Graceful 4-coloring of planar graphs is NP-complete

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Abstract

A graceful k-coloring of a graph is a proper vertex coloring with colors $1, \dots, k$ such that, at every vertex, the incident edges receive distinct differences of their endpoint colors. Cyriac Antony, Laavanya D. and Devi Yamini S. proved that deciding graceful k-colorability is NP-complete for planar graphs when $k \geq 5$, and for $k = 4$ on graphs of maximum degree three, and they left open the complexity of graceful 4-colorability of planar graphs. We prove that this problem is NP-complete, even for planar graphs of maximum degree three (inputs, as in their formulation, may be disconnected). The hardness comes from the reduction of Édouard Bonnet, Dibyayan Chakraborty and Julien Duron, who proved that deciding whether a cubic planar graph has a perfect matching cut (a perfect matching that is exactly the set of edges leaving some vertex set) is NP-hard; we apply their construction to every instance. What we add is an exact correspondence: for every finite cubic graph H, the graph obtained by subdividing each edge of H twice has a graceful 4-coloring if and only if H has a perfect matching cut. This equivalence is proved in Lean from the definitions.

This work has not yet been reviewed by experts. The proofs, the Lean formalization and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

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1 Introduction

All graphs are finite and simple. For a graph G and a map $f\colon V(G) \to \{1,2,3,4\}$, the *difference* of an edge uv is $|f(u) - f(v)|$. The map f is a *graceful 4-coloring* if adjacent vertices receive distinct colors and any two edges sharing an endpoint have distinct differences. Thus a vertex of color 1 with neighbors colored $2, 3, 4$ is fine, while a vertex of color 2 cannot have three neighbors, because only the differences 1 and 2 are available to it. Colors need not be distinct on all of $V(G)$, and not all four need be used. The problem Graceful 4-Colorability asks whether a given graph has a graceful 4-coloring. Graceful colorings were introduced by Bi, Byers, English, Laforge and Zhang [4]. The definition above is the informal definition of Antony, Laavanya and Devi Yamini: a proper coloring whose edge differences form a proper edge coloring [2, Section 1]. Their formal definition can be stronger on graphs with triangles, but the two agree on the triangle-free graphs used here, so Theorem 1 holds for both (Remark 7).

A *perfect matching cut* of a graph H is a perfect matching M of H for which there is a set $A \subseteq V(H)$ such that M is exactly the set of edges with one endpoint in A and one outside A.

Bonnet, Chakraborty and Duron call this "a perfect matching that is also a cutset" [5, Section 2]. The *double subdivision* $S_2(H)$ replaces each edge uv of H by a path $u\,x\,y\,v$ through two new vertices.

1.1 Main results

Theorem 1. Graceful 4-Colorability *is NP-complete on planar graphs. It remains NP-complete on planar graphs of maximum degree at most three.*

We use the construction of Bonnet, Chakraborty and Duron [5] without their splitting step, which is a Turing reduction. This keeps our reduction many-one, but the graphs it produces may be disconnected. This matches the problem as posed, whose input is an arbitrary graph; the connected case is not treated. The proof rests on the following exact equivalence, which holds for all cubic graphs, planar or not.

Theorem 2. *Let H be a finite cubic graph. Then $S_2(H)$ has a graceful 4-coloring if and only if H has a perfect matching cut.*

For example, K_4 has no perfect matching cut (every edge cut of K_4 has zero, three or four edges, while a perfect matching has two), so $S_2(K_4)$ has no graceful 4-coloring. The triangular prism (two triangles joined by three edges) has one, those three joining edges, so its double subdivision has a graceful 4-coloring.

1.2 Where the question comes from

Antony, Laavanya and Devi Yamini proved that "Graceful 5-colorablity [sic] is NP-complete for 3-regular 3-connected planar bipartite graphs" and that "For $k \geq 6$, Graceful k-colorablity is NP-complete for planar bipartite graphs of maximum degree $k-2$" [2, Theorems 4 and 5]. For four colors they proved that "Graceful 4-colorablity is NP-complete for the class of 2-degenerate graphs of maximum degree 3" [2, Theorem 6], by a reduction from Positive Not-All-Equal 3-Sat E4 [6]; the graphs this reduction produces need not be planar. Their abstract ends: "The complexity of checking whether a planar graph is graceful 4-colorable remains open." The same sentence closes the abstract of the published conference version by Cyriac Antony, Jacob Antony, Laavanya D. and Devi Yamini S. [1]. Theorem 1 answers this question.

Pantoja, Dantas and Luiz also proved hardness results for graceful colorings of subcubic graphs; neither concerns planar graphs with four colors. Pantoja's thesis, which presents their three papers [11, 12, 13] among its main contributions [14], attributes to [12] the statement "The G5CP is NP-complete even when restricted to 3-connected bipartite cubic planar graphs" [14, Theorem 10], where G5CP is the graceful 5-coloring problem, and to [13] the statement "The Graceful 4-Coloring problem is NP-complete, even for graphs with maximum degree 3" [14, Theorem 34], together with a polynomial-time algorithm for trees of maximum degree three [14, Theorem 33]. Other recent work on graceful colorings concerns trees [3, 15], corona graphs [7] and a class of bipartite graphs built by an operation called unfolding [8].

1.3 What is new

The input we build on is the Bonnet–Chakraborty–Duron theorem [5, Theorem 1]: "Perfect Matching Cut is NP-hard in 3-connected cubic bipartite planar graphs." Their reduction supplies all of the hardness. Our contribution is Theorem 2, the exact correspondence between graceful 4-colorings of $S_2(H)$ and perfect matching cuts of H. Double subdivision keeps planarity and keeps the maximum degree at three, so the hardness transfers. The step that needs care is the forward direction: a

graceful 4-coloring of $S_2(H)$ must be shown to produce a perfect matching that is an entire cut and not merely contained in one. This is a counting argument on the differences at the original vertices (Section 2).

We use their construction without the extra hypothesis they make for 3-connectivity, which we do not need (Section 3).

2 Proof of Theorem 2

Let H be cubic and let f be a graceful 4-coloring of $S_2(H)$. Call the vertices of H *original*. For an original vertex u and an edge uv of H, the *port* of u toward v is the edge from u to the first new vertex on the path replacing uv, and its difference is the port difference.

Lemma 3. *In a graceful 4-coloring, a vertex with three neighbors has color 1 or 4, its three incident edges have differences $1, 2, 3$, and its neighbors receive the three other colors.*

Proof. A vertex of color c has at most $|\{|c-d| : d \neq c,\ 1 \leq d \leq 4\}|$ distinct positive differences available. This is 3 for $c \in \{1, 4\}$ and 2 for $c \in \{2, 3\}$. Three neighbors need three distinct differences, so $c \in \{1, 4\}$, and then the differences $1, 2, 3$ are realized by the three other colors. □

The first claim of Lemma 3 is the case $\Delta = 3$ of [14, Lemma 6], attributed there to [13]; the same counting appears in [2, proof of Construction 1].

Lemma 4. *Let $u\,x\,y\,v$ be a path whose interior vertices have degree two, colored by a graceful 4-coloring with $f(u), f(v) \in \{1, 4\}$. Up to the symmetry $c \mapsto 5 - c$ and reversing the path, the possibilities are exactly those in Table 1.*

type	$f(u), f(v)$	$f(x), f(y)$	differences along the path	port differences
I	$1, 1$	$2, 4$	$1, 2, 3$	1 and 3
II	$1, 1$	$3, 4$	$2, 1, 3$	2 and 3
III	$1, 4$	$3, 2$	$2, 1, 2$	2 and 2

Table 1: Graceful colorings of a twice subdivided edge with extreme endpoint colors.

Proof. At x the two differences $|f(u) - f(x)|$ and $|f(x) - f(y)|$ must differ, and similarly at y, and consecutive colors must differ. If $f(u) = f(v) = 1$, then $f(x), f(y) \in \{2, 3, 4\}$ are distinct; $(2, 3)$ fails at x (differences $1, 1$) and $(3, 2)$ fails at y, leaving $(2, 4)$, $(3, 4)$ and their reversals. If $f(u) = 1$ and $f(v) = 4$, then $f(x) = 4$ would give y two neighbors of color 4, and $f(y) = 1$ is excluded in the same way, so $f(x), f(y) \in \{2, 3\}$ are distinct; $(2, 3)$ gives differences $1, 1, 1$, and $(3, 2)$ remains. □

Proof of Theorem 2, forward direction. By Lemma 3 every original vertex has color 1 or 4 and exactly one port of each difference $1, 2, 3$. By Lemma 4, a port of difference 1 lies on a path of type I in Table 1, whose other port has difference 3. This pairs each of the $|V(H)|$ ports of difference 1 with a distinct port of difference 3. There are also exactly $|V(H)|$ ports of difference 3, so all of them are used by paths of type I, and no path of type II occurs. Hence every port of difference 2 lies on a path of type III. Let M be the set of edges of H whose paths are of type III. Each original vertex has exactly one port of difference 2, so M is a perfect matching. Let A be the set of original vertices of color 1. Paths of types I and II join endpoints of equal color and paths of type III join endpoints of different colors, so M is exactly the set of edges leaving A. □

Converse direction. Let M be a perfect matching of H equal to the set of edges leaving A. Removing M leaves a graph in which every vertex has degree two, a disjoint union of cycles, and each cycle lies inside A or inside its complement. Orient each cycle cyclically. Give the vertices of A color 1 and the others color 4. Color the path replacing a cycle edge $u \to v$ by $1, 4, 2, 1$ inside A and $4, 1, 3, 4$ outside A; color the path replacing a matching edge from $u \in A$ to $v \notin A$ by $1, 3, 2, 4$. Along these paths the differences are $3, 2, 1$ and $2, 1, 2$, so each new vertex sees two distinct differences, and consecutive colors differ. At an original vertex the outgoing cycle port has difference 3, the incoming cycle port 1 and the matching port 2. This is a graceful 4-coloring of $S_2(H)$. □

3 Proof of Theorem 1

Membership in NP. A certificate is a color in $\{1, 2, 3, 4\}$ for each vertex. Checking that adjacent vertices have distinct colors and that edges at each vertex have distinct differences takes time polynomial in the size of the graph.

Double subdivision.

Lemma 5. *Let H be a graph with n vertices and m edges.*

(a) If H is planar, then $S_2(H)$ is planar.

(b) $S_2(H)$ has $n + 2m$ vertices and $3m$ edges, every new vertex has degree two, and every original vertex keeps its degree. From adjacency lists, $S_2(H)$ is computed in linear time.

Proof. For (a), draw H in the plane, choose two interior points on each edge arc and declare them the new vertices. Part (b) is immediate from the construction. □

The cited hardness theorem. We use [5, Theorem 1] through its proof, which is a reduction from Monotone Not-All-Equal 3SAT-E4. An instance is a set of clauses, each consisting of three distinct variables, with each variable in exactly four clauses; it is *positive* if some truth assignment gives every clause at least one true and at least one false variable. Bonnet, Chakraborty and Duron cite its NP-completeness from Darmann and Döcker [6] [5, Section 3.2]. For an instance I they construct "in polynomial time" a graph $G(I)$ [5, Section 3.2]. The proof of [5, Lemma 13] shows that $G(I)$ is cubic, planar and bipartite, and [5, Lemmas 26 and 28] show that $G(I)$ has a perfect matching cut if and only if I is positive. These two lemmas rest on the preceding results of [5, Section 3]; of these, [5, Lemma 7] is stated for cubic bipartite graphs and [5, Lemma 3] for plane graphs, so the equivalence uses the planarity and bipartiteness of $G(I)$ from the proof of [5, Lemma 13]. Bonnet, Chakraborty and Duron also assume that no variable is a cut vertex of the variable–clause incidence graph of I, justified by a splitting step that they call "a polynomial-time Turing reduction" [5, Section 3.2]. That hypothesis is used only in the proof of [5, Lemma 12], that $G(I)$ is 3-connected, which the proof of their Lemma 13 quotes for 3-connectivity alone. The cubicity, planarity and bipartiteness arguments do not cite their Lemma 12, and no other lemma of theirs uses the hypothesis. The bipartiteness argument shows that every bounded face has even length, counting along exposed paths of the gadgets. The proof of their Lemma 12 records two facts that do not depend on the hypothesis: every gadget is 2-connected, and adjacent gadgets are joined by two disjoint connector edges. Removing one vertex therefore leaves every gadget connected and keeps a connector edge between any two adjacent gadgets, so each component of $G(I)$ is 2-connected, the faces of its drawing are bounded by cycles, and that count shows that each component, and so $G(I)$, is bipartite. (Numbers of lemmas in [5] refer to arXiv:2302.11667v1.)

Proposition 6 (from the proof of [5, Theorem 1])**.** *For every instance J of Monotone Not-All-Equal 3SAT-E4, the graph $G(J)$ is cubic and planar, it is computed from J in polynomial time, and it has a perfect matching cut if and only if J is positive.*

Proof. The construction and its running time are in [5, Section 3.2], cubicity and planarity are shown in the proof of [5, Lemma 13], and the equivalence is [5, Lemmas 26 and 28], which use the bipartiteness shown in the same proof. □

Proof of Theorem 1. Given J, compute $G(J)$ by Proposition 6 and output $S_2(G(J))$. Since $G(J)$ is cubic, Proposition 6 and Theorem 2 give

$$\begin{aligned} J \text{ is positive} &\iff G(J) \text{ has a perfect matching cut} \\ &\iff S_2(G(J)) \text{ has a graceful 4-coloring.} \end{aligned}$$

By Lemma 5, $S_2(G(J))$ is planar, has maximum degree three and is computed in polynomial time. Together with membership in NP, this proves both statements of Theorem 1. □

Remark 7. Antony, Laavanya and Devi Yamini also give a formal version of the definition: the coloring is injective on each closed neighborhood $N[v]$, and the difference labeling is injective on all edges of $G[N[v]]$ [2, Section 1]. On graphs with triangles it can be stronger than the definition used here, since it also separates two edges inside $N[v]$ that share no endpoint. On triangle-free graphs the two agree: the edges of $G[N[v]]$ are then the edges at v, and distinct differences at v force distinct colors on its neighbors. Every double subdivision is triangle-free, so the reduction above proves NP-hardness for both versions. Membership in NP holds for both, because each condition is checked on the closed neighborhoods in polynomial time. Hence Theorem 1 holds under either of their formulations.

4 Formal verification

The Lean project accompanying this paper uses Lean 4 [10] and Mathlib [9]. Its main theorem is

```
theorem Antony24.graceful4_s2_iff_pmc {V : Type*} [Finite V]
    (G : SimpleGraph V) (hG : Cubic G) :
    Graceful4 (S2 G) <-> PMC G
```

Here `SimpleGraph V` is Mathlib's simple graph on the vertex type `V`; the only assumption on `V` is finiteness, so empty and disconnected graphs are included. `Cubic G` says each vertex has exactly three distinct neighbors. `Graceful4 G` says some map to a four-element palette is proper on edges and gives distinct differences to two distinct neighbors of each vertex; a proved lemma shows this is the same as labels $1, \dots, 4$. `S2 G` has vertex type `V` plus the ordered pairs (u, v) of adjacent vertices, with the edge uv becoming the path $u\,(u,v)\,(v,u)\,v$. `PMC G` says some two-sided split of the vertices gives each vertex exactly one neighbor on the other side; a proved lemma shows this is a Mathlib perfect matching equal to the whole cut of a vertex set. So the theorem is Theorem 2, and the project also states it with these unfolded forms on both sides.

The project also proves the counts and degrees of Lemma 5(b) and that every double subdivision is triangle-free, as used in Remark 7; the theorem names are listed in the project's README. The build checks that the main theorems depend only on the axioms `propext`, `Classical.choice` and `Quot.sound`, and checks the two examples of Section 1 by kernel evaluation.

Outside Lean are [5, Theorem 1] and its proof, including its use of [6]; Proposition 6; the planarity and running-time parts of Lemma 5; membership in NP; and the notion of NP-completeness itself.

To reproduce the check, install Lean through elan and run `lake exe cache get` and `lake build` in the `lean` directory. With the downloaded Mathlib cache the build takes under a minute and about 5 GB of memory.

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