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ORBIT EQUIVALENCE UNIVERSALITY OF THE UNITARY GROUP AND THE CLASSIFICATION OF II_1 FACTORS

Abstract. Let $\mathcal{U}(\mathcal{H})$ be the unitary group of a separable infinite-dimensional Hilbert space, with the strong operator topology. We prove that every Polish group is the image of a closed subgroup of $\mathcal{U}(\mathcal{H})$ under a continuous open homomorphism. Consequently $\mathcal{U}(\mathcal{H})$ is orbit equivalence universal: the orbit equivalence relation of every Borel action of every Polish group is Borel reducible to an orbit equivalence relation of $\mathcal{U}(\mathcal{H})$. This answers a question of Gao and Pestov. We then prove that isomorphism of separable factors of any fixed type among II_1, II_∞ and III_λ, $0 \leq \lambda \leq 1$, of all separable factors, and of all separable von Neumann algebras is Borel bireducible with the orbit equivalence relation of a universal Borel action of $\mathcal{U}(\mathcal{H})$, confirming a conjecture of Törnquist. Hence the classification of II_1 factors up to isomorphism is complete for orbit equivalence relations of Polish group actions: it is exactly as complex as the classification of separable C^*-algebras up to isomorphism, or of separable Banach spaces up to isometry, and strictly less complex than the classification of separable Banach spaces up to isomorphism.

For the first result we color the vertices of a countable weighted cube by points of the rational Urysohn space in such a way that every isometry of the rational Urysohn space lifts to an isometry of the cube, and isometries close to the identity have lifts close to the identity. A Baire category argument transfers this to the completions, and the metric of the cube is the square of a Hilbert space distance, so that its isometries act unitarily. For the second we attach to a Hilbertian metric space a Gaussian action of the finitary symmetric group, built from an incomplete infinite tensor product, whose conjugacy class up to automorphisms of the group determines the space, and we transfer these actions to II_1 factors by a W^*-superrigidity theorem of Popa and Vaes.

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1. Introduction

Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space. We equip its unitary group $\mathcal{U}(\mathcal{H})$ with the strong operator topology: $U_j \to U$ means $\|U_j\eta - U\eta\| \to 0$ for every $\eta \in \mathcal{H}$. With this topology $\mathcal{U}(\mathcal{H})$ is a Polish group. This paper has two parts. In the first we show that every Polish group is a quotient of a closed subgroup of $\mathcal{U}(\mathcal{H})$; it follows that the actions of $\mathcal{U}(\mathcal{H})$ are as complicated, in the sense of Borel reducibility, as the actions of arbitrary Polish groups. In the second we show that the isomorphism relation of II_1 factors is exactly as complicated as the actions of $\mathcal{U}(\mathcal{H})$. Together, the two parts determine the complexity of the classification problem for II_1 factors, and for von Neumann algebras with separable predual in general: it is the largest that an orbit equivalence relation of a Polish group action can have.

1.1. The unitary group.

Our first result concerns quotients of closed subgroups of $\mathcal{U}(\mathcal{H})$.

Theorem A. *For every Polish group G there are a closed subgroup $R \leq \mathcal{U}(\mathcal{H})$ and a continuous open surjective homomorphism $R \to G$. Moreover, a single closed subgroup of $\mathcal{U}(\mathcal{H})$ admits such a homomorphism onto every Polish group.*

In other words, every Polish group is isomorphic, as a topological group, to R/N for some closed subgroups $N \trianglelefteq R \leq \mathcal{U}(\mathcal{H})$. (A continuous surjective homomorphism between Polish groups is automatically open, by the open mapping theorem for Polish groups [BK96, Section 1.2]. We keep the word "open" in our statements because the proofs produce openness directly, and at one point depend on it.) Quotients cannot be avoided: not every Polish group is isomorphic to a subgroup of $\mathcal{U}(\mathcal{H})$: there are Polish groups with no nontrivial continuous unitary representations at all, the first (abelian) examples being due to Herer and Christensen [HC75]; see also [SS25]. By a theorem of Megrelishvili [Meg01], the group $\mathrm{Homeo}_+[0,1]$ of increasing homeomorphisms of the interval is such a group.

The theorem has a consequence for the classification of orbit equivalence relations. For a Borel action of a Polish group G on a standard Borel space X we write E_X^G for its orbit equivalence relation. If E and F are equivalence relations on standard Borel spaces X and Y, a *Borel reduction* of E to F is a Borel map $f : X \to Y$ such that

$$x \mathrel{E} x' \quad \iff \quad f(x) \mathrel{F} f(x') \qquad (x, x' \in X);$$

we write $E \leq_B F$ if one exists, and $E \sim_B F$ (*Borel bireducible*) if also $F \leq_B E$. Informally, $E \leq_B F$ says that the F-classes serve as complete invariants for E, computed in a definable way. A Polish group H is *orbit equivalence universal* if for every Borel action of a Polish group G on a standard Borel space X there is a Borel action of H on a standard Borel space Y with $E_X^G \leq_B E_Y^H$.

Corollary B. *The unitary group $\mathcal{U}(\mathcal{H})$ is orbit equivalence universal.*

The corollary follows from Theorem A by two standard steps. An action of G becomes an action of R with the same orbits by composing with $R \to G$, and an action of the closed subgroup R induces an action of $\mathcal{U}(\mathcal{H})$ to whose orbit equivalence relation the original one reduces.

Every Polish group G has a *universal* Borel action, whose orbit equivalence relation is $\leq_B$-above those of all Borel actions of G [BK96, Theorem 2.6.1]. We write E_G for the orbit equivalence relation of a universal Borel action of G; it is well defined up to $\sim_B$. An equivalence relation is *complete for orbit equivalence relations* if it is Borel bireducible with some E_X^G and every E_X^G is Borel reducible to it. In these terms Corollary B says that $E_G \leq_B E_{\mathcal{U}(\mathcal{H})}$ for every Polish group G, that is, that $E_{\mathcal{U}(\mathcal{H})}$ is complete for orbit equivalence relations.

1.2. II_1 factors.

The von Neumann algebras on a fixed separable Hilbert space form a standard Borel space, as shown by Effros [Eff65], and so do the factors among them [Eff65] (see also [Eff66]) and, for each type II_1, II_∞ and III_λ, $0 \leq \lambda \leq 1$, the factors of that type [Eff66, Nie73, HW98]; see Section 7.1. Our second main result concerns their isomorphism relations.

Theorem C. *Isomorphism of separable* II_1 *factors is Borel bireducible with* $E_{\mathcal{U}(\mathcal{H})}$*. The same holds for isomorphism of separable* II_∞ *factors, for isomorphism of separable factors of type* III_λ *for every* $\lambda \in [0,1]$*, for isomorphism of separable factors, and for isomorphism of separable von Neumann algebras.*

Thus the classification problem has the same complexity for every type of factors in the classifications of Murray and von Neumann and of Connes, except for type I, where it is trivial. The upper bounds are due to Sasyk and Törnquist [ST09a], see also [TL13]; our contribution is the lower bound for II_1 factors, from which the lower bounds for factors and for von Neumann algebras follow. The lower bound for II_∞ factors requires in addition that the II_1 factors we construct are not isomorphic to proper amplifications of one another (Proposition 9.9). The lower bound for type III_λ uses moreover that these II_1 factors are full (Lemma 9.10), and the uniqueness theorem of Houdayer, Marrakchi and Verraedt [HMV19] for tensor products of full factors with amenable factors. The proof of Theorem C does not use Theorem A. Combining Theorem C with Corollary B gives:

Corollary D. *For every Borel action of a Polish group* G *on a standard Borel space* X*, the orbit equivalence relation* E_X^G *is Borel reducible to isomorphism of separable* II_1 *factors. Hence the isomorphism relations on separable factors of any fixed type among* II_1*,* II_∞ *and* III_λ*,* $0 \le \lambda \le 1$*, on all separable factors, and on all separable von Neumann algebras are complete for orbit equivalence relations.*

The proof of Theorem C passes through two intermediate classification problems. A *Hilbertian space* is a nonempty separable complete metric space which is isometric to a subset of a Hilbert space. By a theorem of Gao [Gao05, Theorem 4.3], isometry of Hilbertian spaces is Borel bireducible with $E_{\mathcal{U}(\mathcal{H})}$. Let $S = \mathrm{Sym}_{\mathrm{fin}}(\mathbb{N})$ be the countable group of permutations of $\mathbb{N} = \{1, 2, \dots\}$ that move only finitely many points. Two probability measure preserving (pmp) actions α of S on (B, μ) and β of S on (B', μ') are *conjugate up to an automorphism*, $\alpha \simeq_{\mathrm{Aut}} \beta$, if there are $\delta \in \mathrm{Aut}(S)$ and a measure preserving isomorphism $T : B \to B'$ with

$$T \circ \alpha(s) = \beta(\delta(s)) \circ T \quad \text{a.e., for every } s \in S. \tag{1.1}$$

We write $\alpha^{\otimes \mathbb{Z}}$ for the diagonal action $s \mapsto \prod_{k \in \mathbb{Z}} \alpha(s)$ on $(B^{\mathbb{Z}}, \mu^{\mathbb{Z}})$.

Theorem E. *To every Hilbertian space* X*, together with a* reference sequence *(a dense sequence in* X *in which every term occurs infinitely often), we assign a pmp action* G_X *of* S *on a standard nonatomic probability space, its* Gaussian tensor action*, such that for all Hilbertian spaces* X, X'*, and any choice of reference sequences, the following are equivalent:*

(i) *X and X' are isometric;*
(ii) $\mathsf{G}_X \simeq_{\mathrm{Aut}} \mathsf{G}_{X'}$*;*
(iii) $\mathsf{G}_X^{\otimes \mathbb{Z}} \simeq_{\mathrm{Aut}} \mathsf{G}_{X'}^{\otimes \mathbb{Z}}$.

Condition (iii) is included because Theorem F below recovers an action only up to passing to $\alpha^{\otimes \mathbb{Z}}$. The second step transfers actions of S to W*-superrigid actions of a group $\Gamma \supseteq S$. Fix $n \ge 3$, let $u \in \mathrm{PSL}(n, \mathbb{Z})$ be the image of the elementary matrix $1 + E_{1n}$ (E_{1n} the matrix unit) and $\Sigma = \langle u \rangle \cong \mathbb{Z}$, let $\langle \zeta \rangle \cong \mathbb{Z}$ be a further infinite cyclic group, and

$$L = S \times \langle \zeta \rangle, \qquad \Gamma = \mathrm{PSL}(n, \mathbb{Z}) *_\Sigma (\Sigma \times L), \tag{1.2}$$

the amalgamated free product over Σ. A pmp action α of S is regarded as an action of L in which ζ acts trivially, and is then *coinduced* to an action $\mathrm{Ind}(\alpha)$ of Γ (Section 9.1); $M(\alpha)$ denotes the group measure space von Neumann algebra of $\mathrm{Ind}(\alpha)$. Here $\mathrm{PSL}(n, \mathbb{Z})$ supplies property (T), and the functions fixed by ζ single out the base coordinates (Lemma 9.7).

Theorem F. *Let* α, β *be pmp actions of* S *on standard nonatomic probability spaces. Then* $M(\alpha)$ *is a separable* II_1 *factor, and*

$$\alpha \simeq_{\mathrm{Aut}} \beta \implies M(\alpha) \cong M(\beta) \implies \alpha^{\otimes \mathbb{Z}} \simeq_{\mathrm{Aut}} \beta^{\otimes \mathbb{Z}}.$$

By Theorems E and F, X and X' are isometric if and only if $M(\mathsf{G}_X) \cong M(\mathsf{G}_{X'})$, and $X \mapsto M(\mathsf{G}_X)$ is Borel (Section 10); this is the lower bound of Theorem C.

1.3. **Comparison with other classification problems.** Corollary D can be compared with other classification problems of analysis whose complexity is known. On one side are the problems which are complete for orbit equivalence relations. Isomorphism of separable C^*-algebras is Borel reducible to an orbit equivalence relation of a Polish group action by [EFP+13] (see also [FTT14]), and Sabok [Sab16] proved that it is complete, already for simple AI algebras, by proving completeness of affine homeomorphism of Choquet simplices. Linear isometry of separable Banach spaces is complete by a theorem of Melleray [Mel07]; so are isometry of Polish metric spaces, by Gao and Kechris and, independently, Clemens [CGK01] (see also [GK03, Cle12]), and homeomorphism of compact metric spaces, by Zielinski [Zie16]; see also [CG17, Cie19, KV20] for classes of continua. On the other side, Ferenczi, Louveau and Rosendal [FLR09] proved that *isomorphism* of separable Banach spaces, that is, the existence of a bounded linear bijection, is a *complete analytic* equivalence relation: every analytic equivalence relation on a standard Borel space is Borel reducible to it. No orbit equivalence relation has this property. Indeed, by a theorem of Kechris and Louveau [KL97, Section 4], the relation E_1 of eventual agreement between sequences of real numbers is not Borel reducible to any orbit equivalence relation of a Polish group action.

Corollary G.

(a) *The following equivalence relations are pairwise Borel bireducible: isomorphism of separable* II_1 *factors; isomorphism of separable* II_∞ *factors; isomorphism of separable factors of type* III_λ*, for each* $\lambda \in [0,1]$*; isomorphism of separable factors; isomorphism of separable von Neumann algebras; isomorphism of separable* C^**-algebras; linear isometry of separable Banach spaces; isometry of Polish metric spaces; homeomorphism of compact metric spaces.*

(b) *Each of these relations is Borel reducible to isomorphism of separable Banach spaces, and isomorphism of separable Banach spaces is not Borel reducible to any of them.*

Thus separable von Neumann algebras, separable C^*-algebras, and separable Banach spaces up to isometry are equally hard to classify, and separable Banach spaces up to isomorphism are strictly harder; see Figure 1. For von Neumann algebras, part (b) uses only the upper bound of Sasyk and Törnquist; the new information is in part (a).

By Connes's theorem [Con76] all amenable II_1 factors with separable predual are isomorphic, whereas isomorphism of separable nuclear C^*-algebras is already complete [Sab16]. Part (a) of Corollary G says that, nevertheless, classifying II_1 factors is as hard as classifying all separable C^*-algebras: there is a Borel map assigning to every separable C^*-algebra A a separable II_1 factor M_A such that

$$A \cong B \quad \iff \quad M_A \cong M_B$$

for all separable C^*-algebras A and B, and there is a Borel map in the opposite direction with the same property. The map $A \mapsto M_A$ is the composition of the reduction of [EFP+13] to a Polish group action with the reductions of Corollary B and Theorem C, and passes through the unitary group and Hilbertian metric spaces. The same applies with separable Banach spaces up to linear isometry, or Polish metric spaces up to isometry, in place of C^*-algebras.

1.4. **Context.** Gao and Pestov [GP03] proved that every abelian Polish group is a quotient of a closed subgroup of $\mathcal{U}(\mathcal{H})$ (see also [Usp08]), and asked whether every separable metrizable group is a topological factor-group of a subgroup of $\mathcal{U}(\mathcal{H})$. For Polish groups the question is attributed to Kechris in [Pes07, Question 34]; see also [PU16]. Theorem A answers this question positively for Polish groups, with closed subgroups and open homomorphisms. (A separable metrizable group G is a dense subgroup of a Polish group, namely its completion for the two-sided uniformity [PU16, Section 2]. Applying the theorem to the completion and restricting the homomorphism to the preimage of G, we get a positive answer for G as well, with a subgroup that need not be closed.)

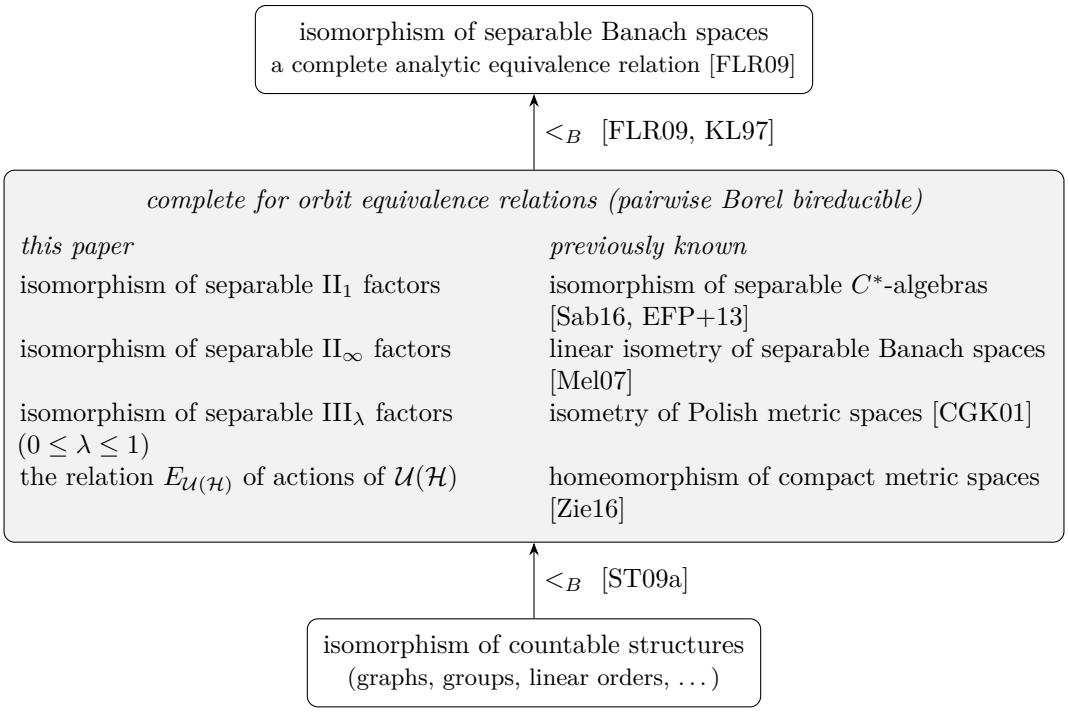


FIGURE 1. Classification problems of equal complexity. The relations in the shaded box are pairwise Borel bireducible, each of them being complete for orbit equivalence relations of Polish group actions; the left column lists the main results of this paper. The two arrows are strict: $E <_B F$ means that $E \le_B F$ and $F \not\le_B E$.

The question whether $\mathcal{U}(\mathcal{H})$ is orbit equivalence universal was posed in the introduction of [GP03]: is there an orbit equivalence relation induced by a Polish group action which is not Borel reducible to any orbit equivalence relation of an action of the unitary group? Gao and Pestov observed there that a positive answer to their question about factor-groups would settle this question, and that is the route taken here; see also [Sab16, Question 9.3], [BCF+23, Section 4.1] and [LP18, p. 1463]. Corollary B shows that there is no such relation, that is, that $\mathcal{U}(\mathcal{H})$ is orbit equivalence universal. By Gao's theorem quoted above, the isometry relation between Hilbertian spaces is then complete for orbit equivalence relations as well. Ding, Li and Peng [DLP26] showed that a Polish group into which not every Polish group embeds as a closed subgroup can induce a complete orbit equivalence relation, answering a question of Sabok [BCF+23]. Since there are Polish groups without nontrivial unitary representations [HC75], not every Polish group embeds into $\mathcal{U}(\mathcal{H})$, so by Corollary B the group $\mathcal{U}(\mathcal{H})$ is another example.

The isomorphism problem for separable von Neumann algebras was placed in the framework of Borel reducibility by Sasyk and Törnquist [ST09a] (see also [ST09b]). Using Popa's rigidity theory [Pop06b, Pop06c], they showed that isomorphism of countable structures is Borel reducible to isomorphism of separable II_1 factors, and that isomorphism of factors of each of the types II_1, II_∞ and III_λ is not classifiable by countable structures (see also [ST10] for Araki–Woods factors). Moroz and Törnquist [MT21] showed that these isomorphism relations are not Borel. Theorem C determines the exact complexity for each of these types. Törnquist conjectured that their upper bound [TL13, Theorem 4.10] is optimal: isomorphism of separably acting factors, and already of II_1 factors, should be universal for orbit equivalence relations of continuous actions of the unitary group on Polish spaces [TL13, Conjecture 4.11]. Since continuous actions on Polish spaces are Borel actions, and the upper bound is witnessed by such an action, Theorem C confirms this conjecture.

For separable C^*-algebras the completeness theorem analogous to Corollary D is due to Sabok [Sab16]; see Section 1.3.

Conjugacy of measure preserving actions of a fixed countable group is known to be complicated: for actions of $\mathbb{Z}$ it is not classifiable by countable structures [FW04] (see also [KLP10] for other groups) and not Borel [FRW11], and for every nonamenable group, conjugacy, orbit equivalence and von Neumann equivalence of free ergodic actions are not Borel [GL26]. Sabok conjectured that conjugacy of measure preserving transformations is complete [BCF+23, Conjecture 4.1], whereas unitary equivalence of unitary representations of a fixed countable group is Borel [HT12]. Theorem E shows that conjugacy *up to automorphisms of the group* of pmp actions of the locally finite group S is Borel above $E_{\mathcal{U}(\mathcal{H})}$, hence, by Corollary B, complete (Corollary 11.2). The twist by the uncountable group $\mathrm{Aut}(S) = \mathrm{Sym}(\mathbb{N})$ cannot be removed by our method, which therefore says nothing about Sabok's conjecture (Remark 11.3).

Theorem F applies W*-superrigidity, from Popa's deformation/rigidity theory [Pop06a, Pop06b, Pop06c, Pop07a]; the first W*-superrigid actions were found by Popa and Vaes [PV10], see also [Pet10]. W*-superrigidity was later established for Bernoulli actions of property (T) groups [Ioa11], for Gaussian actions of rigid groups [Bou13], for all actions of certain amalgamated free products [HPV13], and for coinduced actions [Dri18]. Our group S is locally finite, so these results do not apply to its actions directly; this is why we coinduce to Γ.

1.5. **Outline of the proof of Theorem A.** Let $\mathbb{U}$ be the Urysohn space. By Uspenskij's theorem [Usp90], every Polish group is isomorphic to a closed subgroup of the isometry group $\mathrm{Iso}(\mathbb{U})$. The preimage of a closed subgroup under a continuous open surjection is a closed subgroup which maps openly onto it. So the first assertion of Theorem A reduces to a single group (the second assertion then follows from a theorem of Ding [Din12]):

> *it suffices to find one closed subgroup of $\mathcal{U}(\mathcal{H})$ with a continuous open homomorphism onto* $\mathrm{Iso}(\mathbb{U})$.

We find it as the symmetry group of a metric space $\widehat{C}$ lying over $\mathbb{U}$. More precisely, we construct a complete separable metric space $(\widehat{C}, \rho)$ and a *nonexpansive* map $\widehat{\chi} : \widehat{C} \to \mathbb{U}$, that is, a map which does not increase distances, with two properties.

- (U) *Isometries of $\widehat{C}$ act unitarily.* The metric ρ is the square of a Hilbert space distance. By a classical theorem of Schoenberg there are then unit vectors ξ_x, $x \in \widehat{C}$, in a Hilbert space with $\langle \xi_x, \xi_y \rangle = e^{-\rho(x,y)}$. Every isometry h of $\widehat{C}$ defines a unitary operator U_h by $U_h \xi_x = \xi_{hx}$, and $h \mapsto U_h$ embeds $\mathrm{Iso}(\widehat{C})$ into a unitary group.
- (L) *Isometries of $\mathbb{U}$ lift to $\widehat{C}$.* For every isometry g of $\mathbb{U}$ there is an isometry h of $\widehat{C}$ with $\widehat{\chi} \circ h = g \circ \widehat{\chi}$; and if g is close to the identity, then h can be chosen close to the identity.

The group we are looking for is then the group $\mathrm{Aut}(\widehat{\chi})$ of all pairs (h, g) of isometries with $\widehat{\chi} \circ h = g \circ \widehat{\chi}$. By (U) it embeds into a unitary group via $(h, g) \mapsto U_h$ (a lift h determines g), and the image will be closed. By (L) the projection $p : (h, g) \mapsto g$ maps it openly onto $\mathrm{Iso}(\mathbb{U})$. A Polish group G, viewed as a closed subgroup of $\mathrm{Iso}(\mathbb{U})$, is then the image of the closed subgroup $R = p^{-1}(G)$:

$$\begin{array}{ccccc} R = p^{-1}(G) & \subseteq & \mathrm{Aut}(\widehat{\chi}) & \xrightarrow[\text{closed embedding, by (U)}]{(h,g) \mapsto U_h} & \mathcal{U}(\mathcal{H}) \\ \downarrow & & \downarrow p & \text{open and onto, by (L)} & \\ G & \subseteq & \mathrm{Iso}(\mathbb{U}) & & \end{array}$$

The space $\widehat{C}$ is the completion of a *weighted cube* C. The points of C are the finite subsets of a countable set W of *walls*; each wall w has a weight $r_w > 0$; and the distance between two points s, t

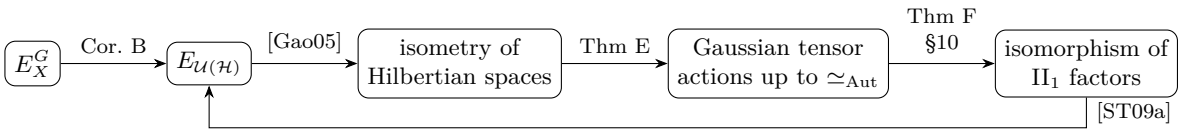
is the total weight of the walls that belong to exactly one of them:

$$\rho(s,t) = \sum_{w \in s \triangle t} r_w = \|1_s - 1_t\|_{L^1} = \|1_s - 1_t\|_{L^2}^2.$$

Here $s \triangle t$ is the symmetric difference, and the norms are taken with respect to the measure on W in which the wall w has mass r_w. The last equality gives (U); it is the familiar fact that wall metrics are of negative type [RS98, Proposition 1.1]; see also [CMV04, CDH10]. For (L), we use the effect of adding a wall w to all points of a finite set $A \subseteq C$, none of which contains w. The set A moves to a parallel copy $\{a \cup \{w\} : a \in A\}$ of itself; every point moves by exactly r_w; and for every point x not containing w the distance from x to the copy $a \cup \{w\}$ of a is $\rho(x,a) + r_w$, the largest value that the triangle inequality allows (see Figure 2 in Section 3). We color the points of C by points of the *rational* Urysohn space $\mathbb{QU}$, by a nonexpansive map $\chi : C \to \mathbb{QU}$, and we arrange that the parallel copies carry all colors that are compatible with χ being nonexpansive: for every finite $A \subseteq C$, every rational $r > 0$, and every nonexpansive map $\psi : A \to \mathbb{QU}$ with $d(\psi(a), \chi(a)) \leq r$ on A, there is a wall w of weight r, belonging to no point of A, such that the copy $a \cup \{w\}$ of each $a \in A$ has the color $\psi(a)$. Such a coloring χ is called *rich*. Rich colorings are constructed in Section 3 by a recursion.

In Section 4 we show that for a rich coloring every isometry g of $\mathbb{QU}$ lifts to an isometry of C, by a back-and-forth argument that handles one wall at a time. The same argument shows that if g moves the colors of the points of a face of the cube by at most a rational number $r > 0$, then g has a lift which moves every point of this face by exactly r. Since every finite set of points lies in a face, the map sending a lift to the isometry it lifts is *open*. We work with the countable spaces C and $\mathbb{QU}$ at this stage because both the construction of χ and the back-and-forth are countable recursions. Section 5 passes to the completions $\widehat{C}$ and $\mathbb{U}$ by a Baire category argument which uses the openness. Section 6 establishes (U) and assembles the proofs of Theorem A and Corollary B.

1.6. **Outline of the proof of Theorem C.** The proof is a chain of Borel reductions, shown schematically below together with Corollary B (an arrow $E \to F$ means $E \leq_B F$; the reductions through Gaussian tensor actions are carried out on codes of Hilbertian spaces, in Section 10):



The device behind (U) above, Schoenberg's Gaussian kernel, is also the starting point here; this time it is applied to the metric of a Hilbertian space itself.

A cloud of unit vectors and its tensor product. Let X be a Hilbertian space with metric d. By Schoenberg's theorem there are unit vectors ξ_x, total in a real Hilbert space $\mathcal{V}_X$, with $\langle \xi_x, \xi_y \rangle = e^{-d(x,y)^2}$; but the Hilbert space $\mathcal{V}_X$ alone does not remember X, since an orthogonal operator does not know which unit vectors belong to the cloud $\{\xi_x\}$. Fix a dense sequence $x_1, x_2, \ldots$ in X in which every term is repeated infinitely often, and let $\mathcal{H}_X$ be the closed span of the elementary tensors $\xi_{q_1} \otimes \xi_{q_2} \otimes \cdots$ with $q_i = x_i$ for all but finitely many i (as the ξ_x are total in $\mathcal{V}_X$, this is von Neumann's incomplete tensor product $\bigotimes_i (\mathcal{V}_X, \xi_{x_i})$ [vN39]). The group S acts by permuting the tensor factors. Isometric spaces give equivalent representations up to an automorphism of S, by a back-and-forth matching of the two reference sequences. Conversely, if v involves only the first m tensor factors and $i, j > m$, then the transposition $\tau_{ij} = (i\ j)$ swaps two untouched factors ξ_{x_i}, ξ_{x_j}, and

$$\langle v, \tau_{ij} v \rangle = \langle \xi_{x_i}, \xi_{x_j} \rangle^2 \, \|v\|^2 = e^{-2d(x_i,x_j)^2} \, \|v\|^2$$

(Figure 4); for arbitrary v this holds up to $o(1)$ as $\min(i,j) \to \infty$. Thus asymptotic coefficients of transpositions encode the distances $d(x_i, x_j)$.

The Gaussian functor and the first chaos. Theorem F needs a measure preserving action, and the Gaussian functor produces one, G_X, whose Koopman representation on mean-zero functions is the sum of the symmetric tensor powers of $\mathcal{H}_X \otimes \mathbb{C}$, the Wiener chaoses; for a unit vector f in the n-th chaos the coefficient $\langle f, \tau_{ij} f\rangle$ is $e^{-2nd(x_i,x_j)^2} + o(1)$ as $\min(i,j) \to \infty$. *A conjugacy need not preserve the first chaos; but for every fixed mean-zero unit vector f the coefficient $\langle f, \tau_{ij} f\rangle$ is at most $e^{-2d(x_i,x_j)^2} + o(1)$, with equality up to $o(1)$ on the first chaos, and this extremal property is preserved by conjugacies.* Hence a conjugacy twisted by $\mathrm{Ad}(\theta)$, $\theta \in \mathrm{Sym}(\mathbb{N})$ (every automorphism of S has this form), satisfies $e^{-2d(x_i,x_j)^2} - e^{-2d'(y_{\theta(i)},y_{\theta(j)})^2} \to 0$ as $\min(i,j) \to \infty$. Since $t \mapsto e^{-2t^2}$ is injective but flat at infinity, $d(x_i, x_j)$ and $d'(y_{\theta(i)}, y_{\theta(j)})$ are asymptotically equal whenever one of them stays bounded. This suffices to define an isometry of X onto X' by $\lim_k x_{i_k} \mapsto \lim_k y_{\theta(i_k)}$, the limits being taken along sequences of distinct indices. The argument applies verbatim to the Gaussian action of $\mathcal{H}_X \otimes \mathcal{K}$ for any nonzero separable real Hilbert space $\mathcal{K}$, and $\mathsf{G}_X^{\otimes\mathbb{Z}}$ is the case $\mathcal{K} = \ell^2(\mathbb{Z})$; this gives (iii)⇒(i) in Theorem E.

From actions to factors. By Popa and Vaes, $\mathrm{Ind}(\alpha)$ is W*-superrigid, since $\mathrm{PSL}(n,\mathbb{Z})$ has property (T) and acts freely on Γ/L, hence by a generalized Bernoulli action. A twisted conjugacy of $\mathrm{Ind}(\alpha)$ and $\mathrm{Ind}(\beta)$ may be normalized to preserve $\Sigma \times L$ and S; the ζ-invariant functions are then the functions of the coordinates indexed by the cosets σL, $\sigma \in \Sigma$, on which S acts by $\alpha^{\otimes\mathbb{Z}}$. As Σ commutes with L, our argument does not separate these coordinates, which is why Theorem F recovers $\alpha^{\otimes\mathbb{Z}}$ rather than α. The theorem of Popa and Vaes applies to isomorphisms onto amplifications as well, and since Γ has no nontrivial finite normal subgroups it shows that $M(\alpha)$ is never isomorphic to a proper amplification of $M(\beta)$; tensoring with $B(\ell^2(\mathbb{N}))$ then gives the II_∞ case. Finally, property (T) of $\mathrm{PSL}(n,\mathbb{Z})$ makes the factors $M(\alpha)$ full, so that by a theorem of Houdayer, Marrakchi and Verraedt $M(\alpha)$ can be recovered, up to stable isomorphism, from its tensor product with a fixed amenable factor of type III_λ; this gives the type III cases.

1.7. **Cited results and organization.** Besides standard facts, the proof of Theorem A and Corollary B rests on Uspenskij's theorem [Usp90], on the density of $\mathrm{Iso}(\mathbb{QU})$ in $\mathrm{Iso}(\mathbb{U})$ [CV06], on Mackey's induced actions [BK96, Theorem 2.3.5], and, for the last assertion of Theorem A only, on Ding's surjectively universal Polish group [Din12, Theorem 3.4]. The proof of Theorem C rests on three published theorems: Gao's theorem [Gao05, Theorem 4.3], the source of the reduction; the W*-superrigidity theorem of Popa and Vaes [PV10, Theorem 6.7(1) and Lemma 6.5], the rigidity input of Theorem F; and the upper bound of Sasyk and Törnquist [ST09a, Theorem 16(i)], see also [TL13, Theorem 4.10]. (Theorem and page numbers for [ST09a, TL13, PV10] are those of the arXiv versions; for [PV10], arXiv:0906.2765v2.) The II_∞ case of Theorem C uses the theorem of Popa and Vaes in its full form, for isomorphisms onto amplifications, and the Borelness of the set of II_∞ factors. The type III_λ cases use in addition property (T) of $\mathrm{PSL}(n,\mathbb{Z})$, the theorem of Houdayer, Marrakchi and Verraedt [HMV19, Theorem E], a theorem of Connes on the type of a tensor product [Con73, Corollaire 3.2.8], and the Borelness of the set of factors of type III_λ, which follows from a theorem of Haagerup and Winsløw [HW98, Theorem 4.9]. Corollary G combines Corollary D with the published results quoted in Section 1.3. In Theorem F, the groups $\mathrm{PSL}(n,\mathbb{Z}) *_\Sigma (\Sigma \times \Lambda)$ and their coinduced actions are those of Popa and Vaes [PV10, Examples 6.8 and 6.9]; what we add is the choice $\Lambda = L = S \times \langle\zeta\rangle$ with ζ acting trivially, the extension of $\mathrm{Aut}(S)$ to Γ, and the recovery of the diagonal power $\alpha^{\otimes\mathbb{Z}}$ of the base action from a conjugacy twisted by an arbitrary automorphism of Γ. The Borel coding of Section 10 uses standard techniques.

The paper is organized as follows. Section 2 collects the preliminaries common to both parts. Sections 3–6 prove Theorem A and Corollary B: the rich coloring is constructed in Section 3, isometries are lifted in Sections 4 and 5, and the unitary representation and the conclusion are in Section 6. Sections 7–11 prove Theorem C and Corollary D: after further preliminaries in Section 7, Theorem E is proved in Section 8 and Theorem F in Section 9, together with its complements on amplifications (Proposition 9.9) and on tensor products with amenable factors (Proposition 9.12);

Section 10 contains the Borel coding, and Section 11 the proofs of the main results and some further consequences. The two parts can be read independently; they meet only in Corollary D and its consequences.

2. Preliminaries

This section fixes notation and collects the standard facts that are used in both parts of the paper. Throughout, $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{Q}_{>0}$ is the set of positive rational numbers. Hilbert spaces are real or complex; real inner products are bilinear, complex ones linear in the second variable, and $\mathrm{i} = \sqrt{-1}$. For a real Hilbert space $\mathcal{V}$ we write $\mathcal{V}_{\mathbb{C}}$ for its complexification and $O_{\mathbb{C}}$ for the unitary extension of an orthogonal operator O. The letter $\mathcal{H}$ always denotes the separable infinite-dimensional complex Hilbert space fixed in the introduction. A map f between metric spaces is *nonexpansive* if $d(f(x), f(y)) \le d(x, y)$ for all x, y. An *isometry* of a metric space X is a distance-preserving bijection of X onto itself, and two metric spaces are *isometric* if there is a distance-preserving bijection of one onto the other.

2.1. **Orbit equivalence relations.** Borel reductions, the relations $\le_B$ and $\sim_B$, the orbit equivalence relation E_X^G of a Borel action, and the relation E_G of a universal Borel action of a Polish group G were defined in the introduction. We use two theorems from the monograph of Becker and Kechris.

Fact 2.1 ([BK96])**.** *Let G be a Polish group.*

(a) ([BK96, Theorem 2.6.1]) *There is a Borel action of G on a standard Borel space Y such that $E_X^G \le_B E_Y^G$ for every Borel action of G on a standard Borel space X.*

(b) (Mackey; [BK96, Theorem 2.3.5], see also [Gao09, Section 3.5]) *Let R be a closed subgroup of G, and let R act in a Borel way on a standard Borel space X. Then there are a Borel action of G on a standard Borel space Y and a Borel injection $i : X \to Y$ such that*

$$x \; E_X^R \; x' \quad \Longleftrightarrow \quad i(x) \; E_Y^G \; i(x') \qquad (x, x' \in X).$$

In (b) one takes for Y the quotient of $G \times X$ by the action $\gamma \cdot (k, x) = (k\gamma^{-1}, \gamma \cdot x)$ of R, on which G acts by left translation of the first coordinate, and lets $i(x)$ be the class of $(1, x)$; that this quotient is a standard Borel space is part of the theorem. Applying (b) to a universal action of R, and then (a), we see that $E_R \le_B E_G$ whenever R is isomorphic to a closed subgroup of G.

2.2. **Polish groups.** A *Polish group* is a topological group whose topology is separable and completely metrizable. We call a map $j : H \to G$ between topological groups a *topological embedding* if it is an injective homomorphism and a homeomorphism onto its image $j(H)$, where $j(H)$ carries the subspace topology.

Fact 2.2. *Let G be a Polish group.*

(a) *A subgroup of G which is Polish in the subspace topology is closed in G. In particular, the image of a topological embedding of a Polish group into G is closed.*

(b) *If N is a closed normal subgroup of G, then G/N is a Polish group* [BK96, Proposition 1.2.3].

Part (a) is a Baire category argument: a Polish subgroup H is a dense G_δ subset of its closure $\operatorname{cl} H$ [Kec95, Theorem 3.11], and so is every coset gH with $g \in \operatorname{cl} H$; two dense G_δ subsets of the Polish space $\operatorname{cl} H$ intersect, so $gH = H$. Recall also that the quotient map $G \to G/N$ is open.

2.3. **Isometry groups.** For a metric space (X, d) we write $\mathrm{Iso}(X)$ for the group of its isometries, with the topology of pointwise convergence: $h_j \to h$ means $d(h_j x, hx) \to 0$ for every $x \in X$. The sets

$$\{h \in \mathrm{Iso}(X) : d(hx, x) < \varepsilon \text{ for all } x \in A\}, \qquad A \subseteq X \text{ finite}, \ \varepsilon > 0,$$

form a neighborhood basis of the identity, and $\mathrm{Iso}(X)$ is a topological group.

Fact 2.3. *Let X be a complete separable metric space.*

(a) $\mathrm{Iso}(X)$ *is a Polish group* [Kec95, Section 9.B].
(b) *Let* $Y \subseteq X$ *be dense. Every isometry of* Y *extends uniquely to an isometry of* X*. This identifies* $\mathrm{Iso}(Y)$ *with a subgroup of* $\mathrm{Iso}(X)$*, and the subspace topology on* $\mathrm{Iso}(Y)$ *is the topology of pointwise convergence on* Y*.*

For (b), an isometry of Y and its inverse extend to nonexpansive maps of X which are inverse to each other. For isometries h_j, h of X and points $x \in X$, $y \in Y$ we have

$$d(h_j x, hx) \le 2d(x,y) + d(h_j y, hy), \tag{2.1}$$

so pointwise convergence on the dense set Y implies pointwise convergence on X. Part (b) will be applied to the two countable spaces of the construction in Sections 3–6 and their completions.

2.4. **The Urysohn space.** The *rational Urysohn space* $\mathbb{QU}$ is a nonempty countable metric space with rational distances which has the following *extension property*.

Let $M \subseteq \mathbb{QU}$ be finite and let $\phi : M \to \mathbb{Q}_{>0}$ satisfy

$$|\phi(m) - \phi(m')| \le d(m,m') \le \phi(m) + \phi(m') \qquad (m, m' \in M). \tag{2.2}$$

Then there is a point $c \in \mathbb{QU}$ with $d(c,m) = \phi(m)$ for all $m \in M$.

The inequalities (2.2) say exactly that M together with one new point at distance $\phi(m)$ from each $m \in M$ is again a metric space. Thus every way of adding one point to a finite subset of $\mathbb{QU}$, with positive rational distances, is realized inside $\mathbb{QU}$. Such a space exists and is unique up to isometry, and its completion is the *Urysohn space* $\mathbb{U}$ [Ury27]; see also [CV06]. By Fact 2.3(b) we regard $\mathrm{Iso}(\mathbb{QU})$ as a subgroup of $\mathrm{Iso}(\mathbb{U})$. Its topology is that of pointwise convergence with respect to the metric of $\mathbb{QU}$; the space $\mathbb{QU}$ is never treated as a discrete set.

We shall use the extension property once, in Lemma 3.2, and the following two theorems. The first is the reason why it is enough to lift isometries of $\mathbb{QU}$ (Section 5), and the second is the reduction to $\mathrm{Iso}(\mathbb{U})$ explained in the introduction.

Fact 2.4.
(a) $\mathrm{Iso}(\mathbb{QU})$ *is dense in* $\mathrm{Iso}(\mathbb{U})$ [CV06].
(b) (Uspenskij [Usp90]) *Every Polish group is isomorphic, as a topological group, to a subgroup of* $\mathrm{Iso}(\mathbb{U})$*. This subgroup is closed by Fact 2.2(a).*

2.5. **Positive definite kernels.** A symmetric kernel $k : P \times P \to \mathbb{R}$ on a set P is *positive definite* when the inequality $\sum_{a,b \le N} c_a c_b\, k(p_a, p_b) \ge 0$ holds for all N, all $p_1, \dots, p_N \in P$ and all $c \in \mathbb{R}^N$. Both parts of the paper turn metric data into Hilbert space data by means of the following two classical facts.

Fact 2.5.
(a) (Kolmogorov; see [BHV08, Theorem C.1.4].) *If* k *is a positive definite kernel on* P*, there are a real Hilbert space* $\mathcal{V}$ *and a map* $\xi : P \to \mathcal{V}$*,* $p \mapsto \xi_p$*, with* $\langle \xi_p, \xi_q \rangle = k(p,q)$ *and total range. If* $(\mathcal{V}', \xi')$ *is the same for a kernel* k' *on* P' *and* $\Phi : P \to P'$ *is a bijection with* $k'(\Phi p, \Phi q) = k(p,q)$*, there is a unique orthogonal operator* $U : \mathcal{V} \to \mathcal{V}'$ *with* $U\xi_p = \xi'_{\Phi(p)}$ *for all* $p \in P$*.*
(b) (Schoenberg [Sch38, Theorem 1].) *If* (P,d) *is a metric space which embeds isometrically into a real Hilbert space, then* e^{-d^2} *is a positive definite kernel on* P*.*

We call $(\mathcal{V}, \xi)$ the *Hilbert space of the kernel* k; by (a) it is unique up to a unique orthogonal operator. The canonical map of a kernel is always denoted by ξ. When $k = e^{-d^2}$ as in (b), the ξ_p are unit vectors and

$$\|\xi_p - \xi_q\|^2 = 2\bigl(1 - e^{-d(p,q)^2}\bigr) \le 2\, d(p,q)^2, \tag{2.3}$$

so $p \mapsto \xi_p$ is continuous, and $\mathcal{V}$ is separable if P is.

3. A rich coloring of the weighted cube

Sections 3–6 are devoted to the proof of Theorem A. The present section constructs the two countable objects on which the proof rests: the weighted cube C, and a coloring $\chi : C \to \mathbb{QU}$ which is *rich* (Definition 3.1). Richness is the only property of χ that Section 4 uses to lift isometries.

3.1. **The weighted cube.** Let $W = \mathbb{Q}_{>0} \times \mathbb{N}$. We call the elements of W *walls*, and the *weight* of the wall $w = (q, n)$ is $r_w = q$. All that matters is that W is countable and that every positive rational number is the weight of infinitely many walls. Let

$$C = \{s \subseteq W : s \text{ finite}\}, \qquad \mathrm{wt}(b) = \sum_{w \in b} r_w \quad (b \subseteq W), \qquad \rho(s,t) = \mathrm{wt}(s \triangle t),$$

where $s \triangle t$ denotes the symmetric difference. Thus $\rho(s,t)$ is the total weight of the walls that belong to exactly one of s and t; we say that these walls *separate* s from t. Since $s \triangle u \subseteq (s \triangle t) \cup (t \triangle u)$ and all weights are positive, ρ is a metric on C, with rational values. One should think of C as the set of vertices of an infinite-dimensional cube, of the sets s as 0–1 vectors with finitely many ones, and of ρ as a weighted Hamming distance. We call (C, ρ) the *weighted cube*.

For a wall w we abbreviate $s \triangle w = s \triangle \{w\}$ and $A \triangle w = \{a \triangle w : a \in A\}$. Passing from s to $s \triangle w$ means crossing the wall w. The map $s \mapsto s \triangle w$ is an isometry of C which moves every point by exactly r_w. The whole construction rests on the following observation about crossing a wall which belongs to neither of two given points.

$$\text{If } w \notin x \text{ and } w \notin y, \text{ then} \quad \rho(x,\, y \triangle w) = \rho(x,y) + r_w. \tag{3.1}$$

Indeed, the walls separating x from $y \triangle w = y \cup \{w\}$ are those separating x from y, and w. Hence, if w belongs to no point of a finite set $A \subseteq C$, then $A \triangle w$ is a copy of A, which we call a *parallel copy*. Every point of A has moved by exactly r_w, and for every point x not containing w the distance $\rho(x, a \triangle w) = \rho(x,a) + r_w$ is the largest that the triangle inequality allows; see Figure 2.

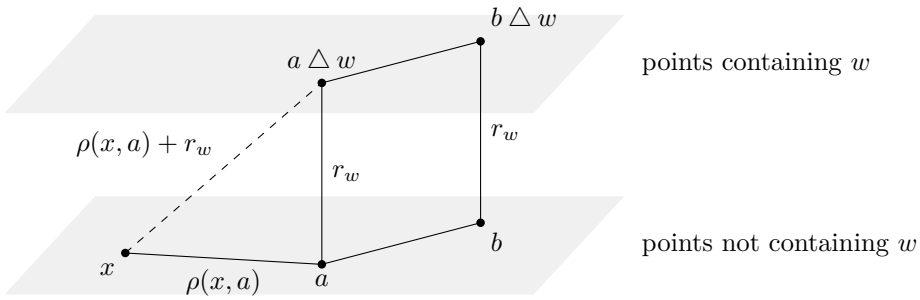


FIGURE 2. Crossing a wall w that belongs to none of the points x, a, b. The set $A = \{a, b\}$ moves to its parallel copy $A \triangle w$: distances inside A are unchanged, each point moves by r_w, and the distance from x to $a \triangle w$ is $\rho(x,a) + r_w$.

3.2. **Colorings.** A *coloring* is a nonexpansive map $\chi : C \to \mathbb{QU}$; we think of $\chi(s)$ as the color of the point s. (Being nonexpansive, a coloring will extend to the completion of C in Section 5.) Suppose χ is a coloring, $A \subseteq C$ is finite, and w is a wall which belongs to no point of A. Consider the colors that χ puts on the parallel copy $A \triangle w$, that is, the map $\psi : A \to \mathbb{QU}$, $\psi(a) = \chi(a \triangle w)$. Two things can be said about ψ in advance. It is nonexpansive, because $a \mapsto a \triangle w$ is an isometry. And it is close to χ: $d(\psi(a), \chi(a)) \le \rho(a \triangle w, a) = r_w$. A coloring is rich if nothing else can be said.

Definition 3.1. A coloring $\chi : C \to \mathbb{QU}$ is *rich* if the following holds. Let $A \subseteq C$ be finite, let $r \in \mathbb{Q}_{>0}$, and let $\psi : A \to \mathbb{QU}$ be a nonexpansive map with

$$d(\psi(a), \chi(a)) \le r \qquad (a \in A).$$

Then there is a wall w of weight $r_w = r$, belonging to no point of A, such that

$$\chi(a \triangle w) = \psi(a) \qquad (a \in A).$$

For example, take $A = \{\varnothing\}$. Richness then says that every point $c \in \mathbb{QU}$ with $d(c, \chi(\varnothing)) \le r$ is the color of a one-element set $\{w\}$ with $r_w = r$: at distance r from the point $\varnothing$ we see all the colors that lie within r of its own color. Richness will be applied in Section 4 in the following way: when a partially defined lift of an isometry g of $\mathbb{QU}$ has to be extended across a wall, ψ will describe the colors that g dictates on the new points, and the wall w provided by richness tells us where to send them.

Rich colorings exist because a wall w that has not been used before puts the copy $A \triangle w$ as far as possible from everything already colored. By (3.1), a point x colored earlier is at distance $\rho(x, a) + r$ from $a \triangle w$, and the two conditions on ψ already give

$$d(\chi(x), \psi(a)) \le d(\chi(x), \chi(a)) + d(\chi(a), \psi(a)) \le \rho(x, a) + r.$$

So the old colors impose no constraint on the new ones beyond the two conditions of the definition, and a recursion can take care of each triple (A, r, ψ) by a wall of its own. Besides this, the recursion must give a color to every point of C. This requires the following lemma, which is the only place where the extension property of $\mathbb{QU}$ is used.

Lemma 3.2 (One-point extension)**.** *Let $A \subseteq C$ be finite, let $\chi_0 : A \to \mathbb{QU}$ be nonexpansive, and let $z \in C \setminus A$. Then χ_0 extends to a nonexpansive map $A \cup \{z\} \to \mathbb{QU}$.*

Proof. We need a point $c \in \mathbb{QU}$ with $d(c, \chi_0(x)) \le \rho(z, x)$ for all $x \in A$. If A is empty, any point will do. Otherwise let $M = \chi_0(A)$ and put

$$\phi(m) = \min_{x \in A} \big(\rho(z, x) + d(\chi_0(x), m)\big) \qquad (m \in M).$$

This is the length of a shortest route from z to m which first travels inside C to a point $x \in A$, then jumps to its color, and then travels inside $\mathbb{QU}$. The values $\phi(m)$ are rational, and positive because $z \notin A$. We verify (2.2). Each function $m \mapsto \rho(z, x) + d(\chi_0(x), m)$ is nonexpansive, hence so is their minimum ϕ, which gives $|\phi(m) - \phi(m')| \le d(m, m')$. If x and y realize the minima defining $\phi(m)$ and $\phi(m')$, then

$$\begin{aligned} \phi(m) + \phi(m') &\ge d(m, \chi_0(x)) + \rho(x, y) + d(\chi_0(y), m') \\ &\ge d(m, \chi_0(x)) + d(\chi_0(x), \chi_0(y)) + d(\chi_0(y), m') \ge d(m, m'), \end{aligned}$$

where we used the triangle inequality in C, then that χ_0 is nonexpansive, then the triangle inequality in $\mathbb{QU}$. By the extension property there is a point $c \in \mathbb{QU}$ with $d(c, m) = \phi(m)$ for all $m \in M$. For $x \in A$ we get $d(c, \chi_0(x)) = \phi(\chi_0(x)) \le \rho(z, x)$, as required. □

The next proposition also records that rich colorings are surjective. This makes the image of the extended coloring dense (Section 5), so that a lift determines the isometry which it lifts (Corollary 6.2).

Proposition 3.3. *There is a rich coloring $\chi : C \to \mathbb{QU}$. Every rich coloring is surjective.*

Proof. Existence. Call a triple (A, r, ψ) a *request* if $A \subseteq C$ is finite, $r \in \mathbb{Q}_{>0}$, and ψ is any map from A to $\mathbb{QU}$. There are countably many requests; enumerate them as (A_k, r_k, ψ_k), $k \ge 0$. We list all maps ψ, and not only those allowed by Definition 3.1, because that definition refers to the coloring, which is not yet built. We construct finite sets $\varnothing = C_0 \subseteq C_1 \subseteq \cdots \subseteq C$ and nonexpansive maps $\chi_k : C_k \to \mathbb{QU}$, each extending the previous one. At stage k we first color the points of A_k, so that the k-th request can be tested against the definition, and then, if it passes the test, we fulfill it on a parallel copy of A_k.

(1) Using Lemma 3.2 finitely many times, extend χ_k to a nonexpansive map χ'_k on $C'_k = C_k \cup A_k$.

(2) Suppose that ψ_k is nonexpansive and that $d(\psi_k(a), \chi'_k(a)) \le r_k$ for all $a \in A_k$. Choose a wall w of weight r_k which belongs to no point of C'_k; this is possible because C'_k is a finite set of finite sets, and infinitely many walls have weight r_k. Let $C_{k+1} = C'_k \cup (A_k \triangle w)$ and let χ_{k+1} extend χ'_k by

$$\chi_{k+1}(a \triangle w) = \psi_k(a) \qquad (a \in A_k).$$

If the supposition fails, let $C_{k+1} = C'_k$ and $\chi_{k+1} = \chi'_k$.

In step (2) the points $a \triangle w$, $a \in A_k$, are pairwise distinct, and they do not lie in C'_k, since they contain w. So χ_{k+1} is well defined, and we must check that it is nonexpansive. For two new points, $d(\psi_k(a), \psi_k(b)) \le \rho(a, b) = \rho(a \triangle w, b \triangle w)$. For an old point $x \in C'_k$ and a new point $a \triangle w$ we have, by the assumption on ψ_k and by (3.1),

$$d(\chi'_k(x), \psi_k(a)) \le d(\chi'_k(x), \chi'_k(a)) + d(\chi'_k(a), \psi_k(a)) \le \rho(x, a) + r_k = \rho(x, a \triangle w).$$

Let χ be the union of the maps χ_k. It is defined on all of C, because every point $z \in C$ belongs to some A_k (there are requests with $A_k = \{z\}$) and $A_k \subseteq C_{k+1}$. It is nonexpansive, because any two points of C lie in a common C_k. To see that χ is rich, let (A, r, ψ) be as in Definition 3.1. It is the request with some number k. At stage k we have $A \subseteq C'_k$ and $\chi'_k = \chi$ on A, so the supposition of step (2) holds, and the wall w chosen there is as required.

Surjectivity. Let χ be rich and $c \in \mathbb{QU}$. Apply richness to the one-point set $A = \{\varnothing\}$, the map $\psi(\varnothing) = c$, and a positive rational $r \ge d(c, \chi(\varnothing))$. We get a wall w with $\chi(\{w\}) = c$. □

4. Lifting isometries to the cube

For the rest of Sections 4–6 we fix a rich coloring $\chi : C \to \mathbb{QU}$. Its symmetries are the pairs

$$\mathrm{Aut}(\chi) = \{(h, g) \in \mathrm{Iso}(C) \times \mathrm{Iso}(\mathbb{QU}) : \ \chi \circ h = g \circ \chi\}.$$

If $(h, g) \in \mathrm{Aut}(\chi)$, we say that h is a *lift* of g: h permutes the points of the cube, and their colors change according to g. The set $\mathrm{Aut}(\chi)$ is a subgroup of $\mathrm{Iso}(C) \times \mathrm{Iso}(\mathbb{QU})$, because $\chi \circ h = g \circ \chi$ and $\chi \circ h' = g' \circ \chi$ imply $\chi \circ hh' = gg' \circ \chi$ and $\chi \circ h^{-1} = g^{-1} \circ \chi$. We give it the subspace topology. The projection

$$p_0 : \mathrm{Aut}(\chi) \to \mathrm{Iso}(\mathbb{QU}), \qquad p_0(h, g) = g,$$

is then a continuous homomorphism. The goal of this section is to prove that p_0 is onto and open (Corollary 4.3). Onto means that every isometry of $\mathbb{QU}$ has a lift. Since p_0 is a homomorphism, open means that isometries sufficiently close to the identity have lifts in any prescribed neighborhood of the identity.

4.1. **Faces and cube maps.** A lift will be constructed in countably many steps, one wall at a time. It cannot be constructed one point at a time, because an isometry between two finite subsets of C need not extend to an isometry of C. For instance, if $r_w = 2$ and $r_{w_1} = r_{w_2} = 1$, then the map $\varnothing \mapsto \varnothing$, $\{w\} \mapsto \{w_1, w_2\}$ preserves the distance, but no isometry of C extends it: the point $\{w_1\}$ lies at distance 1 from both $\varnothing$ and $\{w_1, w_2\}$, whereas no point lies at distance 1 from both $\varnothing$ and $\{w\}$. The isometries of C that we shall use are the following ones. Let $t, t' \in C$, and let σ be a permutation of W which preserves weights, that is, $r_{\sigma(w)} = r_w$ for all w. Then

$$h(s) = t' \triangle \sigma(s \triangle t) \qquad (s \in C) \tag{4.1}$$

is an isometry of C; here and below, $\sigma(b)$ denotes the image of the set b under σ. The map h sends t to t', and if s is reached from t by crossing certain walls, then $h(s)$ is reached from t' by crossing their σ-images. All our lifts will be of this form; in fact C has no other isometries (Remark 6.4). Their finite approximations are defined on *faces*, and richness will supply, for each wall v, a partner $\sigma(v)$ of the same weight for which the colors match.

For $t \in C$ and a finite set of walls $D \subseteq W$, the *face* through t with directions D is

$$F(t, D) = \{t \triangle b : b \subseteq D\}.$$

It consists of the 2^n points that can be reached from t by crossing walls in D, where n is the number of elements of D. Let $F(t, D)$ and $F(t', D')$ be faces and let $\sigma : D \to D'$ be a weight-preserving bijection. The *cube map* $f = [t, t', \sigma]$ is the bijection

$$f : F(t, D) \to F(t', D'), \qquad f(t \triangle b) = t' \triangle \sigma(b) \quad (b \subseteq D).$$

A cube map preserves distances:

$$\rho(t' \triangle \sigma(b_1), t' \triangle \sigma(b_2)) = \mathrm{wt}(\sigma(b_1 \triangle b_2)) = \mathrm{wt}(b_1 \triangle b_2) = \rho(t \triangle b_1, t \triangle b_2).$$

Its inverse is the cube map $[t', t, \sigma^{-1}]$. The same computation shows that (4.1) is an isometry of C, with inverse $s \mapsto t \triangle \sigma^{-1}(s \triangle t')$.

The next lemma extends a cube map which is compatible with the coloring across one more wall; see Figure 3.

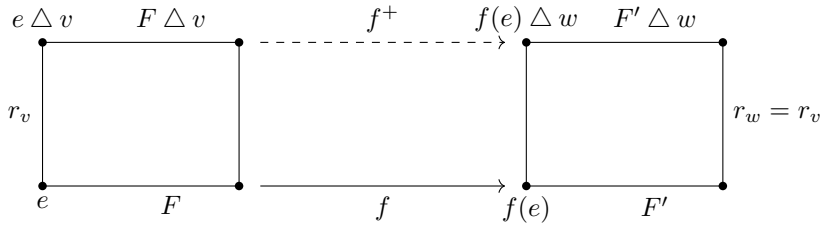


FIGURE 3. Lemma 4.1. The cube map $f : F \to F'$ is extended to $F \triangle v$ by sending $e \triangle v$ to $f(e) \triangle w$. Richness provides a wall w for which $f(e) \triangle w$ has the color $g\chi(e \triangle v)$.

Lemma 4.1 (Adding one wall)**.** *Let $g \in \mathrm{Iso}(\mathbb{QU})$, and let $f = [t, t', \sigma] : F(t, D) \to F(t', D')$ be a cube map with $\chi \circ f = g \circ \chi$ on $F(t, D)$. Let v be a wall not in D. Then there is a wall w not in D', with $r_w = r_v$, such that the cube map*

$$f^+ = [t, t', \sigma \cup \{(v, w)\}] : F(t, D \cup \{v\}) \to F(t', D' \cup \{w\}),$$

which extends f, satisfies $\chi \circ f^+ = g \circ \chi$ on $F(t, D \cup \{v\})$.

Proof. Write $F = F(t, D)$ and $F' = F(t', D')$. The larger face $F(t, D \cup \{v\})$ consists of F and its translate $F \triangle v$. For any wall $w \notin D'$ of weight r_v, the map f^+ agrees with f on F and sends $e \triangle v$ to $f(e) \triangle w$. So we need such a wall w with

$$\chi(f(e) \triangle w) = g\chi(e \triangle v) \qquad (e \in F). \tag{4.2}$$

The right-hand side is the color that g dictates for the point $f(e) \triangle w$. Accordingly, define $\psi : F' \to \mathbb{QU}$ by

$$\psi(f(e)) = g\chi(e \triangle v) \qquad (e \in F).$$

This map is nonexpansive, being the composition of the isometry f^{-1}, the isometry $e \mapsto e \triangle v$, the nonexpansive map χ, and the isometry g. Moreover, since $\chi(f(e)) = g\chi(e)$,

$$d(\psi(f(e)), \chi(f(e))) = d(g\chi(e \triangle v), g\chi(e)) = d(\chi(e \triangle v), \chi(e)) \leq \rho(e \triangle v, e) = r_v.$$

So richness applies to $A = F'$, $r = r_v$ and ψ. It gives a wall w of weight r_v, belonging to no point of F', with $\chi(e' \triangle w) = \psi(e')$ for all $e' \in F'$; this is (4.2). Finally $w \notin D'$, because every wall in D' belongs to some point of F': a wall $w' \in D'$ belongs either to t' or to $t' \triangle w'$. $\square$

The lifting theorem has two parts. Part (a) is the extension property obtained by iterating the lemma. Part (b) is the quantitative statement behind openness: if g moves the colors of a face by at most r, then some lift of g moves the face itself by exactly r.

Theorem 4.2 (Lifting)**.** *Let $g \in \mathrm{Iso}(\mathbb{QU})$.*

(a) *Every cube map $f : F \to F'$ with $\chi \circ f = g \circ \chi$ on F extends to a lift of g.*
(b) *Let F be a face and let $r \in \mathbb{Q}_{>0}$ be such that $d(g\chi(e), \chi(e)) \le r$ for all $e \in F$. Then g has a lift h with $\rho(he, e) = r$ for all $e \in F$.*

Proof. (a) Let $f = [t, t', \sigma_0]$ with $\sigma_0 : D_0 \to D_0'$, and enumerate $W = \{w_1, w_2, \dots\}$. Starting from σ_0, we construct weight-preserving bijections $\sigma_n : D_n \to D_n'$ between finite sets of walls, each extending the previous one, such that each cube map $f_n = [t, t', \sigma_n]$ satisfies $\chi \circ f_n = g \circ \chi$ on its domain $F(t, D_n)$. The odd steps put every wall into the domain of some σ_n, so that the limit map is defined on all of C; the even steps put every wall into the range, so that the limit map is onto.

At step $2m-1$ we make sure that $w_m \in D_{2m-1}$. If w_m already lies in D_{2m-2}, we change nothing. Otherwise we apply Lemma 4.1 to f_{2m-2}, g and $v = w_m$.

At step $2m$ we make sure that $w_m \in D_{2m}'$. If w_m already lies in D_{2m-1}', we change nothing. Otherwise we apply Lemma 4.1 to the inverse cube map $f_{2m-1}^{-1} = [t', t, \sigma_{2m-1}^{-1}]$, which satisfies $\chi \circ f_{2m-1}^{-1} = g^{-1} \circ \chi$, to the isometry g^{-1}, and to $v = w_m$. The lemma gives a wall $w \notin D_{2m-1}$ of weight r_{w_m} such that, for $\sigma_{2m} = \sigma_{2m-1} \cup \{(w, w_m)\}$, the cube map $f_{2m}^{-1} = [t', t, \sigma_{2m}^{-1}]$ satisfies $\chi \circ f_{2m}^{-1} = g^{-1} \circ \chi$; equivalently, $\chi \circ f_{2m} = g \circ \chi$.

The union σ of the maps σ_n is a weight-preserving permutation of W, so $h(s) = t' \triangle \sigma(s \triangle t)$ is an isometry of C as in (4.1). Every point $s \in C$ lies in a face $F(t, D_n)$, because the finite set $s \triangle t$ is contained in some D_n; and on $F(t, D_n)$ the map h coincides with f_n. Hence h extends f, and $\chi \circ h = g \circ \chi$ on all of C.

(b) Let $F = F(t, D)$. The map $\psi = g \circ \chi|_F$ is nonexpansive and satisfies $d(\psi(e), \chi(e)) \le r$ on F. By richness there is a wall w of weight r, belonging to no point of F, with

$$\chi(e \triangle w) = g\chi(e) \qquad (e \in F).$$

As at the end of the proof of Lemma 4.1, $w \notin D$. Therefore the map $f(e) = e \triangle w$ is the cube map $[t, t \triangle w, \mathrm{id}_D]$ from F onto its parallel copy $F \triangle w$, and the displayed equation says that $\chi \circ f = g \circ \chi$ on F. By (a), f extends to a lift h of g, and $\rho(he, e) = \rho(e \triangle w, e) = r$ for $e \in F$. □

Corollary 4.3. *The homomorphism $p_0 : \mathrm{Aut}(\chi) \to \mathrm{Iso}(\mathbb{QU})$ is surjective and open.*

Proof. Onto. Let $g \in \mathrm{Iso}(\mathbb{QU})$. Apply Theorem 4.2(b) to the one-point face $F = \{\varnothing\}$ and a positive rational $r \ge d(g\chi(\varnothing), \chi(\varnothing))$.

Open. Since p_0 is a homomorphism, it is enough to show that $p_0(N)$ is a neighborhood of the identity of $\mathrm{Iso}(\mathbb{QU})$ for every N in some neighborhood basis of the identity of $\mathrm{Aut}(\chi)$. Such a basis is formed by the sets

$$N = \{(h, g) \in \mathrm{Aut}(\chi) : \ \rho(hx, x) < \varepsilon \ (x \in A), \ d(gm, m) < \varepsilon \ (m \in M)\},$$

where $A \subseteq C$ and $M \subseteq \mathbb{QU}$ are finite and $\varepsilon > 0$. Given such an N, let F be a face containing A, for instance $F = F(\varnothing, D)$ with $D = \bigcup_{s \in A} s$, and let r be a rational number with $0 < r < \varepsilon$. The set

$$O = \{g \in \mathrm{Iso}(\mathbb{QU}) : \ d(g\chi(e), \chi(e)) < r \ (e \in F), \ d(gm, m) < \varepsilon \ (m \in M)\}$$

is a neighborhood of the identity. For $g \in O$, Theorem 4.2(b) gives a lift h of g with $\rho(he, e) = r < \varepsilon$ for all $e \in F$. Then $(h, g) \in N$, so $O \subseteq p_0(N)$. □

5. From the rational to the complete Urysohn space

In Section 4 we lifted isometries of the countable space $\mathbb{QU}$ to the countable cube C, and showed that the lifting map p_0 is open. We now pass to the completions, where the groups are Polish, and

prove property (L) of the introduction (Theorem 5.2). The tool is a general lemma about Polish groups (Lemma 5.1).

Let $\widehat{C}$ be the completion of the weighted cube C; it is a complete separable metric space, whose metric we still denote by ρ. Being nonexpansive, the coloring χ is uniformly continuous, so it extends uniquely to a nonexpansive map

$$\widehat{\chi} : \widehat{C} \to \mathbb{U}.$$

Its image contains $\chi(C) = \mathbb{QU}$ (Proposition 3.3) and is therefore dense. In analogy with $\mathrm{Aut}(\chi)$ we define

$$\mathrm{Aut}(\widehat{\chi}) = \{(h, g) \in \mathrm{Iso}(\widehat{C}) \times \mathrm{Iso}(\mathbb{U}) : \ \widehat{\chi} \circ h = g \circ \widehat{\chi}\}, \qquad p(h, g) = g.$$

This is the group that will be realized as a closed subgroup of the unitary group. It is a subgroup of the Polish group $\mathrm{Iso}(\widehat{C}) \times \mathrm{Iso}(\mathbb{U})$, and it is closed: for each $x \in \widehat{C}$, both sides of $\widehat{\chi}(hx) = g\widehat{\chi}(x)$ depend continuously on (h, g). Hence $\mathrm{Aut}(\widehat{\chi})$ is a Polish group, and $p : \mathrm{Aut}(\widehat{\chi}) \to \mathrm{Iso}(\mathbb{U})$ is a continuous homomorphism. By Fact 2.3(b) we may regard $\mathrm{Aut}(\chi)$ as a subgroup of $\mathrm{Aut}(\widehat{\chi})$, with the subspace topology: if $\chi \circ h = g \circ \chi$ on C, then the extensions satisfy $\widehat{\chi} \circ h = g \circ \widehat{\chi}$ on $\widehat{C}$ by continuity. The restriction of p to $\mathrm{Aut}(\chi)$ is p_0.

We know that p_0 maps $\mathrm{Aut}(\chi)$ onto $\mathrm{Iso}(\mathbb{QU})$, a dense subgroup of $\mathrm{Iso}(\mathbb{U})$. This alone does not imply that p is onto. For example, the inclusion $\ell^1 \to \ell^2$ is a continuous homomorphism of Polish groups which maps the dense subgroup of finitely supported sequences of ℓ^1 bijectively onto a dense subgroup of ℓ^2, and it is not onto. What makes the difference is that p_0 is open. The next lemma shows that this is all that is needed. The idea is the following. Openness of a homomorphism φ on a subgroup K means that if $\varphi(a)$ is close to the identity, then a is close to the kernel of φ. So, modulo the kernel, the domain of φ and its image carry the same topology. The image is then a Polish subgroup, hence closed; and if it is dense, it is everything. In the example this fails because a finitely supported sequence can be small in ℓ^2 without being small in ℓ^1.

Lemma 5.1. *Let $\varphi : H \to G$ be a continuous homomorphism between Polish groups, and let $K \leq H$ be a subgroup. Suppose that $\varphi(K)$ is dense in G and that the restriction $\varphi|_K : K \to \varphi(K)$ is an open map, where K and $\varphi(K)$ carry the subspace topologies. Then φ is surjective and open.*

Proof. Assume first that K is dense in H. Let $N = \ker \varphi$, let $Q : H \to H/N$ be the quotient map, and let $\bar{\varphi} : H/N \to G$ be the injective continuous homomorphism with $\varphi = \bar{\varphi} \circ Q$. By Fact 2.2(b), H/N is a Polish group. We claim that $\bar{\varphi}$ is a topological embedding. Then its image $\varphi(H)$ is closed in G by Fact 2.2(a). It contains the dense set $\varphi(K)$, so $\varphi(H) = G$. Thus $\bar{\varphi}$ is an isomorphism of topological groups, and $\varphi = \bar{\varphi} \circ Q$ is onto and open.

To prove the claim, it suffices to show that the inverse of $\bar{\varphi}$, which is a homomorphism defined on $\varphi(H)$, is continuous at the identity: for every open neighborhood O of the identity in H/N we need an open neighborhood P of the identity in G with

$$Q(\varphi^{-1}(P)) \subseteq O. \tag{5.1}$$

Choose an open neighborhood O_1 of the identity in H/N with $\mathrm{cl}\, O_1 \subseteq O$, and let $V = Q^{-1}(O_1)$. Being a full preimage, V satisfies $VN = V$. The set $V \cap K$ is an open neighborhood of the identity in K. As $\varphi|_K$ is open, there is an open neighborhood P of the identity in G with

$$P \cap \varphi(K) \subseteq \varphi(V \cap K).$$

Let $a \in \varphi^{-1}(P) \cap K$. Then $\varphi(a) = \varphi(a')$ for some $a' \in V \cap K$, so $a \in a'N \subseteq VN = V$. This shows $\varphi^{-1}(P) \cap K \subseteq V$. Since $\varphi^{-1}(P)$ is open and K is dense in H,

$$\varphi^{-1}(P) \subseteq \mathrm{cl}(\varphi^{-1}(P) \cap K) \subseteq \mathrm{cl}\, V \subseteq Q^{-1}(\mathrm{cl}\, O_1) \subseteq Q^{-1}(O),$$

which is (5.1).

In general, let H_0 be the closure of K in H. By the case already proved, the restriction of φ to the Polish group H_0 is onto and open. Hence φ is onto. Moreover, if V is a neighborhood of the

identity in H, then $\varphi(V)$ contains $\varphi(V \cap H_0)$, which is a neighborhood of the identity in G; since φ is a homomorphism, this means that φ is open. □

Theorem 5.2. *The homomorphism $p : \mathrm{Aut}(\widehat{\chi}) \to \mathrm{Iso}(\mathbb{U})$, $p(h, g) = g$, is continuous, open and surjective.*

Proof. Apply Lemma 5.1 to $H = \mathrm{Aut}(\widehat{\chi})$, $G = \mathrm{Iso}(\mathbb{U})$, $\varphi = p$ and $K = \mathrm{Aut}(\chi)$. The image $p(K) = \mathrm{Iso}(\mathbb{QU})$ is dense in $\mathrm{Iso}(\mathbb{U})$ by Fact 2.4(a). The restriction $p|_K = p_0$ is open onto $\mathrm{Iso}(\mathbb{QU})$ by Corollary 4.3; here we use that the subspace topologies on $\mathrm{Aut}(\chi)$ and on $\mathrm{Iso}(\mathbb{QU})$ are the topologies of pointwise convergence on C and on $\mathbb{QU}$ (Fact 2.3(b)), which are the ones used in that corollary. □

This is property (L) of the introduction. In particular, every isometry g of $\mathbb{U}$ has a lift to $\widehat{C}$: an isometry h of $\widehat{C}$ with $\widehat{\chi} \circ h = g \circ \widehat{\chi}$.

6. The unitary representation and the proofs of Theorem A and Corollary B

6.1. **Isometries of the cube act unitarily.** It remains to realize $\mathrm{Aut}(\widehat{\chi})$ as a closed subgroup of a unitary group. The cube's metric is an L^1-distance between indicator functions, which is the same as the squared L^2-distance. Regard wt as a measure on W, in which the wall w has mass r_w. For $s, t \in C$,

$$\rho(s,t) = \mathrm{wt}(s \triangle t) = \|1_s - 1_t\|_{L^1(W,\mathrm{wt})} = \|1_s - 1_t\|^2_{L^2(W,\mathrm{wt})}.$$

Hence $s \mapsto 1_s$ is uniformly continuous from (C, ρ) to the real Hilbert space $L^2(W, \mathrm{wt})$, and it extends to a continuous map $\iota : \widehat{C} \to L^2(W, \mathrm{wt})$ with $\|\iota(x) - \iota(y)\|^2 = \rho(x, y)$ for all $x, y \in \widehat{C}$. Thus $\rho^{1/2}$ is a metric on $\widehat{C}$ for which ι is an isometric embedding into a real Hilbert space, and by Fact 2.5(b) the kernel $e^{-\rho}$ is positive definite on $\widehat{C}$. Let $(\mathcal{V}, \xi)$ be the Hilbert space of this kernel (Fact 2.5(a)), and let $\mathcal{H}_C = \mathcal{V}_{\mathbb{C}}$ be its complexification. So $(\xi_x)_{x \in \widehat{C}}$ is a total family of unit vectors in the complex Hilbert space $\mathcal{H}_C$ with

$$\langle \xi_x, \xi_y \rangle = e^{-\rho(x,y)}, \qquad \|\xi_x - \xi_y\|^2 = 2\big(1 - e^{-\rho(x,y)}\big) \le 2\rho(x, y) \qquad (x, y \in \widehat{C}). \tag{6.1}$$

In particular $x \mapsto \xi_x$ is continuous, and $\mathcal{H}_C$ is separable because $\widehat{C}$ is.

Proposition 6.1. *For every $h \in \mathrm{Iso}(\widehat{C})$ there is a unique unitary operator U_h on $\mathcal{H}_C$ with*

$$U_h \xi_x = \xi_{hx} \qquad (x \in \widehat{C}).$$

The map $h \mapsto U_h$ is a topological embedding of $\mathrm{Iso}(\widehat{C})$ into $\mathcal{U}(\mathcal{H}_C)$.

Proof. An isometry h of $\widehat{C}$ is a bijection which preserves the kernel $e^{-\rho}$. By Fact 2.5(a) there is an orthogonal operator on $\mathcal{V}$ which maps ξ_x to ξ_{hx} for every $x \in \widehat{C}$; let U_h be its complexification. A unitary operator on $\mathcal{H}_C$ is determined by its values on the total set $\{\xi_x : x \in \widehat{C}\}$. So U_h is unique, and $U_{hh'} = U_h U_{h'}$.

By (6.1), for all $h, h' \in \mathrm{Iso}(\widehat{C})$ and $x \in \widehat{C}$,

$$\|U_h \xi_x - U_{h'} \xi_x\|^2 = 2\big(1 - e^{-\rho(hx, h'x)}\big).$$

So $U_h = U_{h'}$ implies $h = h'$, and $h_j \to h$ pointwise on $\widehat{C}$ if and only if $U_{h_j} \xi_x \to U_h \xi_x$ for every x. Since the vectors ξ_x span a dense subspace and unitary operators have norm one, the latter is equivalent to $U_{h_j} \to U_h$ in the strong operator topology. □

The group $\mathrm{Aut}(\widehat{\chi})$ consists of pairs (h, g), and we represent a pair by the operator U_h. Nothing is lost, because h determines g; this is where we use that the image of $\widehat{\chi}$ is dense. The resulting corollary is property (U) of the introduction in the form in which we need it.

Corollary 6.2. *The map $(h, g) \mapsto U_h$ is a topological embedding of $\mathrm{Aut}(\widehat{\chi})$ into $\mathcal{U}(\mathcal{H}_C)$, and its image is closed.*

Proof. We first show that the projection $(h, g) \mapsto h$ is a topological embedding of $\mathrm{Aut}(\widehat{\chi})$ into $\mathrm{Iso}(\widehat{C})$. It is a continuous homomorphism. It is injective: if (h, g) and (h, g') belong to $\mathrm{Aut}(\widehat{\chi})$, then $g\widehat{\chi}(x) = \widehat{\chi}(hx) = g'\widehat{\chi}(x)$ for all $x \in \widehat{C}$, so g and g' agree on the dense set $\widehat{\chi}(\widehat{C})$ and hence everywhere. Its inverse is continuous: let (h_j, g_j) and (h, g) be elements of $\mathrm{Aut}(\widehat{\chi})$ with $h_j \to h$. For every $x \in \widehat{C}$,

$$g_j\widehat{\chi}(x) = \widehat{\chi}(h_j x) \longrightarrow \widehat{\chi}(hx) = g\widehat{\chi}(x).$$

So $g_j \to g$ pointwise on the dense set $\widehat{\chi}(\widehat{C})$, and therefore on all of $\mathbb{U}$ by (2.1).

Composing with the embedding of Proposition 6.1, we see that $(h, g) \mapsto U_h$ is a topological embedding. Since $\mathrm{Aut}(\widehat{\chi})$ is a Polish group, its image is closed by Fact 2.2(a). □

Remark 6.3 (The vectors ξ_s, explicitly). For the points of the cube the vectors ξ_s can be written down, which gives a direct proof that $e^{-\rho}$ is positive definite on C, and hence by continuity on $\widehat{C}$, without Schoenberg's theorem. Let $(\omega_w)_{w\in W}$ be the coordinate functions on $\{-1, 1\}^W$ with the product of the uniform measures, a family of independent random variables taking the values ± 1 with probability $\frac{1}{2}$ each. Put $a_w = e^{-r_w}$ and $b_w = (1 - a_w^2)^{1/2}$, so that $a_w + b_w\omega_w$ has mean a_w and mean square 1, and let $\xi'_s = \prod_{w\in s}(a_w + b_w\omega_w) \in L^2(\{-1, 1\}^W)$ for $s \in C$. In the inner product of ξ'_s and ξ'_t, a wall which belongs to both sets contributes the mean square 1, and a wall which separates s from t contributes the mean e^{-r_w}:

$$\langle \xi'_s, \xi'_t\rangle = \prod_{w\in s\cap t} \mathbb{E}[(a_w + b_w\omega_w)^2] \prod_{w\in s\triangle t} \mathbb{E}[a_w + b_w\omega_w] = \prod_{w\in s\triangle t} a_w = e^{-\rho(s,t)}.$$

By Fact 2.5(a), the closed real span of the vectors ξ'_s is a model of the Hilbert space of the kernel $e^{-\rho}$ on C, with ξ'_s in the role of ξ_s.

6.2. **Proofs of Theorem A and Corollary B.** Corollary 6.2 and Theorem 5.2 give one closed subgroup of $\mathcal{U}(\mathcal{H})$ which maps openly onto $\mathrm{Iso}(\mathbb{U})$. Uspenskij's theorem then takes care of every Polish group, and Mackey's induced action gives the corollary.

Proof of Theorem A. The Hilbert space $\mathcal{H}_C$ is separable. Rather than check that it is infinite-dimensional, we enlarge it: $\mathcal{H}_C \oplus \mathcal{H}$ is isomorphic to $\mathcal{H}$, and $U \mapsto U \oplus 1$ is a topological embedding of $\mathcal{U}(\mathcal{H}_C)$ into $\mathcal{U}(\mathcal{H}_C \oplus \mathcal{H}) \cong \mathcal{U}(\mathcal{H})$. Together with Corollary 6.2 this gives a topological embedding j of the Polish group $\mathrm{Aut}(\widehat{\chi})$ into $\mathcal{U}(\mathcal{H})$. Let R_0 be its image; it is closed by Fact 2.2(a). By Theorem 5.2,

$$\bar{p} = p \circ j^{-1} : R_0 \to \mathrm{Iso}(\mathbb{U})$$

is a continuous open surjective homomorphism. Now let G be a Polish group. By Uspenskij's theorem (Fact 2.4(b)) we may assume that G is a closed subgroup of $\mathrm{Iso}(\mathbb{U})$. Then $R = \bar{p}^{-1}(G)$ is a closed subgroup of R_0, hence of $\mathcal{U}(\mathcal{H})$, and $\bar{p}$ maps R onto G. The restriction $\bar{p}|_R : R \to G$ is open, because $\bar{p}(V \cap R) = \bar{p}(V) \cap G$ for every $V \subseteq R_0$.

For the last assertion we use a theorem of Ding [Din12, Theorem 3.4]: there is a Polish group G_0 of which every Polish group is a quotient, that is, which admits a continuous open surjective homomorphism onto every Polish group. By what we have proved, there are a closed subgroup $R \leq \mathcal{U}(\mathcal{H})$ and a continuous open surjective homomorphism $R \to G_0$. Composing it with the homomorphisms from G_0 onto the Polish groups, we obtain continuous open surjective homomorphisms from R onto every Polish group. □

Proof of Corollary B. Let a Polish group G act in a Borel way on a standard Borel space X. By Theorem A there are a closed subgroup $R \leq \mathcal{U}(\mathcal{H})$ and a continuous surjective homomorphism $\varphi : R \to G$. Then $\gamma \cdot x = \varphi(\gamma) \cdot x$, for $\gamma \in R$ and $x \in X$, is a Borel action of R on X, and it has the same orbits as the action of G, because φ is onto. Thus $E_X^R = E_X^G$. By Mackey's induced action (Fact 2.1(b)), applied to the closed subgroup R of $\mathcal{U}(\mathcal{H})$, there are a Borel action of $\mathcal{U}(\mathcal{H})$ on a standard Borel space Y and a Borel reduction of $E_X^G = E_X^R$ to $E_Y^{\mathcal{U}(\mathcal{H})}$. □

6.3. **Remarks.**

Remark 6.4 (The covering group, concretely). The objects of the proof have a description, which the proof does not need. The completion $\widehat{C}$ is the set of all $s \subseteq W$ with $\mathrm{wt}(s) < \infty$, with the metric $\rho(s,t) = \mathrm{wt}(s \mathbin{\triangle} t)$. Every isometry of $\widehat{C}$ has the form

$$s \longmapsto b \mathbin{\triangle} \sigma(s),$$

where $b \in \widehat{C}$ and σ is a weight-preserving permutation of W. To see this, compose with the translation $s \mapsto s \mathbin{\triangle} b$ to arrange that the isometry h fixes $\varnothing$. The identity

$$\rho(\varnothing, s) + \rho(s,t) - \rho(\varnothing, t) = 2\,\mathrm{wt}(s \setminus t)$$

shows that $s \subseteq t$ if and only if s lies metrically between $\varnothing$ and t. So h and h^{-1} preserve inclusion. Hence h permutes the minimal nonempty elements of $\widehat{C}$, which are the singletons: $h(\{w\}) = \{\sigma(w)\}$ for a permutation σ of W, and $r_{\sigma(w)} = \rho(\varnothing, h(\{w\})) = r_w$. Finally $w \in s$ if and only if $\{w\} \subseteq s$, if and only if $\sigma(w) \in h(s)$; that is, $h(s) = \sigma(s)$. The same argument applies to C, where b is finite and the maps are those of (4.1).

Thus $\mathrm{Iso}(\widehat{C})$ is the semidirect product of the abelian group $(\widehat{C}, \mathbin{\triangle})$, which has exponent two, and the group $\prod_{q \in \mathbb{Q}_{>0}} \mathrm{Sym}(\{q\} \times \mathbb{N})$ of weight-preserving permutations of W, a product of countably many copies of the infinite symmetric group. All the lifts constructed in Section 4 have this form with b finite. The proof of Theorem A, through Theorem 5.2 and the first step of the proof of Corollary 6.2, therefore shows that every Polish group is a quotient of a closed subgroup of this particular semidirect product.

Remark 6.5 (Lifts cannot be chosen canonically). By Theorem 5.2 every isometry of $\mathbb{U}$ has a lift to $\widehat{C}$, but the lifts cannot be chosen in a continuous and multiplicative way. Indeed, suppose that $\lambda : \mathrm{Iso}(\mathbb{U}) \to \mathrm{Aut}(\widehat{\chi})$ is a continuous homomorphism with $p \circ \lambda = \mathrm{id}$. Then λ is a topological embedding, since p is a continuous inverse on its image. By Corollary 6.2, $\mathrm{Iso}(\mathbb{U})$ and all its subgroups would then embed into a unitary group. This is impossible: by Uspenskij's theorem $\mathrm{Iso}(\mathbb{U})$ contains a copy of $\mathrm{Homeo}_+[0,1]$, which has no nontrivial continuous unitary representations [Meg01]. In this sense the choices made in the back-and-forth argument of Section 4 cannot be replaced by a canonical construction.

7. Von Neumann algebras, Gaussian actions and the finitary symmetric group

The rest of the paper is devoted to Theorem C. This section collects the additional preliminaries. Measure preserving maps are defined, and equalities between them hold, modulo null sets. Countable groups are discrete. For a pmp action of a countable group Λ on a standard probability space (Y, ν), the *Koopman representation* κ on $L^2(Y,\nu)$ is given by $\kappa(g)f = f \circ g^{-1}$, and $L^2_0(Y,\nu)$ is the orthocomplement of the constants. In Sections 7.2 and 8, which deal with Gaussian actions, all Hilbert spaces are real, and $L^2(Y,\nu)$, $L^2_0(Y,\nu)$ denote the real Hilbert spaces of real-valued functions; in the sections on von Neumann algebras they are complex.

7.1. **Von Neumann algebras and Hilbertian spaces.** A *von Neumann algebra* on the separable Hilbert space $\mathcal{H}$ is a unital *-subalgebra of $B(\mathcal{H})$ closed in the weak operator topology; $(M)_1 = \{x \in M : \|x\| \leq 1\}$ denotes its unit ball. The set $\mathrm{vN}(\mathcal{H})$ of von Neumann algebras on $\mathcal{H}$ carries the *Effros Borel structure*, generated by the sets $\{M : (M)_1 \cap O \neq \varnothing\}$ for weakly open $O \subseteq B(\mathcal{H})$; it is a standard Borel space [Eff65, Theorem 1 and Section 3], see also [TL13, p. 21]. A map $z \mapsto M_z$ is Borel if there are Borel maps f_n into $B(\mathcal{H})$ with its weak Borel structure such that $\{f_n(z)\}_n$ is weakly dense in $(M_z)_1$ for each z [Eff65, Corollary to Theorem 2]. The factors (algebras with trivial center) form a Borel subset [Eff65] (see also [Eff66]), and so do the II_1 factors, the infinite-dimensional factors with a faithful normal tracial state [Eff66], see also [Nie73, HW98]. Isomorphism of separable II_1 factors is the isomorphism relation on the latter set for $\mathcal{H} = \ell^2(\mathbb{N})$. A

factor is of type II_∞ if it is isomorphic to $N \mathbin{\bar\otimes} B(\ell^2(\mathbb{N}))$ for a II_1 factor N. The II_∞ factors and the factors of type III also form Borel subsets of $\mathrm{vN}(\mathcal{H})$ [Nie73], see also [Azo83, Corollary 5.5] and [ST09b, Section 4.1], and isomorphism of separable II_∞ factors is the isomorphism relation on the former set. For $\lambda \in [0,1]$, the factors of type III_λ in the classification of Connes [Con73] form a Borel subset $\mathcal{F}_{\mathrm{III}_\lambda}(\mathcal{H})$ of $\mathrm{vN}(\mathcal{H})$. Indeed, put $A_\mu = \bigcup_{\mu \le t \le 1} \mathcal{F}_{\mathrm{III}_t}(\mathcal{H})$ for $\mu \in [0,1]$. Haagerup and Winsløw proved that A_μ is a G_δ subset of the space of factors, which is itself G_δ in $\mathrm{vN}(\mathcal{H})$, for the Effros–Maréchal topology [HW98, Corollary 3.11(i) and Theorem 4.9]; this topology induces the Effros Borel structure [HW98, Theorem 2.8 and Definition 2.9]. Hence $\mathcal{F}_{\mathrm{III}_\lambda}(\mathcal{H}) = A_\lambda \setminus \bigcup_{q \in \mathbb{Q} \cap (\lambda,1]} A_q$ is Borel. For completeness we include a direct proof (Lemma 7.2 below), based, like the proof of Haagerup and Winsløw, on a theorem of Connes, Haagerup and Størmer. Isomorphism of separable factors of type III_λ is the isomorphism relation on this set. Thus each of these isomorphism relations is an equivalence relation on a standard Borel space.

For a von Neumann algebra N on $\mathcal{H}$ let $S(N)$ be its set of normal states and $\mathcal{U}(N)$ its unitary group. For $\varphi, \psi \in S(N)$ put

$$d_N(\varphi,\psi) = \inf_{u \in \mathcal{U}(N)} \|\varphi \circ \operatorname{Ad} u - \psi\|, \qquad (\varphi \circ \operatorname{Ad} u)(x) = \varphi(uxu^*),$$

the norm being that of the predual N_*, and $D(N) = \sup_{\varphi,\psi \in S(N)} d_N(\varphi,\psi) \in [0,2]$. Since $\varphi \mapsto \varphi \circ \operatorname{Ad} u$ is an isometry of N_*, d_N is a pseudometric, and $\|\varphi \circ \operatorname{Ad} u - \psi \circ \operatorname{Ad} v\| = \|\varphi \circ \operatorname{Ad}(uv^*) - \psi\|$; hence $d_N(\varphi,\psi)$ is the distance between the norm closures of the inner orbits of φ and ψ, and $D(N)$ is the diameter of the normal state space modulo approximate unitary equivalence studied by Connes, Haagerup and Størmer [CHS85], see also [HS90].

Fact 7.1 (Connes–Haagerup–Størmer [CHS85], see also [HS90, Proposition 10.4, pp. 246–247])**.** *If N is a factor of type III_λ with separable predual, $0 \le \lambda \le 1$, then $D(N) = 2\,(1-\sqrt{\lambda})/(1+\sqrt{\lambda})$.*

Lemma 7.2 (the second assertion follows from [HW98, Theorem 4.9])**.** *The map $D\colon \mathrm{vN}(\mathcal{H}) \to [0,2]$ is Borel. Consequently, for every $\lambda \in [0,1]$ the set $\mathcal{F}_{\mathrm{III}_\lambda}(\mathcal{H})$ of factors of type III_λ on $\mathcal{H}$ is a Borel subset of $\mathrm{vN}(\mathcal{H})$.*

Proof. We use the following standard facts. (a) On the unit ball $(B(\mathcal{H}))_1$ the weak, strong and strong* operator topologies are Polish and have the same Borel sets (the identity maps from strong* to strong and from strong to weak are continuous bijections, so their inverses are Borel by [Kec95, Theorem 15.1]). (b) Multiplication is jointly strong* continuous on bounded sets and $x \mapsto x^*$ is strong* continuous; since the exponential series converges uniformly in norm on bounded sets, $h \mapsto e^{i\pi h}$ is strong* continuous on the self-adjoint part of $(B(\mathcal{H}))_1$. (c) Every $u \in \mathcal{U}(N)$ is $e^{i\pi h}$ for a self-adjoint $h \in (N)_1$, namely $h = \pi^{-1}\operatorname{Arg}(u)$, Arg with values in $(-\pi,\pi]$ (Borel functional calculus). (d) Every normal state of N has the form $\sum_n \omega_{\xi_n}|_N$ with $\sum_n \|\xi_n\|^2 = 1$ [KR86, Theorem 7.1.12], hence is the restriction of a normal state of $B(\mathcal{H})$. (e) A normal functional on N is weakly continuous on $(N)_1$.

Selectors. By the definition of the Effros Borel structure, $N \mapsto (N)_1$ is a Borel map into the Effros Borel space of the nonempty closed subsets of the compact metrizable space $((B(\mathcal{H}))_1, \text{weak})$, so by the Kuratowski–Ryll-Nardzewski theorem [Kec95, Theorem 12.13] there are Borel maps a_m from $\mathrm{vN}(\mathcal{H})$ to $(B(\mathcal{H}))_1$ with $\{a_m(N)\}_m$ weakly dense in $(N)_1$. Let (b_k) enumerate the rational convex combinations of the a_m. The set $\{b_k(N)\}_k$ is strong* dense in $(N)_1$: if $x \in (N)_1$ and $\xi_1,\dots,\xi_n \in \mathcal{H}$, the real-linear map $T \mapsto (T\xi_1,\dots,T\xi_n,T^*\xi_1,\dots,T^*\xi_n)$ into $\mathcal{H}^{2n}$ is continuous from the weak operator topology to the weak topology, so the image of x lies in the weak, hence (by convexity) the norm closure of the image of the convex hull of $\{a_m(N)\}_m$, and rational convex combinations are norm dense in the convex hull. Put $v_k(N) = \exp(i\pi(b_k(N) + b_k(N)^*)/2)$. By (b) and (c), $\{v_k(N)\}_k$ is a strong* dense subset of $\mathcal{U}(N)$, and by (a) and (b) each v_k is a Borel map into $(B(\mathcal{H}))_1$.

States. Fix a norm-dense sequence (ω_j) in the normal states of $B(\mathcal{H})$. By (d), and since restriction is norm-contractive, $(\omega_j|_N)_j$ is norm dense in $S(N)$. As $|d_N(\varphi,\psi)-d_N(\varphi',\psi')| \le \|\varphi-\varphi'\|+\|\psi-\psi'\|$, we get $D(N) = \sup_{i,j} d_N(\omega_i|_N, \omega_j|_N)$. If $\omega = \sum_n \omega_{\xi_n}$ with $\sum_n \|\xi_n\|^2 = 1$, then $\|\omega\circ\mathrm{Ad}\,u - \omega\circ\mathrm{Ad}\,v\| \le \sum_n 2\|\xi_n\|\|(u^*-v^*)\xi_n\|$ for unitaries u, v, which tends to 0 as $v \to u$ in the strong* topology; so the infimum defining d_N may be taken over the $v_k(N)$. Finally, by (e) the norm of a normal functional on N is the supremum of its modulus on $\{a_m(N)\}_m$. Altogether

$$D(N) = \sup_{i,j} \inf_k \sup_m |\omega_i(v_k(N)a_m(N)v_k(N)^*) - \omega_j(a_m(N))|.$$

By (a) and (b) each function $N \mapsto \omega_i(v_k(N)a_m(N)v_k(N)^*) - \omega_j(a_m(N))$ is Borel, and so is D.

The type III_λ *factors.* Every factor of type III on $\mathcal{H}$ has separable predual and is of type III_μ for exactly one $\mu \in [0,1]$ [Con73, p. 213]. The function $c(\mu) = 2(1-\sqrt{\mu})/(1+\sqrt{\mu})$ is strictly decreasing on $[0,1]$. By Fact 7.1, $\mathcal{F}_{\mathrm{III}_\lambda}(\mathcal{H}) = \mathcal{F}_{\mathrm{III}}(\mathcal{H}) \cap D^{-1}(\{c(\lambda)\})$, where $\mathcal{F}_{\mathrm{III}}(\mathcal{H})$ is the Borel set of type III factors [Nie73]. □

The intersection with $\mathcal{F}_{\mathrm{III}}(\mathcal{H})$ is needed: let N be a II_1 factor with normalized trace τ. For a nonzero projection $p \in N$ put $t = \tau(p)$ and $\psi_p = \tau(p\,\cdot\,p)/t$. Since τ is invariant under inner automorphisms, $d_N(\psi_p,\tau) = \|\psi_p - \tau\| \ge |(\psi_p-\tau)(2p-1)| = 2(1-t)$. As N has projections of arbitrarily small positive trace, $D(N) = 2 = c(0)$.

Fix a separable infinite-dimensional real Hilbert space $\mathcal{E}$. Every Hilbertian space is isometric to a nonempty closed subset of $\mathcal{E}$, so isometry of Hilbertian spaces is isometry on the standard Borel space $F(\mathcal{E}) \setminus \{\varnothing\}$ of nonempty closed subsets of $\mathcal{E}$.

Fact 7.3 (Gao [Gao05, Theorem 4.3 and the remark on p. 54])**.** *Isometry of nonempty closed subsets of $\mathcal{E}$ is Borel bireducible with $E_{\mathcal{U}(\mathcal{H})}$.*

Gao works with the orthogonal group $O(\mathcal{E})$; his reduction [Gao05, p. 69] takes nonempty, indeed infinite, values. The groups $O(\mathcal{E})$ and $\mathcal{U}(\mathcal{H})$ are each isomorphic to a closed subgroup of the other (realification and complexification), so $E_{O(\mathcal{E})} \sim_B E_{\mathcal{U}(\mathcal{H})}$ by the remark after Fact 2.1.

7.2. **Gaussian actions.** A *Gaussian space* over a separable real Hilbert space $\mathcal{V}$ is a standard probability space $(\Omega,\mathbb{P})$ with a linear isometry $\omega : \mathcal{V} \to L^2_{\mathbb{R}}(\Omega,\mathbb{P})$ whose values are centered Gaussian random variables and generate the σ-algebra (a model is given in Section 10.2).

Fact 7.4 ([Jan97, Chapters 1–4], see also [Kec10, Appendices C–E])**.** *Let $(\Omega,\mathbb{P},\omega)$ and $(\Omega',\mathbb{P}',\omega')$ be Gaussian spaces over $\mathcal{V}$ and $\mathcal{V}'$. Every orthogonal operator $U : \mathcal{V} \to \mathcal{V}'$ induces a measure preserving isomorphism $T_U : \Omega \to \Omega'$, unique modulo null sets, with $\omega'(U\eta) = \omega(\eta)\circ T_U^{-1}$ for $\eta \in \mathcal{V}$; and $T_{UV} = T_U T_V$.*

For an orthogonal representation π of a countable group Λ on $\mathcal{V}$ this gives the *Gaussian action* $g \mapsto T_{\pi(g)}$ on $(\Omega,\mathbb{P})$. If $U\pi(g) = \pi'(\delta(g))U$ for an isomorphism $\delta : \Lambda \to \Lambda'$ and an orthogonal representation π' of Λ' on $\mathcal{V}'$, then $T_U T_{\pi(g)} = T_{\pi'(\delta g)} T_U$: equivalent representations, twisted by δ, have conjugate Gaussian actions, twisted by δ. Over an orthogonal direct sum $\bigoplus_{k\in\mathbb{Z}} \mathcal{V}$, the product $(\Omega^{\mathbb{Z}},\mathbb{P}^{\mathbb{Z}})$ with $\omega((\eta_k)_k) = \sum_k \omega(\eta_k)\circ \mathrm{pr}_k$ is a Gaussian space (the summands are independent centered Gaussians), and the uniqueness in Fact 7.4 shows that the Gaussian action of $\pi \otimes \mathrm{id}_{\ell^2(\mathbb{Z})} = \bigoplus_{k\in\mathbb{Z}} \pi$ is the diagonal action $(T_{\pi(g)})^{\otimes\mathbb{Z}}$.

The Koopman representation of a Gaussian action is described by exponential vectors. (In fact $L^2(\Omega)$ is the symmetric Fock space over $\mathcal{V}$ [Jan97, Theorem 2.6 and Chapter 4]; an isometric embedding into the full Fock space suffices for us, and we prove it.) Let $\mathcal{V}^{\otimes n}$ be the n-th Hilbert tensor power of the real Hilbert space $\mathcal{V}$ ($\mathcal{V}^{\otimes 0} = \mathbb{R}$), let $\mathcal{F}(\mathcal{V}) = \bigoplus_{n\ge 0} \mathcal{V}^{\otimes n}$, and put $\mathrm{Exp}(\eta) = (\eta^{\otimes n}/\sqrt{n!})_{n\ge 0}$ for $\eta \in \mathcal{V}$, so that $\langle \mathrm{Exp}(\eta), \mathrm{Exp}(\eta')\rangle = e^{\langle \eta,\eta'\rangle}$; an orthogonal operator O on $\mathcal{V}$ gives the orthogonal operator $\mathcal{F}(O) = \bigoplus_n O^{\otimes n}$ with $\mathcal{F}(O)\,\mathrm{Exp}(\eta) = \mathrm{Exp}(O\eta)$. The subspace $\mathcal{F}_+(\mathcal{V}) = \bigoplus_{n\ge 1} \mathcal{V}^{\otimes n}$ is the orthocomplement of $\mathrm{Exp}(0)$, and it is invariant under $\mathcal{F}(O)$.

Lemma 7.5 (Fock embedding)**.** *Let π be an orthogonal representation of a group Λ, not necessarily countable, on a separable real Hilbert space $\mathcal{V}$, let $(\Omega, \mathbb{P}, \omega)$ be a Gaussian space over $\mathcal{V}$, and let $\kappa(g)f = f \circ T_{\pi(g)}^{-1}$ be the corresponding Koopman operators. There is a linear isometry $V : L_0^2(\Omega) \to \mathcal{F}_+(\mathcal{V})$ with $V\kappa(g) = \mathcal{F}(\pi(g))V$ for all $g \in \Lambda$. Hence for $f \in L_0^2(\Omega)$ we have $Vf = (v_n)_{n\geq 1}$ with $v_n \in \mathcal{V}^{\otimes n}$, $\|f\|^2 = \sum_n \|v_n\|^2$ and $\langle f, \kappa(g)f\rangle = \sum_n \langle v_n, \pi(g)^{\otimes n} v_n\rangle$.*

Proof. For $\eta \in \mathcal{V}$ put $\phi_\eta = \exp(\omega(\eta) - \frac{1}{2}\|\eta\|^2)$. Since $\omega(\eta)+\omega(\eta')$ is centered Gaussian with variance $\|\eta+\eta'\|^2$, we have $\langle \phi_\eta, \phi_{\eta'}\rangle = e^{\langle \eta,\eta'\rangle}$, so $\sum_k a_k \phi_{\eta_k} \mapsto \sum_k a_k \operatorname{Exp}(\eta_k)$ is a well-defined linear isometry on the span of the ϕ_η. We show below that this span is dense in $L^2(\Omega)$; the isometry then extends to a linear isometry $\tilde{V} : L^2(\Omega) \to \mathcal{F}(\mathcal{V})$. By Fact 7.4, $\kappa(g)\phi_\eta = \phi_{\pi(g)\eta}$, so $\tilde{V}\kappa(g) = \mathcal{F}(\pi(g))\tilde{V}$. As $\tilde{V}1 = \tilde{V}\phi_0 = \operatorname{Exp}(0)$, the isometry $\tilde{V}$ maps $L_0^2(\Omega)$, the orthocomplement of 1, into $\mathcal{F}_+(\mathcal{V})$, the orthocomplement of $\operatorname{Exp}(0)$; let V be its restriction to $L_0^2(\Omega)$. The formulas follow. It remains to show that the ϕ_η are total. Let $f \perp \phi_\eta$ for all η, let (η_k) be dense in $\mathcal{V}$, and let $\mathcal{A}_N$ be the σ-algebra generated by $\omega(\eta_1), \dots, \omega(\eta_N)$. The $\mathcal{A}_N$ generate the σ-algebra of Ω, so $f_N = \mathbb{E}[f \mid \mathcal{A}_N] \to f$. Write $f_N = G(\omega(\eta_1), \dots, \omega(\eta_N))$ with $G \in L^2(\mathbb{R}^N, \lambda)$, λ the centered Gaussian law of $(\omega(\eta_k))_{k\leq N}$. As $\phi_{\sum_k t_k \eta_k}$ is a positive multiple of $e^{\langle t, (\omega(\eta_k))_k\rangle}$, we get $\int G(x) e^{\langle t,x\rangle} d\lambda(x) = 0$ for $t \in \mathbb{R}^N$. This function of t extends to an entire function on $\mathbb{C}^N$, which therefore vanishes; so the Fourier transform of $G\,d\lambda$ vanishes, $G = 0$, and $f = 0$. □

7.3. **Group measure space factors.** Let a countable group Λ act by pmp transformations on a standard probability space (Y, ν), with Koopman representation κ. The *group measure space algebra* $L^\infty(Y) \rtimes \Lambda$ is the von Neumann algebra on $L^2(Y) \otimes \ell^2(\Lambda)$ generated by the operators $f \otimes 1$ ($f \in L^\infty(Y)$) and $u_g = \kappa(g) \otimes \lambda_g$ ($g \in \Lambda$), where $\lambda_g \mathbf{1}_h = \mathbf{1}_{gh}$; thus $u_g(f \otimes 1)u_g^* = (f \circ g^{-1}) \otimes 1$. (This is the model of [ST09a, Section 2]; the unitary $f \otimes \mathbf{1}_g \mapsto \kappa(g)^{-1} f \otimes \mathbf{1}_g$ carries it onto the usual crossed product.) The action is *free* if every $g \neq e$ has a null fixed point set, and *ergodic* if invariant sets are null or conull.

Fact 7.6 ([AP, Proposition 1.4.5 and Section 1.4.3])**.**

(a) *If $\Lambda \curvearrowright (Y, \nu)$ is free and ergodic and Y is nonatomic, then $L^\infty(Y) \rtimes \Lambda$ is a II_1 factor.*
(b) *If $\delta : \Lambda \to \Lambda'$ is a group isomorphism and $\Theta : (Y, \nu) \to (Y', \nu')$ a measure preserving isomorphism with $\Theta(gy) = \delta(g)\Theta(y)$, then $L^\infty(Y) \rtimes \Lambda \cong L^\infty(Y') \rtimes \Lambda'$.*

A free ergodic pmp action $\Lambda \curvearrowright (Y, \nu)$ is *W*-superrigid* if for every free ergodic pmp action $\Lambda' \curvearrowright (Y', \nu')$ of a countable group with $L^\infty(Y) \rtimes \Lambda \cong L^\infty(Y') \rtimes \Lambda'$ there are δ and Θ as in Fact 7.6(b). If Λ acts on a countable set I and (Y_0, ν_0) is a standard probability space, the *generalized Bernoulli action* $\Lambda \curvearrowright (Y_0, \nu_0)^I$ permutes the coordinates: $(g \cdot y)_i = y_{g^{-1}i}$.

Let N be a II_1 factor with trace τ, and let Tr be the usual trace of $B(\ell^2(\mathbb{N}))$. For $t > 0$ the *amplification* N^t is the II_1 factor $p(N \bar{\otimes} B(\ell^2(\mathbb{N})))p$, where p is a projection with $(\tau \otimes \mathrm{Tr})(p) = t$; it is well defined up to isomorphism, and $(N^s)^t \cong N^{st}$ [AP, Section 4.2]. If M is a further II_1 factor and $\theta : M \bar{\otimes} B(\ell^2(\mathbb{N})) \to N \bar{\otimes} B(\ell^2(\mathbb{N}))$ is an isomorphism, then $M \cong N^t$ for some $t > 0$. Indeed, let e_{11} be a minimal projection of $B(\ell^2(\mathbb{N}))$. The projection $1 \otimes e_{11}$ is finite, its corner being M, so $p = \theta(1 \otimes e_{11})$ is a nonzero finite projection of the semifinite factor $N \bar{\otimes} B(\ell^2(\mathbb{N}))$; hence $t = (\tau \otimes \mathrm{Tr})(p)$ is finite [AP, Section 8.3], and θ restricts to an isomorphism of M onto $p(N \bar{\otimes} B(\ell^2(\mathbb{N})))p = N^t$.

7.4. **Automorphisms of the finitary symmetric group.** Recall that $S = \mathrm{Sym}_{\mathrm{fin}}(\mathbb{N})$. Every $\theta \in \mathrm{Sym}(\mathbb{N})$ normalizes S, and $\mathrm{Ad}(\theta)(\tau_{ij}) = \tau_{\theta(i)\theta(j)}$ for the transpositions $\tau_{ij} = (i\ j)$; the converse below is used in Proposition 8.9. The analogous statement for the full symmetric group is due to Schreier and Ulam [SU37]; for the finitary group we include a proof for completeness.

Lemma 7.7. *Every automorphism of S is $\mathrm{Ad}(\theta)$ for a unique $\theta \in \mathrm{Sym}(\mathbb{N})$.*

Proof. Involutions of S are conjugate if and only if they have the same number $k \geq 1$ of 2-cycles; let $\mathcal{C}_k$ be this class. A product of two transpositions has order at most 3. If $k \geq 2$, write $t = (a\ b)(c\ d)r \in \mathcal{C}_k$ with r a product of 2-cycles disjoint from $\{a, b, c, d\}$, pick v outside the support of t, and put $t' = (b\ c)(d\ v)r \in \mathcal{C}_k$; then tt' is a 5-cycle. So $\mathcal{C}_1$ is the only class of involutions in which all products of two elements have order at most 3, and every automorphism δ preserves $\mathcal{C}_1$. Distinct transpositions fail to commute if and only if their supports meet in one point, so a maximal set of pairwise noncommuting transpositions is either a star $\mathcal{S}_n = \{\tau_{nm} : m \neq n\}$, which is infinite, or a triangle $\{(a\ b), (a\ c), (b\ c)\}$. (If $(a\ b), (a\ c)$ belong to such a set, every further member contains a or equals $(b\ c)$, and $(b\ c)$ commutes with $(a\ v)$ for $v \notin \{b, c\}$.) Hence $\delta(\mathcal{S}_n) = \mathcal{S}_{\theta(n)}$ for a bijection θ, and since τ_{nm} is the unique element of $\mathcal{S}_n \cap \mathcal{S}_m$, $\delta(\tau_{nm}) = \tau_{\theta(n)\theta(m)}$. Transpositions generate S, so $\delta = \mathrm{Ad}(\theta)$; only the identity of $\mathrm{Sym}(\mathbb{N})$ commutes with all transpositions. □

8. Hilbertian spaces and Gaussian actions of S

This section proves Theorem E.

8.1. **The tensor representation.** Let (X, d) be a Hilbertian space. A *reference sequence* for X is a dense sequence $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ in X in which every term occurs infinitely often; e.g. $x_i = a_{\varpi(i)}$ for a dense sequence (a_n) and a surjection $\varpi : \mathbb{N} \to \mathbb{N}$ with infinite fibers. Let

$$\ell^2(X; \mathbf{x}) = \Big\{q \in X^{\mathbb{N}} : \sum_i d(q_i, x_i)^2 < \infty\Big\}, \qquad d_2(q, q') = \Big(\sum_i d(q_i, q'_i)^2\Big)^{1/2},$$

the sequences square-summably close to $\mathbf{x}$, with the ℓ^2-sum of the coordinate distances. Call q *m-local* if $q_i = x_i$ for $i > m$, and *local* if it is m-local for some m.

Lemma 8.1.

(a) $(\ell^2(X; \mathbf{x}), d_2)$ *embeds isometrically into a real Hilbert space, so* $k(q, q') = e^{-d_2(q,q')^2}$ *is a positive definite kernel on* $\ell^2(X; \mathbf{x})$.

(b) *The local sequences with all terms in* $\{x_j : j \in \mathbb{N}\}$ *form a countable* d_2*-dense subset of* $\ell^2(X; \mathbf{x})$.

Proof. (a) If $\iota : X \to \mathcal{E}$ is an isometric embedding, $q \mapsto (\iota(q_i) - \iota(x_i))_i$ is an isometric embedding into $\ell^2(\mathbb{N}; \mathcal{E})$; apply Fact 2.5(b). (b) Given q and $\varepsilon > 0$, choose m with $\sum_{i>m} d(q_i, x_i)^2 < \varepsilon^2/2$ and j_i with $d(q_i, x_{j_i})^2 < \varepsilon^2/(2m)$ for $i \leq m$; replace q_i by x_{j_i} for $i \leq m$ and by x_i for $i > m$. □

Definition 8.2. Let $(\mathcal{H}_X, \xi)$ be the Hilbert space of the kernel k on $\ell^2(X; \mathbf{x})$, a real Hilbert space. The group S acts on $\ell^2(X; \mathbf{x})$ by permuting coordinates, $(sq)_i = q_{s^{-1}(i)}$, preserving d_2; the *tensor representation* π_X is the orthogonal representation of S on $\mathcal{H}_X$ with $\pi_X(s)\xi_q = \xi_{sq}$ (Fact 2.5(a)). For a nonzero separable real Hilbert space $\mathcal{K}$, the Gaussian action of $\pi_X \otimes \mathrm{id}_{\mathcal{K}}$ on $\mathcal{H}_X \otimes \mathcal{K}$ is denoted $\mathsf{G}_X^{\mathcal{K}}$, with Gaussian space $(\Omega, \mathbb{P}, \omega)$ and Koopman representation κ on $L^2(\Omega, \mathbb{P})$. The *Gaussian tensor action* is $\mathsf{G}_X = \mathsf{G}_X^{\mathbb{R}}$.

By (2.3), Lemma 8.1(b) shows that the ξ_q with q local and with terms in $\{x_j\}$ are total; so $\mathcal{H}_X$ is separable. (In the outline given in the introduction, ξ_q is the elementary tensor $\bigotimes_i \xi_{q_i}$ of the canonical vectors $\xi_{q_i} \in \mathcal{V}_X$ of the kernel e^{-d^2} on X.) The space Ω is nonatomic, as $\omega(\eta)$ has law $N(0, 1)$ for a unit vector η. As noted after Fact 7.4, $\mathsf{G}_X^{\ell^2(\mathbb{Z})}$ is conjugate to $\mathsf{G}_X^{\otimes\mathbb{Z}}$. Thus Theorem E follows from Proposition 8.7 ((i)⇒(ii); (ii)⇒(iii) is clear) and Proposition 8.9 with $\mathcal{K} = \ell^2(\mathbb{Z})$ ((iii)⇒(i)).

8.2. **Transpositions and the first chaos.** Fix X, $\mathbf{x}$ and $\mathcal{K}$; write $\pi = \pi_X \otimes \mathrm{id}_{\mathcal{K}}$ and $\mathcal{V} = \mathcal{H}_X \otimes \mathcal{K}$. Put $c_{ij} = e^{-2d(x_i,x_j)^2} \in (0, 1]$ for $i \neq j$. We write $f(i, j) \to 0$ *as* $\min(i, j) \to \infty$ if for every $\varepsilon > 0$ there is m with $|f(i, j)| < \varepsilon$ whenever $i \neq j$ and $i, j > m$. Let $\mathcal{V}_m$ be the closed span of the vectors $\xi_q \otimes a$ with q m-local and $a \in \mathcal{K}$.

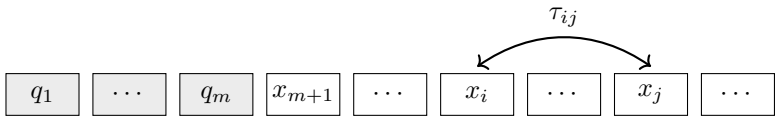


FIGURE 4. An m-local sequence q and a transposition τ_{ij}, $i,j>m$: the matrix coefficient reads off the distance between the two swapped reference points.

Lemma 8.3 (Swap formula)**.**

(a) $\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots$, *with dense union.*

(b) *If* $\eta,\eta' \in \mathcal{V}_m$ *and* $i \neq j$ *are* $> m$, *then* $\langle \eta, \pi(\tau_{ij})\eta'\rangle = c_{ij}\langle \eta,\eta'\rangle$.

Proof. (a) holds because the $\xi_q \otimes a$, q local, are total. (b) Let q, q' be m-local. The sequence $\tau_{ij}q'$ agrees with q' off $\{i,j\}$ and has x_j at i and x_i at j, while q has x_i, x_j there; hence $d_2(q,\tau_{ij}q')^2 = d_2(q,q')^2 + 2d(x_i,x_j)^2$ and $\langle \xi_q\otimes a, \pi(\tau_{ij})(\xi_{q'}\otimes b)\rangle = c_{ij}\langle \xi_q\otimes a, \xi_{q'}\otimes b\rangle$. Extend by bilinearity and continuity. □

Part (b) is not uniform in η,η': a vector of $\mathcal{V}_{m'}$ with $m' \geq \max(i,j)$ may be τ_{ij}-invariant. We pass to the tensor powers $\mathcal{V}^{\otimes n}$, which describe the Gaussian action (Lemma 7.5).

Lemma 8.4 (Tensor powers)**.** *Let* $n \geq 1$. *For every* $v \in \mathcal{V}^{\otimes n}$, $\langle v, \pi(\tau_{ij})^{\otimes n} v\rangle - c_{ij}^n \|v\|^2 \to 0$ *as* $\min(i,j)\to\infty$.

Proof. Write $Q_{ij}(v) = \langle v, \pi(\tau_{ij})^{\otimes n} v\rangle - c_{ij}^n\|v\|^2$, and let $\mathcal{D}_m$ be the span of the elementary tensors $\eta_1\otimes\cdots\otimes\eta_n$ with $\eta_k \in \mathcal{V}_m$. By Lemma 8.3(b), for elementary tensors $\eta = \eta_1\otimes\cdots\otimes\eta_n$ and $\eta' = \eta_1'\otimes\cdots\otimes\eta_n'$ in $\mathcal{D}_m$ and $i,j>m$ we have $\langle \eta, \pi(\tau_{ij})^{\otimes n}\eta'\rangle = \prod_k \langle \eta_k, \pi(\tau_{ij})\eta_k'\rangle = c_{ij}^n\langle \eta,\eta'\rangle$; so $Q_{ij} = 0$ on $\mathcal{D}_m$ for such i,j. The $\mathcal{D}_m$ increase and, by Lemma 8.3(a) and the estimate $\|\eta_1\otimes\cdots\otimes\eta_n - \eta_1'\otimes\cdots\otimes\eta_n'\| \leq \sum_k \|\eta_1\|\cdots\|\eta_{k-1}\|\,\|\eta_k-\eta_k'\|\,\|\eta_{k+1}'\|\cdots\|\eta_n'\|$, their union is dense. Since $|Q_{ij}(v) - Q_{ij}(v')| \leq 2(\|v\|+\|v'\|)\|v-v'\|$, given v and $\varepsilon \in (0,1]$ we choose m and $v' \in \mathcal{D}_m$ with $\|v-v'\| \leq \varepsilon/(2(2\|v\|+1))$, and get $|Q_{ij}(v)| \leq \varepsilon$ for $i,j>m$. □

Proposition 8.5 (Chaos asymptotics)**.**

(a) *For* $\eta \in \mathcal{V}$ *and* $f = \omega(\eta)$, $\langle f, \kappa(\tau_{ij})f\rangle - c_{ij}\|f\|^2 \to 0$ *as* $\min(i,j)\to\infty$.

(b) *For* $f \in L^2_0(\Omega)$ *and* $\varepsilon > 0$ *there is* m *with* $\langle f, \kappa(\tau_{ij})f\rangle \leq c_{ij}\|f\|^2 + \varepsilon$ *whenever* $i \neq j$ *and* $i,j>m$.

Proof. (a) By Fact 7.4, $\kappa(\tau_{ij})\omega(\eta) = \omega(\pi(\tau_{ij})\eta)$, so $\langle f, \kappa(\tau_{ij})f\rangle = \langle \eta, \pi(\tau_{ij})\eta\rangle$; apply Lemma 8.4 with $n=1$. (b) With $Vf = (v_n)$ as in Lemma 7.5, choose N with $\sum_{n>N}\|v_n\|^2 < \varepsilon/2$ and then m such that $\langle v_n, \pi(\tau_{ij})^{\otimes n} v_n\rangle \leq c_{ij}^n\|v_n\|^2 + \varepsilon/(2N)$ for $n \leq N$ and $i,j>m$ (Lemma 8.4). Since $c_{ij}^n \leq c_{ij}$ and the terms with $n>N$ are at most $\|v_n\|^2$, Lemma 7.5 gives the claim. □

Part (b) concerns *all* mean-zero functions, and therefore survives any conjugacy of actions; part (a) shows that the bound is attained on the first chaos $\omega(\mathcal{V})$.

8.3. **Proof of Theorem E.** Let X, X' be Hilbertian spaces with metrics d, d', reference sequences $\mathbf{x} = (x_i)$, $\mathbf{y} = (y_j)$, and coefficients c_{ij}, c'_{ij}; primes refer to X'.

Lemma 8.6 (Matching)**.** *If* $\varphi : X \to X'$ *is a distance-preserving bijection, there is* $\theta \in \mathrm{Sym}(\mathbb{N})$ *with* $\sum_i d'(\varphi(x_i), y_{\theta(i)})^2 < \infty$.

Proof. The sequences $a_i = \varphi(x_i)$ and (y_j) are dense in X' with every term repeated infinitely often, so every ball meets both outside any finite set of indices. Build finite partial injections $\theta_0 \subseteq \theta_1 \subseteq \cdots$: at odd steps k take the least i outside the domain and some j outside the range with $d'(a_i,y_j) < 2^{-k}$; at even steps take the least j outside the range and some i outside the domain with $d'(a_i,y_j) < 2^{-k}$. Then $\theta = \bigcup_k \theta_k$ is a bijection and the sum is at most $\sum_k 4^{-k}$. □

Proposition 8.7. *If X and X' are isometric, there are $\theta \in \mathrm{Sym}(\mathbb{N})$ and an orthogonal operator $U : \mathcal{H}_X \to \mathcal{H}_{X'}$ with $U\pi_X(s) = \pi_{X'}(\theta s \theta^{-1})U$ for $s \in S$. In particular $\mathsf{G}_X \simeq_{\mathrm{Aut}} \mathsf{G}_{X'}$, with $\delta = \mathrm{Ad}(\theta)$.*

Proof. Let φ, θ be as in Lemma 8.6, and define $\Phi(q)_{\theta(i)} = \varphi(q_i)$: the isometry is applied to every term and the terms are relabeled by θ. By the triangle inequality, $\sum_j d'(\Phi(q)_j, y_j)^2 \le 2\sum_i d(q_i, x_i)^2 + 2\sum_i d'(\varphi(x_i), y_{\theta(i)})^2 < \infty$, so Φ maps $\ell^2(X;\mathbf{x})$ into $\ell^2(X';\mathbf{y})$, with inverse $\Psi(q')_i = \varphi^{-1}(q'_{\theta(i)})$ (same estimate), and $d'_2(\Phi q, \Phi q') = d_2(q, q')$. Moreover $\Phi(sq)_j = \varphi(q_{s^{-1}\theta^{-1}(j)}) = ((\theta s \theta^{-1})\Phi(q))_j$. Fact 2.5(a) gives U with $U\xi_q = \xi'_{\Phi(q)}$, which intertwines as claimed; the Gaussian functor (Section 7.2) gives (1.1) with $\delta = \mathrm{Ad}(\theta)$. □

The next lemma is used in Proposition 8.9 and in Corollary 11.2.

Lemma 8.8 (Recovering the space)**.** *If $\theta \in \mathrm{Sym}(\mathbb{N})$ satisfies*

$$c_{ij} - c'_{\theta(i)\theta(j)} \longrightarrow 0 \qquad \text{as } \min(i,j) \to \infty, \tag{8.1}$$

then X and X' are isometric.

Proof. Since θ is a bijection, $\min(i,j) \to \infty$ if and only if $\min(\theta(i),\theta(j)) \to \infty$. For $x \in X$, call a sequence of distinct indices (i_k) *x-admissible* if $x_{i_k} \to x$; such sequences exist by density and repetition (even for isolated x), and $i_k \to \infty$. If (i_k), (j_k) are x-admissible, then along k with $i_k \neq j_k$ we have $c_{i_k j_k} \to 1$, so $c'_{\theta(i_k)\theta(j_k)} \to 1$ by (8.1) and $d'(y_{\theta(i_k)}, y_{\theta(j_k)}) \to 0$. Applied to subsequences of one admissible sequence, this shows that $(y_{\theta(i_k)})_k$ is Cauchy, and its limit $\varphi(x)$ in the complete space X' does not depend on the admissible sequence. If $x \neq x^*$ and (i_k), (j_k) are x- and x^*-admissible, then $c_{i_k j_k} \to e^{-2d(x,x^*)^2} > 0$, so $c'_{\theta(i_k)\theta(j_k)}$ has the same limit and, as $t \mapsto (-\frac{1}{2}\log t)^{1/2}$ is continuous on $(0,1]$, $d'(\varphi(x),\varphi(x^*)) = d(x,x^*)$. With the roles exchanged and θ^{-1} in place of θ we get a distance-preserving map $\psi : X' \to X$ with $\psi(y) = \lim_k x_{\theta^{-1}(j_k)}$ for y-admissible (j_k); then $\theta^{-1}(j_k)$ is $\psi(y)$-admissible and $\varphi(\psi(y)) = \lim_k y_{j_k} = y$. So φ is onto. □

Proposition 8.9. *Let $\mathcal{K}, \mathcal{K}'$ be nonzero separable real Hilbert spaces. If $\mathsf{G}_X^{\mathcal{K}} \simeq_{\mathrm{Aut}} \mathsf{G}_{X'}^{\mathcal{K}'}$, then X and X' are isometric.*

Proof. Let T and δ satisfy (1.1). By Lemma 7.7, $\delta = \mathrm{Ad}(\theta)$ with $\theta \in \mathrm{Sym}(\mathbb{N})$, so $\delta(\tau_{ij}) = \tau_{\theta(i)\theta(j)}$. Since θ is a bijection, $\min(\theta(i),\theta(j)) \to \infty$ as $\min(i,j) \to \infty$. The orthogonal operator $U_T f = f \circ T^{-1}$ maps $L^2_0(\Omega)$ onto $L^2_0(\Omega')$, and $U_T \kappa(s) = \kappa'(\delta s) U_T$. Fix a unit vector $\eta \in \mathcal{H}_X \otimes \mathcal{K}$ and put $f = \omega(\eta)$. By Proposition 8.5(a) for X and Proposition 8.5(b) for X', applied to the fixed unit vector $U_T f \in L^2_0(\Omega')$,

$$c_{ij} + o(1) = \langle f, \kappa(\tau_{ij}) f\rangle = \langle U_T f, \kappa'(\tau_{\theta(i)\theta(j)}) U_T f\rangle \le c'_{\theta(i)\theta(j)} + o(1).$$

The same argument with T^{-1} and $\delta^{-1} = \mathrm{Ad}(\theta^{-1})$ gives $c'_{kl} \le c_{\theta^{-1}(k)\theta^{-1}(l)} + o(1)$ as $\min(k,l) \to \infty$. Hence (8.1), and Lemma 8.8 applies. □

This completes the proof of Theorem E (see the end of Section 8.1). The same argument applies to representations. Recall that $\pi_{X,\mathbb{C}}$ denotes the complexification of the tensor representation, a unitary representation of S on $(\mathcal{H}_X)_{\mathbb{C}}$.

Corollary 8.10. *For Hilbertian spaces X, X', with any choice of reference sequences, the following are equivalent:*

(i) *X and X' are isometric;*
(ii) *π_X and $\pi_{X'} \circ \delta$ are orthogonally equivalent for some $\delta \in \mathrm{Aut}(S)$;*
(iii) *$\pi_{X,\mathbb{C}}$ and $\pi_{X',\mathbb{C}} \circ \delta$ are unitarily equivalent for some $\delta \in \mathrm{Aut}(S)$.*

Proof. (i)⇒(ii) is Proposition 8.7, and (ii)⇒(iii) holds because an orthogonal equivalence extends to a unitary equivalence of the complexifications. For (iii)⇒(i), let W be a unitary operator with

$W\pi_{X,\mathbb{C}}(s) = \pi_{X',\mathbb{C}}(\delta(s))W$, and let $\delta = \mathrm{Ad}(\theta)$ (Lemma 7.7). With the real part of its inner product, $(\mathcal{H}_X)_{\mathbb{C}}$ is the real Hilbert space $\mathcal{H}_X \otimes \mathbb{R}^2$, on which $\pi_{X,\mathbb{C}}$ is $\pi_X \otimes \mathrm{id}_{\mathbb{R}^2}$, and W is orthogonal. So we are in the situation of Section 8.2 with $\mathcal{K} = \mathbb{R}^2$ for both spaces. Fix a unit vector $\eta \in \mathcal{H}_X \otimes \mathbb{R}^2$. By Lemma 8.4 with $n = 1$, applied to η and to the unit vector $W\eta$,

$$c_{ij} + o(1) = \langle \eta, \pi(\tau_{ij})\eta\rangle = \langle W\eta, \pi'(\tau_{\theta(i)\theta(j)})W\eta\rangle = c'_{\theta(i)\theta(j)} + o(1)$$

as $\min(i,j) \to \infty$, where $\pi = \pi_X \otimes \mathrm{id}_{\mathbb{R}^2}$ and $\pi' = \pi_{X'} \otimes \mathrm{id}_{\mathbb{R}^2}$. This is (8.1), and Lemma 8.8 applies. □

9. From pmp actions to II_1 factors

This section proves Theorem F. Let $\Gamma_1 = \mathrm{PSL}(n,\mathbb{Z})$, $\Sigma = \langle u\rangle$, $L = S \times \langle\zeta\rangle$, $\Gamma_2 = \Sigma \times L$ and $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be as in (1.2) (cf. [PV10, Examples 6.8 and 6.9]).

9.1. **Coinduction.** After discarding an invariant null set, a pmp action of a countable group is a Borel action by measure preserving Borel automorphisms. Let α be a pmp action of S on a standard nonatomic probability space (B,μ) and $\tilde\alpha(s\zeta^k) = \alpha(s)$ its extension to L. The coinduced action $\mathrm{Ind}(\alpha)$ is left translation on equivariant functions:

$$Y_\alpha = \{F : \Gamma \to B \ :\ F(\gamma l) = \tilde\alpha(l)^{-1}F(\gamma)\ (\gamma \in \Gamma,\ l \in L)\}, \qquad (\gamma \cdot F)(\gamma') = F(\gamma^{-1}\gamma').$$

For a transversal $\mathcal{T}$ of Γ/L, $F \mapsto (F(t))_{t\in\mathcal{T}}$ is a Borel bijection $Y_\alpha \to B^{\mathcal{T}}$ (with inverse $F(tl) = \tilde\alpha(l)^{-1}b_t$), and ν_α is the measure corresponding to $\mu^{\mathcal{T}}$; it does not depend on $\mathcal{T}$, since another transversal $\{tl_t\}$ changes each coordinate by the measure preserving map $\tilde\alpha(l_t)^{-1}$. In words: Y_α has one independent coordinate with law μ for each coset of L. For $P \subseteq \Gamma/L$ let $\mathcal{B}_P$ be the σ-algebra generated by the maps $F \mapsto F(\gamma)$, $\gamma L \in P$; then $\mathcal{B}_P$ and $\mathcal{B}_{P'}$ are independent when $P \cap P' = \varnothing$, and γ maps $\mathcal{B}_P$ onto $\mathcal{B}_{\gamma P}$, so $\mathrm{Ind}(\alpha)$ is measure preserving. We put $M(\alpha) = L^\infty(Y_\alpha,\nu_\alpha) \rtimes \Gamma$. In the coordinates of a section, $\mathrm{Ind}(\alpha)$ is the coinduced action of [PV10, Section 6.2].

Since Σ commutes with L, for $\sigma,\sigma' \in \Sigma$ and $s \in S$,

$$(s \cdot F)(\sigma) = \alpha(s)F(\sigma), \qquad (\zeta \cdot F)(\sigma) = F(\sigma), \qquad (\sigma' \cdot F)(\sigma) = F(\sigma'^{-1}\sigma), \tag{9.1}$$

because $F(s^{-1}\sigma) = F(\sigma s^{-1}) = \alpha(s)F(\sigma)$. The cosets σL, $\sigma \in \Sigma$, are distinct and are exactly the cosets in Γ_2/L. Hence the restriction map

$$\mathrm{res}_\alpha : Y_\alpha \to B^\Sigma, \qquad \mathrm{res}_\alpha(F) = (F(\sigma))_{\sigma\in\Sigma},$$

carries ν_α to μ^Σ, generates $\mathcal{B}_{\Gamma_2/L}$, and is Γ_2-equivariant when Σ shifts B^Σ, ζ acts trivially, and S acts diagonally, that is, by $\alpha^{\otimes\mathbb{Z}}$ after identifying Σ with $\mathbb{Z}$. Let $A_\alpha = \{f \circ \mathrm{res}_\alpha : f \in L^\infty(B^\Sigma)\}$.

9.2. **Group theory.** In Lemma 9.2, (b) serves Lemma 9.4 (Bernoulli structure, freeness), (c) Lemma 9.7 (ergodicity), (d) Lemma 9.6, and (a), (e) the proof of Theorem F. We use the Bass–Serre tree of Γ [Ser80, Chapter I]: its vertices are the cosets $g\Gamma_1$, $g\Gamma_2$, its edges the cosets $g\Sigma$, joining $g\Gamma_1$ to $g\Gamma_2$; Γ acts by left translation, and the stabilizer of $g\Gamma_i$, $g\Sigma$ is $g\Gamma_i g^{-1}$, $g\Sigma g^{-1}$. An element fixing two vertices fixes the geodesic between them. We write $Z(x)$ for the centralizer in Γ of an element x.

Lemma 9.1.

(a) *If $g \notin \Gamma_2$, then $\Gamma_2 \cap g\Gamma_2 g^{-1} \subseteq \Sigma$.*
(b) *$\Gamma_1 \cap g\Gamma_2 g^{-1} \subseteq g\Sigma g^{-1}$ for every $g \in \Gamma$.*

Proof. (a) An element of $\Gamma_2 \cap g\Gamma_2 g^{-1}$ fixes the distinct vertices Γ_2, $g\Gamma_2$, hence the first edge $\gamma\Sigma$ ($\gamma \in \Gamma_2$) of the geodesic between them, whose stabilizer is $\gamma\Sigma\gamma^{-1} = \Sigma$ because $\Sigma \lhd \Gamma_2$. (b) An element of $\Gamma_1 \cap g\Gamma_2 g^{-1}$ fixes the last edge $g\gamma\Sigma$ ($\gamma \in \Gamma_2$) of the geodesic from Γ_1 to $g\Gamma_2$, with stabilizer $g\Sigma g^{-1}$. □

Lemma 9.2.

(a) *$L \cap gLg^{-1} = \{e\}$ for $g \notin \Gamma_2$, and $Z(c) \subseteq \Gamma_2$ for every $c \in L \setminus \{e\}$.*

(b) Γ_1 *acts freely on* Γ/L*, and every* $\gamma \neq e$ *moves some coset of* L*.*
(c) *If* $c = u^a\zeta^b$ *with* $a \neq 0$*, every* $\langle c\rangle$*-orbit on* Γ/L *is infinite. If* $k \neq 0$*,* ζ^k *fixes the cosets in* Γ_2/L*, and all its other orbits are infinite.*
(d) Γ *has no nontrivial finite normal subgroup.*
(e) *Every* $\delta_S \in \mathrm{Aut}(S)$ *extends to* $\delta \in \mathrm{Aut}(\Gamma)$ *with* $\delta|_{\Gamma_1} = \mathrm{id}$ *and* $\delta(\sigma s\zeta^k) = \sigma\,\delta_S(s)\,\zeta^k$ *on* Γ_2*.*

Proof. (a) $L \cap gLg^{-1} \subseteq \Gamma_2 \cap g\Gamma_2 g^{-1} \cap L \subseteq \Sigma \cap L = \{e\}$ by Lemma 9.1(a). If $xcx^{-1} = c$, then $c \in L \cap xLx^{-1}$, so $x \in \Gamma_2$.

(b) If $h \in \Gamma_1$ fixes gL, then $h \in \Gamma_1 \cap gLg^{-1} \subseteq g(\Sigma \cap L)g^{-1} = \{e\}$ by Lemma 9.1(b). For the second claim take $g \in \Gamma_1 \setminus \Sigma$; then $g \notin \Gamma_2$ and $\bigcap_\gamma \gamma L\gamma^{-1} \subseteq L \cap gLg^{-1} = \{e\}$.

(c) Let $c \in \Sigma \times \langle\zeta\rangle$, the center of Γ_2, and $m \neq 0$. For $g \in \Gamma_2$, $c^m gL = gL$ if and only if $c^m \in L$. For $g \notin \Gamma_2$, $c^m gL = gL$ means $g^{-1}c^m g \in L \cap g^{-1}\Gamma_2 g \subseteq \Sigma \cap L = \{e\}$ (Lemma 9.1(a)), so $c^m = e$. Now $(u^a\zeta^b)^m \in L$ if and only if $am = 0$.

(d) Let x lie in a finite normal subgroup. Its centralizer $Z(x)$ has finite index, so it contains some ζ^m, $m \geq 1$, and some $g \in \Gamma_1 \setminus \Sigma$ (otherwise $Z(x) \cap \Gamma_1 \subseteq \Sigma \cong \mathbb{Z}$ would have finite index in Γ_1, which contains $\mathbb{Z}^2$, e.g. the images of $1 + kE_{1n} + lE_{1,n-1}$). By (a), $x \in Z(\zeta^m) \subseteq \Gamma_2$ and $x \in Z(g\zeta^m g^{-1}) \subseteq g\Gamma_2 g^{-1}$, so $x \in \Sigma$ by Lemma 9.1(a); as x has finite order, $x = e$.

(e) The maps id_{Γ_1} and $\mathrm{id}_\Sigma \times \delta_S \times \mathrm{id}_{\langle\zeta\rangle}$ agree on Σ and define an endomorphism of Γ; δ_S^{-1} gives its inverse. □

9.3. **Freeness, ergodicity and rigidity.** The next lemma detects, from the dynamics, finite orbits on Γ/L.

Lemma 9.3. *Let* $h \in \Gamma$ *and let* $P \subseteq \Gamma/L$ *be nonempty,* h*-invariant, with only infinite* $\langle h\rangle$*-orbits. Then* Y_α *is the product of the* h*-invariant factors* $\mathcal{B}_P$ *and* $\mathcal{B}_{P^c}$*, and* h *acts on* $\mathcal{B}_P$ *as a Bernoulli shift with nontrivial base, hence ergodically. If all* $\langle h\rangle$*-orbits on* Γ/L *are infinite,* $\mathrm{Ind}(\alpha)(h)$ *is ergodic.*

Proof. Let the cosets rL, $r \in \mathcal{R}$, represent the $\langle h\rangle$-orbits in P. Then $\{h^m r : m \in \mathbb{Z},\ r \in \mathcal{R}\}$, together with a transversal of P^c, is a transversal of Γ/L, and in these coordinates $(h \cdot F)(h^m r) = F(h^{m-1}r)$ is the shift of $(B^{\mathcal{R}})^{\mathbb{Z}}$. □

Lemma 9.4. $\mathrm{Ind}(\alpha)$ *is free and ergodic, its restriction to* Γ_1 *is a generalized Bernoulli action* $\Gamma_1 \curvearrowright (B,\mu)^{\Gamma/L}$ *with infinite* Γ_1*- and* Σ*-orbits on* Γ/L*, and* $M(\alpha)$ *is a* II_1 *factor.*

Proof. Bernoulli. Let $\mathcal{R}$ represent the double cosets $\Gamma_1\backslash\Gamma/L$. By Lemma 9.2(b) every coset is γrL for a unique $(\gamma, r) \in \Gamma_1 \times \mathcal{R}$, so $\phi(\gamma rL) = \gamma r$ is a section with $\phi(\gamma i) = \gamma\phi(i)$ for $\gamma \in \Gamma_1$, and in the coordinates $b_i = F(\phi(i))$ we get $(\gamma \cdot b)_i = b_{\gamma^{-1}i}$. The Γ_1- and Σ-orbits are infinite since $\Gamma_1 \supseteq \Sigma$ acts freely.

Ergodic. By Lemma 9.2(c), u has only infinite orbits on Γ/L, so Lemma 9.3 applies.

Free. Let $\gamma \neq e$; by Lemma 9.2(b) there is γ' with $\gamma^{-1}\gamma' L \neq \gamma' L$. If $\gamma F = F$, then $F(\gamma^{-1}\gamma') = F(\gamma')$; writing $\gamma^{-1}\gamma' = t_1 l_1$, $\gamma' = t_2 l_2$ with $t_1 \neq t_2$ in $\mathcal{T}$, this is $b_{t_2} = \tilde{\alpha}(l_2 l_1^{-1})b_{t_1}$ for two independent coordinates with nonatomic law μ, an event of probability 0. Fact 7.6(a) applies as Y_α is nonatomic. □

Fact 9.5 (Popa–Vaes [PV10, Theorem 6.7(1), p. 26, and Lemma 6.5, p. 25])**.** *Let* Γ_1, Γ_2 *be countable groups with a common infinite amenable subgroup* Σ*, proper and normal in* Γ_2*, such that* $\bigcap_i g_i \Sigma g_i^{-1}$ *is finite for some* $g_1, \ldots, g_k \in \Gamma_1$*, and let* $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$*. Assume* Γ_1 *has a nonamenable normal subgroup* N *with the relative property (T). Let* $\Gamma \curvearrowright (Y,\nu)$ *be a free ergodic pmp action whose restriction to* Γ_1 *is a generalized Bernoulli action* $\Gamma_1 \curvearrowright (Y_0,\nu_0)^I$ *with* $N \cdot i$ *and* $\Sigma \cdot i$ *infinite for all* $i \in I$*. If* $\Lambda \curvearrowright (Y',\nu')$ *is a free ergodic pmp action of a countable group and* $L^\infty(Y) \rtimes \Gamma$ *is isomorphic to the amplification* $(L^\infty(Y') \rtimes \Lambda)^t$ *for some* $t > 0$*, then, among other conclusions,* $\Lambda \curvearrowright Y'$ *is induced from an action of a finite index subgroup* Λ_0 *on* $Y_0' \subseteq Y'$*, and there are a finite normal subgroup* $\Gamma_0 \lhd \Gamma$*, an isomorphism* $\delta : \Gamma/\Gamma_0 \to \Lambda_0$ *and a measure space isomorphism* $\Delta : Y/\Gamma_0 \to Y_0'$ *with* $\Delta(gy) = \delta(g)\Delta(y)$*, and* $t = |\Gamma_0|/[\Lambda : \Lambda_0]$*.*

(By [PV10, Theorem 6.7] "all actions appearing in the theorem satisfy the conclusions of Lemma 6.5", of which we state a part; the proof uses [PV10, Theorem 5.2] and [Pop07a, Theorem 0.1].)

Lemma 9.6. $\mathrm{Ind}(\alpha)$ *is W*-superrigid.*

Proof. We check the hypotheses of Fact 9.5. The group $\Sigma \cong \mathbb{Z}$ is amenable, normal and proper in Γ_2. Let $g_0 \in \Gamma_1$ be the image of the matrix which maps e_1 to e_n, e_n to $-e_1$, and fixes the other vectors e_k of the standard basis; it is orthogonal with determinant 1, and conjugates $1 + E_{1n}$ to $1 - E_{n1}$, so $\Sigma \cap g_0 \Sigma g_0^{-1} = \{e\}$. The group $\mathrm{PSL}(n, \mathbb{Z})$, $n \geq 3$, has property (T), as a quotient of $\mathrm{SL}(n, \mathbb{Z})$, and is infinite, hence nonamenable [BHV08, Chapter 1]; take $N = \Gamma_1$. The Bernoulli condition is Lemma 9.4. Now let $M(\alpha) \cong L^\infty(Y') \rtimes \Lambda$ ($t = 1$). By Lemma 9.2(d), $\Gamma_0 = \{e\}$, so $[\Lambda : \Lambda_0] = 1$, $Y_0' = Y'$, $\delta : \Gamma \to \Lambda$ is an isomorphism and Δ conjugates the actions; $\Delta_* \nu_\alpha$ is an invariant probability measure equivalent to ν', so its density is invariant, hence 1 by ergodicity. □

9.4. **Proof of Theorem F.**

Lemma 9.7.

(a) *For $k \neq 0$, the $\mathrm{Ind}(\alpha)(\zeta^k)$-invariant functions in $L^\infty(Y_\alpha)$ are exactly those in A_α; in particular $\mathrm{Ind}(\alpha)(\zeta^k)$ is not ergodic.*
(b) $\mathrm{Ind}(\alpha)(u^a \zeta^b)$ *is ergodic if $a \neq 0$.*

Proof. (a) By Lemmas 9.2(c) and 9.3 with $P = \Gamma/L \setminus \Gamma_2/L$ (nonempty, as $\Gamma \neq \Gamma_2$), $Y_\alpha = B^\Sigma \times Z$, where the first factor is given by res_α, and ζ^k acts trivially on B^Σ by (9.1) and ergodically on Z. By Fubini an invariant function $f(b, z)$ is, for a.e. b, constant in z. Conversely A_α is invariant, and contains nonconstant functions as μ is nonatomic. (b) Lemmas 9.2(c) and 9.3. □

Lemma 9.8. *Let $\delta \in \mathrm{Aut}(\Gamma)$ with $\delta(L) = L$, and let $T_0 : B \to B'$ be a measure preserving isomorphism with $T_0 \circ \tilde{\alpha}(l) = \tilde{\beta}(\delta(l)) \circ T_0$ for $l \in L$, everywhere (after discarding invariant null sets, which changes Y_α, Y_β only by null sets). Then $\Theta(F) = T_0 \circ F \circ \delta^{-1}$ is a measure preserving isomorphism $Y_\alpha \to Y_\beta$ with $\Theta(\gamma \cdot F) = \delta(\gamma) \cdot \Theta(F)$, and $M(\alpha) \cong M(\beta)$.*

Proof. The map Θ relabels the cosets through δ and applies T_0 to each coordinate. With $m = \delta^{-1}(l)$,

$$\Theta(F)(\gamma l) = T_0 \tilde{\alpha}(m)^{-1} F(\delta^{-1}(\gamma)) = \tilde{\beta}(l)^{-1} \Theta(F)(\gamma), \quad \Theta(\gamma \cdot F)(\gamma') = T_0 F(\delta^{-1}(\delta(\gamma)^{-1} \gamma')),$$

so $\Theta(F) \in Y_\beta$ and $\Theta(\gamma \cdot F) = \delta(\gamma) \cdot \Theta(F)$. In the coordinates of transversals $\mathcal{T}$ and $\delta(\mathcal{T})$, Θ is $(b_t)_t \mapsto (T_0 b_t)_{\delta(t)}$, which is measure preserving. Apply Fact 7.6(b). □

Proof of Theorem F. By Lemma 9.4, $M(\alpha)$ is a II_1 factor; it is separable, that is, has separable predual, because it acts on the separable Hilbert space $L^2(Y_\alpha) \otimes \ell^2(\Gamma)$.

First implication. Let $T_0 \alpha(s) = \beta(\delta_S(s)) T_0$, and let δ extend δ_S as in Lemma 9.2(e). Then $\delta(s\zeta^k) = \delta_S(s)\zeta^k$, so $T_0 \tilde{\alpha}(l) = \tilde{\beta}(\delta(l)) T_0$; apply Lemma 9.8.

Second implication. By Lemmas 9.4 and 9.6 there are $\delta \in \mathrm{Aut}(\Gamma)$ and a measure preserving isomorphism $\Theta : Y_\alpha \to Y_\beta$ with $\Theta(\gamma \cdot F) = \delta(\gamma) \cdot \Theta(F)$; Θ^{-1} does the same with δ^{-1}, so all statements below hold symmetrically.

Step 1: $\delta(\Gamma_2)$ is conjugate to Γ_2. By Lemma 9.7(a), $\mathrm{Ind}(\alpha)(\zeta)$ is not ergodic, hence neither is its conjugate $\mathrm{Ind}(\beta)(\delta(\zeta))$, and by (the contrapositive of) Lemma 9.3 $\delta(\zeta)$ has a finite orbit: some $\lambda = \delta(\zeta)^m$, $m \geq 1$, fixes a coset $\gamma_1 L$ ($\lambda \neq e$ as ζ, hence $\delta(\zeta)$, has infinite order), i.e. $\gamma_1^{-1} \lambda \gamma_1 \in L \setminus \{e\}$. As ζ is central in Γ_2, Lemma 9.2(a) gives $\delta(\Gamma_2) \subseteq Z(\lambda) \subseteq \gamma_1 \Gamma_2 \gamma_1^{-1}$. Symmetrically $\delta^{-1}(\Gamma_2) \subseteq \gamma_2 \Gamma_2 \gamma_2^{-1}$, so with $\gamma_0 = \delta(\gamma_2)$ and $g = \gamma_0 \gamma_1$, $\Gamma_2 \subseteq \gamma_0 \delta(\Gamma_2) \gamma_0^{-1} \subseteq g \Gamma_2 g^{-1}$. If $g \notin \Gamma_2$, Lemma 9.1(a) would give $\Gamma_2 \subseteq \Sigma$, which is absurd; so $g \in \Gamma_2$, the inclusions are equalities, and $\delta(\Gamma_2) = \gamma_0^{-1} \Gamma_2 \gamma_0$. Replacing δ by $\mathrm{Ad}(\gamma_0) \circ \delta$ and Θ by $\gamma_0 \cdot \Theta$, we may assume $\delta(\Gamma_2) = \Gamma_2$.

Step 2: δ on Γ_2. The automorphism δ preserves the center $\Sigma \times \langle \zeta \rangle \cong \mathbb{Z}^2$ of Γ_2 (S is centerless). For $s \in S$ write $\delta(s) = c(s)\delta_S(s)$ with $c(s) \in \Sigma \times \langle \zeta \rangle$ and $\delta_S(s) \in S$; both c and δ_S are homomorphisms,

and c is trivial because S is generated by involutions and $\Sigma \times \langle\zeta\rangle$ is torsion-free. So $\delta(S) \subseteq S$, likewise $\delta^{-1}(S) \subseteq S$, and $\delta_S = \delta|_S \in \mathrm{Aut}(S)$. Write $\delta(\zeta) = u^a\zeta^b$; if $a \neq 0$, $\mathrm{Ind}(\beta)(\delta(\zeta))$ would be ergodic (Lemma 9.7(b)), contrary to Step 1. So $\delta(\zeta) = \zeta^b$, $b \neq 0$.

Step 3: the fixed algebras. A function f is $\mathrm{Ind}(\beta)(\zeta^b)$-invariant if and only if $f \circ \Theta$ is $\mathrm{Ind}(\alpha)(\zeta)$-invariant, so $\{f \circ \Theta : f \in A_\beta\} = A_\alpha$ by Lemma 9.7(a). By von Neumann's point realization theorem the resulting isomorphism $L^\infty(B'^\Sigma) \to L^\infty(B^\Sigma)$ is implemented by a measure preserving isomorphism $\Theta_0 : B^\Sigma \to B'^\Sigma$ with $\mathrm{res}_\beta \circ \Theta = \Theta_0 \circ \mathrm{res}_\alpha$ a.e.

Step 4: conclusion. For $s \in S$, by the equivariance of res_α, res_β and $\delta(s) = \delta_S(s)$, $\Theta_0(\mathrm{res}_\alpha(sF)) = \mathrm{res}_\beta(\delta_S(s)\Theta(F)) = \delta_S(s) \cdot \Theta_0(\mathrm{res}_\alpha(F))$. Since $(\mathrm{res}_\alpha)_*\nu_\alpha = \mu^\Sigma$ and S acts on B^Σ, B'^Σ by $\alpha^{\otimes\mathbb{Z}}$, $\beta^{\otimes\mathbb{Z}}$, this says $\alpha^{\otimes\mathbb{Z}} \simeq_{\mathrm{Aut}} \beta^{\otimes\mathbb{Z}}$. □

In particular $M(\alpha) \cong M(\beta)$ if and only if $\alpha \simeq_{\mathrm{Aut}} \beta$ when $\alpha \cong \alpha^{\otimes\mathbb{Z}}$ and $\beta \cong \beta^{\otimes\mathbb{Z}}$, e.g. for $\alpha = \alpha_0^{\otimes\mathbb{N}}$, $\beta = \beta_0^{\otimes\mathbb{N}}$. We do not know whether the second implication of Theorem F holds with α, β in place of $\alpha^{\otimes\mathbb{Z}}, \beta^{\otimes\mathbb{Z}}$.

9.5. **Amplifications.** The theorem of Popa and Vaes also controls isomorphisms onto amplifications (Section 7.3). We use this for the II_∞ case of Theorem C.

Proposition 9.9. *Let α, β be pmp actions of S on standard nonatomic probability spaces, and let $t > 0$. If $M(\alpha) \cong M(\beta)^t$, then $t = 1$. Consequently $M(\alpha) \mathbin{\bar\otimes} B(\ell^2(\mathbb{N})) \cong M(\beta) \mathbin{\bar\otimes} B(\ell^2(\mathbb{N}))$ if and only if $M(\alpha) \cong M(\beta)$, and the fundamental group of $M(\alpha)$ is trivial.*

Proof. By Lemma 9.4 and the verification in the proof of Lemma 9.6, the action $\mathrm{Ind}(\alpha)$ satisfies the hypotheses of Fact 9.5, and $\mathrm{Ind}(\beta)$ is a free ergodic pmp action of the countable group Γ. Fact 9.5, applied with $\Lambda = \Gamma$, gives a finite normal subgroup $\Gamma_0 \lhd \Gamma$ and a subgroup $\Lambda_0 \leq \Gamma$ of finite index with $t = |\Gamma_0|/[\Gamma : \Lambda_0]$. By Lemma 9.2(d), $\Gamma_0 = \{e\}$, so $t = 1/[\Gamma : \Lambda_0] \leq 1$. Since $M(\beta) \cong M(\alpha)^{1/t}$, the same argument with the roles of α and β exchanged gives $1/t \leq 1$. Hence $t = 1$. The second assertion follows, because an isomorphism of the stabilizations gives $M(\alpha) \cong M(\beta)^t$ for some $t > 0$ (Section 7.3); the third is the case $\beta = \alpha$. □

For the application we use the two-sided form of the proposition: triviality of the fundamental groups alone would not exclude that two different factors $M(\alpha)$, $M(\beta)$ are proper amplifications of one another.

9.6. **Tensor products with amenable factors.** For type III we tensor $M(\alpha)$ with a fixed amenable factor and need to recover $M(\alpha)$ from the tensor product. A II_1 factor M with trace τ and norm $\|x\|_2 = \tau(x^*x)^{1/2}$ is *full* if every bounded sequence (x_k) in M with $\|x_k y - y x_k\|_2 \to 0$ for all $y \in M$ satisfies $\|x_k - \tau(x_k)1\|_2 \to 0$; that is, if M does not have property Γ of Murray and von Neumann [MvN43], see [Con74, Corollary 3.8]. The following argument is the standard spectral gap argument, cf. [Cho82].

Lemma 9.10. *The II_1 factor $M(\alpha)$ is full.*

Proof. Write $M = M(\alpha)$, and let u_g, $g \in \Gamma$, be its canonical unitaries.

Step 1: every $g \in \Gamma \setminus \{e\}$ has infinitely many conjugates hgh^{-1} with $h \in \Gamma_1$. Equivalently, the centralizer of g in Γ_1 has infinite index in Γ_1. Let first $g \in \Gamma_1$, the image of a matrix A, and suppose that this centralizer has finite index. Then for all $i \neq j$ it contains the image of $1 + kE_{ij}$ for some $k > 0$, so $A(1 + kE_{ij})A^{-1} = \pm(1 + kE_{ij})$; the sign is $+$, as one sees by taking traces. Hence A commutes with all matrix units E_{ij}, $i \neq j$, and is a scalar matrix, that is, $g = e$. Let now $g \notin \Gamma_1$. An element $h \in \Gamma_1$ which commutes with g fixes the two distinct vertices Γ_1 and $g\Gamma_1$ of the Bass–Serre tree, hence the first edge $\gamma\Sigma$ ($\gamma \in \Gamma_1$) of the geodesic between them. So the centralizer of g in Γ_1 is contained in the cyclic group $\gamma\Sigma\gamma^{-1}$, which has infinite index in Γ_1, because Γ_1 contains $\mathbb{Z}^2$.

Step 2: the unitary representation $h \mapsto \mathrm{Ad}(u_h)$ of Γ_1 on $L^2(M, \tau)$ has no invariant vectors orthogonal to 1. Every $\xi \in L^2(M, \tau)$ has a Fourier expansion $\xi = \sum_{g\in\Gamma} \xi_g u_g$ with $\xi_g \in L^2(Y_\alpha)$ and

$\|\xi\|_2^2 = \sum_g \|\xi_g\|_2^2$, and $u_h \xi u_h^* = \sum_g (\xi_g \circ h^{-1}) u_{hgh^{-1}}$. If ξ is invariant, then $\|\xi_{hgh^{-1}}\|_2 = \|\xi_g\|_2$ for all $h \in \Gamma_1$, so $\xi_g = 0$ for $g \neq e$ by Step 1, and ξ_e is a Γ_1-invariant function. By Lemma 9.2(c) and Lemma 9.3 the element $u \in \Gamma_1$ acts ergodically on Y_α, so ξ_e is constant.

Step 3: conclusion. The group Γ_1 has property (T) [BHV08]. By Step 2 there are therefore a finite set $K \subseteq \Gamma_1$ and $c > 0$ such that $\max_{h \in K} \|u_h \xi u_h^* - \xi\|_2 \geq c\|\xi\|_2$ for all $\xi \in L^2(M, \tau)$ orthogonal to 1. If (x_k) is a bounded sequence in M with $\|x_k y - y x_k\|_2 \to 0$ for all $y \in M$, we apply this to $\xi = x_k - \tau(x_k)1$ and $y = u_h$, $h \in K$, and obtain $\|x_k - \tau(x_k)1\|_2 \to 0$. □

A factor is *amenable* if it is injective, equivalently hyperfinite. The following fact is a special case of the theorem of Houdayer, Marrakchi and Verraedt on the uniqueness of McDuff decompositions; for II_1 factors it goes back to Popa [Pop07b, Theorem 5.1].

Fact 9.11 ([HMV19, Theorem E and Definition 6.2])**.** *Let M_1, M_2 be full II_1 factors with separable predual, and let P be an amenable factor with separable predual which is not of type* I*. If $M_1 \mathbin{\bar\otimes} P \cong M_2 \mathbin{\bar\otimes} P$, then $M_1 \mathbin{\bar\otimes} B(\ell^2(\mathbb{N})) \cong M_2 \mathbin{\bar\otimes} B(\ell^2(\mathbb{N}))$.*

(In the terminology of [HMV19], the factor $\mathcal{M} = M_2 \mathbin{\bar\otimes} P$ is McDuff, since P absorbs the hyperfinite II_1 factor, and by [HMV19, Theorem E] its decomposition as the tensor product of a factor which is not McDuff and an amenable factor not of type I is unique up to stable unitary conjugacy. So if $\theta : M_1 \mathbin{\bar\otimes} P \to \mathcal{M}$ is an isomorphism, there are type I factors F_1, F_2 with separable predual and a unitary in $\mathcal{M} \mathbin{\bar\otimes} F_1 \mathbin{\bar\otimes} F_2$ which conjugates $\theta(M_1 \otimes 1) \mathbin{\bar\otimes} F_1$ onto $M_2 \mathbin{\bar\otimes} F_1$. Thus $M_1 \mathbin{\bar\otimes} F_1 \cong M_2 \mathbin{\bar\otimes} F_1$, and tensoring with $B(\ell^2(\mathbb{N}))$ gives the statement.)

Proposition 9.12. *Let P be an amenable factor with separable predual which is not of type* I*, and let α, β be pmp actions of S on standard nonatomic probability spaces. Then $M(\alpha) \mathbin{\bar\otimes} P \cong M(\beta) \mathbin{\bar\otimes} P$ if and only if $M(\alpha) \cong M(\beta)$.*

Proof. If $M(\alpha) \mathbin{\bar\otimes} P \cong M(\beta) \mathbin{\bar\otimes} P$, then $M(\alpha) \mathbin{\bar\otimes} B(\ell^2(\mathbb{N})) \cong M(\beta) \mathbin{\bar\otimes} B(\ell^2(\mathbb{N}))$ by Lemma 9.10 and Fact 9.11, and $M(\alpha) \cong M(\beta)$ by Proposition 9.9. The converse is clear. □

10. Borel coding

It remains to see that the factors $M(\mathsf{G}_X)$ can be computed in a Borel way from codes of the spaces X.

10.1. **Codes.** Let $\mathcal{M}_H \subseteq \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ be the set of pseudometrics z on $\mathbb{N}$ satisfying $\sum_{a,b\in A} c_a c_b z_{ab}^2 \leq 0$ for all finite $A \subseteq \mathbb{N}$ and real $(c_a)_{a\in A}$ with $\sum_a c_a = 0$. It is closed, and by Schoenberg [Sch38, Section 3] $z \in \mathcal{M}_H$ if and only if the completion X_z of $(\mathbb{N}, z)$ (modulo $z = 0$) is Hilbertian. By the Kuratowski–Ryll-Nardzewski theorem [Kec95, Theorem 12.13] there are Borel maps $\varsigma_a : F(\mathcal{E}) \setminus \{\varnothing\} \to \mathcal{E}$, $a \in \mathbb{N}$, with $\{\varsigma_a(K)\}_a$ dense in K; then $K \mapsto z_K$, $z_K(a,b) = \|\varsigma_a(K) - \varsigma_b(K)\|$, reduces isometry of nonempty closed subsets of $\mathcal{E}$ to isometry of the spaces X_z on $\mathcal{M}_H$. By Fact 7.3, $E_{\mathcal{U}(\mathcal{H})}$ is Borel reducible to the latter. Fix a surjection $\varpi : \mathbb{N} \to \mathbb{N}$ with infinite fibers, e.g. $\varpi(2^{a-1}(2b-1)) = a$; for $z \in \mathcal{M}_H$, $x_i = [\varpi(i)]$ is a reference sequence of X_z with $d(x_i, x_j) = z_{\varpi(i)\varpi(j)}$, and defines G_{X_z} (trivial, on a nonatomic space, if X_z is a point; Theorem F allows this).

10.2. **A fixed model.** Let J be the countable set of maps $\psi : \mathbb{N} \to \mathbb{N}$ with $\psi(i) = i$ for all but finitely many i, and $q^\psi = (x_{\psi(i)})_i$: finitely many reference points are moved onto other reference points. These are exactly the local sequences with terms in $\{x_j\}$, so the ξ_{q^ψ} are total (Lemma 8.1(b)). The group S acts on J by $s \cdot \psi = \psi \circ s^{-1}$, and $q^{s\cdot\psi} = s\, q^\psi$. The kernel

$$k_z(\psi, \psi') = \langle \xi_{q^\psi}, \xi_{q^{\psi'}} \rangle = \exp\Big(-\sum_i z^2_{\varpi(\psi(i))\,\varpi(\psi'(i))}\Big)$$

is a finite sum in the exponent, hence continuous in z, and S-invariant. Let $\mathbb{P}_z$ be the centered Gaussian measure on $\mathbb{R}^J$ with covariance k_z (possibly degenerate), with S acting by $(s\omega)_\psi = \omega_{s^{-1}\cdot\psi}$.

Then $\xi_{q^\psi} \mapsto \omega_\psi$ makes $(\mathbb{R}^J, \mathbb{P}_z)$ a Gaussian space over $\mathcal{H}_{X_z}$, and since $\omega_\psi \circ s^{-1} = \omega_{s\cdot\psi}$ is the variable of $\pi_{X_z}(s)\xi_{q^\psi}$, the shift is G_{X_z} (Fact 7.4).

Take $B = \mathbb{R}^J$ and α the shift. A function $F : \Gamma \to \mathbb{R}^J$ is a function $f(\psi, \gamma) = F(\gamma)_\psi$ on $J \times \Gamma$, and $F \in Y_\alpha$ means that f is invariant under the free right action $(\psi, \gamma) \cdot s\zeta^k = (s^{-1} \cdot \psi, \gamma s \zeta^k)$ of L. So $Y_\alpha = \mathbb{R}^I$ with $I = (J \times \Gamma)/L$, and Γ acts by the coordinate permutation induced by $\gamma \cdot [\psi, \gamma'] = [\psi, \gamma\gamma']$. This action is the same for all z; only the measure $\nu_z = \nu_\alpha$ depends on z. It is the centered Gaussian measure with covariance $K_z([\psi, \gamma], [\psi', \gamma]) = k_z(\psi, \psi')$ and $K_z([\psi, \gamma], [\psi', \gamma']) = 0$ if $\gamma L \neq \gamma' L$ (well defined by S-invariance of k_z, since $[\psi, \gamma] = [s^{-1} \cdot \psi, \gamma s]$), and $M(\mathsf{G}_{X_z}) = L^\infty(\mathbb{R}^I, \nu_z) \rtimes \Gamma$.

10.3. **The crossed product.** For $a \in \mathbb{Q}^{(I)}$ (finitely supported) let $e_a(\omega) = \exp(\mathrm{i}\langle a, \omega\rangle)$, and let $\mathcal{L}_z = L^2(\mathbb{R}^I, \nu_z) \otimes \ell^2(\Gamma)$.

(i) *The e_a generate $L^\infty(\nu_z)$, and the $e_a \otimes \mathbf{1}_\gamma$ are total in $\mathcal{L}_z$.* The map $t \mapsto (e^{\mathrm{i}qt})_{q\in\mathbb{Q}}$ is a continuous injection of $\mathbb{R}$, so by the Lusin–Souslin theorem each coordinate ω_ι, $\iota \in I$, is a Borel function of the e_a. The linear span of the e_a is a unital *-algebra ($e_a e_b = e_{a+b}$, $\overline{e_a} = e_{-a}$), so it is weakly dense in $L^\infty(\nu_z)$, and hence norm dense in $L^2(\nu_z)$.

(ii) *Gram matrix and generators.* $\langle e_a \otimes \mathbf{1}_\gamma, e_b \otimes \mathbf{1}_{\gamma'}\rangle = [\gamma = \gamma'] \exp(-\frac{1}{2}(b-a)^\mathsf{T} K_z (b-a))$ is continuous in z. The algebra $M(\mathsf{G}_{X_z})$ is generated by the operators $e_b \otimes 1$ and u_g, acting by the fixed formulas $(e_b \otimes 1)(e_a \otimes \mathbf{1}_\gamma) = e_{a+b} \otimes \mathbf{1}_\gamma$ and $u_g(e_a \otimes \mathbf{1}_\gamma) = e_{g\cdot a} \otimes \mathbf{1}_{g\gamma}$, where $(g \cdot a)_\iota = a_{g^{-1}\iota}$.

(iii) *Borel Gram–Schmidt.* Enumerate the $e_a \otimes \mathbf{1}_\gamma$ as $v_1, v_2, \ldots$, and discard v_n on the Borel set of z where the Gram determinant of v_n and the previously kept vectors vanishes. On each Borel piece where the kept indices among $v_1, \ldots, v_n$ are fixed, the Gram–Schmidt vectors $o_k = \sum_m c_{k,m}(z) v_m$ have coefficients continuous in finitely many Gram entries. As $\dim \mathcal{L}_z = \infty$, the unitary operator $V_z : o_k \mapsto \mathbf{b}_k$, with $(\mathbf{b}_k)$ the standard basis of $\ell^2(\mathbb{N})$, identifies $\mathcal{L}_z$ with $\ell^2(\mathbb{N})$.

(iv) *Effros measurability.* For a generator x (one of the operators $e_b \otimes 1$, u_g), $\langle \mathbf{b}_k, V_z x V_z^* \mathbf{b}_l\rangle = \sum_{m,m'} \overline{c_{k,m}} c_{l,m'} \langle v_m, x v_{m'}\rangle$ is a finite sum of Borel functions, since $x v_{m'}$ is again in the family. Let $f_n(z)$ enumerate the finite products (including the empty product 1) of these operators and their adjoints and their $(\mathbb{Q} + \mathrm{i}\mathbb{Q})$-linear combinations. Their matrix coefficients are Borel (expand products through the basis), and $\|f_n(z)\| \le r_n$ independently of z; as the weak topology on the ball of radius r_n is compact metrizable and generated by matrix coefficients, each f_n is Borel into it. So is $z \mapsto \|f_n(z)\| = \sup_{k,l} |\langle \eta_k, f_n(z)\eta_l\rangle|$, where (η_k) enumerates the finitely supported vectors of norm ≤ 1 with coordinates in $\mathbb{Q} + \mathrm{i}\mathbb{Q}$. Let A_z be the unital *-algebra generated by the operators $V_z x V_z^*$; the $f_n(z)$ are norm dense in A_z. By the bicommutant theorem A_z is weakly dense in $M_z := V_z M(\mathsf{G}_{X_z}) V_z^*$, and by Kaplansky's density theorem [Kap51] (see also [KR83, Theorem 5.3.5]) $(A_z)_1$ is weakly dense in $(M_z)_1$.

Proposition 10.1. *The map $z \mapsto M_z$ from $\mathcal{M}_H$ to $\mathrm{vN}(\ell^2(\mathbb{N}))$ is Borel, and $M_z \cong M(\mathsf{G}_{X_z})$ is a II_1 factor.*

Proof. Let O be weakly open. We claim $\{z : (M_z)_1 \cap O \neq \varnothing\} = \bigcup_n \{z : \|f_n(z)\| < 1,\ f_n(z) \in O\}$, which is Borel by (iv). The inclusion $\supseteq$ is clear. Conversely, if $(M_z)_1 \cap O \neq \varnothing$, then by (iv) there is $a \in (A_z)_1 \cap O$. Since O is also norm open, there are $t \in (0,1)$ and $\varepsilon > 0$ such that O contains the norm ball of radius ε about $(1-t)a$. Choose n with $\|f_n(z) - (1-t)a\| < \min(t, \varepsilon)$; then $f_n(z) \in O$ and $\|f_n(z)\| < (1-t) + t = 1$. The second claim is Theorem F. □

Corollary 10.2. *Let N be a von Neumann algebra on $\ell^2(\mathbb{N})$. The map $z \mapsto M_z \mathbin{\bar\otimes} N$ from $\mathcal{M}_H$ to $\mathrm{vN}(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}))$ is Borel. For $N = B(\ell^2(\mathbb{N}))$ its values are II_∞ factors.*

Proof. The unit ball of N is separable and metrizable in the weak topology, so there is a countable unital *-algebra N_0 over $\mathbb{Q} + \mathrm{i}\mathbb{Q}$ which is weakly dense in N; for $N = B(\ell^2(\mathbb{N}))$ one can take the $(\mathbb{Q}+\mathrm{i}\mathbb{Q})$-span of 1 and the matrix units. Let B_z be the linear span of the operators $a \otimes b$ with $a \in A_z$, $b \in N_0$. It is a unital *-algebra whose weak closure contains $M_z \otimes 1$ and $1 \otimes N$, and is therefore $M_z \mathbin{\bar\otimes} N$. Let $g_m(z)$ enumerate the $(\mathbb{Q} + \mathrm{i}\mathbb{Q})$-linear combinations of the operators $f_n(z) \otimes b$, $b \in N_0$.

They are norm dense in B_z, are bounded independently of z, and have Borel matrix coefficients, hence Borel norms as in (iv). By Kaplansky's density theorem the unit ball of B_z is weakly dense in that of $M_z \mathbin{\bar\otimes} N$, and the proof of Proposition 10.1 applies verbatim with the g_m in place of the f_n. (Alternatively, $M \mapsto M \mathbin{\bar\otimes} N$ is Borel by [ST09a, Lemma 10].) For $N = B(\ell^2(\mathbb{N}))$ the values are II_∞ factors because the M_z are II_1 factors. □

11. Proofs of Theorem C and Corollary D, and further consequences

Proof of Theorem C. Upper bound (Sasyk and Törnquist [ST09a, Theorem 16(i)], see also [TL13, Theorem 4.10]). The map $M \mapsto M \otimes 1 \in \mathrm{vN}(\mathcal{H} \otimes \mathcal{H})$ is Borel (by [Eff65, Theorem 2 and its Corollary], as $x \mapsto x \otimes 1$ is isometric and weakly continuous on bounded sets), and reduces isomorphism on all of $\mathrm{vN}(\mathcal{H})$ to unitary conjugacy, as $M \otimes 1$ has properly infinite commutant and isomorphisms of von Neumann algebras with properly infinite commutants are spatial; unitary conjugacy is the orbit equivalence relation of a Borel action of $\mathcal{U}(\mathcal{H} \otimes \mathcal{H}) \cong \mathcal{U}(\mathcal{H})$ (both facts are due to Effros [Eff66], see also [ST09a, proof of Corollary 15]).

Lower bound. By Section 10.1, $E_{\mathcal{U}(\mathcal{H})}$ is Borel reducible to isometry of the spaces X_z on $\mathcal{M}_H$. Writing $\cong$ also for the isometry of metric spaces, we have by Theorems E and F

$$X_z \cong X_{z'} \Rightarrow \mathsf{G}_{X_z} \simeq_{\mathrm{Aut}} \mathsf{G}_{X_{z'}} \Rightarrow M(\mathsf{G}_{X_z}) \cong M(\mathsf{G}_{X_{z'}}) \Rightarrow \mathsf{G}_{X_z}^{\otimes\mathbb{Z}} \simeq_{\mathrm{Aut}} \mathsf{G}_{X_{z'}}^{\otimes\mathbb{Z}} \Rightarrow X_z \cong X_{z'},$$

and $z \mapsto M_z$ is Borel by Proposition 10.1. So $E_{\mathcal{U}(\mathcal{H})}$ is Borel reducible to isomorphism of separable II_1 factors. As the II_1 factors form a Borel subset of the factors and of $\mathrm{vN}(\mathcal{H})$ (Section 7.1), the lower bound passes to both.

II_∞ factors. The upper bound is the restriction of the upper bound for von Neumann algebras to the Borel set of II_∞ factors (Section 7.1). For the lower bound, fix a unitary operator from $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ onto $\ell^2(\mathbb{N})$. By Corollary 10.2, $z \mapsto M_z \mathbin{\bar\otimes} B(\ell^2(\mathbb{N}))$ is then a Borel map from $\mathcal{M}_H$ into the II_∞ factors on $\ell^2(\mathbb{N})$. If X_z and $X_{z'}$ are isometric, then $M_z \cong M_{z'}$ by the implications displayed above, so the stabilizations are isomorphic. Conversely, if the stabilizations of M_z and $M_{z'}$ are isomorphic, then $M_z \cong M_{z'}$ by Proposition 9.9, and X_z, $X_{z'}$ are isometric, again by the implications displayed above.

Factors of type III_λ. Fix $\lambda \in [0,1]$ and an amenable factor P_λ of type III_λ on $\ell^2(\mathbb{N})$: for instance the Powers factor if $0 < \lambda < 1$, the Araki–Woods factor of type III_1 if $\lambda = 1$, and a Krieger factor of type III_0 if $\lambda = 0$. If M is a II_1 factor, then $S(M \mathbin{\bar\otimes} P_\lambda) = S(P_\lambda)$ for the invariant S of Connes [Con73, Corollaire 3.2.8], so $M \mathbin{\bar\otimes} P_\lambda$ is a factor of type III_λ. The upper bound is again the restriction of the upper bound for von Neumann algebras, now to the Borel set $\mathcal{F}_{\mathrm{III}_\lambda}(\ell^2(\mathbb{N}))$ ([HW98], see Lemma 7.2). For the lower bound, by Corollary 10.2 the map $z \mapsto M_z \mathbin{\bar\otimes} P_\lambda$, composed with the fixed unitary from $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ onto $\ell^2(\mathbb{N})$, is a Borel map from $\mathcal{M}_H$ into the factors of type III_λ on $\ell^2(\mathbb{N})$. By Proposition 9.12, $M_z \mathbin{\bar\otimes} P_\lambda \cong M_{z'} \mathbin{\bar\otimes} P_\lambda$ if and only if $M_z \cong M_{z'}$, that is, if and only if X_z and $X_{z'}$ are isometric. □

Proof of Corollary D. By Corollary B and the definition of $E_{\mathcal{U}(\mathcal{H})}$, every E_X^G is Borel reducible to $E_{\mathcal{U}(\mathcal{H})}$, which by Theorem C is Borel bireducible with each of the isomorphism relations in question. These are therefore complete for orbit equivalence relations, $E_{\mathcal{U}(\mathcal{H})}$ being itself an orbit equivalence relation. □

Proof of Corollary G. (a) Isomorphism of separable C^*-algebras, linear isometry of separable Banach spaces, isometry of Polish metric spaces and homeomorphism of compact metric spaces are complete for orbit equivalence relations, by [Sab16] (with [EFP+13]), [Mel07], [CGK01] (see also [GK03, Cle12]) and [Zie16] respectively. So are the isomorphism relations of Corollary D, and any two relations which are complete for orbit equivalence relations are Borel bireducible.

(b) Let F be the relation of isomorphism of separable Banach spaces, on the standard Borel space of closed linear subspaces of $C[0,1]$. By [FLR09], every analytic equivalence relation on a standard Borel space is Borel reducible to F. The orbit equivalence relation of a Borel action of a Polish

group G on a standard Borel space X is analytic, being the image of the Borel map $(g,x)\mapsto(x,g\cdot x)$ on $G\times X$. Hence $E_{\mathcal{U}(\mathcal{H})}\le_B F$, and by (a) and Theorem C every relation in (a) is Borel reducible to F. Conversely, suppose that F is Borel reducible to one of the relations in (a), hence to $E_{\mathcal{U}(\mathcal{H})}$. Let E_1 be the equivalence relation on $\mathbb{R}^{\mathbb{N}}$ in which two sequences are equivalent if they agree in all but finitely many terms. It is Borel, so $E_1\le_B F\le_B E_{\mathcal{U}(\mathcal{H})}$. This contradicts the theorem of Kechris and Louveau [KL97, Section 4] that E_1 is not Borel reducible to the orbit equivalence relation of any Borel action of a Polish group. □

11.1. **Actions and representations.** For a countable group Λ, $A(\Lambda,Y,\nu)$ is the Polish space of pmp actions of Λ on (Y,ν). As Gao's reduction takes infinite values (Section 7.1), $E_{\mathcal{U}(\mathcal{H})}$ is Borel reducible to isometry of the spaces X_z on the open set $\mathcal{M}_H^{(2)}\subseteq\mathcal{M}_H$ of codes of spaces with at least two points.

Lemma 11.1.

(a) *If X has at least two points, $\mathcal{H}_X$ is infinite-dimensional.*

(b) *There are Borel maps $z\mapsto\pi_z$ from $\mathcal{M}_H^{(2)}$ to the orthogonal representations of S on $\ell^2_{\mathbb{R}}(\mathbb{N})$, and $z\mapsto a_z\in A(S,\mathbb{R}^{\mathbb{N}},\mathsf{n}^{\mathbb{N}})$, $\mathsf{n}=N(0,1)$, with π_z equivalent to π_{X_z} and a_z conjugate to G_{X_z}.*

Proof. (a) If $\dim\mathcal{H}_X<\infty$, then $\mathcal{V}_m=\mathcal{H}_X$ for some m (Lemma 8.3(a) with $\mathcal{K}=\mathbb{R}$), so by Lemma 8.3(b) the orthogonal operators $\pi_X(\tau_{ij})$, $i,j>m$, equal $c_{ij}\cdot 1$; hence $c_{ij}=1$, i.e. $x_i=x_j$ for all $i,j>m$, and $X=\overline{\{x_i:i>m\}}$ is a point.

(b) Borel Gram–Schmidt (item (iii) of Section 10.3) applied to the ξ_{q^ψ}, $\psi\in J$, with Gram matrix k_z, gives by (a) orthogonal operators $U_z:\mathcal{H}_{X_z}\to\ell^2_{\mathbb{R}}(\mathbb{N})$ such that $\pi_z(s)=U_z\pi_{X_z}(s)U_z^{-1}$ has Borel matrix coefficients. Let $a_z(s)=T_{\pi_z(s)}$ on the Gaussian space $(\mathbb{R}^{\mathbb{N}},\mathsf{n}^{\mathbb{N}})$ with coordinate variables. By Lemma 7.5, applied to the identity representation of the orthogonal group of $\ell^2_{\mathbb{R}}(\mathbb{N})$, the Koopman operator of T_O on mean-zero functions is $V^*\mathcal{F}(O)V$, which depends continuously on the orthogonal operator O in the strong operator topology; as the topology of $\mathrm{Aut}(\mathbb{R}^{\mathbb{N}},\mathsf{n}^{\mathbb{N}})$ is induced by the Koopman representation, $z\mapsto a_z$ is Borel, and T_{U_z} conjugates G_{X_z} to a_z. □

Corollary 11.2. *Each of the following relations is Borel above $E_{\mathcal{U}(\mathcal{H})}$; by Corollary B it is Borel bireducible with $E_{\mathcal{U}(\mathcal{H})}$, hence complete for orbit equivalence relations:*

(a) *conjugacy up to $\mathrm{Aut}(S)$ of pmp actions of S on a standard nonatomic probability space;*

(b) *unitary equivalence up to $\mathrm{Aut}(S)$ of unitary representations of S on $\mathcal{H}$ ($\pi\sim\pi'$ if $\pi'(\delta(s))=U\pi(s)U^*$ for some $U\in\mathcal{U}(\mathcal{H})$, $\delta\in\mathrm{Aut}(S)$ and all s), and its orthogonal analogue.*

Proof. Upper bounds (using Corollary B). By the isomorphism theorem for measures [Kec95, Theorem 17.41], all standard nonatomic probability spaces are isomorphic, so in (a) we may take $(Y,\nu)=(\mathbb{R}^{\mathbb{N}},\mathsf{n}^{\mathbb{N}})$. By Lemma 7.7, these are the orbit equivalence relations of the continuous actions of $\mathrm{Aut}(Y,\nu)\times\mathrm{Sym}(\mathbb{N})$ on $A(S,Y,\nu)$ and of $\mathcal{U}(\mathcal{H})\times\mathrm{Sym}(\mathbb{N})$ on the representations of S, by $((T,\theta)\cdot a)(s)=T\,a(\theta^{-1}s\theta)\,T^{-1}$ and $((U,\theta)\cdot\pi)(s)=U\,\pi(\theta^{-1}s\theta)\,U^*$; apply Corollary B.

Lower bounds. (a) $z\mapsto a_z$ is a reduction by Theorem E. (b) By Corollary 8.10, X_z and $X_{z'}$ are isometric if and only if $\pi_z\sim\pi_{z'}$, if and only if their complexifications are unitarily equivalent up to an automorphism of S; so $z\mapsto\pi_z$ is a reduction for the orthogonal relation, and $z\mapsto(\pi_z)_{\mathbb{C}}$ for the unitary one. □

Remark 11.3 (Limits of the method). Our reduction is detected by Koopman representations up to the uncountable group $\mathrm{Aut}(S)$ (Proposition 8.9). Up to a countable group of automorphisms this is impossible: for a countable group Λ and a countable subgroup $\Lambda_1\le\mathrm{Aut}(\Lambda)$, the relation "$\kappa_a\cong\kappa_b\circ\delta$ for some $\delta\in\Lambda_1$" between the Koopman representations of pmp actions a,b of Λ is Borel [HT12], whereas $E_{\mathcal{U}(\mathcal{H})}$ is not, being $\ge_B$ isomorphism of II_1 factors, which is not Borel [ST09a, Corollary 15]; so the method says nothing about Sabok's conjecture [BCF+23, Conjecture 4.1].

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