Competitive equilibrium can fail near equal budgets for three additive agents

Peter Graham
https://orcid.org/0009-0007-8493-1505

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Abstract

In a Fisher market with indivisible goods, each agent has a budget of artificial money and a competitive equilibrium is an allocation of all the goods with prices at which every agent gets a most preferred bundle within budget. With exactly equal budgets one good and two agents already have no equilibrium. Babaioff, Nisan and Talgam-Cohen proved that two agents with additive preferences over any number of goods have a competitive equilibrium whenever their budgets are sufficiently close to equal but not equal, and asked whether this extends to more than two agents (their Open Problem 1). It does not. We give three agents with strict additive valuations over five goods such that for every $0 < \delta < 1/17$ no competitive equilibrium exists at the budgets $(1-3\delta, 1+2\delta, 1+\delta)$, nor at any budget vector near them. So every neighborhood of equal budgets contains a set of positive measure on which no equilibrium exists. Adding goods of small value, as in Segal-Halevi's Lemma 5, extends this to every number of goods $m \geq 5$. This answers the case of three additive agents and $m \geq 5$ goods that Segal-Halevi left open: the budgets with no competitive equilibrium can have positive measure. The proof gives, for each allocation of the goods, an exact Farkas (linear-programming duality) certificate that no prices support it; an independent exact linear-programming computation confirms it.

This work has not yet been reviewed by experts. The proofs, the computer checks and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

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1 Introduction

1.1 Main result

A set M of m indivisible goods is to be divided among n agents. Agent i has an additive valuation V_i: a positive value $V_i(g)$ for each good g, and $V_i(S) = \sum_{g \in S} V_i(g)$ for each bundle $S \subseteq M$. Agent i strictly prefers S to T when $V_i(S) > V_i(T)$, and the preferences are *strict* when the 2^m bundle values of each agent are pairwise distinct. Each agent i has a budget $t_i > 0$. An *allocation* $X = (X_1, \ldots, X_n)$ is a partition of M into n bundles, some possibly empty; prices $p_g > 0$ extend to bundles by $p(S) = \sum_{g \in S} p_g$.

Definition 1.1. A *competitive equilibrium* (CE) at budgets $t = (t_1, \ldots, t_n)$ is a pair (X, p) of an allocation and prices such that for every agent i: (1) $p(X_i) \le t_i$, and (2) $p(Y) > t_i$ for every bundle Y with $V_i(Y) > V_i(X_i)$.

This is Segal-Halevi's Definition 1 [7], specialized to additive preferences; Babaioff, Nisan and Talgam-Cohen [1, Definition 2.1] use the same conditions with nonnegative prices. The theorem below holds under either convention: the proof rules out even nonnegative prices with weak inequalities in (2) (Lemma 2.1). Let $e = (1, 1, 1)$ denote equal budgets.

	v	w	x	y	z
agent 1	6	12	11	9	13
agent 2	14	4	1	8	20
agent 3	8	6	9	16	12

Table 1: The values $V_i(g)$ of the three agents for the five goods.

Theorem 1.2. *Consider three agents with the additive valuations of Table 1 over the goods $M = \{v, w, x, y, z\}$; their preferences are strict. There is an open set $U \subset \mathbb{R}^3_{>0}$, closed under multiplication by positive scalars, such that no CE exists at any budget vector in U, and U contains*

$$e + \delta(-3, 2, 1) = (1 - 3\delta,\ 1 + 2\delta,\ 1 + \delta) \qquad \text{for every } 0 < \delta < 1/17.$$

Consequently every neighborhood of e, and of each positive multiple of e, contains a set of budget vectors of positive Lebesgue measure at which no CE exists.

For example, no CE exists at the budgets $(97, 102, 101)$ or $(17, 22, 21)$, the multiples of $e + \delta(-3, 2, 1)$ for $\delta = 1/100$ and $\delta = 1/20$. The theorem does not say that every budget vector near e lacks a CE: near e, the set U is exactly the set of budgets with $t_1 < t_3 < t_2$ and $t_1 + t_2 < 2t_3$ (Remark 3.1).

Corollary 1.3. *For every $m \ge 5$ there are three agents with strict additive valuations over m goods and a nonempty open set of budget vectors, closed under positive scaling and with e in its closure, at which no CE exists.*

1.2 The two questions

Competitive equilibrium from equal incomes is a classical rule for fair and efficient division [5, 8]. With indivisible goods it can fail even for one good and two agents, since whoever doesn't get the good can afford it. Babaioff, Nisan and Talgam-Cohen credit Budish [4] with initiating "the study of competitive equilibrium with almost-equal budgets", with "existence results for an approximate such equilibrium notion", and "take the idea of almost-equal budgets a step further", to exact equilibria with generic budgets [1, Section 1]. Their main result is [1, Theorem 1.1]: "Consider 2 agents with additive preferences over multiple items. If the budgets are sufficiently close to being equal, but not precisely so, then a CE exists" (in full, [1, Theorem 7.1]). They then pose [1, Open Problem 1]:

> Consider $n > 2$ agents with additive preferences over multiple items. Does there exist a CE when budgets are sufficiently close to being equal?

Open Problem 1 appears from the second arXiv version of their paper (September 25, 2018). The first (March 23, 2017) had asked more broadly [2, Section 1.4]: "Does there always exist a CE for any number of players with additive preferences and generic budgets?" Segal-Halevi's Theorem 5 [7] answers that question negatively with four agents, and Theorem 1.2 does so with three. Babaioff, Nisan and Talgam-Cohen note that Segal-Halevi's four-agent result "does not rule out that for more than 2 agents, equilibrium exists for generic budgets that are almost equal" [1, Section 1], where generic existence means existence "for every vector of budgets except for a zero-measure subset" [1, Section 2.2.2]. They also report a computer search: on instances from the Spliddit website with additive preferences and more than two agents, "after perturbing the budgets to make them almost equal, a CE was found" [1, Section 9.2 and Appendix B.2]. The example here is not one of those instances.

Open Problem 1 can be read as asking for a CE at every unequal budget vector near e, or at almost every one (generic existence). Theorem 1.2 answers both readings negatively: for the three agents of Table 1 there is no $\varepsilon > 0$ such that a CE exists at almost all budget vectors within ε of e.

Segal-Halevi's Definition 3 [7] reads: "Given integers $n \geq 2$ and $m \geq 1$, a CE exists for almost all incomes if the set of income-vectors for which no CE exists has a zero measure", where by Definition 2 a CE exists for an income vector t if one exists "for all preference-profiles" of the class. For additive preferences Segal-Halevi proved that three agents and four goods have a CE for almost all incomes [7, Theorem 4], and that with four or more agents and four or more goods there are additive preferences and a positive-measure set of incomes without a CE [7, Theorem 5]. For monotone, non-additive preferences, two agents and five goods may have no CE on a positive-measure set of incomes [3, Theorem 3.5], [7, Theorem 6]. Section 10.1 of [7] states: "for $n = 2$ or $n = 3$ agents with additive preferences, when there are $m \geq 5$ goods, it is open whether CE exists for almost all incomes." Wikipedia's list of unsolved problems in fair division records as an open conjecture that "When there are three agents, even with additive valuations, CE for almost all incomes might not exist" [9]. Corollary 1.3 settles the case $n = 3$: the answer is no for every $m \geq 5$, as that conjecture expects (Corollary 4.1(b)). With the results of [7] for $m \leq 4$ [7, table in Section 1, Theorem 4], this completes the case of three additive agents, and $m = 5$ is the least number of goods for which a CE can fail on a positive-measure set of incomes.

What this paper adds is the example and its proof. The definitions, the questions and the lifting argument for more goods are those of Babaioff, Nisan and Talgam-Cohen and of Segal-Halevi. The difficult step was finding an additive example: additivity rules out the preferences used for monotone valuations (Segal-Halevi explains why the profile of [7, Theorem 6] is not additive [7, Section 10.1]), and the example here was found by computer search. This paper does not address two additive agents with $m \geq 5$ goods, which Segal-Halevi lists as open, or Open Problem 1 for a fixed number of agents $n \geq 4$.

1.3 Proof overview

At a fixed budget vector, deciding whether a CE exists is finite: there are $3^5 = 243$ allocations, and for each one the CE conditions are linear inequalities in the prices. We drop strictness, allowing $p \geq 0$ and $p(Y) \geq t_i$, and give for each allocation nonnegative multipliers whose combination of these inequalities gives a linear inequality on the budgets alone, $\sum_i w_i t_i \geq 0$ (Lemma 2.1). Where the budgets violate it, that allocation has no CE prices. Along the ray $e + \delta(-3, 2, 1)$ the inequality $\sum_i w_i t_i < 0$ is linear in δ, and the 243 certificates together cover exactly $0 < \delta < 1/17$. Each certificate excludes an open half-space of budgets, which gives the open set U. Two kinds of obstruction appear. In the first, an agent prefers part of another agent's bundle to its own, which forces a comparison of budgets that fails on the ray (Example 2.2). In the second, an agent's

preferred bundles together use the goods of other agents' bundles, so their prices, each above that agent's budget, are bounded by the other budgets (Example 2.3); an obstruction of this kind bounds δ by $1/17$.

2 Certificates

Lemma 2.1. *Let X be an allocation and t a budget vector. Let P be the set of pairs (i, Y) with $V_i(Y) > V_i(X_i)$. Suppose $\alpha_i \geq 0$ for each agent and $\lambda_{i,Y} \geq 0$ for $(i, Y) \in P$ satisfy, for every good g,*

$$\sum_{i:\, g \in X_i} \alpha_i \;\geq \sum_{(i,Y) \in P:\, g \in Y} \lambda_{i,Y}, \tag{1}$$

and let $w_i = \alpha_i - \sum_{Y:(i,Y)\in P} \lambda_{i,Y}$. If $\sum_i w_i t_i < 0$, then no prices $p \geq 0$ satisfy $p(X_i) \leq t_i$ for all i and $p(Y) \geq t_i$ for all $(i, Y) \in P$. In particular there is no CE (X, p) at t, nor at any budget vector t' with $\sum_i w_i t'_i < 0$.

Proof. Suppose such p exist. Multiply $t_i - p(X_i) \geq 0$ by α_i and $p(Y) - t_i \geq 0$ by $\lambda_{i,Y}$, and add:

$$\sum_i w_i t_i \;\geq\; \sum_g p_g \Big(\sum_{i:\, g \in X_i} \alpha_i - \sum_{(i,Y) \in P:\, g \in Y} \lambda_{i,Y} \Big) \;\geq\; 0,$$

by (1) and $p \geq 0$, contradicting $\sum_i w_i t_i < 0$. A CE (X, p) at t would satisfy these relaxed inequalities, since $p > 0$ and $p(Y) > t_i$ imply $p \geq 0$ and $p(Y) \geq t_i$; so none exists, and the same holds at any t' with $\sum_i w_i t'_i < 0$. □

By Farkas' lemma [6], such multipliers exist whenever the relaxed system has no solution, so nothing is lost by asking for them. We call $w = (w_1, w_2, w_3)$ the *weight* of the certificate. Two examples show how the certificates read.

Example 2.2. Let $X = (vwx,\, z,\, y)$, writing bundles as words. Agent 3 values y at 16 and vx at 17, so it prefers part of agent 1's bundle to its own. In a CE, $p(vx) \leq p(vwx) \leq t_1$ while $p(vx) > t_3$, so $t_1 > t_3$. The certificate is $\alpha_1 = \lambda_{3,vx} = 1/2$, with weight $(1/2, 0, -1/2)$. Along the ray, $t_1 - t_3 = -4\delta < 0$, so this allocation is excluded for every $\delta > 0$. Seventeen allocations are excluded by a linear inequality between budgets that holds with equality at e (Appendix A).

Example 2.3. Let $X = (w,\, vz,\, xy)$. Agent 1 values w at 12 and prefers vx (17), vy (15) and z (13). In a CE each of these costs more than t_1, and since $vx + vy + 2z$ uses each of v, x, y, z as often as $2\,vz + xy$ does,

$$4t_1 < p(vx) + p(vy) + 2p(z) = 2p(vz) + p(xy) \leq 2t_2 + t_3.$$

On the ray, $4t_1 - 2t_2 - t_3 = 1 - 17\delta$, which is positive for $\delta < 1/17$. This certificate ($\alpha_2 = 2/7$, $\alpha_3 = 1/7$, $\lambda_{1,vx} = \lambda_{1,vy} = 1/7$, $\lambda_{1,z} = 2/7$) is the one that limits the theorem to $\delta < 1/17$.

3 Proof of Theorem 1.2

Write $d = (-3, 2, 1)$. For each of the $3^5 = 243$ allocations X, a certificate as in Lemma 2.1, with weight w^X, is listed in the certificate file that accompanies this paper (Section 5 says where); the counts and the bound $1/17$ below are checked by the programs described there. Any direction d with $d_1 < d_3 < d_2$ and $d_1 + d_2 < 2d_3$ would serve for small δ (Remark 3.1); the certificates were computed for $d = (-3, 2, 1)$. Each weight is of one of two kinds:

(A) $w^X \cdot e < 0$ (226 allocations). Then $w^X \cdot (e + \delta d) = w^X \cdot e + \delta\, w^X \cdot d < 0$ for $0 \le \delta < r_X$, where $r_X = -w^X \cdot e / w^X \cdot d$ if $w^X \cdot d > 0$ and $r_X = \infty$ otherwise.

(B) $w^X \cdot e = 0$ and $w^X \cdot d < 0$ (17 allocations). Then $w^X \cdot (e + \delta d) < 0$ for every $\delta > 0$.

The least r_X is $1/17$, attained by Example 2.3.

Proof of Theorem 1.2. The preferences are strict and the valuations strictly monotone (Section 5). Let

$$U = \{t \in \mathbb{R}^3_{>0} : \ w^X \cdot t < 0 \text{ for all 243 allocations } X\}.$$

It is a finite intersection of open half-spaces with $\mathbb{R}^3_{>0}$, so it is open, and it is closed under positive scaling. At $t \in U$, every allocation X is excluded by Lemma 2.1 with its certificate, so no CE exists. By (A) and (B), $e + \delta d \in U$ for every $0 < \delta < 1/17$. Given a positive multiple ce and $\varepsilon > 0$, choose δ with $0 < \delta < 1/17$ and $c\delta\|d\| < \varepsilon$; then $c(e + \delta d) \in U$ is within ε of ce, and since U is open, a small ball around it lies in U and in the ε-ball around ce. That ball has positive measure. □

Remark 3.1. Near e, the set U is cut out by the kind-(B) inequalities alone. The 243 certificates have 51 distinct weights up to positive scaling, four of kind (B). These four give the inequalities $t_1 < t_3$, $t_3 < t_2$, $t_1 < t_2$ and $t_1 + t_2 < 2t_3$ (Appendix A), and the kind-(A) inequalities hold strictly at e. So there is a neighborhood N of e in which $U \cap N$ is the intersection of N with the open polyhedral cone defined by $t_1 < t_3 < t_2$ and $t_1 + t_2 < 2t_3$ (the inequality $t_1 < t_2$ follows from these).

Remark 3.2. The certificates use only which bundles each agent strictly prefers to which. So the theorem holds for every additive profile that ranks the bundles as Table 1 does, an open set of valuation profiles.

4 More goods, and the two questions

Proof of Corollary 1.3. We follow Segal-Halevi's proof of [7, Lemma 5(a)], which adds a "low-value item" that changes no comparison between the original bundles. Let $k = m - 5$ and give each agent value 2^{-j} for the new good g_j, $j = 1, \dots, k$. Any set Z of new goods has value less than 1. The original values are integers and the preferences are strict, so $V_i(W) > V_i(Y)$ implies $V_i(W) \ge V_i(Y) + 1 > V_i(Y \cup Z)$ for original bundles W, Y. Bundles of the larger market with different original parts therefore have different values, and bundles with the same original part differ by distinct sums of distinct powers of 2; so the preferences stay strict. Let $t \in U$ and suppose (X, p) is a CE of the larger market at t. Put $X'_i = X_i \cap M$. Then $p(X'_i) \le p(X_i) \le t_i$, and if $W \subseteq M$ has $V_i(W) > V_i(X'_i)$ then $V_i(W) > V_i(X_i)$, so $p(W) > t_i$. Thus $(X', p|_M)$ is a CE of the five-good market at t, contradicting Theorem 1.2. So no CE exists at any $t \in U$. □

Corollary 4.1. *(a) The answer to [1, Open Problem 1] is no: for the three additive agents of Table 1 there is no $\varepsilon > 0$ such that a CE exists for almost all budget vectors within ε of equal budgets. (b) For three agents with additive preferences and $m \ge 5$ goods, a CE need not exist for almost all incomes in the sense of [7, Definition 3].*

Proof. Both follow from Theorem 1.2 and Corollary 1.3: one profile has no CE on a set of positive measure within every ε-ball around e. Normalizing budgets to sum to 1, as [1, Section 2.1] does, changes nothing: the coordinates of d sum to 0, and U is closed under scaling. □

5 Computational verification

Three short programs in exact rational arithmetic check the theorem from the market and the certificate file. The programs, the certificate file `data/certificate.json` and the command `sh checks/check.sh` that runs them are in the release accompanying this paper, https://github.com/6/clank-lab/tree/near-equal-budgets-v1/results/near-equal-budgets; its README describes the file format and the requirements. The first checks the hypotheses: the values are positive integers, each agent's 32 bundle values are distinct, and the market in the certificate file is that of Table 1. The second reads the certificate for each of the 243 allocations; checks that every $\lambda_{i,Y}$ is attached to a bundle Y that agent i strictly prefers to X_i, that the multipliers are nonnegative and satisfy (1), and that the weight is of kind (A) or (B); and computes the least r_X, which is $1/17$. It runs in well under a second.

The third program does not read the certificates. For each allocation it solves, with its own exact two-phase simplex method, the linear program in the prices $p \geq 0$ and $\delta \geq 0$ with $p(X_i) \leq 1 + \delta d_i$ and $p(Y) \geq 1 + \delta d_i$ for every bundle Y agent i strictly prefers to X_i. The feasible values of δ form a closed interval, and the program computes its left end δ_X. It finds 214 allocations infeasible for every $\delta \geq 0$, 17 feasible only at $\delta = 0$ (the allocations of kind (B)), and 12 whose feasible interval has left end $\delta_X \geq 1/17$, the least δ_X being exactly $1/17$. So the two programs agree on the bound $1/17$, and by Farkas' lemma no certificate as in Lemma 2.1 exists at $\delta = 1/17$ for at least one allocation. Whether a CE exists at some $\delta \geq 1/17$ is not decided here. The program takes about four minutes on one core.

As sanity checks, the second program rejects the certificates with one multiplier doubled. With agent 1's value for v changed from 6 to 1, the second program rejects the original certificates, and the third finds an allocation feasible for all δ in $[0, 2/11]$. The example was found by a randomized search with floating-point linear programs and reduced to small integers; the search is not part of the proof.

Appendix A The seventeen allocations of kind (B)

Each allocation below, written (X_1, X_2, X_3), is excluded at budgets t by the single inequality shown. The middle column names the bundles Y whose prices the certificate compares with the budgets; each is strictly preferred by the named agent to its own bundle. For instance, in the first row agent 3 prefers vx ($V_3 = 17 > 16$), which lies inside X_1. All seventeen are excluded on the ray for every $\delta > 0$, since $t_1 - t_3 = -4\delta$, $t_3 - t_2 = -\delta$, $t_1 - t_2 = -5\delta$ and $t_1 + t_2 - 2t_3 = -3\delta$.

allocation	preferred bundles used	excluded where
(vwx, z, y), (vx, wz, y)	agent 3: vx	$t_1 < t_3$
(wy, z, vx)	agent 3: wy	$t_1 < t_3$
(vw, xz, y)	agent 2: wz; agent 3: vx	$t_1 < t_3$
(wxz, v, y), (wz, vx, y), (wz, v, xy), (xz, vw, y)	agent 2: z	$t_1 < t_2$
(z, vw, xy), (xz, v, wy), (z, vx, wy), (wz, y, vx)	agent 2: z	$t_1 < t_2$
(wx, v, yz)	agent 2: z; agent 3: wxy	$t_1 < t_2$
(wx, z, vy)	agent 2: vy	$t_3 < t_2$
(vw, z, xy), (vx, z, wy)	agent 1: wx; agent 2: vy	$t_3 < t_2$
(wx, vz, y)	agent 3: vx, wz	$t_1 + t_2 < 2t_3$

For example, for (vw, z, xy): agent 1 prefers wx ($23 > 18$) and agent 2 prefers vy ($22 > 20$); together wx and vy use the goods of $X_1 \cup X_3 = vwxy$ once each, so in a CE $t_1 + t_2 < p(wx) + p(vy) =$

$p(vw) + p(xy) \le t_1 + t_3$. For (wx, vz, y), agent 3 prefers vx ($17 > 16$) and wz ($18 > 16$), which together use the goods of $X_1 \cup X_2$ once each, so a CE needs $2t_3 < p(vx) + p(wz) = p(wx) + p(vz) \le t_1 + t_2$.

References

[1] M. Babaioff, N. Nisan and I. Talgam-Cohen, *Competitive equilibrium with indivisible goods and generic budgets*, Math. Oper. Res. **46** (2021), 382–403; doi:10.1287/moor.2020.1062; arXiv:1703.08150v2, the version cited.

[2] M. Babaioff, N. Nisan and I. Talgam-Cohen, *Competitive equilibria with indivisible goods and generic budgets*, arXiv:1703.08150v1 (March 23, 2017), the first version of [1].

[3] M. Babaioff, N. Nisan and I. Talgam-Cohen, *Competitive equilibrium with generic budgets: beyond additive*, arXiv:1911.09992v1 (2019).

[4] E. Budish, *The combinatorial assignment problem: approximate competitive equilibrium from equal incomes*, J. Political Econ. **119** (2011), 1061–1103; doi:10.1086/664613.

[5] D. K. Foley, *Resource allocation and the public sector*, Yale Econ. Essays **7** (1967), 45–98.

[6] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, Chichester, 1986.

[7] E. Segal-Halevi, *Competitive equilibrium for almost all incomes: existence and fairness*, Auton. Agents Multi-Agent Syst. **34** (2020), article 26; doi:10.1007/s10458-020-09444-z; arXiv:1705.04212v6, the version cited.

[8] H. R. Varian, *Equity, envy, and efficiency*, J. Econom. Theory **9** (1974), 63–91; doi:10.1016/0022-0531(74)90075-1.

[9] Wikipedia, *List of unsolved problems in fair division*, section "Competitive equilibrium for almost all incomes", `https://en.wikipedia.org/wiki/List_of_unsolved_problems_in_fair_division`, revision 1360704274, accessed November 5, 2026.