The magic of free spheres

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Abstract

We study integration, diffusion and vibration on the classical, half-liberated and free spheres. The latitude polynomials recover the free hyperspherical law and describe heat released at a pole outside its stationary support. On the free ball two invariant integrals admit Weingarten formulas. The divergence integral selects a radial completion on which we construct a closed energy, a completely positive Poisson extension and its boundary normal derivative. On the 2 quantum balls, degree 3 already gives nonpolynomial harmonic extensions; the free boundary spectrum has exponential multiplicities. We distinguish this radial theory from projected spherical heat and the boundary obstruction on the universal ball. Stereographic charts and mixed coordinate laws lead back to the sphere, whose wave frequencies we compare with the quantum sphere played by Connes.

Dedicated to Alain

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Introduction

On an ordinary sphere the same spherical harmonics serve two purposes. They describe how a temperature distribution relaxes to equilibrium and which frequencies occur when the sphere vibrates. What becomes of this connection when the coordinates cease to commute? Connes' reconstruction theorem [13] recovers a compact Riemannian manifold from suitable spectral data. For our spheres we begin with the coordinate algebra and its invariant integral. Together they let us separate a polynomial of a given degree from all polynomials of lower degree.

We work with the classical, half-liberated and free real spheres of [4]. All 3 have self-adjoint coordinates satisfying $\sum_i x_i^2 = 1$. Their additional relations range from commutation, through half-commutation, to no further relations. The corresponding orthogonal quantum groups provide invariant integrals and Weingarten formulas. Section 1 recalls these constructions and develops the harmonic decomposition. On the free sphere, taking the common kernel of adjacent contractions gives both the degree spaces V_r° and the orthogonal polynomials of a single coordinate.

To pass from harmonics to heat we must choose a Laplacian. We use the intrinsic spherical semigroup of [16]. In each degree r, let Q_r be the orthogonal polynomial of one coordinate, normalized by $Q_r(1) = 1$. Its slope at the pole, $\lambda_r = Q_r'(1)$, is the eigenvalue of the positive Laplacian L on the entire harmonic space V_r°. With this convention $\lambda_1 = 1$ and the heat operator at time s is e^{-sL}. The successive gaps have a particularly simple form (Section 2).

Theorem A. *For $N > 2$, let λ_r be the eigenvalue of L on V_r°, put $g_r = \lambda_{r+1} - \lambda_r$, and write $N = 2\cosh(2u)$ with $u > 0$. We have $\lambda_0 = 0$, $g_0 = 1$, and*

$$\begin{aligned} g_r^{cl} &= 1 + 2r/(N-1), && r \geq 1, \\ g_{2m-1}^{*} = g_{2m}^{*} &= 1 + 2m/(N-1), && m \geq 1, \\ g_{2m-1}^{+} = g_{2m}^{+} &= \tanh((2m+1)u)/\tanh u, && m \geq 1. \end{aligned}$$

In particular, with $c_N = \coth u$ and $l_N = \sum_{j\geq 0}(g_j^+ - c_N)$, we have $\lambda_r^+ = c_N r + l_N + O_N(e^{-2ur})$. The classical and half-liberated eigenvalues grow quadratically with the degree.

The central process on O_N^+ gives a different second eigenvalue even after matching the first (Section 2). The polynomials Q_r also determine the equilibrium law μ_N of the latitude x_N under the invariant trace h. In Section 3 their orthogonality measure recovers the free hyperspherical law of [3].

Theorem B. *For $N > 2$, let $-1 < q < 0$ satisfy $q + q^{-1} = -N$ and put $a = 2/\sqrt{N+2}$. The monic orthogonal polynomials of $W = \sqrt{N+2}\, x_N$ are the monic normalizations of the continuous q-ultraspherical polynomials $C_r(w/2; -q \,|\, q)$. The negative parameter is covered by Theorem 3.4. The support of μ_N is $[-a, a]$, and W converges in law to the standard semicircle as $N \to \infty$.*

In equilibrium the latitude lies in $[-a, a]$, with $a < 1$. On the first-row algebra $A_u \subset C_u(O_N^+)$ we can still evaluate x_N at the ordinary north pole, where its value is 1. Since spherical heat acts on A_u, we can release heat at this pole and ask how the latitude law enters its stationary interval.

Theorem C. *Write $N = 2\cosh(2u) > 2$, $c_N = \coth u$, $a = 1/\cosh u$ and $s_* = u/c_N$. For $0 < s < s_*$ the latitude law of intrinsic heat started at the north pole character on A_u has exactly one atom, at*

$$b(s) = \frac{\cosh(u - c_N s)}{\cosh u}.$$

The remaining measure has a continuous density relative to μ_N. The atom reaches a with vanishing mass. At $s = s_*$ the law is absolutely continuous, with density asymptotic to a positive constant times $(a - y)^{-1}$ at the upper edge. This density belongs to $L^1(\mu_N)$ but not to $L^2(\mu_N)$. For $s > s_*$ it is continuous.

Section 4 computes the atom's mass for every exterior starting latitude. For a source at the pole, $s_* - \frac{1}{2}\log N \to 0$ as N grows, so the atom reaches the edge on the intrinsic cutoff scale of [23].

We then pass from the sphere to the ball. Let $C(B_+^N)$ be the universal algebra of N self-adjoint coordinates with $\sum_i x_i^2 \leq 1$. Deleting one sphere coordinate leaves the defect $d = 1 - \sum_i x_i^2$; adjoining a self-adjoint square root of d recovers the sphere. Section 5 expresses this as

$$C(S_+^N) \simeq C(B_+^N) *_{C([0,1])} C([-1, 1]),$$

where the common interval coordinate is sent to d and t^2, respectively. Choosing the positive square root gives the upper hemisphere. For integration on the ball there is a further choice. Rotations fix the squared radius, so their invariance cannot specify its distribution. Two classical descriptions of uniform ball volume suggest different invariant integrals in the free case.

Theorem D. *There is a unique state τ_{rad} on $C(B_+^N)$ satisfying*

$$(N+k)\tau_{rad}(p) = Nh_N(p|_{S_+^{N-1}}) \qquad (\deg p = k)$$

for every homogeneous polynomial p, where h_N is the invariant spherical trace. It is an invariant tracial state. Deleting two coordinates of S_+^{N+1} gives another invariant trace $\tau_{N,2}$. Both have Weingarten formulas and give x_1^2 the integral $1/(N+2)$, while

$$\tau_{rad}(x_1^4) = \frac{2}{(N+1)(N+4)}, \qquad \tau_{N,2}(x_1^4) = \frac{2}{(N+2)(N+3)}$$

for every $N \geq 1$.

For the divergence integral τ_{rad} the radius ρ becomes a central variable, and at each positive radius the angular variables generate the reduced spherical algebra A. Section 6 identifies the resulting ball algebra with

$$B = \{F \in C([0,1], A) \mid F(0) \in \mathbb{C}1\}$$

and equips it with the polar-coordinate energy. Evaluation at $\rho = 1$ is restriction to the boundary, so we can ask how to extend data from A harmonically to the ball. For the projected integral $\tau_{N,2}$ there is a different starting point, namely heat on the ambient sphere.

Theorem E. *For $N \geq 2$, the radial energy*

$$E(F) = N \int_0^1 \left(\|F'\|_2^2 + \frac{N-1}{\rho^2} \|L^{1/2} F\|_2^2 \right) \rho^{N-1}\, d\rho$$

is closed. The Poisson extension $H : A \to B$ is unital and completely positive, and on finite harmonic expansions it is

$$Hf(\rho) = \sum_k \rho^{\alpha_k} f_k, \qquad \alpha_k = \sqrt{\frac{(N-2)^2}{4} + (N-1)\lambda_k} - \frac{N-2}{2}.$$

It is the unique continuous harmonic extension. Its boundary normal derivative L_∂ acts by α_k in degree k. For finite-energy boundary data f, every extension F satisfies

$$E(F) = N\|L_\partial^{1/2} f\|_2^2 + E(F - Hf).$$

Thus Hf is the unique extension of least energy.

Classically $\alpha_k = k$. On both quantum balls $2 < \alpha_3 < 3$, so a nonzero degree-3 harmonic has no polynomial Poisson extension. Averages of spherical heat operators give complete positivity [8, 25, 29]. On the free sphere with $N > 2$, the boundary semigroup e^{-tL_∂} is compact for $t > 0$. Its trace diverges because the angular multiplicities grow exponentially.

For projected heat the defect d measures the squared length of the deleted coordinates. Section 7 computes its evolution and tests coordinate-preserving Poisson extension on the universal disk.

Theorem F. *Let $N, j \geq 1$ and $M = N + j$. Intrinsic heat on S_+^{M-1} preserves the polynomials in the first N coordinates. On their norm closure in the spherical trace representation it gives a Markov semigroup preserving $\tau_{N,j}$. For $d = 1 - \sum_{i=1}^N x_i^2$ we have*

$$T_s d = \frac{j}{M} + e^{-2Ms/(M-1)} \left(d - \frac{j}{M} \right)$$

for every $s \geq 0$. For the universal disk B_+^2 there is no unital completely positive map $C(S_+^1) \to C(B_+^2)$ sending both boundary coordinates to the corresponding disk coordinates.

The obstruction comes from a 3×3 matrix pair with nonzero defect which splits off from each of its disk dilations. Section 7 relates it to the matrix boundary [18, 19, 20] and compares the self-adjoint setting with Poisson transforms on Fock space [27, 28].

Classically, dividing the equatorial coordinates by one minus the height sends the punctured sphere onto Euclidean space. On the free sphere we must decide on which side to divide. Section 8 characterizes the image of right division by a relation on the chart coordinates themselves.

Theorem G. *In a unital C^*-algebra, let $t, x_1, \dots, x_N$ be self-adjoint with $t^2 + \sum_i x_i^2 = 1$ and $1 - t$ invertible. The correspondence $z_i = x_i(1-t)^{-1}$ is a bijection onto families satisfying*

$$(1+R)z_i = z_i^*(1+R), \qquad R = \sum_i z_i^* z_i.$$

Its inverse is

$$t = (R-1)(R+1)^{-1}, \qquad x_i = 2z_i(R+1)^{-1},$$

with $1 - t = 2(1+R)^{-1}$.

Symmetric division keeps the new coordinates self-adjoint at the cost of losing the height. Already on the free circle, different heights can give the same symmetric coordinate. The right chart also poses an integration question. Its squared radius is a function of the latitude alone, while its direction involves the other coordinates. Section 9 evaluates the Fourier coefficients of the latitude density by a single residue and recovers the finite moment formulas of [3]. We then examine which mixed moments are needed for block radii and for the joint law of latitude and direction.

Connes' performance of *Au clair de la lune* on a quantum sphere [12] suggests a question we can ask of our three spheres. Once their first notes have been tuned alike, how far into the scale must we go to hear a difference? We take the scalar wave frequency $\omega_r = \sqrt{\lambda_r}$ as the pitch of degree r. There are $\dim V_r^\circ$ independent modes at that pitch.

For chords we compare ratios of frequencies. The major third and perfect fifth in just intonation have ratios $5/4$ and $3/2$ to the root. We measure the signed error of a ratio x from its target y by $1200 \log_2(x/y)$ cents. An octave contains 1200 cents and an equal-tempered semitone contains 100. An error of 3.837 cents amounts to a relative frequency error of about 0.22%.

Theorem H. *For $N \geq 3$, pitch first distinguishes the classical sphere from each of the half-liberated and free spheres in degree 3, and distinguishes the latter pair in degree 4. Multiplicity distinguishes these pairs in degrees two and three, respectively. On S^2_+ the degrees $8, 12, 17$ give frequency ratios*

$$1 : 1.24723335\ldots : 1.50005245\ldots,$$

with maximum error below 3.837 cents from the major triad $1 : 5/4 : 3/2$. For the quantum-integer pitches $[r]_p = (p^r - p^{-r})/(p - p^{-1})$, with $p = 2^{1/12}$, the infimum of this error over all triples of distinct positive degrees is $13.68628\ldots$ cents.

The quantum-integer scale in this comparison is the angular Dirac scale of [15], with the parameter fixed in the theorem. Connes credits the spin geometry of [6], whose full operator contains an additional term. Section 10 explains how the choice $p = 2^{1/12}$ relates to the intervals in his melody.

The growth of the spherical eigenvalues also tells us where to look for chords in higher degrees. On the free sphere λ_r grows linearly, so for every positive integer b we have

$$\frac{\omega_{b^2 m}}{\omega_m} \to b \qquad (m \to \infty).$$

Thus the degrees $16m, 25m, 36m$ approach the frequency ratios $4 : 5 : 6$. On the classical and half-liberated spheres, degrees $4m, 5m, 6m$ have the same limit because their eigenvalues grow quadratically. On the quantum-integer scale it is a fixed shift of degree that approaches a fixed equal-tempered interval.

Figure 1 gives two ways of picturing the free two-sphere. On the left, a unit vector in the Pauli representation assigns an expectation to each coordinate. As the vector varies, the three expectations trace a sphere of radius $1/\sqrt{3}$ (Proposition 1.3). On the right, the three scales show the frequencies of the scalar wave equation. Their first two pitches coincide.

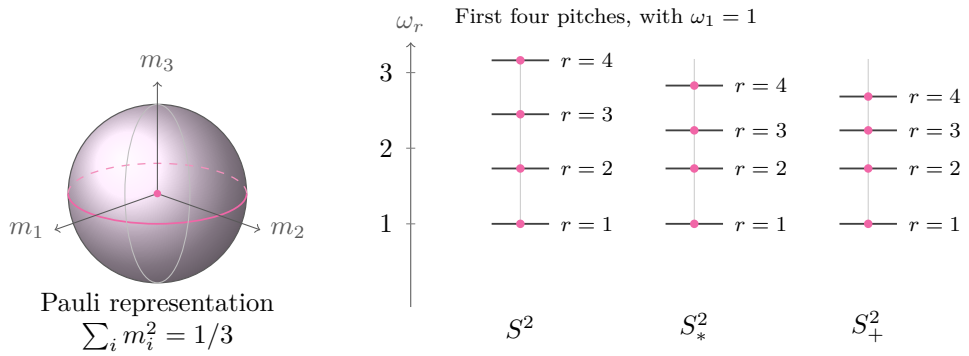

Figure 1: A matrix portrait and a musical portrait. Left: pure-state expectations in the Pauli representation of S^2_+. Right: the intrinsic scalar-wave pitches for $N = 3$; their squares are $(1, 3, 6, 10)$, $(1, 3, 5, 8)$ and $(1, 3, 5, 36/5)$, respectively.

In the figure the classical and quantum pitches first separate at the third note. To account for this difference we begin with the coordinate relations and the harmonic spaces they determine.

1 Spheres and spherical harmonics

The coordinate functions on the ordinary sphere in $\mathbb{R}^N$ satisfy $\sum_i x_i^2 = 1$ and commute in pairs. Keeping the sphere equation while relaxing commutation gives two further spheres. The difference between them concerns the order of the letters in a coordinate word.

Definition 1.1. *The classical, half-liberated and free real spheres have universal unital C^*-algebras generated by $x_i = x_i^*$ with $\sum_i x_i^2 = 1$ and the additional relations*

space	*additional relations*
S^{N-1}	$x_i x_j = x_j x_i$
S^{N-1}_*	$x_i x_j x_k = x_k x_j x_i$
S^{N-1}_+	*none*

for all coordinate indices. We use N for the number of coordinates.

Since $0 \leq x_i^2 \leq 1$, each generator is a contraction and the universal algebras exist. We count coordinates in the same way for the balls B^N. Unless specified otherwise we take $N \geq 2$ for spheres. The ball with one coordinate has the two-point sphere as its boundary.

Proposition 1.2. *The 3 spheres have the same characters, parametrized by S^{N-1}. Adding half-commutation and then commutation gives surjections*

$$C(S^{N-1}_+) \to C(S^{N-1}_*) \to C(S^{N-1}),$$

which induce the identity on this common character space.

Proof. A character sends the generators to real scalars whose squares add to one. Such scalars satisfy all the additional relations. Conversely every ordinary sphere point gives a character on each algebra. The quotient maps impose exactly the additional relations indicated above. □

Characters see only the ordinary sphere. In a matrix representation we can instead take the expectation of each coordinate in the same unit vector. For the Pauli matrices on $\mathbb{C}^2$ this gives the smaller sphere in Figure 1.

Proposition 1.3. *For the Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the assignment $x_i \to A_i = \sigma_i/\sqrt{3}$ defines a 2-dimensional representation of $C(S^2_+)$. Its pure-state expectations $m_i = \langle A_i v, v\rangle$, for $\|v\| = 1$, range over the sphere $\sum_i m_i^2 = 1/3$.

Proof. Each σ_i is self-adjoint and has square I, so $\sum_i A_i^2 = I$. Every rank-one projection on $\mathbb{C}^2$ has the form

$$vv^* = \frac{1}{2}(I + s \cdot \sigma), \qquad s \in \mathbb{R}^3,$$

because $I, \sigma_1, \sigma_2, \sigma_3$ form a basis of the self-adjoint matrices and $Tr(vv^*) = 1$. The identity $(s\cdot\sigma)^2 = |s|^2 I$ shows that this is a rank-one projection exactly when $|s| = 1$. Since $Tr(\sigma_i\sigma_j) = 2\delta_{ij}$, we obtain $m_i = Tr(vv^*A_i) = s_i/\sqrt{3}$. Every unit s gives a rank-one projection, proving that the whole expectation sphere occurs. □

To integrate coordinate words we use the orthogonal symmetries of each sphere. Their matrix entries obey the same commutation rules as the sphere coordinates.

Definition 1.4. *The groups O_N, O_N^*, O_N^+ have fundamental matrices $U = (U_{ij})$ with self-adjoint entries, $UU^t = U^tU = 1$, and respectively commutation, half-commutation or no additional relations. Their coproduct is*

$$\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj},$$

extended as a unital homomorphism.

The free orthogonal quantum group O_N^+ was constructed in [32]. For the free unitary group and its representation theory see [1]. The first row of an orthogonal matrix satisfies the sphere equation. This lets us integrate words on each sphere by taking the Haar state [33] on the corresponding group, as in [4].

Proposition 1.5. *The map $x_i \to U_{1i}$ defines a state h on each sphere by restriction of the Haar state. It is the invariant tracial state of [4]. All coordinates have the same law, determined by their moments.*

Proof. The first row satisfies the sphere relation and the relevant commutation rules, giving a homomorphism on the universal algebra. The orthogonal quantum groups are of Kac type, so their Haar states are traces. First-row invariance is the construction in [4]. Permutation symmetry identifies the coordinate laws, and compact support makes their moment problems determinate. □

The sphere equation fixes the second moment of a coordinate on all three spheres. Its fourth moment already detects the passage from commuting to half-commuting coordinates.

Proposition 1.6. *For each sphere $h(x_1) = 0$ and $h(x_1^2) = 1/N$. In degree 4,*

$$h_{class}(x_1^4) = \frac{3}{N(N+2)}, \qquad h_*(x_1^4) = h_+(x_1^4) = \frac{2}{N(N+1)},$$

with h_{class}, h_*, h_+ denoting the respective invariant traces.

Proof. Sign changes and $\sum_i x_i^2 = 1$ give the first 2 moments. Normalizing a real Gaussian vector gives x_1^2 the beta law with parameters $1/2, (N-1)/2$ in the classical case. Its second moment gives the first displayed fourth moment.

For the half-liberated sphere take a uniform vector w on the unit sphere of $\mathbb{C}^N$ and use the matrix model

$$M(w_i) = \begin{pmatrix} 0 & w_i \\ \bar{w}_i & 0 \end{pmatrix}.$$

Integrated normalized matrix trace gives h_* by [4, 16]. The matrices half-commute and $M(w_1)^2 = |w_1|^2 I$. Normalized complex Gaussians give $|w_1|^2$ the beta law with parameters $1, N-1$. Its second moment is the required value.

For the free sphere the two degree 4 pairing vectors have Gram matrix

$$G = \begin{pmatrix} N^2 & N \\ N & N^2 \end{pmatrix}.$$

Their coordinates against $e_1^{\otimes 4}$ are both 1. The pairing formula of [4], recalled in Proposition 1.10, gives $h_+(x_1^4) = (1,1)G^{-1}(1,1)^t = 2/[N(N+1)]$. □

Taking these moments uses only the trace. Evaluation at a scalar point, which we shall need for heat started at the pole, also depends on the choice of C^*-completion.

Definition 1.7. *The universal sphere algebra is specified by its generators and relations. Write A_u for the closure of its first-row image in $C_u(O_N^+)$, and A for the image in the spherical trace representation. Use the corresponding groups for the classical and half-liberated spheres.*

The canonical maps go from the universal sphere algebra to A_u and then to A. Every ordinary sphere point gives a character of A_u: complete it to the first row of an orthogonal matrix and evaluate there. The trace is faithful on A. The north-pole character need not descend to A, as Section 3 will show. All scalar-point heat laws below are evaluated on A_u, before passage to the trace representation.

One familiar symmetry survives even before we take the trace. On S^3, simultaneous rotation of its two complex coordinates gives the Hopf fibration. The same rotations act on S^3_+, where one can describe the fixed words explicitly. This example uses only the sphere equation. Other quantum three-spheres arise from equations for the Chern character of a unitary [14].

Proposition 1.8. *On the ordinary $S^3 \subset \mathbb{C}^2$, the map $z \to zz^*$ identifies the quotient by common phase with S^2. On S^3_+, put $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$. The rule $z_i \to e^{i\theta} z_i$ defines a circle action whose fixed algebra is the closed linear span of words with equally many starred and unstarred letters.*

Proof. For a unit vector $z = (z_1, z_2)^t$, the matrix

$$P(z) = zz^* = \begin{pmatrix} |z_1|^2 & z_1\bar{z}_2 \\ z_2\bar{z}_1 & |z_2|^2 \end{pmatrix}$$

is the projection onto its line. Equality of these projections means that the two vectors differ by a common phase. Write a trace-one self-adjoint matrix as $\frac{1}{2}\begin{pmatrix} 1+a & b+ic \\ b-ic & 1-a \end{pmatrix}$. Its determinant is zero exactly when $a^2 + b^2 + c^2 = 1$, proving the classical identification.

In the free case rotating each real coordinate pair preserves the sum of its squares without using commutation. This gives an automorphism, with inverse rotation by $-\theta$. The sphere relation becomes

$$\frac{1}{2}\sum_{i=1}^{2}(z_i^* z_i + z_i z_i^*) = 1.$$

Averaging a word over the circle keeps it when its letter counts agree and kills it otherwise. Polynomial density and continuity of averaging give the stated closed span. □

Circle averaging selects words by the balance of starred and unstarred letters. For the Laplacian we need a different decomposition, available in every dimension. We group words by length and use the invariant trace to remove their lower-degree components.

Definition 1.9. *Let $V_r \subset L^2(h)$ be the span of coordinate words of length at most r, with $V_{-1} = 0$. The degree-r harmonic space is*

$$V_r^\circ = V_r \ominus V_{r-1},$$

with orthogonality taken for the invariant trace.

This is the filtration of [4]. On the ordinary sphere removing traces from homogeneous polynomials gives the same spaces. On the free sphere the positions of the contractions matter.

Let $V = \mathbb{C}^N$ with its usual basis. The matrix $U^{\otimes r}$ has entries $U_{i_1 j_1} \cdots U_{i_r j_r}$ in this order. For a vector v in $V^{\otimes r}$ we write $[v]_r = \langle U^{\otimes r} v, v\rangle$, with the inner product linear in the first variable. Thus $[e_1]_1 = x_1$ and $[v \otimes e_1]_{r+1} = [v]_r x_1$. In this tensor calculation U denotes the free orthogonal matrix.

A noncrossing pairing joins ordered points in pairs by disjoint arcs above a line. Its tensor has coordinate 1 when the indices agree on each pair and coordinate 0 otherwise.

Proposition 1.10. *On the free orthogonal group, $(id \otimes h)U^{\otimes r}$ is the orthogonal projection onto the span of the noncrossing pairing tensors. If their columns form a matrix B, this projection is*

$$B(B^*B)^{-1}B^*,$$

for even r and $N \geq 2$; it is zero in odd degree.

Proof. The pairing formula of [4, Sections 2 and 4] identifies these tensors with the Haar-invariant vectors. Haar averaging is the orthogonal projection onto that space. The displayed matrix is the projection onto the column span of B. In odd degree there are no pairings. □

In degree two the only pairing tensor is $\eta = \sum_i e_i \otimes e_i$. In degree four the two pairings (12)(34) and (14)(23) give the Gram matrix used in Proposition 1.6. In higher degrees we can avoid inverting these matrices one by one. Every nonempty noncrossing pairing has an innermost arc joining neighbors, so contracting adjacent positions detects the traces we must remove.

Definition 1.11. *Regard $\eta = \sum_i e_i \otimes e_i$ as a map from $\mathbb{C}$ to $V \otimes V$. Put $H_0 = \mathbb{C}$, $H_1 = V$, and let H_r be the common kernel of the $r-1$ adjacent contractions on $V^{\otimes r}$. Write f_r for the orthogonal projection onto H_r.*

The basic contraction $\eta^*(e_i \otimes e_j) = \delta_{ij}$ is the tensor analogue of taking a trace. A tensor in $H_r \otimes V$ is already traceless in its first r places. To project it onto H_{r+1} only the last adjacent pair remains to be considered.

Lemma 1.12. *Set $k_1 = N$, $k_r = N - 1/k_{r-1}$ for $r \geq 2$. For $r \geq 1$ define*

$$T_r : H_{r-1} \to H_r \otimes V, \qquad T_r v = (f_r \otimes 1)(v \otimes \eta).$$

Then $T_r^* T_r = k_r 1$ and we have

$$f_{r+1} = f_r \otimes 1 - k_r^{-1} T_r T_r^*$$

on $V^{\otimes(r+1)}$. Writing $d_r = \dim H_r$, we also have $d_0 = 1$, $d_1 = N$, $d_{r+1} = N d_r - d_{r-1}$ and $k_r = d_r/d_{r-1}$.

Proof. Orthogonality gives $U^{\otimes 2}\eta = \eta$ and $\eta^* U^{\otimes 2} = \eta^*$. Contractions, insertions and their kernel projections therefore intertwine the tensor representations. Matrix coefficients between orthogonal invariant summands consequently vanish. On $H_r \otimes V$, contraction of the last two places takes values in H_{r-1}, because all earlier contractions vanish. Its adjoint is T_r. Its kernel is H_{r+1}. It remains to compute the squared length of T_r. For $r = 1$ it is the squared length of η, namely N.

For the induction write $Av = v \otimes \eta$ for $v \in H_{r-1}$. The inductive projection formula gives

$$T_r^* T_r = A^*(f_r \otimes 1)A = N1 - k_{r-1}^{-1} A^*(T_{r-1}T_{r-1}^* \otimes 1)A.$$

The last compression is 1. In fact $(T_{r-1}^* \otimes 1)Av = v$, viewed in $H_{r-2} \otimes V$. To see this, write $v = \sum_j v_j \otimes e_j$. The projection onto H_{r-1} fixes v, and contracting e_j with the first leg of $\sum_i e_i \otimes e_i$ leaves e_j on its second leg. This gives the claimed identity and hence the recursion for k_r. For $N \geq 2$ induction gives $k_r > 1$. Consequently $T_r T_r^*/k_r$ is the projection onto the orthogonal complement of the kernel inside $H_r \otimes V$. Taking dimensions gives $d_{r+1} = Nd_r - d_{r-1}$, and comparison with the recurrence for k_r gives $k_r = d_r/d_{r-1}$. □

This is the Jones–Wenzl recursion in ordinary tensor notation. The spaces H_r live in tensor powers, while V_r° lives in $L^2(h)$. Their matrix coefficients will give the spherical harmonics.

Fix the first coordinate x_1. In each degree there is a unique monic orthogonal polynomial for its law. We seek its coefficients from the tensor description alone. If $e = e_1$, the coefficient of $e^{\otimes r}$ is just x_1^r. The vector

$$v_r = f_r e^{\otimes r}, \qquad v_0 = 1$$

retains the part killed by every adjacent contraction. Since $H_{r+1} \subset H_r \otimes V$, we have $v_{r+1} = f_{r+1}(v_r \otimes e)$. The coefficient of the lower-degree term is determined by contracting the last leg of v_r against e.

Lemma 1.13. *Contraction of the last leg of v_r against e gives $c_r v_{r-1}$, where*

$$c_1 = 1, \qquad c_{r+1} = 1 - c_r/k_r.$$

Moreover $\|v_r\|^2 = c_r \|v_{r-1}\|^2$.

Proof. The assertion starts with $v_1 = e$. Under the induction hypothesis, $T_r^*(v_r \otimes e) = c_r v_{r-1}$. Lemma 1.12 therefore reads

$$v_{r+1} = v_r \otimes e - (c_r/k_r) T_r v_{r-1}.$$

Apply $1 \otimes \langle \cdot, e \rangle$ to the last leg. On $T_r v_{r-1}$ this leaves $f_r(v_{r-1} \otimes e) = v_r$, proving the recursion. Finally $\|v_r\|^2 = \langle v_{r-1} \otimes e, v_r \rangle$, which gives the norm formula. In particular $0 < c_r \leq 1$ by induction. □

Proposition 1.14. *The coefficient $[v_r]_r$ is a monic polynomial $p_r(x_1)$ of degree r. These polynomials are orthogonal for the coordinate law and satisfy*

$$p_0 = 1, \quad p_1 = x, \qquad x p_r = p_{r+1} + \frac{c_r^2}{k_r} p_{r-1}$$

for $r \geq 1$. Moreover $p_r(x_1) \in V_r^\circ$.

Proof. Decompose $v_r \otimes e = v_{r+1} + (c_r/k_r) T_r v_{r-1}$. The 2 summands belong to orthogonal invariant subspaces. The coefficient of $T_r v_{r-1}$ is $k_r [v_{r-1}]_{r-1}$ because T_r intertwines the representations and $T_r^* T_r = k_r 1$. Taking coefficients gives precisely the monic recurrence. For $r > 0$, $h([v_r]_r) = 0$. Indeed every nonempty noncrossing pairing has an adjacent pair, and v_r is killed by the corresponding contraction. The pairing projection above now applies. If $j < r$, iterating the recurrence expands $x^j p_r$ in polynomials of index at least $r - j > 0$. Integrating gives zero, which proves orthogonality.

In fact $p_r(x_1)$ is orthogonal to every word of degree $j < r$, so it belongs to V_r°. In the integral of the product, the pairing projection acts on $r + j$ places. In even total degree a noncrossing pairing must join two of the first r places, and an innermost such arc joins neighbors within that block. Its contraction kills v_r. In odd total degree the pairing projection vanishes. Thus the 1-variable orthogonality just proved is also visible in the full sphere. □

The polynomial $p_r(x_1)$ gives one harmonic in degree r. To count all the harmonics in that degree, we allow an arbitrary coefficient tensor in H_r in place of the one obtained from the first coordinate.

Proposition 1.15. *On the free sphere the map*

$$H_r \to V_r^\circ, \qquad v \to p_v = \sum_{i_1,\dots,i_r} v_{i_1\cdots i_r} x_{i_1}\cdots x_{i_r}$$

is a linear isomorphism. In particular $\dim V_r^\circ = d_r$*, with* $d_0 = 1$*,* $d_1 = N$ *and* $d_{r+1} = Nd_r - d_{r-1}$*.*

Proof. The pairing argument just used shows that p_v is orthogonal to every word of smaller degree. Also the orthogonal complement of H_r is the sum of the images of adjacent insertions of η. Each insertion becomes $\sum_i x_i^2 = 1$ in the polynomial algebra, so it gives a word of degree $r-2$. Projecting the coefficient tensor of any word onto H_r therefore gives its highest-degree harmonic part. This proves surjectivity.

For injectivity apply the degree-$2r$ Weingarten formula to $h(p_v^* p_w)$. Any column pairing with a pair internal to either block of r places has an adjacent internal pair; its contraction kills v or w. The unique remaining noncrossing pairing joins the two blocks in reverse order. Since the adjoint already reverses the first word, its contribution is $\langle w, v\rangle$. All row indices are 1, so there is a scalar w_r such that

$$h(p_v^* p_w) = w_r \langle w, v\rangle.$$

This scalar is strictly positive. Indeed, for $v = v_r$ the first-row coefficient p_{v_r} equals $[v_r]_r = p_r(x_1)$: the invariant projection f_r can be applied to either side of the coefficient. Its squared norm is positive by the monic recurrence, whose coefficients c_r^2/k_r are strictly positive. Thus $w_r > 0$, and the map is injective. The recurrence for d_r was obtained from Lemma 1.12. □

In degree 2 there is just one contraction, and its kernel is the orthogonal complement of $\eta = \sum_k e_k \otimes e_k$. The projection formula gives

$$f_2(e_i \otimes e_j) = e_i \otimes e_j - \frac{\delta_{ij}}{N}\,\eta, \qquad p_{f_2(e_i\otimes e_j)} = x_i x_j - \frac{\delta_{ij}}{N}\,1.$$

Thus every off-diagonal word $x_i x_j$ is already harmonic. The diagonal harmonics $x_i^2 - 1/N$ sum to zero by the sphere relation. Since the N^2 projected words span a space of dimension $N^2 - 1$, this accounts for every linear relation among them.

Monic normalization made the highest-degree term easy to track. For the Laplacian it will be more convenient to give every latitude polynomial the value one at the pole.

Proposition 1.16. *Put* $Q_r = p_r/p_r(1)$*,* $a_{-1} = 0$ *and* $a_r = d_r - a_{r-1}$*. Then* $p_r(1) = \|v_r\|^2 > 0$*, and*

$$d_r x Q_r = a_r Q_{r+1} + a_{r-1} Q_{r-1},$$

where the last term vanishes for $r = 0$*. In particular* $Q_r(1) = 1$*.*

Proof. Evaluation at $x = 1$ substitutes the identity for U, giving $p_r(1) = \|v_r\|^2$. Lemma 1.13 makes these values strictly positive. Put $a_{r-1} = d_{r-1}c_r$ for $r \geq 1$; its recursion gives $a_r = d_r - a_{r-1}$. Divide the monic recurrence by $p_r(1)$ and use $k_r = d_r/d_{r-1}$ to obtain the formula. Starting from $Q_0 = 1$ gives

$$Q_1(x) = x, \qquad Q_2(x) = \frac{Nx^2 - 1}{N-1}.$$

These initial values also display the normalization directly. □

This is the spherical recurrence of [16, Proposition 7.12]. It involves two ways of evaluating a coordinate polynomial. The group identity gives its value at the pole, and Haar integration gives its inner products. The next two sections use these to compute the Laplacian eigenvalues and the stationary coordinate law, respectively.

2 The spherical Laplacians

In each harmonic degree we know a polynomial Q_r of one coordinate and its value at the pole. The intrinsic heat flow of [16] is determined by how fast these polynomials vary there.

Definition 2.1. *Let Q_r be the latitude polynomial normalized by $Q_r(1) = 1$. The intrinsic spherical Laplacian L is the spectral operator*

$$Lf_r = \lambda_r f_r, \qquad \lambda_r = Q_r'(1), \qquad f_r \in V_r^\circ,$$

with domain given by $\sum_r \lambda_r^2 \|f_r\|_2^2 < \infty$.

In this normalization $\lambda_0 = 0$ and $\lambda_1 = 1$. The Markov generator is $-L$, so the heat equation is $\partial_s f = -Lf$. It is instructive to compare this invariant operator with the derivative obtained by following the circle action of Section 1. The latter acts on the coordinates themselves.

Lemma 2.2. *Simultaneous rotation of the coordinate pairs on S_+^{2m-1} has infinitesimal generator $\delta(x) = -y$, $\delta(y) = x$ on each pair. On the polynomial algebra,*

$$h(\delta a) = 0, \qquad \varphi(a, b) = h(a\,\delta b)$$

defines an antisymmetric cyclic 1-cocycle.

Proof. The rotation preserves each sum $x^2 + y^2$. Differentiating gives the stated formulas and $\delta(x^2 + y^2) = 0$ by the Leibniz rule. Invariance of h gives $h(\delta a) = 0$. Applying this to ab and using the trace property proves antisymmetry. A second application of the Leibniz rule and cyclicity of h gives

$$\varphi(ab, c) - \varphi(a, bc) + \varphi(ca, b) = 0,$$

which is the cocycle identity. □

The cocycle is an example of the differential calculus of group actions [10, 11]. Constructing it required choosing coordinate planes to rotate. For spherical heat we impose invariance under the whole orthogonal quantum group.

The homogeneous-space point of view in [16] explains why a single coordinate controls the calculation. The sphere polynomial algebra is the first-row subalgebra of the orthogonal quantum group G. Its coproduct Δ has one leg in that subalgebra, and an idempotent state on the group gives the associated averaging maps. Averaging on both sides leaves the polynomial algebra generated by U_{11}. For a free sphere this description uses an idempotent state even though there is no corresponding quotient by a stabilizer quantum subgroup [16, Proposition 7.1]. An invariant transition therefore has one scalar multiplier per harmonic degree; its value is read from the action on the latitude polynomial of that degree.

For the intrinsic heat flow the multiplier in degree r is $e^{-sQ_r'(1)}$. We shall need this flow both on the algebra containing the pole character and on the algebra carrying the faithful trace.

Proposition 2.3. *The operator L is positive and $T_s = e^{-sL}$ extends to a strongly continuous invariant unital completely positive semigroup on A_u and on A. It preserves h and is symmetric on $L^2(h)$. Classically, $L = -div\,grad/(N-1)$ and*

$$\lambda_r^{cl} = \frac{r(r+N-2)}{N-1}$$

for every degree r.

Proof. Use the pair $(b, \nu) = (1, 0)$ in [16, Theorem 7.5]. The resulting polynomial semigroup acts on V_r° by $e^{-sQ_r'(1)}$. Its construction in [16, Sections 3–4] uses a convolution semigroup of states φ_s on the group polynomial algebra, starting at its spherical idempotent state. These states

extend to $C_u(O_N^+)$: in their GNS representations, the orthogonal matrix relations bound every generator. Their left convolution maps are therefore unital and completely positive. On the first-row algebra the construction gives

$$T_s a = (\varphi_s \otimes id)\Delta(a),$$

which preserves that algebra and hence its norm closure A_u. At $s = 0$ the idempotent averaging acts as the identity on A_u. Polynomial density and contractivity give strong continuity.

Haar invariance gives $hT_s = h$. The Schwarz inequality implies $h((T_s a)^* T_s a) \le h(a^* a)$, so the kernel of the trace representation is preserved. The maps descend to A and extend as contractions on $L^2(h)$. Their real scalar multipliers make them symmetric. Contractivity and $e^{-sQ_r'(1)} > 0$ imply $Q_r'(1) \ge 0$. The other 2 spheres use the same argument. On the ordinary sphere the unscaled positive Laplace–Beltrami operator has eigenvalues $r(r + N - 2)$; division by $N - 1$ normalizes the first eigenvalue to 1. □

Corollary 7.13 of [16] expresses the free eigenvalues as sums of quotients of dimension sums, and Corollary 7.14 gives linear bounds. The following closed form evaluates those quotients and refines the asymptotic estimate.

Proposition 2.4. *Write $g_r = \lambda_{r+1} - \lambda_r$ and suppose $N > 2$. On all 3 spheres $g_0 = 1$. For the higher increments we have*

$$\begin{aligned} g_r^{cl} &= 1 + 2r/(N-1), && r \ge 1, \\ g_{2m-1}^* = g_{2m}^* &= 1 + 2m/(N-1), && m \ge 1, \\ g_{2m-1}^+ = g_{2m}^+ &= \tanh((2m+1)u)/\tanh u, && m \ge 1, \end{aligned}$$

where $u > 0$ is determined by $N = 2\cosh(2u)$.

Proof. The first formula follows from $g_r^{cl} = \lambda_{r+1}^{cl} - \lambda_r^{cl}$. For the half-liberated sphere, Section 1 identifies the law of the squared latitude $v = x^2$: its density is $(N-1)(1-v)^{N-2}$. By symmetry the degree-$2m$ orthogonal polynomial is even and the degree-$(2m+1)$ one is odd. They have the forms $P_m(x^2)$ and $xP_m(x^2)$, with a different P_m in each case. The resulting weights for P_m are $w(v) = v^b(1-v)^{N-2}$ for $b = 0, 1$, respectively. Normalize $P_m(1) = 1$. The operator

$$f \to w^{-1}(v(1-v)wf')' = v(1-v)f'' + [b + 1 - (N+b)v]f'$$

preserves polynomial degrees and is symmetric by integration by parts, since $v(1-v)w$ vanishes at both ends. Thus P_m is an eigenvector. Comparing leading coefficients gives its eigenvalue $-m(m+N+b-1)$. At $v = 1$ the second derivative drops out, giving $P_m'(1) = m(m+N+b-1)/(N-1)$. Differentiating with respect to x and using $\lambda_r = Q_r'(1)$ yields

$$\lambda_{2m}^* = \frac{2m(m+N-1)}{N-1}, \qquad \lambda_{2m+1}^* = 1 + \frac{2m(m+N)}{N-1}$$

These formulas recover [16, Corollary 7.11], with $-\lambda_r^*$ as the eigenvalues of the Markov generator $-L$. Taking $g_r^* = \lambda_{r+1}^* - \lambda_r^*$ gives the second row.

For the free sphere differentiate the recurrence $d_r x Q_r = a_r Q_{r+1} + a_{r-1} Q_{r-1}$ at 1. Since $a_r + a_{r-1} = d_r$, $Q_r(1) = 1$ and $Q_r'(1) = \lambda_r$, we obtain

$$a_r g_r^+ = d_r + a_{r-1} g_{r-1}^+, \qquad g_r^+ = \frac{d_0 + \cdots + d_r}{a_r}, \qquad a_r = d_r - d_{r-1} + \cdots + (-1)^r d_0$$

The first equation starts with $a_0 g_0^+ = 1$. To evaluate the quotient use $d_j = \sinh(2(j+1)u)/\sinh(2u)$. When $r = 2m - 1$ the numerator and denominator are, respectively,

$$\frac{\sinh(2mu)}{\sinh(2u)} \cdot \frac{\sinh((2m+1)u)}{\sinh u}, \qquad \frac{\sinh(2mu)}{\sinh(2u)} \cdot \frac{\cosh((2m+1)u)}{\cosh u}$$

These equalities follow by summing the geometric progressions for the two exponentials in each d_j. When $r = 2m$ the same sums have common factor $\sinh(2(m+1)u)/\sinh(2u)$ in place of $\sinh(2mu)/\sinh(2u)$. Cancelling gives the last row. □

For $N > 2$ the free gaps approach $\coth u$ exponentially fast. Consequently their deviations from this limit contribute a finite constant when summed over all degrees.

Corollary 2.5. *For $N > 2$, put $N = 2\cosh(2u)$, $c_N = \coth u$ and $l_N = \sum_{j\geq 0}(g_j^+ - c_N)$. Then*

$$\lambda_r^+ = c_N r + l_N + O_N(e^{-2ur}), \qquad c_N = \sqrt{\frac{N+2}{N-2}}.$$

The series defining l_N converges absolutely. The classical and half-liberated eigenvalues grow quadratically; for $N = 2$ the free and half-liberated spectra agree in every degree.

Proof. Proposition 2.4 and $1 - \tanh v = 2/(e^{2v}+1)$ give $g_j^+ - c_N = O_N(e^{-2uj})$. Thus

$$\lambda_r^+ - c_N r - l_N = -\sum_{j\geq r}(g_j^+ - c_N),$$

and summing the geometric bound proves the stated error. The other 2 increment sequences grow linearly. At $N = 2$ the squares of the 2 coordinates are central, so the free circle satisfies half-commutation and has the half-liberated spectrum. □

There is a second way to put heat on the free sphere. One can start with a central process on O_N^+ and restrict it to the first row. To compare the two generators on the same clock we give both first eigenvalue one.

Proposition 2.6. *Let $U_0(s) = 1$, $U_1(s) = s$ and $U_{r+1}(s) = sU_r(s) - U_{r-1}(s)$. The central group process of [9], restricted to the free sphere and normalized in degree 1, has parameters $\lambda_r^{group} = NU_r'(N)/U_r(N)$. In degree 2,*

$$\lambda_2^{group} = \frac{2N^2}{N^2-1}, \qquad \lambda_2^+ = \frac{2N}{N-1},$$

so their degree-2 gap is $2N/(N^2-1)$. It tends to zero as $N \to \infty$, as suggested in [16, Remark 7.16].

Proof. The restriction formula is [16, Remark 7.16]. The factor N makes its first parameter equal to 1. Substitute $U_2(s) = s^2 - 1$ for the group parameter and differentiate $Q_2(x) = (Nx^2-1)/(N-1)$ for the intrinsic parameter. □

We use the intrinsic operator below. Its derivative at 1 acts on latitude polynomials. A Gaussian generating functional on the whole group must satisfy $\gamma(abc) = 0$ for $a, b, c \in \ker\epsilon$ [30]. On O_N^+ this condition obstructs the construction of spherical Brownian motion from a bi-invariant Gaussian functional [17, Theorem 3.23 and Remark 3.24]. Indeed, every central Gaussian functional on O_N^+ vanishes for $N \geq 2$ [22, Proposition 5.2]. For the general Gaussian part see [21].

The central process above is also called Brownian in the cutoff literature. In the original group clock its first rate is $1/N$ and the projected cutoff occurs at $N\log N/2$. The intrinsic spherical cutoff occurs at $\log N/2$ [23, Propositions 3.20 and 3.21]. To compute the intrinsic latitude kernel we now identify its stationary measure.

3 The invariant coordinate law

The pole normalization of the latitude polynomials gave the heat eigenvalues. We now use their monic recurrence to identify the equilibrium measure on the free sphere. Its support and its generating function are what we need for the evolution of heat. We return to its individual moments in Section 9.

Take $N > 2$, let $-1 < q < 0$ satisfy $q + q^{-1} = -N$, and put

$$\beta = -q, \qquad W = \sqrt{N+2}\, x_N$$

We use $w = z + z^{-1}$ to pass between symmetric Laurent polynomials on the circle and ordinary polynomials on the interval $[-2, 2]$.

Set $P_r(w) = (N+2)^{r/2} p_r(w/\sqrt{N+2})$, the monic orthogonal polynomial for W.

Lemma 3.1. *We have*

$$wP_r = P_{r+1} + b_r P_{r-1}, \qquad b_r = \frac{(1-q^r)(1-q^{r+1})}{(1+q^r)(1+q^{r+1})}$$

for $r \geq 1$.

Proof. Put $t = -q$. Solving the dimension recurrence gives $d_r = t^{-r}(1-t^{2r+2})/(1-t^2)$. For the other factor in Proposition 1.14 we claim

$$c_r = \frac{1-q^{r+1}}{(1-q)(1+q^r)}.$$

At $r = 1$ the value is 1. For the induction use $1/k_r = t(1-q^{2r})/(1-q^{2r+2})$. This gives $c_r/k_r = -q(1-q^r)/((1-q)(1+q^{r+1}))$, whose complement is the claimed formula at $r+1$. Finally $b_r = (N+2)c_r^2/k_r$ and $(N+2)t = (1-q)^2$. Cancelling factors proves the assertion. □

The two consecutive factors involving q^r in b_r suggest the continuous q-ultraspherical polynomials. To identify the latitude law we first match the monic recurrences, then determine the orthogonality weight for our negative value of q.

Definition 3.2. *Write $(y;q)_\infty = \prod_{j\geq 0}(1-yq^j)$ and $(y;q)_r = \prod_{j=0}^{r-1}(1-yq^j)$. For $\beta = -q$, define*

$$B(y) = \frac{(\beta y;q)_\infty}{(y;q)_\infty}, \qquad G(z,t) = B(tz)B(t/z) = \sum_{r\geq 0} C_r(z)t^r.$$

Here t is a scalar series variable, and the C_r are the coefficients of the generating function G.

We keep the letter β to show the origin of the 2 factors in b_r. Each C_r is a symmetric Laurent polynomial in z, hence a polynomial in $w = z + z^{-1}$. In the following calculations $|t|$ can be taken small enough for all series to converge near the circles used.

Lemma 3.3. *The recurrence coefficient of C_r is*

$$\frac{(1-q^r)(1-\beta^2 q^{r-1})}{(1-\beta q^{r-1})(1-\beta q^r)}$$

in monic normalization, for $r \geq 1$.

Proof. Shifting the product once gives $B(qy)/B(y) = (1-y)/(1-\beta y)$. Consequently

$$(1-\beta wt+\beta^2 t^2)G(z,qt) = (1-wt+t^2)G(z,t).$$

Comparing terms of degree $r+1$ gives, with $C_{-1} = 0$,

$$(1-\beta q^r)wC_r = (1-q^{r+1})C_{r+1} + (1-\beta^2 q^{r-1})C_{r-1}.$$

The leading coefficient of C_r is $(\beta;q)_r/(q;q)_r$, by induction from $C_0 = 1$. Dividing by the respective leading coefficients gives the stated result. □

Setting $\beta = -q$ recovers b_r, so the two monic polynomial families agree. To recover the density of [3] from this identification we need Rogers orthogonality for $-1 < q < 0$. We give the contour argument because a square root of q is then imaginary and the contour shift includes a rotation.

Theorem 3.4. *The law of W is the image under $z \to z + z^{-1}$ of normalized Lebesgue measure on $\mathbb{T}$ weighted by*

$$K(z) = \left| \frac{(z^2;q)_\infty}{(\beta z^2;q)_\infty} \right|^2, \qquad \beta = -q,$$

and divided by its integral. Its monic orthogonal polynomials are the continuous q-ultraspherical (Rogers) polynomials, namely the monic normalizations of $C_r(w/2; -q \,|\, q)$.

Proof. The lemma at $\beta = -q$ gives exactly Lemma 3.1. To identify the measure it remains to prove orthogonality for the proposed weight. Away from the unit circle use the meromorphic continuation

$$K(z) = \frac{(z^2;q)_\infty (z^{-2};q)_\infty}{(\beta z^2;q)_\infty (\beta z^{-2};q)_\infty}.$$

Choose a square root s of q, so $|s| < 1$, and define the Rogers difference operator

$$A(z) = \frac{1 - \beta z^2}{1 - z^2}, \qquad R_q f(z) = A(z) f(sz) + A(1/z) f(z/s).$$

For a symmetric Laurent polynomial the apparent poles at $z = \pm 1$ cancel. First we check that R_q is symmetric for the bilinear integral $\oint f(z) g(z) K(z)\, dz/(2\pi i z)$. The product shift gives the identity

$$K(z) A(z) = K(sz) A(1/(sz)).$$

In the term containing $f(sz)$ substitute $y = sz$. The resulting integrand on the circle $|y| = |s|$ contains the product

$$f(y) g(y/s) K(y) A(1/y).$$

The differential dy/y is unchanged. Move this circle back to $|y| = 1$. The innermost boundary in this annulus contains the possible poles $y^2 = \beta$, exactly cancelled by $1 - \beta/y^2$. All remaining poles from that family satisfy $|y| < |s|$, and the reciprocal family is outside the unit circle. At $y = \pm 1$ the numerator of K cancels the pole of $A(1/y)$. Convergence of the products is uniform on compact annuli away from their poles. Thus the adjoint of the first shift term is the second shift term, proving symmetry.

Next, C_r is an eigenfunction of R_q with eigenvalue $s^{-r} + \beta s^r$. Apply R_q to $G(z, st)$ and divide by $G(z, t)$. The product-shift identity reduces the calculation to

$$A(z) \frac{1 - tz}{1 - \beta tz} + A(1/z) \frac{1 - t/z}{1 - \beta t/z} = 1 + \beta \frac{1 - tz}{1 - \beta tz} \cdot \frac{1 - t/z}{1 - \beta t/z}.$$

Clearing denominators verifies the identity using $A(z) + A(1/z) = 1 + \beta$ and $zA(z) + z^{-1} A(1/z) = \beta(z + z^{-1})$. Multiplying back gives $R_q G(z, st) = G(z, t) + \beta G(z, qt)$, as required. The eigenvalues are distinct: equality for $r \neq j$ would imply $(s^r - s^j)(\beta - s^{-r-j}) = 0$, which is impossible by absolute values. Symmetry therefore gives $\int C_r C_j K = 0$ for $r \neq j$.

The measure obtained by normalizing K and the law of W now agree on $C_0 = 1$ and vanish on every C_r with $r > 0$. These polynomials span all polynomials. Both measures have compact support, which proves equality. □

The zeros of the circular weight and its limit as $q \to 0$ can both be read from the product. They give the support at fixed N and the limiting density as the number of coordinates grows.

Proposition 3.5. *For $N > 2$, the support of the latitude law and the reduced spectrum are*

$$supp\,(law(x_N)) = \left[-\frac{2}{\sqrt{N+2}},\ \frac{2}{\sqrt{N+2}}\right].$$

The north-pole character does not descend to the reduced algebra, and $\sqrt{N+2}\,x_N$ converges in law to the standard semicircle as $N \to \infty$.

Proof. The weight K is positive on every open arc away from $1, -1$, so its image has support $[-2, 2]$. Faithfulness of the reduced trace identifies this support with the coordinate spectrum. Its upper endpoint is less than 1, excluding the north-pole character. As $N \to \infty$, we have $q \to 0$ and uniform convergence of the products on the circle, with $K(e^{i\theta}) \to 4\sin^2\theta$. The normalized image density is $\sqrt{4-w^2}/(2\pi)$, giving the semicircular limit. □

At $N = 2$ the parameter reaches $q = -1$, outside the product formula. In this dimension the sphere relation itself gives a shorter route to the law.

Lemma 3.6. *The free and half-liberated circles coincide. Their coordinate law is*

$$d\mu_2(x) = |x|\,dx, \qquad -1 \le x \le 1,$$

so the squared coordinate is uniform on $[0, 1]$.

Proof. The relation $x_1^2 + x_2^2 = 1$ makes both squares central. Every 3-letter word either equals its reversal or differs by moving a square, proving half-commutation. The complex-sphere model in Proposition 1.6 makes x_1^2 uniform on $[0, 1]$. Coordinate sign symmetry then gives the displayed density. □

We shall also recover this circle law from the limiting moments in Section 9. For $N > 2$ a source at the pole starts outside the stationary interval. We now ask how long any mass can remain outside that interval.

4 Heat and waves on the sphere

Consider a source at latitude x. Under the intrinsic heat flow the degree-r latitude polynomial acquires the factor $e^{-s\lambda_r}$. This gives all the polynomial moments of the temperature distribution at time s. The completion A_u matters here. Its scalar points allow us to put the source anywhere in $[-1, 1]$, including the part of the interval outside the reduced spectrum.

Proposition 4.1. *For every $x \in [-1, 1]$ and $s \ge 0$, there is a unique probability measure $K_s(x, dy)$ satisfying*

$$\int Q_r(y)\,K_s(x, dy) = e^{-s\lambda_r} Q_r(x)$$

for all r. We have $K_0(x, dy) = \delta_x$ and weak convergence to μ_N as $s \to \infty$. Its first 2 ordinary moments are

$$e^{-s}x, \qquad \frac{1}{N} + e^{-2Ns/(N-1)}\left(x^2 - \frac{1}{N}\right),$$

respectively.

Proof. Proposition 2.3 gives a unital completely positive semigroup on A_u, preserving the latitude algebra because $Q_r(x_N) \in V_r^\circ$. Choose an ordinary sphere point with last coordinate x and evaluate T_s there. Positivity gives a probability measure on the spectrum $[-1, 1]$ of the latitude in A_u. Its moments are the displayed multipliers and depend only on x. Polynomial approximation proves uniqueness and the initial condition. Since $\lambda_r > 0$ for $r > 0$, the moments converge to those of μ_N, and compact support gives weak convergence. Finally use $Q_1(y) = y$ and $Q_2(y) = (Ny^2 - 1)/(N - 1)$. □

For heat released at the pole the mean is e^{-s}. As long as $s < \log(1/a)$, this exceeds the stationary edge $a = 2/\sqrt{N+2}$, so some mass must still lie above a. To find its location and mass we use the higher-degree moments, whose growth outside $[-a, a]$ is quite different from their growth inside.

Fix $N > 2$. Use u and c_N from Corollary 2.5, so that $q = -e^{-2u}$, $\beta = -q$, $a = 1/\cosh u$ and $c_N = \coth u$. Put

$$x(v) = a\cosh v, \qquad \kappa = \frac{(\beta;q)_\infty}{(q;q)_\infty}, \qquad A(v) = \kappa B(e^{-2v}) \quad (v > 0)$$

Here $x(u) = 1$. Denote by $J_r(y)$ the polynomial $C_r(z)$ of Section 3 written in the variable $y = a(z + z^{-1})/2$. Thus $Q_r(y) = J_r(y)/J_r(1)$. Put

$$n_r = \int J_r(y)^2 \, d\mu_N(y)$$

Lemma 4.2. *There is a number $n_\infty > 0$ with $n_r = n_\infty + O(|q|^r)$. Uniformly on $[-a, a]$ we have $|J_r| \le C_N(r+1)$. For each $0 < v \le u$ there exists $\rho_v < 1$ such that*

$$J_r(x(v)) = A(v)e^{vr} + O_v(\rho_v^r), \qquad \lambda_r = c_N r + l_N + O_N(e^{-2ur})$$

as $r \to \infty$.

Proof. The leading coefficient of C_r in $z + z^{-1}$ is $\kappa_r = (\beta;q)_r/(q;q)_r$. By the monic recurrence, consecutive squared norms have ratio b_r in monic normalization. Hence $n_r = \kappa_r^2 b_1 \cdots b_r$. Both κ_r/κ and b_r equal $1 + O(|q|^r)$, and all factors are positive. The norm assertion follows. The identity $(1 - \beta y)B(qy) = (1 - y)B(y)$ gives $B(y) = \sum_{j\ge 0} \kappa_j y^j$. At $|z| = 1$, the coefficient of t^r in $B(tz)B(t/z)$ is therefore bounded by $(r+1)\sup_j \kappa_j^2$.

At $z = e^v$ the same product has a simple pole at $t = e^{-v}$, with coefficient $A(v)$ in front of $(1 - te^v)^{-1}$, since $\lim_{y\to 1}(1 - y)B(y) = \kappa$. Every other possible pole has modulus at least $\min(e^v, e^{2u-v}) > 1$. After subtracting the simple-pole term, Cauchy's coefficient estimate on a circle of radius strictly between 1 and that bound gives the stated remainder. The eigenvalue estimate is Corollary 2.5. □

Outside the stationary interval, the leading term of $J_r(x(w))$ grows like e^{wr}. Multiplication by the heat factor changes the exponent to $(w - c_N s)r$. As long as $w - c_N s > 0$, this is the growth of the polynomials at another exterior point. Subtracting an atom at that point leaves moments that can be summed on $[-a, a]$.

Theorem 4.3. *For the laws on A_u from Proposition 4.1, suppose $x = x(w)$, with $0 < w \le u$, and set $s_w = w/c_N$. For $0 < s < s_w$, write $v = w - c_N s$. Then*

$$K_s(x, dy) = m_w(s)\,\delta_{x(v)}(dy) + f_{s,x}(y)\,\mu_N(dy), \qquad m_w(s) = e^{-sl_N}\frac{A(w)}{A(v)}$$

where $f_{s,x}$ is continuous on $[-a, a]$. For $s > s_w$ the full law has a continuous density relative to μ_N on $[-a, a]$. For $x = -x(w)$ reflect the formula. For $|x| \le a$ the full law has a continuous density for every $s > 0$.

Proof. The J_r-moment of $K_s(x, dy)$ is $e^{-s\lambda_r}J_r(x)$, since J_r is a scalar multiple of Q_r. For $s < s_w$, the leading term is $e^{-sl_N}A(w)e^{(w-c_N s)r}$ by Lemma 4.2. Subtract from the moment the quantity $m_w(s)J_r(x(v))$. Since $v = w - c_N s$, the leading terms cancel. The errors from the parameters λ_r have order $e^{(v-2u)r}$, hence the remaining moments e_r decay exponentially. Then

$$f_{s,x}(y) = \sum_{r\ge 0} \frac{e_r}{n_r} J_r(y)$$

converges uniformly on the stationary interval. Orthogonality gives exactly the moments e_r. The equality of finite signed measures on $[-1,1]$ follows from polynomial density. Positivity of K_s implies positivity of its continuous part and $0 < m_w(s) \le 1$. The stationary measure is atomless by Theorem 3.4; this accounts for every atom.

If $s > s_w$, the original moments already decay exponentially. For $|x| \le a$ they have order $O((r+1)e^{-sc_N r})$ at each $s > 0$. The same series therefore works directly. Reflection follows from $J_r(-y) = (-1)^r J_r(y)$ and the symmetry of μ_N. □

As $s \to s_w^-$, the moving point approaches a and its mass tends to zero because $A(v) \sim \kappa^2/(2v)$ as $v \to 0^+$. At the critical time the orthogonal moments no longer decay. We can still recover the measure by summing the generating function at its boundary.

Proposition 4.4. *At $x = x(w)$ and $s = s_w$ the law is absolutely continuous with respect to μ_N. Its density near $y = a$ is a positive constant times $(a-y)^{-1}$ to leading order. In particular it belongs to $L^1(\mu_N)$ and does not belong to $L^2(\mu_N)$. The assertion at the negative edge follows by reflection.*

Proof. At the critical time $s = s_w$ the normalized coefficients satisfy

$$\frac{e^{-s_w\lambda_r} J_r(x(w))}{n_r} = c_w + O(\rho^r), \qquad c_w = \frac{e^{-s_w l_N} A(w)}{n_\infty} > 0$$

for some $0 < \rho < 1$. The exponentially small terms give a uniformly convergent correction as in Theorem 4.3. For the constant term multiply the r-th coefficient by t^r with $0 < t < 1$. With $y = a\cos\theta$ the corresponding sum is exactly

$$c_w B(te^{i\theta})B(te^{-i\theta}) = c_w |B(te^{i\theta})|^2$$

The stationary measure in θ is comparable to $\sin^2\theta\, d\theta$ on $[0,\pi]$, since only the first numerator factor of K in Theorem 3.4 vanishes. The pole at $z = 1$ is removable in $(1-z)B(z)$; the remaining product is continuous and nonzero on $|z| \le 1$. Also $|1 - te^{i\theta}|^2 = (1-t)^2 + t|1-e^{i\theta}|^2$. For $1/2 \le t < 1$ these facts give an integrable bound independent of t after multiplication by the stationary weight. Dominated convergence therefore identifies the limiting density as $c_w|B(e^{i\theta})|^2$ plus the continuous correction. Integrating against J_r recovers the critical moments. At $\theta = 0$ the first term is equivalent to $c_w\kappa^2/|1-e^{i\theta}|^2$, while the correction stays bounded. Since $|1-e^{i\theta}|^2 = 2(a-y)/a$, the claimed asymptotic and both integrability assertions follow. □

For a source at the north pole, $w = u$. This gives an explicit time at which the last exterior mass disappears and allows us to compare that time across dimensions.

Corollary 4.5. *For heat released at the north pole, let $b(s)$ be the location of the moving atom. Then*

$$s_* = u\tanh u, \qquad b'(0) = -1, \qquad s_* - \frac{1}{2}\log N \to 0$$

as $N \to \infty$, with $N = 2\cosh(2u)$.

Proof. Theorem 4.3 gives $s_* = u/c_N$ and $b(s) = \cosh(u - c_N s)/\cosh u$, where $c_N = \coth u$. Differentiating gives $b'(0) = -c_N \tanh u = -1$. Also

$$u = \frac{1}{2}\log\left(\frac{N + \sqrt{N^2-4}}{2}\right).$$

Thus $u - \frac{1}{2}\log N \to 0$ and $u(1 - \tanh u) \to 0$, proving the limit. □

The same arrival time is the Hilbert–Schmidt threshold for intrinsic heat on the full space $L^2(h)$ of the free sphere when $N > 2$. Indeed, the squared Hilbert–Schmidt norm is finite exactly when $\sum_{r\geq 0} d_r e^{-2s\lambda_r}$ converges. By Proposition 1.15 and Corollary 2.5,

$$d_r = \frac{\sinh(2(r+1)u)}{\sinh(2u)}, \qquad d_r e^{-2s\lambda_r} \sim \frac{e^{2u-2sl_N}}{2\sinh(2u)} e^{2(u-sc_N)r}.$$

Thus the sum converges for $s > s_* = u/c_N$ and diverges for $0 \leq s \leq s_*$. At arrival its terms tend to a positive constant.

For $N = 3$ the atom reaches the edge at time 0.21520 to five decimal places. Halfway there its height is 0.92044 and its mass is 0.85968. The logarithmic asymptotic in the corollary concerns a different limit, as N grows. It gives the intrinsic cutoff scale of [23, Proposition 3.21]. The group-character calculation in [23, Proposition 3.14] and the random-walk interpretation in [5] suggest isolating the atom. For the latitude it is the Rogers generating function that gives its trajectory and the density at the time of arrival.

To pass from heat to waves we return to all the angular harmonics. On an ordinary sphere these are restrictions of homogeneous harmonic polynomials in the ambient coordinates. On the free sphere an angular harmonic comes from an adjacent-traceless tensor. The following short word explains why the ambient directional Laplacian of [24] gives a different test.

Lemma 4.6. *For $N \geq 2$, adjacent-traceless homogeneous polynomials need not be harmonic for the directional Laplacian $\sum_i \partial_i^2 p(x)[a, a]$ of [24], where a is a further noncommuting letter.*

Proof. The word $p = x_1x_2x_1$ is killed by both adjacent contractions, since the neighboring indices differ. Differentiating twice in x_1 gives $2ax_2a$, while the other second derivatives vanish. Thus the directional Laplacian does not vanish on this word. □

For waves we use the intrinsic L on these same angular spaces. A mode which heat damps by $e^{-s\lambda_r}$ oscillates with angular frequency $\sqrt{\lambda_r}$. The constant component has to be treated separately because its initial velocity can be nonzero.

Proposition 4.7. *The kernel of L in $L^2(h)$ consists of the constants. For initial displacement $a = \sum_r a_r$ and velocity $b = \sum_r b_r$ with finitely many harmonic components, the solution of $w'' + Lw = 0$ is*

$$w(s) = a_0 + sb_0 + \sum_{r\geq 1} \left(\cos(s\sqrt{\lambda_r})a_r + \frac{\sin(s\sqrt{\lambda_r})}{\sqrt{\lambda_r}} b_r \right),$$

with $w(0) = a$ and $w'(0) = b$.

Proof. All eigenvalues except λ_0 are strictly positive, proving the kernel assertion. In each positive degree the equation is the scalar oscillator equation; in degree zero it is $w_0'' = 0$. Solving these finite-dimensional equations gives the formula and uniqueness. □

There is another way to pose a temperature question on the free sphere. Take the space of 2-dimensional representations of $C(S^2_+)$ and assign an energy to each triple of coordinate matrices. Commutators give an energy whose ground states have an ordinary geometric interpretation. We use normalized matrix trace, with $tr(I) = 1$.

Definition 4.8. *For three self-adjoint 2×2 matrices $A = (A_1, A_2, A_3)$ satisfying $\sum_i A_i^2 = I$, their commutator energy is*

$$E(A) = \sum_{i<j} tr([A_i, A_j]^*[A_i, A_j]).$$

A ground state is a family with $E(A) = 0$.

At zero energy the matrices commute, so simultaneous diagonalization produces two ordinary sphere points. Their angle extends to a continuous observable on the whole matrix model.

Proposition 4.9. *Write $A_i = a_i I + v_i \cdot \sigma$ in the Pauli basis. The sphere relation and energy become*

$$\sum_i (a_i^2 + |v_i|^2) = 1, \qquad \sum_i a_i v_i = 0, \qquad E(A) = 4 \sum_{i<j} |v_i \times v_j|^2.$$

The continuous function $C(A) = 2\sum_i (tr(A_i))^2 - 1$ is the cosine of the angle between the two sphere points at every ground state.

Proof. Use $(v \cdot \sigma)(w \cdot \sigma) = (v \cdot w)I + i(v \times w) \cdot \sigma$. The scalar and vector parts of $\sum_i A_i^2 = I$ give the first 2 equations; the commutator identity gives the third. At zero energy the vectors v_i are collinear, so write $v_i = b_i u$ with $|u| = 1$, allowing $b = 0$. The two joint eigenvalue vectors are $a + b$ and $a - b$. Their squared norms equal one, and their inner product is $|a|^2 - |b|^2 = 2|a|^2 - 1 = C(A)$. □

Thus $C(A)$ can be measured throughout the matrix model using only the traces of its coordinates. At a ground state this measurement recovers the angle through its cosine. To average $C(A)$ at positive temperature one would also have to choose a reference measure for the Gibbs distribution.

5 The free ball and its invariant integrals

Write a point of the free sphere in the order $(t, x_1, \ldots, x_N)$, where $t = x_{N+1}$ is its height. After we forget t, the remaining coordinates satisfy $\sum_i x_i^2 \leq 1$. There are two questions about this projection. Which operator families arise in this way, and what integral do they inherit from the sphere? The first has a universal algebraic answer. For the second, even the number of coordinates we delete will matter.

Definition 5.1. *The universal self-adjoint free ball has algebra*

$$C(B_+^N) = C^* \left(1, x_1, \ldots, x_N \;\middle|\; x_i = x_i^*,\ \sum_i x_i^2 \leq 1 \right).$$

Its defect is $d = 1 - \sum_i x_i^2$.

Each generator is a contraction, so the universal norm exists. Reconstructing a height amounts to taking a self-adjoint square root of the defect. Requiring the height to be positive makes this square root unique. Allowing every choice leads to an amalgamated product.

Proposition 5.2. *The universal algebra of the upper hemisphere $t \geq 0$ is $C(B_+^N)$. The universal algebra of the full sphere is the unital full amalgamated product*

$$C(S_+^N) \simeq C(B_+^N) *_{C([0,1])} C([-1, 1])$$

where the coordinate function on $[0, 1]$ is sent to d in the ball algebra and to t^2 in the interval algebra.

Proof. Any point of the ball has the unique positive completion $t = \sqrt{d}$. For the full sphere the amalgamation permits every self-adjoint completion t with $t^2 = d$. All such heights are contractions. Thus a representation of either side is precisely a choice of the same $N+1$ operators with the same relation, giving inverse homomorphisms on generators. The two maps from $C([0, 1])$ are injective: scalar ball points realize every value of d, and t^2 ranges over $[0, 1]$. The substitutions $t = \pm\sqrt{d}$ give the two hemisphere quotients and each splits the inclusion of $C(B_+^N)$. □

For scalar coordinates the two signs of $\sqrt{d}$ cover the sphere. For operators, a self-adjoint square root can have both signs in a single representation. Already on the circle this affects a commutator.

Proposition 5.3. *The hemisphere quotients do not jointly detect the commutator of the two free-circle coordinates. Moreover, if $J = \ker(C(B_+^1) \to C(S^0))$, then $\mathbb{C}1 + J$ is not isomorphic to $C(S_+^1)$. Thus ordinary boundary collapse does not recover the free circle.*

Proof. For $0 < a < 1$, the sphere relation is satisfied by

$$x = a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad t = \sqrt{1-a^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Their commutator is nonzero. In either hemisphere $t = \pm\sqrt{1-x^2}$ is a function of x, so the commutator vanishes. Also $C(B_+^1) = C([-1,1])$, making $\mathbb{C}1 + J$ commutative, whereas the matrix example proves that $C(S_+^1)$ is not. □

More generally, let J be the ideal generated by d in the ball algebra. It is the kernel of restriction to the sphere, so the elements of $\mathbb{C}1 + J$ have scalar boundary value. Classically this algebra describes $B^N/\partial B^N = S^N$. On the free circle the commutator above distinguishes boundary collapse from adjoining a height.

The invariant trace allows us to ask what can be inferred about the missing height from the retained coordinates. Write τ for the spherical trace, $B = W^*(x_1, \ldots, x_N)$ in its von Neumann algebra, and E_B for the trace-preserving conditional expectation.

Lemma 5.4. *For every $f \in C([-1,1])$, the missing height satisfies*

$$E_B(f(t)) = \frac{f(\sqrt{d}) + f(-\sqrt{d})}{2},$$

where $d = t^2 = 1 - \sum_i x_i^2$.

Proof. Reflection fixes B pointwise and sends t to $-t$. Testing against $b \in B$ and using invariance gives $E_B(f(t)) = E_B(f(-t))$. The even part of $f(t)$ is the displayed function of $t^2 = d$, which already belongs to B and is fixed by E_B. □

This determines every conditional moment of the height alone. Inserting a ball coordinate between two heights also detects the order of the factors. The free circle supplies an explicit joint law.

Proposition 5.5. *In the invariant trace of the free circle the coordinates have joint law*

$$(x, t) = (\sqrt{1-T}\, a, \sqrt{T}\, b)$$

where T is central, uniform on $[0,1]$ and classically independent of the pair (a,b). The signs a, b are freely independent self-adjoint unitaries of trace zero. In particular $\tau(xtxt) = 0$ and $\tau(x^2t^2) = 1/6$.

Proof. The relation $x^2 + t^2 = 1$ makes both squares central. Hence every half-commutation relation $ijk = kji$ among the 2 generators holds: a word of length three either agrees with its reversal or requires only commutation with a square. We can use the half-liberated sphere integral of [4, Section 4]. Explicitly, take (w_1, w_2) uniform on the unit sphere of $\mathbb{C}^2$ and represent each coordinate by

$$M(w_j) = \begin{pmatrix} 0 & w_j \\ \bar{w}_j & 0 \end{pmatrix}.$$

The integrated normalized matrix trace gives the spherical moments: odd words have trace zero, and in an even word the two diagonal entries are an alternating product of $w_i, \bar{w}_j$ and its conjugate. Their integrals agree by conjugation symmetry and are precisely the integrals in [4].

Write $w_1 = \sqrt{1-T}e^{i\alpha}$ and $w_2 = \sqrt{T}e^{i\beta}$. Normalizing two independent standard complex Gaussians shows that their phases are independent and uniform, independently of T. Their squared lengths before normalization are independent unit exponentials. The change $(a_1, a_2) = (s(1-u), su)$ has Jacobian s, so the joint density becomes $se^{-s}\,ds\,du$, proving uniformity of T. The signs are $M(e^{i\alpha}), M(e^{i\beta})$. Odd alternating words have zero trace and the normalized matrix trace of an even alternating word of length $2k$ is $\cos(k(\alpha-\beta))$. It integrates to zero for $k \geq 1$. Every centered polynomial of either sign is a scalar multiple of that sign, so these are all the freeness tests. Finally $xtxt = T(1-T)abab$ has trace zero by freeness, while $x^2t^2 = T(1-T)$ has trace $\int_0^1 u(1-u)\,du = 1/6$. □

For example, the two words $xtxt$ and x^2t^2 have the same letters and different integrals. In higher dimensions one encounters the same ordering issue in conditional moments $E_B(tct)$, where c is a word in the retained coordinates. We return to the representations of the free circle in Section 7, using the classification in [20, Proposition 4.1].

We now consider integration on the ball in its own right. Write h_N for the invariant trace on S_+^{N-1}, with coordinates u_i. On the sphere the sum of their squares is fixed at one. On the ball its value can vary, and orthogonal invariance leaves its law undetermined.

Proposition 5.6. *The rule $x_i \to \sum_j v_{ij} \otimes x_j$ defines an O_N^+ coaction on $C(B_+^N)$ fixing $R^2 = \sum_i x_i^2$. Every probability law on $[0,1]$ occurs as the law of R^2 in an invariant tracial state.*

Proof. Orthogonality gives

$$\sum_i \left(\sum_j v_{ij} \otimes x_j \right)^2 = 1 \otimes \sum_j x_j^2.$$

Thus the rule respects the ball relation and fixes R^2; its coaction identity follows from the coproduct of v. For a prescribed scalar random variable $T \in [0,1]$, independent of the spherical algebra, use $x_i = \sqrt{T}\,u_i$. Integrating the spherical trace against the law of T gives an invariant trace and makes $R^2 = T$. □

To select a volume we use the classical divergence theorem. For a homogeneous polynomial p of degree k, the radial vector field $x\,p(x)$ has divergence $(N+k)p$ and outward flux p on the unit sphere. If both volume and surface area are normalized to probability, their ratio introduces a factor N on the boundary side. The resulting identity makes sense for an ordered word because it uses only its degree and its restriction to the sphere.

Proposition 5.7. *There is a unique state on $C(B_+^N)$ satisfying $(N+k)\tau_{rad}(p) = N h_N(p|_{S_+^{N-1}})$ for every homogeneous polynomial p of degree k. It is the invariant trace*

$$\tau_{rad}(p) = N \int_0^1 r^{N-1} h_N(p(ru_1, \ldots, ru_N))\,dr$$

Its odd moments vanish. In degree $2m$ each ordered moment equals the corresponding sphere moment multiplied by $N/(N+2m)$.

Proof. Sending x_i to the continuous function $r \to ru_i$ respects the ball relation. Composing the resulting homomorphism with the integral of the fiber traces gives a state and proves traciality and invariance. For homogeneous p the scalar integral is $N\int_0^1 r^{N+k-1}dr = N/(N+k)$. Since polynomial words are norm dense, the identities determine the state. □

In this representation R^2 acts by multiplication by the scalar function r^2. Commuting with the coordinates is an additional relation imposed by the radial model, as one sees on two matrices.

Lemma 5.8. *For $N \geq 2$, the element $R^2 = \sum_i x_i^2$ is not central in $C(B_+^N)$. In particular the radial representation is not faithful.*

Proof. On the disk take

$$x_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad x_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad R^2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix}.$$

These matrices satisfy the ball inequality and $[R^2, x_1] \neq 0$. Append zero coordinates for larger N. The same commutator vanishes in every scalar radial model, proving the last assertion. □

A second classical construction of uniform volume is to retain the first N coordinates of a point uniformly distributed on the sphere in $N + 2$ coordinates. The same restriction makes sense for any number j of deleted coordinates on a free sphere.

Proposition 5.9. *Let $j \geq 1$ and $M = N + j$. The first N coordinates define an injection $C(B_+^N) \to C(S_+^{M-1})$. Restriction of h_M gives an O_N^+-invariant trace $\tau_{N,j}$. In the classical construction, $j = 2$ gives ordinary probability volume on the unit ball.*

Proof. Any ball representation extends to the sphere by putting back $\sqrt{1 - R^2}$ and $j - 1$ zero coordinates; hence the homomorphism is injective. The subgroup with block matrix $diag(v, I_j)$ acts by the ball coaction on the visible coordinates, proving invariance. For the classical assertion, normalize $N + 2$ independent real Gaussians. The retained squared length has beta parameters $N/2, 1$, independently of its uniform direction. These are precisely the radial ingredients of ordinary ball volume. □

For a fixed word in the retained coordinates the two prescriptions use the same noncrossing pairings. Projection changes the dimension in their Gram matrix. Radial integration keeps the dimension N and contributes a scalar depending only on word length.

Proposition 5.10. *Let $P = NC_2(2m)$ and let $\delta_\pi(I)$ be the indicator of equality of the 2 indices on each pair of π. Set $G_d(\pi, \sigma) = d^{|\pi \vee \sigma|}$, where the exponent counts components after superimposing the pairings. For $I = (i_1, \ldots, i_{2m})$ with $i_l \leq N$, write $x_I = x_{i_1} \cdots x_{i_{2m}}$. Then*

$$\tau_{N,j}(x_I) = \sum_{\pi,\sigma \in P} \delta_\pi(I)(G_{N+j}^{-1})_{\pi\sigma}, \qquad \tau_{rad}(x_I) = \frac{N}{N + 2m} \sum_{\pi,\sigma \in P} \delta_\pi(I)(G_N^{-1})_{\pi\sigma}$$

The second inverse-matrix formula is for $N \geq 2$. At $N = 1$ its spherical factor is the even moment 1.

Proof. In the first-row realization every row index is 1, so the row pairing conditions are automatic. The Gram projection of Section 1 gives the first formula. At $N \geq 2$ the sphere of dimension $N - 1$ gives the second by Proposition 5.7. At $N = 1$ its coordinates are the two scalar signs with equal weights. Reflection kills odd moments. This use of the rectangular block requires only the ordinary spherical Weingarten matrix. See also [2] for integration on homogeneous spaces. □

Codimension two is the classical prescription for uniform volume. To compare it with the divergence prescription on the free ball, it suffices to integrate powers of a single coordinate.

Corollary 5.11. *For every $N \geq 1$, both τ_{rad} and $\tau_{N,2}$ give x_1^2 the integral $1/(N + 2)$, whereas*

$$\tau_{rad}(x_1^4) = \frac{2}{(N + 1)(N + 4)}, \qquad \tau_{N,2}(x_1^4) = \frac{2}{(N + 2)(N + 3)},$$

so these invariant traces are distinct.

Proof. The sphere moments in d coordinates are $1/d$ and $2/[d(d + 1)]$. For the radial integral multiply their values at $d = N$ by $N/(N + 2)$ and $N/(N + 4)$; for the projected integral set

$d = N + 2$. The resulting fourth moments differ for every $N \geq 1$. At $N = 1$ they are $1/5$ and $1/6$, respectively. □

The fourth moments distinguish these two traces. One could still ask whether a different distribution of the scalar radius would reproduce the projected trace. Every such model makes $[R^2, x_1]$ vanish, so a single mixed moment tests all radial distributions at once.

Proposition 5.12. *For $N \geq 2$, $j \geq 1$ and $M = N + j$, the projected integral satisfies*

$$\tau_{N,j}([R^2, x_1]^*[R^2, x_1]) = \frac{2(N-1)j}{M(M+1)(M^2-2)}.$$

It cannot be a mixture of scalar dilations of the spherical trace. For $N = j = 2$ the displayed value is $1/70$.

Proof. To see this, use the tensors of Section 1 in $\mathbb{C}^M$, and put

$$w_k = \sum_{i=1}^{k} (e_i \otimes e_i \otimes e_1 - e_1 \otimes e_i \otimes e_i)$$

The corresponding first-row coefficient against $e_1^{\otimes 3}$ is a commutator. It vanishes for w_M by the sphere relation. Contracting the first 2 or the last two places of w_k gives $(k-1)e_1$ or $-(k-1)e_1$. Hence the same coefficient for w_N is obtained from $z = w_N - (N-1)w_M/(M-1) \in H_3$. Directly, $\|z\|^2 = 2(N-1)j/(M-1)$. Apply the degree 6 Weingarten formula to the squared norm of the coefficient of z. A column pairing containing an internal cap in either block of three contracts adjacent legs of z and gives zero. The only survivor joins the two blocks in reverse order; the adjoint reverses the first block, so its contribution is $\|z\|^2$. All row indices are 1. It remains to find the row sum of G_M^{-1} at this pairing.

The 5 pairings of six points have 2 cyclic orbits. Two pairings use three cyclically adjacent pairs, and the other three include the surviving pairing. Denote the respective row sums of G_M^{-1} by b and c. Using $G_M(G_M^{-1}\mathbf{1}) = \mathbf{1}$, count the loops against one representative from each orbit to obtain

$$(M^3 + M)b + 3M^2c = 1, \qquad 2M^2b + (M^3 + 2M)c = 1$$

For example the other pairing in the first orbit makes one loop with the first representative, and each member of the second orbit makes two. Solving gives $c = (M-1)/[M(M+1)(M^2-2)]$. Multiplication by $\|z\|^2$ gives the stated commutator moment. It is strictly positive, whereas R^2 is central in every scalar radial model. Such models therefore cannot realize this projected volume. □

For $M > 2$ each retained coordinate has norm at most $2/\sqrt{M+2}$ in the projected trace, by Theorem 3.4. Thus evaluation at the scalar point $x_1 = 1$ cannot pass to this completion. There is also a question at the opposite end of the ball. Can the projected radius come arbitrarily close to zero? When a single coordinate has been deleted, its latitude law gives the answer.

Proposition 5.13. *For one deletion from S_+^N, with $N \geq 2$, the retained radius has support*

$$\mathit{supp}\left(\mathit{law}_{\tau_{N,1}}(R)\right) = \left[\sqrt{\frac{N-1}{N+3}},\, 1\right].$$

Under the radial state τ_{rad} we have $P(R \leq r) = r^N$ for $0 \leq r \leq 1$.

Proof. We have $R^2 = 1 - x_{N+1}^2$. Proposition 3.5 in $N+1$ coordinates gives the spectrum of x_{N+1} as $[-2/\sqrt{N+3}, 2/\sqrt{N+3}]$. Continuous spectral mapping and faithfulness of the trace give the first formula. The radial integral has density $N\rho^{N-1}$, whose integral up to r is r^N. □

Deleting one coordinate therefore leaves a gap around the origin for every $N \geq 2$, and scaling the ball scales both endpoints of the radial support. For the disk obtained by retaining two of four coordinates, the corresponding question is whether $0 \in \mathit{supp}\,(law_{h_4}(x_1^2 + x_2^2))$. The first two moments computed in Section 9 leave this open.

We can also vary the radius in the defining inequality itself. Write $B_+^N(R)$ for the ball defined by $\sum_i x_i^2 \leq R^2$.

Proposition 5.14. *Every bounded self-adjoint family belongs to one of these balls. Its least radius is $\|[x_1 \ \cdots \ x_N]\|$, and scalar translation by $a \in \mathbb{R}^N$ increases that radius by at most $|a|$. For $R' \geq R$ there is a coordinate-preserving surjection*

$$C(B_+^N(R')) \to C(B_+^N(R)),$$

interpreted as restriction to the smaller ball.

Proof. The row norm squared is $\|\sum_i x_i^2\|$, giving the least radius. The triangle inequality for rows gives the translation bound. A representation of the smaller ball satisfies the larger-radius inequality, so universality gives the homomorphism. Its image contains all coordinate generators, hence it is surjective. □

Varying the bound R includes tuples with noncentral squared radius, such as the one in Lemma 5.8. Scalar dilations of sphere points alone would miss them. The restriction maps give an inverse system of bounded pieces of free affine space, with a pro-C^* limit as in [26]. In the algebraic description of [20, Section 1], points are matricial, bounded or unbounded representations of the free $*$-algebra $\mathbb{C}\langle x_1, \ldots, x_N\rangle$ on self-adjoint generators. The balls $B_+^N(R)$ exhaust its bounded representations.

6 Radial geometry and Poisson extension

For the divergence integral the representation $x_i(\rho) = \rho u_i$ puts all the angular variables on a single sphere and makes the radius central. We can therefore prescribe boundary data at $\rho = 1$ and ask how each spherical harmonic extends to smaller radii. First we identify the algebra in which continuity at the center is to be understood.

Throughout this section $N \geq 2$. Let A be the reduced algebra of one of our 3 spheres S^{N-1}, with generators u_i and faithful invariant trace h. Write τ_{rad} for the radial state defined by the formula of Proposition 5.7 with this choice of h.

Definition 6.1. *The radial completion and its coordinates are*

$$B = \{F \in C([0,1], A) \mid F(0) \in \mathbb{C}1\}, \qquad x_i(\rho) = \rho u_i,$$

where ρ is a scalar radius.

An element is a continuous family of angular observables, with only a scalar value at the center. All directions collapse there, while the whole algebra A remains at radius one. For the ordinary sphere this is the usual closed ball.

Proposition 6.2. *The elements $x_i(\rho) = \rho u_i$ generate B. The state*

$$\tau(F) = N \int_0^1 \rho^{N-1} h(F(\rho))\, d\rho$$

is faithful, and B is the norm closure of the universal ball algebra in the representation associated with τ_{rad}. Evaluation at 1 is a surjective homomorphism $B \to A$, whereas evaluation at 0 takes values in $\mathbb{C}$.

Proof. The ball relation gives a homomorphism from the universal ball algebra into B. Its image contains $\rho^2 1 = \sum_i x_i^2$, and therefore $\rho 1$ and every continuous scalar function of ρ. For a

word $u_I = u_{i_1} \cdots u_{i_m}$ it contains $\rho^m u_I$. If a scalar function ϕ vanishes near zero, multiplication by the continuous function $\phi(\rho)/\rho^m$ shows that ϕu_I also belongs to the image. Polynomial approximation in A, followed by a cutoff near zero, gives every element of $C_0((0,1], A)$. Adding the constant function 1 gives exactly B.

If $F \geq 0$ and $\tau(F) = 0$, the continuous function $\rho \to h(F(\rho))$ vanishes for every $\rho > 0$. Faithfulness of h gives $F(\rho) = 0$ there, and continuity gives $F(0) = 0$. Thus τ is faithful. On a homogeneous word of length m, the formula for τ is the spherical integral multiplied by $N/(N+m)$. It is consequently the radial state τ_{rad} specified above. The associated faithful representation of the image identifies its norm closure with B. Finally ρa has boundary value a for every $a \in A$, proving surjectivity at 1. □

Two features of this completion enter the boundary problem. The identity $\sum_i x_i^2 = \rho^2 1$ makes the radius central, and evaluation at $\rho = 1$ takes values in the reduced spherical algebra. In the free case the first property already forces a quotient of the universal ball by Lemma 5.8.

To define energy, recall that our normalization $\lambda_1 = 1$ gives $(N-1)L$ as the usual positive angular Laplacian on the classical unit sphere. At radius ρ the corresponding angular energy carries the factor ρ^{-2}.

Definition 6.3. *On the Hilbert space with norm*

$$\|F\|_{2,B}^2 = N \int_0^1 \rho^{N-1} \|F(\rho)\|_2^2 \, d\rho,$$

define the radial energy by

$$E(F) = N \int_0^1 \left(\|F'(\rho)\|_2^2 + \frac{N-1}{\rho^2} \|L^{1/2} F(\rho)\|_2^2 \right) \rho^{N-1} \, d\rho.$$

Its domain consists of square-integrable functions locally absolutely continuous in $L^2(h)$ on $(0,1]$, belonging to $Dom(L^{1/2})$ almost everywhere and having finite displayed energy.

The radial volume weight vanishes at the center, where a finite-energy function need not have a value. Near the outer boundary the weight stays bounded away from zero. This gives a boundary value in $L^2(h)$ even for data in the form domain.

Lemma 6.4. *Evaluation at 1 defines a bounded map from $Dom(E)$, with its form norm, to $L^2(h)$. More precisely,*

$$\|F(1)\|_2^2 \leq C_N (\|F\|_{2,B}^2 + E(F)),$$

for a constant depending only on N.

Proof. On $[1/2, 1]$ the radial weight is bounded above and below. The Hilbert-valued Sobolev trace estimate bounds $\|F(1)\|_2^2$ by the integrals of $\|F\|_2^2$ and $\|F'\|_2^2$ there. These are controlled by the displayed norm and energy. □

We can now impose zero boundary values by taking the kernel of this trace map. Keeping the full form domain gives the other usual realization of the same interior Laplacian.

Proposition 6.5. *The energy E is a densely defined closed quadratic form with $E(F) \geq 0$. It gives a symmetric, trace-preserving, completely positive heat semigroup on the finite von Neumann algebra generated by B. On smooth functions its interior differential expression is*

$$L_B F = -F'' - \frac{N-1}{\rho} F' + \frac{N-1}{\rho^2} LF.$$

The operator associated with the full form has the natural Neumann condition $F'(1) = 0$, interpreted on smooth functions. Restricting the form domain to $F(1) = 0$ gives the Dirichlet realization of the same interior expression.

Proof. Expand F in the orthogonal degree spaces of $L^2(h)$. Its norm and energy are sums of scalar radial norms and energies, with one copy for each vector in each degree. The derivative form

and multiplication by ρ^{-2} are closed; their sum is closed on their common domain. Equivalently, convergence in the form norm gives convergence in ordinary Sobolev norm on every $[\epsilon, 1]$ and convergence of the weighted derivative and angular terms. The local limits agree, and recover the stated domain. Functions with finitely many angular degrees and smooth radial coefficients supported away from zero are dense in $L^2(B, \tau)$. This proves density and closedness.

For later use, the form can be approximated by finite angular sums whose nonconstant coefficients vanish near zero and whose scalar coefficient is constant there. To see this, first truncate the degree expansion. For a fixed degree $k > 0$, the angular term controls $\int_0^1 \rho^{N-3} \|F_k\|_2^2 \, d\rho$. A cutoff with derivative bounded by C/ρ on $[\epsilon, 2\epsilon]$ therefore has vanishing error as $\epsilon \to 0$. The scalar coefficient can first be truncated to a bounded function. For $N > 2$ the cost of a cutoff at the center is $O(\epsilon^{N-2})$. For $N = 2$ use a logarithmic cutoff on $[\epsilon^2, \epsilon]$; its cost is $O(1/|\log \epsilon|)$. Ordinary smooth approximation on the remaining interval finishes the argument. The same approximation can preserve a zero boundary value at 1.

We give the positivity argument as well. The spherical heat semigroup is symmetric and completely positive, so its energy decreases under clipping a self-adjoint element to $[0, 1]$, at every matrix level. This is the Dirichlet-form correspondence recalled in [8]. The radial derivative energy has the same property. Indeed, for self-adjoint elements a, b in a finite tracial algebra and a real function ϕ with Lipschitz constant at most one,

$$\|\phi(a) - \phi(b)\|_2 \le \|a - b\|_2.$$

One can verify this inequality with the commuting left and right spectral resolutions of a and b on L^2: integrate $|\phi(s) - \phi(t)|^2 \le |s - t|^2$ against their positive joint measure at the vector 1. Difference quotients give the derivative assertion. Integrating both inequalities proves the clipping property for E and for every matrix amplification. Its closed form therefore generates a completely positive Markov semigroup. Since $1 \in Dom(E)$ and $E(1) = 0$, the semigroup is unital; symmetry then gives preservation of τ.

Finally, integration by parts on the smooth functions just described gives

$$E(F, G) = \tau((L_B F)^* G) + N h(F'(1)^* G(1)).$$

The term at zero vanishes. This identifies the interior expression and the natural condition at 1. The boundary estimate makes the zero-boundary subspace closed in form norm, giving the Dirichlet realization. □

On the ordinary ball this is the positive Laplacian in polar coordinates. On all three balls the radial derivative terms are the same, so the difference in their interior equations enters through the angular eigenvalues. For harmonic extension we prescribe the value at radius one and allow the normal derivative there to be determined by the solution.

Definition 6.6. *An element $F \in B$ is harmonic in the interior if*

$$N \int_0^1 \rho^{N-1} h(F(\rho)^* L_B G(\rho)) \, d\rho = 0$$

for every smooth G with compact support in $0 < \rho < 1$ and finitely many angular degrees.

For smooth functions this is the equation $L_B F = 0$. Projecting onto one angular degree reduces the weak equation to an ordinary differential equation.

Lemma 6.7. *For $f_k \in V_k^\circ$, the bounded radial harmonic solution with boundary value f_k is $\rho^{\alpha_k} f_k$, where*

$$\alpha_k = \sqrt{\frac{(N-2)^2}{4} + (N-1)\lambda_k} - \frac{N-2}{2}.$$

We have $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = 2$.

Proof. For $F(\rho) = v(\rho) f_k$, the equation becomes

$$v'' + \frac{N-1}{\rho}\, v' - \frac{(N-1)\lambda_k}{\rho^2}\, v = 0.$$

The trial power ρ^α gives $\alpha(\alpha + N - 2) = (N-1)\lambda_k$. Its larger root is the displayed value. For $k > 0$, the other solution is $\rho^{-\alpha_k - N + 2}$ and is unbounded at zero. In degree zero the solutions are $1, \rho^{2-N}$ for $N > 2$ and $1, \log \rho$ for $N = 2$. Boundedness again leaves only the constant. The first 3 roots follow from $\lambda_0 = 0$, $\lambda_1 = 1$, $\lambda_2 = 2N/(N-1)$. □

Boundedness at the center has selected one power ρ^{α_k} in each degree. Writing $\rho = e^{-t}$ turns these powers into semigroup multipliers $e^{-t\alpha_k}$. To extend arbitrary continuous boundary data, we need the resulting operators to act on A as well as on $L^2(h)$.

Definition 6.8. *Put $b = (N-2)/2$. The boundary operator and its semigroup are*

$$L_\partial = \sqrt{b^2 + (N-1)L} - b, \qquad P_t = e^{-tL_\partial},$$

with square root defined by spectral calculus on $L^2(h)$.

The eigenvalue of L_∂ on V_k° is precisely the radial exponent α_k. The extension method for functions of a Laplacian [25, 29] suggests writing P_t as an average of spherical heat operators. Positivity of the averaging measure will then give positivity at every matrix level.

Lemma 6.9. *For $b \geq 0$, $x \geq 0$ and $t > 0$, we have*

$$e^{-t(\sqrt{b^2+x} - b)} = \frac{te^{bt}}{2\sqrt{\pi}} \int_0^\infty v^{-3/2} e^{-t^2/(4v) - b^2 v} e^{-vx}\, dv.$$

The measure multiplying e^{-vx} is a probability measure. Consequently

$$P_t = \frac{te^{bt}}{2\sqrt{\pi}} \int_0^\infty v^{-3/2} e^{-t^2/(4v) - b^2 v} T_{(N-1)v}\, dv$$

is an average of unital completely positive maps on A.

Proof. Put $I(a) = \int_0^\infty e^{-z^2 - a^2/z^2}\, dz$ for $a \geq 0$. For $a > 0$, differentiation and $w = a/z$ give $I'(a) = -2I(a)$. Since $I(0) = \sqrt{\pi}/2$ and monotone convergence gives continuity at zero, $I(a) = (\sqrt{\pi}/2)e^{-2a}$. Substituting $v = t^2/(4z^2)$ in the stated integral and taking $a = t\sqrt{b^2 + x}/2$ proves the identity, including $b = x = 0$. At $x = 0$ the integral is one. Apply spectral calculus with $x = (N-1)L$. The probability integral converges in norm on each element of A, since the spherical heat maps are contractions. Positivity at every matrix level and preservation of the unit pass through the integral. □

For $0 < \rho < 1$ the choice $t = -\log \rho$ now gives a contraction on A. To obtain an element of the closed ball algebra B, we must also identify its limits at the center and at the boundary.

Theorem 6.10. *There is a unital completely positive map $H : A \to B$ given by*

$$(Hf)(\rho) = P_{-\log\rho} f \quad (0 < \rho \le 1), \qquad (Hf)(0) = h(f)1.$$

It has boundary value $(Hf)(1) = f$, reproduces the coordinates $H(u_i) = x_i$, and is the unique norm-continuous extension harmonic in the interior. On a finite harmonic expansion it is

$$(Hf)(\rho) = \sum_k \rho^{\alpha_k} f_k,$$

with $f_k \in V_k^\circ$.

Proof. The integral formula proves complete positivity, preservation of 1 and contractivity for every P_t. Its action on a polynomial is the finite sum of its harmonic components multiplied by $e^{-t\alpha_k}$. Consequently $P_t f \to f$ in norm as $t \to 0$, and $P_t f \to h(f)1$ in norm as $t \to \infty$, first for polynomials and then for every $f \in A$ by contractivity and polynomial approximation. This proves continuity at both ends and shows that $Hf \in B$. Positivity in $M_m(B)$ is tested at each radius, so H is completely positive. Since $\alpha_1 = 1$, the formula for the coordinates follows.

For general $f \in A$, the spectral expansion converges in $L^2(h)$. On each compact subinterval of $0 < \rho < 1$, the factors ρ^{α_k} dominate every fixed power of λ_k. The series can therefore be differentiated in L^2 and satisfies the radial equation in each degree. In particular it is harmonic in the weak sense, tested against smooth functions with compact radial support and finitely many angular degrees.

Conversely, let $F \in B$ be weakly harmonic with boundary value f. Projection onto each finite-dimensional space V_k° gives the radial differential equation derived above. Its distributional solutions are the two displayed radial solutions. Boundedness at zero excludes the singular one, including the logarithm when $N = 2, k = 0$. The boundary value fixes the remaining coefficient to f_k. Completeness of the degree decomposition gives $F = Hf$ in L^2 at each radius, hence in A by faithfulness of h. □

The heat average preserves the spherical mean at each radius. Its pointwise order bound and its integrated L^2 bound describe the size of the extension for continuous and square-integrable data.

Corollary 6.11. *If $0 \le f \le 1$ in $M_m(A)$, then $0 \le Hf \le 1$ in $M_m(B)$. At every radius $h(Hf(\rho)) = h(f)$, hence $\tau(Hf) = h(f)$. The extension also defines a contraction on L^2 boundary data, with*

$$\|Hf\|_{2,B}^2 = N \sum_k \frac{\|f_k\|_2^2}{2\alpha_k + N} \le \|f\|_2^2,$$

for every $f \in L^2(h)$.

Proof. The maximum principle follows from unital complete positivity. Each spherical heat map preserves h, so its probability mixture does too. For finite harmonic sums, orthogonality and integration of $\rho^{2\alpha_k+N-1}$ give the displayed norm formula. Completion extends it to all L^2 data, and $\alpha_k \ge 0$ gives contractivity. □

Every L^2 boundary value thus has a square-integrable extension. Requiring finite energy places a stronger condition on its higher-degree components. We can read the exact weight from the energy of a single radial power.

Theorem 6.12. *The boundary values of $Dom(E)$ are exactly the functions $f \in Dom(L_\partial^{1/2})$. For every such f, the unique extension of least energy is Hf, and*

$$E(Hf) = N \sum_k \alpha_k \|f_k\|_2^2 = N \|L_\partial^{1/2} f\|_2^2.$$

More precisely, every $F \in Dom(E)$ with $F(1) = f$ satisfies

$$E(F) = E(Hf) + E(F - Hf),$$

so equality in the minimum occurs only at $F = Hf$.

Proof. For a smooth coefficient v in degree k, constant near zero when $k = 0$ and zero there otherwise, completing a square gives

$$\int_0^1 \left(\|v'\|_2^2 + \frac{(N-1)\lambda_k}{\rho^2} \|v\|_2^2 \right) \rho^{N-1} \, d\rho = \alpha_k \|v(1)\|_2^2 + \int_0^1 \left\| v' - \frac{\alpha_k}{\rho} \, v \right\|_2^2 \rho^{N-1} \, d\rho.$$

The cross term is integrated by parts, and the coefficient identity is $\alpha_k^2 + (N-2)\alpha_k = (N-1)\lambda_k$. There is no contribution at zero. Sum over the degrees and use the form approximation from Proposition 6.5. Continuity of the boundary trace and closedness of the weighted degree sum show that every boundary value satisfies

$$N \sum_k \alpha_k \|f_k\|_2^2 \le E(F).$$

This proves one inclusion in the asserted boundary space.

For a finite harmonic expansion, direct substitution of $v(\rho) = \rho^{\alpha_k} f_k$ gives

$$N \int_0^1 (\alpha_k^2 + (N-1)\lambda_k) \rho^{2\alpha_k + N - 3} \, d\rho = N\alpha_k$$

when $k > 0$; the constant mode has zero energy. Thus the asserted formula holds for finite sums. If $\sum_k \alpha_k \|f_k\|_2^2 < \infty$, their extensions converge in the form norm by this identity and the preceding L^2 estimate. The limit has boundary value f, proving the reverse inclusion and the energy formula.

For a smooth zero-boundary function G, integration by parts against $\rho^{\alpha_k} f_k$ gives $E(Hf, G) = 0$, first for finite sums. Approximation in the form norm gives the same identity for every $G \in Dom(E)$ with $G(1) = 0$. Apply it to $G = F - Hf$ to obtain the Pythagorean identity. A zero-energy function has zero radial derivative and only a constant angular component. Its zero boundary value makes it zero, which proves uniqueness. □

The factor N in the minimum energy comes from the normalized radial density $N\rho^{N-1}$ at $\rho = 1$, just as in the divergence identity of Section 5. For a single harmonic the outward derivative of $\rho^{\alpha_k} f_k$ at that radius is $\alpha_k f_k$. Passing from one harmonic to general L^2 data identifies the domain of the Dirichlet-to-Neumann operator.

Proposition 6.13. *For $f \in L^2(h)$ the derivative $\partial_\rho Hf(\rho)$ has an L^2 limit as $\rho \to 1^-$ exactly when $f \in Dom(L_\partial)$. In that case the limit is $L_\partial f$. Moreover,*

$$L_\partial^2 + (N-2)L_\partial = (N-1)L$$

on $Dom(L)$, and $P_t f = Hf(e^{-t})$.

Proof. The derivative multiplier is $\alpha_k \rho^{\alpha_k - 1}$. For $k > 0$ we have $\alpha_k \ge 1$, so its absolute value is bounded by α_k near 1 and converges to that value. Dominated convergence gives the limit when $\sum_k \alpha_k^2 \|f_k\|_2^2$ is finite. Conversely, an L^2 limit has degree-k component $\alpha_k f_k$, forcing this sum to be finite. The quadratic relation holds on each degree by Lemma 6.7; spectral calculus gives the stated operator identity and its domain. The last assertion is the defining formula for H. □

A homogeneous word of length k acquires the radial power ρ^k under $x_i = \rho u_i$. The quadratic identity now lets us ask when this algebraic extension of a degree-k harmonic is itself harmonic in the ball.

Proposition 6.14. *On the classical ball $\alpha_k = k$ in every degree. On the half-liberated and free balls the Poisson extension agrees with the usual homogeneous extension in degrees zero, one and two, but in degree 3*

$$\alpha_3 = \frac{\sqrt{N^2 + 8N + 8} - N + 2}{2}, \qquad 2 < \alpha_3 < 3.$$

The extension of a nonzero degree 3 boundary harmonic is therefore not a polynomial in the ball coordinates, for every $N \geq 2$.

Proof. The classical eigenvalue is $k(k+N-2)/(N-1)$, and substitution in the quadratic relation gives $\alpha_k = k$. All 3 spectra have $\lambda_0 = 0$, $\lambda_1 = 1$ and $\lambda_2 = 2N/(N-1)$, giving $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = 2$. On each of the other two spheres the third eigenvalue is $(3N+1)/(N-1)$. The root is the displayed one. Since $2N < 3N + 1 < 3N + 3$ for $N \geq 2$, monotonicity of $\alpha(\alpha + N - 2)$ on $[0, \infty)$ places it strictly between 2 and 3.

A polynomial in $x_i = \rho u_i$ has, after projection onto a fixed angular degree, a finite sum of integer powers of ρ with vector coefficients. A nonzero coefficient times ρ^{α_3} cannot have this form. For instance apply a linear functional which does not vanish on that coefficient and use the linear independence of distinct real powers on $(0, 1)$. This proves the last assertion.

In degree 2 the extension is especially transparent:

$$H(u_i u_j) = \rho^2(u_i u_j - \delta_{ij}/N) + \delta_{ij}/N = x_i x_j + \frac{\delta_{ij}}{N}(1 - \rho^2).$$

This follows by separating the constant and degree 2 components. □

The degree-two formula will reappear as a polynomial lift to the universal ball in Section 7. In degree three the radial solution already requires a noninteger power. The effect of the exponent is visible in Figure 2, where we fix a degree-eight mode of unit boundary amplitude on each of the three balls with $N = 3$. At a given interior radius the free mode retains the largest amplitude, since it has the smallest angular eigenvalue.

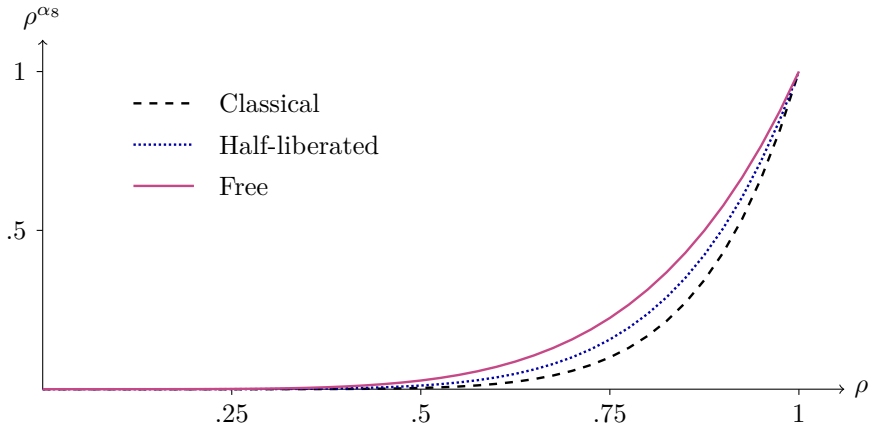

Figure 2: Radial attenuation of a degree-eight harmonic with unit boundary amplitude, for $N = 3$. The curves are the exact spectral powers ρ^{α_8} sampled along the radius.

For $N > 2$ put

$$c_N = \sqrt{\frac{N+2}{N-2}}, \qquad \gamma = \frac{N + \sqrt{N^2 - 4}}{2}.$$

For the free sphere these constants govern two different quantities. The spherical eigenvalue grows like $c_N k$, whereas the number of harmonics in degree k grows as a constant times γ^k. Counting boundary modes requires both rates.

Proposition 6.15. *On the free sphere with $N > 2$, the eigenvalues satisfy*

$$\alpha_k = \sqrt{(N-1)c_N k} - \frac{N-2}{2} + O_N(k^{-1/2}).$$

The operator L_∂ has compact resolvent. If $n_\partial(a)$ counts its eigenvalues at most a, with multiplicity, then

$$\lim_{a\to\infty} \frac{\log n_\partial(a)}{a^2} = \frac{\log \gamma}{(N-1)c_N}.$$

For every $t > 0$, the operator e^{-tL_∂} on $L^2(h)$ is compact but is not Hilbert–Schmidt and is not of trace class.

Proof. Corollary 2.5 gives $\lambda_k = c_N k + O_N(1)$. Substitution into the square-root formula proves the eigenvalue asymptotic. The dimension of V_k° is

$$d_k = \frac{\gamma^{k+1} - \gamma^{-k-1}}{\gamma - \gamma^{-1}},$$

by Proposition 1.15 and its dimension recurrence. Thus $\log \sum_{j=0}^{k} d_j = k \log \gamma + O_N(1)$. The increments of λ_k are positive, and so are those of α_k. The greatest degree with $\alpha_k \le a$ is $k = a^2/((N-1)c_N) + O_N(a)$, by the preceding asymptotic. Substitution proves the counting formula. Finite multiplicities and $\alpha_k \to \infty$ prove compactness of the resolvent and of each positive-time boundary heat operator.

For any $p > 0$ the terms in $\sum_k d_k e^{-pt\alpha_k}$ fail even to tend to zero: their logarithms are $k \log \gamma - O_N(\sqrt{k})$. In particular the sums of eigenvalues and of their squares diverge. This gives the last two assertions. □

At a fixed interior radius $\rho = e^{-t}$ the heat average constructs the Poisson map on A without summing over a basis. A Hilbert–Schmidt norm on $L^2(h)$ counts the squared multiplier $e^{-2t\alpha_k}$ once for each of the d_k independent modes. It is this sum that diverges.

7 Projected heat and universal boundary

Return to the projected integral $\tau_{N,j}$, obtained by retaining N of the $N + j$ coordinates of a free sphere. Heat is already defined on the ambient sphere. To obtain a heat equation on the ball we must check that, if the initial observable depends only on the retained coordinates, this remains true at every later time. The defect $d = 1 - \sum_{i=1}^{N} x_i^2$ is the simplest observable that records the contribution of the missing coordinates.

Proposition 7.1. *Put $M = N + j$ with $j \ge 1$, and use the intrinsic spherical heat flow in M coordinates. The polynomials in the first N coordinates form an invariant subspace. On their norm closures in both A_u and A, the restriction is a Markov semigroup. On the latter it preserves $\tau_{N,j}$. For the ball defect $d = 1 - \sum_{i=1}^{N} x_i^2$ its action is*

$$T_s d = \frac{j}{M} + e^{-2Ms/(M-1)} \left(d - \frac{j}{M} \right)$$

for every $s \ge 0$.

Proof. Write $p_r(e_{i_1} \otimes \cdots \otimes e_{i_r}) = x_{i_1} \cdots x_{i_r}$ and extend linearly, using this notation only in this proof. For an index string I the polynomial $p_r(f_r e_I)$ lies in V_r° of the ambient sphere. Indeed, when paired with a word of smaller degree, every noncrossing pairing in even total degree has an

adjacent cap inside the block of r indices; the projection f_r kills that contraction. In odd total degree there are no pairings.

By Lemma 1.12, f_r is the identity plus a linear combination of noncrossing diagrams with fewer than r through strings. This follows also by expanding its recursion in adjacent contractions and insertions: any term besides the identity factors through at most $r-2$ tensor places, and concatenation only removes closed loops at the scalar cost M. In evaluating such a diagram on e_I, a cap tests equal input indices. On the output, an innermost cup sums an adjacent pair of letters as $\sum_{i=1}^{M} x_i^2 = 1$. Erase those cups successively. The letters left are an ordered subword of I. Thus $p_r(f_r e_I)$ is the original word plus a linear combination of shorter words in its visible indices. Induction on length decomposes every visible word into visible harmonic components. The spherical heat multiplier scales each of these by a scalar. Complete positivity, contractivity and stationarity on the two closures follow by restriction of Proposition 2.3.

Finally $Q_2(x_i)$ is a degree 2 eigenvector. It follows that $T_s(x_i^2) = M^{-1} + e^{-s\lambda_2}(x_i^2 - M^{-1})$. Sum this formula over the N visible indices and use $\lambda_2 = 2M/(M-1)$. □

At an equatorial scalar point the missing coordinates are initially zero. Evaluating $T_s d$ at that point measures how much squared length they have acquired on average after time s.

Corollary 7.2. *Evaluate projected heat on A_u at an ordinary sphere point $(x, 0)$ with $\sum_{i=1}^{N} x_i^2 = 1$. Its mean defect is*

$$E_s(d) = \frac{j}{M}\left(1 - e^{-2Ms/(M-1)}\right) > 0$$

for every $s > 0$. Its equilibrium defect is j/M.

Proof. This character is defined on A_u. Evaluate the formula in Proposition 7.1 there, where $d = 0$, and then let $s \to \infty$. In particular the equilibrium values for one and 2 deletions are $1/(N+1)$ and $2/(N+2)$. □

On the ordinary ball one can see the effect of projection in the second-order part of the generator. A derivative in a retained coordinate must first be projected onto the tangent plane of the ambient sphere.

Proposition 7.3. *On the first N coordinates of the ordinary sphere in $M = N+j$ real coordinates, the positive projected Laplacian is*

$$L_{proj} f = -\frac{1}{M-1} \sum_{i,k=1}^{N} (\delta_{ik} - x_i x_k)\, \partial_i \partial_k f + \sum_{i=1}^{N} x_i \partial_i f,$$

with the first spherical eigenvalue normalized to one.

Proof. The unscaled positive tangential Laplacian is $-\sum_{i=1}^{M} \partial_i^2 + (x \cdot \nabla)^2 + (M-2) x \cdot \nabla$. Expand the square on a function of the visible coordinates and divide by $M-1$. Applying its negative to $1 - |x|^2$ also recovers the defect evolution above. □

Even for $j = 2$, when the equilibrium measure is ordinary uniform volume, the diffusion coefficient in the radial direction depends on the distance from the center.

There is one calculation in which the projected and radial pictures agree. Let $\pi : C(B_+^N) \to C(S_+^{N-1})$ set $d = 0$, and write u_i for the boundary coordinates. A quadratic word on the boundary splits into a constant and a traceless quadratic. These two pieces lift to the ball separately.

Proposition 7.4. *The rule*

$$H(u_i u_j) = x_i x_j + \frac{\delta_{ij}}{N}\, d$$

extends the quadratic boundary relation and agrees with radial Poisson extension. Its nonconstant part is annihilated by the degree 2 contraction and is a degree 2 eigenvector for projected heat.

Proof. On the sphere, split $u_i u_j$ into $u_i u_j - \delta_{ij}/N$ and its constant part. Lift the first term to $x_i x_j - (\delta_{ij}/N) \sum_l x_l^2$ and the constant unchanged. This gives the displayed formula and respects

$\sum_i u_i^2 = 1$. The contraction sends $x_i x_j$ to δ_{ij}, so it kills the nonconstant term. That term is also traceless in the ambient sphere, hence projected heat multiplies it by $e^{-s\lambda_2}$. The radial formula agrees because $\alpha_2 = 2$. □

Higher adjacent-traceless tensors likewise lift as polynomials, although by Section 6 their homogeneous lifts can already fail to be radially harmonic in degree three. There is a more basic question about the universal ball. Can we extend even the coordinate functions by a single unital completely positive map defined on the whole boundary algebra?

On the operator row ball the free holomorphic calculus [27] and the pluriharmonic Dirichlet theorem [28, Theorem 4.1] use a Poisson transform on full Fock space. Our boundary data instead lie in the self-adjoint sphere algebra, with restriction given by π. The relevant finite matrix sets over the Euclidean ball are studied in [19]. In two variables our set is the wild disk of [18, Section 7.2], where the following notion of boundary applies.

Definition 7.5. *A finite-dimensional point of a matrix convex set belongs to its Arveson boundary if every dilation within the set splits off the original family as a reducing summand.*

A coordinate-preserving extension would dilate every disk point to a sphere point by Stinespring's theorem. It therefore suffices to find an Arveson boundary point of the disk with nonzero defect.

Proposition 7.6. *For the universal self-adjoint disk there is no unital completely positive map $H : C(S^1_+) \to C(B^2_+)$ sending its 2 coordinates u_i to x_i.*

Proof. We make the boundary phenomenon in [18, Section 7.2] explicit with a 3×3 witness, then use it to obstruct the coordinate-preserving unital completely positive map. With $p = 1/2$ and $q_0 = \sqrt{1-p^2}$ take the pair $A = (A_1, A_2)$ given by

$$A_1 = \begin{pmatrix} 0 & 0 & p \\ 0 & q_0 & 0 \\ p & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} q_0 & 0 & 0 \\ 0 & 0 & p \\ 0 & p & 0 \end{pmatrix}$$

Their defect is $diag(0, 0, 1 - 2p^2)$. Suppose this pair is the compression of a disk representation $Y_i = \left(\begin{smallmatrix} A_i & B_i \\ B_i^* & C_i \end{smallmatrix}\right)$. Positivity of the top-left corner of the larger defect gives $\sum B_i B_i^* \le I - \sum A_i^2$. Hence both B_i have range in $\mathbb{C}e_3$. That same corner kills e_1, e_2; positivity of the full defect forces its off-diagonal corner also to have zero first and second rows. This corner is $-\sum(A_i B_i + B_i C_i)$. Since $A_1 e_3 = p e_1$ and $A_2 e_3 = p e_2$, its first 2 rows force $B_1 = B_2 = 0$. Thus every disk dilation splits off the original pair.

If H existed, compose it with evaluation at A and apply Stinespring's theorem. The resulting dilation is a representation of $C(S^1_+)$, so its coordinates satisfy $\sum Y_i^2 = I$. On the reducing 3-dimensional subspace this would give $\sum A_i^2 = I$, contrary to the computed defect. □

For the pair in the proof, the defect is a multiple of the projection onto the third coordinate axis, and A_1 moves that axis. The squared radius is therefore noncentral. This explains precisely where the example drops out on passing to the radial completion of Theorem 6.10.

One can also distinguish the two boundaries by the dimensions of their irreducible representations. For the sphere quotient these can be read from the free-circle classification in [20, Section 4].

Proposition 7.7. *Every irreducible representation of $C(S^1_+)$ has dimension at most two. With $z = u_1 + iu_2$, its central parameter $z^2 = a$ gives one unitary-equivalence class of 2-dimensional irreducible representations for $|a| < 1$ and two characters for $|a| = 1$. The disk point in Proposition 7.6 is instead an irreducible 3-dimensional Arveson boundary point.*

Proof. The sphere relation gives

$$z^* z + z z^* = 2, \qquad z^2 z^* = z^* z^2.$$

Thus z^2 is central. After setting $z^2 = a$, every word reduces to the span of $1, z, z^*, zz^*$, so an irreducible image is a matrix algebra of dimension at most four. Its representation dimension is at most two. The parameter classification is [20, Proposition 4.1].

For the disk matrices of Proposition 7.6,

$$I - A_1^2 - A_2^2 = \frac{1}{2}\, e_{33}, \qquad A_1 e_{33} = \frac{1}{2}\, e_{13}, \qquad A_2 e_{33} = \frac{1}{2}\, e_{23}.$$

These elements and their adjoints generate every matrix unit, proving irreducibility. The dilation computation in that proposition proves the Arveson boundary assertion. □

Writing the free circle in terms of $z = u_1 + iu_2$ places it at one end of a family of quantum conics [20]. The scalar points form the same circle throughout the family. The irreducible operator representations change at the endpoint.

Definition 7.8. *For $-1 \le t < 1$, the quantum disc is the universal C^*-algebra generated by $z = x + iy$ with x, y self-adjoint and*

$$z^*z - tzz^* = 1 - t.$$

The parameter t is fixed for each algebra.

At $t = -1$ the defining equation is $z^*z + zz^* = 2$, the free-circle relation just used. As soon as $-1 < t < 1$, an infinite-dimensional weighted shift appears.

Proposition 7.9. *For $-1 < t < 1$, the quantum-disc algebra is the Toeplitz algebra. Its irreducible representations are the circle characters and the Fock representation*

$$ze_m = \sqrt{1 - t^{m+1}}\, e_{m+1}, \qquad m \ge 0,$$

up to unitary equivalence. At every parameter $-1 \le t < 1$, its classical points form the unit circle.

Proof. The classification and faithfulness of the Fock model for $-1 < t < 1$ are given in [20]. The weights tend to one, so its image is contained in the Toeplitz algebra. Conversely, spectral calculus on $1 - zz^*$ gives the rank-one projection onto e_0; multiplying by the shift and its adjoint gives all matrix units. The difference between z and the unilateral shift is compact, so the image contains that shift and is the Toeplitz algebra. The scalar relation reduces to $(1 - t)|z|^2 = 1 - t$, giving the circle. □

For the universal self-adjoint disk the relation is the inequality $z^*z + zz^* \le 2$. Its characters fill the closed disk, while those of the quantum discs above lie on its circumference.

The sphere and ball relations share another property that matters when one tries to extend these constructions to other free varieties. Each bounds the norm of every coordinate in every representation. For the sphere this follows from $1 - \sum x_i^2 = 0$ modulo the defining ideal, and for the ball it follows from the defining inequality. The elliptic quadrics in [20, Sections 1 and 5] show how these boundedness properties can come apart. Compactness at each matrix size need not give a uniform norm bound, and a uniform bound on bounded representations need not admit an algebraic sum-of-squares certificate. For a differential equation on such a variety one must specify the operator domain as well as the integration state.

8 Stereographic projection

The balls of Section 5 describe bounded pieces of affine space by an inequality on the coordinates. Stereographic projection suggests another way to obtain a free plane, starting with a sphere and

deleting its north pole. To see which features of the ordinary plane such a chart should recover, recall its descriptions in Cartesian and polar coordinates.

Proposition 8.1. *Ordinary Euclidean space has the function-algebra descriptions*

$$C_0(\mathbb{R}^N) = \bigotimes_{i=1}^{N} C_0(\mathbb{R}), \qquad C_0(\mathbb{R}^N) = \{F \in C_0([0,\infty), C(S^{N-1})) \mid F(0) \in \mathbb{C}1\}.$$

It is also the quotient of the tautological line bundle over $\mathbb{R}P^{N-1}$ obtained by collapsing its zero section.

Proof. The first formula is the function algebra of a Cartesian product. For the second, write $v = ru$: the radius and direction determine every nonzero vector, and all directions collapse at zero. The condition on $F(0)$ is exactly continuity at that point. For the last description, a pair (l, v) with $v \in l$ is determined by v away from zero. The map $(l, v) \to v$ is proper and identifies precisely the zero section, so the induced bijection from its quotient is a homeomorphism. □

The Cartesian description uses commuting factors, and the polar description uses a central radius. There is also a description in terms of a single character. A function on the punctured sphere vanishes at infinity precisely when it extends over the missing pole with value zero. The kernel of evaluation at the pole makes sense on the universal free sphere as well.

Lemma 8.2. *In the universal algebra $C(S^N_+)$, let ϵ_p be the north-pole character $p = (1, 0, \dots, 0)$. Then*

$$I_p = \ker \epsilon_p = \overline{C(S^N_+)(1-t)C(S^N_+)},$$

where the bar denotes closed linear span.

Proof. The ideal on the right is killed by ϵ_p. In its quotient $t = 1$, so $\sum_i x_i^2 = 0$. Positivity forces each $x_i = 0$, and the quotient is exactly $\mathbb{C}$. Hence the ideal is the kernel of that character. □

Classically $I_p = C_0(S^N - \{p\})$. Stereographic projection identifies it with $C_0(\mathbb{R}^N)$ and thus transports Euclidean addition to the punctured sphere. It is worth keeping track of where the addition coproduct takes its values, since the sum of two large opposite vectors can remain bounded.

Proposition 8.3. *With $r^2 = \sum_i v_i^2$, the maps*

$$\Phi(v) = \left(\frac{r^2-1}{1+r^2}, \frac{2v}{1+r^2}\right), \qquad \Psi(t, x) = \frac{x}{1-t}$$

are inverse homeomorphisms between $\mathbb{R}^N$ and the sphere with its north pole removed. Transported addition gives $\Delta_{add} f(P, Q) = f(\Phi(\Psi(P) + \Psi(Q)))$, with values in the multiplier algebra of $I_p \otimes I_p$.

Proof. Substitution gives the sphere equation and $1 - t = 2/(1+r^2)$; substituting this in Ψ proves both inverse identities. For $x = \sqrt{1-t^2}\, u$ the chart is $v = \sqrt{(1+t)/(1-t)}\, u$, so it preserves direction. Euclidean addition is associative, hence its transport defines the coproduct. For $f \in C_0(\mathbb{R}^N)$, the function $f(v + w)$ is bounded and continuous, so it belongs to $M(C_0(\mathbb{R}^N) \otimes C_0(\mathbb{R}^N)) = C_b(\mathbb{R}^{2N})$. It need not vanish at infinity: along $(v, -v)$ it equals $f(0)$. □

For operator coordinates the expression $x_i/(1-t)$ must be given an order before we can ask about addition or integration. Write $b = 1 - t$ and suppose that b is invertible in a unital C^*-algebra. Dividing on the right will let us recover the height from the positive element $\sum_i z_i^* z_i$, even when the chart coordinates z_i themselves are not self-adjoint.

Theorem 8.4. *The map $z_i = x_i b^{-1}$ is a bijection from sphere representations with b invertible onto families satisfying*

$$(1+R)z_i = z_i^*(1+R), \qquad R = \sum_i z_i^* z_i.$$

Its inverse is

$$t = (R-1)(R+1)^{-1}, \qquad x_i = 2z_i(R+1)^{-1}$$

with $b = 1 - t = 2(1+R)^{-1}$.

Proof. In the forward direction,

$$R = b^{-1}(1-t^2)b^{-1} = (1+t)(1-t)^{-1}, \qquad b = 2(1+R)^{-1}.$$

Writing $x_i = z_i b$, self-adjointness is precisely $z_i b = b z_i^*$, which becomes the displayed condition after multiplication by b^{-1} on both sides. Conversely put $b = 2(1+R)^{-1}$ and use that condition to make each x_i self-adjoint. Then $\sum_i x_i^2 = \sum_i x_i^* x_i = bRb = 2b - b^2$ and $t = 1 - b$ completes the family. The two substitutions are inverse. □

For scalars the chart relation simply says that each z_i is real. On the reduced free sphere the support of the latitude stays away from both poles. This makes the chart bounded and also allows us to divide its coordinate column by its radius.

Corollary 8.5. *On the reduced S_+^N with $N \geq 2$, both $1 - t$ and $1 + t$ are invertible. The right chart has squared radius $R = (1+t)/(1-t)$ and polar column*

$$u_i = z_i R^{-1/2} = x_i(1-t^2)^{-1/2}, \qquad \sum_i u_i^* u_i = 1.$$

The law of R is the image of the latitude law under $t \to (1+t)/(1-t)$.

Proof. Proposition 3.5 in $N+1$ coordinates gives $\|t\| = 2/\sqrt{N+3} < 1$. The inverse functions are therefore bounded. Theorem 8.4 gives R; its factors commute because they are functions of t. This proves the polar formula, and $\sum_i x_i^2 = 1 - t^2$ gives the isometry identity. Functional calculus gives the law assertion. □

The joint law of this direction column and the latitude is considered in Section 9. There is a second natural order of division in the chart itself. Placing $b^{-1/2}$ on each side of x_i gives a self-adjoint coordinate, but recovering the height now requires solving an equation for b.

Proposition 8.6. *A self-adjoint family y_i is of the form $y_i = b^{-1/2} x_i b^{-1/2}$ for sphere coordinates with $b = 1 - t$ invertible exactly when there is a positive invertible solution of*

$$b + \sum_i y_i b y_i = 2.$$

Each such solution gives $x_i = b^{1/2} y_i b^{1/2}$ and $t = 1 - b$. The solution b need not be unique, already for the free circle.

Proof. Conjugating $\sum_i x_i^2 = 2b - b^2$ by $b^{-1/2}$ gives the equation. Conversely the equation implies $b \leq 2$ and $\sum_i x_i^2 = 2b - b^2$, proving the sphere relation. For nonuniqueness, put

$$y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad b_s = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}, \qquad -1 < s < 1.$$

Every b_s is positive and invertible, and $y b_s y = b_{-s}$, so it solves the equation. The heights $1 - b_s$ are distinct, but the symmetric chart coordinate is always y. □

We turn to the volume carried by the ordinary chart. Starting with round probability gives a probability density on the plane. Starting with Lebesgue measure gives an infinite measure on the punctured sphere. Both follow from the same length factor in the derivative of Φ. Write dA for ordinary area on S^N and A_N for its total area.

Proposition 8.7. *For measurable $f \geq 0$ on the punctured sphere,*

$$\int_{\mathbb{R}^N} f(\Phi(v))\, dv = \int_{S^N - \{p\}} f(t,x)(1-t)^{-N}\, dA(t,x).$$

In the other direction the image of dA/A_N on the plane has density $A_N^{-1}(2/(1+|v|^2))^N$ with respect to Lebesgue measure.

Proof. Put $b = 2/(1+|v|^2)$, so $\Phi(v) = (1-b, bv)$. For a vector w based at v put $c = db(w) = -b^2\langle w, v\rangle$. The squared length of $d\Phi(w)$ is

$$b^2|w|^2 + 2bc\langle w, v\rangle + (1+|v|^2)c^2 = b^2|w|^2.$$

Thus every tangent length is multiplied by b and volume by b^N. Since $b = 1-t$, both changes of variables follow. The pole has zero round area and the transported volume has infinite mass. □

The probability density on the plane is radial. It can also be obtained by dividing a Gaussian vector by the length of an independent one.

Corollary 8.8. *Let G, H be independent standard real Gaussian vectors in $\mathbb{R}^N$. The law of $G/|H|$ is the image of round probability under stereographic projection. For $N = 2$ its density is $1/(\pi(1+|v|^2)^2)$.*

Proof. Conditioning on $|H| = s$, the density of G/s contributes the factor $s^N e^{-s^2|v|^2/2}$. The radial Gaussian density of $|H|$ is proportional to $s^{N-1}e^{-s^2/2}$. Integrating their product gives

$$\int_0^\infty s^{2N-1} e^{-(1+|v|^2)s^2/2}\, ds = \frac{2^{N-1}\Gamma(N)}{(1+|v|^2)^N}.$$

The resulting probability density is proportional to $(1+|v|^2)^{-N}$, hence agrees with Proposition 8.7 by normalization. For $N = 2$, the area $A_2 = 4\pi$ gives the stated density. □

On the reduced free sphere we likewise know the law of the squared chart radius, since $R = (1+t)/(1-t)$. Integrating a general polynomial in the chart coordinates requires the joint law of radius and direction. This is a question about the spherical trace, even before one asks for an invariant infinite-volume integral on a free plane.

9 Moments and spherical coordinates

The product formula of Section 3 describes the law of one coordinate on the free sphere. To get its moments we now compute the Fourier coefficients of the circular weight. The calculation recovers the finite formulas in [3] and gives a point of comparison for the mixed moments that arise when several coordinates form a block.

For $N > 2$ recall that $W = \sqrt{N+2}\, x_N$, where $q + q^{-1} = -N$ and $-1 < q < 0$. By Theorem 3.4 its law is the image under $z \to z + z^{-1}$ of normalized circle measure weighted by $K(z)/\int_{\mathbb{T}} K$, with

$$K(z) = \left| \frac{(z^2;q)_\infty}{(-qz^2;q)_\infty} \right|^2$$

Since $K(-z) = K(z)$, only even Fourier indices occur. Put $v = z^2$ and write the normalized density as $F(v) = \sum_{j\in\mathbb{Z}} F_j v^j$. We can leave the integral of K unevaluated, since the normalization is fixed by $F_0 = 1$. Introduce the products

$$\Theta(v) = (v;q)_\infty (q/v;q)_\infty, \qquad Z(v) = \frac{\Theta(v)}{\Theta(-v)}.$$

Shifting v by a factor of q gives a simple relation for Z.

Lemma 9.1. *The products above satisfy*

$$\Theta(qv) = -v^{-1}\Theta(v), \qquad Z(qv) = -Z(v), \qquad K(z) = (v - v^{-1})Z(v),$$

where $v = z^2$. The last product has a removable singularity at $v = -1$.

Proof. Move the first factor outside each infinite product to obtain the identity for Θ. Taking the quotient at v and $-v$ gives the identity for Z. Finally use $(v^{-1};q)_\infty = (1 - v^{-1})(q/v;q)_\infty$ and $(-v;q)_\infty = (1 + v)(-qv;q)_\infty$ to recover K. The factor $v - v^{-1}$ cancels the simple pole of Z at -1. □

The identity $Z(qv) = -Z(v)$ lets us compare Laurent coefficients on two circles whose radii differ by a factor of $|q|$. Only one pole lies between them, and the same residue enters every coefficient.

Theorem 9.2. *For every integer j the circular coefficient of index $2j$ for W is*

$$F_j = (-1)^j \frac{q^{j-1}(1+q)^2}{(1+q^{j-1})(1+q^{j+1})}.$$

In particular the odd moments vanish and the even moments are

$$h(W^{2k}) = \sum_{j=-k}^{k} \binom{2k}{k+j} F_j$$

for $k \geq 0$.

Proof. On $|q| < |v| < 1$, let b_j be the Laurent coefficients of Z. Fix $|q| < R < 1$. Integrate $Z(v)v^{-j-1}\,dv/(2\pi i)$ first at radius R, then at radius $R/|q|$. The first integral is b_j. In the second put $y = qv$. Rotation through π preserves orientation and $Z(y/q) = -Z(y)$, so the second integral is $-q^j b_j$. The poles of Z are simple and located at $-q^m$, $m \in \mathbb{Z}$. Only -1 lies between the two circles. If b is the residue of Z there, the difference of the integrals gives

$$-(1+q^j)b_j = b(-1)^{-j-1}, \qquad b_j = b\frac{(-1)^j}{1+q^j}.$$

Since the product $(v - v^{-1})Z(v)$ is regular across the unit circle, its Laurent coefficients there are $b_{j-1} - b_{j+1}$. This difference is

$$b(1-q^2)\frac{(-1)^j q^{j-1}}{(1+q^{j-1})(1+q^{j+1})}.$$

Divide by its value at $j = 0$. This proves the density formula of [3, Theorem 5.1]. Finally, integrating $(z + z^{-1})^{2k}$ against $F(z^2)$ selects the constant term. It gives the asserted sum since $F_{-j} = F_j$. The odd powers have no constant term. □

The coefficient F_j occurs in the proof as a difference of two neighboring Laurent coefficients. Moving that difference onto the binomial factors gives the finite sum of [3, Theorem 5.2].

Corollary 9.3. *For $k \geq 0$, the even moment is*

$$h(W^{2k}) = \frac{1+q}{q-1} \cdot \frac{1}{k+1} \sum_{j=-k-1}^{k+1} (-1)^j \binom{2k+2}{k+1+j} \frac{j}{1+q^j},$$

with the same parameter $-1 < q < 0$.

Proof. Put $g_j = (-1)^j/(1+q^j)$. The proof of Theorem 9.2 gives $F_j = (g_{j-1} - g_{j+1})/(g_{-1} - g_1)$, with $g_{-1} - g_1 = (1-q)/(1+q)$. Shift the two binomial sums by one index, extending binomial coefficients by zero outside their range. The coefficient of g_j becomes

$$\binom{2k}{k+j+1} - \binom{2k}{k+j-1} = -\frac{j}{k+1} \binom{2k+2}{k+1+j}.$$

This identity follows from factorials, including at the extreme indices. Substitution gives the stated sum. □

For $N = 2$ we obtained the density $|x|$ from the half-liberated circle model. The same density can now be recovered coefficient by coefficient as $q \to -1$.

Proposition 9.4. *As $q \to -1$ from above, $F_j \to 0$ for odd j and $F_j \to 1/(1-j^2)$ for even j. The limiting coordinate law is $|x|\,dx$ on $[-1,1]$, so its square is uniform on $[0,1]$.*

Proof. Allow $N = t + t^{-1}$ to be real in the scalar recurrences, with $0 < t < 1$. Their product measures extend continuously in each fixed moment to $t = 1$: only finitely many recurrence steps are involved, and the coefficients k_r remain positive. At the endpoint the recurrence is the one proved for the free circle.

For odd j, the numerator $(1+q)^2$ in F_j tends to zero while the two denominator factors stay away from zero. For even j, use $(1+q^m)/(1+q) \to m$ for odd integers m, including negative m. This gives $1/(1-j^2)$. These are exactly the Fourier coefficients of $(\pi/2)|\sin(2\theta)|$ in $v = e^{2i\theta}$. Indeed, integration of $\sin\phi\cos(j\phi)$ on $(0,\pi)$ gives $2/(1-j^2)$ for even j, while the second half-period cancels odd indices. Since $x = \cos\theta$ at $N = 2$, the image density is $|x|$. □

One latitude suffices for all the calculations above. In spherical coordinates one also groups several coordinates together, first taking the length of a block and then dividing by that length to obtain a direction. The squared length makes sense for every block on the free sphere.

Definition 9.5. *For $1 \leq k < N$, the squared radius of the first block is*

$$E_k = r_k^2 = \sum_{i=1}^{k} x_i^2.$$

The complementary block has squared radius $1 - E_k$.

The identity $\sum_i x_i^2 = 1$ expresses the complementary squared radius in terms of the first. It also determines the covariance of two distinct coordinate squares once the variance of one square is known.

Proposition 9.6. *On the free sphere E_{N-k} has the same law as $1 - E_k$. Moreover*

$$h(E_k) = \frac{k}{N}, \qquad Var(E_k) = \frac{k(N-k)}{N^2(N+1)}$$

with both moments taken in the invariant state.

Proof. The complement identity and permutation symmetry give the law assertion. Put $Y_i = x_i^2$. By permutation symmetry the variances have a common value v, and the covariances for ordered

distinct indices have a common value c. Taking the covariance of Y_1 with $\sum Y_i = 1$ gives $v + (N-1)c = 0$. The degree 4 calculation in Section 1 gives

$$v = \frac{2}{N(N+1)} - \frac{1}{N^2} = \frac{N-1}{N^2(N+1)}$$

The sum of k variances and $k(k-1)$ covariances gives the result. □

For the ordinary and half-liberated spheres one can identify the whole block law by normalizing Gaussian vectors. This gives a way to compare their variances with the free value just obtained.

Proposition 9.7. *The law of E_k on the ordinary sphere is beta with parameters $k/2, (N-k)/2$. On the half-liberated sphere it is beta with parameters $k, N-k$. At $N = 4$, $k = 2$, their variances are respectively $1/12$ and $1/20$, the latter also being the free variance.*

Proof. Normalizing independent real Gaussians gives the first beta law. In the complex-sphere matrix model, $E_k = (\sum_{i=1}^{k} |w_i|^2)I$; normalized complex Gaussians give the second. Substitute the parameters in the beta variance formula and use Proposition 9.6 for the free variance. □

Consider the block of two coordinates at $N = 4$. Its free variance agrees with the half-liberated variance, but its higher moments involve a sum over words $x_{i_1}^2 \cdots x_{i_m}^2$ with $i_j \in \{1, 2\}$. For a single latitude all tensor indices were equal. Here the pairing conditions depend on the sequence of indices being summed. This block is also the squared radius of the projected disk of Section 5. In particular, its first two moments leave open whether its support contains zero.

For the block complementary to a single latitude t, the squared radius is just $1 - t^2$. Its square root is invertible in the reduced algebra when $N > 2$, so in this case we can carry out the second step and define the direction.

Proposition 9.8. *On the reduced free sphere with $N > 2$, put $t = x_N$ and $r = (1 - t^2)^{1/2}$. Then r is invertible, and*

$$u_i = x_i r^{-1}, \qquad \sum_{i<N} u_i^* u_i = 1, \qquad x_i = u_i r, \qquad u_i r = r u_i^*,$$

for each visible coordinate $i < N$.

Proof. Proposition 3.5 gives $\|t\| < 1$, hence invertibility of r. Since $\sum_{i<N} x_i^2 = r^2$, conjugation by r^{-1} gives the isometry identity. The last two identities follow from the definition of u_i and self-adjointness of x_i. □

The latitude law fixes the distribution of r. To integrate words in the original coordinates, however, we must know how r occurs between the factors u_i and u_i^*. Already on S_+^2 this asks for mixed moments of t and (u_1, u_2). The noncrossing moment–cumulant relations of [31] give a test for any proposed freeness relation.

On the free circle this calculation can be carried through. There $r = |x_1|$, and the direction on its support is the sign of x_1. Proposition 5.5 gives the joint law explicitly. A single central uniform variable determines both absolute values, and the two signs are freely independent.

10 The music of the spheres

We return to the wave equation of Section 4. On V_r° it reads $f'' + \lambda_r f = 0$, so one hears a single frequency from every mode in this space. There are two pieces of spectral information here, the pitch and the number of independent modes that can produce it. The same distinction enters the spectral action [7], where a trace counts each Dirac eigenvalue with its multiplicity.

Definition 10.1. *With the first positive pitch tuned to 1, the degree-r pitch is $\omega_r = \sqrt{\lambda_r}$ and its multiplicity is $\dim V_r^\circ$.*

Tuning the first pitches alike leaves the higher pitches available for comparison. One can also

distinguish two spheres when they have different numbers of harmonics at a pitch they share.

Theorem 10.2. *For $N \geq 3$ the first degrees distinguishing each pair are as follows:*

	by pitch	*by multiplicity*
S^{N-1}, S^{N-1}_*	3	2
S^{N-1}, S^{N-1}_+	3	2
S^{N-1}_*, S^{N-1}_+	4	3

From degree 4 onwards the classical pitch is strictly highest and the free pitch strictly lowest.

Proof. For the pitch comparison use Proposition 2.4. The function $m \to \tanh((2m+1)u)/\tanh u$ is strictly concave on $[0, \infty)$. Its values at 0 and 1 are 1 and $(N+1)/(N-1)$. The straight line through them is $1 + 2m/(N-1)$. Thus the free increments agree with the half-liberated ones through $r = 2$ and are strictly smaller for $r \geq 3$. The half-liberated increments agree with the ordinary ones through $r = 1$ and are strictly smaller for $r \geq 2$. Summing the increments gives λ_r, so taking square roots proves the first column of the table. The first squared-pitch gaps are

$$\lambda_3^{cl} - \lambda_3^* = \frac{2}{N-1}, \qquad \lambda_4^* - \lambda_4^+ = \frac{2(N-2)(N+1)}{(N-1)(N^2-N-1)}$$

where the second difference also follows by using $g_3^+ = (N^2+N-1)/(N^2-N-1)$.

All 3 multiplicities start with $1, N$. At degree 2, removing the trace gives $N(N+1)/2 - 1$ symmetric quadratics on the ordinary sphere and $N^2 - 1$ quadratics on each quantum sphere. At degree 3 the free multiplicity is $d_3 = N^3 - 2N$ by the splitting in Section 1 and the first-row harmonic decomposition in [16, Section 7.4]. For the half-liberated count, use its complex sphere model from the same reference. The upper matrix entry of $x_i x_j x_k$ is $z_i \bar{z}_j z_k$, and the lower entry is determined by conjugating the variables. The two holomorphic positions are symmetric. The bidegree $(2,1)$ monomials are independent on the complex unit sphere by homogeneity, giving $N\binom{N+1}{2}$ words after symmetrization. The lower-degree part consists of the N linear modes, embedded by multiplying by $\sum_j |z_j|^2$. Subtracting these leaves $N\binom{N+1}{2} - N$ harmonics. The free multiplicity exceeds this by $N(N-2)(N+1)/2$. □

For the two-spheres of Figure 1, the comparison is particularly easy to read from the squared pitches.

Corollary 10.3. *For $N = 3$, the squared pitches are*

degree	1	2	3	4
S^2	1	3	6	10
S^2_*	1	3	5	8
S^2_+	1	3	5	36/5

in the common tuning $\omega_1 = 1$. At $N = 2$ the classical circle separates from the other two in degree 3 by pitch and degree 2 by multiplicity; the half-liberated and free circles coincide.

Proof. Insert $N = 3$ in Proposition 2.4 and sum the increments. For example the free increments are $1, 2, 2, 11/5$. At $N = 2$ the classical and half-liberated eigenvalue formulas first differ in degree three. Their degree 2 multiplicities are 2 and 3, respectively. The equality of the two quantum circles is Lemma 3.6. □

At $N = 1$ all three algebras describe the two-point sphere. For a musical comparison beyond these three families we turn to the quantum sphere played by Connes.

At the Collège de France on 13 October 2011, Alain Connes played *Au clair de la lune* on a quantum sphere [12]. He began on mode 25. Then he asked the audience for another starting number greater than ten. Someone called out 13, and he began the tune again:

$$13,\ 13,\ 13,\ 15,\ 17,\ 15,\ 13,\ 17,\ 15,\ 15,\ 13$$

In the transcript Connes credits Podleś, Brain and Landi, and refers to a well-chosen deformation parameter. The spin geometry of [6] contains an angular Dirac operator with quantum-integer spectrum as in [15]. The full Dirac operator also has another spectral term [6, Corollary 5.5], and the transcript leaves the calibration unspecified. For an explicit comparison we take the angular scale

$$[n]_p = \frac{p^n - p^{-n}}{p - p^{-1}}, \qquad p = 2^{1/12}, \qquad n \geq 1.$$

The choice $p = 2^{1/12}$ lets us interpret a shift of one degree at high pitch as a semitone. Thus the two-degree steps $13 \to 15 \to 17$ in the melody suggest looking at ratios of notes a fixed number of degrees apart.

Lemma 10.4. *Replacing p by p^{-1} leaves $[n]_p$ unchanged. For an integer $j > 0$,*

$$R_{n,j} = \frac{[n+j]_p}{[n]_p} = p^j \frac{1 - p^{-2(n+j)}}{1 - p^{-2n}} \to p^j$$

as $n \to \infty$. Thus a fixed shift of j degrees approaches an interval of j equal-tempered semitones.

Proof. Inversion of p changes the signs of both numerator and denominator in $[n]_p$. Dividing the two quantum integers and factoring out their positive powers of p gives the displayed ratio. Its correction factor tends to one, and $p^j = 2^{j/12}$ is the stated musical interval. □

A fixed pattern of degree shifts therefore approaches the corresponding equal-tempered melody as its starting degree rises. A chord poses a different tuning question. In a pure major triad the three frequencies are in the ratios $4 : 5 : 6$, so the third and the fifth must approximate $5/4$ and $3/2$ above the same root.

Definition 10.5. *The signed tuning error of a frequency ratio x against a target y is $1200 \log_2(x/y)$ cents. The error of a major triad is the larger absolute error of its third and fifth against $5/4$ and $3/2$.*

An octave contains 1200 cents, and an equal-tempered semitone contains 100. For the much smaller discrepancies that occur below, it is helpful to convert a cent error back to a relative frequency error.

Lemma 10.6. *An error of c cents corresponds to relative frequency error*

$$\frac{x}{y} - 1 = 2^{c/1200} - 1 = \frac{\log 2}{1200}\, c + O(c^2)$$

as $c \to 0$.

Proof. Exponentiate $c = 1200 \log_2(x/y)$ and expand the exponential at zero. □

One cent is about 0.058% in frequency, and 3.837 cents is roughly 0.22%. A negative signed error means the interval is flat, a positive error that it is sharp. The following comparison uses the larger of the two absolute errors in each chord.

Proposition 10.7. *On S^2_+ the modes $8, 12, 17$ give a major triad with error below 3.837 cents. On the quantum-integer scale $[n]_{2^{1/12}}$, the infimum of the error over all triples of distinct positive modes is*

$$1200 \log_2 \left(\frac{2^{1/3}}{5/4} \right) = 13.68628\ldots \text{ cents}$$

and is approached as the root tends to infinity.

Proof. For $N = 3$ compute $d_0, d_1, \ldots = 1, 3, 8, 21, 55, \ldots$ and $a_r = d_r - a_{r-1}$, starting from $a_{-1} = 0$. Summing $(d_0 + \cdots + d_r)/a_r$ for $0 \leq r < k$ gives the following values of λ_k for

$k = 8, 12, 17$, respectively:

$$\frac{17787}{1105}, \qquad \frac{573778224}{22914385}, \qquad \frac{80853025082059}{2232250643545}$$

Taking square roots of the ratios λ_k/λ_8 gives $1 : 1.24723335\ldots : 1.50005245\ldots$. The signed errors are -3.83602 and $+0.06054$ cents to five decimal places. On a piano the corresponding errors are $+13.68629$ and -1.95500 cents.

For the quantum integers put $b = p^4/(5/4)$. To attain an error at most $1200 \log_2 b$, the third and fifth would both have to belong to

$$I_3 = [(5/4)/b, (5/4)b], \qquad I_5 = [(3/2)/b, (3/2)b],$$

respectively. Write the 3 degrees as $n, n+d, n+e$, with $0 < d < e$. The formula for $R_{n,d}$ makes it strictly decreasing in n and strictly increasing in d, with $R_{n,d} > p^d$. Since the upper end of I_3 is p^4, only $d = 1, 2, 3$ are possible. For these three d, the values of n with $R_{n,d} \in I_3$ are, respectively, 4, 9, and 16, 17. To reproduce this short check, substitute each listed n and its two neighbors in $R_{n,d}$ and use monotonicity. The table gives the two consecutive ratios straddling the target fifth for each of these roots.

n	d	j	$R_{n,j}$	$R_{n,j+1}$
4	1	1	1.256243	1.516679
9	2	3	1.379705	1.514906
16	3	5	1.444322	1.546393
17	3	5	1.430428	1.529618

Here $I_3 \subset (1.24015, 1.25993)$ and $I_5 \subset (1.48818, 1.51191)$. All displayed ratios in the table are rounded to six decimal places. The comparisons can be checked by rational bounds using $1.059463094359 < p < 1.059463094360$, obtained by taking twelfth powers. Thus $R_{n,e} \in I_5$ is impossible. Finally $R_{n,4} \to p^4$ and $R_{n,7} \to p^7$. The limiting fifth misses by less than two cents, so the limiting third gives the asserted infimum. □

For $N = 3$ we can also minimize the triad error under a bound on the highest degree. The first half-liberated triple to beat the free $(8, 12, 17)$ has degrees $(22, 28, 34)$ and error 3.35641 cents. As the degree bound runs from 17 to 161, the successive records on the free sphere are attained at

$$(8 + 16k,\ 12 + 25k,\ 17 + 36k), \qquad 0 \le k \le 4.$$

For fixed outer degrees $i < l$ the fifth is fixed, so only the third needs to be optimized. It suffices to test the allowed eigenvalues λ_j, $i < j < l$, nearest to $(25/16)\lambda_i$ on each side, including equality. For each interval, if x is its squared ratio divided by its squared target, its absolute error is $600 \log_2 \max(x, 1/x)$ cents. Thus all comparisons can be made exactly before taking logarithms.

The program `coltrane.py` reproduces these records with `-records 161`. Its `-check` option independently recovers $Q'_r(1)$ from noncrossing-pairing moments for $2 \le N \le 6$ and $1 \le r \le 5$. The program `mozart.py` computes the radial exponents and checks the heat mixture and the radial energy by quadrature.

For a single angular mode we can measure both the rate of decay into the ball and the frequency of vibration on the sphere. Both are determined by the same spherical eigenvalue.

Corollary 10.8. *Boundary exponents and wave frequencies satisfy*

$$\alpha_k = \sqrt{(N-1)\omega_k^2 + (N-2)^2/4} - (N-2)/2.$$

They agree when $N = 2$, while for fixed $N > 2$,

$$\alpha_k = \sqrt{N-1}\,\omega_k - \frac{N-2}{2} + O_N(\omega_k^{-1})$$

as $k \to \infty$.

Proof. Substitute $\lambda_k = \omega_k^2$ in Lemma 6.7. For $N = 2$ the correction terms vanish. Otherwise expand the square root at infinity, using $\omega_k \to \infty$. □

For the disk the attenuation exponent of a mode is exactly its wave frequency. In higher dimensions their leading ratio is $\sqrt{N-1}$. Both quantities come from the scalar Laplacian. A Dirac interpretation of the boundary derivative would require additional first-order structure.

Finally, the growth of the wave frequencies tells us how to choose degrees for any chord with integer frequency ratios. On the free sphere the pitch grows like the square root of the degree, while on the other two spheres it grows linearly.

Corollary 10.9. *Fix $N \geq 3$ and positive integers $b_1, \ldots, b_s$. On the free sphere, the modes $b_1^2 m, \ldots, b_s^2 m$, tuned to a common first note, converge in frequency ratios to $b_1 : \cdots : b_s$. On the classical and half-liberated spheres, the modes $b_1 m, \ldots, b_s m$ have the same limit as $m \to \infty$.*

Proof. For the free sphere $\lambda_r = c_N r + O_N(1)$, so the ratio of the pitches in degrees $b^2 m$ and $b_1^2 m$ tends to b/b_1. The classical formula gives $\lambda_r = r^2/(N-1) + O_N(r)$, and the half-liberated formula gives $\lambda_r = r^2/(2(N-1)) + O_N(r)$. Taking square roots at degrees bm and $b_1 m$ gives the same limit. □

Thus degrees $16m, 25m, 36m$ on the free sphere, and $4m, 5m, 6m$ on the other 2 spheres, approach $4 : 5 : 6$. The finite comparison is sharper: the free major triad already starts in degree 8 and beats the quantum-integer benchmark of Proposition 10.7. Summarizing, in just intonation, our spheres sound better than Connes'.

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