No holeless tetrad of four congruent pentagons

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Abstract

A tetrad is a set of four plane regions with disjoint interiors, every two of which share a boundary segment of positive length; it is holeless if the complement of their union is connected. Holeless tetrads of four congruent hexagons appear in Martin Gardner's account of the puzzle, one of them found by Scott Kim, and Juris Čerņenoks and Andrejs Cibulis list the least number of sides of such a polygon as an open problem. Four congruent pentagons can form a tetrad around a hole. Walter Trump asked in 2020 whether they can form one without a hole, judging it extremely unlikely, and showed that convex tiles cannot form a tetrad at all. We prove that no four congruent simple pentagons, convex or not and mirror images allowed, form a holeless tetrad, and that triangles and quadrilaterals cannot either; so six is the least number of sides. The proof reduces every such tetrad to one of three plane maps and finitely many corner placements, decides the linear conditions on angles and side lengths in exact arithmetic, and excludes each of the 116 remaining two-parameter families by an exact certificate that the prototile cannot close up as a simple pentagon. A separately written verifier checks the certificates.

This work has not yet been reviewed by experts. The proofs, the checking programs and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

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1 Introduction

A *tetrad* is a set of four closed regions in the plane with pairwise disjoint interiors such that every two of them share a boundary segment of positive length. A tetrad is *holeless* if the complement of the union of its four regions is connected. All polygons below are simple: a *polygon with n sides* is a closed region bounded by a simple closed polygonal curve with exactly n corners, none of angle 180°. Two polygons are *congruent* if an isometry of the plane, possibly orientation-reversing, maps one onto the other.

Theorem 1. *There are no four congruent polygons with five sides that form a holeless tetrad.*

The tiles in Theorem 1 need not be convex, and mirror images are allowed. The same proof, applied to three and four sides, gives the following.

Corollary 2. *No four congruent triangles or quadrilaterals form a holeless tetrad. Hence six is the least number of sides of a polygon whose congruent copies form a holeless tetrad.*

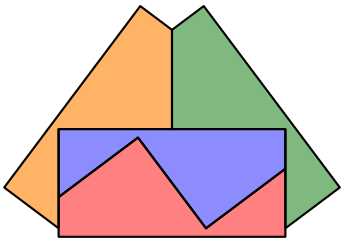

Figure 1: A holeless tetrad of four congruent hexagons of the type found by Scott Kim, with rational corner coordinates. The tile in the middle (blue) touches each of the other three along a segment.

The second sentence uses Scott Kim's holeless tetrad of four congruent hexagons [8]; Figure 1 shows a tetrad of that type with rational coordinates. A pentagon whose corners include straight angles is a triangle or a quadrilateral, so Theorem 1 and Corollary 2 together also exclude pentagons in that weaker sense.

History. The Wikipedia article on the puzzle [8] records, citing Gardner [3], that the tetrad was named by Michael R. W. Buckley in 1975 in the *Journal of Recreational Mathematics*, that whether the four regions can be congruent, with or without holes, became a puzzle, and that Scott Kim found a holeless solution made of hexagons. Čerņenoks and Cibulis [1, Section 6, Problem P1] write: "In the book (Gardner, 1989) two full tetrads made of 6-gons are given, and the question what the smallest n-gon is like has been asked" (a *full* tetrad is a holeless one). Pentagons can form a tetrad around a hole: Walter Trump [5] shows five such tetrads, the oldest sent to Gardner in 1979 and one by Livio Zucca. On whether pentagons can do without the hole, Trump writes [5]: "It is extremely unlikely that such a tetrad without hole exists, but I have not seen a proof." His answer on hexagons is stated conditionally on "the conjecture that holeless pentagons cannot form a tetrad" [6]. David Eppstein asked the same question in 2021 ("Is the hole necessary?") [2], and the Wikipedia article lists it as unknown [8]. Trump also shows that convex tiles cannot form a tetrad, even with holes [7], and states that four quadrilaterals cannot form a tetrad, asking for a short proof [4]. Theorem 1 proves Trump's conjecture, and Corollary 2 answers the question of the smallest n for holeless tetrads of congruent polygons. For quadrilaterals our argument covers only holeless tetrads, so it does not settle Trump's statement about quadrilaterals in general.

What is new. The convex case is settled by Trump's argument [7]. Without convexity a tile may have reflex corners, a corner of one tile may lie inside a side of another, and two tiles may meet along several segments. The first step of our proof bounds how the tiles fit together, by Euler's formula. The resistant step is the last one: in the 116 families of angle and length data that satisfy every linear condition, the difficulty lies at degenerate points on the boundary of the parameter region, where many of the relevant trigonometric polynomials vanish at once, some to high order. Interval arithmetic alone cannot give a strict sign on a cell containing such a point; we use exact higher-order Taylor bounds and resultants there.

The proof depends on a computation, described in Section 6; the reduction to that computation (Sections 3 and 4) is a written argument.

2 Proof overview

Fix a holeless tetrad of four congruent polygons with n sides, copies of a prototile P. The union of the tile boundaries is a connected plane graph whose faces are the four tile interiors and one outer face. After removing vertices of degree two, Euler's formula leaves a plane map with at most six vertices and nine edges in which every two tiles share an edge. There are exactly three such maps (Lemma 4). For each map we enumerate how the corners of each tile sit on it and which corner of P each one is. The angles of P must then satisfy linear equations (the angles at a vertex sum to 360°) and strict inequalities, and the side lengths must satisfy linear equations (a side of one tile is covered by segments shared with others). Deciding these systems exactly leaves, for $n = 5$, 116 cases and none for $n = 3, 4$ (Lemma 5).

In each surviving case the angles form a two-parameter family, an open polygon R in the plane of two free angles (p, q), and the admissible side lengths form the relative interior of a polyhedral cone with at most three generators. The remaining condition is that P closes up, a pair of trigonometric equations, and is simple. Relabeling P maps the 116 cases onto one another in 22 classes. For one case in each class a certificate subdivides R into cells and shows on each cell, by exact and interval arithmetic, that no admissible lengths close P into a simple pentagon (Lemma 6).

3 The boundary map

Let $T_1, \dots, T_4$ be the tiles of a holeless tetrad of congruent polygons with n sides, and let N be the set of all corners of all tiles. Cut every side of every tile at the points of N on it; the resulting segments are the *pieces*.

Lemma 3. *Two pieces of different tiles are either equal or meet only at common endpoints. The pieces form a connected plane graph G with vertex set N, whose faces are the four tile interiors and one outer face. At each vertex x of G, each tile containing x occupies exactly one of the sectors cut out by the edges at x, with angle equal to its corner angle at x or to 180°; the sector angles at x sum to 360°, and at most one tile sector at a vertex of degree at least three has angle 180°.*

Proof. Let s and t be pieces of T_i and T_j, $i \neq j$, sharing a point x that is not an endpoint of both. If they are collinear and overlap, both are maximal segments between consecutive points of N on a common line, so they coincide. If x is interior to both and they cross, then near x the interior of T_i is an open half-disk on one side of s, while the interior of T_j has points on both sides of s near x; the interiors would meet. If x is an endpoint of t and interior to s, then $x \in N$ and s would have been cut at x. So G is a plane graph. It is connected because each tile boundary is connected and every two boundaries meet.

The interior of each T_i is connected and open, misses every other boundary (a point of it on ∂T_j would have interior points of T_j nearby), and has its frontier in G; so it is a face. Every other point off G lies outside the union U of the tiles, so the remaining faces are the components of $\mathbb{R}^2 \setminus U$: one, since the tetrad is holeless. Near a vertex x on ∂T_i, the tile T_i is a single closed angular sector, and no edge of G enters its interior; so it is one sector of G at x, of angle equal to its corner angle or 180°. Two straight tile sectors would fill 360° and leave only two sectors. □

A vertex of G of degree two is a corner of some tile; we call it a *bend*. Its two sectors are not both outer, since every piece has a tile on one side. If both are tile sectors, the bend is *interior*: the two angles sum to 360° and neither is 180°, so it is a corner of both tiles. If one sector is outer, the bend is *exterior*: it is a corner of the other tile, and the outer sector has positive angle.

Suppress every bend, interior or exterior. The result is a connected plane multigraph M with the same five faces and all degrees at least three. Each edge of M carries the bends suppressed on it: interior bends on an edge between two tiles, exterior bends on an edge between a tile and the outer face. Each edge of M has two different faces on its sides, at least one a tile; each tile boundary is a closed walk through each vertex at most once; and every two tiles share an edge, since a common boundary segment of positive length contains an interior point of a common piece. With V, E and $F = 5$ the numbers of vertices, edges and faces, Euler's formula gives $\sum_v(\deg v - 2) = 2E - 2V = 2(F - 2) = 6$, so $V \le 6$ and $E \le 9$.

Lemma 4. *Up to homeomorphism of the plane, M is one of three maps:*

(W) one tile is surrounded by the other three, each of which meets the outer face along an edge;

(R) two tiles meet the outer face and share two edges, enclosing the other two tiles between them;

(R_0) as (R), except that one of the two shared edges shrinks to a vertex at which one enclosed tile touches the outer face.

Proof. This is a finite enumeration. The plane dual D of M has the five faces as vertices; it is connected, loopless and plane; every two tile vertices are adjacent; and every face of D (a vertex of M) has length at least three and contains each tile at most once. Enumerating all rotation systems on at most nine edges satisfying these conditions gives exactly three maps up to isomorphism, including mirror images. A separate enumeration of the primal maps M directly, with $V \le 6$ vertices of degree at least three and the conditions above, gives the same three, with degree sequences $3, 3, 3, 3, 4$ for (R_0) and $3, 3, 3, 3, 3, 3$ for (R) and (W). □

4 Angle and length conditions

Let $a_0, \dots, a_{n-1}$ be the angles of P and $\ell_0, \dots, \ell_{n-1}$ its side lengths, side i joining corners i and $i + 1$ in counterclockwise order (indices mod n). A *placement* records, for each tile, which of its sectors at vertices of M are corners and which are straight, and how many bends lie on each edge of M, so that each tile has n corners; an interior bend counts as a corner of both its tiles, and an exterior bend as a corner of its one tile. A *labeling* identifies the corners of each tile with those of P: if the tile is a direct copy of P, corner k of the tile is corner $s + k$ of P and the side after it is side $s + k$; if the tile is a mirror copy, corner k is corner $s - k$ and the side after it is side $s - k - 1$. Applying an isometry to the whole tetrad, we may take the tile on a fixed face of the representative map to be P itself, so the labelings of the other three tiles range over the $2n$ identifications each. Finally a *reflex pattern* says which a_i exceed $180°$.

Each tetrad yields a map, a placement, a labeling and a reflex pattern for which the following hold:

- *angles*: $\sum_i a_i = (n - 2) \cdot 180°$; the tile sectors at a vertex of M sum to $360°$ when no sector is outer and to less than $360°$ otherwise; the two corners at an interior bend sum to $360°$, while an exterior bend imposes no equation; each a_i lies in $(0°, 180°)$ or $(180°, 360°)$ as the pattern says;

- *lengths*: on an edge of M with b bends, interior or exterior, there are $b+1$ segments of positive length, one between each two consecutive vertices of G on it; each side of each tile is a run of such segments through vertices where the tile is straight, and its length is the sum of theirs and equals ℓ_i for its label i.

Both systems are linear, with strict inequalities, and share no variables.

Lemma 5. *For $n = 3$ and $n = 4$ no map, placement, labeling and pattern has a feasible angle system. For $n = 5$ there are* 54,938 *placements; exactly* 2,850 *of their labeled, patterned versions have a feasible angle system, and exactly* 116 *of these also have a feasible length system.*

Proof. A depth-first search assigns labelings tile by tile, pruning by interval bounds on the angles, and decides each complete system by linear programming in exact rational arithmetic: it maximizes a common slack t in all strict inequalities and accepts if and only if the optimum is positive. Appendix A lists the counts by map. A separate implementation, written from a specification of the conditions without access to the search and using exact polyhedra, reproduces every count by map for $n = 3, 4, 5$. □

For each of the 116 cases the angle equalities have rank $n - 2 = 3$, so the angles form a two-parameter family $a_i = A_i p + B_i q + C_i$ with integers A_i, B_i, C_i and two free angles p, q; the strict inequalities cut out an open convex polygon R in the (p, q)-plane. The admissible length vectors ℓ are the relative interior of the projection L of the cone of nonnegative solutions of the length equalities, since a strictly positive solution exists. In every case L is generated by $m \le 3$ integer rays $r_1, \dots, r_m$, so the admissible ℓ are the sums $\sum_j \mu_j r_j$ with all $\mu_j > 0$.

5 Closure and simplicity

Let $\theta_0 = 0$ and $\theta_{i+1} = \theta_i + 180^\circ - a_{i+1}$ be the directions of the sides of P, and $u_i = \cos\theta_i + \sqrt{-1}\sin\theta_i$. The prototile closes up if and only if $\sum_i \ell_i u_i = 0$, that is, $\sum_j \mu_j v_j = 0$ with $v_j = \sum_i r_{j,i} u_i$. It is simple if and only if no two non-adjacent sides meet. With $Z = \cos\frac{p}{2} + \sqrt{-1}\sin\frac{p}{2}$ and $W = \cos\frac{q}{2} + \sqrt{-1}\sin\frac{q}{2}$, each u_i is $\pm Z^{2s} W^{2t}$, where s and t are the integer coefficients of p and q in θ_i, so every quantity in these conditions is a Laurent polynomial in Z and W restricted to the torus $|Z| = |W| = 1$.

Closure with all $\mu_j > 0$ splits into branches. For $m = 1$ it needs $v_1 = 0$. For $m = 2$ there are two branches: the degenerate one, $v_1 = v_2 = 0$, on which every $\mu > 0$ closes P; and the generic one, v_1 and v_2 nonzero with $v_1 \times v_2 = 0$ and $v_1 \cdot v_2 < 0$, on which $\mu = (-v_1 \cdot v_2,\ |v_1|^2)$ up to a positive factor. For $m = 3$ write $c_{12} = v_1 \times v_2$, $c_{23} = v_2 \times v_3$ and $c_{31} = v_3 \times v_1$. On the generic branch the three are all positive or all negative, and $\mu = \sigma(c_{23}, c_{31}, c_{12})$ up to a positive factor, σ being their common sign. On the degenerate branch all three vanish, so v_1, v_2, v_3 are collinear and μ is not determined. Only on the generic branches (and for $m = 1$, where μ is a single positive number) is μ substituted into the conditions that sides i and j meet, which then become sign conditions on finitely many trigonometric polynomials in p, q. A degenerate branch is excluded from necessary conditions alone, without choosing μ: it requires two of the closure quantities ($|v_1|^2$ and $|v_2|^2$ for $m = 2$; two of the cross products for $m = 3$) to vanish at a common point.

Lemma 6. *For each of the 116 cases, no $(p, q) \in R$ and $\mu > 0$ make P a closed simple pentagon.*

Proof. Relabeling the corners of P (rotating or reflecting its labels) maps cases to cases and preserves closure and simplicity. Under it the 116 cases fall into 22 classes, and for every case an explicit relabeling onto a chosen member of its class is checked to carry its angle family, region and length cone exactly onto that member's. It is enough to treat the chosen members.

For 21 classes the chosen member has a certificate. It lists, for every closure and simplicity quantity Q, an identity $Q = K \prod_k g_k^{e_k}$ on the torus with K a nonzero real constant and each g_k a real Laurent polynomial with Gaussian rational coefficients or a product $f f^* = |f|^2$; and a

subdivision of a box containing R into cells, each with a reason why no sign pattern of the factors that is possible on the cell satisfies the closure branch and simplicity. Appendix B lists the sign rules, including how degenerate closure branches are excluded.

The remaining class has $m = 1$ and is closed by an identity. Its angles are $a = (180 - p - q,\ 180 + q,\ 180 - q,\ p,\ q)$ with $0 < p + q < 180$, and its ray is $r = (1, 2, 1, 2, 1)$. A direct computation gives

$$\operatorname{Re}(ZW\, v_1) = 3 \cos \frac{\pi(p+q)}{360},$$

which is positive on R, so $v_1 \neq 0$ and P cannot close. □

Proof of Theorem 1 and Corollary 2. A holeless tetrad of four congruent polygons with n sides gives, by Lemmas 3 and 4 and Section 4, one of the enumerated maps, placements, labelings and patterns, with angles and lengths satisfying the linear systems, and with P closed and simple. For $n = 3, 4$ Lemma 5 leaves no case. For $n = 5$ it leaves the 116 cases, and Lemma 6 excludes each. With Kim's hexagonal tetrad (Figure 1), six is the least number of sides. □

6 Computational verification

The computer checks Lemmas 4, 5 and 6 and the relabeling reduction; Lemma 3 and the translation into placements and linear systems are the written argument above. The certificates are checked by a verifier written separately from the programs that produced them, from a specification of the conditions alone, with its own interval arithmetic (outward-rounded intervals of the `mpmath` library rather than the ball arithmetic used to produce them). It rederives each case's angle family, region and length cone, checks every factorization identity symbolically, replays the subdivision, and rejects corrupted certificates. The identity for the remaining class is checked by two separate symbolic computations.

With the accompanying code and data, checking the map enumerations, the cover and every certificate takes about three minutes on two cores; rerunning the exact angle screen takes about five CPU-hours. Appendix B also describes tests of the reduction.

Appendix A Counts

	(R_0)	(R)	(W)
placements, $n = 3$	284	190	722
placements, $n = 4$	1,986	2,250	7,698
placements, $n = 5$	7,078	12,040	35,820
angle-feasible, $n = 5$	60	36	2,754
also length-feasible, $n = 5$	4	4	108

Table 1: Placements and feasible labeled cases by map. For $n = 3, 4$ no case is angle-feasible. The 116 length-feasible cases form 22 classes under relabeling of P: four of size two and eighteen of size six.

Appendix B Sign rules and tests

The certificates of Lemma 6 decide signs on a cell as follows. The sign of a factor $\rho\cos(k_p p+k_q q+\delta)$ on a cell follows from the exact range of its phase; other factors are bounded by interval arithmetic. Near a point c where several factors vanish (a vertex of R or a point inside it), the certificate uses polar cells around c and the exact order k of vanishing of each factor at c (all partial derivatives of order below k vanish, tested in the cyclotomic field) together with the Taylor remainder of order $k+1$. On a degenerate closure branch, and wherever else two factors would have to vanish at a common point of a cell, either one of them has a strict sign on the cell or the resultants in W and in Z show that they have no common zero there.

As a test of the reduction, the same programs applied to hexagons find holeless tetrads of congruent hexagons, one of Kim's type, without being given an example; and drawings of Kim's tetrad and of several pentagon tetrads with a hole are found by the placement and labeling search with their actual labeling.

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