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# RESOLUTIONS OF CONJECTURES ON SURFACE WEBS

Abstract. We prove the coefficient-localization, boundary-direction, and boundary-color conjectures for the native $SL_3$ web cluster structures on bordered surfaces. We also give a six-flag biweb with a tree representation which is not a cluster monomial, disproving the forest implication in the biweb conjecture.

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## 1. Introduction

The tensor-diagram construction of [2, 3] attaches cluster structures to rings of invariants of vectors, covectors, and unimodular matrices. Its extension from a disk to a bordered surface introduces two features. Closed curves give trace invariants, and a boundary proxy may pass around a boundary component before its defining caterpillar terminates. Consequently the ordinary invariant ring and the cluster algebra need not be identified by the disk argument alone.

We work with the native vector–covector decorations of [3]. In particular, a black marked point carries one vector and a white marked point carries one covector. Affine-flag decorations occur only in the last part of the paper. Let $S$ be a connected oriented surface of genus $g$ with $h > 0$ boundary components and no punctures. Every boundary has marked points, and its cyclic black–white word is nonalternating. Write $a, b$ for the numbers of white and black points and $c = 2g + h - 1$. With $V = \mathbb{C}^3$ endowed with a volume form, put

$$\mathcal{R}(S) = \mathbb{C}[(V^*)^a \times V^b \times \mathrm{SL}(V)^c]^{\mathrm{SL}(V)}.$$

Let $\mathcal{A}(S)$ and $\mathcal{U}(S)$ denote the cluster and upper cluster algebras of [3]. Original coefficient variables are not inverted in $\mathcal{A}(S)$. Let $\mathcal{D}$ be the multiplicative set they generate. The small polynomial disk cases are described in Section 2.4.

**Theorem 1.1** (Coefficient localization)**.** *For every such surface in the native tensor-diagram construction,*

$$\mathcal{A}(S)[\mathcal{D}^{-1}] = \mathcal{R}(S)[\mathcal{D}^{-1}] = \mathcal{U}(S)[\mathcal{D}^{-1}].$$

*Thus Conjecture 17 of* [3] *holds.*

**Theorem 1.2** (Boundary directions)**.** *Reverse the direction used to construct special-invariant proxies on any chosen subset of boundary components. The resulting cluster structure is the same structure in the same invariant field, with the same actual coefficient functions. Thus Conjecture 24 of* [3] *holds.*

**Theorem 1.3** (Boundary colors)**.** *Complement all colors on each boundary component in any chosen subset. The mutable exchange quivers before and after this change belong to the same mutation class. Thus Conjecture 25 of* [3] *holds.*

The two invariance assertions have different meanings. The direction assertion identifies actual cluster functions and their compatibility. The color assertion concerns the mutation class of the mutable quiver; the two native invariant fields can have different numbers of vector and covector arguments.

The coefficient-localization proof uses explicit frames whose determinants are products of original coefficients. For several boundary components, cross-boundary arcs recover holonomy matrices. On one boundary, root derivations and finite principal-open covers recover the missing columns. The one-mark case requires a separate argument: two boundary proxies recover all matrix entries up to

the two boundary traces, and valuation theory followed by a finite cover recovers those traces. The second proxy is supplied by the direction theorem. The proof of that theorem is independent of localization; there is no circular use of Theorem 1.1.

Our final result concerns the bivariant, affine-flag version of the construction.

**Theorem 1.4** (A forest counterexample)**.** *In the* $\mathrm{SL}_3$ *Fock–Goncharov cluster algebra of a disk with six affine flags, there is a non-elliptic single biweb invariant which has a connected tree tensor representation but is not a cluster monomial. It is not even a frozen Laurent monomial times a cluster monomial. Consequently the forest-to-cluster-monomial implication in the second clause of Conjecture 28 of* [3] *is false.*

The construction in Theorem 1.4 comes from the diagrammatic phenomenon already displayed in [2, Section 10, Figure 26]. That example establishes that decomposable web invariants with a tree representation need not be cluster monomials. Here we place the diagram in the six-flag space, verify the additional biweb conditions and the flag relations, and use the actual quasi-isomorphism of [4, Theorem 7.2] to exclude all frozen Laurent multiples of cluster monomials. The elementary prime-factor argument is given in full. The example is decomposable; it does not contradict a conjecture with an additional indecomposability hypothesis. No conclusion about the other clauses of Conjecture 28 is needed.

We first recall the tensor and seed conventions. The boundary symmetries are proved before coefficient localization. We then treat the ordinary boundary configurations, the one-mark case, and the upper-cluster-algebra comparison. The final section and its appendix give the explicit six-flag counterexample and a finite tensor-contraction certificate.

## 2. Tensor and cluster preliminaries

2.1. **Cross products and transport.** The volume form defines alternating maps

$$V \times V \longrightarrow V^*, \quad (u,v) \longmapsto u \times v, \qquad V^* \times V^* \longrightarrow V, \quad (\alpha,\beta) \longmapsto \alpha \times \beta.$$

Our convention is $(u \times v)(w) = \det(u,v,w)$ and its dual. For example,

$$\alpha \times (u \times v) = \alpha(v)u - \alpha(u)v, \tag{2.1}$$

$$(u \times v) \times (w \times z) = \det(u,v,z)w - \det(u,v,w)z. \tag{2.2}$$

All tensor identities below follow from these formulas and multilinearity. An element $G \in \mathrm{SL}(V)$ transports vectors by $u \mapsto Gu$ and covectors by $\alpha \mapsto \alpha G^{-1}$. Thus cross products commute with transport. At a cut crossed in the opposite direction we use $G^{-1}$.

Choose a lift of each boundary interval. If its period is $m$ and boundary holonomy is $H$, write

$$u_{i+m} = Hu_i, \qquad \alpha_{i+m} = \alpha_i H^{-1}.$$

The native vector is $u_i$ at a black point and the native covector is $\alpha_i$ at a white point. The other half is the proxy. With the forward boundary convention the recursion is

$$\alpha_i = u_i \times u_{i+1} \quad (i \text{ black}), \qquad u_i = \alpha_i \times \alpha_{i+1} \quad (i \text{ white}). \tag{2.3}$$

The recursion stops at the first adjacent pair of equal colors. This is precisely why nonalternation is assumed. Products across a cut are transported before they are taken. In particular $\alpha_i(u_i) = 0$.

A special bipod along a simple arc is the pairing of the transported proxies at its endpoints. A special tripod is the determinant of the three proxies at the corners of a contractible triangle. Corners can be different lifts of the same marked point. Their cyclic order is the order in that actual triangle, not an arbitrary ordering of group words. A tensor diagram can have crossings; a web has none. Non-ellipticity excludes a contractible loop, an internal digon, and an internal four-cycle. Splitting a boundary clasp means separating its incident half-edges into different leaves.

2.2. **Published structural inputs.** We use the following precise parts of the native construction [3, Propositions 6–14 and Theorems 15–19], with the disk factorization rules of [2, Sections 6–8].

For a triangulation $T$, form $K(T)$ from the two directed bipods on each edge and the vector tripod in each triangle. Factor its nonzero elements into indecomposable special invariants. The set of factors, without repetitions, is an extended cluster $\mathbf{z}(T)$ in $\operatorname{Frac}\mathcal{R}(S)$. Indecomposable special invariants are irreducible elements of $\mathcal{R}(S)$. The distilled three-term skein identities give the actual exchange relations. In particular, the one-step exchange partners of this seed are also indecomposable special invariants. Triangulation seeds are mutation equivalent, $\mathcal{A}(S) \subset \mathcal{R}(S)$, and every invariant is Laurent in every extended cluster. For a nondisk surface,

$$|\mathbf{z}(T)| = 3(a+b) + 8(c-1), \qquad |\mathbf{d}| = a+b, \tag{2.4}$$

where $\mathbf{d}$ is the set of original coefficient functions. On disks the exceptional coefficient lists are retained.

Deleting a marked point gives the natural injection of invariant rings; replacing a same-colored adjacent pair by its cross product gives another injection. If both signatures remain admissible, each injection realizes the smaller native cluster structure by freezing some mutable vertices and deleting vertices with zero columns to the remaining mutable set. The actual functions, and not just the quivers, are part of these statements. Section 3 explains the descent argument we will use for such substructures.

We also use that $\mathcal{R}(S)$ is a unique factorization domain with unit group $\mathbb{C}^*$, as in the proof of [3, Theorem 15]; see [8, Theorem 3.17]. It is a finitely generated normal domain. The invariant generators consist of traces of matrix words, pairings of transported vectors and covectors, and determinants of three transported vectors or covectors [3, Theorem 1]. Thus a frame with invertible determinant, all native coordinates in that frame, and all generating holonomy matrices in that frame generate the localized invariant ring. This last observation is simply substitution into the displayed generators, and does not require any identification of a skein algebra with an invariant ring.

For clarity, when passing to invariant frames we use the following elementary form of this observation. If $F = (f_1, f_2, f_3)$ is an equivariant matrix with invariant unit $\delta = \det F$, then $G = F\operatorname{diag}(1, 1, \delta^{-1})$ lies in $\mathrm{SL}_3$. Sending a configuration to $G$ and to its transform by $G^{-1}$ gives an isomorphism of its frame-open set with $\mathrm{SL}_3 \times Y$, where $Y$ is the slice with normalized frame. The inverse sends $(G, y)$ to $G \cdot y$. Both maps are regular because $\delta$ is inverted. Taking invariants therefore gives $\mathbb{C}[Y]$.

2.3. **Laurent conventions.** For a seed $t$ let $\mathbf{x}_t$ be its mutable cluster. Set

$$L_t = \mathbb{C}[\mathbf{d}^{\pm 1}, \mathbf{x}_t^{\pm 1}], \qquad \mathcal{U}_{\mathcal{D}} = \bigcap_t L_t \ \subset\ \operatorname{Frac}\mathcal{R}(S).$$

This notation places all original coefficients in the coefficient torus before taking the intersection. We will prove equality with $\mathcal{R}(S)[\mathcal{D}^{-1}]$ directly. The sandwich

$$\mathcal{A}(S)[\mathcal{D}^{-1}] \subseteq \mathcal{U}(S)[\mathcal{D}^{-1}] \subseteq \mathcal{U}_{\mathcal{D}} \tag{2.5}$$

is valid for either of the usual polynomial or Laurent original-coefficient conventions for $\mathcal{U}(S)$. We never interchange localization with an infinite intersection without this argument.

A cluster monomial in this paper has nonnegative integer exponents in an extended cluster. A *localized* cluster monomial can in addition have negative exponents in variables explicitly designated as inverted. These two notions will be distinguished in the root-derivation argument and in the six-flag counterexample.

2.4. **The elementary small disks.** The general triangulation count is not used on disks with at most four marked points. Their invariant rings are polynomial rings, with the elementary generators regarded as coefficients. With three monochromatic points the generator is their determinant; with a mixed nonalternating triple the generators are its two vector–covector pairings. With four points the generators are the four triple determinants in the monochromatic case, the three pairings and one determinant in the $(1,3)$ or $(3,1)$ case, and the four pairings in the $(2,2)$ case. They are algebraically independent: in the determinant cases choose three inputs as a frame, and in the pairing cases choose the independent inputs first and solve for the other ones. For at most two admissibly colored points the ring is $\mathbb{C}$. These descriptions also follow immediately from the elementary invariant generators. Their mutable quivers are empty, direction reversal leaves the listed functions unchanged, and color complementation dualizes the list. All three assertions of the paper are immediate in these degenerate cases. For the five-point alternating-tail disk, we keep the six coefficients of [2, Remark 6.8], instead of the generic five-coefficient count.

## 3. Reversing boundary directions

The direction used to construct a proxy is separate from the orientation of the surface. In particular, changing that direction does not reflect a volume tensor. We prove the following assertion with this convention.

**Theorem 3.1.** *Reversing the proxy construction direction on any collection of boundary components leaves the Fomin–Pylyavskyy cluster structure unchanged, as a cluster structure in the same invariant field with the same coefficient functions.*

We first give a descent argument. It will also be used for changes of color.

3.1. **Freezing and deletion.** A substructure obtained by *freezing and deletion* has the precise meaning of [2, Definition 8.7]: in an ambient seed, freeze some mutable vertices and then delete vertices not adjacent to any remaining mutable vertex. The retained variables are the original rational functions. No deleted function is specialized to a number.

**Lemma 3.2.** *Suppose that two such substructures of one ambient cluster structure have the same embedded fraction field and the same final frozen variables. Then they have the same extended variables and the same clusters. Their exchange data agree in the sense of the cluster-structure convention of* [2, Definition 7.10].

*Proof.* Choose an ambient seed presenting the second substructure. Let $I$ be its retained mutable indices, $J$ all the other ambient mutable indices, $E$ the deleted extended coordinates, and $Z$ the retained extended coordinates. The defining condition is

$$b_{ei} = 0 \qquad (e \in E,\ i \in I). \tag{3.1}$$

The matrix mutation formula shows that this condition persists under mutations in $I$. Such mutations are therefore precisely the ambient mutations with the deleted coordinates omitted. In particular the embedded field of the second substructure is $K = \mathbb{C}(Z)$.

If an ambient cluster variable $x$ belongs to $K$, its Laurent expansion in the presenting ambient seed is independent of $E$. Indeed,

$$\mathbb{C}[Z^{\pm 1}, E^{\pm 1}] \cap \mathbb{C}(Z) = \mathbb{C}[Z^{\pm 1}],$$

by the linear independence of Laurent monomials in the algebraically independent coordinates $E$. Its denominator coordinate at each deleted ambient mutable variable is thus zero. The denominator criterion [1, Theorem 6.3] implies that $x$ is compatible with each such variable. The possibility that $x$ is itself a deleted variable is excluded by algebraic independence.

Now take an extended variable $x$ of the first substructure. If it is ambient frozen, the same field argument shows that it is retained in the second substructure. If it is ambient mutable, it is

compatible with every retained member of $J$: these are among the common final frozen variables and occur together with $x$ in an ambient lift of a cluster of the first substructure. It is compatible with the deleted members of $J$ by the preceding paragraph. A finite pairwise compatible collection of cluster variables lies in a cluster, by [1, Theorem 7.4]. Moreover, clusters containing a fixed collection are connected by mutations retaining that collection, by [1, Theorem 6.2]. Apply these statements to $x \cup J$. A path retaining $J$ uses only $I$; by (3.1) it descends to the second substructure. Hence $x$ belongs to that substructure. Interchanging the two structures gives equality of their extended variable sets.

The same argument works with any finite set $S$ compatible in the first substructure. Its ambient lift is compatible, and every member of $S$ is compatible with $J$. The two results just quoted place $S \cup J$ in an ambient cluster and connect it to the presenting cluster while retaining $J$. Thus $S$ is compatible in the second substructure. The converse follows on interchanging the two. Maximal compatible sets, with the common frozen variables adjoined, are consequently the same extended clusters.

The exchange partner is determined by the adjacent clusters. Its product with the old variable has a unique expression as the sum of the two exchange monomials in the common algebraically independent extended cluster. This determines each exchange column up to sign; skew symmetry makes the choices consistent on each connected mutable component. These are exactly the component reversals allowed in the cited convention. □

When an exact comparison of matrices is needed, we use the following strengthening. Choose a common retained extended cluster $Z$ and ambient seeds realizing it in the two substructures. The fixed-face connectivity theorem connects these ambient seeds while retaining every ambient mutable member of $Z$. Every mutation on this path is at a deleted vertex. Initially, every such vertex has zero adjacency to every retained mutable vertex. The zero block persists along this path as well, since mutation at a vertex of a zero block cannot create an entry across that block. The entries between retained mutable vertices, and their entries with retained frozen vertices, are unchanged. Thus the induced source matrices agree in the inherited ambient conventions. This observation will prevent an unwarranted identification of a quiver with its negative.

3.2. **The coefficient set and a fixed fork embedding.** On a boundary lift, let $U_i \in V$ and $\alpha_i \in V^*$ denote the two completed proxies. At a black point $U_i$ is native, and at a white point $\alpha_i$ is native. The cross products are normalized by $(u \times v)(w) = \det(u, v, w)$ and its dual. Native and proxy tensors are transported by the actual surface holonomies.

**Lemma 3.3.** *The set of original coefficient functions on a boundary is unchanged by reversing its proxy direction.*

*Proof.* For an opposite-color boundary edge the nonzero coefficient is the native vector–covector pairing, which is unchanged. Let $i$ be a same-color edge and let $j > i$ be the next same-color edge on the universal cover. The coefficient associated with $i$ in the forward convention is the unrooted caterpillar with end forks at $(i, i+1)$ and $(j, j+1)$ and with the intervening alternating native leaves. Reading this same tree from the other end is the coefficient associated with $j$ in the reverse convention. Its evaluation is unchanged: re-rooting reassociates contractions, and the elementary reassociation

$$\gamma(\alpha \times \beta) = \det(\gamma, \alpha, \beta) = \beta(\gamma \times \alpha)$$

is cyclic and has positive sign. The vector version is identical.

If there is only one same-color edge in a period, its next occurrence is its holonomy translate. Transport the whole tree back by that holonomy; its determinant is one, so the evaluations are still exactly equal. For a single black mark this is the identity

$$\det(v, Hv, H^2 v) = \det(H^{-1}v, v, Hv).$$

The dual identity handles a white mark. The coefficient definition in [3, Proposition 9], or its factor-set version in the exceptional disk cases, gives precisely these functions and their indecomposable factors. Equality of the unrooted functions and uniqueness of their normalized special factorization also gives equality of the factor sets. □

Replace every white point by two adjacent black points through the *fixed* physical substitution $\alpha = a \times b$. The cyclic order of the fork is that of the oriented surface. Reversing the proxy search direction does not change this substitution or its sign. The map on configuration spaces is surjective: every covector is a cross product of two vectors, including zero. Its pullback $\Phi$ is therefore injective. The functoriality theorems [3, Theorems 18–19] realize both source direction choices by freezing and deletion through this same $\Phi$. The operations compose, since the zero block (3.1) persists. The fraction-field assertion of [3, Theorem 15] gives their common embedded field $\Phi(\mathrm{Frac}\,\mathcal{R})$; Lemma 3.3 gives their common final frozen set. Consequently, once the all-black direction assertion is proved in a given topology, Lemma 3.2 proves it for all admissible signatures in that topology. No equality $\mathcal{A} = \mathcal{R}$ or $\mathcal{A} = \mathcal{U}$ is used here.

3.3. **A collar based on another boundary.** Suppose that there are at least two boundary components and all marks are black. Choose a marked point $q$ on a component different from the one being changed. A thin neighborhood of the selected boundary and a simple stem from $q$ gives a collar with a single seam based at $q$. Its fan triangles, in the universal cover, are

$$(u, v_i, v_{i+1}) \quad (1 \le i \le m), \qquad (u, Hv_1, Hu), \qquad v_{i+m} = Hv_i.$$

Here $u$ is the native vector at $q$, $H$ is the selected boundary holonomy, and $\beta$ is the actual unchanged covector proxy at $q$. The last triangle closes the collar. Its two incidences at $q$ are distinct lifted corners. Introduce

$$\begin{aligned} A_i &= \det(u, v_i, v_{i+1}) && (0 \le i \le m+1), & B_i &= \beta(v_i) && (1 \le i \le m+1),\\ d_i &= \det(v_i, v_{i+1}, v_{i+2}) && (1 \le i \le m), & C_0 &= \det(u, Hv_1, Hu),\\ E &= \beta(Hu), & & & F &= (H\beta)(u). \end{aligned}$$

The $d_i$ are the original coefficients on the selected boundary. The forward fan has $A_1, \dots, A_{m+1}$, whereas the backward fan has $A_0, \dots, A_m$. All the other displayed functions are common. Indeed, a spoke contributes its proxy pairing, while the boundary triangle determinant contributes $A_i$ in both conventions.

Set $u_0 = H^{-1}u$ and $v_0 = H^{-1}v_m$. The five-vector determinant relation, with the common vector $v_1$, gives

$$A_{m+1}A_0 = A_1A_m + d_mC_0. \tag{3.2}$$

For example, the five factors other than $A_1$ can be written

$$A_{m+1} = \det(u_0, v_1, v_2), \quad A_0 = \det(u, v_0, v_1), \quad A_m = \det(u_0, v_0, v_1),$$

$$d_m = \det(v_0, v_1, v_2), \qquad C_0 = \det(v_1, u, u_0).$$

These identities also prove (3.2) when $m = 1$, where its first term on the right is $A_1^2$.

We record the entire matrix calculation to identify this relation as an actual seed mutation. The seven slots of an $\mathrm{SL}_3$ triangle are $210, 120, 201, 102, 021, 012, 111$. We may include its three side arrows: they cancel on internal edges and vanish at the zero boundary slot. The two boundary-triangle tables are

| | 210 | 120 | 201 | 102 | 021 | 012 | 111 |
|---|---|---|---|---|---|---|---|
| + | $B_i$ | $A_i$ | $B_{i+1}$ | $A_{i+1}$ | 0 | $d_i$ | $A_i$ |
| − | $B_i$ | $A_{i-1}$ | $B_{i+1}$ | $A_i$ | $d_{i-1}$ | 0 | $A_i$ |

$(d_0 = d_m)$.

These are exactly the exposed triangle blocks of [3, Figures 8–9]. Their arrows after deletion and identification are, respectively,

$$B_i \to B_{i+1}, \quad B_{i+1} \to A_i, \quad A_i \to A_{i+1}, \quad A_{i+1} \to B_{i+1}, \quad A_{i+1} \to d_i \to A_i; \tag{3.3}$$

$$B_i \to B_{i+1}, \quad B_i \to A_{i-1}, \quad A_{i-1} \to A_i, \quad A_i \to B_i, \quad A_i \to d_{i-1} \to A_{i-1}. \tag{3.4}$$

The closing triangle has tables

| | 210 | 120 | 201 | 102 | 021 | 012 | 111 |
|---|---|---|---|---|---|---|---|
| + | $B_{m+1}$ | $A_{m+1}$ | $E$ | $F$ | $A_1$ | $B_1$ | $C_0$ |
| − | $B_{m+1}$ | $A_m$ | $E$ | $F$ | $A_0$ | $B_1$ | $C_0$. |

Its forward arrows are

$$B_{m+1} \to E, \quad B_{m+1} \to A_{m+1} \to C_0 \to B_{m+1}, \quad E \to C_0 \to F \to E,$$
$$A_1 \to A_{m+1}, \quad C_0 \to A_1 \to B_1 \to C_0, \quad F \to B_1.$$

The backward list replaces $A_{m+1}, A_1$ by $A_m, A_0$. On summing the forward blocks, the column at $A_{m+1}$ is exactly

$$A_1, A_m \longrightarrow A_{m+1} \longrightarrow d_m, C_0. \tag{3.5}$$

Mutation reverses these arrows and adds $A_1 \to d_m$, $A_1 \to C_0$, $A_m \to d_m$, and $A_m \to C_0$. The middle two appropriate arrows cancel $C_0 \to A_1$ and $d_m \to A_m$. Comparing (3.3) and (3.4) after shifting the index shows that all interior terms agree; the surviving additions are precisely the two end changes. The closing block supplies the changes at $B_1, B_{m+1}$. Thus mutation at $A_{m+1}$ gives the whole backward matrix and replaces that function by $A_0$ according to (3.2). For $m = 1$ the repeated incoming vertex gives multiplicity two; one copy of each relevant added arrow cancels and the other remains.

Here the functions really are the factors in the original seed. Choose the fundamental-group generators to include $H$ and omit the anchor boundary generator. Then $H$ is an independent $\mathrm{SL}_3$ matrix. For $m \geq 2$ each $A_i$ is a three-vector determinant after an invertible change of a native vector by $H$. For $m = 1$ it is linear in $u$, with coefficient vector $H^i v \times H^{i+1} v$. For $v \neq 0$ the vanishing of this coefficient imposes the two conditions $Hv \in \langle v \rangle$; the locus $v = 0$ has codimension three. Thus the coefficients have no common divisor and the linear polynomial is irreducible. The same argument applies to $C_0$, linear in $v_1$ with coefficient obtained from $u \times H^{-1}u$ by an invertible transport. Finally $B_i$ is linear in its target native vector. The zero locus of its coefficient $\beta$ has codimension at least two, both when $\beta = u \times w$ for distinct native vectors and when $\beta = u \times Ku$ for a one-point anchor. In the latter case $K$ is an invertible change of the free $H$ coordinate, with all other group coordinates fixed. This proves the needed primitivity of these functions.

**Lemma 3.4.** *The displayed collar factor list has no further identifications, apart from the annulus folds in* (3.6)*. The same assertion holds if the exterior has arbitrary colors and the black anchor is chosen in a same-color adjacent pair, or is its boundary's only mark.*

*Proof.* Such an anchor still has the form $\beta = u \times w$ with an independent native neighbor, or the corresponding ordered cross product of $u$ and $Ku$ at a one-mark boundary. The preceding primitivity proof is unchanged. We check distinctness of the positive-target-degree factors. The original target coefficients $d_i$ have anchor degree zero, whereas $A_i, B_i, C_0$ have positive anchor degree. The pairwise distinctness of the $d_i$ is already part of the original coefficient-set theorem. The $A_i$ have total target degree two, whereas $B_i, C_0$ have target degree one. Different physical target natives distinguish all the $B_i$, except the two end spokes $B_1, B_{m+1}$. For $m \geq 3$, the unordered pair of target natives similarly distinguishes the $A_i$ except $A_1, A_{m+1}$.

The remaining comparisons can be made on the legal family $H = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1\lambda_2\lambda_3 = 1$. Since $A_{m+1} = \det(H^{-1}u, v_1, v_2)$, its coefficient of a fixed coordinate monomial involving $u_j$ is $\lambda_j^{-1}$

times the corresponding coefficient of $A_1$. These are different nonconstant characters of the diagonal torus. When $m = 2$, take $u = e_1, v_1 = e_2, v_2 = e_3$; the three values are

$$A_1 = 1, \qquad A_2 = -\lambda_2, \qquad A_3 = \lambda_1^{-1},$$

so no two are constant associates. When $m = 1$, the coefficient vectors of $A_1, A_2$ as linear functions of $u$ are $v \times Hv$ and $H^{-T}(v \times Hv)$. For generic $v$ all three coordinates of the first are nonzero, and the second multiplies them by the distinct characters $\lambda_j^{-1}$. They cannot be constant associates either. The backward list has the same argument after shifting the lifted indices.

For the two end $B$ spokes the coefficient covectors are $\beta$ and $H^T\beta$. On the diagonal family, choose the anchor data so that all three components of $\beta$ are nonzero. With an independent neighbor this is a generic choice of $u, w$. With a one-point anchor it is a generic choice of $u$ and diagonal $K$; even in the annulus $K = H^{\pm 1}$ gives this property. The three coefficients are multiplied by $\lambda_1, \lambda_2, \lambda_3$, excluding a constant association.

Only $B_1, B_{m+1}$ can have the same target multidegree as $C_0$. If the anchor has an independent neighbor $w$, the $B$ spokes have positive $w$-degree and $C_0$ has none. If the anchor has one mark and the surface is not an annulus, its boundary holonomy can be varied while fixing $H$ on a legal diagonal family. For three or more boundaries, vary another free boundary generator. For two boundaries and positive genus, keep all but one handle trivial and vary its commutator. Taking a diagonal handle matrix $A$ and a cyclic permutation matrix $P \in \mathrm{SL}_3$, the commutators $[A, P]$ range over the diagonal torus: the three diagonal ratios can be prescribed with product one, and a common scalar adjusts $\det A$ to one. The boundary relation therefore lets $K$ range over that torus while $H$ stays fixed. The covector $\beta$ then varies nontrivially, whereas $C_0 = \det(u, Hv_1, Hu)$ is unchanged. Thus neither $B$ spoke can be a constant associate of $C_0$. For the annulus, direct substitution gives exactly the two folds in (3.6); the end-spoke comparison just made excludes any additional one.

Finally every exterior factor has target native degree zero and so cannot equal any factor just considered. This remains true for a mixed exterior. Factorization of an exterior function cannot acquire a factor with positive target native degree, since that grading is nonnegative. Its actual exterior factor list is consequently unchanged when only the selected boundary is recolored. □

The seam variables $E, F$ have zero entry in (3.5); an arbitrary exterior therefore attaches without affecting the mutation. This remains true under exterior distillation. Explicitly, if an interface special $p_j$ factors as $\prod_a z_a^{L_{aj}}$, the exponent-difference rows of its exchange monomials are multiplied by $L$. Each such row is zero in the mutated column, so this operation commutes with the mutation. The exterior factor list has target-boundary degree zero, while all factors in (3.2) have positive target-boundary degree. There is consequently no hidden identification with the mutated variable.

The two degenerations when the whole surface is an annulus with only one mark on the anchor boundary deserve mention. They are

$$\begin{array}{c|ll} \beta = u \times H^{-1}u & F = 0 & C_0 = B_1 \\ \beta = Hu \times u & E = 0 & C_0 = B_{m+1}. \end{array} \tag{3.6}$$

The indicated zero vertex has no adjacency to $A_{m+1}$. Among the two vertices being merged, only $C_0$ has nonzero net adjacency to that vertex. Hence the positive-part mutation formula commutes with both folds. These cover both choices of the anchor direction. We have proved the all-black assertion for at least two boundary components, and may repeat the argument one selected component at a time.

3.4. **One boundary and positive genus.** An exterior anchor is unavailable in this case. Instead we change the fan in the link of every marked point, using the actual lifted corners. At a lift $v$ of a marked point list its link neighbors in order as $u_0, \ldots, u_r$. Define

$$[i, j] = \det(v, u_j, u_i), \quad x_j = [j, r], \quad y_j = [0, j], \quad T_j = [j-1, j], \quad d = [0, r].$$

The endpoints satisfy

$$x_0 = d, \quad x_{r-1} = T_r, \quad x_r = 0, \qquad y_0 = 0, \quad y_1 = T_1, \quad y_r = d.$$

The forward proxy gives $x_1, \dots, x_{r-2}$; the reverse proxy gives $y_2, \dots, y_{r-1}$. In a one-mark component the two endpoint neighbors are $H^{-1}v, Hv$, where $H$ is the actual product of handle commutators.

In the two-dimensional quotient $V/\langle v \rangle$, the determinant relation is

$$x_k y_{k+1} = y_k x_{k+1} + dT_{k+1}. \tag{3.7}$$

Consequently the successive mutations at $x_1, \dots, x_{r-2}$ replace the forward fan by the backward fan. To specify all arrows, write $(a, b, c)$ for $a \to b \to c \to a$. The two link matrices are

$$\sum_{j=0}^{r-2} (x_j, x_{j+1}, T_{j+1}), \qquad \sum_{j=1}^{r-1} (y_j, y_{j+1}, T_{j+1}). \tag{3.8}$$

At each mutation their column gives the two monomials in (3.7); induction proves the displayed final matrix, including the rows of the $T_j$ and $d$.

The physical triangle labels $T_j$ need not be distinct occurrences. To justify merging them, first give each occurrence a separate label. Merging vertices other than a mutated vertex $k$ commutes with mutation whenever the entries in column $k$ of each merging class have the same weak sign: positive and negative parts are then additive in the mutation formula. At an interior step of the fan sequence the only triangle occurrence in this column is $T_{k+1}$. At the first step the possible opposite-side occurrences are $T_1, T_2$, and at the last they are $T_{r-1}, T_r$. These are distinct physical triangles if no interior edge is self-adjacent. If $r = 3$, $T_1 = T_r$ is allowed, but the two occurrences lie on the same side and give a square. Copies from every other point link have zero in the current column. Thus every physical merging commutes with this sequence; no common exchange factor is canceled.

For completeness, the interior triangle with slots

$$(a, b, c, e, f, h, T) = (210, 201, 120, 021, 102, 012, 111)$$

has the three corner cycles $(a, b, T), (e, c, T), (f, h, T)$ and the side arrows $a \to c, e \to h, f \to b$ in the side-augmented convention. Its three point links are these corner cycles. If the clockwise triangle order is used for its actual determinant, the link order above gives their simultaneous opposites; use that one global opposite arrow convention throughout (3.8). This changes no exchange binomial, since $\mu_k(-B) = -\mu_k(B)$. No determinant is multiplied by a sign. Across an interior arc the side arrows cancel. At an exposed black side one coordinate is zero, and the prescribed spoke–center identification is the actual boundary block of [3, Figures 8–9], also described as $Q_3^1$ in [5, Section 3.5]. The full surface matrix is therefore the sum of the point-link matrices, with physical triangle labels merged. Concatenating all the link sequences proves the comparison of the full matrices.

We now verify that these are actual seeds for every required genus and number of marks. Start with the polygon having edge word

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} C.$$

Triangulate it from vertex 0, with triangles $(0, i, i+1)$ for $1 \le i \le 4g-1$, and glue the handle sides in reversed pairs. All polygon corners are the one physical marked point. Each interior edge has two distinct physical incident triangles and there is just one exposed side. Subdividing that side by additional genuine marked points and subdividing its triangle into a fan gives the required triangulation for every number $m$ of marks. Each triangle has at most one exposed side and no internal edge is self-adjacent.

Every nonzero bipod and native tripod in this all-black $K(T)$ has native degree three. Every nonconstant special factor has degree at least three; in particular there are no degree-zero loop factors in the special families defining $z(T)$. Thus these entries are indecomposable specials. They are nonzero uniformly: take a faithful Schottky representation of the free surface group in $\mathrm{PSL}_2$,

lift to $\mathrm{SL}_2$, and compose with $\mathrm{Sym}^2$. Choose generic projective native decorations. The finitely many distinct lifts occurring in the triangulation then give distinct points on a nonsingular conic, any three of which are linearly independent. The tripod corners are distinct lifts. A nonzero bipod likewise uses three distinct lifts: otherwise its internal arc would have the same lifted endpoints as the neighboring boundary interval and would be boundary-parallel. For one mark the boundary word is a nontrivial element of a torsion-free free group, so $H^{-1}v, v, Hv$ are distinct projective lifts for a generic decoration.

There are $E = 6g + m - 3$ internal arcs and $t = 4g + m - 2$ triangles. Before the prescribed exposed identifications there are $2E + m + t$ nonzero degree-three entries, and each of the $m$ exposed sides mandates one spoke–center identification. The resulting formal list has

$$2E + t = 16g + 3m - 8$$

entries. The proved cardinality theorem [3, Theorem 12] gives exactly this number of distinct elements of $z(T)$. Since the entries are already nonzero and indecomposable and there are no additional factors, any further equality would contradict that theorem. This establishes all the seed identifications used above. The actual-seed theorem [3, Theorem 15] and the triangle blocks now apply. For a single mark the link has $12g - 3$ corners and the sequence has $12g - 5$ steps; with $m$ marks the total is $12g + m - 6$ steps. These formulas are not needed for the proof, but display its uniform finite nature.

3.5. **The disk and completion of the proof.** For an all-black disk, the original structure is Scott's Grassmannian structure, by [2, Theorem 8.5]. Signed reversal of the ordered vectors is its twisted reflection, which preserves clusters and the frozen cyclic minors and reverses the exchange arrows [4, Section 5]. Here signed reversal means $(v_1, \dots, v_N) \mapsto (-v_N, \dots, -v_1)$. It is exactly the reflection of the tensor construction: if a diagram has $t$ internal trivalent vertices, total boundary degree $d$, and $e$ edges, then $3t + d = 2e$ gives $(-1)^t = (-1)^d$. Reflection contributes the first sign by reversing each volume tensor, while negating all arguments contributes the second. Hence it sends the forward construction to the backward one with the actual normalized tensor functions. This proves the all-black disk assertion. The mixed disk assertion follows from the same fork embedding and Lemma 3.2; the factor-set statement in Lemma 3.3 retains the exceptional five-point coefficient lists. In the elementary polynomial conventions for smaller disks described in the preliminary section, the frozen generators are the native pairings and volume determinants; reversing direction preserves this same set directly.

We have proved the all-black assertion in all topologies where the original construction is defined. The fixed fork embedding and descent prove it for every admissible mixed signature. For multiple boundaries the collar argument can be repeated while leaving all other direction choices fixed. For one boundary its single reversal is the entire assertion. This proves Theorem 3.1.

## 4. Changing boundary colors

**Theorem 4.1.** *Complementing all native colors on any collection of boundary components preserves the mutation type of the principal mutable quiver.*

The assertion concerns principal quivers. It does not identify the invariant fields or the coefficient functions before and after a partial color change. We first establish the global color comparison with the actual native seed construction.

4.1. **Native and opposite triangle centers.** For a fixed signature and fixed proxy directions, let $\mathcal{A}^b$ denote the standard construction, whose triangle centers are determinants of vector proxies. Let $\mathcal{A}^w$ denote the construction using covector-proxy centers. The proxies themselves are unchanged. The second construction is the transport by global tensor duality of the standard construction for the complementary signature; it is consequently a cluster structure in the same invariant field.

Global duality uses the contragredient action on all holonomies. In particular, the original Laurent inclusion for $\mathcal{R}$ holds for both structures.

**Proposition 4.2.** *The native-center and opposite-center constructions give the same cluster structure. Their matrices agree in compatible ambient arrow conventions. Consequently global tensor duality transports the standard structure for a signature to the standard structure for its complement.*

We need an elementary factor argument.

**Lemma 4.3.** *Let $R$ be a factorial domain with units $\mathbb{C}^*$, containing a cluster algebra and contained in the Laurent ring of every cluster. Suppose the extended seed variables are pairwise nonassociate prime elements of $R$. Let $xw = M_1 + M_2 = P$ be a mutation at $x$, where $P$ is irreducible in the Laurent ring of the other seed coordinates. If no initial frozen variable divides $w$ in $R$, then $w$ is irreducible in $R$.*

*Proof.* Localizing $R$ at all initial coordinates gives exactly their Laurent ring: one inclusion is the assumed Laurent inclusion, and the other follows because those coordinates belong to $R$. In that ring $w = P/x$ is irreducible. If $w = fh$ in $R$, one factor, say $f$, becomes a Laurent unit. Factoriality and the primeness of the initial coordinates imply that $f$ is a constant times a monomial in them with nonnegative exponents.

No initial mutable coordinate can divide $w$ in $R$. For $x$ itself, the expression $w/x = w^2/P$ is not Laurent in the mutated seed, because $P$ is a nonunit independent of $w$. For another initial mutable coordinate $y$, mutate $y$ in the seed already containing $w$. Writing $yy' = P_y$, the expression $w/y = wy'/P_y$ cannot be Laurent there: a nonunit Laurent binomial cannot divide a Laurent monomial. The exceptional possibility $P_y = 2$ would make $y$ a unit of $R$, contrary to its assumed primeness as a nonunit. The hypothesis excludes all initial frozen divisors as well. Hence $f$ is constant. □

*Proof of Proposition 4.2.* The disk statement is [2, Theorem 8.11]. Suppose first that the surface is not a disk and has at least three black native marks on every boundary. Choose one anchor on each boundary, triangulate using the anchors, and then subdivide each exposed side by the remaining marks. We choose a regular taut triangulation: no triangle is self-folded and every triangle has at most one exposed side. Such triangulations exist in this setting [5, Section 2.2]; for a single boundary the handle-polygon construction in Section 3 supplies one directly. Subdivide the triangle at an exposed side into the fan from its opposite corner. All nonexposed triangles then have only anchor corners.

The degree, nonvanishing, and cardinality argument used above works unchanged for this all-black triangulation. Explicitly, for $N$ marks and $h$ boundaries there are

$$E = 6g + 3h + N - 6, \qquad t = 4g + 2h + N - 4$$

internal arcs and triangles. The nonzero degree-three entries number $2E + N + t$ before the $N$ exposed identifications, and therefore

$$2E + t = 3N + 8(2g + h - 2)$$

after them. Nonvanishing follows from a Schottky representation and the conic argument, also for the rank-one free group of an annulus. The degree-three entries cannot have nonconstant special factors of smaller degree. The exact cardinality theorem of [3, Theorem 12] excludes all further identifications. In particular, a nonexposed center and its six surrounding arc coordinates are distinct actual seed variables. The original irreducibility assertion for indecomposable specials makes them prime in the factorial invariant ring $\mathcal{R}$.

For an unexposed triangle with lifted corners $p, q, r$, write

$$a = \alpha_p(U_q),\ b = \alpha_p(U_r),\ c = \alpha_q(U_p), \quad e = \alpha_q(U_r),\ f = \alpha_r(U_p),\ h = \alpha_r(U_q).$$

The vanishing $\alpha_p(U_p) = \alpha_q(U_q) = \alpha_r(U_r) = 0$ gives the determinant identity

$$TT^* = aef + bch, \qquad T = \det(U_p, U_q, U_r), \quad T^* = \det(\alpha_p, \alpha_q, \alpha_r). \tag{4.1}$$

This is the actual center exchange. Its six coordinates are distinct, so its two monomials are coprime and their exponent difference is primitive. A Laurent binomial with primitive exponent difference is irreducible: a unimodular change of exponent coordinates makes it a unit times $1 + z$. Thus Lemma 4.3 applies once frozen divisors are excluded.

Let $u_j$ be the anchor and $w_j$ its immediate neighbor in the selected proxy direction on boundary $j$. A nonexposed $T^*$ uses only these two physical native vectors on that boundary, through transported copies of $u_j \times w_j$. Every coefficient on this stabilized all-black boundary is a determinant of three consecutive natives. Since there are at least three genuine marks, those involve three distinct physical native coordinates. Each coefficient therefore has positive degree in a native coordinate in which $T^*$ has degree zero. The invariant ring has a nonnegative native multigrading, so that coefficient cannot divide $T^*$. The lemma proves that the whole opposite center is irreducible. This argument is the reason for adding genuine marks before treating small boundary components.

Distinct unexposed centers are not adjacent, and their mutations commute. At an exposed side from $p$ to $q$, the actual identity $\alpha_p = U_p \times U_q$ instead gives

$$a = 0, \qquad T = b, \qquad T^* = ch.$$

The opposite center adds no new factor. With the reverse proxy direction the identities are $c = 0$, $T = e$, $T^* = af$. Consequently the opposite-center factor list is precisely the list after mutating all unexposed centers.

We also compare the full matrices. Use the printed nine-arrow interior block

$$Q = (a, b, T) + (e, c, T) + (f, h, T),$$

and let $S$ be the three side arrows $a \to c, e \to h, f \to b$. Let $P$ interchange the two arc slots on each side, namely $P = (a\ c)(b\ f)(e\ h)$. Applying the mutation formula to the three incoming and three outgoing center arrows gives

$$PSP^{-1} = -S, \qquad \mu_T(Q) = PQP^{-1} - S, \qquad P\mu_T(Q)P^{-1} = Q + S. \tag{4.2}$$

For instance each length-two path through $T$ supplies an arrow between one member of $\{a, e, f\}$ and one of $\{b, c, h\}$; three cancel the old corner arrows and the other six give exactly the right side of the middle equality. The opposite-color block pulled back by duality is $PQP^{-1}$, so the difference from the mutated source is $S$.

At an exposed triangle this is an identity of the actual distilled blocks. For $a = 0, T = b$, their arrow lists are

| | |
|---|---|
| native center | $e \to c,\ c \to b,\ b \to e,\ f \to h,\ h \to b,\ b \to f$ |
| opposite center | $e \to c,\ c \to b,\ b \to e,\ f \to h,\ h \to b,\ e \to h.$ |

Their difference is $e \to h + f \to b$, the surviving part of $S$. This is the white exposed block of [3, Figure 9] after duality; it is not a mutation at the reducible function $ch$. The fold $c = 0, T = e$ gives the same conclusion in the reverse convention. The contributions $S$ cancel on every internal side, and on a boundary side one endpoint is the prescribed zero function. Hence the entire glued matrices agree, including coefficient rows. Merging repeated arc labels commutes with the center mutations, since the center occurs in only its own block, its six neighboring functions are distinct, and copies from other blocks have zero entry in its column.

For an arbitrary original signature, expand each white mark through the fixed physical fork and add genuine black marks until every boundary has at least three. Denote the resulting injective map by $\Phi$. The functoriality of the standard construction gives $\Phi(\mathcal{A}^b)$ as a freeze/delete substructure of the stabilized $\mathcal{A}^b$. Dual transport of the same functoriality gives $\Phi(\mathcal{A}^w)$ as a substructure of the stabilized $\mathcal{A}^w$, through the *same* $\Phi$: dualizing a fork interchanges the two allowed fork operations,

with its cyclic port order fixed, and deleting a native coordinate dualizes to deleting its dual native coordinate. This uses only the definition of $\mathcal{A}^w$, not the assertion being proved. The stabilized ambient structures are equal by the preceding construction. The embedded fraction fields of the two original structures are $\Phi(\operatorname{Frac}\mathcal{R})$, and their final coefficients are the same boundary bipod factors, whose definition does not involve the choice of center. Lemma 3.2 gives equality of the original structures; the fixed-face observation following that lemma gives their exact induced matrices. Together with the disk theorem, this proves the proposition. □

4.2. **Primitive cross-boundary functions.** For the relative color change assume there are at least two boundary components. On an exterior boundary choose a same-color pair as the anchor, using global duality simultaneously in the two models if necessary to make it black. Direction invariance allows either neighbor to be used. Its actual flag has the form

$$(u, \beta), \qquad \beta = u \times w \quad \text{or} \quad \beta = u \times Ku. \tag{4.3}$$

The second case is a one-point anchor. Choose the selected boundary holonomy $H$ as a free generator and omit the anchor boundary holonomy. Then $K = G_1 H^{-1} G_2$, with the remaining holonomies in $G_1, G_2$. For fixed other data $H \mapsto K$ is an invertible change of $\mathrm{SL}_3$ coordinates. We retain this dependence throughout.

Work first on the unreduced configuration variety, a product of vector spaces, covector spaces, and $\mathrm{SL}_3$ factors. It is smooth, irreducible and factorial. The zero locus of $\beta$ has codimension at least two. This is immediate for $u \times w$. For $u \times Ku$, over $u \neq 0$ the condition $Ku \in \langle u \rangle$ has codimension two in $\mathrm{SL}_3$, while $u = 0$ has codimension three. The coordinate change just described proves the same statement with shared holonomy. An invertible transport does not alter this zero locus.

**Lemma 4.4.** *Let $Z$ be a vector- or covector-valued polynomial with zero locus of codimension at least two, and let $z$ be a new independent native of the same type. The zero locus of $z \times Z$ has codimension at least two. This remains true for every alternating cross-product chain obtained by adjoining independent native leaves, beginning with a native $p$ or an invertible transport of $p$.*

*Proof.* Over $Z \neq 0$, the equation says $z \in \langle Z \rangle$, two conditions on the three independent coordinates of $z$. Over $Z = 0$ the base already has codimension at least two. Induction gives the assertion; the initial native or transported native has zero locus $p = 0$ of codimension three. □

**Lemma 4.5.** *Let $p$ be a native target covector and $t$ a covector involving only exterior natives and holonomies, independent of the target natives, with $\{t = 0\}$ of codimension at least two. Let $U$ be the actual vector proxy rooted at $p$. Assume that the target has more than one mark, and, if its caterpillar wraps back to its first physical mark, assume*

$$\det(H^{\pm 1} p, p, t) \not\equiv 0, \tag{4.4}$$

*where the action on covectors is the dual action. Then $t(U)$ is irreducible. The assertion with vectors and covectors interchanged also holds.*

*Proof.* Read the caterpillar to its first same-color pair. After invertible changes of native coordinates, all intermediate leaves are independent; only the last one can repeat the first as $H^{\pm 1} p$. If the first pair has the same color, $U = p \times r$ with $r$ an independent native covector. The coefficients of the linear function $t(U)$ in $r$ vanish only where $p, t$ are proportional or $t = 0$, a set of codimension at least two.

Otherwise the next native is an independent vector $v$ and $U = p \times (v \times X)$. The contraction is

$$t(U) = v \cdot \big(p(X)t - t(X)p\big). \tag{4.5}$$

The locus where $p, t$ are dependent again has codimension at least two. On its complement put $L = p \times t \neq 0$. The three coefficients in (4.5) vanish precisely when $X \in \langle L \rangle$. If $X$ is an independent native vector, this imposes two conditions. Otherwise $X = r \times Y$, where $r$ is a new independent

covector and $Y$ is the remaining covector chain, independent of $v, r$. Lemma 4.4 treats $Y = 0$. For $Y \neq 0$ the map $r \mapsto r \times Y$ has a one-dimensional kernel and image $\ker Y$. If $Y(L) \neq 0$, the line $\langle L \rangle$ meets this plane only at zero, so there are two conditions on $r$. If $Y(L) = 0$, there is one condition on $r$ and one on the base, provided $Y(L)$ is not identically zero.

That nonidentity follows from free tail leaves, or from (4.4) if no such leaves remain. For clarity, two alternating free leaves allow a covector $w \times (s \times Z)$. For a prescribed covector $A$ not proportional to a nonzero $Z$, choose $w \in \ker A$ with $Z(w) \neq 0$ and $s = A/Z(w)$. The triple-product identity gives $w \times (s \times Z) = A$. The tail therefore takes values in a dense set and need not annihilate $L$. A free terminal native has the same property directly. Thus the common-zero set of the linear coefficients has codimension at least two in every case.

Off that set, the hypersurface $t(U) = 0$ is the rank-two kernel bundle of a nonzero linear functional on a trivial rank-three bundle over an irreducible open base. It is irreducible and reduced there. Over the exceptional base set the full fiber has dimension at most one less than a hypersurface component, so no extra component can occur. Equivalently, the linear polynomial is primitive and remains linear over the base fraction field, and Gauss's lemma applies. This proves irreducibility. Interchanging the two volume forms gives the stated dual proof with exactly the same dimension estimates. □

The condition (4.4) holds for the actual cross-boundary spokes. For a two-native anchor, $\beta = u \times w$ varies freely on a dense set independently of target natives and $H$. For a one-point anchor set the other group variables to the identity, so $K = H^{-1}$. The legitimate specialization

$$H = \operatorname{diag}(1, 2, 1/2), \quad u = (1, 1, 1), \quad p = (1, 1, 0), \quad \beta = (3/2, -1, -1/2)$$

gives

$$\det(H^{-T} p, p, \beta) = -1/4, \qquad \det(H^{T} p, p, \beta) = 1/2.$$

The extra end-spoke transport specializes to a power of this diagonal $H$ and retains a nonzero third coordinate of $\beta$, so the same nonidentity holds there. For the dual spokes with exterior native $u$, it is the elementary nonidentity $\det(H^{\pm 1} v, v, u) \not\equiv 0$. We conclude that every raw cross-boundary spoke is irreducible for arbitrary mixed targets. When a spoke uses a native of the correct type it is simply a primitive linear pairing. This also covers monochromatic targets with at least two marks.

4.3. **A single white target mark.** The case of one white native has no independent intermediate leaf, and requires its own argument. Let its covector be $\alpha$. Its reverse vector proxy is $w = (H^T \alpha) \times \alpha$. The two spokes are the quadratics

$$q_t(\alpha) = \det(t, H^T \alpha, \alpha), \qquad t = \beta \quad \text{or} \quad t = H^T \beta.$$

For nonzero $t$, an identically zero quadratic forces

$$H^T = aI + t\ell^T. \tag{4.6}$$

Indeed, in a basis with $t = e_3$, write $H^T = (h_{ij})$. The five coefficients are those of

$$-h_{21}x^2 + h_{12}y^2 + (h_{11} - h_{22})xy - h_{23}xz + h_{13}yz.$$

Their vanishing is precisely (4.6). If $t(u_*) = 0$, this identity implies $Hu_* = au_*$. For the first spoke take $u_* = u$, and for the second take $u_* = H^{-1}u$; in the second case $t(u_*) = \beta(u) = 0$. Each coefficient-zero locus is therefore contained in

$$\{t = 0\} \ \cup \ \{Hu \in \langle u \rangle\},$$

which has codimension at least two, including the shared-holonomy anchor in (4.3). Notice that the translated pairing uses $H^T \beta$, not the dual group action $H^{-T} \beta$.

Both quadratics have generic rank three. At $H = \operatorname{diag}(1, 2, 1/2)$, $u = (1, 1, 1)$ and $\beta = u \times H^{-1}u = (3/2, -1, -1/2)$, they are

$$\tfrac{1}{2}xy + \tfrac{1}{2}xz + \tfrac{9}{4}yz, \qquad \tfrac{1}{4}xy + xz + \tfrac{9}{4}yz.$$

Their Hessian determinants are both $9/8$. This one nonzero determinant proves generic rank three. A ternary quadratic of rank three is irreducible even over the algebraic closure of the base fraction field; the absence of a coefficient-content divisor excludes a vertical factor. Thus both actual spokes are irreducible.

The closing function has the exact form

$$C = \det(u, Hw, Hu) = (u \times H^{-1}u)(w) = \alpha(Hu)\alpha(H^{-1}u) - \alpha(u)^2. \tag{4.7}$$

It is the same quadratic with $t = u \times H^{-1}u$ and has the same content and generic-rank proof. It is consequently irreducible as well. For at least two white target marks, the two covectors in a proxy are independent, and the primitive linear argument of the preceding subsection proves the corresponding statement.

4.4. **The relative color comparison.** For a mixed target, choose a cut at a transition from white to black. Number the old marks so that $p_0$ is white and $p_1$ black, and use forward proxies. In the complemented signature use reverse proxies and begin the fan at the old $p_0$, now black. Thus both closing triangles use a native black target vector. Shifting the starting spoke does not change the seam based at the exterior anchor. For a monochromatic target use the same comparison without a distinguished transition; the one-step shift has no effect on its constant color word.

Write the actual completed flag at target mark $i$ as $(U_i, \alpha_i)$, put $a_i = \alpha_i(u)$, $b_i = \beta(U_i)$, and let $c$ denote the nonzero coefficient on a boundary triangle. The forward recursion is

$$\varepsilon_i = B:\ \alpha_i = U_i \times U_{i+1}, \qquad \varepsilon_i = W:\ U_i = \alpha_i \times \alpha_{i+1}.$$

For reverse proxies the corresponding equations use the right endpoint:

$$\varepsilon_{i+1} = B:\ \alpha_{i+1} = U_i \times U_{i+1}, \qquad \varepsilon_{i+1} = W:\ U_{i+1} = \alpha_i \times \alpha_{i+1}.$$

The four spoke slots $210, 120, 201, 102$ are $(b_i, a_i, b_{i+1}, a_{i+1})$. The remaining actual slots are

| | 021 | 012 | 111 | dual center |
|---|---|---|---|---|
| forward, left $B$ | 0 | $c$ | $a_i$ | $cb_{i+1}$ |
| forward, left $W$ | $c$ | 0 | $ca_{i+1}$ | $b_i$ |
| reverse, right $B$ | $c$ | 0 | $a_{i+1}$ | $cb_i$ |
| reverse, right $W$ | 0 | $c$ | $ca_i$ | $b_{i+1}$. |

(4.8)

These equations follow from the cross-product recursions and (4.1). The white rows use the actual opposite-center exposed block; their reducible native center is not treated as a mutable variable.

We verify the factor list before comparing matrices. All spokes are irreducible by Lemma 4.5 and the one-white-mark argument. For the mixed cut the closing variable $C = \det(u, HU_1, Hu)$ is linear in the independent native black $U_1$, and its coefficient vector is an invertible transport of $u \times H^{-1}u$. Its coefficient-zero locus has codimension at least two, proving its irreducibility. The selected boundary has exactly its original coefficient factors, by [3, Proposition 9]; every exterior factor is unchanged. Every spoke has positive selected-boundary native degree, while all seam and exterior factors have degree zero there, so they cannot coincide. The only prescribed closing folds are (3.6), which are identities for every completed target vector and hence occur identically on both sides.

There are no additional hidden identifications. Compare the formal factor list with that of the all-black target collar on the same marked surface and with the same, possibly mixed, exterior. Lemma 3.4 identifies the latter list without any additional coincidences. It consists of $2(m+1)$ spokes, one closing variable, the $m$ target coefficients, and that exterior factor list, with exactly the same prescribed annulus fold. Equations (4.8) express every entry of $K(T)$ in these factors, and all the factors occur. The all-black count is the exact cardinality $3N + 8(2g + h - 2)$ of [3, Theorem 12]. The topology and $N$ have not changed. An additional equality of factors in the mixed list would reduce its cardinality, contradicting that theorem. This argument also excludes equality of the two

holonomy-translated end spokes. It establishes the actual seed list before any matrix comparison is made.

For explicit arrows set $a = a_i$, $A = a_{i+1}$, $b = b_i$, $B = b_{i+1}$. In the side-augmented convention define

$$P_0 : b \to B, \quad B \to a, \quad a \to A, \quad A \to B,$$
$$R_0 : b \to B, \quad A \to b, \quad b \to a, \quad a \to A.$$

The four rows of (4.8) have the following complete distilled boundary fragments:

(4.9)

| | principal fragment | coefficient arrows |
|---|---|---|
| forward $B$ | $P_0$ | $A \to c \to a$ |
| forward $W$ | $R_0$ | $B \to c \to b$ |
| reverse $B$ | $R_0$ | $A \to c \to a$ |
| reverse $W$ | $P_0$ | $B \to c \to b$. |

They are the one-exposed-side rule of [2, Figure 47], which depends only on the indicated endpoint color and permits the other endpoint to have either color. For the printed nine-interior-arrow convention, subtract the side arrows $b \to a$ and $A \to B$ on both sides. Its two lists become

$$P_{mint} : b \to B, B \to a, a \to b, a \to A; \qquad R_{mint} : b \to B, A \to b, a \to A, B \to A.$$

Thus the actual slot positions, not just abstract quiver shapes, are fixed by the comparison.

We provide the local exchange calculation including transitions. At an internal spoke use subscripts $-, 0, +$ for its neighboring positions, suppress 0, and write $c_-, c_+$ for the two boundary coefficients. The numerators of the two exchange relations are

(4.10)

| $(\varepsilon_{i-1}, \varepsilon_i)$ | column at $a$ | column at $b$ |
|---|---|---|
| $BB$ | $c_-a_+b + c_+a_-b_+$ | $a_-b_+ + ab_-$ |
| $BW$ | $c_-a_+ + a_-$ | $a_-b_+ + c_+a_+b_-$ |
| $WB$ | $a_+b_- + c_+a_-b_+$ | $c_-b_+ + b_-$ |
| $WW$ | $a_+b_- + a_-b$ | $c_-ab_+ + c_+a_+b_-$. |

To derive them, put

$$T_- = \det(u, U_{i-1}, U_i), \quad T_+ = \det(u, U_i, U_{i+1}),$$
$$D_- = \det(\beta, \alpha_{i-1}, \alpha_i), \quad D_+ = \det(\beta, \alpha_i, \alpha_{i+1}).$$

The vector and covector determinant identities are

$$b \det(u, U_{i-1}, U_{i+1}) = b_-T_+ + b_+T_-,$$
$$a \det(\beta, \alpha_{i-1}, \alpha_{i+1}) = a_-D_+ + a_+D_-.$$

Substituting the four center values in (4.8) gives (4.10). For the $BW$ column at $a$, the original special relation has the common factor $b$; for the $WB$ column at $b$ it has the common factor $a$. Cancel these prescribed special factors. There is no such factor in the other six columns. This is precisely the distillation of the local skein relations in the original construction, now with all their factors identified. The same local relations hold in contractible lifts of surface triangles; no claim about arbitrary specialization of full-flag seeds is involved.

After complementing colors and shifting the cut, the right endpoint color in the new reverse triangle is the complement of the old left endpoint color. Therefore (4.9) gives equality of the principal boundary fragments one by one. The closing triangle has the same seven slots in both models:

$$(210, 120, 201, 102, 021, 012, 111) = (b_{m+1}, a_{m+1}, E, F, a_1, b_1, C). \tag{4.11}$$

Its whole block, both seam slots, and the exterior attachments agree. The same exterior factor substitutions and the same folds (3.6) are applied to equal principal matrices, so preserve their equality. Triangle side arrows cancel along internal arcs as before. This proves equality of the actual

principal quivers for the forward original collar and the reverse complemented collar, with the entire exterior attached.

*Proof of Theorem 4.1.* If there is only one boundary, complementing its colors is global duality, so Proposition 4.2 proves the claim. For at least two boundaries the relative comparison just constructed handles a chosen component, whether monochromatic or mixed. The single white case is supplied by (4.7) and its quadratic argument. Theorem 3.1 puts the complemented construction back into any requested proxy convention. The exterior anchor can be chosen as in (4.3) by applying global duality simultaneously to the two models; the proved global comparison transports the resulting sequence back to their standard structures. Repeating this one-component comparison handles any selected collection of boundary components. The elementary small disk cases have empty principal mutable quivers, as in the conventions of the preliminary section. This completes the proof. □

*Remark* 4.6. The restriction to mutation type for a partial color change is essential to this proof. On an all-black collar the substitution $\alpha_i = v_i \times v_{i+1}$ gives

$$\alpha_{i-1} \times \alpha_i = d_{i-1} v_i, \qquad \det(\alpha_i, \alpha_{i+1}, \alpha_{i+2}) = d_i d_{i+1}.$$

For even $m$, the exponent map $d_i \mapsto d_i d_{i+1}$ has an alternating kernel. It cannot be used as an isomorphism of the full coefficient fields. The proof instead compares actual irreducible seed factors and principal exchange matrices directly.

## 5. Generation with ordinary boundary decorations

In this section we prove the localized equality except when there is a single boundary component with a single marked point. We write

$$B = \mathcal{A}[\mathcal{D}^{-1}], \qquad R_D = \mathcal{R}[\mathcal{D}^{-1}],$$

where $\mathcal{D}$ is generated by the original coefficient variables. All intermediate membership assertions refer to $\mathcal{A}$ or to an explicitly displayed localization of $B$. In particular, a coefficient of a smaller surface is not automatically a unit of $B$.

5.1. **Boundary frames and simple-arc coordinates.** The frame arguments below concern surfaces that are not disks; the disk case is recalled separately at the end. This excludes the small-disk factorization exceptions from every assertion about a nonzero adjacent coefficient.

Fix a trivialization along a lift of a boundary component. At its $i$th root let $u_i \in V$ and $a_i \in V^*$ be the two canonical proxies. With the volume conventions fixed in the preliminaries their recursion is

$$\begin{cases} u_i = v_i, & a_i = u_i \times u_{i+1}, \quad i \text{ black}, \\ a_i = \phi_i, & u_i = a_i \times a_{i+1}, \quad i \text{ white}. \end{cases} \tag{5.1}$$

The recursion terminates at the first pair of equal colors. If the boundary has $n$ roots and monodromy $C$, then

$$u_{i+n} = C u_i, \qquad a_{i+n} = a_i C^{-1}. \tag{5.2}$$

These identities, including their transports, are used throughout.

Let $c_i$ be the nonzero special invariant of the boundary side $(i, i+1)$; explicitly

$$c_i = a_{i+1}(u_i) \quad (i \text{ black}), \qquad c_i = a_i(u_{i+1}) \quad (i \text{ white}).$$

The special factorization theorem says that its factors are original coefficients. Thus every $c_i$ is a unit of $B$.

**Lemma 5.1** (Boundary frames)**.** *Suppose that $i, i+1$ have the same color. The column frame $F = (u_i, u_{i+1}, u_{i+2})$ and row frame $G = (a_i; a_{i+1}; a_{i+2})$ have determinants that are original coefficient monomials. For a black pair their determinants are $c_i$ and $c_i c_{i+1}$; for a white pair they are $c_i c_{i+1}$ and $c_i$, respectively.*

*Proof.* For a black pair put $\delta = \det(u_i, u_{i+1}, u_{i+2})$ and $\epsilon = a_{i+2}(u_{i+1})$. Formula (5.1) gives $\delta = c_i$, $\epsilon = c_{i+1}$ and

$$(u_i \times u_{i+1}) \times (u_{i+1} \times u_{i+2}) = \delta u_{i+1}.$$

Pairing this vector with $a_{i+2}$ proves $\det G = \delta\epsilon$. For a white pair the identical tensor identity on $V^*$ gives

$$\det G = c_i, \qquad u_i \times u_{i+1} = c_i a_{i+1}, \qquad \det F = c_i c_{i+1}.$$

All assertions concern the transported proxies on the specified lift. □

We record the geometric coordinate constructions used below. A *positive port convention* for a simple based loop means that its two germs, after the basepoint is opened into a small boundary interval, are ordered so that the returning germ can be moved forward along that interval without crossing the outgoing germ. Each unoriented simple loop has one of its orientations in this convention. The conventions are unchanged by a boundary Dehn twist.

**Lemma 5.2** (One strip and a later frame)**.** *Let $x_1, x_2, x_3$ be distinct physical roots in a proper boundary interval, and put $F = (u_{x_1}, u_{x_2}, u_{x_3})$. If $H$ is the transport of a simple based loop in the positive port convention, then the three signed Cramer numerators of $Hu_{x_1}$ in $F$ belong to $\mathcal{A}$. More generally, suppose that $y, x_1, x_2, x_3$ occur in this order in a proper lifted boundary interval, allowing $y = x_1$. The three Cramer numerators of $Hu_y$ in $F$ also belong to $\mathcal{A}$. Whenever $\det F$ is a unit of the ring under consideration, the corresponding column belongs to that ring.*

*Proof.* Move the auxiliary basepoint to the beginning of the interval. A thin regular neighborhood of the interval and the loop is an annular strip. On its universal cover the relevant roots have order

$$x_1, x_2, x_3, Hx_1 \quad \text{or} \quad y, x_1, x_2, x_3, Hy.$$

The triangles joining $Hx_1$, or $Hy$, to each pair among $x_1, x_2, x_3$ can be drawn inside one open fundamental strip. Their interiors therefore project injectively to the surface. An unused boundary root is placed outside a side by pushing that side slightly into the collar. When two corners have the same physical root, their incidences are distinct; their joining side is the noncontractible based arc. These are precisely the repeated-corner triangles permitted in the special-invariant construction of [3, Section 6]. Each triangle extends to a triangulation. After attaching its canonical caterpillars, its tensor is the appropriate lower special invariant. Its irreducible special factors belong to that triangulation seed. Its value, and hence any signed Cramer numerator, belongs to $\mathcal{A}$. The caterpillars may overlap; this does not change the definition of a special invariant.

The transport assertion is also literal. To move a based loop along a boundary interval, take two parallel copies of the interval in a thin collar, attach them to the two loop germs in their noncrossing order, and round the corners. The loop remains simple. Move the frame along the same interval. If $F' = FP$ in the original trivialization, the new matrix is $P^{-1}HP$. Thus returning by $P$ measures the same $H$, not a holonomy with one endpoint transported independently. Formula (5.2) records a crossing of the chosen boundary cut. □

We shall also use the dual version of Lemma 5.2 inside the *same* original algebra. Here is its seed interface. An upper special triangle with an exposed side $(i, i+1)$ reduces to a bipod when $i$ is white. When $i$ is black, the identity

$$a_i \times a_{i+1} = a_{i+1}(u_i)u_{i+1}$$

expresses it as the product of the adjacent coefficient and the appropriate bipod from $i+1$ to the third corner. For a triangle with no exposed side, the ordinary triangle relation of [3, equation (5)

and Figure 8], after removal of its common special factors, is the actual mutation at its center. The upper triangle consists of the new mutable factor and the already present factors. It consequently belongs to $\mathcal{A}$ as an ordinary cluster monomial. This argument also permits repeated physical corners. It proves the dual strip lemma using upper triangles without any assumption about global color reversal.

The following two observations will make the subsequent generation arguments transparent. First, Cramer coordinates of boundary decorations against a fixed unit proxy frame are short boundary triangles and bipods. When their roots lie in one interval of length at most one period, with only its first and last roots possibly identified, the same strip construction applies. Thus the coordinates lie in $B$. Second, if $\delta = \det F$ is an original coefficient unit, then

$$g_F = \operatorname{diag}(1, 1, \delta)F^{-1} \in \mathrm{SL}_3(R_D). \tag{5.3}$$

If the coordinates of all native decorations and all free holonomy generators in $F$ lie in $B$, they do so in this determinant-one gauge. The invariant-theoretic generators in [3, Theorem 1]—traces of holonomy words, word-decorated pairings, and vector and covector volumes—then belong to $B$. This proves $R_D = B$. Matrix inverses introduce no denominator because they are adjugates.

5.2. **At least two boundary components.** Suppose first that the surface has at least two boundary components. Choose roots $p, q$ on different components and a simple arc $\eta : p \longrightarrow q$. There are simple arcs $\gamma_1, \ldots, \gamma_c : p \longrightarrow q$ for which $\gamma_j\eta^{-1}$ generate the fundamental group. To see this, take a standard cut graph based at $p$, with simple free-generator petals and a tree branch $\eta$ leading to $q$. For each petal, choose its orientation so that its returning germ can be joined to the branch in the boundary half-disk without crossing its outgoing germ. Rounding this junction gives $\gamma_j$. Each arc is required to be simple individually; no pairwise disjointness assertion is needed.

The endpoint of a cross-boundary arc can be moved to any specified root and any specified lift. A partial turn is a collar slide and a whole turn is a Dehn twist about that boundary. The two collars are disjoint, so the two endpoint choices are independent. Write $T_\gamma : V_q \to V_p$ for inverse parallel transport along $\gamma$. Normalize the frames of Lemma 5.1 to determinant one by multiplying one column or row by an original coefficient unit. The cross-boundary bipods give

$$G_pT_\gamma F_q \in \mathrm{SL}_3(B). \tag{5.4}$$

Each entry is an actual special bipod on the endpoint-slid simple arc, multiplied by a coefficient unit.

Assume that one component, chosen as that of $p$, has at least three physical roots. Its frame $F_p$ uses three consecutive proxy roots following an equal-color pair. For each cross-boundary arc $\gamma$, the three Cramer numerators for each column of $T_\gamma F_q$ are special tripods. Here one may take the interval of two boundary sides supporting $F_p$ and the initial stem of $\gamma$. Its thin closed regular neighborhood is a disk. Inside it, any two specified frame roots and the other endpoint of $\gamma$ bound the required triangle. The omitted root is put outside the triangle. This uses the fixed two-side interval, which is shorter than a period, rather than its complement. Moving the $q$-end lift moves this entire construction by a twist. Consequently

$$L_\gamma = F_p^{-1}T_\gamma F_q \in \mathrm{SL}_3(B). \tag{5.5}$$

The matrices $L_{\gamma_j}L_\eta^{-1}$ are all free generators in the $F_p$ gauge.

For completeness, both types of native decoration can be recovered without treating $G_p$ as $F_p^{-1}$. Vectors off the $p$-component use the Cramer construction just given. For a vector $v$ on that component, pair it along a cross-boundary arc with $G_q$, and multiply the resulting column by $(G_qT_\eta^{-1}F_p)^{-1}$. A native covector off the $q$-component can first be paired along an arc with $F_q$, then its row is multiplied by $L_\eta^{-1}$. Covectors on the $q$-component pair directly across to $F_p$. The paths used in these four constructions may differ from the original cut-system paths by holonomy words,

all of whose matrices have already been obtained. Thus all native coordinates belong to $B$, and (5.3) finishes this case.

It remains to treat the case when every component has one or two roots. The nonalternating condition makes each such component monochromatic. If the $p$-component is black, use the determinant-one frame $G_p^{-1}$. Put $K_\gamma = G_p T_\gamma F_q$. Equation (5.4) gives $K_\gamma \in \mathrm{SL}_3(B)$ and $K_{\gamma_j} K_\eta^{-1}$ gives the free generators. Pair each required covector proxy $a_r$ on the $p$-component across to $F_q$, then multiply by $K_\eta^{-1}$. This recovers its row in the chosen gauge, including the predecessor lifts. The black recursion gives

$$u_r = \frac{a_{r-1} \times a_r}{\det(u_{r-1}, u_r, u_{r+1})}. \tag{5.6}$$

Its denominator is an original adjacent coefficient unit. Hence all native vectors of the base component and its proxy frame are recovered. Vectors on other components pair directly across with $G_p$; covectors there pair with the recovered $F_p$, whose known determinant is a unit.

If the base component is white, use $F_p$ instead. The matrices $Q_\gamma = G_q T_\gamma^{-1} F_p$ belong to $\mathrm{SL}_3(B)$, and

$$Q_\eta^{-1} Q_\gamma = F_p^{-1} T_\eta T_\gamma^{-1} F_p$$

generate the same free group, with generators possibly inverted. Cross-boundary pairings recover the required $u_r$ on the base component. The dual identity

$$a_r = \frac{u_{r-1} \times u_r}{\det(a_{r-1}, a_r, a_{r+1})}$$

then recovers its native covectors and $G_p$. Covectors on the other components pair with $F_p$, and vectors pair with this recovered $G_p$. This is a calculation in the original white decoration algebra. In every case the criterion following (5.3) proves

$$B = R_D \qquad \text{if there are at least two boundary components.} \tag{5.7}$$

5.3. **A root slice in the ordinary cluster algebra.** The remaining arguments use a particular consequence of the freezing theorem. We state all assumptions, including those controlling coefficient inverses. Let $\mathcal{A} \subset R \subset U_{\mathrm{ext}}$ be domains, where $\mathcal{A}$ is an ordinary geometric-type cluster algebra and $U_{\mathrm{ext}}$ is the intersection of Laurent rings with the original coefficients also inverted. Suppose an actual seed contains a frozen variable $s$ and a mutable variable $\ell$, and that a locally nilpotent derivation $E$ of $R$ satisfies

$$E = \ell \frac{\partial}{\partial s} \tag{5.8}$$

in its rational function field. In particular it fixes all other seed coordinates.

**Lemma 5.3** (Ordinary root slice)**.** *Under these assumptions the mutable support of the $s$-row is contained in $\{\ell\}$. The finite Taylor operation*

$$\pi_E(z) = \sum_{k \geq 0} \frac{(-s/\ell)^k}{k!} E^k z \tag{5.9}$$

*is a homomorphism $R[\ell^{-1}] \to (\ker E)[\ell^{-1}]$ and satisfies $\pi_E(\mathcal{A}) \subset \mathcal{A}[\ell^{-1}]$. For an ordinary cluster monomial $z$ of maximal $E$-degree $d$,*

$$S_E(z) = \ell^d \pi_E(z) \in \mathcal{A}, \qquad E^{(d)} z = E^d z / d! \in \mathcal{A}. \tag{5.10}$$

*When the $s$–$\ell$ entry is nonzero, each nonzero expression in* (5.10) *is an ordinary cluster monomial up to a nonzero scalar. No assertion is made here about intermediate derivatives of arbitrary elements of $\mathcal{A}$.*

*Proof.* If another mutable $x_j$ has a nonzero $s$-row entry of absolute value $m$, its exchange numerator has one term $s^m M$ and one term independent of $s$. Applying the maximal divided derivative to the new variable gives $\ell^m M/x_j \in R$. In the $j$-mutated seed this has the nonunit exchange binomial in its denominator and a Laurent monomial in its numerator. Such a binomial does not divide a Laurent monomial. This contradicts $R \subset U_{\mathrm{ext}}$. The row-support assertion follows and persists along all mutations avoiding $\ell$.

Assume first that the remaining entry is nonzero. Reversing all arrows does not change any exchange relation; choose that convention so that $s$ has positive exponent in $\widehat{y}_\ell$. In the separation formula $z = x^g F_z(\widehat{y})$ the lowest $s$-face is precisely the $n_\ell = 0$ face of $F_z$. The freezing theorem [9, Theorem 5.11] says that this face is a localized cluster monomial reached by a mutation word avoiding $\ell$. This application does not require a rank assumption on the original matrix. Indeed, append an identity matrix of new frozen rows, so that the extended matrix has full column rank. Lift the given mutation word and its cluster monomial, apply the full-rank theorem, and specialize every new coefficient to 1. Each exchange relation specializes to the original relation, and taking the $n_\ell = 0$ face commutes with this specialization. The resulting mutation word still avoids $\ell$.

Local nilpotence and (5.8) imply that $z$ is a polynomial in $s$ over the field generated by the other seed coordinates. Thus its lowest $s$-exponent is nonnegative. If it is positive, (5.9) is zero; otherwise (5.9) is the freezing face. Its only potentially negative mutable exponent in the output seed is that of $\ell$.

We check the original frozen exponents separately. For an original frozen variable $d_0$, take the formal $d_0$-adic valuation in the initial rational function field. Under a mutation the other residual coordinates remain algebraically independent. If the $d_0$-row entry is nonzero, the residual exchange numerator is one nonzero monomial; if it is zero, it is the sum of two positive monomials in independent residual coordinates. It is nonzero in either case. The new residual variable is that numerator divided by the old residual variable, which is a birational change of coordinates. Induction proves that each mutable has $d_0$-value zero, and the valuation of a Laurent polynomial in any seed is its lowest $d_0$-exponent. Ordinary cluster monomials have nonnegative such exponents. Selecting an $s$-face does not introduce any negative exponent of another original frozen variable. The constant face has $s$-exponent zero. Hence the frozen exponents of the output monomial are nonnegative, proving $\pi_E(z) \in \mathcal{A}[\ell^{-1}]$.

Since $E(s/\ell) = 1$, the finite Taylor formula is a homomorphism by Leibniz's rule. This proves the assertion for all of $\mathcal{A}$. The homogeneous expression in (5.10) belongs to $R$, as is clear from its finite sum with polynomial coefficients. It is a localized cluster monomial in the output seed. If its $\ell$-exponent were negative, mutation at $\ell$ would introduce a nonunit exchange-binomial denominator that cannot divide its monomial numerator, contradicting $R \subset U_{\mathrm{ext}}$. The formal valuations already exclude negative original frozen exponents. This proves the first assertion of (5.10).

Apply the same freezing theorem with the opposite arrow convention to select the highest $s$-face. Its monomial is $s^d$ times the leading $s$-coefficient. Multiplying by $\ell^d s^{-d}$ gives $E^{(d)}z$. In the initial Laurent expansion this deletes the highest $s$-power and introduces only a nonnegative power of $\ell$. All original frozen valuations are still nonnegative. This expression belongs to $R$; the same neighboring-seed argument excludes a negative $\ell$-exponent in the output seed. It is an ordinary cluster monomial up to scalar.

Finally, if the $s$-row is zero, every mutable variable in every seed is independent of $s$. A cluster monomial has the form $s^d M$ with $M$ independent of $s$. Its slice is 0 for $d > 0$ and $M$ for $d = 0$; its highest divided derivative is $\ell^d M \in \mathcal{A}$. This proves exactly the membership assertions stated, without claiming compatibility of $\ell$ with the factors of $M$. □

We will use the following recognition principle only when polynomial Laurentness in the original coefficients has been separately verified.

**Lemma 5.4** (A regular binomial)**.** *Assume* (5.8)*, with nonzero $s$–$\ell$ entry, and suppose $R$ is contained in every Laurent ring with the original coefficients kept polynomial. Let $z$ be an ordinary cluster monomial with $E(z) = 0$. Suppose a nonnegative grading has $\deg z = 0$ and positive degrees on $s, \ell, d_0$, where $d_0$ is an original frozen variable fixed by $E$. If*

$$\ell w = d_0^k + sz, \qquad k > 0, \quad w \in R, \tag{5.11}$$

*then $w \in \mathcal{A}$.*

*Proof.* Lemma 5.3, applied to the invariant monomial $z$, places $z$ in a seed reached without mutating $\ell$. It is an ordinary monomial there. The grading excludes $s, \ell, d_0$ from that monomial. Let $P$ be the actual exchange polynomial at $\ell$ in this seed. Laurentness of $w$ in the adjacent seed implies that $P$ divides $Q = d_0^k + sz$ in the Laurent polynomial ring in all the other variables. The exponent difference of the two terms of $Q$ has $s$-coordinate 1. It is primitive, so $Q$ is an irreducible Laurent binomial. The nonzero $s$–$\ell$ entry makes $P$ a nonunit. Thus $P$ and $Q$ are associates and $w$ is a Laurent monomial in the adjacent seed. Polynomial Laurentness excludes its negative frozen exponents. Any negative exponent at a nonzero mutable column produces a nonunit exchange denominator in its neighbor, which a monomial numerator cannot cancel. At an isolated column the exchange is $xx' = 2$, and a negative power is a scalar multiple of a power of $x'$. Therefore $w$ belongs to the ordinary cluster algebra in all cases. □

5.4. **A monochromatic boundary with at least three roots.** Let the only boundary be black with $n \geq 3$ roots, and write $v_{i+n} = Cv_i$. Put $F_i = (v_i, v_{i+1}, v_{i+2})$ and $\delta_i = \det F_i$. All $\delta_i$ are original cyclic coefficients. The transition matrix is

$$P_i = F_i^{-1} F_{i+1} = \begin{pmatrix} 0 & 0 & \delta_{i+1}/\delta_i \\ 1 & 0 & -\det(v_i, v_{i+2}, v_{i+3})/\delta_i \\ 0 & 1 & \det(v_i, v_{i+1}, v_{i+3})/\delta_i \end{pmatrix}. \tag{5.12}$$

Its two additional numerators are boundary triangles. For $n = 3$ their first and fourth roots are the same physical point on consecutive lifts; they are the repeated-corner triangles in the boundary collar. For $n \geq 4$ the four roots lie in a short interval. Thus $P_i$ lies in $\mathrm{GL}_3(B)$, with determinant $\delta_{i+1}/\delta_i$. In particular

$$F_1^{-1} C F_1 = P_1 \cdots P_n \in \mathrm{SL}_3(B).$$

Take a simple based-loop free generating system from the bands of a disk-and-bands presentation, and orient its loops in the positive port convention. For its matrix $H$ in the $F_1$ gauge, set $Q_1 = P_1$ and $Q_2 = P_1 P_2$. Lemma 5.2, with simultaneous movement of the loop and frame, gives the three columns

$$c_0 = He_1, \qquad c_1 = (Q_1^{-1} H Q_1)e_1, \qquad c_2 = (Q_2^{-1} H Q_2)e_1$$

in $B^3$. Since $Q_1 e_1 = e_2$ and $Q_2 e_1 = e_3$,

$$H = (c_0 \mid Q_1 c_1 \mid Q_2 c_2) \in \mathrm{SL}_3(B). \tag{5.13}$$

The other decorations have short-boundary Cramer coordinates in the same frame. Equation (5.3) gives $R_D = B$. For a white boundary carry out these formulas in $V^*$ with transport $H^{-T}$ and $C^{-T}$. The upper-triangle argument following Lemma 5.2 places every numerator in the original algebra. The resulting inverse-transpose matrices recover the original matrices by adjugate and transpose. Thus the same conclusion holds for a white boundary, without a color-duality assumption.

5.5. **Actual boundary roots at a color change.** Suppose now that the single boundary is mixed and has at least three roots. At a color change let $x$ be the native decoration at the center of three consecutive roots and let $y$ be the proxy of the same representation type at the following root. Its caterpillar terminates at a same-color pair before returning to $x$. Hence it is independent of $x$, and

$$E = y \cdot \partial_x \tag{5.14}$$

is an equivariant locally nilpotent derivation of the native invariant ring. Every outside caterpillar reaching $x$ contains it only through $x \times y$, so all outside proxies are fixed.

Here is the actual ear computation. If $x$ is black, let the preceding native covector be $a$, and the following covector proxy be $b$, with $b(y) = 0$. Set

$$c = x \times y, \quad v = a \times c, \quad s = a(x), \quad \ell = a(y), \quad t = b(x).$$

Then $v = \ell x - sy$, and $E$ fixes $a, b, y, c, v, \ell, t$ while $E(s) = \ell$. The nonzero ear bipods are $s, t, \ell, \ell t$; the lower triangle is zero. If $x$ is white, the preceding decoration $a$ is a vector and $b$ is the following vector proxy. Set

$$c = x \times y, \quad v = a \times c, \quad s = x(a), \quad \ell = y(a), \quad t = x(b).$$

Now $y(b) = 0$, and the opposite diagonal and lower triangle both equal $v(b) = \ell t$. The other nonzero bipods again factor through $s, t, \ell$. These identities give the same conclusion in the original white algebra.

The short diagonal $\ell$ is the irreducible direction of the special bipod in the local factorization and surface seed rules of [2, 3]; its reverse direction has the factor $t$. For three distinct physical roots in positive genus it is a mutable factor. Complete this ear to a triangulation. The kernel of a locally nilpotent derivation in characteristic zero is factorially closed: the maximal derivative degrees of nonzero products add. Thus the fixed outside special invariants have fixed irreducible factors. The actual seed has exactly $E = \ell \partial_s$. Lemma 5.3 applies.

At the start of a black run write the consecutive roots as $w, p, q, r$, where $w$ is white and $p, q$ are black. Put

$$F = (p, q, u), \quad u = u_r, \quad \delta = \det F, \quad \epsilon = a_r(q), \quad \lambda = a_r(p). \tag{5.15}$$

Both $\delta$ and $\epsilon$ are old coefficient units, and $a_r = (\lambda, \epsilon, 0)$ in this frame. The ear at $p$ gives the actual root $q\partial_p$, fixing $q, u, a_r, p \times q$. The strip lemma gives $Hp$. In addition the following are actual ordinary cluster monomials, up to their orientation signs:

$$X_1 = \det(Hp, q, u), \qquad X_3 = \det(p, q, Hp), \qquad Y = a_r Hp.$$

Each has degree at most one for this root. Their maximal derivatives are $\det(Hq, q, u)$, $\det(p, q, Hq)$ and $a_r Hq$. By Lemma 5.3 these belong to $\mathcal{A}$. Consequently

$$Hp, Hq \in B^3, \qquad (Hq)_2 = \frac{a_r Hq - \lambda (Hq)_1}{\epsilon}. \tag{5.16}$$

All matrices here refer to the same fixed boundary trivialization.

If $r$ is white, write $u = a_r \times a_{r+1}$. The based special bipod $a_r Hu$ is an ordinary cluster monomial for either of the two endpoint-incidence directions. The white root $a_{r+1}\partial_{a_r}$ fixes $u$ and has maximal derivative $a_{r+1} Hu$. Write the two row coordinates as $a_r = (\lambda, \epsilon, 0)$ and $a_{r+1} = (\mu, \nu, 0)$. The physical volume of $F$ is $\delta$, so $\lambda\nu - \epsilon\mu = \delta$. Hence

$$(Hu)_1 = \frac{\nu(a_r Hu) - \epsilon(a_{r+1} Hu)}{\delta}, \qquad (Hu)_2 = \frac{\lambda(a_{r+1} Hu) - \mu(a_r Hu)}{\delta} \tag{5.17}$$

belong to $B$. This supplies two entries, not a third independent row.

5.6. **The three-root signature** $BBW$**.** Let the native decorations be $p, q, \phi$ in that order. In positive genus the boundary loop is noncontractible. Write

$$u = \phi \times C(p \times q), \quad F = (p, q, u), \quad \delta = \det F, \quad s = \phi(q), \quad t = \phi(Cp), \quad \ell = \phi(p).$$

Here $C(p \times q)$ denotes the transported covector $(p \times q)C^{-1}$. The three original coefficients are $\delta, s, t$. In the $F$ gauge, $\phi = (\ell, s, 0)$. The first column of $C$, its reverse-ear bipod $\beta = \phi Cq$, and its based

white-root bipod $\theta = \phi C u$ belong to $B$ by their actual one-period arcs. Write the first column as $(a, d, g)^T$. The identity $C^{-1}u = \beta p - tq$, together with $\det C = 1$, gives

$$C = \begin{pmatrix} a & \beta a/t & a\theta/t - s \\ d & \beta d/t & d\theta/t + \ell \\ g & (\beta g - 1)/t & i \end{pmatrix}, \qquad \ell a + sd = t. \tag{5.18}$$

For example the top two entries $c, f$ of its third column are determined by $af - dc = t$ and $\ell c + sf = \theta$. All entries except $i$ are therefore known in $B$.

Use the ear $(p, q, \phi)$ and its actual root $E = p\partial_q$. Outside this ear the proxies are $p, p \times q, \phi, u$, all fixed by $E$; its local nonzero factors are $\delta, s, \ell$. The pairing $\ell$ is irreducible and mutable, so this is an actual seed with $E = \ell\partial_s$. Consider the simple $q$-to-$q$ arc parallel to the boundary, with the incoming vector on the previous lift and covector proxy $q \times u$ on the current lift. Its special bipod is

$$J(q) = \det(q, u, C^{-1}q) \in \mathcal{A}. \tag{5.19}$$

The arc is drawn by two disjoint radial approaches to the inner edge of a thin collar; it spans one period. The caterpillar $u$ is an endpoint decoration, not a third endpoint requiring a longer triangular interval. Completing this arc to a triangulation proves that $J(q)$ is an ordinary cluster monomial.

Since $E(u) = 0$,

$$J(q + kp) = J(q) + kJ_\times + k^2 J(p),$$

where

$$J(p) = \det(p, u, C^{-1}p), \qquad J_\times = \det(p, u, C^{-1}q) + \det(q, u, C^{-1}p).$$

Lemma 5.3 gives $J(p) \in \mathcal{A}$ and

$$S = \ell^2 J(q) - \ell s J_\times + s^2 J(p) \in \mathcal{A}.$$

If the degree drops, the displayed expression is a nonnegative power of $\ell$ times the corresponding homogeneous slice, so this membership assertion remains valid. Define

$$L = (\ell^2 J(q) + s^2 J(p) - S)/s = \ell J_\times \in B.$$

Using entries $a, b, c; d, e, f; g, h, i$ for $C$, expansion of the inverse matrix gives

$$J(p)/\delta = di - fg, \quad J_\times/\delta = (e - a)i + cg - fh, \quad e = \beta d/t.$$

Consequently all quantities on the right in the following formulas belong to $B$:

$$D_0 = J(p)/\delta + fg = di, \quad A_0 = \ell(\beta/t)D_0 - L/\delta + \ell(cg - fh) = \ell a i, \quad i = (A_0 + sD_0)/t. \tag{5.20}$$

Only $\delta, s, t$ have been inverted. Thus $C \in \mathrm{SL}_3(B)$.

To recover a general simple handle matrix $H$, use the reverse ear $(\phi C, p, q)$. The actual root is $q\partial_p$; its active coefficient is $t$ and its diagonal is $\beta = \phi C q$. Its outside proxies are fixed because $p \times q$ is fixed. The diagonal is an irreducible pairing after the invertible coordinate change $q \mapsto Cq$, and the actual seed has $q\partial_p = \beta\partial_t$. Formula (5.16), now with $a_r = \phi$, gives $He_1, He_2$ in $B^3$. A boundary Dehn twist gives the simple matrix $T = CHC^{-1}$ with the same two columns known. The third row of $(5.18)^{-1}$ is $(-d/t, a/t, 0)$. Put

$$V_j = C^{-1}Te_j - (C^{-1})_{1j}He_1 - (C^{-1})_{2j}He_2, \qquad j = 1, 2.$$

Then $V_1 = -(d/t)He_3$ and $V_2 = (a/t)He_3$. Therefore

$$He_3 = -sV_1 + \ell V_2 \in B^3. \tag{5.21}$$

All native decorations have known coordinates in $F$. Equation (5.3) proves $R_D = B$ for $BBW$. The calculation for $WWB$ takes place in $V^*$, using the directly checked white ear and original upper special triangles. Every displayed formula is a tensor identity there; it recovers $H^{-T}$ and hence $H$. It is therefore a proof in the original $WWB$ algebra.

5.7. **Two disjoint equal-color edges.** Suppose a mixed boundary has two vertex-disjoint monochromatic edges. If it has a black edge, choose $p, q$ at the start of a black run and use $F = (p, q, u_r)$ from (5.15). Choose a second monochromatic pair $s, t$ disjoint from $p, q$, occurring after $r$ and before $p$ returns. Let $v$ follow $t$. By Lemma 5.1, $F' = (u_s, u_t, u_v)$ is a vector proxy frame with original coefficient determinant, whether the second pair is black or white. The root $r$ precedes $s$, or equals it. The interval from $r$ to $v$ is proper; at worst $v = Cp$, still before the next lift of $r$. Lemma 5.2 therefore recovers $Hu_r$ in $F'$. Every Cramer entry of $P = F^{-1}F'$ is a boundary triangle using roots between $p$ and $Cp$ inclusive. Its determinant is a ratio of original coefficient units. Thus $P \in \mathrm{GL}_3(B)$ and $Hu_r$ returns to a known column in the original $F$ gauge. Together with (5.16), this recovers every simple free holonomy matrix. Short-boundary coordinates and (5.3) finish the proof.

If there is no black edge, choose a white run and use the row frames of Lemma 5.1. The white-root computation above and the original upper-triangle version of Lemma 5.2 give the same three columns for the inverse-transpose transport. Adjugate and transpose recover the original matrices. This also treats cases with two white edges and isolated black roots.

5.8. **One run of length two and an arbitrary alternating tail.** The remaining length-two black-run signatures have order

$$p, q, a_1, v_1, a_2, v_2, \ldots, a_{m-1}, v_{m-1}, a_m,$$

with black vectors $p, q, v_j$ and white covectors $a_j$. The case $m = 1$ is the $BBW$ case already proved. Suppose $m \geq 2$. Set

$$\begin{aligned} b_0 &= p \times q, & b &= Cb_0 = (Cp) \times (Cq), \\ u_m &= a_m \times b, & u_j &= a_j \times (v_j \times u_{j+1}) \quad (1 \leq j < m). \end{aligned}$$

Use $F = (p, q, u_1)$ and put

$$\Delta = \det F, \quad s = a_1(q), \quad r_j = a_j(v_j), \quad \beta_j = a_{j+1}(v_j), \quad t_m = a_m(Cp).$$

These are original coefficient units. All native decoration coordinates in $F$ lie in $B$, by the short-boundary constructions following Lemma 5.2. In particular the covector proxy immediately after $a_1$ has known coordinates. Equations (5.16) and (5.17) show that every entry of the boundary matrix $C$ except

$$G = C_{33}$$

lies in $B$. They give the same partial entries for every positive-port simple matrix $H$. The physical volume of $F$ is $\Delta$; in these coordinates $b$ is $\Delta$ times the usual coordinate cross product of $Cp, Cq$. In particular

$$(a \times b)_F = a(Cq)Cp - a(Cp)Cq \tag{5.22}$$

for any covector $a$.

Define the following known elements:

$$\eta_j = b(v_j), \quad \mu_i = b_0(a_i \times b), \quad t_i = a_i(Cp), \quad s_i = a_i(q), \quad \beta = a_1(Cq), \quad k = \prod_{j=1}^{m-1} (-\beta_j).$$

The element $k$ is an original coefficient unit. We need an identity inside $B$, rather than only a relation modulo an ideal extended to $R_D$. Temporarily represent covectors as columns and put

$$S_j = a_{j+1} v_j^T - \beta_j I.$$

The covector proxy immediately after $a_1$ is $b_1 = S_1 \cdots S_{m-1} b$. The elementary telescoping identity is

$$b_1 = kb + \sum_{j=1}^{m-1} \eta_j \left( \prod_{l>j} (-\beta_l) \right) S_1 \cdots S_{j-1} a_{j+1}. \tag{5.23}$$

Indeed expand the product by selecting, at its last non-scalar factor, $a_{j+1}v_j^T$ instead of $-\beta_j I$. All coefficient vectors in this expression have coordinates in $B$. Taking the cross product with $a_1$ and using (5.22) yields explicitly defined vectors $R_j \in B^3$ with

$$e_3 = k(\beta Cp - t_1 Cq) + \sum_j \eta_j R_j. \tag{5.24}$$

Pairing with $b_0$ gives

$$\Delta = k\mu_1 + \sum_j \eta_j d_j, \qquad d_j = b_0(R_j) \in B. \tag{5.25}$$

In particular $(\eta_1, \dots, \eta_{m-1}, \mu_1) = B$.

We now give a finite cover on which $G$ belongs to the corresponding localization of $B$. Its elements are

$$\eta_1, \dots, \eta_{m-1}, \quad f_1 = \mu_1 t_1, \quad x = C_{12}, \quad \rho = (a_m)_3 (v_{m-1})_3, \quad f_m = \mu_m s_m. \tag{5.26}$$

An identically zero member gives an empty open set.

On the $\eta_j$-open, use the frame $(v_j, Cp, Cq)$. It has determinant $\eta_j$, and all of its columns are already known in $F$. The input root $a_1$ precedes the three frame roots in the proper interval $[a_1, Cq]$. Lemma 5.2 recovers $Hu_1$ for any simple $H$, so it recovers all of $H$, including $G$ for $H = C$.

Retain the smaller signature $(p, q, a_1)$. Its three frozen functions are $\mu_1, s, t_1$, and its ordinary algebra embeds in $\mathcal{A}$ by the deletion theorem [3, Theorem 18]. The successive deletions remain nonalternating because they retain $p, q$. The $BBW$ result therefore puts $\operatorname{tr} C$ in $B[(\mu_1 t_1)^{-1}]$. The retained signature $(p, q, a_m)$ similarly puts $\operatorname{tr} C$ in $B[(\mu_m s_m)^{-1}]$, since its third frozen $t_m$ is already a unit. Subtracting the known $C_{11} + C_{22}$ gives $G$ on these two opens.

The based boundary-parallel bipod at $q$, with the same incidences as (5.19), is

$$J_q = \det(q, u_1, C^{-1}q) \in \mathcal{A}.$$

Its arc has one period and its long caterpillar is attached at its endpoint. The cofactor identity is

$$J_q/\Delta = (C^{-1})_{12} = C_{13}C_{32} - C_{12}G. \tag{5.27}$$

This proves $G \in B[x^{-1}]$. Finally let $v = v_{m-1}$ and $a = a_m$. Since $v$ precedes $a$, the interval from $a$ to $Cv$ is shorter than one period. Its inner parallel arc is an actual distinct-endpoint special bipod, with native decorations at both ends. Thus $Z = aCv \in \mathcal{A}$. Expanding it gives

$$Z = K + \rho G, \qquad K = \sum_{(i,j)\neq(3,3)} a_i C_{ij} v_j \in B.$$

Therefore $G \in B[\rho^{-1}]$.

We prove that (5.26) generates the unit ideal of $B$. Suppose that a prime ideal contains all its entries, and compute in the fraction field of the quotient. Every function used below is already known to lie in $B$; $G$ will not occur. If all $\eta_j$ vanish, (5.25) gives $k\mu_1 = \Delta$, so $\mu_1 \neq 0$. The vanishing of $f_1$ forces $t_1 = 0$. Equation (5.24) becomes $e_3 = k\beta Cp$, whence

$$Cp = (0, 0, c)^T, \qquad c = (k\beta)^{-1} \neq 0.$$

The vanishing of $x$ lets us write $Cq = (0, d, h)^T$. Since $a_1 = (\ell, s, 0)$, we have $\beta = sd$ and $d \neq 0$. The original unit $t_m = a_m(Cp) = c(a_m)_3$ makes $(a_m)_3 \neq 0$. The vanishing of $\rho$ forces $v_3 = 0$. Now $b = \Delta(-cd, 0, 0)$ and $\eta_{m-1} = b(v) = 0$, so also $v_1 = 0$. The original unit $\beta_{m-1} = a_m(v) = (a_m)_2 v_2$ makes $s_m = (a_m)_2 \neq 0$. Formula (5.22) gives

$$\mu_m = \Delta cd (a_m)_2 \neq 0.$$

This contradicts $f_m = 0$. No prime contains the entire list, so it generates $B$. This argument uses exact $B$-valued identities and requires no assertion about extending primes from $B$ to $R_D$. Membership on a finite principal cover implies membership in the ring: for $z \in \operatorname{Frac} B$ with

$z \in B[f_i^{-1}]$, choose a common exponent $N$ with $f_i^N z \in B$; the $f_i^N$ still generate the unit ideal. We conclude that $G \in B$, hence $C \in \mathrm{SL}_3(B)$.

It remains to recover every handle matrix, not just the boundary matrix. Set $N = C^{-1} \in \mathrm{Mat}_3(B)$ and $I = (\eta_1, \dots, \eta_{m-1})$. Applying $e_3^T N$ to (5.24) gives $N_{33} \in I$. The $(3,3)$ entry of $NC = I$ consequently gives

$$(\eta_1, \dots, \eta_{m-1}, N_{31}, N_{32}) = B. \tag{5.28}$$

For a simple $H$, the boundary twist $T = CHC^{-1}$ is simple with the same ports. The first two columns of both matrices are known, and

$$N_{3j} He_3 = NTe_j - N_{1j} He_1 - N_{2j} He_2, \qquad j = 1, 2. \tag{5.29}$$

This recovers $He_3$ on the two $N_{3j}$-opens. The whole $H$ was already recovered on the $\eta_j$-opens. Cover (5.28) gives $H \in \mathrm{SL}_3(B)$ globally. The gauge criterion proves $R_D = B$ for every length-two black run.

For a unique white run, execute this proof in $V^*$ with transports $C^{-T}, H^{-T}$. The two retained $BBW$ calculations become the original $WWB$ calculation of Section 5.6. Every triangular probe is an original upper triangle, every nonlinear ear is the white ear of Section 5.5, and every other probe is a bipod in the original algebra. The telescope and prime-ideal argument are tensor identities in that representation. They give all inverse-transpose matrices and therefore all original matrices. This proves the length-two white-run case as well.

5.9. **One run of length three.** Consider the order

$$p, q, r, a_1, v_1, a_2, \dots, v_{m-1}, a_m$$

with its initial three black roots. Use $F = (p, q, r)$, $\Delta = \det F$, and let $u = (u_1, u_2, u_3)^T$ be the vector proxy at $a_1$. The original adjacent coefficient $\epsilon = \det(q, r, u) = \Delta u_1$ shows that $u_1$ is a unit of $B$. Write $u = a_1 \times b_1$, where $b_1$ is the following covector proxy. All decoration and proxy coordinates in this frame belong to $B$. The root at the beginning of the black run recovers $Hp, Hq$ by (5.16), using the fixed proxy $r \times u$ and the coefficient $\epsilon$. In particular $Cp, Cq$ are known.

The proper long-collar bipod $X = a_1 Cr$ belongs to $\mathcal{A}$. The root $b_1 \partial_{a_1}$ fixes $r, C, b_1, u$, and its maximal degree-one derivative gives $Y = b_1 Cr \in \mathcal{A}$. Put

$$a = a_m, \qquad t = a(Cp), \quad \beta = a(Cq), \quad Z = a(Cr).$$

The element $t$ is an original coefficient unit; $\beta, Z$ are distinct-endpoint long-collar bipods. Retaining $(p, r, a)$ gives an ordinary coefficient of that smaller algebra, whose image

$$\mu_{pr} = \det(p, r, a \times C(p \times r))$$

belongs to $\mathcal{A}$ by [3, Theorem 18]. The triple-product identity is

$$a \times C(p \times r) = ZCp - tCr, \qquad \mu_{pr} = \Delta(tC_{23} - ZC_{21}).$$

Thus

$$C_{23} = (\mu_{pr}/\Delta + ZC_{21})/t \in B. \tag{5.30}$$

The three rows $a_1, b_1, e_2^T$ have determinant $\Delta u_2$ in $F$ coordinates. Their values on $Cr$ are $X, Y, C_{23}$. Hence $C$ is generated on the $u_2$-open.

We construct a complementary open with controlled coefficients. Merge $p, q$ into the native covector $b_0 = p \times q$, using the actual cluster-subalgebra map of [3, Theorem 19]. The new signature

$$b_0, r, a_1, v_1, \dots, a_m$$

has a unique white run of length two across the cut. The tail proxies are unchanged: they use the same terminal covector $C(p \times q)$. The covector proxy at $r$ is $r \times u$, and the vector proxy at the new white root is

$$u_{b_0} = b_0 \times (r \times u) = \Delta(u_3 r - u). \tag{5.31}$$

All coefficients of the merged signature are original units except possibly

$$\kappa = a_m(C u_{b_0}).$$

Indeed its adjacent coefficients are $b_0(r) = \Delta$, $a_1(r)$, the unchanged internal pairings, and this last white–white coefficient. Using (5.31) and $a_m C = (t, \beta, Z)$ gives the exact cancellation

$$\kappa = -\Delta(u_1 t + u_2 \beta), \qquad 1 = -\frac{\kappa}{\Delta u_1 t} - \frac{u_2 \beta}{u_1 t}. \tag{5.32}$$

This proves $(u_2, \kappa) = B$ before the third column is known.

On the $\kappa$-open, use the consecutive covector frame of the merged white run. In the original $F$ gauge its rows are

$$K = (a_m C\,;\, b_0\,;\, r \times u), \qquad \det K = \Delta\kappa.$$

Its entries are known in $B$: the first row is $(t, \beta, Z)$ and the other two are known proxies. The length-two white-run result puts $KHK^{-1}$ in $B[\kappa^{-1}]$ for each generating holonomy $H$, because these entries are invariant rational functions of the merged data and have only merged frozen denominators. Returning by the known matrix $K$ puts $H$ in $\mathrm{Mat}_3(B[\kappa^{-1}])$. In particular $C$ is generated there. The $u_2$-open and (5.32) now give $C \in \mathrm{SL}_3(B)$.

Set $N = C^{-1}$ and $\eta_r = \det(r, Cp, Cq)$. The cofactor identity $N_{33} = \eta_r/\Delta$ and $(NC)_{33} = 1$ give

$$(\eta_r, N_{31}, N_{32}) = B.$$

On $\eta_r \neq 0$ the frame $(r, Cp, Cq)$ is known and invertible, and the later-frame strip construction gives $Hr$. On each $N_{3j}$-open, (5.29) gives that column from the first two columns of $H$ and its actual boundary twist. The cover therefore recovers every $H$ in $B$. Equation (5.3) proves $R_D = B$. The unique white run of length three is proved in $V^*$: the merge is the native white–white merge, the rows become columns, and all triangular probes are the original upper triangles. The displayed identities and the two coefficient units are unchanged in the dual representation, so the same argument recovers $C^{-T}, H^{-T}$ in the original algebra and hence $C, H$.

5.10. **Two marked points on one boundary.** The only nonalternating two-root signatures are $BB$ and $WW$. Take $BB$ first, with native vectors $p, q$ and actual boundary matrix $C$. The lifted order is $p, q, Cp, Cq$. Cut off the contractible ear $(p, q, Cp)$ along the boundary-parallel $p$-to-$p$ arc; the two $p$ corners are distinct incidences. Set

$$\delta = \det(p, q, Cp), \quad s = \det(q, Cp, Cq), \quad \ell = \det(p, Cp, Cq).$$

The original coefficients are $\delta, s$. The nonzero ear special functions factor through $\delta, s, \ell$, with lower triangle $\delta$. The complement has endpoints only at $p$, whose proxies are $p, p \times q$ and their transports. The actual locally nilpotent root $E = p\partial_q$ fixes those proxies, fixes $\delta, \ell$, and satisfies $E(s) = \ell$.

The function $\ell$ is a single mutable special factor. Indeed every nonconstant special invariant on an all-black boundary has native degree divisible by 3, by the action of the center of $\mathrm{SL}_3$, and therefore has degree at least 3. The degree of $\ell$ is 3, so it cannot split into two nonconstant special factors. It is distinct from $s$ by its $q$-degree, and from $\delta$ as a polynomial function. For an explicit witness take $C = \mathrm{diag}(1, 2, 1/2)$ and $p = (1, 1, 1)^T$. For $q = e_1, e_2$ respectively one obtains $(\delta, \ell) = (3/2, -3/2), (-1/2, 1)$, so $\ell$ is nonzero and is not a constant multiple of $\delta$. This $C$ is an actual $\mathrm{SL}_3$ commutator: if $P$ cyclically permutes $e_1, e_2, e_3$, then $C = [2^{-1/3}\,\mathrm{diag}(1, 2, 1), P]$. The

original seed and irreducibility theorem consequently makes it a mutable variable. We have an actual seed with $E = \ell\partial_s$. The cyclic ear $(q, Cp, Cq)$ gives the other actual root

$$E' = (C^{-1}q)\partial_p, \qquad E'(\delta) = \ell_2 = \det(q, Cp, C^{-1}q), \qquad E'(s) = 0.$$

The same degree argument makes $\ell_2$ a mutable variable. Its outside proxies $q, q \times Cp$ are fixed.

We verify the polynomial-coefficient Laurent condition needed in Lemma 5.4. For any $f \in \mathcal{R}$, local nilpotence of $E$ and the rational-coordinate formula $E = \ell\partial_s$ imply that $f$ is polynomial in $s$ over the field of the other coordinates: a rational function with a finite pole cannot have a terminating derivative sequence in characteristic zero. Thus its formal $s$-valuation is nonnegative. The other ear proves the same assertion for $\delta$. The formal-valuation invariance in the proof of Lemma 5.3, together with the published full Laurent inclusion $\mathcal{R} \subset U_{\mathrm{ext}}$, now shows that every Laurent expansion of $f$ has nonnegative exponents of both original frozen variables. Thus $\mathcal{R}$ is contained in the polynomial-coefficient upper algebra here.

Use $F = (p, q, Cp)$. Its determinant is $\delta$, and in its coordinates

$$C = \begin{pmatrix} 0 & a & d \\ 0 & b & e \\ 1 & c & f \end{pmatrix}, \qquad ae - bd = 1, \quad a = s/\delta, \quad b = -\ell/\delta. \tag{5.33}$$

Write $K = dc - af = \ell_2/\delta$. Deleting $q$ gives an ordinary cluster-subalgebra embedding from the one-black-root surface, by [3, Theorem 18]. Its coefficient

$$\Delta_p = \det(p, Cp, C^2p) = -\delta e$$

is therefore an ordinary cluster monomial in the present algebra. This uses only the already constructed one-root cluster structure, not its localized equality. The regular polynomial identity is

$$\ell \det(q, Cp, C^2p) = \delta^2 + s\Delta_p. \tag{5.34}$$

The $s$–$\ell$ entry in the adapted seed is necessarily nonzero. If it were zero, the row-support assertion of Lemma 5.3 would make the entire $s$-row zero. The actual exchange polynomial $P$ at $\ell$ would be independent of $s$. Since $E(\Delta_p) = 0$, the function $\Delta_p$ is also independent of $s$, both before and after this mutation. The $s$-constant coefficient of the regular quotient in (5.34), in the mutated seed, would be $\delta^2\ell'/P$. Laurentness forces $P$ to divide the Laurent monomial $\delta^2\ell'$. This is impossible for a nonunit exchange binomial. The only unit case is the isolated exchange $\ell\ell' = 2$, also impossible because $\ell$ is a nonunit of the invariant ring while both variables belong to that ring. The entry is therefore active.

Apply Lemma 5.4 with $z = \Delta_p$, $d_0 = \delta$, $k = 2$, and the native $q$-degree. This degree is nonnegative on the ordinary algebra, is zero on $\Delta_p$, and is positive on $\ell, s, \delta$. All its assumptions have just been verified. Thus $\det(q, Cp, C^2p) = \delta d$ belongs to $\mathcal{A}$, and $d \in B$. Also $e = -\Delta_p/\delta \in B$.

For the cyclic frame $P_0 = (q, Cp, Cq)$ the transition is

$$F^{-1}P_0 = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}, \qquad \det(F^{-1}P_0) = a.$$

The entries of $C$ in this frame satisfy

$$a' = a^{-1}, \quad b' = -K/a, \quad c' = d/a, \quad d' = c/a, \quad e' = a - cK/a, \quad f' = b + cd/a.$$

Apply the same cyclic argument using the coefficient of the surface obtained by deleting $p$. It gives $sd' = \delta c \in \mathcal{A}$. Thus $c \in B$, and finally $f = (dc - K)/a \in B$. All entries of $C$ have been recovered using only the two original units $\delta, s$.

We finish by specifying the actual arc probes for an arbitrary free handle generator. Give it the positive port orientation. Move its returning germ from $p$ to $q$ along an inner collar track, then move its outgoing germ along a second track nearer the boundary. At the new root the port order is unchanged. Before a second slide, shrink the collar so that the old horizontal tracks lie outside it

and only the two endpoint germs enter it. This permits the next slide from $q$ to the lift $Cp$, again over a proper interval, with no crossing. All these curves are simple. In the common trivialization along $p, q, Cp, Cq$ they measure the same $H$; when brought from the lift $Cp$ to physical $p$, the matrix is $C^{-1}HC$. An actual boundary Dehn twist realizes this conjugation and preserves the ports.

The first column is obtained from

$$P_1 = \det(q, Cp, Hp), \quad P_2 = \det(p, Cp, Hp), \quad P_3 = \det(p, q, Hp), \qquad [Hp]_F = \delta^{-1}(P_1, -P_2, P_3)^T.$$

Here $P_3$ is the original based $p$-bipod, $P_1$ is the bipod after the first endpoint slide, and $P_2$ is the based $p$-bipod from the one-root source after deleting $q$. Each is an ordinary cluster monomial by the actual special construction or deletion embedding. In the next frame the three probes

$$Q_1 = \det(Cp, Cq, Hq), \quad Q_2 = \det(q, Cq, Hq), \quad Q_3 = \det(q, Cp, Hq)$$

are, respectively, the endpoint-slid bipod, the one-root source bipod after deleting $p$, and the original based $q$-bipod. Hence

$$[Hq]_F = (F^{-1}P_0)(Q_1, -Q_2, Q_3)^T/s \in B^3.$$

For the simple twisted matrix $T = C^{-1}HC$, its already recovered first column yields $H(Cp) = C(Tp)$. Thus the complete $H$ is in $\mathrm{SL}_3(B)$. The determinant-one gauge (5.3) recovers all invariants with only $\delta, s$ inverted and proves $R_D = B$.

For $WW$, regard the two native covectors as vectors in $V^*$ and use inverse-transpose transport. The two repeated ears have exposed white–white sides; their upper triangles are the original special bipods. Their degree-three factor argument, actual locally nilpotent roots, deletion bipods, and regular identity (5.34) therefore all lie in the original $WW$ algebra. The same polynomial-coefficient valuation argument applies. It recovers every inverse-transpose holonomy matrix, and then its adjugate transpose recovers the original matrix. This establishes the second two-root signature without using Conjecture 25.

### 5.11. **Exhaustion of the signatures.**

**Proposition 5.5.** *For every admissible surface other than a positive-genus surface with one boundary component and one marked point,*

$$\mathcal{A}[\mathcal{D}^{-1}] = \mathcal{R}[\mathcal{D}^{-1}].$$

*Proof.* For a disk with at least five roots the ordinary equality $\mathcal{A} = \mathcal{R}$ is [2, Theorem 8.1]. The small admissible disk cases have the polynomial, rank-zero conventions described in the preliminaries. Surfaces with at least two boundary components are covered by (5.7). For one boundary and positive genus, monochromatic signatures with at least three roots are covered by Section 5.4, and the two-root cases by Section 5.10.

Consider a mixed cyclic word with at least three roots. If it has two vertex-disjoint equal-color edges, Section 5.7 applies. Otherwise all its equal-color edges lie in a single run: edges in different runs are disjoint. This run has length at most three, because a run of length four has disjoint first and third edges. Its length is at least two, since the word is nonalternating. All other runs have length one. These are exactly the length-two and length-three alternating-tail families, in one of the two colors, proved in Sections 5.8 and 5.9. This exhausts every asserted case. □

## 6. One boundary component with one marked point

We treat the case of a surface of genus $g \geq 1$ with one boundary component and one marked point. The only coefficient is a cyclic determinant. The proof uses both choices of boundary proxy from Theorem 3.1. Its main ingredient is a generation statement that does not involve cluster algebras.

Let $W$ be a three-dimensional complex volume space, and set

$$X = \mathrm{SL}(W)^{2g} \times W, \qquad H = \prod_{i=1}^{g} [A_i, B_i], \qquad \Delta = \det(p, Hp, H^2p). \tag{6.1}$$

Here $p \in W$ and $A_i, B_i$ are independent coordinates on $X$, representing a standard system of based generators of the fundamental group. In particular $H$ is the actual boundary word, not an additional independent matrix. Put

$$R = \mathbb{C}[X]^{\mathrm{SL}(W)}[\Delta^{-1}], \qquad t = \operatorname{tr} H, \qquad u = \operatorname{tr} H^{-1}.$$

An oriented simple based arc has its interior in the interior of the surface and its ends at the marked point, with distinct ports there. A word denoting such an arc will also denote its holonomy.

**Theorem 6.1.** *Suppose that a complex subalgebra $B \subset R$ contains $\Delta^{\pm 1}$ and, for every nonseparating simple based arc in both directions, contains*

$$b_+(M) = \det(p, Hp, Mp), \qquad b_-(M) = \det(p, H^{-1}p, Mp). \tag{6.2}$$

*Then $B = R$.*

No finite generation or normality assumption on $B$ is made. We first establish the topological operations used in the proof. Products of words are taken in the order displayed in (6.1).

6.1. **Ports, complementary arcs, and twists.** Replace a small half-disc at the marked point by its inner semicircle. The two ends of an arc become distinct ports on a small boundary interval. Order this interval by the boundary orientation used to read $H$. Call the direction from its first port to its second port *compatible*. This selects one direction of each unoriented arc and is preserved by homeomorphisms fixing the boundary pointwise.

**Lemma 6.2.** *If $M$ is a compatible nonseparating simple based arc, then $M^{-1}H$ is another such arc. Every boundary conjugate $H^{-k}MH^k$, $k \in \mathbb{Z}$, is also compatible and nonseparating. There is a free generating system of $2g$ compatible nonseparating based arcs.*

*Proof.* The regular neighborhood of the boundary and a narrow band along $M$ is a pair of pants. Its ribbon spine has half-edge order $(h^+, m^+, m^-, h^-)$. If $\sigma$ is this cyclic order and $\iota$ exchanges the two ends of an edge, the boundary cycles of $\sigma\iota$ are

$$(h^+), \qquad (m^-), \qquad (m^+, h^-).$$

They read $H$, $M^{-1}$, and $MH^{-1}$. Traverse each inner boundary in the opposite direction. Use two disjoint sides of a narrow channel from the old boundary interval to that inner boundary, and omit a small joining interval of the inner boundary. This gives embedded based arcs $M$ and $K_0 = HM^{-1}$. The two channel sides are distinct; there is no overlapping out-and-back path. The ribbon order shows that both directions run from the first port to the second port.

A Dehn twist in a collar of the old boundary, with the appropriate sign, acts on based words by conjugation by $H^{-1}$. It changes $K_0$ into

$$K = H^{-1}K_0H = M^{-1}H.$$

This proves simplicity and compatibility. Since the unique boundary has zero homology class, $[K] = -[M] \neq 0$. Here a nonseparating arc has nonzero homology because it admits a closed dual curve of intersection number one. A separating simple based arc has zero homology class, so $K$ is nonseparating. Boundary twists give the assertion about all conjugates.

For the last assertion, a standard rose spine has successive half-edge blocks

$$(A_i^+, B_i^-, A_i^-, B_i^+), \qquad 1 \le i \le g.$$

The boundary permutation reads $A_i^+, B_i^+, A_i^-, B_i^-$ and proceeds to the next block. It therefore gives $\prod_i [A_i, B_i]$, while the first-port directions of its free arcs are $A_i$ and $B_i^{-1}$. □

**Lemma 6.3.** *For every compatible nonseparating arc $M$ there exist compatible nonseparating arcs $N, K$, with $K = N^{-1}H$, for which*

$$NM, \quad NM^2, \quad MK, \quad M^2K \tag{6.3}$$

*are compatible nonseparating simple arcs.*

*Proof.* Homeomorphisms fixing the boundary pointwise act transitively on nonseparating arcs between two individually fixed points of the same boundary. This is the vertex case of [11, Proposition 2.2(1)]. To recall its relevant proof, cutting along such an arc gives a connected surface of genus $g-1$ with two boundary components. Retain the labels on the two cut copies and the old boundary intervals. The classification of oriented surfaces gives a homeomorphism of two such cut surfaces preserving these labels. Boundary collars make it pointwise fixed on the old boundary intervals. It glues to the required homeomorphism; fixing the two endpoints individually retains their order.

Apply this fact to separated ports and then collapse the small interval. A compatible nonseparating arc can consequently be the first arc in an actual standard handle system with its based ports retained. For

$$A = HMH^{-1}$$

choose such a system, writing $H = [A, B_0]J$. Its arcs $A$ and $B_0^{-1}$ are compatible. The Dehn twist $T$ about the $A$ core, supported in this handle, has the based action

$$T(A) = A, \qquad T(B_0) = B_0A, \qquad T(J) = J, \qquad T(H) = H. \tag{6.4}$$

The multiplication side can be read after cutting along the $A$ core: the $B_0$ strand crosses the cut once and the indicated twist inserts $A$ on its terminal side. The boundary identity is also checked directly by $[A, B_0A] = [A, B_0]$.

Set

$$N = B_0H, \qquad K = H^{-1}B_0^{-1}H = N^{-1}H.$$

The first is the complementary arc to $B_0^{-1}$ from Lemma 6.2; the second is a boundary conjugate of $B_0^{-1}$. Since $M = H^{-1}AH$, equation (6.4) gives

$$T(M) = M, \qquad T(N) = NM, \qquad T(K) = M^{-1}K.$$

Hence $NM^j = T^j(N)$ and $M^jK = T^{-j}(K)$ are actual compatible nonseparating arcs for all integers $j$. Take $j = 1, 2$. □

6.2. **Recovering matrices up to the boundary traces.** We now assume the hypotheses of Theorem 6.1. The cyclic frame $F = (p, Hp, H^2p)$ is invertible in $R$. Cayley–Hamilton gives

$$C = F^{-1}HF = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -u \\ 0 & 1 & t \end{pmatrix}, \qquad C^{-1} = \begin{pmatrix} u & 1 & 0 \\ -t & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.5}$$

Write $N_M = F^{-1}MF$. Superscript $\mathsf{T}$ denotes transpose, and $e_1, e_2, e_3$ are the coordinate vectors in the cyclic frame.

**Lemma 6.4.** *For every compatible nonseparating arc $M$,*

$$N_M = \begin{pmatrix} X_M & a_M & b_M \\ Y_M - tc_M & P_M - ua_M & E_M - ub_M \\ c_M & d_M & Q_M - uc_M + td_M \end{pmatrix}, \tag{6.6}$$

*where all nine named coefficients belong to $B$. The same assertion holds for every integer boundary conjugate of $M$.*

*Proof.* For every integer $k$, Lemma 6.2 and the hypotheses give

$$f_k = \Delta^{-1}b_+(H^{-k}MH^k) \in B, \qquad g_k = \Delta^{-1}b_-(H^{-k}MH^k) \in B.$$

Determinant invariance gives $p \times H^{-1}p = -(p \times Hp)H$. Thus, with $N = N_M$,

$$f_k = e_3^{\mathsf{T}}C^{-k}NC^ke_1, \qquad g_k = -e_3^{\mathsf{T}}C^{1-k}NC^ke_1. \tag{6.7}$$

The matrices in (6.5) give

$$f_0 = N_{31}, \qquad f_1 = N_{12}, \qquad -g_1 = N_{32}, \qquad -g_2 = N_{13}, \tag{6.8}$$

$$-g_0 = N_{21} + tN_{31}, \qquad f_2 = N_{23} + uN_{13}. \tag{6.9}$$

These determine $c_M, a_M, d_M, b_M, Y_M, E_M$ in $B$.

The complementary arc $K = M^{-1}H$ is simple and nonseparating. The probe hypothesis applies to its reverse as well, and

$$(N_K^{-1})_{31} = (C^{-1}N)_{31} = N_{11}.$$

Hence $X_M = N_{11} \in B$. Apply this argument to $H^{-1}MH$ and $HMH^{-1}$. Their $(1,1)$ entries are

$$(C^{-1}NC)_{11} = N_{22} + uN_{12}, \qquad (CNC^{-1})_{11} = N_{33} + uN_{31} - tN_{32}.$$

They give $P_M, Q_M$. Boundary twisting preserves all the hypotheses, so the same argument applies to every conjugate. □

**Lemma 6.5.** *One has $R = B[t, u]$.*

*Proof.* Choose the compatible free arcs of Lemma 6.2, with cyclic matrices $N_1, \dots, N_{2g}$. We show that their entries and $\Delta^{\pm 1}$ generate the entire invariant ring.

Put $D_\Delta = \mathrm{diag}(1, 1, \Delta)$. Let $Y$ be the affine variety with coordinates $N_1, \dots, N_{2g} \in \mathrm{SL}(W)$ and $\Delta \in \mathbb{C}^*$, subject to the equations that the boundary word $C$ in these matrices satisfies $Ce_1 = e_2$ and $Ce_2 = e_3$. These are polynomial equations, since inverses of determinant-one matrices are adjugates. There is an explicit regular isomorphism

$$X_\Delta \longrightarrow \mathrm{SL}(W) \times Y, \qquad (M_i, p) \longmapsto (FD_\Delta^{-1}, (F^{-1}M_iF)_i, \Delta). \tag{6.10}$$

Here $M_i$ denotes the compatible free system, obtained by inverting selected $B_i$ in (6.1). The inverse is

$$(G, (N_i), \Delta) \longmapsto ((GD_\Delta N_i D_\Delta^{-1} G^{-1})_i,\ Ge_1). \tag{6.11}$$

Indeed, the reconstructed boundary takes $Ge_1$ to $Ge_2$ and then to $\Delta Ge_3$. Its cyclic frame is $GD_\Delta$ and its cyclic determinant is $\Delta$, verifying both inverse identities. The group $\mathrm{SL}(W)$ acts only by left multiplication on the first factor of (6.10). Therefore $R = \mathbb{C}[Y]$ is generated by the stated coordinates. In particular it is a finitely generated complex algebra, and every complex point is represented by actual matrices and a cyclic vector. Lemma 6.4 puts all its generators in $B[t, u]$; the reverse inclusion is immediate. □

6.3. **Integrality from bounded boundary conjugation.** We prove integrality before descending any unit ideal from $R$ to $B$. The argument includes valuations of arbitrary rank.

**Lemma 6.6.** *Let $K$ be a field with a rank-one valuation. Let $G = \langle N_1, \dots, N_r \rangle \subset \mathrm{SL}_3(K)$ and $C \in [G, G]$. Suppose, for each $i$, that all entries of $C^{-k}N_iC^k$ have valuations bounded below independently of $k \in \mathbb{Z}$. Then all eigenvalues of $C$ have valuation zero. In particular $\operatorname{tr} C$ and $\operatorname{tr} C^{-1}$ have nonnegative valuation.*

*Proof.* Extend the valuation to a finite splitting field of $C$ and choose a fixed Jordan basis. The valuation is still rank one, and the fixed change of basis changes the lower bound by at most a constant. If $C$ is diagonalizable, the $(a, b)$ entry of a conjugate is

$$(\lambda_b/\lambda_a)^k (N_i)_{ab}.$$

When $v(\lambda_a) \neq v(\lambda_b)$, one direction $k \longrightarrow \pm\infty$ violates the lower bound unless $(N_i)_{ab} = 0$. Thus each generator preserves the decomposition by eigenvalue valuation.

In dimension three the only additional case with different valuations is a Jordan block of size two for $\lambda$ and a third eigenvalue $\zeta$ of different valuation. Write

$$C = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \zeta \end{pmatrix}, \qquad C^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} & 0 \\ 0 & \lambda^k & 0 \\ 0 & 0 & \zeta^k \end{pmatrix}.$$

The four entries between the two blocks of $C^{-k}NC^k$ are

$$(23) = (\zeta/\lambda)^k N_{23}, \qquad (13) = (\zeta/\lambda)^k (N_{13} - kN_{23}/\lambda),$$
$$(31) = (\lambda/\zeta)^k N_{31}, \qquad (32) = (\lambda/\zeta)^k (N_{32} + kN_{31}/\lambda).$$

The $(2,3)$ and $(3,1)$ entries force $N_{23} = N_{31} = 0$; the other two then force $N_{13} = N_{32} = 0$. No asymptotic cancellation claim is needed. If all eigenvalues have the same valuation, including a single Jordan block of size three, that valuation is zero because $\det C = 1$.

Each generator therefore preserves every constant-valuation block $W_s$. On it, $L \mapsto \det(L|_{W_s})$ is a character of $G$, so it takes $C \in [G,G]$ to 1. If $\dim W_s = d_s$, taking valuations gives $d_s s = 0$, hence $s = 0$. The trace assertions follow by summing the eigenvalues and their inverses. □

**Lemma 6.7.** *The elements $t, u$ are integral over $B$. Thus $R$ is finite and integral over $B$.*

*Proof.* Let $v : \mathrm{Frac}(R)^* \to \Gamma$ be any valuation whose ring contains $B$. If a trace vanishes, omit it from a maximum involving its value. Suppose that one trace has negative value, and put

$$m = \max\{0, -v(t), -v(u)\} > 0.$$

For every compatible free generator $N_i$ and all $k \in \mathbb{Z}$, formula (6.6) holds for $C^{-k}N_iC^k$, with every named coefficient in $B$. Consequently

$$v\big((C^{-k}N_iC^k)_{ab}\big) \geq -m \qquad (i,k,a,b). \tag{6.12}$$

The same bound works for all $k$, although the coefficients vary.

Let

$$\Gamma_m = \{\gamma \in \Gamma : |\gamma| \leq nm \text{ for some } n \geq 1\}$$

be the principal convex subgroup generated by $m$. Coarsen $v$ by $\Gamma/\Gamma_m$. The conjugates in (6.12) and the companion matrix $C$ have integral entries for this coarsened valuation. Their determinants are 1, so their reductions are invertible. Reduction commutes with products, inverses, and conjugation. Denote the residue field by $K_0$ and reduction by a bar. The original valuation induces a residual valuation $v_0$ on $K_0$ with values in $\Gamma_m$. A nonzero residue of a representative of coarse value zero has its original value. Thus the reduced conjugates still have the uniform lower bound $-m$.

At least one of $t, u$ has value $-m$. It has coarse value zero, survives reduction, and is the corresponding trace of $\overline{C}$ or $\overline{C}^{-1}$. Its residual value is $-m$. Now quotient by the convex subgroup

$$\Gamma_{\inf} = \{\gamma \in \Gamma_m : n|\gamma| < m \text{ for every } n \geq 1\}.$$

The quotient $\Gamma_m/\Gamma_{\inf}$ is nonzero and Archimedean: every positive element outside the subgroup has a multiple at least $m$, whereas every element of $\Gamma_m$ is bounded by a multiple of $m$. The image of $m$ is strictly positive. Composing $v_0$ with this quotient gives a rank-one valuation on $K_0$.

The reduced generators and boundary still satisfy the commutator relation, and their conjugates retain a uniform lower bound. Lemma 6.6 says that both reduced traces have nonnegative value. This contradicts the negative image of $-m$. Hence every valuation ring of $\mathrm{Frac}(R)$ containing $B$ contains $t$ and $u$.

We recall why the valuative criterion now applies without a Noetherian hypothesis. For any subring $B$ of a field $K$, its integral closure in $K$ is the intersection of all valuation rings of $K$ containing $B$. Integral elements lie in these rings because valuation rings are integrally closed. Conversely, if $z$ is not integral, then $z^{-1}$ is a nonunit of $B[z^{-1}]$: otherwise an equation $1 = \sum_{j \geq 1} b_j z^{-j}$ gives a monic

equation for $z$. Choose a maximal ideal containing $z^{-1}$ and a valuation ring of $K$ dominating the corresponding local ring. In that valuation ring $z^{-1}$ lies in the maximal ideal, so $z$ is absent. This proves the criterion, using the usual existence of a dominating valuation ring. It shows that $t, u$ are integral over $B$. Finally $R = B[t, u]$ makes the extension finite. □

6.4. **A local trace identity.** For a compatible arc $L$, keep the notation of (6.6); thus $c_L = (N_L)_{31} = \Delta^{-1} b_+(L) \in B$. Multiplying the third row of one reconstruction matrix by the first column of another gives

$$c_{L_1 L_2} = c_{L_1} X_{L_2} + d_{L_1} Y_{L_2} + Q_{L_1} c_{L_2} - u c_{L_1} c_{L_2}. \tag{6.13}$$

The terms involving $t$ cancel. If the product is an actual nonseparating simple arc, its coefficient on the left belongs to $B$.

**Lemma 6.8.** *For every compatible nonseparating arc $M$, one has $u \in B[c_M^{-1}]$.*

*Proof.* Take $N, K$ from Lemma 6.3. Apply (6.13) to

$$(N, M), \qquad (NM, M), \qquad (M, K), \qquad (M, MK).$$

All factors and products are compatible nonseparating simple arcs. It follows that

$$u c_M c_N, \quad u c_M c_{NM}, \quad u c_M c_K, \quad u c_M c_{MK} \quad \text{belong to } B. \tag{6.14}$$

The four functions $c_N, c_{NM}, c_K, c_{MK}$ have no common zero in $\operatorname{Spec} R[c_M^{-1}]$. Indeed, the explicit slice (6.10) represents every complex point by actual holonomies and a cyclic vector $p$. At such a point put

$$q = Hp, \qquad \alpha = \Delta^{-1}(p \times Hp), \qquad U = \ker \alpha = \operatorname{span}(p, q), \qquad V_0 = \operatorname{span}(p, Mp).$$

Both spaces are planes, since $\Delta \neq 0$ and $\alpha(Mp) = c_M \neq 0$. If $c_N = c_{NM} = 0$, the independent vectors $Np, NMp$ lie in $U$, so $NV_0 = U$. Consequently

$$w = N^{-1} q \in V_0 \setminus \{0\}.$$

Since $K = N^{-1} H$, the other two vanishings say $\alpha(w) = \alpha(Mw) = 0$. On the basis $(p, Mp)$ of $V_0$, the forms $\alpha, \alpha M$ have matrix

$$\begin{pmatrix} 0 & c_M \\ c_M & \alpha(M^2 p) \end{pmatrix}.$$

Its determinant is $-c_M^2 \neq 0$, contradicting $w \neq 0$.

The ring $R[c_M^{-1}]$ is a finitely generated complex algebra. The Nullstellensatz gives

$$(c_N, c_{NM}, c_K, c_{MK}) R[c_M^{-1}] = R[c_M^{-1}].$$

By Lemma 6.7, this ring is integral over $B[c_M^{-1}]$. Lying-over therefore gives

$$(c_N, c_{NM}, c_K, c_{MK}) B[c_M^{-1}] = B[c_M^{-1}]. \tag{6.15}$$

A maximal ideal of the latter ring containing all four coefficients would otherwise lift to a prime of the former containing them. Multiply a linear combination giving 1 in (6.15) by $u$. Equation (6.14), after inverting $c_M$, puts every summand in $B[c_M^{-1}]$, proving the lemma. □

6.5. **A finite cover by actual arc coefficients.** We now remove the temporary localizations in Lemma 6.8. Let $N_1, \dots, N_{2g}$ be compatible free generators in cyclic coordinates, and set $K_i = N_i^{-1} C$. Form the following set of at most $36g$ elements of $B$:

$$\mathcal{C} = \{(C^{-k} N_i C^k)_{31},\ (C^{-k} K_i C^k)_{31} : 1 \le i \le 2g,\ 0 \le k \le 8\}. \tag{6.16}$$

Every element is the normalized probe of an actual compatible nonseparating arc.

**Lemma 6.9.** *The set $\mathcal{C}$ generates the unit ideal in $R$ and in $B$.*

*Proof.* Suppose that its elements vanish at a complex point of $\operatorname{Spec} R$, represented by matrices and a cyclic vector using (6.10). The linear operator

$$T : \operatorname{End}(W) \longrightarrow \operatorname{End}(W), \qquad T(L) = C^{-1}LC$$

is invertible on a nine-dimensional space. Its characteristic polynomial is monic of degree nine with nonzero constant term. Every sequence $\ell(T^k L)$, for a linear form $\ell$, satisfies the corresponding recurrence for all integers $k$. Nine consecutive zero terms force all terms to vanish, using the leading coefficient forward and the nonzero constant coefficient backward. Our assumption consequently gives

$$(C^{-k}N_iC^k)_{31} = (C^{-k}K_iC^k)_{31} = 0 \qquad (1 \le i \le 2g,\ k \in \mathbb{Z}). \tag{6.17}$$

Set $p_j = C^j e_1$ and $\ell_j = \mathbb{C}p_j$. Every three consecutive $p_j$ form a basis. Fix $N = N_i$. The first vanishing in (6.17) gives unique scalars $a_j, b_j$ with

$$Np_j = a_j p_j + b_j p_{j+1}. \tag{6.18}$$

The second, using $K_i = N^{-1}C$, says $N^{-1}p_{j+1} \in \operatorname{span}(p_j, p_{j+1})$, or equivalently $p_{j+1} \in \operatorname{span}(Np_j, Np_{j+1})$. Taking the determinant and substituting (6.18) gives

$$0 = -a_j b_{j+1} \det(p_j, p_{j+1}, p_{j+2}), \qquad a_j b_{j+1} = 0 \quad (j \in \mathbb{Z}).$$

If some $a_j \neq 0$, then $b_{j+1} = 0$. Invertibility of $N$ forces $a_{j+1} \neq 0$, and repetition gives $b_n = 0$ for all $n \geq j+1$. But $b_n = (C^{-n}NC^n)_{21}$ is another matrix-coefficient sequence of the same invertible nine-dimensional operator. Its zero right tail forces its entire two-sided sequence to vanish. Thus $N$ fixes every line $\ell_j$. If all $a_j = 0$, invertibility forces every $b_j \neq 0$, and $N$ takes $\ell_j$ to $\ell_{j+1}$ for every $j$. We have proved the uniform dichotomy

$$N \text{ fixes all } \ell_j, \text{ or } N\ell_j = \ell_{j+1} \text{ for all } j. \tag{6.19}$$

It holds at every spectral type of $C$, including nonsemisimple and periodic configurations.

Let

$$D_0 = \{L \in \mathrm{GL}(W) : L\ell_j = \ell_j \text{ for every } j \in \mathbb{Z}\}.$$

This is diagonal in the basis $(p_0, p_1, p_2)$, hence abelian. It is normalized by $C$, which shifts the lines. By (6.19), each $N_i$ belongs to $D_0 \cup CD_0$. Their generated group $G$ lies in $\langle D_0, C\rangle$, whose quotient by $D_0$ is cyclic. Therefore $[G, G] \subset D_0$. The actual boundary relation gives $C \in [G, G]$, but $C \notin D_0$ because $C\ell_0 = \ell_1 \neq \ell_0$. This contradiction rules out a common zero.

The Nullstellensatz yields $(\mathcal{C})R = R$. Integrality and lying-over yield $(\mathcal{C})B = B$, exactly as in (6.15). □

**Corollary 6.10.** *One has $u \in B$.*

*Proof.* Lemma 6.8 applies to each coefficient in $\mathcal{C}$. For every $c \in \mathcal{C}$, choose a power $c^{n_c}$ with $c^{n_c}u \in B$. These powers generate an ideal with the same radical as $(\mathcal{C})B = B$, so they themselves generate the unit ideal. Multiply a $B$-linear combination giving 1 by $u$. □

### 6.6. **Reading the boundary in the other direction.**

*Proof of Theorem 6.1.* To recover $t$, read the same surface with the opposite orientation, so that $H' = H^{-1}$. Keep the same subring $B \subset R$. The new cyclic determinant and traces are

$$\Delta' = \det(p, H^{-1}p, H^{-2}p) = -\Delta, \qquad t' = u, \qquad u' = t. \tag{6.20}$$

For every actual nonseparating arc in either direction, the new probes are $b'_+(M) = b_-(M)$ and $b'_-(M) = b_+(M)$; both already belong to the same $B$. The reversed interval order selects the opposite compatible directions. Lemmas 6.2 and 6.3 apply with this order and with $H'$. The boundary is still a product of commutators of a genuine free system.

For an explicit frame check, Cayley–Hamilton gives

$$F^{-1}(p, H^{-1}p, H^{-2}p) = \begin{pmatrix} 1 & u & u^2 - t \\ 0 & -t & 1 - tu \\ 0 & 1 & u \end{pmatrix}. \tag{6.21}$$

This matrix has determinant $-1$ and conjugates $C^{-1}$ to the companion matrix with parameters $(u, t)$. Hence the reconstruction, local trace calculation, and finite cover give $u' = t \in B$. Integrality of $R$ over $B$, needed for the unit-ideal descents, is already proved. Together with $u \in B$ and $R = B[t, u]$, this gives $R = B$.

This change of boundary direction changes how the argument is read. It does not assume that an orientation-reversing homeomorphism acts on a cluster algebra: both families of ordinary functions were hypotheses in the same subalgebra from the beginning. □

### 6.7. **Application to the native cluster algebra.**

**Theorem 6.11.** *For either color of a single marked point on the unique boundary of a surface of genus $g \geq 1$,*

$$\mathcal{A}[\mathcal{D}^{-1}] = \mathcal{R}[\mathcal{D}^{-1}].$$

*Proof.* First suppose the point is black, with native vector $p \in W$. The sole original coefficient is $\Delta = \det(p, Hp, H^2p)$, up to the fixed volume-order sign; see [3, Proposition 9] and the explicit one-point construction in [5, Section 3.5]. Put $B = \mathcal{A}[\Delta^{-1}]$.

Every nonseparating simple based arc extends to an ideal triangulation. Its bipod in the forward proxy construction is $b_+(M)$, with the fixed tensor-order sign if needed, and is an actual special invariant attached to that triangulation. It has native degree three. Every nonconstant special factor has a boundary leaf and hence positive native degree; invariance under the center of $\mathrm{SL}_3$ makes that degree a positive multiple of three. Thus a nonzero such bipod cannot split into two nonconstant special factors. The special factorization and seed theorem put it in the ordinary cluster algebra.

For clarity, its nonvanishing is a polynomial assertion that can be checked uniformly. Give the free fundamental group a faithful Schottky representation in $\mathrm{PSL}_2(\mathbb{C})$, lift its free generators to $\mathrm{SL}_2(\mathbb{C})$, and apply $\mathrm{Sym}^2$. Such a representation is obtained from generators with disjoint attracting and repelling discs by taking sufficiently high powers; the ping-pong argument makes the group free. The elements $1, H, M$ are distinct, as are $1, H^{-1}, M$: a nonseparating $M$ has nonzero homology class, whereas $[H] = 0$. Choose a point of $\mathbb{P}^1$ outside the finite fixed-point sets of the nontrivial pairwise quotients of these elements. Its three images are distinct. Their Veronese images on a nonsingular conic in $\mathbb{P}(W)$ are not collinear, so the determinant of corresponding lifts is nonzero. Choosing $p$ on this conic proves the required nonvanishing for both probes.

Apply the same reasoning to the reverse direction of every arc. Theorem 3.1 identifies the reverse boundary-proxy construction with the same cluster structure. Its bipod is

$$(H^{-1}p \times p)(Mp) = -b_-(M).$$

Consequently both probes belong to the same ordinary complex algebra. Theorem 6.1 now gives $B = \mathcal{R}[\Delta^{-1}]$.

For a white point, work instead in the volume space $W = V^*$ with native vector equal to the original covector, and replace each holonomy by $M^{-\mathsf{T}}$. This is a group representation and retains the actual commutator boundary relation. The unique original white coefficient is its cyclic determinant. The actual white bipods again have degree three, and their nonvanishing and special factorization follow in that volume space by the same conic argument. The two boundary directions provided by Theorem 3.1 place both probe families in this original white algebra. Theorem 6.1 applies directly. No color-complementation theorem is needed. □

*Remark* 6.12. The coefficient $\Delta$ is a nonzero nonconstant polynomial for every $g \geq 1$. Choose distinct nonzero $\lambda_1, \lambda_2, \lambda_3$ with product 1, a three-cycle permutation matrix $P$, and a diagonal matrix $D \in \mathrm{SL}_3$ with $[D, P] = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. The required ratios determine the diagonal entries of $D$ up to a common scalar, which can be chosen to give determinant 1. Take the first handle to be $(D, P)$, all other handles to be the identity, and $p = (1, 1, 1)^{\mathsf{T}}$. Then $\Delta$ is a nonzero Vandermonde determinant. It has degree three in $p$ and vanishes at $p = 0$, so is not a unit. Representations with $\Delta = 0$ are outside precisely the original coefficient localization in the theorem. No other principal open remains in the final statement.

## 7. Completion of the localization theorem

Theorems 5.5 and 6.11 give

$$\mathcal{A}(S)[\mathcal{D}^{-1}] = \mathcal{R}(S)[\mathcal{D}^{-1}] \tag{7.1}$$

in all admissible cases. The elementary disk conventions were settled in Section 2.4. We finish by identifying the upper algebra; the argument also records why no additional arc variable needs to be inverted in the conclusion.

**Lemma 7.1.** *For every native surface cluster structure, $\mathcal{U}_{\mathcal{D}} = \mathcal{R}(S)[\mathcal{D}^{-1}]$.*

*Proof.* Write $R = \mathcal{R}(S)[\mathcal{D}^{-1}]$ and choose a triangulation seed $(x_1, \ldots, x_n; \mathbf{d})$. Let $x_i'$ be its one-step exchange partner at $i$. Before localization, all $x_i, x_i'$ are irreducible special invariants in the UFD $\mathcal{R}(S)$. Distinct initial variables are nonassociate because they are algebraically independent and the only units are constants. Moreover $x_i'$ is not associate to $x_i$: otherwise $x_i x_i' = M_i^+ + M_i^-$ would give an algebraic relation between the initial coordinates. None of these irreducibles becomes a unit after inverting $\mathbf{d}$, since none is an associate of an original coefficient: $x_i'$ and all original coefficients belong to the algebraically independent coordinates of the $i$-adjacent seed.

Let $\mathfrak{p}$ be a height-one prime of $R$. If no $x_i$ belongs to $\mathfrak{p}$, the initial Laurent chart is contained in $R_{\mathfrak{p}}$. If $x_i \in \mathfrak{p}$, unique factorization implies that no other $x_j$, and no $x_i'$, belongs to $\mathfrak{p}$. Thus the Laurent chart of the $i$-adjacent seed is contained in $R_{\mathfrak{p}}$. Every element of $\mathcal{U}_{\mathcal{D}}$ belongs to all these finitely many charts and hence to every $R_{\mathfrak{p}}$. Normality gives

$$\mathcal{U}_{\mathcal{D}} \subseteq \bigcap_{\mathrm{ht}\mathfrak{p}=1} R_{\mathfrak{p}} = R.$$

Conversely every invariant is Laurent in every seed by [3, Theorem 16], so $R \subseteq \mathcal{U}_{\mathcal{D}}$. □

Combining (7.1), Lemma 7.1, and the sandwich (2.5) proves Theorem 1.1. Only original coefficients are inverted in that equality. All other local inversions used during the proof were removed by the explicit unit-ideal arguments.

## 8. A six-flag forest counterexample

An affine flag for $\mathrm{SL}_3$ is a pair $F = (v, \alpha)$ with $v \neq 0$, $\alpha \neq 0$, and $\alpha(v) = 0$. Consider six such flags in a disk, in cyclic order, and write

$$a = \alpha_1, \quad x = v_2, \quad b = \alpha_3, \quad y = v_4, \quad c = \alpha_5, \quad z = v_6. \tag{8.1}$$

No expression below uses the other half of any of these flags. The six inputs in (8.1) vary independently on a dense open set: any nonzero vector can be completed by a nonzero annihilating covector, and conversely. In particular the six incidence relations impose no extra identity on these inputs.

Define

$$h = a(y)b(z)c(x) - a(y)b(x)c(z) - a(z)b(y)c(x) - a(x)b(z)c(y), \tag{8.2}$$

$$d = \det(a, b, c), \qquad e = b(z)c(x) - b(x)c(z), \qquad Z = hde. \tag{8.3}$$

We will prove that $Z$ satisfies Theorem 1.4.

8.1. **Membership and the two tensor diagrams.** Each pairing in (8.2)–(8.3) joins different flags. The corresponding chord can be extended to a triangulation, in which the pairing is a cluster or coefficient variable. The covector tripod $d = \det(\alpha_1, \alpha_3, \alpha_5)$ is also a cluster variable. Indeed, triangulate the hexagon with central triangle $(1, 3, 5)$. If $T = \det(v_1, v_3, v_5)$ and $q_{ij} = \alpha_i(v_j)$, its center exchange is

$$Td = q_{13}q_{35}q_{51} + q_{15}q_{31}q_{53}.$$

The six side variables are distinct, so this is the ordinary center mutation. Thus $h, d, e, Z$ belong to the ordinary six-flag cluster algebra; no equality with an invariant ring is being assumed.

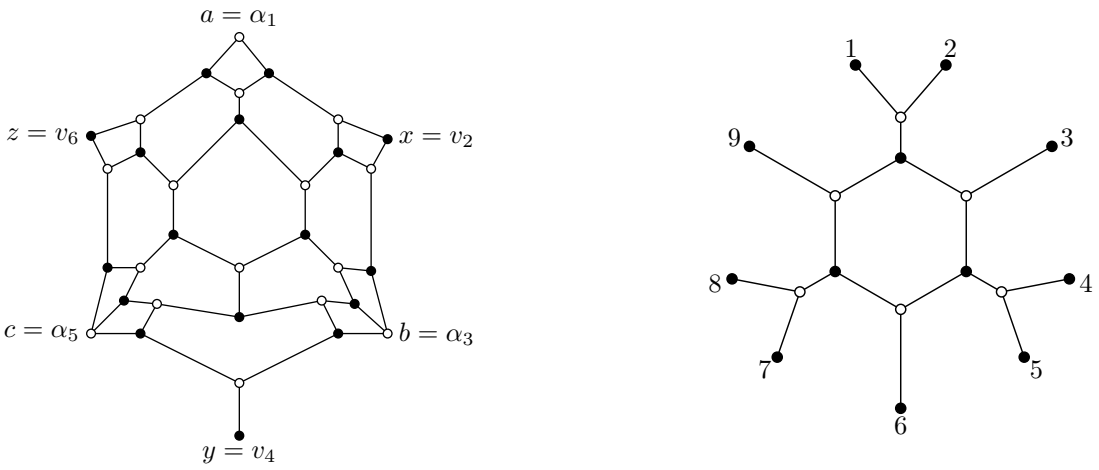


FIGURE 1. Left: the single biweb for $Z = hde$, obtained by reflecting the web in [2, Figure 26] and assigning its six boundary points to the six different flags. Right: the single-cycle nine-vector web underlying the prime factor $H$ of its Grassmannian pullback. Black and white internal vertices denote the two volume tensors.

**Lemma 8.1.** *The left diagram in Figure 1 is a non-elliptic single biweb, and its normalized invariant is $Z$.*

*Proof.* The graph has 27 internal trivalent vertices and 47 edges, with boundary degrees $(2, 2, 3, 1, 3, 2)$ in the order $(a, x, b, y, c, z)$. Its bounded faces away from the boundary are hexagons. There is no digon or loop and no internal four-cycle. These assertions can be read directly from the diagram; the complete rotation system is given in Appendix A. At each of the six flags only one of its two colors is used. Hence the graph remains bipartite even at the boundary, and has no boundary three-cycle. It therefore meets the additional non-ellipticity conditions for biwebs in [3, Section 12].

Here is a finite contraction computation fixing the scalar and sign. Set $a = e_1^*$, $b = e_2^*$, $c = e_3^*$. Then

$$d = 1, \qquad e = x_3z_2 - x_2z_3, \qquad h = ey_1 - x_3z_1y_2 - x_1z_2y_3. \tag{8.4}$$

Expand every internal volume tensor as $\varepsilon_{ijk}$ in the cyclic order of the diagram. A nonzero term assigns the three distinct indices $1, 2, 3$ to the three edges at each internal vertex. The edges ending at $a, b, c$ are forced to have indices $1, 2, 3$, respectively. The 12 remaining assignments, listed explicitly in Appendix A, give

$$\begin{aligned}[W] = {} & x_1x_2y_3z_2z_3 - x_1x_3y_3z_2^2 + x_2^2y_1z_3^2 \\ & - 2x_2x_3y_1z_2z_3 + x_2x_3y_2z_1z_3 + x_3^2y_1z_2^2 - x_3^2y_2z_1z_2.\end{aligned}$$

This is *he* by (8.4). This calculation includes the volume signs: reflection reverses the cyclic order of each of the 27 internal vertices, changing the sign of the unreflected contraction.

To pass from this slice to arbitrary $a, b, c$, let $A$ have rows $a, b, c$ and suppose $d \neq 0$. The matrix $G = \mathrm{diag}(1, 1, d^{-1})A \in \mathrm{SL}_3$ sends the three covectors to $e_1^*, e_2^*, de_3^*$. Both $[W]$ and $hde$ are $\mathrm{SL}_3$-invariant and have the same multidegree

$$(\deg_a, \deg_b, \deg_c; \deg_x, \deg_y, \deg_z) = (2, 3, 3; 2, 1, 2).$$

Thus the slice identity proves $[W] = hde$ when $d \neq 0$. Polynomiality proves it everywhere. □

**Lemma 8.2.** *The invariant $Z$ has a connected tree representation after splitting its boundary clasps.*

*Proof.* Take separate copies of every occurrence of a repeated subtree in

$$\begin{aligned} &k = c \times b, && A_R = a \times (z \times k), && A_L = (k \times x) \times a, \\ &L = b \times (x \times A_R), && R = (A_L \times z) \times c. \end{aligned} \tag{8.5}$$

We claim

$$\det(R, L, y) = hde. \tag{8.6}$$

On the slice of (8.4), the successive cross products give

$$L = (-x_1 z_2, 0, -e), \qquad R = (-x_3 z_1, -e, 0).$$

Expanding their determinant with $y$ gives

$$\det(R, L, y) = e^2 y_1 - e x_3 z_1 y_2 - e x_1 z_2 y_3 = eh.$$

The same $\mathrm{SL}_3$ transformation and multidegree argument as in Lemma 8.1 proves (8.6) without any restriction on the inputs.

Every cross product in (8.5) is a binary vertex. The outer determinant joins three disjoint argument trees at a new trivalent vertex. With the two occurrences of $k$ represented separately, the result has eleven internal trivalent vertices and thirteen leaves. It is a connected tree before equal boundary labels are clasped. Crossings are allowed in a tensor diagram, so its leaf order imposes no further restriction. This is precisely a forest representation in the sense of Conjecture 28. □

8.2. **The actual cluster map to nine vectors.** Let $w_1, \ldots, w_9 \in V$ and put $P_{ijk} = \det(w_i, w_j, w_k)$ for increasing indices. The map of [4, (25)–(27), Theorem 7.2] is

$$\begin{aligned} F_{2j-1} &= (w_{3j-2},\ w_{3j-2} \times w_{3j-1}), \\ F_{2j} &= (w_{3j},\ w_{3j-1} \times w_{3j}), \qquad 1 \leq j \leq 3. \end{aligned} \tag{8.7}$$

The incidence relations hold because a cross product annihilates each of its arguments. On the consecutive-generic open sets, the pullback $\Psi^*$ is a cluster quasi-isomorphism: it sends all variables of one cluster to variables of one target cluster multiplied by frozen Laurent monomials. It also sends frozen variables to frozen monomials. We use this theorem with its coefficient factors, not as an ordinary ring isomorphism.

For completeness the adjacent frozen pairings pull back as follows:

| $i$ | $\Psi^*(\alpha_i(v_{i+1}))$ | $\Psi^*(\alpha_{i+1}(v_i))$ |
|---|---|---|
| 1 | $P_{123}$ | $P_{123}$ |
| 2 | $P_{234}$ | $P_{345}$ |
| 3 | $P_{456}$ | $P_{456}$ |
| 4 | $P_{567}$ | $P_{678}$ |
| 5 | $P_{789}$ | $P_{789}$ |
| 6 | $P_{189}$ | $P_{129}$ |

Here $i+1$ is read cyclically. All entries are cyclic frozen Plücker coordinates; no additional mutable denominator is concealed in this map.

Substitution into (8.1) gives

$$a = w_1 \times w_2, \quad x = w_3, \quad b = w_4 \times w_5, \quad y = w_6, \quad c = w_7 \times w_8, \quad z = w_9.$$

Consequently

$$H := \Psi^* h = P_{126}P_{459}P_{378} - P_{126}P_{345}P_{789}$$

$$- P_{129}P_{456}P_{378} - P_{123}P_{459}P_{678}, \tag{8.8}$$

$$D := \Psi^* d = P_{127}P_{458} - P_{128}P_{457}, \tag{8.9}$$

$$E := \Psi^* e = P_{459}P_{378} - P_{345}P_{789}, \qquad \Psi^* Z = HDE. \tag{8.10}$$

8.3. **A prime factor in the full polynomial ring.** The following argument is independent of any cluster algebra.

**Lemma 8.3.** *The polynomial $h$ is irreducible in the polynomial ring $P_0$ of the eighteen coordinates of $a, b, c, x, y, z$.*

*Proof.* Let $A$ again have rows $a, b, c$, and localize at $d = \det A$. The coordinate change $X = Ax$, $Y = Ay$, $Z' = Az$ identifies this localization with

$$\mathbb{C}[A_{ij}, d^{-1}][X_1, X_2, X_3, Y_1, Y_2, Y_3, Z'_1, Z'_2, Z'_3].$$

In these coordinates $h$ becomes

$$h_0 = Y_1(X_3Z'_2 - X_2Z'_3) - Y_2X_3Z'_1 - Y_3X_1Z'_2.$$

Over the UFD $\mathbb{C}[A_{ij}, d^{-1}, X_1, X_2, X_3, Z'_1, Z'_2, Z'_3]$, its three $Y$-coefficients have greatest common divisor 1: the monomials $X_3Z'_1$ and $X_1Z'_2$ already have no common irreducible factor. It is therefore primitive and is irreducible by Gauss's lemma, since it is a nonconstant linear polynomial over the fraction field.

The determinant $d$ is irreducible in $\mathbb{C}[A_{ij}]$. For example, singular $3 \times 3$ matrices form the closure of the image of $\mathrm{Mat}_{3,2} \times \mathrm{Mat}_{2,3} \to \mathrm{Mat}_{3,3}$, $(U, W) \mapsto UW$, so this hypersurface is irreducible; its determinant has a nonzero derivative at $\mathrm{diag}(1, 1, 0)$. Hence a factor of $h$ that becomes a unit after inverting $d$ must be a constant times a power of $d$. But $d$ does not divide $h$: at $a = b = c = e_1^*$ one has $d = 0$ and $h = -2x_1y_1z_1 \neq 0$. A factorization of $h$ in $P_0$ is consequently trivial. □

**Lemma 8.4.** *Let $\mathrm{SL}_2$ act by right multiplication on the columns $u, v$ of a generic $3 \times 2$ matrix. The invariant ring is the polynomial ring in the three coordinates of $u \times v$. The same assertion holds with any additional polynomial variables fixed by the action.*

*Proof.* Put $p_{ij} = u_iv_j - u_jv_i$. If $p_{12} \neq 0$, the unique determinant-one column operation making the first two rows $\mathrm{diag}(1, p_{12})$ gives

$$\begin{pmatrix} 1 & 0 \\ 0 & p_{12} \\ -p_{23}/p_{12} & p_{13} \end{pmatrix}.$$

Indeed multiply the matrix on the right by $B^{-1}\,\mathrm{diag}(1, p_{12})$, where $B$ is its top $2 \times 2$ block. Its determinant is 1. An invariant polynomial is thus a polynomial in the three minors with $p_{12}$ inverted. Using the rows $1, 3$ gives the same statement with $p_{13}$ inverted. The three minors are algebraically independent, as the displayed normal form realizes their generic values. The intersection

$$\mathbb{C}[p_{12}, p_{13}, p_{23}, p_{12}^{-1}] \cap \mathbb{C}[p_{12}, p_{13}, p_{23}, p_{13}^{-1}] = \mathbb{C}[p_{12}, p_{13}, p_{23}]$$

follows from unique factorization. The reverse inclusion is immediate. The argument is unchanged over a polynomial ring in fixed variables. □

**Lemma 8.5.** *If a connected algebraic group $G$ with no nontrivial algebraic characters acts on a polynomial ring $P$, each irreducible factor in $P$ of a nonzero $G$-invariant polynomial is itself $G$-invariant.*

*Proof.* Unique factorization lets $G$ permute the finitely many associate classes of factors. Connectedness makes this permutation constant. Thus $gq = \chi(g)q$ for each irreducible factor $q$ and a scalar $\chi(g)$. Comparing a nonzero coefficient of $q$ shows that $\chi$ is a regular character. It is therefore trivial. □

**Proposition 8.6.** *$H$ is prime in $P = \mathbb{C}[w_{ri} : 1 \le r \le 3,\ 1 \le i \le 9]$.*

*Proof.* Let $G = (\mathrm{SL}_2)^3$ mix the column pairs $(1,2)$, $(4,5)$, $(7,8)$, respectively, and fix columns $3, 6, 9$. Applying Lemma 8.4 three times identifies $P^G$ with $P_0$ under

$$(a, b, c, x, y, z) = (w_1 \times w_2, w_4 \times w_5, w_7 \times w_8, w_3, w_6, w_9).$$

The generators are independent by the same normal forms. Hence $H$ is irreducible in $P^G$ by Lemma 8.3. Each $\mathrm{SL}_2$ is connected and has no nontrivial characters: it is generated by its two unipotent one-parameter subgroups, and an algebraic homomorphism from the additive group to $\mathbb{C}^*$ is trivial. Lemma 8.5 implies that any factorization of $H$ in $P$ would be a factorization in $P^G$. Thus $H$ is irreducible, and is prime because $P$ is a polynomial UFD. □

8.4. **Excluding every frozen Laurent multiple.** The right diagram in Figure 1 is the alternating six-cycle with its three covector leaves replaced by the forks $(1,2)$, $(4,5)$, $(7,8)$. Its polynomial is $H$, up to the fixed volume sign. One direct verification before the forks are inserted is

$$h = \mathrm{tr}\big(C(a)C(x)C(b)C(y)C(c)C(z)\big), \qquad C(u)w = u \times w,$$

using coordinate identifications to compose the alternating maps. The identity $C(u)C(v) = vu^t - (u^t v)I$ gives exactly the four terms of (8.2). Inserting the three ordered forks gives (8.8).

This is a non-elliptic single web with an internal cycle, and no arborization move can start. In the planar situation an initial arborizing move requires a boundary four-cycle; see [4, Example 9.7] or [2, Definitions 10.2–10.3]. All nine boundary vertices here are univalent. Equivalently, its only nontrivial rooted tree branches are the three forks, with different leaf sets, so none is a matching sibling pair. The web is therefore not arborizable.

Let $R_9 = P^{\mathrm{SL}_3}$, the coordinate ring of the affine Grassmannian cone for $\mathrm{Gr}(3, 9)$. Scott's theorem identifies its ordinary coordinate-ring cluster algebra with $R_9$ [10, Theorem 3]. The frozen variables are the cyclic three-column minors. Fraser's theorem states that every mutable variable is an indecomposable arborizable web invariant [4, Theorem 9.10(1)]. The linear independence of the normalized web basis [7] implies that no mutable variable is a nonzero constant multiple of $H$. Nor is $H$ a frozen variable: it has total coordinate degree 9, whereas a frozen minor has degree 3.

Suppose now that

$$HDE = M_f \prod_j X_j^{n_j}, \qquad n_j \ge 0, \tag{8.11}$$

where $M_f$ is a frozen Laurent monomial and the $X_j$ are mutable cluster variables. We do not even need them to lie in one cluster. Clear the frozen denominators in (8.11). By Proposition 8.6, $H$ divides the polynomial on the right in the full coordinate ring $P$. Its degree excludes divisibility of any frozen minor, so $H$ divides some $X_j$. Write $X_j = HQ$ with $Q \in P$. Invariance gives $H(gQ - Q) = 0$ for every $g \in \mathrm{SL}_3$, whence $Q \in R_9$.

Mutable variables are irreducible in the ordinary cluster algebra with noninverted coefficients [6, Theorem 3.1, with $p = n$]. Thus the factorization $X_j = HQ$ in $R_9$ forces $Q$ to be a unit. A unit of $R_9 \subset P$ is a nonzero constant, so $X_j$ is a nonzero constant multiple of $H$, a contradiction. Notice that this argument needs no factoriality assertion about $R_9$ and no assertion that an arbitrary cluster variable is irreducible in $P$.

Finally, if $Z$ were a six-flag cluster monomial, the quasi-isomorphism (8.7) would give (8.11). The same conclusion holds if $Z$ is multiplied or divided by a frozen monomial, since the frozen pullbacks are the minors listed above. This is impossible. Lemmas 8.1 and 8.2 now prove Theorem 1.4.

## Appendix A. The finite biweb contraction

This appendix specifies the left diagram of Figure 1 without depending on drawing resolution. Number its boundary vertices

$$0 = x, \quad 1 = a, \quad 2 = z, \quad 3 = b, \quad 4 = c, \quad 5 = y.$$

The boundary order is $(1, 0, 3, 5, 4, 2)$. Internal black vertices are

$$6, 7, 9, 10, 11, 12, 13, 14, 15, 24, 25, 28, 29, 31;$$

the other internal vertices are white. The following triples give the cyclic order used by each volume tensor. They also determine every edge. Boundary adjacency lists are

$$0 : (22, 23),\ 1 : (6, 7),\ 2 : (20, 21),\ 3 : (10, 11, 12),\ 4 : (13, 14, 15),\ 5 : (32).$$

| $v$ | cyclic neighbors | $v$ | cyclic neighbors |
|---|---|---|---|
| 6 | $(22, 8, 1)$ | 20 | $(25, 2, 7)$ |
| 7 | $(8, 20, 1)$ | 21 | $(15, 2, 25)$ |
| 8 | $(9, 7, 6)$ | 22 | $(0, 24, 6)$ |
| 9 | $(27, 26, 8)$ | 23 | $(12, 24, 0)$ |
| 10 | $(32, 16, 3)$ | 24 | $(23, 27, 22)$ |
| 11 | $(3, 16, 17)$ | 25 | $(26, 21, 20)$ |
| 12 | $(3, 17, 23)$ | 26 | $(29, 25, 9)$ |
| 13 | $(32, 4, 18)$ | 27 | $(28, 9, 24)$ |
| 14 | $(18, 4, 19)$ | 28 | $(17, 30, 27)$ |
| 15 | $(4, 21, 19)$ | 29 | $(30, 19, 26)$ |
| 16 | $(11, 10, 31)$ | 30 | $(31, 29, 28)$ |
| 17 | $(12, 11, 28)$ | 31 | $(18, 30, 16)$ |
| 18 | $(31, 13, 14)$ | 32 | $(5, 13, 10)$ |
| 19 | $(14, 15, 29)$ | | |

For the contraction, edges are ordered lexicographically by their two endpoint numbers:

$$(0, 22),\ (0, 23),\ (1, 6),\ (1, 7),\ (2, 20),\ (2, 21),\ (3, 10),\ (3, 11)$$
$$(3, 12),\ (4, 13),\ (4, 14),\ (4, 15),\ (5, 32),\ (6, 8),\ (6, 22),\ (7, 8)$$
$$(7, 20),\ (8, 9),\ (9, 26),\ (9, 27),\ (10, 16),\ (10, 32),\ (11, 16),\ (11, 17)$$
$$(12, 17),\ (12, 23),\ (13, 18),\ (13, 32),\ (14, 18),\ (14, 19),\ (15, 19),\ (15, 21)$$
$$(16, 31),\ (17, 28),\ (18, 31),\ (19, 29),\ (20, 25),\ (21, 25),\ (22, 24),\ (23, 24)$$
$$(24, 27),\ (25, 26),\ (26, 29),\ (27, 28),\ (28, 30),\ (29, 30),\ (30, 31)$$

At $a = e_1^*, b = e_2^*, c = e_3^*$, each incident edge at boundary vertices $1, 3, 4$ is forced to have color $1, 2, 3$, respectively. Requiring the three colors at each internal vertex to be distinct leaves the following 12 colorings. A color word has one digit per edge in the displayed order. Its sign is the product of the 27 alternating-tensor signs in the cyclic orders above.

| sign | edge colors (in lexicographic edge order) |
|---|---|
| - | 121132222333323321323113131221212233132132113 21 |
| - | 231113222333223321321331312112122233311232113 21 |
| + | 221133222333123321231331311221212233121323113 21 |
| + | 211132222333323321323113131221212233131232113 21 |
| + | 231131222333223321321331312112122233131232113 21 |
| - | 231132222333123321321331311221212233131232113 21 |
| + | 121123222333332231233113131221212233123123113 21 |
| + | 321113222333232231231331312112122233211323113 21 |
| - | 321123222333132231231331311221212233123123113 21 |
| - | 311122222333332231323113131221212233131232113 21 |
| - | 331121222333232231321331312112122233131232113 21 |
| + | 331122222333132231321331311221212233131232113 21 |

At boundary vertices $0, 5, 2$, multiply the coordinates of $x, y, z$ indicated by the incident edge colors. Grouping the twelve terms gives:

| monomial | positive terms | negative terms | coefficient |
|---|---|---|---|
| $x_1x_2y_3z_2z_3$ | 2 | 1 | 1 |
| $x_1x_3y_3z_2^2$ | 0 | 1 | $-1$ |
| $x_2^2y_1z_3^2$ | 1 | 0 | 1 |
| $x_2x_3y_1z_2z_3$ | 0 | 2 | $-2$ |
| $x_2x_3y_2z_1z_3$ | 2 | 1 | 1 |
| $x_3^2y_1z_2^2$ | 1 | 0 | 1 |
| $x_3^2y_2z_1z_2$ | 0 | 1 | $-1$ |

This is exactly the slice polynomial in the proof of Lemma 8.1. Exhaustiveness can also be checked without polynomial expansion: start with the eight forced boundary half-edges, and at a vertex with two known colors force the third; whenever only one color is known, choose the order of the two remaining colors. Discard a choice immediately if it repeats a color at a neighboring vertex. The rotation and edge tables make this a finite three-color calculation.

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