Counts of maximal matchings in a graph need not be log-concave

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Abstract

The numbers of inclusion-maximal matchings of each size in a graph need not form a log-concave sequence. We give an explicit counterexample on 76 vertices: the join of six disjoint edges with two disjoint cliques of order 32. Its maximal matchings have sizes 35 through 38, and the square of the coefficient at size 36 is smaller than the product of its two neighbors. This answers the general log-concavity question posed by Došlić and Short and by Došlić and Zubac. The proof uses the independent uncovered-set viewpoint discussed by Došlić and Short, retaining the number of perfect matchings for each uncovered set. For the stated graph, this reduces the coefficient calculation to a short recurrence on residual graphs. A second computation evaluates an explicit factorial sum, and both use exact integer arithmetic. The result concerns log-concavity for arbitrary graphs; the displayed coefficient sequence is unimodal.

This work has not yet been reviewed by experts. The proofs, the code and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

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1 Introduction

Write K_t for the complete graph on t vertices, $\sqcup$ for disjoint union, and $X \vee Y$ for the join, which adds every edge between the vertices of X and Y. A matching is maximal if no edge can be added to it. Let $m_k(G)$ count the maximal matchings of G with k edges.

Theorem 1. *For the graph $G = (6K_2) \vee (K_{32} \sqcup K_{32})$, the sequence $(m_k(G))_{k=0}^{38}$ is not log-concave. More precisely,*

$$m_{36}(G)^2 < m_{35}(G)m_{37}(G).$$

The order 76 is not claimed to be minimal.

For example, the four-vertex path has one maximal matching of size one and one of size two, so its maximal matching polynomial is $x + x^2$. In general this polynomial is $\sum_k m_k(G)x^k$. A nonnegative sequence (a_k) is log-concave when $a_k^2 \geq a_{k-1}a_{k+1}$ at every interior index. Maximality refers to inclusion, whereas a maximum matching has the largest possible size.

For ordinary matchings, log-concavity follows from the Heilmann–Lieb root theorem: the monomer–dimer partition function has its "zeros on the imaginary axis" [3, abstract]. Equivalently, the generating polynomial by matching size has real negative roots, so Newton's inequalities give log-concavity. Burnham and Erey showed that "maximal matching polynomials can have non-real roots" [4, abstract]. Thus real-rootedness cannot establish log-concavity in general for maximal matchings; non-real roots alone do not refute log-concavity.

After discussing log-concavity for ordinary matching polynomials, Došlić and Short ask, "Is this valid also for maximal matching polynomials?" [1, p. 20 of the preprint]. Došlić and Zubac similarly ask whether "their coefficients form log-concave or unimodal sequences for all graphs" [2, p. 275]. Theorem 1 gives a negative answer to the log-concavity question. The four positive coefficients below increase through size 37 and then decrease, so the unimodality question is unaffected by this example. The proof depends on the exact computations described in Section 4.

The key simplification is to fix the uncovered vertices of a matching. The discussion of nice independent sets in [1, p. 13 of the preprint] motivates this viewpoint. Each uncovered set must be weighted by the number of perfect matchings of its complement. In our graph the join restricts the possible uncovered sets, and deleting one leaves a graph described by four small integers. We calculate the weights by a pivot recurrence and check them separately by a factorial sum. The uncovered-set identity in Lemma 2 is elementary. Our contribution is the explicit graph and the exact coefficient comparison that disproves log-concavity.

2 Counting by uncovered vertices

Let $\mathrm{pm}(H)$ be the number of perfect matchings of a finite graph H, with $\mathrm{pm}(\varnothing) = 1$.

Lemma 2. *If H is a graph on n vertices, then*

$$m_k(H) = \sum_{\substack{U \subseteq V(H) \text{ independent} \\ |U| = n-2k}} \mathrm{pm}(H - U).$$

Proof. A matching is maximal exactly when there is no edge between two uncovered vertices. Its uncovered set U is therefore independent, and its edges form a perfect matching of $H - U$. Conversely, every perfect matching of $H - U$ for independent U is maximal in H. The uncovered set of a matching is unique, so these contributions are disjoint. □

Partition $V(G)$ into A, B, C, where $G[A] = 6K_2$ and B, C are the two 32-vertex cliques. Figure 1 displays the construction. An independent set lies entirely in A or entirely in $B \cup C$, because every vertex of A is adjacent to every vertex of $B \cup C$. Within A, an independent set of size u chooses one endpoint from each of u distinct edges, giving $\binom{6}{u}2^u$ choices for $0 \le u \le 6$. Within $B \cup C$, a nonempty independent set is a singleton (64 choices) or a pair with one vertex in each clique (1024 choices). The empty set is counted with the subsets of A. Thus this description contains $3^6 + 64 + 1024 = 1817$ independent sets, including those of odd size.

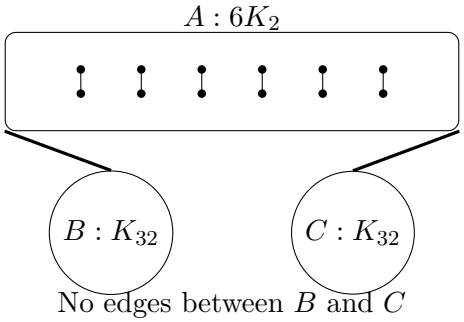

Figure 1: The graph G. Each thick line denotes all edges between the two indicated parts. The circles denote complete graphs.

Let $H(p,q,c,d)$ count perfect matchings in the join of p disjoint edges and q isolated vertices with $K_c \sqcup K_d$. Define

$$D(t) = \begin{cases} t!/(2^{t/2}(t/2)!), & t \text{ even}, \\ 0, & t \text{ odd}. \end{cases}$$

Here all parameters are nonnegative integers.

Lemma 3. *The values of H satisfy $H(0,0,c,d) = D(c)D(d)$ and*

$$H(p,q,c,d) = H(p-1,q,c,d) + cH(p-1,q+1,c-1,d) + dH(p-1,q+1,c,d-1) \qquad (p>0), \tag{1}$$

$$H(0,q,c,d) = cH(0,q-1,c-1,d) + dH(0,q-1,c,d-1) \qquad (q>0). \tag{2}$$

Terms with multiplier zero are omitted. States of odd total order have value zero.

Proof. When $p > 0$, fix one endpoint of an intact edge. Its partner is either the other endpoint, one of the c vertices of the first clique, or one of the d vertices of the second clique. A partner in a clique leaves the other endpoint isolated in the first part. These disjoint choices give (1). When $p = 0 < q$, fix an isolated vertex of that part; its partner must lie in a clique. When the first part is empty, the two cliques are matched separately. A clique of even order t has $D(t)$ perfect matchings, obtained by ordering its vertices and dividing by the orders of the pairs and of their list. An odd number of vertices cannot be perfectly matched. □

3 The coefficient comparison

Proof of Theorem 1. An uncovered independent set in A of size u leaves $6-u$ intact edges and u isolated vertices in that part. Odd u contributes zero. The nonempty even independent sets outside A delete one vertex from each clique. Lemma 2 therefore gives

$$\begin{aligned} m_{35} &= 64H(0,6,32,32), \\ m_{36} &= 240H(2,4,32,32), \\ m_{37} &= 60H(4,2,32,32) + 1024H(6,0,31,31), \\ m_{38} &= H(6,0,32,32). \end{aligned}$$

Every other coefficient is zero, since an independent set has at most six vertices. Evaluating Lemma 3 in exact integers gives Table 1. Squaring the entry for 36 and multiplying the entries for 35 and 37 gives the strict inequality in the theorem. The ratio is

$$\frac{m_{36}^2}{m_{35}m_{37}} \approx 0.3098,$$

so the failure has a substantial margin. Both integer operations, as well as all four entries, are reproduced by the supplied computation. □

For a separate evaluation, write $(a)_j = a!/(a-j)!$ for $0 \le j \le a$. After choosing s of the p intact edges to belong to a perfect matching, exactly $t = 2p+q-2s$ vertices of the first part must be paired into the cliques. Choosing which ℓ go to the first clique and injecting them into the two cliques gives

$$H(p,q,c,d) = \sum_{s=0}^{p} \binom{p}{s} \sum_{\ell=\max(0,t-d)}^{\min(c,t)} \binom{t}{\ell} (c)_\ell (d)_{t-\ell} D(c-\ell)D(d-t+\ell), \qquad t = 2p+q-2s. \tag{3}$$

k	$m_k(G)$
35	22987373923658324122444493100869760000000000
36	10255434507320368171719093557648511240000000000
37	1476626532666126480980309672739646583672700000000
38	5459186978290764902678436212139255344183789062 5

Table 1: All nonzero coefficients of the maximal matching polynomial of G.

The residual clique vertices are matched internally. Each perfect matching determines the chosen internal edges, the two injections, and the residual clique matchings uniquely. Conversely, each such choice yields a perfect matching. This bijection proves the formula.

4 Computational verification

The certificate specifies the labeled graph and the complete coefficient sequence, including zero entries. Vertices 0 through 11 form the six edges $\{2i, 2i+1\}$ for $0 \le i < 6$; vertices 12 through 43 form B; and vertices 44 through 75 form C. The computation compares every adjacency bit with this definition.

Two counting methods reproduce every coefficient. One uses the independent-set multiplicities above and (3). The other enumerates the actual labeled independent sets, extracts the residual parameters from the adjacency data, and applies Lemma 3. Their full independent-set counts agree, including 428 odd sets with zero contributions. An altered edge fails the adjacency check before either counting method runs; an altered coefficient or odd-set count is rejected by each counting method.

The same methods agree with a generic computation on labeled vertex subsets when both cliques have order two or four. This generic computation uses Lemma 2 and checks the residual reductions and Lemma 3; the uncovered-set identity itself is justified by its written proof. It also reproduces x for K_2, $3x^2$ for K_4, $6x^2$ for $K_{2,3}$, and $x + x^2$ for the four-vertex path. The complete check takes less than a second on a typical contemporary CPU. The accompanying README gives the command and requirements. The proof relies on the written combinatorial reductions and on the execution of Python's integer arithmetic; it is not formalized in a proof assistant.

References

[1] T. Došlić and T. Short, *Maximal matchings in polyspiro and benzenoid chains*, Applicable Analysis and Discrete Mathematics **15** (2021), 179–200. doi:10.2298/AADM161106003D. Preprint arXiv:1511.00590v1, November 2, 2015.

[2] T. Došlić and I. Zubac, *Counting maximal matchings in linear polymers*, Ars Mathematica Contemporanea **11** (2016), 255–276. doi:10.26493/1855-3974.851.167. Published online November 30, 2015.

[3] O. J. Heilmann and E. H. Lieb, *Theory of monomer-dimer systems*, Communications in Mathematical Physics **25** (1972), 190–232. doi:10.1007/BF01877590.

[4] M. Burnham and A. Erey, *On the roots of maximal matching polynomials*, Discrete Mathematics **348** (2025), 114547. doi:10.1016/j.disc.2025.114547.