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THE SHAPE OF MATHEMATICAL UNIVERSES

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Abstract. What is the shape of the space of all mathematical universes? We take classifying toposes whose points are elementary toposes or models of set theory, and look at their geometry. The space of all toposes is contractible, yet its open parts contain every countable homotopy type: spheres and tori live among mathematical universes. Set theory has a genuine hole, built from two constructibility degrees that no definition can tell apart, and arithmetic cannot see it.

1. Introduction

What does the totality of mathematics look like? We regard each mathematical universe as a point of a space, and we ask for the shape of this space. Topos theory gives such spaces: a classifying topos is a space whose points are the models of a theory [14]. The new point is the choice of the theory. We take theories whose models are themselves mathematical universes: the theory of elementary toposes, and set theory.

The first answer is disappointing. The theory of elementary toposes has an initial model, so its classifying topos $\mathcal{M}$ is contractible (Theorem A(1)). But the open parts of $\mathcal{M}$ have every shape: every countable homotopy type, and nothing else, is the shape of an open subtopos (Theorem A). The vertices are toposes whose object Ω of truth values has exactly $m+1$ elements, and the simplices are their mixtures. So a sphere or a torus lives in the space of mathematical universes.

Set theory has a hole that arithmetic cannot see. Groszek and Laver, and Fuchs, Gitman and Hamkins, found two "twin" constructibility degrees that no definition can tell apart [10, 8]. Theorem B turns the twins into a double cover that does not split over the classifying topos $\mathcal{Z}$ of ZFC. For every arithmetic theory consistent with ZFC, it splits over one completion and not over another with the same arithmetic, and $H^1(\mathcal{Z};\mathbb{F}_2)$ has dimension $\aleph_0$. The axiom $V = \mathrm{HOD}$ fills all holes (Proposition 5.1).

What is the fundamental group of the space of set-theoretic universes, and which axioms open or fill its holes? We hope that the geometry of the totality of mathematics becomes a field of its own.

2. Main results

Theorem A. *Let $\mathcal{M}$ be the classifying topos of the theory of nondegenerate elementary toposes with chosen structure, with or without a chosen natural numbers object* (Section 4).

1. *The shape of $\mathcal{M}$ is contractible, and $H^n(\mathcal{M};A) = 0$ for every abelian group object A of $\mathcal{M}$ and every $n > 0$.*
2. *A homotopy type is the shape of an open subtopos of $\mathcal{M}$ if and only if it is the homotopy type of a countable CW complex.*
3. *A homotopy type is the shape of a quasi-compact open subtopos of $\mathcal{M}$ if and only if it is the homotopy type of a finite CW complex.*

For example, the boundary of a tetrahedron gives a quasi-compact open subtopos $\mathcal{U}$ of $\mathcal{M}$ with $H^2(\mathcal{U};\mathbb{F}_2) \cong \mathbb{F}_2$ (Proposition 4.3). We prove Theorem A in Section 4.

Let $\mathcal{Z}$ be the classifying topos of ZFC set theory, in classical first-order logic (Section 6). Its points are models of ZFC, and its morphisms of points are elementary embeddings. Let $\mathcal{R}$ be the classifying topos of second-order arithmetic. Taking the natural numbers and the

sets of natural numbers of a model of ZFC gives a geometric morphism $r\colon \mathcal{Z} \to \mathcal{R}$. For a set A of sentences, we write $\mathcal{Z}_A$ for the subtopos of $\mathcal{Z}$ classifying models of $\mathrm{ZFC} + A$.

Theorem B. *Assume that* ZFC *is consistent. There is an $\mathbb{F}_2$-torsor Q on $\mathcal{R}$, defined from the two minimal constructibility degrees of a Sacks extension (Construction 6.2), with the following properties.*

(1) *Let A be a set of sentences of first-order arithmetic such that $\mathrm{ZFC}+A$ is consistent. Then there are two complete theories B_0 and B_1 that contain $\mathrm{ZFC} + A$ and have the same arithmetic sentences, such that r^*Q is trivial on $\mathcal{Z}_{B_0}$ and nontrivial on $\mathcal{Z}_{B_1}$. In particular, r^*Q is nontrivial on $\mathcal{Z}_A$.*
(2) *The groups $H^1(\mathcal{R};\mathbb{F}_2)$ and $H^1(\mathcal{Z};\mathbb{F}_2)$, and the image of r^* between them, have dimension $\aleph_0$ over $\mathbb{F}_2$.*
(3) *On $\mathcal{Z}_{\{V=\mathrm{HOD}\}}$, every torsor under every abelian group object is trivial, and all positive-degree cohomology vanishes.*

Part (3) is a special case of Proposition 5.1, which characterizes the classifying toposes whose terminal object is projective. We prove Theorem B in Section 6.

3. Toposes of models

We fix notation and recall standard facts. For a small category $\mathcal{D}$, we write $[\mathcal{D}, \mathbf{Set}]$ for the topos of functors $\mathcal{D} \to \mathbf{Set}$ and $N\mathcal{D}$ for the nerve of $\mathcal{D}$.

3.1. **Presheaf toposes and their shape.** A *cosieve* of $\mathcal{D}$ is a set D of objects such that if $X \in D$ and $X \to Y$ is a morphism, then $Y \in D$. We identify D with the corresponding full subcategory. The subterminal objects of $[\mathcal{D}, \mathbf{Set}]$ are the cosieves of $\mathcal{D}$, and the open subtopos of a cosieve D is $[D, \mathbf{Set}]$. If D is a cosieve, then the remaining objects form a full subcategory $\mathcal{D} \setminus D$ closed under domains of morphisms, and the complementary closed subtopos is $[\mathcal{D} \setminus D, \mathbf{Set}]$. A functor on $\mathcal{D} \setminus D$ extends to $\mathcal{D}$ by taking the value 1 on D. See [14, A4.5]. For an object P of $\mathcal{D}$, we write $V_P = \{X : \mathrm{Hom}(P, X) \neq \emptyset\}$ for the image of the functor represented by P. Every cosieve D is the union of the V_P with $P \in D$. Representable functors are quasi-compact, so a cosieve is quasi-compact if and only if it is a finite union of cosieves V_P.

The *shape* of $[\mathcal{D}, \mathbf{Set}]$ is the homotopy type of the classifying space $|N\mathcal{D}|$ [18]. In particular, for an abelian group A, the cohomology of $[\mathcal{D}, \mathbf{Set}]$ with coefficients in the constant sheaf A is the singular cohomology $H^*(|N\mathcal{D}|; A)$. We use two facts about classifying spaces.

Fact 3.1 ([21, Section 1])**.** *Let $\Phi\colon \mathcal{D} \to \mathcal{D}'$ be a functor between small categories.*

(a) A natural transformation between two functors $\mathcal{D} \to \mathcal{D}'$ induces a homotopy between the induced maps $|N\mathcal{D}| \to |N\mathcal{D}'|$. Hence, if Φ has an adjoint, then $|N\Phi|$ is a homotopy equivalence.
(b) (Quillen's Theorem A) *If the comma category J/Φ has a contractible classifying space for every object J of $\mathcal{D}'$, then $|N\Phi|$ is a homotopy equivalence.*

For a simplicial complex K, the nerve of the poset of nonempty faces of K is the barycentric subdivision $\mathrm{sd}\, K$, so its classifying space is homeomorphic to $|K|$. Every countable CW complex is homotopy equivalent to a countable simplicial complex, and every finite CW complex to a finite one [12, Theorem 2C.5].

3.2. **Essentially algebraic theories.** Let $\mathbb{T}$ be a finitary essentially algebraic theory in a countable signature, and let $\mathcal{C}$ be a skeleton of the category of finitely presentable models of $\mathbb{T}$. The category $\mathcal{C}$ is countable, and $[\mathcal{C}, \mathbf{Set}]$ is the classifying topos of $\mathbb{T}$ [14, D3.1].

3.3. **Classical first-order theories.** Let T be a consistent classical first-order theory in a countable, possibly many-sorted, language. We write $\mathcal{E}_T$ for the classifying topos of the full classical Morleyization of T [14, D1.5], [1, Section 1]. Its points are the models of T, and its morphisms of points are elementary embeddings. It is the topos of coherent sheaves on the syntactic pretopos of T. We use the following two facts.

Fact 3.2. *The Yoneda embedding of the syntactic pretopos into $\mathcal{E}_T$ is fully faithful, and the syntactic pretopos is countable. In particular, a morphism $[\vec{x}.\varphi] \to [\vec{y}.\psi]$ in $\mathcal{E}_T$ between objects defined by formulas is given by a formula that T proves to be the graph of a function* [1, Section 1].

Fact 3.3. *The terminal object of $\mathcal{E}_T$ is quasi-compact. The complemented subterminal objects are given by sentences up to T-provable equivalence. For a set A of sentences, the topos $\mathcal{E}_{T+A}$ is the subtopos of $\mathcal{E}_T$ obtained by intersecting the open subtoposes of the sentences in A.*

An interpretation of one theory in another induces a functor between their syntactic pretoposes, and hence a geometric morphism between their classifying toposes [14, Part D].

An $\mathbb{F}_2$-torsor on a topos is trivial if and only if it has a global section, and isomorphism classes of $\mathbb{F}_2$-torsors form the group $H^1(-;\mathbb{F}_2)$ [23, Tags 03AG and 03AJ].

4. Open subtoposes of the space of elementary toposes

4.1. **The space of nondegenerate toposes.** Let $\mathbb{T}$ be the essentially algebraic theory of elementary toposes with chosen finite limits and chosen power objects [7, Section 2.3], [6]. Its models in **Set** are small elementary toposes with this chosen structure. Morphisms of models are the functors that preserve the chosen structure exactly. Everything below also works if we add a chosen natural numbers object with chosen recursion to $\mathbb{T}$, so we do not repeat this.

The constants $\top$ and $\bot$ are closed terms $1 \to \Omega$ of $\mathbb{T}$. A model is *degenerate* if $\top = \bot$ in it. Let $\mathcal{M}$ be the classifying topos of $\mathbb{T}$ with the axiom $(\top = \bot) \vdash$ false.

Lemma 4.1. *Let $\mathcal{C}_{\mathrm{nd}} \subseteq \mathcal{C}$ be the full subcategory of nondegenerate finitely presentable models. Then $\mathcal{M} \simeq [\mathcal{C}_{\mathrm{nd}}, \mathbf{Set}]$, and $\mathcal{C}_{\mathrm{nd}}$ has an initial object.*

Proof. A morphism of models preserves $\top$ and $\bot$. So the degenerate models form a cosieve of $\mathcal{C}$, and the axiom defines its closed complement $[\mathcal{C}_{\mathrm{nd}}, \mathbf{Set}]$ (Section 3). The initial model of $\mathbb{T}$ is finitely presentable. It is nondegenerate because it maps to a nondegenerate model, for example the topos of sets in V_ω (or, with a natural numbers object, the topos of sets in $V_{\omega+\omega}$). □

Proof of Theorem A(1). By Lemma 4.1, $\mathcal{M} \simeq [\mathcal{C}_{\mathrm{nd}}, \mathbf{Set}]$ and $\mathcal{C}_{\mathrm{nd}}$ has an initial object I. So $N\mathcal{C}_{\mathrm{nd}}$ is contractible. The global sections functor sends X to $\lim X \cong X(I)$, so it is exact. Hence all positive-degree cohomology vanishes. □

4.2. **Counting truth values.** We label the vertices of our simplicial complexes by the number of truth values. For $k \geq 1$, let α_k be the sentence of the internal language

$$\alpha_k \;:=\; \forall p_0, \ldots, p_k \in \Omega.\; \bigvee_{0 \leq i < j \leq k} (p_i = p_j).$$

Informally, α_k says that Ω has at most k elements. For $m \geq 1$, we put

$$\sigma_m \;:=\; \alpha_{m+1} \wedge \neg \alpha_m,$$

which informally says that Ω has exactly $m+1$ elements. Let $b_m \colon 1 \to \Omega$ be the truth value of σ_m. It is a closed term of $\mathbb{T}$: equality, connectives and quantifiers of higher-order intuitionistic logic are given by terms built from finite limits and power objects, and morphisms of models preserve them [16, Part II]. For a finite set I of positive integers, we put $b_I := \bigvee_{i \in I} b_i$, so that $b_\emptyset = \bot$.

Lemma 4.2. *In every model of $\mathbb{T}$, we have $b_m \wedge b_n = \bot$ for $m \neq n$. For $m \geq 1$, there is a nondegenerate model W_m of $\mathbb{T}$ with $b_m = \top$.*

Proof. Intuitionistically, α_k implies α_{k+1}: apply α_k to the first $k+1$ of $k+2$ elements. If $m < n$, then σ_m gives α_{m+1}, hence α_n, which contradicts $\neg\alpha_n$ in σ_n. Intuitionistic logic is sound in every topos, so $b_m \wedge b_n = \bot$.

For the second claim, let L_m be the m-element chain as a monoid under min, and let W_m be the topos of right L_m-sets in $V_{\omega+\omega}$ (or in V_ω if there is no natural numbers object). At the unique stage, Ω is the set of right ideals of L_m, that is, the down-sets of the chain, and there are $m+1$ of them [17, I.4]. The site has one object, so by Kripke–Joyal semantics the quantifier in α_k ranges over Ω^{k+1} at that stage, and equality and finite disjunction are evaluated there [17, VI.7]. By the pigeonhole principle, α_{m+1} is true, while α_m fails for $m+1$ distinct down-sets. The terminal object of W_m has only two subobjects, so the truth value of α_m is $\bot$ and $\neg\alpha_m$ is true. Hence $b_m = \top$ in W_m. □

4.3. **Realizing simplicial complexes.** For a finite set I of positive integers, let F_I be the model of $\mathbb{T}$ freely generated by the equation $b_I = \top$, and let

$$U_I := \{X \in \mathcal{C}_{\mathrm{nd}} : b_I = \top \text{ in } X\}.$$

The model F_I is finitely presentable, and $\mathrm{Hom}(F_I, X)$ has one element if $X \in U_I$ and is empty otherwise. If $I \neq \emptyset$, then F_I maps to W_i for $i \in I$, so F_I is nondegenerate and U_I is the representable cosieve of F_I. By Lemma 4.2 and the distributive law,

$$U_I \cap U_J = U_{I\cap J}, \qquad U_\emptyset = \emptyset. \tag{1}$$

Indeed, $b_I \wedge b_J = \bigvee_{i\in I,\, j\in J}(b_i \wedge b_j) = b_{I\cap J}$, and $b_\emptyset = \bot \neq \top$ in a nondegenerate model.

Let K be a simplicial complex whose vertices are positive integers. Let P_K be the poset of nonempty faces of K, with $I \leq J$ if $I \supseteq J$, and let

$$D_K := \bigcup_{I\in K} U_I.$$

This is a cosieve of $\mathcal{C}_{\mathrm{nd}}$, so it gives an open subtopos $[D_K, \mathbf{Set}]$ of $\mathcal{M}$.

Proposition 4.3. *For $X \in D_K$, the faces I of K with $X \in U_I$ have a least element $S(X)$, and $X \in U_I$ if and only if $S(X) \subseteq I$. The assignments $I \mapsto F_I$ and $X \mapsto S(X)$ are functors $F\colon P_K \to D_K$ and $S\colon D_K \to P_K$ with $F \dashv S$. Hence $|ND_K| \simeq |K|$.*

Proof. By (1), the faces I with $X \in U_I$ are closed under intersection. Each face is finite, so there is a least one, $S(X)$. If $S(X) \subseteq I$, then $b_{S(X)} \leq b_I$, so $X \in U_I$. A morphism $X \to Y$ preserves $b_I = \top$, so $S(Y) \subseteq S(X)$, and S is a functor. For $I \in P_K$ and $X \in D_K$, the sets $\mathrm{Hom}(F_I, X)$ and $\mathrm{Hom}_{P_K}(I, S(X))$ both have one element if $S(X) \subseteq I$ and are empty otherwise. This gives the adjunction $F \dashv S$. By Fact 3.1, $|ND_K| \simeq |NP_K| \cong |K|$. □

Remark 4.4. We have $SF = \mathrm{id}$, since F_I maps to W_i for every $i \in I$. The vertex m corresponds to the property "Ω has exactly $m+1$ elements". The condition $b_I = \top$ is weaker than $b_i = \top$ for some $i \in I$. For instance, the product $W_m \times W_n$ satisfies $b_m \vee b_n = \top$, but neither $b_m = \top$ nor $b_n = \top$.

4.4. **Quasi-compact open subtoposes.** From now on, for $P \in \mathcal{C}_{\mathrm{nd}}$ we write $V_P = \{X \in \mathcal{C}_{\mathrm{nd}} : \mathrm{Hom}(P, X) \neq \emptyset\}$, the cosieve of Section 3 for $\mathcal{D} = \mathcal{C}_{\mathrm{nd}}$.

Lemma 4.5. *Each $|NV_P|$ is contractible. For $P, P' \in \mathcal{C}_{\mathrm{nd}}$, the intersection $V_P \cap V_{P'}$ is $V_{P\sqcup P'}$ if the coproduct $P \sqcup P'$ is nondegenerate, and is empty otherwise.*

Proof. Coproducts of finitely presentable models are finitely presentable. If $X \in V_P$, then a morphism $P \to X$ gives a morphism $P \sqcup X \to X$, so $P \sqcup X$ is nondegenerate and lies in V_P. The functor $P \sqcup (-)\colon V_P \to V_P$ receives natural transformations from the identity and from the constant functor at P. By Fact 3.1, $|NV_P|$ is contractible. For the second claim,

X lies in $V_P \cap V_{P'}$ exactly when there is a morphism $P \sqcup P' \to X$; in that case, $P \sqcup P'$ is nondegenerate. □

Proof of Theorem A(2) and (3). Every open subtopos of $\mathcal{M}$ is $[D, \mathbf{Set}]$ for a cosieve D of $\mathcal{C}_{\mathrm{nd}}$, and its shape is $|ND|$. Since $\mathcal{C}_{\mathrm{nd}}$ is countable, ND is a countable simplicial set, and $|ND|$ is a countable CW complex. If D is quasi-compact, then $D = V_{P_1} \cup \cdots \cup V_{P_n}$ for some $P_1, \dots, P_n \in \mathcal{C}_{\mathrm{nd}}$ (Section 3). Let $\mathcal{P}$ be the poset of nonempty sets $J \subseteq \{1, \dots, n\}$ with $\bigcap_{j \in J} V_{P_j} \neq \emptyset$, ordered by inclusion. It is the poset of faces of a finite simplicial complex, the nerve of the cover. The assignment $\Phi(X) = \{i : X \in V_{P_i}\}$ is a functor $\Phi \colon D \to \mathcal{P}$, and for $J \in \mathcal{P}$ the comma category J/Φ is the full subcategory $\bigcap_{j \in J} V_{P_j}$. Since this intersection is nonempty, Lemma 4.5 and induction on the size of J show that the coproduct P of the P_j with $j \in J$ is nondegenerate and the intersection is V_P. So J/Φ has a contractible classifying space. By Fact 3.1(b), $|ND| \simeq |N\mathcal{P}|$, which is a finite simplicial complex.

Conversely, a countable CW complex is homotopy equivalent to a countable simplicial complex K whose vertices are positive integers. By Proposition 4.3, the open subtopos $[D_K, \mathbf{Set}]$ has shape $|K|$. If K is finite, then D_K is a finite union of representable cosieves U_I (for maximal faces I), so it is quasi-compact. The empty space is the shape of the empty open subtopos. □

Remark 4.6. Campion, Cousins and Ye show that the Lascar group of a complete first-order theory is the fundamental group of the classifying space of its category of models, and they ask which homotopy types arise in this way [3, Question 5.1]. Every homotopy type arises from a suitable accessible category [3, Theorem 5.5] and from a suitable abstract elementary class [4]. In Theorem A, all countable homotopy types already appear as shapes of open subtoposes of one fixed classifying topos.

5. Definable witnesses and projectivity

Let T be a consistent classical first-order theory in a countable, possibly many-sorted, language. We say that T has *definable witnesses* if for every formula $\varphi(\vec{x})$ without parameters there is a formula $\psi(\vec{x})$ without parameters such that

$$T \vdash \exists \vec{x}\, \varphi(\vec{x}) \to \exists! \vec{x}\, \big(\psi(\vec{x}) \wedge \varphi(\vec{x})\big).$$

By the Tarski–Vaught test, this implies that in every model of T the elements definable without parameters form an elementary submodel. The converse also holds: if this is true in every model, then by compactness there are finitely many formulas $\psi_1, \dots, \psi_n$ such that T proves that whenever $\exists \vec{x}\, \varphi$ holds, some ψ_k defines a unique element satisfying φ. We then take the first such ψ_k.

Proposition 5.1. *The terminal object of $\mathcal{E}_T$ is projective if and only if T has definable witnesses. In this case, every torsor under every abelian group object of $\mathcal{E}_T$ is trivial, and $H^n(\mathcal{E}_T; A) = 0$ for every abelian group object A and every $n > 0$.*

Proof. Suppose that 1 is projective, and let $\varphi(\vec{x})$ be a formula. The morphism $[\vec{x}.\varphi] \sqcup [\neg\exists \vec{x}\, \varphi] \to 1$ is an epimorphism, so it has a section. On the subterminal object $[\exists \vec{x}\, \varphi]$, the section is a morphism $[\exists \vec{x}\, \varphi] \to [\vec{x}.\varphi]$. By Fact 3.2, it is given by a formula ψ as required.

Conversely, suppose that T has definable witnesses, and let $E \to 1$ be an epimorphism. Since 1 is quasi-compact and the objects $[\vec{x}.\varphi]$ generate $\mathcal{E}_T$, there are finitely many formulas $\varphi_1, \dots, \varphi_n$ with morphisms $[\vec{x}_i.\varphi_i] \to E$ such that $T \vdash \bigvee_i \exists \vec{x}_i\, \varphi_i$. We replace φ_i by $\varphi_i \wedge \bigwedge_{j<i} \neg\exists \vec{x}_j\, \varphi_j$, so that the sentences $\exists \vec{x}_i\, \varphi_i$ are pairwise incompatible. A definable witness ψ_i for φ_i gives a morphism $[\exists \vec{x}_i\, \varphi_i] \to [\vec{x}_i.\varphi_i]$. These glue to a morphism $1 = \coprod_i [\exists \vec{x}_i\, \varphi_i] \to E$ over 1. So every epimorphism onto 1 splits. Since epimorphisms are stable under pullback, 1 is projective.

If 1 is projective, then the global sections functor is exact on abelian group objects: we lift a global section of the target of an epimorphism by pulling back along that section. So

positive-degree cohomology vanishes, and every torsor, being locally trivial, has a global section. □

Example 5.2. (1) Every extension T of Peano arithmetic in the language of arithmetic has definable witnesses: we take the least witness, which exists by induction.

(2) Let T be a consistent extension of ZFC in the language $\{\in\}$. Then T has definable witnesses if and only if $T \vdash V = \mathrm{HOD}$. Indeed, $V = \mathrm{HOD}$ gives a definable well-order of the universe, and we take the least witness. Conversely, we take a definable witness for the formula $x \notin \mathrm{HOD}$: an element definable without parameters is ordinal definable, so T proves that no element lies outside HOD. In terms of models, this is a theorem of Dorais and Hamkins [5, Theorem 1].

6. The Sacks double cover

Let $\mathcal{Z} = \mathcal{E}_{\mathrm{ZFC}}$, and let $\mathcal{R} = \mathcal{E}_{Z_2}$, where Z_2 is second-order arithmetic with full comprehension, viewed as a two-sorted first-order theory. The interpretation of Z_2 in ZFC by $(\omega, \mathcal{P}(\omega))$ gives $r\colon \mathcal{Z} \to \mathcal{R}$ (Section 3). Its inverse image r^* restricts to a functor between the syntactic pretoposes, so it preserves the objects and quotients defined by formulas. We call the elements of the second sort *reals*.

Fact 6.1. *There is a formula $\rho(x, y)$ of second-order arithmetic, without parameters, such that* ZFC *proves that $\rho(x, y)$ holds if and only if $x \in L[y]$ for all reals x and y* [13, p. 494, after Theorem 25.26].

We fix such a formula ρ. Let γ be the sentence of second-order arithmetic saying that ρ is a preorder on the reals and the quotient partial order has a least element and exactly two minimal elements above it. Over γ, we call the two minimal elements *the two minimal degrees*.

Construction 6.2. We define an object Q of $\mathcal{R}$ to be the set of the two minimal degrees (a quotient of an object by an equivalence relation, both defined by formulas) over the clopen subtopos of γ, and $1 \sqcup 1$ over the clopen subtopos of $\neg\gamma$. Exchanging the two elements makes Q an $\mathbb{F}_2$-torsor.

We use the following facts from set theory.

Fact 6.3 (Groszek and Laver [10]; Fuchs, Gitman and Hamkins [8, Theorem 4.6])**.** *Let N be a countable model of* ZFC $+ V = L$*, and let a, b be mutually generic Sacks reals over N. In $N[a, b]$, the constructibility degrees of reals have a least element and exactly two minimal elements above it, and these two degrees satisfy the same formulas without parameters.*

See also [19] for the constructibility degrees in this model. For ill-founded N, see Section A.

Fact 6.4. *Let M be a countable model of* ZFC*, possibly ill-founded. Then L^M is a model of* ZFC $+ V = L$*, by Gödel's theorem applied inside M. Forcing over a countable model of* ZFC*, well-founded or not, yields models of* ZFC [11]*. These models have the same natural numbers as M (Section A), so they satisfy the same sentences of first-order arithmetic as M.*

Proof of Theorem $B(1)$. Let M be a countable model of ZFC $+ A$, let a, b be mutually generic Sacks reals over L^M, and put $B_0 = \mathrm{Th}(L^M)$ and $B_1 = \mathrm{Th}(L^M[a, b])$. By Fact 6.4, both contain ZFC $+ A$ and have the same arithmetic sentences.

In L^M, every real is constructible, so the quotient partial order has only one element and γ is false. Thus r^*Q is $1 \sqcup 1$ on $\mathcal{Z}_{B_0}$ and hence trivial.

Suppose that r^*Q has a global section on $\mathcal{Z}_{B_1}$. In ZFC, the degree of a real x is the set $d(x) = \{z \subseteq \omega : z \in L[x] \text{ and } x \in L[z]\}$. By Fact 6.1, ZFC proves that $x \mapsto d(x)$ induces a bijection from the reals modulo the equivalence relation of ρ to the set of degrees. Since

r^* preserves quotients, over γ the object r^*Q is $[d.\mu(d)]$, where $\mu(d)$ says that d is one of the two minimal degrees; the same holds after restriction to $\mathcal{Z}_{B_1} = \mathcal{E}_{B_1}$. By Fact 6.3, γ lies in B_1. By Fact 3.2, the global section is given by a formula $\psi(d)$ without parameters such that B_1 proves that exactly one minimal degree satisfies ψ. This contradicts Fact 6.3. So r^*Q is nontrivial on $\mathcal{Z}_{B_1}$. A global section on $\mathcal{Z}_A$ would restrict to one on $\mathcal{Z}_{B_1}$, so r^*Q is also nontrivial on $\mathcal{Z}_A$. □

Proof of Theorem B(2). Let B_{ar} be the Boolean algebra of sentences of first-order arithmetic modulo equivalence over ZFC, regarded as an $\mathbb{F}_2$-vector space under symmetric difference. A sentence θ gives an idempotent $\chi_\theta \in H^0(\mathcal{Z}; \mathbb{F}_2)$, and the map

$$B_{\mathrm{ar}} \to H^1(\mathcal{Z}; \mathbb{F}_2), \qquad \theta \mapsto \chi_\theta \cdot [r^*Q]$$

is $\mathbb{F}_2$-linear. Its image lies in the image of r^*, since $\chi_\theta \cdot [Q]$ is defined on $\mathcal{R}$ by the same sentence. This map is injective: if $\theta \neq 0$ in B_{ar}, then ZFC + θ is consistent, and the restriction of $\chi_\theta \cdot [r^*Q]$ to $\mathcal{Z}_{\{\theta\}}$ is $[r^*Q]$, which is nonzero by part (1). For nonzero θ, the arithmetic consequences of ZFC + θ form a consistent recursively enumerable extension of Robinson arithmetic, so a Rosser sentence for this extension [22] splits θ into two nonzero parts. Since ZFC is consistent, $B_{\mathrm{ar}} \neq 0$, so B_{ar} is infinite. Thus the image of r^* has dimension at least $\aleph_0$, and so do both groups.

For the upper bound, we consider $\mathcal{Z}$; the argument for $\mathcal{R}$ is the same. An $\mathbb{F}_2$-torsor is trivialized along an epimorphism onto 1, which we may refine to a map from a finite coproduct U of objects defined by formulas, since 1 is quasi-compact. By descent, the torsor is determined up to isomorphism by U and a cocycle $U \times U \to 1 \sqcup 1$. This cocycle is a morphism of the syntactic pretopos by Fact 3.2, and the syntactic pretopos is countable. So there are only countably many isomorphism classes of $\mathbb{F}_2$-torsors. □

Proof of Theorem B(3). By Example 5.2(2), the theory ZFC + V = HOD has definable witnesses. We apply Proposition 5.1. □

Remark 6.5. The torsor Q is a sheaf-theoretic form of the failure of Ehrenfeucht's lemma in set theory [8]. Groszek and Laver study more general finite groups of conjugates under ordinal definability [10]. For other relations between cohomology and definability, see [20] for Galois cohomology in model theory and [2] for the axiom of choice.

Use of AI tools

The mathematics and the text of this paper were produced with substantial help from AI systems. On 19 November 2026, GPT-6.2 Astra (OpenAI, in ChatGPT) proposed the Sacks double cover and a realization of finite simplicial complexes by open subtoposes. On 22 November 2026, GPT-6 Astra Pro (OpenAI, in ChatGPT) checked these arguments and extended them to countable complexes and to a fixed arithmetic theory, and GPT-6 in ChatGPT (Pro setting) proposed the simplifications used here: the toposes of actions of finite chains, the adjunction of Proposition 4.3, the converse for quasi-compact open subtoposes, the form of Theorem B(1), and several references. Claude Opus 5.5 (Anthropic, in Claude Code) wrote the text, including the introduction and the abstract, surveyed the literature, and wrote the arguments of Section A; a separate instance of Claude Opus 5.5 refereed the proofs. Gemini 4 Neon (Google, in Antigravity) polished the English. The AI research notes of 19 and 22 November 2026 are posted on the forum of the page *Moduli spaces of mathematics* of the author's wiki duaLab, https://dualab.ryuyahora.jp/Moduli+spaces+of+mathematics/forum. The author chose the problem and the main results and directed the work. The author understands the topos-theoretic setting of the paper (classifying toposes, their open subtoposes, and the idea of studying spaces of mathematical universes) and why the results are interesting. He has not checked the proofs, and he does not claim to understand the homotopy-theoretic part of Theorem A or the set-theoretic part (Theorem B and Section A).

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Appendix A. Sacks extensions of ill-founded models

In the proof of Theorem B(1), the model M of $\mathrm{ZFC}+A$ may be ill-founded, for instance when A contains a false arithmetic sentence. We explain why Fact 6.3 still applies to $N = L^M$.

Let $\mathbb{S}$ be Sacks forcing, let $\mathbb{P} = \mathbb{S} \times \mathbb{S}$, and let $\dot{a}, \dot{b}$ be the canonical names for the two generic reals. We need two statements about the forcing relation of $\mathbb{P}$, each provable in $\mathrm{ZFC} + V = L$.

(a) Every condition forces that the constructibility degrees of reals have a least element and exactly two minimal elements above it, namely the degrees of $\dot{a}$ and $\dot{b}$.
(b) For each formula $\psi(d)$ without parameters, every condition forces $\psi(\deg \dot{a}) \leftrightarrow \psi(\deg \dot{b})$.

Statement (a) is stated in [8, Theorem 4.6], where the fusion argument is sketched. A complete fusion argument, carried out with conditions in the ground model and hence a proof in $\mathrm{ZFC} + V = L$ about the forcing relation, is given by Groszek and Hamkins [9, Section 5]: in a product Sacks extension, a real that lies in none of the extensions by a single generic real constructs at least two minimal nonzero degrees (the Technical Lemma, Lemma 8 in the arXiv version). For two factors, together with the minimality of a single Sacks real [13, Theorem 15.34], this shows that every condition forces that the nonconstructible degrees of reals are the degree of $\dot{a}$, the degree of $\dot{b}$, and degrees above both, which gives (a). See also [15, Section 5] for finite products of Sacks forcing.

Statement (b) has a short internal proof. For Sacks conditions p and q, there is a constructible order isomorphism from the conditions below p to the conditions below q. It is induced by a constructible homeomorphism h between the corresponding perfect trees, so it forces $L[h(\dot{a})] = L[\dot{a}]$; that is, it preserves the degree of the generic real. If f is such an isomorphism, then $\langle r, s\rangle \mapsto \langle f^{-1}(s), f(r)\rangle$ is an automorphism of the conditions of $\mathbb{P}$ below $\langle p, q\rangle$, and it exchanges the degrees of $\dot{a}$ and $\dot{b}$. Since the forcing relation is invariant under automorphisms acting on names [13, Lemma 14.37], $\langle p, q\rangle \Vdash \psi(\deg \dot{a})$ holds if and only if $\langle p, q\rangle \Vdash \psi(\deg \dot{b})$. So no condition forces $\psi(\deg \dot{a}) \wedge \neg\psi(\deg \dot{b})$, and this gives (b).

Now let N be a countable model of $\mathrm{ZFC} + V = L$, possibly ill-founded. By [11], an N-generic filter for $\mathbb{P}^N$ exists, and the model $N[a, b]$ of names modulo the filter is a model of ZFC that satisfies every statement forced by a condition in it. Hence (a) and (b) give the conclusion of Fact 6.3 in $N[a, b]$. Every name for a natural number is decided by a condition in the filter to be a natural number of N, so N and $N[a, b]$ have the same natural numbers and satisfy the same arithmetic sentences.

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