A Steiner triple system with a broken chromatic spectrum

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Abstract

A bicoloring of a Steiner triple system colors its points so that every block receives exactly two colors, and the feasible set of the system is the set of integers k for which some bicoloring uses exactly k colors. Mario Gionfriddo conjectured in 2003 that the feasible set of every bicolorable Steiner triple system has no gaps; Tuza and Voloshin listed the question as open in 2008, and Gionfriddo restated the conjecture in 2025. In 2014 Gionfriddo, Guardo, Milazzo and Voloshin left open whether a system of order 39 has feasible set $\{3, 5\}$. We give a Steiner triple system of order 39 that has bicolorings with exactly three and exactly five colors but none with exactly four, so the conjecture is false. With their theorem that bicolorings of systems of order 39 use three, four or five colors, its feasible set is $\{3, 5\}$. The class sizes of the two bicolorings come from the tables of Colbourn, Dinitz and Rosa. The absence of a four-color bicoloring is proved by computer: a SAT solver's clausal refutation of a direct encoding, checked by DRAT-trim and by a separate unit-propagation checker.

This work has not yet been reviewed by experts. The proofs, the search code, the certificate checkers and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

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1 Introduction

A *Steiner triple system* of order v, an STS(v), is a pair $(X, \mathcal{B})$ where X is a set of v points and $\mathcal{B}$ is a family of 3-element subsets of X, the *blocks*, such that every pair of distinct points lies in exactly one block. Such a system exists exactly when $v \equiv 1$ or $3 \pmod 6$. The smallest nontrivial example is the Fano plane, with 7 points and 7 blocks.

Definition 1. Let $S = (X, \mathcal{B})$ be a Steiner triple system and k a positive integer. A *k-bicoloring* of S is a surjective map $f\colon X \to \{1, \dots, k\}$ such that every block $B \in \mathcal{B}$ satisfies $|f(B)| = 2$. The *feasible set* $\Omega(S)$ is the set of all k for which S has a k-bicoloring. A *gap* of $\Omega(S)$ is an integer $k \notin \Omega(S)$ with elements of $\Omega(S)$ both below and above k.

For example, coloring a point of the Fano plane 1, the two other points of one block through it 2, and the remaining four points 3 gives a 3-bicoloring, and the Fano plane has feasible set $\{3\}$. Bicolorings are the strict colorings of the system viewed as a mixed hypergraph in Voloshin's sense [8], in which every block is both a C-edge (it has two points of the same color) and a D-edge (it has two points of different colors). Since a block is never monochromatic, each color class of a

bicoloring is an independent set, and a Steiner triple system with more than three points has no 1- or 2-bicoloring [2, Section 1]. The feasible set of a mixed hypergraph can have gaps: Jiang, Mubayi, Tuza, Voloshin and West showed this in *The chromatic spectrum of mixed hypergraphs* [6].

For Steiner triple systems, Gionfriddo conjectured the opposite in a lecture at Messina in 2003, published in [3]. Tuza and Voloshin's survey of open problems on mixed hypergraphs, in its Section 7 on Steiner systems, records that "M. Gionfriddo conjectured in [21] that every BSTS has a gap-free chromatic spectrum" (their [21] is [3]), where a BSTS is a Steiner triple system whose blocks are both C-edges and D-edges, and lists "Prove or disprove that all BSTS have gap-free chromatic spectrum" as Open problem 2 [7]. Gionfriddo's 2025 survey states the conjecture again, as Conjecture [M. Gionfriddo 2003] in its Section 4 [4], in the terminology of chromatic spectra: the spectrum is *broken* when the feasible set has a gap. Gionfriddo, Guardo, Milazzo and Voloshin [5] wrote that "The problem to find a bicolorable STS with a gap in its feasible set still remains an open problem", listed "Determine whether there exist systems S with a gap in their $\Omega(S)$" as problem 4 of their conclusion, and, in the table closing their Section 3, listed $\Omega(S) = \{3, 5\}$ for an STS(39) as open.

1.1 Main results

Let S_{39} be the Steiner triple system on the points $0, 1, \ldots, 38$ whose 247 blocks are listed in Appendix A.

Theorem 1. *The Steiner triple system S_{39} has a 3-bicoloring and a 5-bicoloring, and it has no 4-bicoloring. Hence 4 is a gap of $\Omega(S_{39})$, and Gionfriddo's conjecture is false.*

The bicolorings in Theorem 1 have color classes of sizes $10, 12, 17$ and $1, 2, 4, 16, 16$. The nonexistence of a 4-bicoloring is proved by computer, by a clausal proof of unsatisfiability checked by two independent programs (Section 4).

Gionfriddo, Guardo, Milazzo and Voloshin [5, Theorem 3.7] state that the union of the feasible sets of all STS(39)s is $\{3, 4, 5\}$. Together with Theorem 1 this gives the following.

Corollary 2. $\Omega(S_{39}) = \{3, 5\}$.

This is the entry left open for order 39 in the table of [5]. The analogous question for the union of the feasible sets over all systems of one order was settled earlier: Colbourn, Dinitz and Rosa found that this union is $\{3, 5\}$ for order 31 and remarked that it "is not necessarily an interval" [2, Section 4], while Gionfriddo, Guardo, Milazzo and Voloshin proved that no single STS(31) has feasible set $\{3, 5\}$ [5, Theorem 3.1]. Theorem 1 concerns one system with both colorings at once.

The class sizes we used come from Colbourn, Dinitz and Rosa [2]. Their list of 3-splits in Section 2 contains $(17, 12, 10)$ for $v = 39$, and their Table 2 of 5-splits contains $(16, 16, 4, 2, 1)$, each marked as realized by some STS(39). These realizations are separate systems. The STS(39) with a 5-bicoloring described in the proof of [5, Theorem 3.7] "does not admit two distinct disjoint independent sets of cardinalities 9, 14, 16 and 10, 12, 17, i.e., it is not 3-bicolorable." What is new here is a single system that carries a 3-bicoloring and a 5-bicoloring together while admitting no 4-bicoloring; the difficult step is the last one, a nonexistence statement over all colorings of 39 points.

2 The construction

We looked for the system and its two bicolorings at the same time. Fix the 5-coloring g of $\{0, \ldots, 38\}$ with the classes

$$\{0\}, \quad \{1, 2\}, \quad \{3, 4, 5, 6\}, \quad \{7, \ldots, 22\}, \quad \{23, \ldots, 38\}.$$

A Boolean variable for each triple of points receiving exactly two colors under g records whether the triple is a block; the constraints require every pair of points to lie in exactly one chosen triple. Further variables describe a second coloring h with three classes of sizes 10, 12 and 17, and every chosen triple must receive exactly two colors under h. Every solution is therefore a Steiner triple system with the 5-bicoloring g and a 3-bicoloring h. The SAT solver CaDiCaL [1] returned a solution, and S_{39} is that solution, with h the coloring of Appendix A.

Nothing in this search excludes a 4-bicoloring. That is proved separately, for the system found, by the argument of the next section.

3 Proof of Theorem 1

We encode k-bicolorings as satisfying assignments of a propositional formula. For a Steiner triple system $S = (X, \mathcal{B})$ with $X = \{0, \dots, v-1\}$ and an integer $k \geq 1$, let $\Phi_k(S)$ be the formula in the variables $p_{x,c}$ ($x \in X$, $1 \leq c \leq k$), read "x has color c", with the clauses

$$
\begin{aligned}
&p_{x,1} \vee \cdots \vee p_{x,k} \quad \text{and} \quad \neg p_{x,c} \vee \neg p_{x,c'} && \text{for } x \in X \text{ and } c < c',\\
&p_{0,c} \vee p_{1,c} \vee \cdots \vee p_{v-1,c} && \text{for } 1 \leq c \leq k,\\
&\neg p_{x,a} \vee \neg p_{y,b} \vee \neg p_{z,c} && \text{for } \{x,y,z\} \in \mathcal{B} \text{ and colors } a,b,c \text{ with } |\{a,b,c\}| \in \{1,3\}.
\end{aligned}
$$

In the last line each block is written once as $\{x, y, z\}$, and (a, b, c) runs over the assignments of colors to x, y, z that give the block one or three distinct colors; for $k = 4$ there are $4 + 24 = 28$ of them.

Lemma 3. *The satisfying assignments of $\Phi_k(S)$ correspond bijectively to the k-bicolorings of S, by $f(x) = c$ exactly when $p_{x,c}$ is true.*

Proof. The clauses of the first line hold exactly when each point has exactly one color, so they correspond to maps $f\colon X \to \{1, \dots, k\}$. Given such a map, the clauses of the second line hold exactly when f is surjective. For a block $\{x, y, z\}$, the clauses of the third line hold exactly when $(f(x), f(y), f(z))$ is not an assignment with one or three distinct colors, that is, when $|f(\{x, y, z\})| = 2$. □

Proof of Theorem 1. The first two assertions are finite checks. The 247 blocks of Appendix A cover every pair of points exactly once, and the two colorings given there are surjective and give every block exactly two colors; both facts are verified directly from the definitions (Section 4).

For the third assertion take $k = 4$. The formula $\Phi_4(S_{39})$ has 156 variables and $39 \cdot 7 + 4 + 247 \cdot 28 = 7{,}193$ clauses, and by Lemma 3, S_{39} has a 4-bicoloring if and only if $\Phi_4(S_{39})$ is satisfiable. CaDiCaL produced a proof of unsatisfiability of $\Phi_4(S_{39})$ in the DRAT format [9]: a sequence of clause additions and deletions, ending with the empty clause, in which every added clause follows from the formula and the clauses kept so far by unit propagation. This proof is checked as described in Section 4. Hence $\Phi_4(S_{39})$ is unsatisfiable and S_{39} has no 4-bicoloring. □

The encoding contains no symmetry breaking, no fixed colors and no assumption on class sizes, so the refutation covers every map from the 39 points onto four colors.

4 Computational verification

The system, its two bicolorings, the formula $\Phi_4(S_{39})$ in DIMACS format and the DRAT proof are distributed with this paper, together with the programs that check them; the result's README

gives the single command that runs every check, in a few seconds on a laptop. The checks are as follows.

1. A direct check that the 247 blocks are distinct 3-subsets of $\{0, \dots, 38\}$ covering every pair exactly once, and that both colorings are surjective and give every block exactly two colors.

2. An independent construction of $\Phi_4(S_{39})$ from the blocks, in which the block clauses are produced by listing all 4^3 color assignments of each block and keeping those with one or three colors. It is compared, clause by clause as a multiset, with the distributed formula.

3. The DRAT proof is checked against the formula by DRAT-trim [9]. Every lemma in its verified core is a unit-propagation consequence, so the proof needs no resolution-asymmetric-tautology step.

4. The same proof is checked by a short separate program that accepts an added clause only when unit propagation from its negation, over the formula and the clauses accepted before it, yields a conflict, and that ignores deletions. It accepts all 6,374 nonempty added clauses and the final empty clause. Ignoring deletions keeps the check sound.

As calibration, the same definitions give feasible set $\{3\}$ for the Fano plane and for the affine plane of order 3, by listing all set partitions of their points, in agreement with [5, table at the end of Section 3]. For the Fano plane and $k = 2, 3, 4$, the formula Φ_k holds at exactly the k-bicolorings among all k^7 colorings. Both proof checkers reject a proof consisting only of the empty clause, the first half of the proof, and the proof applied to the satisfiable formula $\Phi_5(S_{39})$; the design checker rejects altered blocks and altered colorings.

Beyond the correctness of these programs and of the C compiler and Python interpreter running them, the proof relies on nothing further. The system was found by search, but the search is not part of the proof. Corollary 2 relies in addition on [5, Theorem 3.7].

Acknowledgments

The question is Mario Gionfriddo's, and the colorings were guided by the class-size tables of Charles Colbourn, Jeffrey Dinitz and Alexander Rosa and by the study of small feasible sets by Mario Gionfriddo, Elena Guardo, Lorenzo Milazzo and Vitaly Voloshin. The SAT solver CaDiCaL is by Armin Biere and collaborators; DRAT-trim is by Nathan Wetzler, Marijn Heule and Warren Hunt.

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Appendix A The system and its bicolorings

The row for x lists the pairs $y\ z$ with $x < y < z$ such that $\{x, y, z\}$ is a block of S_{39}; every block appears once, in the row of its smallest point.

x	blocks $\{x, y, z\}$ with $x < y < z$, written $y\ z$
0	1 2; 3 6; 4 5; 7 9; 8 15; 10 13; 11 17; 12 21; 14 19; 16 22; 18 20; 23 33; 24 32; 25 26; 27 30; 28 29; 31 37; 34 35; 36 38
1	3 5; 4 6; 7 17; 8 9; 10 12; 11 22; 13 16; 14 15; 18 19; 20 21; 23 38; 24 31; 25 32; 26 33; 27 28; 29 37; 30 34; 35 36
2	3 4; 5 6; 7 20; 8 14; 9 16; 10 19; 11 21; 12 17; 13 15; 18 22; 23 26; 24 25; 27 35; 28 37; 29 30; 31 32; 33 36; 34 38
3	7 21; 8 11; 9 13; 10 15; 12 22; 14 16; 17 18; 19 20; 23 30; 24 37; 25 31; 26 36; 27 32; 28 33; 29 34; 35 38
4	7 15; 8 22; 9 20; 10 18; 11 12; 13 17; 14 21; 16 19; 23 36; 24 35; 25 37; 26 29; 27 31; 28 34; 30 32; 33 38
5	7 11; 8 17; 9 18; 10 22; 12 16; 13 14; 15 20; 19 21; 23 24; 25 30; 26 28; 27 29; 31 36; 32 35; 33 34; 37 38
6	7 22; 8 18; 9 12; 10 14; 11 16; 13 20; 15 19; 17 21; 23 35; 24 30; 25 33; 26 34; 27 38; 28 36; 29 31; 32 37
7	8 31; 10 25; 12 26; 13 37; 14 24; 16 33; 18 23; 19 28; 27 36; 29 38; 30 35; 32 34
8	10 38; 12 35; 13 33; 16 27; 19 24; 20 34; 21 36; 23 28; 25 29; 26 32; 30 37
9	10 26; 11 23; 14 29; 15 35; 17 31; 19 38; 21 37; 22 36; 24 27; 25 34; 28 32; 30 33
10	11 34; 16 29; 17 30; 20 32; 21 23; 24 36; 27 37; 28 31; 33 35
11	13 35; 14 37; 15 31; 18 38; 19 27; 20 30; 24 26; 25 28; 29 33; 32 36
12	13 30; 14 31; 15 36; 18 34; 19 25; 20 27; 23 37; 24 29; 28 38; 32 33
13	18 28; 19 34; 21 27; 22 24; 23 32; 25 38; 26 31; 29 36
14	17 23; 18 32; 20 38; 22 33; 25 27; 26 30; 28 35; 34 36
15	16 32; 17 29; 18 24; 21 33; 22 28; 23 25; 26 37; 27 34; 30 38
16	17 37; 18 36; 20 26; 21 25; 23 34; 24 38; 28 30; 31 35
17	19 32; 20 28; 22 35; 24 34; 25 36; 26 38; 27 33
18	21 29; 25 35; 26 27; 30 31; 33 37
19	22 26; 23 29; 30 36; 31 33; 35 37
20	22 25; 23 31; 24 33; 29 35; 36 37
21	22 30; 24 28; 26 35; 31 34; 32 38
22	23 27; 29 32; 31 38; 34 37

The 3-bicoloring has the color classes

$$\{6, 8, 11, 14, 15, 17, 20, 24, 27, 36\},$$
$$\{0, 3, 10, 16, 18, 19, 26, 30, 32, 34, 37, 38\},$$
$$\{1, 2, 4, 5, 7, 9, 12, 13, 21, 22, 23, 25, 28, 29, 31, 33, 35\},$$

and the 5-bicoloring is the coloring g of Section 2, with the classes $\{0\}$, $\{1, 2\}$, $\{3, 4, 5, 6\}$, $\{7, \ldots, 22\}$ and $\{23, \ldots, 38\}$.