh)$, $w_4 = -1/(2L^4Dh)$.

Lemma 4.1 *In $\mathbb{C}(E)[r]$ we have*

$$kP_0^2 - Q^2 = w_1 r^8 + w_2 + w_3(r+1)^8 + w_4(r+L)^8.$$

Proof Multiplying by $2L^4Dh^2$ and using $(L-h)^2 = 2LA$, we can write the identity as

$$28L^3AP_0^2 - hA^2r^8 - L^4hA^2 + L^4h(r+1)^8 + h(r+L)^8 = 2D\hat{Q}^2,$$

where $\hat{Q} = L^2hQ = hr^4 + 2L(L+1)r^3 + (7L^2 - Lh)r^2 + 2L^2(L+1)r + L^2h$. Both sides are polynomials of degree 8 in r with coefficients in $\mathbb{Z}[L,h]$, and both are unchanged under $F(r) \mapsto L^{-4}r^8F(L/r)$: this holds for P_0^2 and $\hat{Q}^2$ because $P_0(L/r) = L^2r^{-3}(r+1)(r+L)$ and $\hat{Q}(L/r) = L^2r^{-4}\hat{Q}(r)$, and the substitution exchanges hA^2r^8 with L^4hA^2 and $L^4h(r+1)^8$ with $h(r+L)^8$. It therefore suffices to compare the coefficients of $r^8, \dots, r^4$. Reducing with $h^2 = LT$ and $A^2 = L^4 + 1 - 2Dh$, one finds that on both sides they are

$$2Dh^2, \quad 8L(L^3+1)h, \quad 28L^2D(L+h), \quad 56L^3(L^3+1),$$
$$28L^3\big(L^4 + 3L^3 - 2L^2 + 3L + 1 - Dh\big).$$

For example, on the left the coefficients of r^8 and r^7 are $h(L^4 + 1 - A^2) = 2Dh^2$ and $8h(L^4 + L) = 8L(L^3+1)h$, and on the right they are $2Dh^2$ and $2D \cdot 2h \cdot 2L(L+1) = 8L(L^3+1)h$. □

For $e = (L,h) \in E$ with $L \notin \{0, 1, \infty\}$ and $r \in \mathbb{C} \setminus \{0, -1, -L\}$ define

$$\begin{aligned} x_1 &= -\frac{\mathrm{i}\, r^6(L-h)^2}{2L^3(r+1)^2(r+L)^2}, \quad & x_2 &= -\frac{\mathrm{i}\, L(L-h)^2}{2r^2(r+1)^2(r+L)^2}, \\ x_3 &= \frac{2\mathrm{i}\, L^3(r+1)^6}{r^2(L-h)^2(r+L)^2}, & x_4 &= \frac{2\mathrm{i}\,(r+L)^6}{Lr^2(L-h)^2(r+1)^2}, \\ Y &= \frac{2\mathrm{i}}{\sqrt{7}}\,\frac{(L-1)Dh\,Q}{(L-h)P_0}. \end{aligned} \tag{4.2}$$

Here $L \neq h$, since $h = L$ would give $T = L$, whereas $T - L = 2(L-1)^2 \neq 0$.

Corollary 4.2 *With these definitions $x_1x_2x_3x_4 = 1$,*

$$1 - \frac{u}{28}\Phi(x) = \frac{Q^2}{kP_0^2}, \qquad \text{and} \qquad Y^2 = (1-u^4)\Big(1 - \frac{u}{28}\Phi(x)\Big).$$

Thus (u,x,Y) is a point of X_{t_0}.

Proof Let $\psi = 28/u = -28\mathrm{i}L/h$ and $\ell = (\ell_1, \ell_2, \ell_3, \ell_4) = (r, 1, r+1, r+L)$. Using $(L-h)^2 = 2LA$ one checks that $x_i = \psi w_i \ell_i^8/(kP_0^2)$ for $i = 1, \dots, 4$. Since $\ell_1\ell_2\ell_3\ell_4 = P_0$ and $\psi^4 = (28L/h)^4 = k^4/(w_1w_2w_3w_4)$, the product of the x_i equals 1. As $u\psi = 28$, Lemma 4.1 gives $\frac{u}{28}\Phi(x) = \sum_i w_i\ell_i^8/(kP_0^2) = 1 - Q^2/(kP_0^2)$. Finally, by (4.1),

$$(1-u^4)\frac{Q^2}{kP_0^2} = -\frac{4(L-1)^2D}{L^2} \cdot \frac{L^2Dh^2Q^2}{7(L-h)^2P_0^2} = Y^2.$$

□

Remark 4.3 Lemma 4.1 has the following geometric meaning. Let $s = u/28$, and consider the octic threefold $\widetilde{Z}$ of §3.2 with $\psi = 1/s$. Choose $a \in \mathbb{C}^*$ and $\alpha_1, \dots, \alpha_4$ with $\alpha_i^8 = aw_i$ and $(\alpha_1\alpha_2\alpha_3\alpha_4)^2 = ak/\psi$; this is possible because $\psi^4 = k^4/(w_1w_2w_3w_4)$. On the plane $\{(z_0 : \alpha_1 r : \alpha_2 : \alpha_3(r+1) : \alpha_4(r+L))\}$ the equation of $\widetilde{Z}$ becomes $z_0^8 = a\,(kP_0^2 - \sum_i w_i\ell_i^8) = aQ^2$. So the plane section splits into the two quartic curves $z_0^4 = \pm\sqrt{a}\,Q(r)$, and their images in Z_s are the two curves $v = \pm Q/(\sqrt{k}P_0)$ lying over the rational curve $r \mapsto x$ of (4.2). Thus this rational curve meets the branch surface $\{\Phi = 1/s\}$ only at the four zeros of Q, each with even intersection multiplicity: it is totally tangent to the branch surface. Such splitting planes form a one-parameter family, parametrised by E, and the corresponding curves sweep out the surface S constructed below.

4.2 The surface S

Let $E^\circ \subset E$ be the complement of the points with $L \in \{0, 1, \infty\}$ or $D = 0$. For $e \in E^\circ$ we have $u(e)^4 \neq 1$ by (4.1). Let Ψ be the map from $\{(e, r) : e \in E^\circ,\ r \in \mathbb{C} \setminus \{0, -1, -L\}\}$ to X_{t_0} given by $u = u(e)$ and (4.2). Its image avoids $\operatorname{Sing} X_{t_0}$, which lies over $u^4 = 1$.

Let S be the closure of the image of Ψ in $X_{t_0}^\circ$. It is an irreducible algebraic surface, and Ψ is birational onto S by Corollary 4.7 below.

Remark 4.4 As e tends to one of the four points of E with $D = 0$ (where $u = \pm 1$), the function k tends to infinity while the x_i stay bounded for r away from $0, -1, -L, \infty$. Hence $1 - t_0u\Phi(x) = Q^2/(kP_0^2) \to 0$, and the curves degenerate into $\operatorname{Sing} X_{t_0}$. In particular S is not closed in X_{t_0}: its closure meets $\operatorname{Sing} X_{t_0}$ over $u = \pm 1$.

Lemma 4.5 *The involution $\varsigma(L, h, r) = (1/L,\ h/L^2,\ 1/r)$ preserves E° and the function u, and $\Psi \circ \varsigma = (12) \circ \iota \circ \Psi$, where (12) exchanges x_1 and x_2.*

Proof One has $(h/L^2)^2 = T/L^3 = L^{-1}\,T(L^{-1})$, and u is unchanged since $\mathrm{i}(h/L^2)/L^{-1} = \mathrm{i}h/L$. Under ς we get $L - h \mapsto (L-h)/L^2$, $r+1 \mapsto (r+1)/r$, $r+L \mapsto (r+L)/(rL)$, $P_0 \mapsto (r+1)(r+L)/(Lr^3)$ and $Q \mapsto L^2Q/r^4$. Substituting into (4.2) gives $x_1 \leftrightarrow x_2$, $x_3 \mapsto x_3$, $x_4 \mapsto x_4$ and $Y \mapsto -Y$. □

4.3 The intersection number

Proposition 4.6 *The surface S and the cycle γ of Lemma 2.3 (for $t = t_0$) meet in exactly two points p_+, p_-, both lying over $u = \mathrm{i}$. At each of them S is smooth and the intersection is transverse with local intersection number $+1$. In particular $S \cdot \gamma = 2$.*

Proof Step 1: points in the image of Ψ. Suppose that $\Psi(e,r)\in\gamma$. Then $|x_j|=1$ for all j, and $u=u(e)$ lies in $[0,1)\cup\mathrm{i}[0,1)$, since γ lies over $[0,1]\cup\mathrm{i}[0,1]$ and $u(e)^4\neq 1$. Thus $L+L^{-1}=\frac{3-u^2}{2}$ is real and lies in $(1,2)$, so $|L|=1$ and $L\neq\pm1$. Write $L=e^{\mathrm{i}\phi}$ with $0<|\phi|<\pi$, and $c=\cos(\phi/2)\in(0,1)$. Since $x_1/x_2=r^8/L^4$ and $x_3/x_4=L^4\big((r+1)/(r+L)\big)^8$, we get $|r|=1$ and $|r+1|=|r+L|$, so $r=\pm e^{\mathrm{i}\phi/2}$, $|r+1|^2=2\pm2c$, and $|x_2|=1$ becomes $|L-h|^2=2|r+1|^4=8(1\pm c)^2$.

If $u=\varrho\in[0,1)$, then $h=-\mathrm{i}uL=-\mathrm{i}\varrho L$ and $|L-h|^2=1+\varrho^2$. From $2\cos\phi=\frac{3-\varrho^2}{2}$ we get $c^2=\frac{1+\cos\phi}{2}=\frac{7-\varrho^2}{8}$, so $1+\varrho^2=8(1-c^2)$. The condition becomes $(1\pm c)^2=(1-c)(1+c)$, which forces $c\in\{0,1\}$, contradicting $0<c<1$.

If $u=\mathrm{i}\varrho$ with $\varrho\in[0,1)$, then $h=\varrho L$, $|L-h|^2=(1-\varrho)^2$ and $c^2=\frac{7+\varrho^2}{8}$. The sign $+$ is impossible because $8(1+c)^2>8>(1-\varrho)^2$. For the sign $-$ we get $c=1-\frac{1-\varrho}{2\sqrt2}$. Squaring and comparing with $c^2=\frac{7+\varrho^2}{8}=1-\frac{1-\varrho}{4}+\frac{(1-\varrho)^2}{8}$ gives $(1-\varrho)\big(\frac{1}{\sqrt2}-\frac14\big)=0$, i.e. $\varrho=1$, which is excluded.

Consequently the image of Ψ does not meet γ.

Step 2: limit points. Let $p\in S\cap\gamma$. Then $p=\lim\Psi(e_n,r_n)$ for a sequence in the domain of Ψ, and after passing to a subsequence $(e_n,r_n)\to(e_*,r_*)\in E\times\mathbb{P}^1$. Since $u(e_n)\to u(p)$, which is finite, we have $L_*\notin\{0,\infty\}$. By Step 1, (e_*,r_*) is not in the domain of Ψ, so one of the following holds.

(i) $e_*\in E^\circ$ and $r_*\in\{0,-1,-L_*,\infty\}$. These four points are distinct, and near each of them one coordinate x_j has a zero of order 6 and the other three have poles of order 2, with coefficients that are bounded and bounded away from 0 because $L-h\neq0$ near e_*. So some $|x_j(e_n,r_n)|$ tends to 0 or ∞, contradicting $p\in\gamma$.

(ii) $D(e_*)=0$. The argument of (i) applies near e_* (there $L\neq1$ and $L-h\neq0$) and excludes $r_*\in\{0,-1,-L_*,\infty\}$; so, as in Remark 4.4, $p\in\operatorname{Sing}X_{t_0}$, which is impossible because $\gamma\subset X^\flat_{t_0}$ (Lemma 2.3(a)).

(iii) $L_*=1$. Then $u(p)^2=3-2(L_*+L_*^{-1})=-1$, so $u(p)=\mathrm{i}$ (as $p\in\gamma$) and $h_*=1$. Put $L=1+\mu$. On the branch of E with $h(0)=1$ we have $h=1+\mu+\mu^2-\frac{\mu^4}{2}+O(\mu^5)$, so $L-h=-\mu^2\big(1+O(\mu^2)\big)$. From $|x_1/x_2|\to1$ we get $|r_*|=1$, and from $|x_2|\to1$ and $|L-h|^2\sim|\mu|^4$ we get $|(r_n+1)(r_n+L_n)|\sim|\mu_n|^2/\sqrt2$, so $r_*=-1$. Writing $r_n=-1+b_n\mu_n$, so that $r_n+L_n=(b_n+1)\mu_n$, the conditions $|x_2|\to1$ and $|x_3/x_4|\to1$ become $|b_n(b_n+1)|\to2^{-1/2}$ and $|b_n|/|b_n+1|\to1$, so the b_n stay in a compact subset of $\mathbb{C}\setminus\{0,-1\}$. Consider $\widehat\Psi(\mu,b)=\Psi\big(1+\mu,h(\mu),-1+b\mu\big)$. The negative powers of μ cancel in (4.2), so $\widehat\Psi$ extends holomorphically to a neighbourhood of $\{0\}\times(\mathbb{C}\setminus\{0,-1\})$, with

$$\widehat\Psi(0,b)=\Big(\mathrm{i},\ -\frac{\mathrm{i}}{2b^2(b+1)^2},\ -\frac{\mathrm{i}}{2b^2(b+1)^2},\ \frac{2\mathrm{i}b^6}{(b+1)^2},\ \frac{2\mathrm{i}(b+1)^6}{b^2},\ 0\Big),$$

in the coordinates (u,x_1,x_2,x_3,x_4,Y). Passing to a further subsequence, we may assume $b_n\to b_*$; then $p=\widehat\Psi(0,b_*)$. The point $\widehat\Psi(0,b)$ lies over $\mathbb{T}_\mathbb{R}$ if and only if $2|b|^2|b+1|^2=1$, $2|b|^6=|b+1|^2$ and $2|b+1|^6=|b|^2$, that is, $|b|^2=|b+1|^2=2^{-1/2}$. So b_* is one of $b_\pm=-\frac12\pm\mathrm{i}c_0$, where $c_0=\frac12\sqrt{2\sqrt2-1}$. For $b=b_\pm$ we have $b(b+1)=-2^{-1/2}$ and $b+1=-\bar b$, so $x_1=x_2=-\mathrm{i}$, $x_3=4\mathrm{i}b^8$ and $x_4=4\mathrm{i}\bar b^8$. Using $b_\pm^4=(1-\sqrt2)(1\mp\mathrm{i}c_0)$ one finds $\Phi=-2\mathrm{i}+8\mathrm{i}\,\mathrm{Re}(b^8)$ with $\mathrm{Re}(b^8)=\frac{23-16\sqrt2}{4}$, hence

$$1-t_0\,\mathrm{i}\,\Phi=\frac{18-8\sqrt2}{7}=v_0^2,\qquad v_0=\frac{4-\sqrt2}{\sqrt7}>0. \tag{4.3}$$

The two points $p_\pm = \widehat{\Psi}(0, b_\pm)$ are distinct, since $x_3 - x_4 = -8\,\mathrm{Im}(b^8) = \pm 16 c_0(3 - 2\sqrt{2})$. They lie on S, as limits of $\widehat{\Psi}(\mu, b_\pm)$ for $\mu \to 0$, and in $X^\circ_{t_0}$ because $1 - t_0 \mathrm{i}\Phi \neq 0$. Since $u(p_\pm) = \mathrm{i}$, $Y = 0$ and $x \in \mathbb{T}_\mathbb{R}$, they lie on γ.

Step 3: local coordinates. Near $p_\pm$ we use $\xi_j = \log x_j$ $(j \leq 3)$ and $y = Y/\sqrt{1 - t_0 u \Phi(x)}$, with the principal square root, which is holomorphic there by (4.3). Then $y^2 = 1 - u^4$, so (y, ξ_1, ξ_2, ξ_3) are holomorphic coordinates near $p_\pm$. On γ near $p_\pm$ we have $u = \mathrm{i}\varrho$ with $\varrho \in [0, 1]$, $x \in \mathbb{T}_\mathbb{R}$ and $y = \pm\sqrt{1 - \varrho^4}$, where the sign is $+$ on Λ_i^+ and $-$ on Λ_i^-. Hence, near $p_\pm$, the support of γ is $\{\mathrm{Im}\, y = 0,\ \mathrm{Re}\,\xi_1 = \mathrm{Re}\,\xi_2 = \mathrm{Re}\,\xi_3 = 0\}$, and $\gamma = \Lambda_1 - \Lambda_\mathrm{i}$ coincides with $-\Lambda_\mathrm{i}^+ + \Lambda_\mathrm{i}^-$. On Λ_i^+ the parameter ϱ increases while y decreases, and on Λ_i^- both increase. So γ is oriented near $p_\pm$ by the ordered basis $(\partial_{\mathrm{Re}\, y}, \partial_{\mathrm{Im}\,\xi_1}, \partial_{\mathrm{Im}\,\xi_2}, \partial_{\mathrm{Im}\,\xi_3})$.

Step 4: transversality and orientation. Expanding (4.2) along $\widehat{\Psi}$ to first order in μ gives, with $m = b(b+1)$,

$$\partial_\mu(\xi_1, \xi_2, \xi_3)\big|_{\mu=0} = \big(-3(2b+1),\ 2b+1,\ 2b+3\big),$$
$$\partial_b(\xi_1, \xi_2, \xi_3)\big|_{\mu=0} = \frac{2}{m}\big(-(2b+1),\ -(2b+1),\ 2b+3\big),$$
$$Y = \frac{\mathrm{i}(2m^2 + 8m + 1)}{\sqrt{7}\, m}\,\mu + O(\mu^2).$$

At $b = b_\pm$ we have $m = -2^{-1/2}$, so the coefficient of μ in Y equals $2\mathrm{i}v_0$ and $y = 2\mathrm{i}\mu + O(\mu^2)$. Write $b_\pm = -\frac{1}{2} + \mathrm{i}\tau$ with $\tau = \pm c_0$, so that $2b + 1 = 2\mathrm{i}\tau$. The derivatives of the functions $(\mathrm{Im}\, y, \mathrm{Re}\,\xi_1, \mathrm{Re}\,\xi_2, \mathrm{Re}\,\xi_3)$ with respect to $(\mathrm{Re}\,\mu, \mathrm{Im}\,\mu, \mathrm{Re}\, b, \mathrm{Im}\, b)$ at $(0, b_\pm)$ form the matrix

$$\mathsf{N} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 6\tau & 0 & -4\sqrt{2}\tau \\ 0 & -2\tau & 0 & -4\sqrt{2}\tau \\ 2 & -2\tau & -4\sqrt{2} & 4\sqrt{2}\tau \end{pmatrix}, \qquad \det \mathsf{N} = -512\tau^2 < 0.$$

Therefore $\widehat{\Psi}$ is an immersion at $(0, b_\pm)$, and its image is transverse to γ. Consider the ordered basis

$$(\partial_{\mathrm{Re}\, y}, \partial_{\mathrm{Im}\,\xi_1}, \partial_{\mathrm{Im}\,\xi_2}, \partial_{\mathrm{Im}\,\xi_3}, \partial_{\mathrm{Im}\, y}, \partial_{\mathrm{Re}\,\xi_1}, \partial_{\mathrm{Re}\,\xi_2}, \partial_{\mathrm{Re}\,\xi_3}),$$

consisting of an oriented basis of $T\gamma$ followed by the remaining coordinate vectors. It differs from the complex orientation $(\mathrm{Re}\, y, \mathrm{Im}\, y, \mathrm{Re}\,\xi_1, \mathrm{Im}\,\xi_1, \dots)$ by a permutation with nine inversions. In this basis the tangent vectors $\partial_{\mathrm{Re}\,\mu}, \partial_{\mathrm{Im}\,\mu}, \partial_{\mathrm{Re}\, b}, \partial_{\mathrm{Im}\, b}$ of S, followed by those of γ, form the matrix $\left(\begin{smallmatrix} * & I_4 \\ \mathsf{N} & 0 \end{smallmatrix}\right)$, whose determinant is $\det \mathsf{N} < 0$. Since (μ, b) are complex coordinates on S, this basis of $T_{p_\pm} S$ is positively oriented; the two signs cancel, so the local intersection number at $p_\pm$ is $+1$.

Step 5: multiplicity one. By Step 2, every sequence in the domain of Ψ whose image converges to $p_\pm$ eventually lies in the chart (μ, b) near $(0, b_\pm)$, where $\widehat{\Psi}$ is an injective immersion. Thus, near $p_\pm$, S is the smooth surface $\widehat{\Psi}$(neighbourhood of $(0, b_\pm)$), and the local intersection numbers computed in Step 4 are those of S. Together with Steps 1 and 2, this proves $S \cdot \gamma = 2$. □

Corollary 4.7 *The map Ψ has degree one onto S: a general point of S has exactly one preimage. In particular Ψ is birational onto S.*

Proof By Step 5 of the proof of Proposition 4.6, every point of the image of Ψ close to p_+, for instance $\widehat{\Psi}(\mu, b_+)$ with small $\mu \neq 0$, has exactly one preimage. Since the number of preimages is constant on a dense Zariski-open subset of S, the claim follows. □

5 The normal function and its Picard–Fuchs equation

5.1 The curves in the fibres

Let $E^{\circ\circ} \subset E^\circ$ be the complement of the points where $h = 0$, $L = -1$ or $u^4 = 7^4$, or where Q has a multiple root. These are finitely many points, since the discriminant of Q is a nonzero rational function on E, as one checks at a single point. The roots of Q lie in $\mathbb{C} \setminus \{0, -1, -L\}$, since Q has degree 4, $Q(0) = 1$, $Q(-1) = A/L^2$, $Q(-L) = A$, and $A = (L-h)^2/(2L) \neq 0$ on E°. By the proof of Lemma 4.5, the roots of Q at $\varsigma(e)$ are the inverses of those at e, so $E^{\circ\circ}$ is stable under ς. For $e \in E^{\circ\circ}$ put $s(e) = t_0 u(e) = u(e)/28$; then $s(e) \in \mathcal{P}$. The function k has no zeros or poles on $E^{\circ\circ}$; fix a square root $\sqrt{k}$ of $k(e)$. By Corollary 4.2,

$$1 - s(e)\,\Phi\big(x(e,r)\big) = \varphi_e(r)^2, \qquad \varphi_e(r) = \frac{Q(r)}{\sqrt{k}\,P_0(r)}. \tag{5.1}$$

So the preimage in $Z_{s(e)}$ of the rational curve $r \mapsto x(e,r)$ is the union of the two curves $C_e^\pm = \{(x(e,r), \pm\varphi_e(r)) : r \in \mathbb{C} \setminus \{0,-1,-L\}\}$, which intersect in the four points over the roots of Q, the *points of tangency*. The four points $r \in \{0, -1, -L, \infty\}$, where the curve leaves the torus, are its *punctures*. The curves $C_e^\pm$ are closed in $Z_{s(e)}$, since $x(e,r)$ leaves every compact subset of $\mathbb{T}$ as r tends to a puncture. By Remark 4.3, their closures $\overline{C}_e^\pm$ in $\overline{Z}_{s(e)}$ are the images of the two quartic curves $z_0^4 = \pm\sqrt{a}\,Q(r)$ in a plane section of $\widetilde{Z}$; in particular they are compact. At the punctures $z_0 \neq 0$, because Q has degree 4 and no roots in $\{0,-1,-L\}$. Hence $\overline{C}_e^\pm \setminus C_e^\pm$ consists of four points of $\overline{U}_{s(e)} \setminus U_{s(e)}$, and $\overline{C}_e^\pm$ meets $\overline{R}_{s(e)}$ only at the points of tangency.

With $u = u(e)$ we choose the square root δ of $1 - u^4$ determined by

$$\frac{\delta}{\sqrt{k}} = \frac{2\mathrm{i}(L-1)Dh}{\sqrt{7}\,(L-h)}; \tag{5.2}$$

this defines a square root of $1 - u^4$, because the square of the right-hand side is $(1-u^4)/k = -\frac{4(L-1)^2D^2h^2}{7(L-h)^2}$ by (4.1). Then $Y(e,r) = \delta\varphi_e(r)$, so under the identification (3.1) the curve $\Psi(e,\cdot)$ corresponds to C_e^+.

Lemma 5.1 *Let $e \in E^{\circ\circ}$ and $u = u(e)$. Then Ψ is proper over a neighbourhood of u. Identify $\mathrm{pr}^{-1}(u)$ with $Z_{s(e)}$ by (3.1), with δ as in (5.2). Then the fibre of S over u is the cycle $C_e^+ + (12)_* C_e^-$, and $e_{\chi_0}[C_e^+ + (12)_* C_e^-] = e_{\chi_0}[C_e^+ - C_e^-]$.*

Proof The points of E over u are e and $\varsigma(e)$, because u determines L up to $L \mapsto L^{-1}$ by (4.1), and then $h = -\mathrm{i}uL$; both lie in $E^{\circ\circ}$. A point of S over u is a limit of points $\Psi(e_n, r_n)$ with $u(e_n) \to u$. After passing to a subsequence, e_n tends to e or to $\varsigma(e)$, and r_n cannot tend to a puncture, since the limit would then not lie in $\mathbb{C} \times \mathbb{T} \times \mathbb{C}$ (see Step 2(i) of the proof of Proposition 4.6). Hence the fibre of S over u is the union of the images of $\Psi(e,\cdot)$ and $\Psi(\varsigma(e),\cdot)$. By Lemma 4.5, $\Psi(\varsigma(e),\cdot) = (12)\iota\Psi(e,\cdot)$, which corresponds to $(12)_*\iota_* C_e^+ = (12)_* C_e^-$. The same argument shows that Ψ is proper over a neighbourhood of u. As Ψ has degree one

(Corollary 4.7), the projection formula shows that the fibre of S over u, as a cycle, is the push-forward under Ψ of the fibre of $(e',r)\mapsto u(e')$ over u. That fibre is the reduced curve $(\{e\}\cup\{\varsigma(e)\})\times\mathbb{P}^1$, with the punctures removed, because $u\colon E\to\mathbb{P}^1$ is unramified at e and $\varsigma(e)$ (its ramification points are the four points with $L=\pm1$). Moreover $r\mapsto\Psi(e,r)$ is generically injective, since $x_1/x_2=r^8/L^4$ determines r up to an eighth root of unity and x_2 then determines it for general r; so both curves occur in the fibre with multiplicity one. Finally $e_{\chi_0}(12)_*=\operatorname{sgn}(12)\,e_{\chi_0}=-e_{\chi_0}$, so $e_{\chi_0}[(12)_*C_e^-]=-e_{\chi_0}[C_e^-]$. □

5.2 The cycles Σ_s

Fix $e_*\in E^{\circ\circ}$ and a small disc W around $u(e_*)$ such that $u^{-1}(W)$ contains a disc $D_*\subset E^{\circ\circ}$ around e_* which is mapped isomorphically onto W, and choose a holomorphic branch of $\sqrt{k}$ on D_*. For $u\in W$ let $e=e(u)$ be the corresponding point of D_*, and put $s=u/28=s(e)$. Let $\mathcal{M}\subset\mathbb{T}$ be the surface swept out by the curves $r\mapsto x(e',r)$, $e'\in D_*$; it does not depend on s. Let $\mathcal{M}_s$ be the closure in $\overline{U}_s$ of the preimage of $\mathcal{M}$ in U_s. By (5.1),

$$s\,\Phi\big(x(e',r)\big)=\frac{s}{s(e')}\big(1-\varphi_{e'}(r)^2\big). \tag{5.3}$$

At the points of tangency we join C_e^+ and C_e^- by cylinders, as follows. Let r_p be a root of Q (for e). Differentiating (5.3) with respect to $u'=u(e')$ at $e'=e$, $r=r_p$ gives $\partial_{u'}\big(s\Phi(x(e',r))\big)=-1/u\neq0$. By the implicit function theorem, near (r_p,e) the equation $v^2=1-s\Phi(x(e',r))$ determines u' as a holomorphic function of (r,v^2), so (r,v) are local coordinates on the part of the preimage of $\mathcal{M}$ in Z_s swept out by the curves with (e',r) near (e,r_p). Here we use that $(r,u')\mapsto x(e',r)$ is an immersion at (r_p,u): $\partial_r x\neq0$, as the $\partial_r\log x_i$ have no common zero for $L\neq0,1$, and $\partial_r x$ is tangent to the level set of $s\Phi$ at r_p, since $\partial_r(s\Phi(x(e,r)))=-\partial_r(\varphi_e^2)$ vanishes there, whereas $\partial_{u'}x$ is not. For small $\epsilon>0$ let $\mathcal{C}_p$ be the cylinder consisting of the points with $|r-r_p|=\epsilon$ and $v=\zeta\varphi_e(r)$, ζ running along the upper half of the unit circle from -1 to 1. It lies in $\mathcal{M}_s$ and joins C_e^- to C_e^+; we orient it by the anticlockwise circle followed by the direction of ζ.

Let $\overline{C}^\pm_{e,\epsilon}$ be the curves $\overline{C}^\pm_e$ with the discs $|r-r_p|<\epsilon$ around the four points of tangency removed. Then

$$\Sigma_s=\overline{C}^+_{e,\epsilon}-\overline{C}^-_{e,\epsilon}-\sum_{Q(r_p)=0}\mathcal{C}_p$$

is a closed 2-cycle in $\overline{U}_s$: near r_p the boundary of $\overline{C}^+_{e,\epsilon}-\overline{C}^-_{e,\epsilon}$ is the anticlockwise circle on C_e^- minus that on C_e^+, and so is $\partial\mathcal{C}_p$.

Lemma 5.2*(a) $e_{\chi_0}\Sigma_s$ is the boundary of a rational 3-chain Γ_s in $\overline{U}_s$.*

(b) Regard $\overline{C}^\pm_e\cap\overline{U}_s$, the curves without the points of tangency, as locally finite 2-cycles in $\overline{U}_s$. For each r_p let $K_p\subset\mathcal{M}_s$ be the set of points with coordinates $(r,\zeta\varphi_e(r))$, where $0<|r-r_p|\le\epsilon$ and ζ lies on the upper half of the unit circle. With a suitable orientation, $K=\sum_p K_p$ is a locally finite 3-chain in $\overline{U}_s$ with $\partial K=(\overline{C}^+_e-\overline{C}^-_e)-\Sigma_s$.

Proof (a) The cycle $e_{\chi_0}\Sigma_s$ is ι-anti-invariant, and $H_2(\overline{U}_s,\mathbb{Q})^- = 0$ by Proposition 3.1 and the universal coefficient theorem.

(b) Near r_p, the cycle $(\overline{C}_e^+ - \overline{C}_e^-) - \Sigma_s$ is $D_p^+ - D_p^- + \mathcal{C}_p$, where $D_p^\pm$ is the set of points with coordinates $(r, \pm\varphi_e(r))$, $0 < |r - r_p| < \epsilon$. The set K_p is swept out by the half-circles joining D_p^- to D_p^+, so its boundary is $\pm(D_p^+ - D_p^- + \mathcal{C}_p)$; there is no boundary at $r = r_p$, since this end tends to $\overline{R}_s$, which is not in $\overline{U}_s$. □

5.3 The inhomogeneous Picard–Fuchs equation

By Lemma 5.2(a) we can choose rational 3-chains Γ_s in $\overline{U}_s$ with $\partial\Gamma_s = e_{\chi_0}\Sigma_s$, depending piecewise smoothly on $u \in W$: choose Γ_{s_1} for one s_1, transport it along a trivialisation of the family $\{\overline{U}_s\}$, and add the chain swept out by a homotopy from the transported cycle to $e_{\chi_0}\Sigma_s$. For $j \geq 0$ put $\nu_j(u) = (2\pi\mathrm{i})^{-3}\int_{\Gamma_s}\vartheta^j\eta_s$, where $s = u/28$; on functions of u we have $\vartheta = s\,\partial_s = u\,\partial_u$.

Proposition 5.3*(a)* $\vartheta\nu_j = \nu_{j+1}$ *for all* $j \geq 0$.
(b) $\mathcal{L}\nu_0 = -(2\pi\mathrm{i})^{-3}\sum_{Q(r_p)=0}\int_{\mathcal{C}_p}\beta_s$.

Proof We trivialise the family $\{\overline{U}_s\}$ locally, keeping the torus coordinates x fixed on U_s (near $\overline{U}_s \setminus U_s$ this amounts to keeping z_1, z_2, z_3 fixed in the chart $z_4 = 1$ of §3.2); then ∂_s acts on forms as in §3.3. If the chains Γ_s are moved by a vector field whose component tangent to the fibres is $\mathcal{V}$, then for any family of closed 3-forms ω_s

$$\partial_s\int_{\Gamma_s}\omega_s = \int_{\Gamma_s}\partial_s\omega_s + \int_{\partial\Gamma_s}\mathcal{V}\lrcorner\,\omega_s.$$

In these coordinates the boundary $\partial\Gamma_s = e_{\chi_0}\Sigma_s$ moves inside the fixed complex surfaces $\sigma(\mathcal{M}_s)$, $\sigma \in \mathfrak{S}_4$, so $\mathcal{V}$ can be chosen tangent to them along $\partial\Gamma_s$. The forms $\vartheta^j\eta_s$ are of type $(3,0)$, and a $(3,0)$-form vanishes on any three vectors lying in a complex subspace of dimension 2. Hence $\mathcal{V}\lrcorner\,\vartheta^j\eta_s$ vanishes on $\partial\Gamma_s$, and we obtain $\vartheta\nu_j = \nu_{j+1}$ for all $j \geq 0$. This proves (a). By Proposition 3.2 and Stokes' theorem, it follows that

$$\mathcal{L}\nu_0 = \frac{1}{(2\pi\mathrm{i})^3}\int_{\Gamma_s}\mathrm{d}\beta_s = \frac{1}{(2\pi\mathrm{i})^3}\int_{e_{\chi_0}\Sigma_s}\beta_s = \frac{1}{(2\pi\mathrm{i})^3}\int_{\Sigma_s}\beta_s,$$

where the last step uses $e_{\chi_0}\beta_s = \beta_s$ (Proposition 3.2). Since β_s is holomorphic on $\overline{U}_s$, it restricts to zero on the complex curves $\overline{C}_{e,\epsilon}^\pm$, including near the punctures, so only the cylinders $\mathcal{C}_p$ contribute to $\int_{\Sigma_s}\beta_s$. This proves (b). □

5.4 The contribution of a point of tangency

Near the points of tangency we use (r, u'), $u' = u(e')$, as coordinates on $\mathcal{M}$ and write the restriction of β_s to $\mathcal{M}_s$ as $\beta_s = b(r,u')\,v^{-7}\,\mathrm{d}r\wedge\mathrm{d}u'$. Explicitly, $b = \frac{1}{8}(N_1m_1 - N_2m_2 + N_3m_3)$, where m_i is the 2×2 minor obtained by deleting the ith column of the matrix $\begin{pmatrix}\partial_r\log x_1 & \partial_r\log x_2 & \partial_r\log x_3\\ \partial_{u'}\log x_1 & \partial_{u'}\log x_2 & \partial_{u'}\log x_3\end{pmatrix}$. Let $\Delta(r,u') = \partial_{u'}\big(s\Phi(x(e',r))\big)$, the derivative being taken at fixed r and s, and put

$$j_0 = -\frac{2b}{\Delta}, \qquad j_1 = -\frac{\partial_{u'}j_0}{\Delta}, \qquad j_2 = \frac{(\partial_{u'}^2 j_0)\Delta - (\partial_{u'}j_0)(\partial_{u'}\Delta)}{2\Delta^3}, \tag{5.4}$$

all evaluated at $u' = u$, so that they become meromorphic functions of r. Let

$$\mathcal{H}_e = -\frac{2j_0}{5\varphi_e^5} - \frac{4j_1}{15\varphi_e^3} - \frac{16j_2}{15\varphi_e}.$$

Lemma 5.4 *For every root r_p of Q we have $\int_{\mathcal{C}_p} \beta_s = 2\pi\mathrm{i}\,\mathrm{Res}_{r=r_p} \mathcal{H}_e\,\mathrm{d}r$.*

Proof On $\mathcal{M}_s$ we have $v^2 = 1 - s\Phi(x(e', r))$, hence $2v\,\mathrm{d}v = -\Delta\,\mathrm{d}u'$ at fixed r, and $\mathrm{d}r \wedge \mathrm{d}u' = -(2v/\Delta)\,\mathrm{d}r \wedge \mathrm{d}v$. Therefore $\beta_s = J(r, v^2)\, v^{-6}\,\mathrm{d}r \wedge \mathrm{d}v$, where $J(r, v^2) = -2b/\Delta$ is evaluated at the point $u' = u'(r, v^2)$ of §5.2. At fixed r we have $\partial_{v^2} = -\Delta^{-1}\partial_{u'}$, and $v^2 = \varphi_e(r)^2$ corresponds to $u' = u$. By Taylor's formula, $J(r, v^2) = \sum_{n\ge 0} j_n(r)(v^2 - \varphi_e^2)^n$, with j_0, j_1, j_2 as in (5.4) and all j_n holomorphic near r_p; the series converges on $\mathcal{C}_p$ for small ϵ, because there $|v^2 - \varphi_e^2| \le 2|\varphi_e|^2$ and $\varphi_e = O(\epsilon)$. Each term $(v^2 - \varphi_e^2)^n v^{-6}$ is a Laurent polynomial in v with only even powers, so its integral from $-\varphi_e$ to φ_e does not depend on the path, and

$$\int_{-\varphi_e}^{\varphi_e} \frac{(v^2 - \varphi_e^2)^n}{v^6}\,\mathrm{d}v = -\frac{2}{5\varphi_e^5},\ -\frac{4}{15\varphi_e^3},\ -\frac{16}{15\varphi_e} \qquad (n = 0, 1, 2),$$

while for $n \ge 3$ the result is a constant times φ_e^{2n-5}, which is holomorphic at r_p. Integrating over the circle $|r - r_p| = \epsilon$ gives the claim. □

Proposition 5.5 *For $u \in W$,*

$$\mathcal{L}\nu_0(u) = \frac{1}{4\pi^2} \sum_{Q(r_p)=0} \mathrm{Res}_{r=r_p} \mathcal{H}_e\,\mathrm{d}r.$$

Proof Combine Proposition 5.3(b) with Lemma 5.4, using $-2\pi\mathrm{i}/(2\pi\mathrm{i})^3 = 1/(4\pi^2)$. □

5.5 Evaluation of the residues

Put $\mathcal{R} = (\sqrt{k})^{-1} \sum_{Q(r_p)=0} \mathrm{Res}_{r=r_p} \mathcal{H}_e\,\mathrm{d}r$. Since j_0, j_1, j_2 do not involve $\sqrt{k}$, while φ_e occurs in $\mathcal{H}_e$ only with the odd exponents $-5, -3, -1$ and φ_e^{-m} is $k^{m/2}$ times a rational function of r, $\mathcal{R}$ does not depend on the choice of $\sqrt{k}$. As a symmetric function of the roots of Q, it lies in the function field $\mathbb{C}(E) = \mathbb{C}(L, h)$.

Proposition 5.6 *We have*

$$\frac{(L - h)\,\mathcal{R}}{(L - 1)Dh} = \frac{32u^2\,(3u^{10} - 105u^8 - 1614u^6 - 7574u^4 - 2709u^2 - 2401)}{343\,(u^4 - 1)^3}. \tag{5.5}$$

Consequently, with δ as in (5.2),

$$\delta\,\pi^2\,\mathcal{L}\nu_0(u) = \mathrm{i}\sqrt{7}\,r_0(u), \tag{5.6}$$

where

$$r_0(u) = \frac{4u^2\,(3u^{10} - 105u^8 - 1614u^6 - 7574u^4 - 2709u^2 - 2401)}{343\,(u^4 - 1)^2}.$$

Proof Both sides of (5.5) are explicit elements of $\mathbb{C}(E)$, and we verified the identity by a finite computation in exact arithmetic, described in Appendix A. For (5.6), we have $\mathcal{L}\nu_0 = \sqrt{k}\,\mathcal{R}/(4\pi^2)$ by Proposition 5.5. By (5.2),

$$\delta\sqrt{k} = \frac{\delta^2}{\delta/\sqrt{k}} = -\frac{\mathrm{i}\sqrt{7}\,(1-u^4)(L-h)}{2(L-1)Dh},$$

so $\delta\pi^2\mathcal{L}\nu_0 = \frac{1}{4}\delta\sqrt{k}\,\mathcal{R} = -\frac{\mathrm{i}\sqrt{7}}{8}(1-u^4)\frac{(L-h)\mathcal{R}}{(L-1)Dh}$, and (5.6) follows from (5.5). □

6 The class of S

6.1 Set-up

Let $B = \{u \in \mathbb{C} : |u| < 2\}$ and $X_B = \mathrm{pr}^{-1}(B) \cap X^\flat_{t_0}$. For $u \in B$ with $u \neq 0$ and $u^4 \neq 1$ we have $0 < |t_0 u| < 1/4$, so $t_0 u \in \mathcal{P}$ and the fibre of X_B over u is identified with $U_{t_0 u}$ by (3.1). The cycle γ of Lemma 2.3 (for $t = t_0$) lies in X_B. As S is closed in $X^\circ_{t_0}$, the surface $S \cap X_B$ is closed in X_B; we write $\mathrm{PD}(S) \in H^4(X_B, \mathbb{Q})$ for its Poincaré dual.

Let $\mathcal{Z} \subset B$ be a finite set, stable under $u \mapsto \mathrm{i}u$, containing 0, the fourth roots of unity, and all points $u(e) \in B$ with $e \in E \setminus E^{\circ\circ}$. Let $B^\circ = B \setminus \mathcal{Z}$ and $X_{B^\circ} = \mathrm{pr}^{-1}(B^\circ) \cap X_B$. Both X_B and X_{B° are stable under G.

Lemma 6.1 *The restriction map $H^4(X_B, \mathbb{C})^{\chi_0} \to H^4(X_{B^\circ}, \mathbb{C})^{\chi_0}$ is injective.*

Proof For $z \in \mathcal{Z}$ let $\mathcal{F}_z$ be the reduced fibre of X_B over z. These are smooth divisors:

- for $z^4 = 1$, $\mathcal{F}_z = \{u = z,\ Y = 0\} \cong \mathbb{T} \setminus \{\Phi = 1/(t_0 z)\}$, and $u - z$ is a unit times Y^2 near $\mathcal{F}_z$;
- $\mathcal{F}_0 = \{u = 0,\ Y = \pm 1\}$ is the disjoint union of two copies of $\mathbb{T}$;
- for the other z, $\mathcal{F}_z \cong U_{t_0 z}$.

Since the $\mathcal{F}_z$ are disjoint, the Gysin sequence for $X_{B^\circ} = X_B \setminus \bigcup_z \mathcal{F}_z$ shows that the kernel of the restriction map in degree 4 is the image of $\bigoplus_z H^2(\mathcal{F}_z, \mathbb{C})$. The Gysin maps are compatible with $\mathfrak{S}_4 \times \langle\iota\rangle$, which acts holomorphically, so it suffices to show $H^2(\mathcal{F}_z, \mathbb{C})^{\chi_0} = 0$. For $z^4 = 1$ this holds because ι acts trivially on $\mathcal{F}_z$. For $z = 0$ the involution ι exchanges the two tori, so $H^2(\mathcal{F}_0)^- \cong H^2(\mathbb{T})$ as a representation of $\mathfrak{S}_4$. Here $H^1(\mathbb{T})$ is the standard representation St of $\mathfrak{S}_4$, so $H^2(\mathbb{T}) = \Lambda^2\mathrm{St} \cong \mathrm{St} \otimes \mathrm{sgn}$ contains no copy of sgn. For the remaining z we use $H^2(U_{t_0 z})^- = 0$ (Proposition 3.1). □

6.2 Cohomology classes on X_{B°

By Proposition 3.1(c), the map $\mathrm{pr}\colon X_{B^\circ} \to B^\circ$ is a locally trivial fibration; its fibre over u is identified with $U_{t_0 u}$ by (3.1), for either choice of δ. The two identifications differ by ι, which extends to $\overline{U}_s$; hence the partial compactifications $\overline{U}_{t_0 u}$, $u \in B^\circ$, glue to a locally trivial family $\overline{X}_{B^\circ} \to B^\circ$ that contains X_{B° as an open subset. For the fibres of both families the χ_0-part of the cohomology vanishes outside degree 3 by Proposition 3.1(a),(b); in degree 3 it is the ι-anti-invariant part, on which $\mathfrak{S}_4$ acts by sgn (Lemma 3.3(c)). Since B° is homotopy equivalent to a graph, in total degree 4

the χ_0-part of the Leray spectral sequence of either family consists of the single term $E_2^{1,3} = H^1(B^\circ, R^3\mathrm{pr}_*\mathbb{C})^{\chi_0}$; in particular $H^4(\overline{X}_{B^\circ}, \mathbb{C})^{\chi_0} \cong E_2^{1,3}$.

We represent classes on X_{B° by differential forms, using the coordinates (u, x) and the forms $\eta = \eta_{t_0 u}$ on the fibres. Let f be holomorphic on B° and $j \geq 0$. Since $\vartheta^j\eta/\delta$ is δ^{2j}/Y^{2j+1} times a polynomial in $t_0 u\Phi$, times ω_0, the 4-form $\frac{\mathrm{d}u}{u} \wedge \frac{f(u)}{\delta}\,\vartheta^j\eta$ on X_{B° does not depend on the choice of δ. It is holomorphic of top degree, hence closed, and it extends holomorphically to $\overline{X}_{B^\circ}$ by Proposition 3.2. As $\eta/\delta = \omega_0/Y$ and ρ^* fixes $\mathrm{d}u/u$, ω_0 and Y, we have

$$\rho^*\Big(\frac{\mathrm{d}u}{u} \wedge \frac{f(u)}{\delta}\eta\Big) = \frac{\mathrm{d}u}{u} \wedge \frac{f(\mathrm{i}u)}{\delta}\eta. \tag{6.1}$$

Lemma 6.2 *For f holomorphic on B° let $\vartheta^\dagger f = -\delta\,\vartheta(f/\delta) = -\vartheta f - \frac{2u^4}{1-u^4}f$, where $\vartheta = u\,\mathrm{d}/\mathrm{d}u$.*

(a) For $j \geq 1$ the forms $\frac{\mathrm{d}u}{u} \wedge \frac{f}{\delta}\vartheta^j\eta$ and $\frac{\mathrm{d}u}{u} \wedge \frac{\vartheta^\dagger f}{\delta}\vartheta^{j-1}\eta$ differ by an exact form on X_{B°.

(b) On X_{B° the class of $\theta^j\Omega_{t_0}$ is represented by $\frac{\mathrm{d}u}{u} \wedge \frac{b_j}{\delta}\eta$ with $b_j = 4^{-j}(\vartheta^\dagger)^j(u^2)$. In particular $[P(\theta)\Omega_{t_0}]$ is represented by $\frac{\mathrm{d}u}{u} \wedge \frac{b_P}{\delta}\eta$, where

$$b_P = 15b_0 + 304b_1 + 1920b_2 = \frac{u^2\,(343 + 1082u^4 + 15u^8)}{(1-u^4)^2}.$$

Proof (a) The 3-form $\frac{f}{\delta}\vartheta^{j-1}\eta$ is well defined on X_{B°, and since $u\,\partial_u$ acts on η as ϑ,

$$\mathrm{d}\Big(\frac{f}{\delta}\,\vartheta^{j-1}\eta\Big) = \frac{\mathrm{d}u}{u} \wedge \Big(\vartheta\Big(\frac{f}{\delta}\Big)\vartheta^{j-1}\eta + \frac{f}{\delta}\,\vartheta^j\eta\Big).$$

(b) By (3.2), $\theta^j\Omega_{t_0} = \frac{\mathrm{d}u}{u} \wedge \frac{u^2}{\delta}4^{-j}\vartheta^j\eta$, because $\theta = \frac{t}{4}\partial_t$ and $t\partial_t$ acts on η_{tu} as ϑ. The claim follows by applying (a) j times. For the formula for b_P, we have $b_1 = \frac{1}{4}\vartheta^\dagger(u^2) = -\frac{u^2}{2(1-u^4)}$ and $b_2 = \frac{1}{4}\vartheta^\dagger(b_1) = \frac{u^2(1+2u^4)}{4(1-u^4)^2}$. □

6.3 The class of S on X_{B°

Proposition 6.3 *In $H^4(X_{B^\circ}, \mathbb{C})$,*

$$(2\pi\mathrm{i})^2\,e_{\chi_0}\,\mathrm{PD}(S) = \Big[\frac{\mathrm{d}u}{u} \wedge \frac{\mathrm{i}\sqrt{7}\,r_0(u)}{32\,\delta}\,\eta_{t_0 u}\Big],$$

with r_0 as in Proposition 5.6.

Proof Let $\overline{S}$ be the closure of $S \cap X_{B^\circ}$ in $\overline{X}_{B^\circ}$. Since $\mathrm{PD}(\overline{S})$ restricts to $\mathrm{PD}(S)$ on X_{B° and the form on the right-hand side extends to $\overline{X}_{B^\circ}$, it suffices to prove the identity on $\overline{X}_{B^\circ}$ with $\overline{S}$ in place of S. We use currents of integration: for a locally finite chain G in $\overline{X}_{B^\circ}$ let $[G]$ be the current with $\int \xi \wedge [G] = \int_G \xi$ for compactly supported forms ξ of degree $\dim G$. For a closed cycle Z the current $[Z]$ represents $\mathrm{PD}(Z)$, and Stokes' theorem gives $[\partial G] = -\mathrm{d}[G]$ if $\dim G$ is odd.

Local chains. Cover B° by convex discs W_α as in §5.2. Over W_α choose Σ_s^α and K_s^α as in §5.2, and Γ_s^α and ν_j^α as in §5.3. Replacing Γ_s^α by $e_{\chi_0}\Gamma_s^\alpha$, which does not change the ν_j^α, we may assume that Γ_s^α is χ_0-isotypic. Put $G_s^\alpha = \Gamma_s^\alpha + e_{\chi_0}K_s^\alpha$. By Lemma 5.2(b), $\partial G_s^\alpha = e_{\chi_0}(\overline{C}_e^+ - \overline{C}_e^-)$, which by Lemma 5.1 is the image under e_{χ_0} of the fibre of $\overline{S}$ over u. Since Ψ is proper of degree one over W_α (by the same lemma), these fibres make up $\overline{S}\cap \mathrm{pr}^{-1}(W_\alpha)$. Let G^α be the 5-chain swept out by the chains G_s^α, $u\in W_\alpha$, oriented by the base followed by the fibre; since the base has even dimension, ∂G^α is swept out by the ∂G_s^α with the same convention. Hence $\partial G^\alpha = e_{\chi_0}\overline{S}$ in $\mathrm{pr}^{-1}(W_\alpha)$, and

$$e_{\chi_0}[\overline{S}] = -\mathrm{d}[G^\alpha] \qquad \text{on } \mathrm{pr}^{-1}(W_\alpha). \tag{6.2}$$

Over $W_\alpha\cap W_{\alpha'}$ the two constructions may use different radii ϵ and different points e, $\varsigma(e)$ of E over u. By Lemma 4.5, the chains built from $\varsigma(e)$ are the images under $(12)\iota$ of those built from e with the circles $|r-r_p|=\epsilon$ replaced by the closed curves $|r^{-1}-r_p^{-1}|=\epsilon$, and $e_{\chi_0}(12)_*\iota_* = e_{\chi_0}$. Therefore $e_{\chi_0}K_s^\alpha$ and $e_{\chi_0}K_s^{\alpha'}$ coincide near the points of tangency, and their difference is a compact 3-chain contained in the surfaces $\sigma(\mathcal{M}_s)$, $\sigma\in\mathfrak{S}_4$, on which the forms $\vartheta^j\eta$ vanish. Consequently $c_{\alpha\alpha'} = G_s^\alpha - G_s^{\alpha'}$ is a continuous family of compact 3-cycles in $\overline{U}_{t_0u}$, which are χ_0-isotypic since G_s^α and $G_s^{\alpha'}$ are. These chains depend on $\sqrt{k}$ and δ only through $\delta/\sqrt{k}$, whereas replacing $(\sqrt{k},\delta)$ by $(-\sqrt{k},-\delta)$ changes the signs of η, β_{t_0u} and the ν_j^α. Computing η and the $\nu_j^{\alpha'}$ with the same δ as over W_α, we have $\int_{c_{\alpha\alpha'}}\vartheta^j\eta = (2\pi\mathrm{i})^3(\nu_j^\alpha - \nu_j^{\alpha'})$.

Local primitives. On $\mathrm{pr}^{-1}(W_\alpha)$ we identify the fibres with $\overline{U}_{t_0u}$ by (3.1), with δ as in (5.2), and regard $\mathbf{E} = (\eta,\vartheta\eta,\vartheta^2\eta,\vartheta^3\eta)^{\mathsf{T}}$ as a vector of 3-forms on $\mathrm{pr}^{-1}(W_\alpha)$ that involve only the fibre coordinates. Put $\boldsymbol{\nu}^\alpha = (\nu_0^\alpha,\dots,\nu_3^\alpha)^{\mathsf{T}}$ and

$$\tilde{a}_\alpha = -\frac{(2\pi\mathrm{i})^3}{\kappa}\,(\boldsymbol{\nu}^\alpha)^{\mathsf{T}}\mathsf{B}\,\mathbf{E} = -\frac{1}{128}(\boldsymbol{\nu}^\alpha)^{\mathsf{T}}\mathsf{B}\,\mathbf{E}, \qquad \mu_\alpha = \frac{\nu_0^\alpha}{128}\,\frac{\mathrm{d}u}{u}\wedge\beta_{t_0u}.$$

These forms are unchanged if $(\sqrt{k},\delta)$ is replaced by $(-\sqrt{k},-\delta)$, as they are bilinear in $(\boldsymbol{\nu}^\alpha,\mathbf{E})$ and in $(\nu_0^\alpha,\beta_{t_0u})$. By (3.5), which also holds on $\overline{U}_{t_0u}$, the restriction of $\tilde{a}_\alpha-\tilde{a}_{\alpha'}$ to each fibre represents $\mathrm{PD}(c_{\alpha\alpha'})$. By Proposition 3.2, $\vartheta\mathbf{E} = \mathsf{M}\mathbf{E} + \frac{1}{1-\lambda}(0,0,0,\mathrm{d}\beta_{t_0u})^{\mathsf{T}}$, where $\mathrm{d}\beta_{t_0u}$ is the exterior derivative along the fibres. By Proposition 5.3(a) and the definition of $\mathcal{L}$, $\vartheta\boldsymbol{\nu}^\alpha = \mathsf{M}\boldsymbol{\nu}^\alpha + \frac{\mathcal{L}\nu_0^\alpha}{1-\lambda}(0,0,0,1)^{\mathsf{T}}$. Using (3.6) and the fact that the last row and the last column of B are $(-(1-\lambda),0,0,0)$ and $(1-\lambda,0,0,0)^{\mathsf{T}}$, we obtain

$$\vartheta\tilde{a}_\alpha = \frac{\mathcal{L}\nu_0^\alpha}{128}\,\eta - \frac{\nu_0^\alpha}{128}\,\mathrm{d}\beta_{t_0u}, \qquad \text{hence} \qquad \mathrm{d}\tilde{a}_\alpha = \frac{\mathrm{d}u}{u}\wedge\vartheta\tilde{a}_\alpha = \Theta + \mathrm{d}\mu_\alpha,$$

with

$$\Theta = \frac{\mathrm{d}u}{u}\wedge\frac{\mathcal{L}\nu_0^\alpha}{128}\,\eta = \frac{\mathrm{i}\sqrt{7}}{128\pi^2}\,r_0(u)\,\frac{\mathrm{d}u}{u}\wedge\frac{\eta}{\delta}.$$

The last expression, which follows from Proposition 5.6, shows that Θ is a holomorphic 4-form on $\overline{X}_{B^\circ}$ that does not depend on α.

Comparison. The closed current $T = e_{\chi_0}[\overline{S}] + \Theta$ satisfies $T = \mathrm{d}P_\alpha$ on $\mathrm{pr}^{-1}(W_\alpha)$, where $P_\alpha = \tilde{a}_\alpha - \mu_\alpha - [G^\alpha]$, by (6.2) and the formula for $\mathrm{d}\tilde{a}_\alpha$. On $\mathrm{pr}^{-1}(W_\alpha\cap W_{\alpha'})$ the difference $P_\alpha - P_{\alpha'} = (\tilde{a}_\alpha-\tilde{a}_{\alpha'}) - (\mu_\alpha-\mu_{\alpha'}) - [C_{\alpha\alpha'}]$ is closed; here $C_{\alpha\alpha'}$ is the closed 5-chain swept out by the cycles $c_{\alpha\alpha'}$. Since $W_\alpha\cap W_{\alpha'}$ is contractible, the homology of $\mathrm{pr}^{-1}(W_\alpha\cap W_{\alpha'})$ in degree 3 is spanned by cycles $\{u\}\times c'$, where c' is a 3-cycle in the fibre over u. On such a cycle the forms μ vanish, the integral of $\tilde{a}_\alpha-\tilde{a}_{\alpha'}$ is $c_{\alpha\alpha'}\cdot c'$, and so is that of $[C_{\alpha\alpha'}]$, because $C_{\alpha\alpha'}\cdot(\{u\}\times c') = c_{\alpha\alpha'}\cdot c'$ (the base directions come first in both orientations). Hence $P_\alpha - P_{\alpha'}$ is exact. Both $e_{\chi_0}[\overline{S}]$ and Θ are χ_0-isotypic, and by §6.2 the χ_0-part of $H^4(\overline{X}_{B^\circ},\mathbb{C})$ is identified with $E_2^{1,3}$. As T is exact on each $\mathrm{pr}^{-1}(W_\alpha)$, the image of $[T]$ in $E_2^{1,3}$ is, up

to a universal sign, the Čech cocycle formed by the classes of the $P_\alpha - P_{\alpha'}$ (cf. [20, 21]), and this cocycle vanishes. Therefore T is exact, that is, $e_{\chi_0}\,\mathrm{PD}(\overline{S}) = -[\Theta]$. Multiplying by $(2\pi\mathrm{i})^2 = -4\pi^2$ and using the second expression for Θ gives the assertion. □

6.4 The class of S on X_B

For a class $\Xi \in H^4(X_B,\mathbb{C})^{\chi_0}$ and $\zeta \in \{\pm 1, \pm\mathrm{i}\}$ let $\Xi_\zeta = \frac{1}{4}\sum_{k=0}^{3}\zeta^{-k}(\rho^*)^k\Xi$ denote its ζ-eigencomponent for ρ^*.

Theorem 6.4 *Let $\Xi = e_{\chi_0}\,\mathrm{PD}(S) \in H^4(X_B,\mathbb{C})^{\chi_0}$. Then $\Xi_{\mathrm{i}} = \Xi_{-\mathrm{i}} = 0$ and*

$$(2\pi\mathrm{i})^2\,\Xi_{-1} = \frac{1}{56\sqrt{-7}}\,\big[P(\theta)\Omega_{t_0}\big] \qquad \text{in } H^4(X_B,\mathbb{C}).$$

Proof By Lemma 6.1 it suffices to prove the identities after restriction to X_{B°, which commutes with ρ^*. Write $r_0(u) = \sum_m c_m u^m/(u^4-1)^2$ and let f_ζ be the sum of the terms with $\mathrm{i}^m = \zeta$. As $(u^4-1)^2$ is invariant under $u \mapsto \mathrm{i}u$, Proposition 6.3 and (6.1) show that on X_{B° the class $(2\pi\mathrm{i})^2\Xi_\zeta$ is represented by $\frac{\mathrm{d}u}{u} \wedge \frac{\mathrm{i}\sqrt{7}}{32\delta} f_\zeta\,\eta$. Since r_0 is even, $f_{\pm\mathrm{i}} = 0$. The numerator of r_0 is $4(3u^{12} - 105u^{10} - 1614u^8 - 7574u^6 - 2709u^4 - 2401u^2)$; keeping the terms with $m \equiv 2 \pmod 4$ gives

$$f_{-1} = \frac{4(-105u^{10} - 7574u^6 - 2401u^2)}{343(u^4-1)^2} = -\frac{4}{49}\,\frac{u^2(343 + 1082u^4 + 15u^8)}{(1-u^4)^2} = -\frac{4}{49}\,b_P.$$

Comparing with Lemma 6.2(b), we see that $(2\pi\mathrm{i})^2\Xi_{-1}$ is represented by $-\frac{4}{49}\cdot\frac{\mathrm{i}\sqrt{7}}{32} = -\frac{\mathrm{i}\sqrt{7}}{392} = \frac{1}{56\sqrt{-7}}$ times $P(\theta)\Omega_{t_0}$. □

7 Proof of Theorem 1.1

By Lemma 2.3(b) the series in Theorem 1.1 equals $\frac{1}{\pi(2\pi\mathrm{i})^3}\int_\gamma P(\theta)\Omega_{t_0}$, and γ is a compact cycle in X_B. Let Ξ be as in Theorem 6.4. Since $e_{\chi_0}\gamma = \gamma$ by Lemma 2.3(c), Proposition 4.6 gives $\int_\gamma \Xi = \int_{e_{\chi_0}\gamma}\mathrm{PD}(S) = \int_\gamma \mathrm{PD}(S) = S\cdot\gamma = 2$. By Theorem 6.4, $\Xi = \Xi_1 + \Xi_{-1}$. As Ξ_1 is ρ-invariant, we have $\int_\gamma \Xi_1 = \frac{1}{4}\int_{\gamma+\rho_*\gamma+\rho_*^2\gamma+\rho_*^3\gamma}\Xi_1$, which vanishes by Lemma 2.3(c). Hence $\int_\gamma \Xi_{-1} = 2$, and Theorem 6.4 yields

$$\int_\gamma P(\theta)\Omega_{t_0} = 56\sqrt{-7}\,(2\pi\mathrm{i})^2\int_\gamma \Xi_{-1} = 112\,\mathrm{i}\sqrt{7}\,(2\pi\mathrm{i})^2 = -448\,\mathrm{i}\sqrt{7}\,\pi^2.$$

Since $\pi(2\pi\mathrm{i})^3 = -8\mathrm{i}\pi^4$, the series equals $\frac{-448\mathrm{i}\sqrt{7}\,\pi^2}{-8\mathrm{i}\pi^4} = \frac{56\sqrt{7}}{\pi^2}$. □

Supplementary information. The file `verify_computations.py` performs the computer calculation described in Appendix A. It requires Python 3 with SymPy (≥ 1.12) and mpmath.

Declarations

- **Competing interests.** The authors declare no competing interests.
- **Code availability.** The script verifying the computation in Appendix A is provided as supplementary material.
- **Use of AI tools.** Claude Opus 5.5 (Anthropic) was used to revise the draft, including language editing, the correction of errors and the rewriting of several proofs, and to write the script `verify_computations.py`. The authors take full responsibility for the content of this article.

Appendix A The computer calculation

We explain how the identity (5.5) reduces to finite polynomial arithmetic in the function field $K = \mathbb{Q}(L)[h]/(h^2 - L(2L^2 - 3L + 2))$, and how it was verified.

The parameter w. Put $w = \mathrm{i}u = -h/L$. Replacing the coordinate u' on E by $w' = \mathrm{i}u'$ multiplies both b and Δ in §5.4 by the same constant, so the functions j_0, j_1, j_2 of (5.4) are unchanged if $\partial_{u'}$ is replaced by $\partial_{w'}$. With this choice all data are defined over $\mathbb{Q}$. On K the derivation $\partial = \mathrm{d}/\mathrm{d}w$ is determined by $\mathrm{d}L/\mathrm{d}w = -Lh/(L^2 - 1)$, which follows from $\mathrm{d}u/\mathrm{d}L = \mathrm{i}(L^2 - 1)/(Lh)$.

Homogeneity. Let $\hat{q}_i = \frac{u'}{28} x_i(e', r) = \frac{w'}{28}\, c_i\, \ell_i^8 / P_0^2$ with $(c_1, c_2, c_3, c_4) = (-A/L^2,\ -L^2A,\ L^2/A,\ 1/(L^2A))$ and $\ell = (r, 1, r+1, r+L)$. At fixed s we have $q_i = s x_i = (w/w')\,\hat{q}_i$. Let $N_i^{(d)}$ $(d = 1, 2, 3)$ be the homogeneous part of degree d of N_i as a polynomial in q. Then $b = \sum_d (w/w')^d b^{(d)}$ and $\Delta = (w/w')\Delta_1$, where $b^{(d)} = \frac{1}{8}\big(N_1^{(d)}(\hat{q})m_1 - N_2^{(d)}(\hat{q})m_2 + N_3^{(d)}(\hat{q})m_3\big)$ and $\Delta_1 = \partial_{w'}\big(\sum_i \hat{q}_i\big) - \frac{1}{w'}\sum_i \hat{q}_i$ do not involve s. Put $J_d = -2b^{(d)}/\Delta_1$ and $\alpha = -1/w$. At fixed s we have $-2b/\Delta = \sum_d (w/w')^{d-1} J_d$; differentiating with respect to w' and evaluating at $w' = w$ gives

$$j_0 = \sum_d J_d, \qquad j_1 = -\frac{J^{[1]}}{\Delta_1}, \qquad j_2 = \frac{J^{[2]}\Delta_1 - J^{[1]}(\partial\Delta_1 + \alpha\Delta_1)}{2\Delta_1^3},$$

$$J^{[1]} = \sum_d \big(\partial + (d-1)\alpha\big) J_d,$$

$$J^{[2]} = \sum_d \big(\partial^2 + 2(d-1)\alpha\partial + (d-1)\partial\alpha + (d-1)^2\alpha^2\big) J_d,$$

where ∂ acts on coefficients, at fixed r.

Polynomial form. The logarithmic derivatives are $\partial_r \log x_i = \mathsf{e}_i/P_0$ and $\partial_{w'} \log x_i = \mathsf{f}_i/(r+L)$ with

$$\mathsf{e}_1 = 6L + 4(L+1)r + 2r^2, \qquad \mathsf{e}_2 = -2L - 4(L+1)r - 6r^2,$$
$$\mathsf{e}_3 = -2L + (4L-4)r + 2r^2, \qquad \mathsf{e}_4 = -2L + (4-4L)r + 2r^2,$$
$$\mathsf{f}_i = \frac{\partial c_i}{c_i}(r+L) - 2\,\partial L + 8\,\delta_{i4}\,\partial L.$$

Hence the minors are $m_i = \mathsf{m}_i/(P_0(r+L))$ with polynomials $\mathsf{m}_1 = \mathsf{e}_2\mathsf{f}_3 - \mathsf{f}_2\mathsf{e}_3$, $\mathsf{m}_2 = \mathsf{e}_1\mathsf{f}_3 - \mathsf{f}_1\mathsf{e}_3$, $\mathsf{m}_3 = \mathsf{e}_1\mathsf{f}_2 - \mathsf{f}_1\mathsf{e}_2$ of degree 3. Also $\Delta_1 = \mathcal{D}/P_0^3$ with a polynomial $\mathcal{D}$ of degree 11, and $J_d = -\mathcal{N}_d/(4P_0^{2d-2}(r+L)\mathcal{D})$, where $\mathcal{N}_d = \sum_{i=1}^{3}(-1)^{i+1}N_i^{(d)}(P_0^2\hat{q})\,\mathsf{m}_i$ is a polynomial of degree $8d+3$.

Residues. Let $Q_\mathrm{m} = L^2Q$, which is monic of degree 4 in r. For $\mathsf{G} \in K[r]$ of degree less than $4n$,

$$\sum_{Q_\mathrm{m}(r_*)=0} \mathrm{Res}_{r=r_*} \frac{\mathsf{G}(r)}{Q_\mathrm{m}(r)^n}\,\mathrm{d}r = [r^{4n-1}]\,\mathsf{G},$$

the coefficient of r^{4n-1}, because the left-hand side is minus the residue at ∞. Since $k^{-1/2}\varphi_e^{-m} = k^{(m-1)/2}L^{2m}P_0^m/Q_\mathrm{m}^m$ for odd m, we get $\mathcal{R} = L^{10}[r^{19}]\mathsf{G}_5 + L^6[r^{11}]\mathsf{G}_3 + L^2[r^3]\mathsf{G}_1$, where G_5, G_3 and G_1 are the remainders of $-\frac{2}{5}k^2P_0^5j_0$, $-\frac{4}{15}kP_0^3j_1$ and $-\frac{16}{15}P_0j_2$ modulo Q_m^5, Q_m^3 and Q_m. Here the denominators P_0, $r+L$ and $\mathcal{D}$ are inverted modulo powers of Q_m. This is possible because the roots of Q are not punctures and $\mathcal{D} \equiv \alpha P_0^3$ modulo Q_m; the latter holds because $\Delta_1 = \partial_{w'}(s\Phi) = -1/w$ at $w' = w$ and $r = r_p$, by the computation in §5.2. The coefficients of Q_m depend on w, so reduction modulo Q_m^n does not commute with ∂. All derivatives are therefore taken before reducing: it suffices to know J_d modulo Q_m^5, ∂J_d modulo Q_m^3 and $\partial^2 J_d$ modulo Q_m.

Carrying out these operations in exact arithmetic in K gives (5.5). We implemented the computation with SymPy, representing elements of K as pairs of rational functions of L and inverting modulo powers of Q_m by the extended Euclidean algorithm and Newton iteration; it takes about half an hour on a laptop. The supplementary script `verify_computations.py` also contains an independent numerical check. It evaluates the residues in Proposition 5.5 directly, by numerical contour integration around the roots of Q with 60-digit precision, at both points of E over each of three sample values of u; here the parameter u is used, and the derivatives with respect to u' are computed from power series in $u' - u$. The results agree with (5.5) to about 40 significant digits. The script also re-checks several identities that are proved by hand in the paper (Proposition 3.2, Lemma 4.1, (3.6) and the formulas for b_P and f_{-1}) and evaluates the series to 45 digits; the option `--quick` omits the exact computation.

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