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A remark on the density of blow-up data for the Navier–Stokes equations

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Abstract. Alpöge and Buckmaster have shown that a nonzero smooth backward discretely self-similar solution of the Navier–Stokes equations with the decay of a Type I singularity gives smooth data on $\mathbb{T}^3$ whose solutions blow up in finite time. We show that, given such a solution, their construction yields smooth data that equal a given smooth divergence-free field b on $\mathbb{T}^3$ outside a prescribed ball and converge to b in L^p for every $1 \le p < 3$ and in H^s for every $s < 1/2$, with solutions that blow up at times tending to zero. Smooth data whose solutions blow up in finite time are therefore dense among the divergence-free fields of these spaces. The ranges cannot be extended to $p = 3$ or $s = 1/2$. By the theory of Koch and Tataru, for every time before the solution from b ceases to be smooth, data close enough to b in L^3, $\dot{H}^{1/2}$ or BMO^{-1} remain smooth up to that time, and for all times if that solution is global.

1. Introduction

Alpöge and Buckmaster [1] prove that smooth solutions of the Navier–Stokes equations can blow up in finite time, on $\mathbb{R}^3$ from compactly supported data and on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. Their Theorem 1.1 derives the blow-up from a profile, a nonzero smooth backward discretely self-similar solution with the decay of a Type I singularity, and their Section 7 obtains one from a certificate. We use only their proof of Theorem 1.1, and a profile is the hypothesis of Theorem 1.2 below. With viscosity $\nu > 0$ the equations are

$$\partial_t u + (u \cdot \nabla)u = \nu \Delta u - \nabla p, \qquad \operatorname{div} u = 0. \tag{1.1}$$

Smooth solutions are those of [1, Subsection 1.1], periodic on the torus. On $\mathbb{T}^3$ any two with the same datum agree, and the *lifespan* $T_*(a) \in (0, \infty]$ of a smooth divergence-free field a is the length of the largest interval $[0, T)$ on which there is one with datum a. If w is a smooth solution with $\nu = 1$ on $[0, T) \times \mathbb{T}^3$ and $m, x_0 \in \mathbb{R}^3$, then $m + \nu\, w(\nu t, x - x_0 - mt)$ is one with viscosity ν on $[0, T/\nu) \times \mathbb{T}^3$, and conversely. This change of variables multiplies lifespans by ν^{-1} and the constants of bounds $c\,(T-t)^{-1/4} \le \|\nabla w(t)\|_{L^2} \le C\,(T-t)^{-1/4}$ by $\nu^{3/4}$.

We recall the terms of [1, Subsection 1.1], at viscosity one. Let $S > 0$ and let R be a linear isometry of $\mathbb{R}^3$. A *profile* with period S and isometry R is a pair (W, q) of smooth functions $W \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, not identically zero, and $q \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ such that, on $\mathbb{R} \times \mathbb{R}^3$ and for a constant C_W,

$$\partial_s W + \tfrac{1}{2} W + \tfrac{1}{2}(y \cdot \nabla)W - \Delta W + (W \cdot \nabla)W + \nabla q = 0, \qquad \operatorname{div} W = 0,$$

$$W(s + S, y) = R W(s, R^{-1} y), \quad |W(s,y)| \le C_W (1 + |y|)^{-1}, \quad |\nabla W(s,y)| \le C_W (1 + |y|)^{-2}.$$

For $T > 0$ the field

$$u_T^*(t,x) = (T-t)^{-1/2}\, W\Big(\log \frac{T}{T-t}, \frac{x}{\sqrt{T-t}}\Big) \qquad (t < T,\ x \in \mathbb{R}^3) \tag{1.2}$$

then solves (1.1) with $\nu = 1$ on $(-\infty, T) \times \mathbb{R}^3$. Let B_r be the open ball of radius r about 0. On $\mathbb{R}^3$, [1, Theorem 1.1] gives a smooth, compactly supported, divergence-free datum (we write $C^\infty_{c,\mathrm{div}}$ for the space of such fields) whose smooth solution u with $\nu = 1$ has finite energy and $c\,(1-t)^{-1/4} \le \|\nabla u(t)\|_{L^2} \le C\,(1-t)^{-1/4}$; on the torus it reads as follows.

Theorem 1.1 (Alpöge–Buckmaster [1, Theorem 1.1])**.** *Suppose that there exists a profile (W, q). Then there are $T \in (0, 1]$, a divergence-free datum in $C^\infty(\mathbb{T}^3; \mathbb{R}^3)$, a smooth solution u^{per} of (1.1) with $\nu = 1$ on $[0, T) \times \mathbb{T}^3$ with this datum, a number $\kappa \in (0, \frac{1}{4})$ and constants $0 < c \le C$ such that, for $0 \le t < T$,*

$$c\,(T-t)^{-1/4} \le \|u^{\mathrm{per}}(t)\|_{H^1(\mathbb{T}^3)} \le C\,(T-t)^{-1/4}, \quad \|u^{\mathrm{per}}(t) - u^*_T(t)\|_{H^1(B_{1/16})} \le C\,(T-t)^{\kappa - 1/4}.$$

We show that such data approach any smooth divergence-free field in supercritical norms.

Theorem 1.2. *Suppose that there exists a profile. Let $\nu > 0$, let b be a smooth divergence-free field on $\mathbb{T}^3$, and let $B \subset \mathbb{T}^3$ be an open ball. Then there are smooth divergence-free fields u_n on $\mathbb{T}^3$ with $u_n = b$ outside B such that $u_n \to b$ in $L^p(\mathbb{T}^3)$ for every $1 \le p < 3$ and in $H^s(\mathbb{T}^3)$ for every $s < \frac{1}{2}$, whose lifespans $T_n = T_*(u_n)$ tend to 0, and whose solutions $u^{(n)}$ satisfy*

$$c\,(T_n - t)^{-1/4} \le \|\nabla u^{(n)}(t)\|_{L^2(\mathbb{T}^3)} \le C\,(T_n - t)^{-1/4} \qquad (0 \le t < T_n),$$

where $0 < c \le C$ depend only on the profile and on ν.

Since smooth divergence-free fields are dense in those of $L^p(\mathbb{T}^3)$, $1 \le p < 3$, and of $H^s(\mathbb{T}^3)$, $s < \frac{1}{2}$, so are smooth data whose solutions blow up at this rate before any given time.

Idea of the proof. A blow-up that survives small changes of its datum in $\dot{H}^1$, up to finitely many directions, can be inserted at a small scale into any smooth field b, since b at the scale of the profile is small in $\dot{H}^1$. On $\mathbb{R}^3$, Theorem 1.2 holds for $b \in C^\infty_{c,\mathrm{div}}(\mathbb{R}^3)$, with blow-up times in place of lifespans. Indeed, let $\bar{u}_D(0)$ be $W(0, \cdot)$ cut off at the radius D through a vector potential [1, (5.1)], and put $w_0 = \bar{u}_D(0) + r\,b(r\,\cdot)$. Since $\|\nabla(r\,b(r\,\cdot))\|_{L^2} = r^{1/2}\|\nabla b\|_{L^2}$, [1, Lemma 5.1] puts w_0 in the ball about $W(0, \cdot)$ of [1, Proposition 2.1] for D large and r small. That proposition gives $\gamma_r \in E$, with $\|\nabla \gamma_r\|_{L^2} \le C + \varepsilon_0$ in its notation by the second bound of [1, (2.5)] at $t = 0$, and a solution from $w_0 + \gamma_r$, smooth by [1, Lemma 2.3]. The rescaling $u \mapsto r^{-1}u(r^{-2}t, r^{-1}x)$ turns it into a solution from b plus the packet $\bar{u}_D(0) + \gamma_r$ at the scale r.

On the torus they work at blow-up time 1 on $\mathbb{T}^3_L = \mathbb{R}^3/L\mathbb{Z}^3$, with L large, and return to $\mathbb{T}^3$ by the dilation $u \mapsto Lu(L^2 t, Lx)$ [1, Subsection 1.3]. Their approximate solution u^a is the periodization of u^*_1 cut off at the radius $L/8$, and a contraction about u^a gives a solution once the datum is corrected in a fixed space E of finite dimension [1, Subsection 6.4]. The lemma behind the contraction, [1, Lemma 4.8], also allows any change χ of the datum that is small in $\dot{H}^1$ on a period cell Q_L. They take $\chi = 0$ [1, (6.17)], so that [1, Proposition 2.2] gives one datum, and we need the lemma itself. For b of mean zero we prescribe

$$\chi = b_L + \Pi_L(\bar{u}_D(0) - \bar{u}_{L/8}(0)), \qquad b_L(y) = L^{-1}b(y/L), \tag{1.3}$$

where Π_L is periodization, $\bar{u}_{L/8}(0)$ is the same cutoff of $W(0, \cdot)$ at the radius $L/8$, and D is a large fixed radius. The datum becomes $b_L + \Pi_L(\bar{u}_D(0) + \gamma)$ with $\gamma \in E$: the field b at the scale of the profile, unchanged, plus one packet in each period cell. After the dilation the datum is b plus the periodization of $h_r = r^{-1}h(\cdot/r)$, where $r = 1/L$ and $h = \bar{u}_D(0) + \gamma$ lies in a bounded set of fields supported in one ball, and, for $s \ge 0$,

$$\|h_r\|_{L^p(\mathbb{R}^3)} = r^{3/p-1}\|h\|_{L^p(\mathbb{R}^3)}, \qquad \|h_r\|_{\dot{H}^s(\mathbb{R}^3)} = r^{1/2-s}\|h\|_{\dot{H}^s(\mathbb{R}^3)}. \tag{1.4}$$

The exponents are positive exactly when $p < 3$ and $s < \frac{1}{2}$, and the blow-up time is r^2.

The field b_L needs no cutoff. Its interactions with u^a and with itself are terms of the equation that the contraction solves, and only its size enters, which is small because $\dot{H}^1$ is subcritical: $\|\nabla b_L\|_{L^2(Q_L)} = L^{-1/2}\|\nabla b\|_{L^2(\mathbb{T}^3)}$.

The radius D is fixed for two reasons. With the cutoff at $L/8$ the datum on $\mathbb{T}^3$ equals $u_T^*(0)$, $T = L^{-2}$, on $B_{1/16}$ away from a ball of radius of order $1/L$ [1, Subsection 6.4], so that the change of the datum is spread over a ball of fixed radius; with D the packet lies in a ball of radius of order D/L. And [1, Lemma 5.1] allows $\|\bar{u}_{L/8}(0)\|_{L^2}$ to be of order $L^{1/2}$, which the dilation turns into order one in $L^2(\mathbb{T}^3)$, while $\|\bar{u}_D(0)\|_{L^2}$ is bounded.

For the equations with an external force, Cao, Chi and Nie [3, Theorems 3.1 and 4.1] found the same threshold: at fixed viscosity and zero initial velocity, the smooth forces producing breakdown by a given time are dense in the topology of $L^1_t H^s_x$, on $\mathbb{T}^3$ and on $\mathbb{R}^3$, exactly when $s < \frac{1}{2}$; their constructions change the force, and here there is none.

At $p = 3$ and $s = \frac{1}{2}$, and in BMO^{-1}, the norms are invariant under the scaling of (1.1), and the theory of Koch and Tataru [6] gives stability in place of density. On $\mathbb{T}^3$ we take these norms on fields of mean zero, writing $f^\circ = f - \fint_{\mathbb{T}^3} f$, and the norm of BMO^{-1} is that of [6, Section 1], applied to the periodic extension.

Proposition 1.3. *Let $\nu > 0$, let b be a smooth divergence-free field on $\mathbb{T}^3$, and let $\|\cdot\|$ be the norm of $L^3(\mathbb{T}^3)$, of $\dot{H}^{1/2}(\mathbb{T}^3)$ or of BMO^{-1}. For every $\tau < T_*(b)$ there is $\delta > 0$ such that every smooth divergence-free field a on $\mathbb{T}^3$ with $\|a^\circ - b^\circ\| \le \delta$ has $T_*(a) > \tau$. If $T_*(b) = \infty$, there is $\delta > 0$ such that every such a has $T_*(a) = \infty$.*

Smooth data whose lifespans tend to 0 therefore cannot converge to b in these norms, nor in any norm that controls one of them on fields of mean zero, such as the norms of L^p for $3 \le p \le \infty$, of H^s for $s \ge \frac{1}{2}$ and of C^α. In particular, the ranges in Theorem 1.2 cannot be extended, and smooth data whose solutions blow up are not dense in any of these norms: by the second statement of Proposition 1.3 for $b = 0$, every smooth divergence-free field a with $\|a^\circ\| \le \delta$ has $T_*(a) = \infty$. The proof applies [6, Theorems 2 and 3] to periodic extensions. On $\mathbb{R}^3$ the first statement of Proposition 1.3 is known in L^3 from Kawanago [5, Corollary 4.1], and the second in L^3 from the same paper [5, Corollary 1.2 and Proposition 4.2], in L^3 and $\dot{H}^{1/2}$ from Gallagher, Iftimie and Planchon [4, Theorems 0.1 and 3.1], and in BMO^{-1} from Auscher, Dubois and Tchamitchian [2].

Section 2 carries out the construction of [1] on a torus of large period for any small χ, Section 3 applies it to the field (1.3) and proves Theorem 1.2, and Section 4 proves Proposition 1.3.

2. The construction on a torus of large period

Lemma 2.1 below is the analogue on $\mathbb{T}^3_L$, uniform in L, of [1, Proposition 2.1]: the periodic form of the stability in finite codimension that Alpöge and Buckmaster state on $\mathbb{R}^3$ [1, Subsection 1.1]. We keep the notation of [1, Sections 4–6] for the profile of Theorem 1.2, at viscosity one. For $L \ge 1$ let $\mathbb{T}^3_L = \mathbb{R}^3/L\mathbb{Z}^3$ and $Q_L = [-\frac{L}{2}, \frac{L}{2})^3$, and let $\mathcal{X}_L = \dot{H}^1_{\mathrm{div}}(\mathbb{T}^3_L)$ be the space of L-periodic divergence-free fields of mean zero with the norm $\|u\|_{\mathcal{X}_L} = \|\nabla u\|_{L^2(Q_L)}$ [1, (3.15)]; solutions in $\mathcal{X}_L$ are the mild solutions of [1, Subsection 3.6]. For a field f on $\mathbb{R}^3$ that vanishes outside $\bar{B}_{L/8}$ let $\Pi_L f = \sum_{j\in\mathbb{Z}^3} f(\cdot - Lj)$; if moreover $f \in C^\infty_{c,\mathrm{div}}$, then $\Pi_L f \in \mathcal{X}_L$ and $\|\Pi_L f\|_{\mathcal{X}_L} = \|\nabla f\|_{L^2(\mathbb{R}^3)}$ [1, (6.1)]. For maps $v\colon [0,1) \to \mathcal{X}_L$ let $\|v\|_\kappa = \sup_{0\le t<1}(1-t)^{1/4-\kappa}\|v(t)\|_{\mathcal{X}_L}$ [1, (4.5)].

The number $\kappa \in (0, \frac{1}{4})$ and the space $E \subset C^\infty_{c,\mathrm{div}}$ of finite dimension, whose elements vanish outside the ball $\bar{B}_{r_E}$ of [1, Subsection 6.3], are those of [1, Proposition 4.3]; the constant $c_* > 0$ of [1, (4.4)], the constants C_1 and C_2 of [1, Lemmas 6.2 and 6.3], and the number ε_1 of [1, Lemma 4.8] on the spaces $\mathcal{X}_L$ are those of [1, Section 6]; none of them depends on L. For $\sigma > 0$ let $\bar{u}_\sigma$ be the field [1, (5.1)] with radius σ, the field u_1^* cut off at the radius σ through a vector potential. Since $u_1^*(0) = W(0,\cdot)$ by (1.2), [1, Lemma 5.1] gives a constant C_3 such that $\bar{u}_\sigma(0) \in C^\infty_{c,\mathrm{div}}$ vanishes outside B_σ and

$$\|\nabla(\bar{u}_\sigma(0) - W(0,\cdot))\|_{L^2} \le C_3\, \sigma^{-1/2} \qquad (\sigma > 0). \tag{2.1}$$

The approximate solution of [1, Subsection 6.2] is $u^a = \Pi_L \bar{u}_{L/8}$, and F^a is its residual [1, (4.16)] in $\mathcal{X}_L$.

Lemma 2.1. *There are $L_0 \ge 1$, $\varepsilon_0 > 0$ and C_0 such that, for every $L \ge L_0$ and every $\chi \in \mathcal{X}_L$ with $\|\chi\|_{\mathcal{X}_L} \le \varepsilon_0$, there are $\gamma \in E$ with $\|\nabla\gamma\|_{L^2} \le C_0$ and a solution u in $\mathcal{X}_L$ on $[0,1)$ with $u(0) = \Pi_L(\bar{u}_{L/8}(0) + \gamma) + \chi$ and*

$$\tfrac{1}{2} c_*\, (1-t)^{-1/4} \le \|u(t)\|_{\mathcal{X}_L} \le C_0\, (1-t)^{-1/4} \qquad (0 \le t < 1). \tag{2.2}$$

Proof. We follow the proof of [1, Proposition 2.2] in [1, Subsection 6.4], with χ in place of 0. Let $C_R = 2C_2$, let $\varepsilon_0 = \min\{\varepsilon_1/(2C_R^2),\, c_*/(16C_R)\}$, and choose $L_0 \ge \max\{1, 8r_E\}$ with

$$C_2 L_0^{-1/2} \le \tfrac{1}{2}, \qquad C_R^2 C_1 L_0^{-1/2} \le \tfrac{1}{2}\varepsilon_1, \qquad C_1(1 + 2C_R) L_0^{-1/2} \le \tfrac{3}{8} c_*; \tag{2.3}$$

these are the conditions [1, (6.16)] with ε_1 and $\frac{1}{2}c_*$ lowered to $\frac{1}{2}\varepsilon_1$ and $\frac{3}{8}c_*$, and the room left is spent on χ. Let $L \ge L_0$. By the first condition of (2.3), [1, Lemma 6.3] and the Neumann series of [1, Subsection 6.4], the hypothesis [1, (4.22)] of [1, Lemma 4.8] holds on $\mathcal{X}_L$ for u^a, E and $J = \Pi_L$, with the constant C_R. This right inverse, u^a and F^a are built before χ is chosen, and χ enters [1, Lemma 4.8] only through $e^{t\Delta}\chi$, with $\|e^{t\Delta}\chi\|_\kappa \le \|\chi\|_{\mathcal{X}_L}$ by [1, (4.20)]. With [1, Lemma 6.2],

$$d = \|e^{t\Delta}\chi - F^a\|_\kappa \le \|\chi\|_{\mathcal{X}_L} + C_1 L^{-1/2},$$

so that the smallness condition [1, (4.23)], $C_R^2 d \le \varepsilon_1$, holds by the choice of ε_0 and the second condition. Thus [1, Lemma 4.8] gives $\gamma \in E$ and z with $\|z\|_\kappa \le 2C_R d$ [1, (4.24)] such that $u = u^a + z$ is a solution in $\mathcal{X}_L$ on $[0,1)$ with $u(0) = u^a(0) + \chi + \Pi_L\gamma = \Pi_L(\bar{u}_{L/8}(0) + \gamma) + \chi$.

Since $\|z(t)\|_{\mathcal{X}_L} \le (1-t)^{\kappa - 1/4}\|z\|_\kappa \le (1-t)^{-1/4}\|z\|_\kappa$, the bounds [1, (6.5)] for u^a give

$$\|u(t)\|_{\mathcal{X}_L} \ge (c_* - C_1(1+2C_R)L^{-1/2} - 2C_R\varepsilon_0)(1-t)^{-1/4} \ge \tfrac{1}{2}c_*\,(1-t)^{-1/4},$$

by the third condition and $2C_R\varepsilon_0 \le \frac{1}{8}c_*$, and $\|u(t)\|_{\mathcal{X}_L} \le (C_1 + 2C_R d)(1-t)^{-1/4}$. At $t = 0$ the residual and the time integrals in [1, (4.16)–(4.17)] vanish, so that [1, (4.17)] reads $z(0) - \Pi_L\gamma = \chi$, and by [1, (6.1)], $\|\nabla\gamma\|_{L^2} = \|\Pi_L\gamma\|_{\mathcal{X}_L} \le \|z\|_\kappa + \|\chi\|_{\mathcal{X}_L}$. These bounds are uniform in $L \ge L_0$ and in χ. Since $d \le \varepsilon_0 + C_1$, the lemma holds with C_0 depending only on C_1, C_R and ε_0. □

3. Proof of Theorem 1.2

Proof. By the change of variables after (1.1), with $m = \int_{\mathbb{T}^3} b$ and x_0 the center of B, we may take $\nu = 1$, b of mean zero and $B = B_\rho$, where ρ is the radius of B; the constants c and C at viscosity ν are $\nu^{3/4}$ times those found below.

Step 1: the radius D and the period L. Let ε_0, L_0 and C_0 be as in Lemma 2.1, and fix $D > 0$ with $C_3 D^{-1/2} \le \frac{1}{2}\varepsilon_0$. Let $A = \max\{D, r_E\}$, and let $L_1 \ge \max\{L_0, 8A\}$ be such that, for $L \ge L_1$,

$$L^{-1/2}\big(\|\nabla b\|_{L^2(\mathbb{T}^3)} + \sqrt{8}\, C_3\big) \le \tfrac{1}{2}\varepsilon_0, \qquad A/L < \rho.$$

For $L \ge L_1$ let χ be the field (1.3). The field b_L is smooth, L-periodic, divergence-free and of mean zero, and $\bar u_D(0) - \bar u_{L/8}(0) \in C^\infty_{c,\mathrm{div}}$ vanishes outside $\bar B_{L/8}$, so that $\chi \in \mathcal{X}_L$ by [1, (6.1)], and by the change of variables $y = Lx$ in [1, (3.15)], by [1, (6.1)] and by (2.1) for the radii D and $L/8$,

$$\|\chi\|_{\mathcal{X}_L} \le L^{-1/2}\|\nabla b\|_{L^2(\mathbb{T}^3)} + C_3 D^{-1/2} + C_3 (L/8)^{-1/2} \le \varepsilon_0.$$

Step 2: the solution on $\mathbb{T}^3_L$. For $L \ge L_1$, Lemma 2.1 for the field (1.3) gives $\gamma_L \in E$ with $\|\nabla\gamma_L\|_{L^2} \le C_0$ and a solution u_L in $\mathcal{X}_L$ on $[0,1)$ that satisfies (2.2), with $u_L(0) = \Pi_L(\bar u_{L/8}(0) + \gamma_L) + \chi = b_L + \Pi_L h_L$, where $h_L = \bar u_D(0) + \gamma_L$. Since E has finite dimension and $\|\nabla\cdot\|_{L^2}$ is a norm on it, the fields γ_L lie in a bounded subset of E. Hence the fields h_L vanish outside $\bar B_A$ and are bounded in every $L^p(\mathbb{R}^3)$ and in $\dot H^1(\mathbb{R}^3)$, uniformly in L.

Step 3: return to $\mathbb{T}^3$. Let $T = L^{-2}$ and $r = 1/L$. The field $\tilde h_L = L\, h_L(L\,\cdot)$ vanishes outside $\bar B_{Ar}$, where $Ar < \rho$ and $Ar \le \frac{1}{8}$, so that the translates in $H = \Pi_1 \tilde h_L$ have disjoint supports. Let $w(t,x) = L u_L(L^2 t, Lx)$. By [1, (3.21)], as in [1, Subsection 6.4], w is a solution in $\dot H^1_{\mathrm{div}}(\mathbb{T}^3)$ on $[0,T)$ with datum $w(0) = b + H$. This datum is smooth, so that w is a smooth solution on $[0,T)\times\mathbb{T}^3$ [1, Lemma 2.3]. By [1, (3.21)], $\|\nabla w(t)\|_{L^2(\mathbb{T}^3)} = L^{1/2}\|u_L(L^2 t)\|_{\mathcal{X}_L}$, and $L^{1/2}(1 - L^2 t)^{-1/4} = (T-t)^{-1/4}$, so that (2.2) becomes

$$\tfrac{1}{2} c_*\,(T-t)^{-1/4} \le \|\nabla w(t)\|_{L^2(\mathbb{T}^3)} \le C_0\,(T-t)^{-1/4} \qquad (0 \le t < T).$$

A smooth solution on $[0,T]\times\mathbb{T}^3$ has bounded gradient, so that $T_*(w(0)) = T$.

Since $Ar < \rho$, $w(0) = b$ outside B_ρ. By (1.4) with $h = h_L$, for which $h_r = \tilde h_L$, the norms $\|H\|_{L^p(\mathbb{T}^3)}$ and $\|\nabla H\|_{L^2(\mathbb{T}^3)}$ are $r^{3/p-1}\|h_L\|_{L^p(\mathbb{R}^3)}$ and $r^{-1/2}\|\nabla h_L\|_{L^2(\mathbb{R}^3)}$, and by interpolation $\|H\|_{H^s} \le \|H\|_{L^2}^{1-s}\|H\|_{H^1}^{s} \le C' r^{1/2-s}$ for $0 \le s \le 1$, with C' independent of L, while $\|H\|_{H^s} \le \|H\|_{L^2} \le C' r^{1/2}$ for $s < 0$. For a sequence $L_n \to \infty$ with $L_n \ge L_1$, the data $u_n = w(0)$ for $L = L_n$ therefore have the properties stated in Theorem 1.2, with $T_n = L_n^{-2}$, $c = \frac{1}{2}c_*$ and $C = C_0$, which depend only on the profile. □

Remark 3.1. The data u_n converge to b also in $W^{s,p}(\mathbb{T}^3)$ for $1 \le p < 3$ and $0 \le s < 3/p - 1$, with the Slobodeckij norm for non-integer s; for $s = 0$ and for $p = 2$ this is contained in Theorem 1.2. Indeed, by the reduction at the start of the proof and by Step 3, $\|\nabla^j(u_n - b)\|_{L^p(\mathbb{T}^3)} = \nu\, r_n^{3/p-1-j}\|\nabla^j h_{L_n}\|_{L^p(\mathbb{R}^3)}$ for every integer $j \ge 0$, where $r_n = 1/L_n$ and the fields h_{L_n} lie in a bounded subset of the finite-dimensional affine space $\bar u_D(0) + E$ of smooth fields; for non-integer s one interpolates between neighboring integers.

4. Proof of Proposition 1.3

Fields on $\mathbb{T}^3$ are identified with their periodic extensions. For $R > 0$ let BMO^{-1}_R and X_R be the local versions of BMO^{-1} and of the space X of [6, (3)], in which only balls of radius at most R, and times below R^2, enter [6, Section 1]; the norm of X_R bounds $\sup_{0<t<R^2} t^{1/2}\|u(t)\|_{L^\infty}$.

Proof. Reductions. By the change of variables after (1.1), applied to a and to b with their own means, and since the three norms are homogeneous, we may take $\nu = 1$ and a, b of mean zero. On fields f of mean zero, $\|f\|_{\mathrm{BMO}^{-1}} \le C\|f\|_{L^3} \le C\|f\|_{\dot H^{1/2}}$. The second inequality is

the Sobolev embedding. For the first, let $w(t) = e^{t\Delta} f$ and $I = |B(x,R)|^{-1} \int_0^{R^2} \int_{B(x,R)} |w|^2$. At radii $R \le 1$ the ball $B(x,R)$ meets at most 27 cells and the heat flow contracts $L^3(\mathbb{T}^3)$, so that, by Hölder's inequality, $I \le |B(x,R)|^{-2/3} R^2 \sup_t \|w(t)\|^2_{L^3(B(x,R))} \le C\|f\|^2_{L^3(\mathbb{T}^3)}$. At radii $R \ge 1$, $I \le C\|f\|^2_{L^2(\mathbb{T}^3)}$, because $\|w(t)\|_{L^2(\mathbb{T}^3)} \le e^{-4\pi^2 t}\|f\|_{L^2(\mathbb{T}^3)}$ and a ball of radius R meets at most CR^3 cells. So we may take $\|\cdot\|$ to be the norm of BMO^{-1}.

Small data. By [6, Theorems 2 and 3] there is $\varepsilon > 0$ such that, for every $R > 0$, divergence-free data u_0 with $\|u_0\|_{\mathrm{BMO}^{-1}_R} \le \varepsilon$ have a unique small solution in X_R, and data with $\|u_0\|_{\mathrm{BMO}^{-1}} \le \varepsilon$ a unique small solution in X, for all times. Their proof, through the fixed-point form [6, (11)] of the equations, [6, Lemmas 3.1 and 3.2] and the bound [6, (22)] for the heat flow, shows also, after ε is decreased, that the small solutions u, $\tilde{u}$ from two such data satisfy

$$\sup_{0<t<R^2} t^{1/2} \|u(t) - \tilde{u}(t)\|_{L^\infty} \le K \, \|u_0 - \tilde{u}_0\|_{\mathrm{BMO}^{-1}_R}, \tag{4.1}$$

with K independent of R by scaling. For smooth periodic data the small solution is periodic, by uniqueness, and it is the smooth solution v, which solves [6, (11)] because its pressure is periodic and the operators there act on periodic fields as the Fourier multipliers of the torus. The two agree for small times, where v is small in $X_{R'}$ as $\|v\|_{X_{R'}} \le 2R' \sup_{t<R'^2} \|v(t)\|_{L^\infty}$, and then, by continuity in R', as long as v exists below R^2. And v exists below R^2, since $t^{1/2}\|v(t)\|_{L^\infty}$ is then bounded and $\frac{d}{dt}\|\nabla v\|^2_{L^2} \le \|v\|^2_{L^\infty}\|\nabla v\|^2_{L^2}$, so that Grönwall's inequality, applied from a positive time, keeps $\|\nabla v\|_{L^2}$ bounded and the local theory in H^1 recalled in [1, Appendix A.2] continues v. In the global case this holds for all times.

Finite times. Let V be the solution from b, let $0 < \tau < T_*(b)$ and $M = \max_{[0,\tau]} \|V(t)\|_{L^\infty}$, and fix an integer $N \ge 1$ such that $RM \le \frac{1}{2}\varepsilon$ for $\theta = \tau/N$ and $R = (2\theta)^{1/2}$. By the maximum principle for the heat equation, $\|f\|_{\mathrm{BMO}^{-1}_R} \le R\|f\|_{L^\infty}$. Let a be smooth, divergence-free and of mean zero, with $\|a - b\|_{\mathrm{BMO}^{-1}} \le \delta \le \frac{1}{2}\varepsilon$, and let u be its solution. Suppose that, for some $0 \le j < N$, u exists on $[0, j\theta]$ and $d_j = \|u(j\theta) - V(j\theta)\|_{\mathrm{BMO}^{-1}_R} \le \frac{1}{2}\varepsilon$. Then $u(j\theta)$ and $V(j\theta)$ have norms at most ε in BMO^{-1}_R, so that u exists on $[0, j\theta + 2\theta)$, and (4.1) at $t = \theta$ gives $e_{j+1} = \|u((j+1)\theta) - V((j+1)\theta)\|_{L^\infty} \le \theta^{-1/2} K d_j$. Here $d_0 \le \delta$ and $d_j \le R e_j$ for $j \ge 1$, so that $e_j \le (\sqrt{2}K)^{j-1}\theta^{-1/2} K\delta$. For δ small, depending on ε, N and K, the induction runs to $j = N - 1$: then u exists on $[0, (N+1)\theta)$, so that $T_*(a) > \tau$, and

$$\|u(\tau) - V(\tau)\|_{L^\infty} \le (\sqrt{2}K)^{N-1}\theta^{-1/2} K \, \delta.$$

Global solutions. Let $T_*(b) = \infty$. By the energy equality $\|V(t)\|^2_{L^2} + 2\int_0^t \|\nabla V\|^2_{L^2} = \|b\|^2_{L^2}$, the norm $\|\nabla V(\tau)\|_{L^2}$ is as small as we please for some $\tau > 0$; we take it so small that $\|V(\tau)\|_{\mathrm{BMO}^{-1}} \le C\|V(\tau)\|_{L^3} \le C\|\nabla V(\tau)\|_{L^2} \le \frac{1}{2}\varepsilon$. For δ small, the preceding paragraph gives, for every a as there, $T_*(a) > \tau$ and $\|u(\tau) - V(\tau)\|_{\mathrm{BMO}^{-1}} \le C\|u(\tau) - V(\tau)\|_{L^\infty} \le \frac{1}{2}\varepsilon$. Then $\|u(\tau)\|_{\mathrm{BMO}^{-1}} \le \varepsilon$, and by [6, Theorem 2] the solution from $u(\tau)$ exists for all times, so that $T_*(a) = \infty$. □

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