KPP fronts in the 1-1-1 ABC flow speed up linearly in every direction

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First published November 9, 2026; this version November 9, 2026 (v2)
Zenodo archive (Lean development in `lean`, `checks`): https://doi.org/10.5281/zenodo.24081425

hexagon:2611.00854v1

Abstract

Let $V(x,y,z) = (\sin z + \cos y,\ \sin x + \cos z,\ \sin y + \cos x)$ be the Arnold–Beltrami–Childress (ABC) flow with $A = B = C = 1$, and let $c_e^*(K)$ be the minimal speed of pulsating fronts of $T_t + K\,V \cdot \nabla T = \Delta T + f(T)$ in the direction e, for a reaction f of KPP type. We prove that $\lim_{K\to\infty} c_e^*(K)/K$ exists and is positive for every unit vector e. Zlatoš proved that this limit exists and equals a supremum over first integrals of the flow, and Xin, Yu and Zlatoš, who constructed periodic orbits of this flow, pointed out that a positive-measure tube of orbits moving roughly linearly would make the limit positive, but that a rigorous proof of such tubes was not known. We prove that such a tube exists. Two independent interval-arithmetic computations certify an elliptic periodic orbit with nonzero twist; Pöschel's KAM theorem with frequencies as parameters then gives invariant curves of the return map with a Lipschitz label, which yield a Lipschitz first integral supported on the tube. The deduction from the certified facts is checked in Lean, with Zlatoš's and Pöschel's theorems as cited inputs.

This work has not yet been reviewed by experts. The proofs, the Lean formalization and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

2020 Mathematics Subject Classification. 35K57, 37J40, 76F25, 65G30. *Keywords.* ABC flow, KPP fronts, front speed-up, KAM theory, computer-assisted proof.

1 Introduction

1.1 Main result

Let $V\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the ABC flow with $A = B = C = 1$,

$$V(x,y,z) = (\sin z + \cos y,\ \sin x + \cos z,\ \sin y + \cos x).$$

It is 2π-periodic in each variable, divergence-free and of mean zero. Let $f\colon [0,1] \to \mathbb{R}$ satisfy, for some $\varepsilon_0 > 0$,

$$\begin{aligned} f \in C^{1,\varepsilon_0}([0,1]), \qquad & f(0) = f(1) = 0, \\ f \text{ non-increasing on } (1-\varepsilon_0, 1), \qquad & 0 < f(s) \le f'(0)s \text{ for } s \in (0,1). \end{aligned} \tag{1}$$

For $K > 0$ consider

$$T_t + K\,V \cdot \nabla T = \Delta T + f(T) \qquad \text{on } \mathbb{R} \times \mathbb{R}^3. \tag{2}$$

A *pulsating front* in the direction of a unit vector e with speed c is a classical solution with values in $[0,1]$ of the form $T(t,\xi) = U(\xi \cdot e - ct, \xi)$, where $U(s,\cdot)$ is 2π-periodic for every s, and $U(s,\xi) \to 1$

as $s \to -\infty$ and $U(s,\xi) \to 0$ as $s \to +\infty$, uniformly in ξ. The minimal front speed $c_e^*(K)$ is the infimum of the speeds of pulsating fronts; in this KPP setting fronts exist exactly for $c \geq c_e^*(K)$ [1].

Theorem 1. *For every unit vector $e \in \mathbb{R}^3$ and every f satisfying (1), the limit $\lim_{K\to\infty} c_e^*(K)/K$ exists and is positive.*

The proof uses a computer-assisted theorem about the streamlines of V (Theorem 2), two published theorems, of Zlatoš (Theorem 8) and Pöschel (Theorem 9), quoted exactly in Appendix A, and a deduction that is checked in Lean (Section 7).

1.2 The question and prior work

Zlatoš [18, Theorem 1.1] proved, for 1-periodic flows u and f as in (1), that $\lim_{A\to\infty} c_e^*(A)/A$ exists and equals a supremum over the real first integrals $w \in H^1$ of u (Theorem 8); by [18, Remark 3] it is positive "precisely when there exists $w \in I$ such that $\int_{\mathbb{T}^n}(u\cdot e)w\,dx \neq 0$". El Smaily and Kirsch [5, Theorem B and Remark 2] extended the formula to periodic domains in three or more dimensions and stated the same criterion. For the 1-1-1 ABC flow, Xin, Yu and Zlatoš [17] proved that V has periodic orbits with rotation vectors parallel to the coordinate axes, which gives linear speed-up for the G-equation model. In their introduction they wrote: "numerical simulations suggest that there are 'vortex tubes' composed of orbits which travel roughly linearly in the $\pm x$, $\pm y$, and $\pm z$ directions [6], suggesting that $c^*(p,K)$ should also grow linearly in K for each p. One could expect to find such vortex tubes, if they indeed exist, in the vicinity of the 'periodic' orbits constructed in this paper, but a rigorous proof of their existence is currently not known." Here [6] is Dombre, Frisch, Greene, Hénon, Mehr and Soward [4]. McMillen, Xin, Yu and Zlatoš [10, Section 1], working near the integrable regime $0 < A \ll 1$, $B = C = 1$, wrote: "We are not able to prove the existence of such tubes ... even in the KAM region since KAM type theorems do not provide any regularity of the set of quasi-periodic orbits." Shen, Xin and Zhou [13] computed KPP front speeds numerically and found that "in ABC flows however, the enhancement is $O(\delta)$ which can be attributed to the presence of principal vortex tubes". Huang, Zhao and Dai [6] study invariant tori and chaotic streamlines in a near-integrable ABC regime, using volume-preserving KAM and Melnikov methods.

1.3 What is new

We prove that a vortex tube exists, and we obtain the regularity that [10] asks for from a Lipschitz family of invariant curves rather than from the set of quasi-periodic orbits itself. The proof has three new ingredients. One is a computer-assisted proof that V has an elliptic periodic orbit with nonzero twist (Theorem 2). Breden and Lessard [2] gave computer-assisted proofs for ABC flows: "for ABC flows, we prove existence of ballistic spiral orbits". Kapela and Simó [8] proved rigorous KAM results around periodic orbits, and Wilczak and Barrio [15] proved stable elliptic periodic orbits with surrounding invariant tori, both by computer. Another is the passage from Pöschel's Theorem A [12], a KAM theorem whose estimates are explicit in all its parameters, to a positive-measure family of invariant curves of the return map whose label is Lipschitz on their union (Proposition 4). This was the hardest step: theorems for a fixed nondegenerate twist map do not survive the rescaling near an elliptic point, and Pöschel's theorem applies only after a Floquet reduction, a time-periodic Birkhoff normal form, and a choice of parameters that keeps the perturbation below Pöschel's threshold. The last is a graph-form construction of the first integral (Lemma 5). Recently Szumiński and Llibre [14] studied the ABC flow "in perturbative regimes near its three elementary integrable coordinate axes" and proved the existence of elliptic and hyperbolic periodic solutions there; their work concerns other parameter values and other questions.

2 Proof overview

By Zlatoš's theorem, after rescaling to period 1, the limit exists, and it is positive as soon as V has a Lipschitz, 2π-periodic first integral W with $\int (V\cdot e)W \neq 0$. By the cyclic symmetry $(x,y,z)\mapsto(y,z,x)$ of V it suffices to find one $W\geq 0$, supported where $V_z>0$, with $\int V_x W = \int V_y W = 0$ and $\int V_z W > 0$. Such a W comes from a tube of orbits that move monotonically in z and stay in a bounded region in x and y. We find the tube around an elliptic periodic orbit: its first-crossing map P from $\{z=0\}$ to $\{z=2\pi\}$ has an elliptic fixed point p with nonzero twist (certified by computer), so KAM theory gives a positive-measure family of P-invariant closed curves around p. The measure of the set of labels of curves surrounding a point gives a P-invariant Lipschitz function w on the plane, and carrying w along the flow gives W. The one delicate point is regularity: w is Lipschitz only because the label of the curves is Lipschitz on their union, and that is what Pöschel's estimates provide.

3 The certified periodic orbit

For $a\in\mathbb{R}^2$ let $P(a)\in\mathbb{R}^2$ be the point where the streamline of V from $(a,0)$ first reaches the plane $z=2\pi$ (no reduction modulo 2π). Let $\mathcal{X}$ be the closed square with half-width 2^{-20} and center

$$(6533234620664,\ 1727108826179)\cdot 2^{-40}\approx(5.941942,\ 1.570796).$$

Theorem 2 (computer-assisted)**.** *There is $p\in\mathcal{X}$ such that:*

1. *the streamline γ of V with $\gamma(0)=(p,0)$ satisfies $\gamma(T)=(p,2\pi)$ for some $T\in[3.2352,3.2354]$, and $\sin y+\cos x\geq 1.92$ along γ on $[0,T]$; in particular $P(p)=p$;*
2. *P is differentiable at p and $\operatorname{tr} DP(p)\in[-1.9650,-1.9640]$;*
3. *the first Birkhoff coefficient of P at p is nonzero.*

Since P preserves the area form of Lemma 3 below and fixes p, $\det DP(p)=1$; as the trace lies in $(-2,-1)$, the eigenvalues of $DP(p)$ are $e^{\pm 2\pi i\nu}$ with $\nu\in(1/3,1/2)$, so $\lambda^q\neq 1$ for $q=1,\dots,4$. The first Birkhoff coefficient is defined in Appendix B from the 3-jet of P at p; it is the coefficient of $w^2\bar{w}$ after the quadratic terms have been removed in a complex eigencoordinate w, and its vanishing does not depend on the coordinates.

Two programs prove Theorem 2 independently. The first uses the CAPD library [3]: it computes the C^3 jet of P over all of $\mathcal{X}$, proves a unique fixed point in $\mathcal{X}$ by the interval Newton method, encloses the trace in $[-1.96455,-1.96451]$ and the return time in $[3.2352771,3.2352775]$, and computes the Birkhoff normal form with CAPD's routine, whose twist coefficient has real part in $[-0.5787,-0.5685]$. The second, written independently, uses ball arithmetic from Arb [7] with its own Taylor integrator. It proves by the Krawczyk method a unique fixed point in a square of half-width 2^{-100} inside $\mathcal{X}$, encloses the trace in $-1.96453154969901637\pm 4\cdot 10^{-18}$ and $\det DP(p)$ in $1\pm 3\cdot 10^{-20}$, and computes the twist coefficient of $\Phi\circ P\circ\Phi^{-1}$ for the area-preserving coordinates Φ of Lemma 3, with real part $0.29530841922827626555718$, with error below $3\cdot 10^{-24}$. Both check $\sin y+\cos x>0$ on every integration step of the whole orbit segment, with lower bounds 1.9215 and 1.9378. As a test, each program reproduces the exact values for an integrable flow and rejects three altered certificates (Appendix D). Appendix D gives details.

4 Invariant curves with a Lipschitz label

Lemma 3. *Let $X = x\sin y + \sin x$, $Y = y$, and $H(x,y,z) = y\sin z + \sin y + \cos x - x\cos z$. Where $\sin y + \cos x \neq 0$, the streamlines of V, parametrized by z, satisfy $dX/dz = \partial_Y H$, $dY/dz = -\partial_X H$, with H written in the variables (X,Y).*

Proof. With X fixed, $\partial x/\partial Y = -x\cos y/(\sin y + \cos x)$ and $\partial x/\partial X = 1/(\sin y + \cos x)$. Hence $\partial_Y H = V_x + x\cos y\, V_y/V_z = dX/dz$ and $-\partial_X H = V_y/V_z = dY/dz$. □

So near the certified orbit the streamlines form a 2π-periodic Hamiltonian system with one degree of freedom, and P is its period map, area-preserving in (X,Y).

Proposition 4. *Let p be as in Theorem 2 and $\delta > 0$. There are numbers $\rho, L > 0$, a homeomorphism h from a neighborhood of the closed disk $\bar{B}(0,\rho)$ onto an open subset of $\mathbb{R}^2$ with $h(0) = p$, $h(\bar{B}(0,\rho)) \subset B(p,\delta)$ and h bi-Lipschitz with constant L on $\bar{B}(0,\rho)$, a compact set $\mathcal{K} \subset \mathbb{R}$ of positive measure, and continuous 2π-periodic functions g_s, $s \in \mathcal{K}$, with $0 < g_s < \rho$, such that the closed curves $C_s = h(\{g_s(\psi)(\cos\psi, \sin\psi) : \psi \in \mathbb{R}\})$ satisfy $P(C_s) = C_s$ and*

$$|s - s'| \le L\,|g_s(\psi)(\cos\psi,\sin\psi) - g_{s'}(\psi')(\cos\psi',\sin\psi')| \quad \text{for all } s, s' \in \mathcal{K},\ \psi, \psi' \in \mathbb{R}.$$

Outline of proof. The steps, each proved in Lean, are given in Appendix C. Translate to the periodic solution and pass to a real Floquet frame, in which the linear part is the rotation with frequency ν. Since $\lambda^q \neq 1$ for $q \le 4$, two time-periodic symplectic changes of variables bring the Hamiltonian to $\nu\, p + b\, p^2 + R$ with $p = |w|^2/2$ and $R = O(|w|^5)$, analytic on a polydisc times a strip $|\operatorname{Im} t| < s$. The twist coefficient is a conjugacy invariant up to a positive factor, so Theorem 2(3) gives $b \neq 0$. For small $\varepsilon > 0$ write $p = p_0(\omega) + I$ with $p_0(\omega) = (\omega_1/\omega_2 - \nu)/(2b)$ and treat the frequencies $\omega = (\omega_1, \omega_2)$ as parameters, following [12, §2a]: the time-t Hamiltonian becomes $e(\omega) + \omega_1 I + \omega_2 J + P_\omega$ in action–angle variables $(I, \theta_1, J, \theta_2 = t)$, with ω in $\Omega = \{|\omega_2 - 1| < \frac{1}{2},\ \omega_1/\omega_2 \in \nu + 2b(\varepsilon/2, \varepsilon)\}$. Apply Theorem 9 with $n = \tau = 2$, $r = \varepsilon^{5/4}$, $\alpha = \varepsilon^{9/8}$ and $h = \alpha s^3$: the perturbation is $O(\varepsilon^{5/2})$, below Pöschel's threshold $\gamma\alpha r s^3 \sim \varepsilon^{19/8}$ for small ε. The time row of the invariance equation forces $\omega_2 = 1$ to be preserved, so for each σ with $(\sigma, 1) \in \Omega_\alpha$ the torus meets $\{t = 0\}$ in a P-invariant curve with rotation number σ. The labels σ form a compact set of measure at least $|b|\varepsilon(1 - 8\alpha) - 16\alpha(2 + |\nu|) > 0$. Pöschel's Lipschitz estimates make each curve a radial graph over the angle and bound its displacement in σ below by $|\sigma - \sigma'|/(4|b|)$, minus terms of relative size $O(\varepsilon^{1/8})$; this gives the Lipschitz label. □

5 The invariant function and the tube

Lemma 5. *In the setting of Proposition 4, with δ small, let $D_s = h(\{r(\cos\psi, \sin\psi) : 0 \le r < g_s(\psi)\})$ and $w(a) = \operatorname{Leb}\{s \in \mathcal{K} : a \in D_s\}$. Then $w \ge 0$ is Lipschitz on $\mathbb{R}^2$, $w(p) > 0$, w vanishes outside $h(B(0,\rho))$, and $w(P(a)) = w(a)$ for a in $h(\bar{B}(0,\rho))$.*

Proof. $w(p) = \operatorname{Leb}\mathcal{K} > 0$, since $p = h(0)$ lies in every D_s. Let $a, b \in \mathbb{R}^2$. If $a \in D_s$ and $b \notin D_s$ for some s, the segment between their chart preimages meets the radial graph of g_s. So, for a, b in $h(\bar{B}(0,\rho))$, the labels s counted in exactly one of $w(a)$, $w(b)$ are labels of points of one segment of length at most $L|a - b|$, and by the Lipschitz label they form a set of diameter at most $L^2|a - b|$. Hence $|w(a) - w(b)| \le L^2|a - b|$ there; since w vanishes outside $h(B(0,\rho))$, the bound extends to all of $\mathbb{R}^2$ with a larger constant. For invariance: near p, P is a homeomorphism onto its image (its inverse is the first-crossing map in negative time), $P(C_s) = C_s$ and $P(p) = p$. So $P(D_s)$ is connected, misses C_s and contains p; as $h(\bar{B}(0,\rho))$ is a closed disk on which h is a homeomorphism and C_s is the boundary of D_s, this forces $P(D_s) \subseteq D_s$, and the same argument for P^{-1} gives equality. □

Radial graphs in a chart make the interior D_s explicit, so no Jordan curve theorem is needed.

Proposition 6. *There is a function $W\colon \mathbb{R}^3 \to [0,\infty)$, not identically zero, that is Lipschitz and 2π-periodic in each variable, satisfies $V \cdot \nabla W = 0$ almost everywhere, satisfies $V_z \geq c_0 > 0$ wherever $W \neq 0$, and whose support has x- and y-projections contained, modulo 2π, in closed intervals of length less than 2π.*

Proof. By Theorem 2(1) and Gronwall's inequality, $V_z \geq 1.9$ along every streamline from $(a,0)$ with a near p for $|t| \leq 4$, so each such streamline crosses the planes $z = 0$ and $z = \pm 2\pi$ exactly once in that time window, transversally. Choose δ in Proposition 4 small enough that w from Lemma 5 is supported in this neighborhood. For ξ on the streamline through $(a,0)+\ell$, with a in the support of w and $\ell \in 2\pi\mathbb{Z}^3$, set $W(\xi) = w(a)$, and set $W = 0$ elsewhere. This is well defined, because the crossings of the streamline with the planes $z \in 2\pi\mathbb{Z}$ are the points $P^k(a)$ and $w \circ P = w$ there; it is constant on streamlines; and it is Lipschitz, because the crossing point depends in a Lipschitz way on ξ (the flow is smooth and the crossings are transversal). A Lipschitz function constant along streamlines satisfies $V \cdot \nabla W = 0$ wherever it is differentiable, hence almost everywhere. On the support of W we have $V_z = \sin y + \cos x \geq 1.9$, which forces $\cos x \geq 0.9$ and $\sin y \geq 0.9$; so, modulo 2π, the x-projection of the support lies in $[-\pi+1, \pi-1]$ and the y-projection in $[-\pi/2+1, 3\pi/2-1]$. □

Lemma 7. *For W as in Proposition 6 and $Q = [0,2\pi)^3$, $\int_Q V_x W = \int_Q V_y W = 0$ and $\int_Q V_z W > 0$.*

Proof. $\int_Q V_z W \geq c_0 \int_Q W > 0$. For V_x, let χ be a 2π-periodic Lipschitz function of x alone that equals x on the x-projection of the support of W (a sawtooth). Then $V_x W = (V \cdot \nabla\chi) W$, and, integrating by parts over the torus with $\operatorname{div} V = 0$, $\int_Q (V \cdot \nabla\chi) W = -\int_Q \chi\, V \cdot \nabla W = 0$. The same argument applies to V_y. □

6 Proof of Theorem 1

Let $u(\xi) = V(2\pi\xi)$ and $\hat{f} = (2\pi)^2 f$. If T is a front of (2) with speed c, then $\hat{T}(t,\xi) = T((2\pi)^2 t, 2\pi\xi)$ is a 1-periodic front of $\hat{T}_t + 2\pi K\, u \cdot \nabla\hat{T} = \Delta\hat{T} + \hat{f}(\hat{T})$ with speed $2\pi c$, and conversely. Hence $c_e^*(K)/K = \hat{c}_e^*(2\pi K)/(2\pi K)$, where $\hat{c}^*$ refers to u and $\hat{f}$. The flow u satisfies Zlatoš's hypothesis (1.3) and $\hat{f}$ Zlatoš's (1.2), so by Theorem 8 the limit exists and equals $\ell(e) = \sup_w \int (u \cdot e) w^2 / \|w\|_2^2$ over Zlatoš's set.

For positivity, let σ be the cyclic permutation $(x,y,z) \mapsto (y,z,x)$, which satisfies $V \circ \sigma = \sigma \circ V$, and let W be as in Proposition 6. The functions W, $W \circ \sigma$ and $W \circ \sigma^{-1}$ are again Lipschitz periodic first integrals, and by Lemma 7 and a change of variables, $\int_Q (V \cdot e) W = e_z \int_Q V_z W$, $\int_Q (V \cdot e)\, W \circ \sigma = e_x \int_Q V_z W$ and $\int_Q (V \cdot e)\, W \circ \sigma^{-1} = e_y \int_Q V_z W$. Choose the one whose coefficient is a nonzero component of e, and let $\hat{W}(\xi)$ be it at $2\pi\xi$. Then $\hat{W}$ lies in Zlatoš's set I of real H^1 first integrals of u, and $\beta = \int_{[0,1)^3} (u \cdot e)\hat{W} \neq 0$. For small $\eta > 0$ the function $w_\eta = 1 + \eta \operatorname{sgn}(\beta)\hat{W}$ is in I, satisfies $\|\nabla w_\eta\|_2^2 = \eta^2 \|\nabla\hat{W}\|_2^2 \leq \hat{f}'(0)\|w_\eta\|_2^2$, and, since $\int u \cdot e = 0$, $\int (u \cdot e) w_\eta^2 = 2\eta|\beta| + \eta^2 \int (u \cdot e)\hat{W}^2 > 0$. Hence $\ell(e) > 0$. □

7 Formal verification and reproducibility

The deduction of Theorem 1 from Theorem 2 and the two cited theorems is proved in Lean 4 [11] with Mathlib [9]. The formal theorem is

```
theorem ABC.abc_kpp_linear_speedup (f : ℝ → ℝ) (hf :  ZlatosKPP f) (e :
E 3) (he : ‖e‖ = 1) : ∃ ℓ : ℝ, 0 < ℓ ∧ Tendsto (fun K => cstar (2 * Real.pi)
abcField f e K / K) atTop (𝓝 ℓ)
```

Here `abcField` is V; `ZlatosKPP f` is (1) (an exponent $\varepsilon \in (0,1]$, f of class C^1 on $[0,1]$ with ε-Hölder derivative, $f'(0)$ one-sided); and `cstar (2 * Real.pi) abcField f e K` is the infimum of the speeds of pulsating fronts as defined in Section 1, for classical solutions with T, T_t, ∇T and D^2T jointly continuous. The proof depends on Lean's three standard axioms and on three further axioms: `ABC.Cert.fixed_point`, which states Theorem 2 with its numbers as exact rationals (the uniqueness of the fixed point, which both programs also prove, is not used); `ABC.Zlatos2010.theorem_1_1`; and `ABC.Poschel2001.theorem_A`. The last two state Theorems 8 and 9 with all their hypotheses, and the Lean files list how each phrase of the quoted statements is read (for instance, in Pöschel's theorem, the sup-norms, "real analytic" as holomorphic and real on real points, and the inclusions $\varphi(\Omega_\alpha) \subset O_h$, which hold after decreasing its constant γ). Every hypothesis of the two cited theorems is proved in Lean for this flow; nothing else is assumed. The build checks the axiom list. A separate program checks that the rationals in the certificate axiom are those the two programs certify.

What lies outside Lean is the correctness of the two interval computations, including CAPD's normal-form routine and the Arb library, and the exact transcription of the cited theorems, quoted in Appendix A from their arXiv versions (Zlatoš v2 and Pöschel's version 1.3, which notes small corrections relative to the printed text). The commands to rebuild the Lean proof and rerun both programs, with running times, are in the README of the accompanying files.

Appendix A The cited theorems

Theorem 8 (Zlatoš [18, Theorem 1.1]). *With (1.1) "$T_t + Au\cdot\nabla T = \Delta T + f(T)$ on $\mathbb{R}^n$", (1.2) "$f \in C^{1,\varepsilon}([0,1])$, $f(0) = f(1) = 0$ and f is non-increasing on $(1-\varepsilon, 1)$ for some $\varepsilon > 0$, $0 < f(s) \le sf'(0)$ for $s \in (0,1)$", (1.3) "The 1-periodic flow $u\colon \mathbb{T}^n \to \mathbb{R}^n$ satisfies $u \in C^{1,\varepsilon}(\mathbb{T}^n)$, $\nabla\cdot u \equiv 0$, $\int_{\mathbb{T}^n} u\,dx = 0$", and (1.5) "$I \equiv \{w \in H^1(\mathbb{T}^n) \mid \operatorname{Im} w = 0 \text{ and } u\cdot\nabla w = 0\}$": "If u and f satisfy (1.2) and (1.3) and $|e| = 1$, then*

$$\lim_{A\to\infty} \frac{c_e^*(A)}{A} = \sup_{w\in I,\ \|\nabla w\|_2^2 \le f'(0)\|w\|_2^2} \frac{\int_{\mathbb{T}^n} (u\cdot e)w^2\,dx}{\|w\|_2^2}.$$

In particular, the limit exists."

Pulsating fronts are defined there as "a solution of (1.1) of the form $T(t,x) = U(x\cdot e - ct, x)$, with c the front speed, and U 1-periodic in x and such that $\lim_{s\to-\infty} U(s,x) = 1$, $\lim_{s\to+\infty} U(s,x) = 0$, uniformly in x". In the formalization, $u\cdot\nabla w = 0$ holds almost everywhere, as in the proof of Zlatoš's Theorem 1.2.

Theorem 9 (Pöschel [12, Theorem A]). *"Let $H = N + P$. Suppose P is real analytic on $D_{r,s}\times O_h$ with $|P|_{r,s,h} \le \gamma\alpha r s^\nu$, $\alpha s^\nu \le h$, where $\nu = \tau + 1$ and γ is a small constant depending only on n and τ. Suppose also that $r, s, h \le 1$. Then there exists a Lipschitz continuous map $\varphi\colon \Omega_\alpha \to \Omega$ close to the identity and a Lipschitz continuous family of real analytic torus embeddings $\Phi\colon \mathbb{T}^n \times \Omega_\alpha \to B \times \mathbb{T}^n$ close to Φ_0, such that for each $\omega \in \Omega_\alpha$ the embedded tori are Lagrangian and $X_H|_{\varphi(\omega)} \circ \Phi = D\Phi\cdot X_N$. Moreover, Φ is real analytic on $T_* = \{\theta : |\operatorname{Im}\theta| < s/2\}$ for each ω, and*

$$|W(\Phi - \Phi_0)|,\ \alpha s^\nu |W(\Phi-\Phi_0)|_{\mathfrak{L}} \le \frac{c}{\alpha r s^\nu}|P|_{r,s,h}, \qquad |\varphi - \mathrm{id}|,\ \alpha s^\nu |\varphi - \mathrm{id}|_{\mathfrak{L}} \le \frac{c}{r}|P|_{r,s,h},$$

uniformly on $T_*\times\Omega_\alpha$ and Ω_α, respectively, where c is a constant depending only on n and τ, and $W = \operatorname{diag}(r^{-1}\mathrm{Id}, s^{-1}\mathrm{Id})$."

Here, in Pöschel's notation, $N = e(\omega) + \langle \omega, I \rangle$; Ω_α is the set of $\omega \in \Omega$ with $|\langle k, \omega \rangle| \geq \alpha/|k|^\tau$ for all $0 \neq k \in \mathbb{Z}^n$ and at distance at least α from the boundary of Ω, with $\tau > n - 1$ fixed; $D_{r,s} = \{|I| < r\} \times \{|\operatorname{Im} \theta| < s\}$ and $O_h = \{\omega : |\omega - \Omega_\alpha| < h\}$, with sup-norms; and $\Phi_0(\theta, \omega) = (0, \theta)$. The Lean statement takes $n \geq 2$, Pöschel's standing assumption.

Appendix B The twist coefficient

Let F be a C^3 map of the plane with $F(p) = p$ and $\det DF(p) = 1$, and let $v \in \mathbb{C}^2$ be an eigenvector of $DF(p)$ for an eigenvalue $\lambda \notin \mathbb{R}$ with $\lambda^2 \neq \lambda$ and $\lambda^2 \neq \bar{\lambda}$; then $|\lambda| = 1$. In the coordinate w given by $x = p + vw + \bar{v}\bar{w}$, write $F = \sum_{j+k \leq 3} a_{jk} w^j \bar{w}^k + o(|w|^3)$, so $a_{10} = \lambda$, $a_{01} = 0$. The first Birkhoff coefficient is nonzero when

$$\beta = a_{21} + \frac{2a_{20}a_{11}}{|\lambda|^2 - \lambda} + \frac{a_{11}a_{20}}{\lambda^2 - \lambda} + \frac{|a_{11}|^2}{|\lambda|^2 - \bar{\lambda}} + \frac{2|a_{02}|^2}{\lambda^2 - \bar{\lambda}} \neq 0,$$

the coefficient of $w^2\bar{w}$ after the change $w \mapsto w + q(w)$ with $q_{jk} = a_{jk}/(\lambda^j \bar{\lambda}^k - \lambda)$ that removes the quadratic terms. The twist coefficient of both programs is $-i\beta/\lambda$ up to a positive factor and complex conjugation; CAPD's routine uses only $\bar{\lambda} = 1/\lambda$, which holds at p since $\det DP(p) = 1$. Lean proves that, under this determinant-one hypothesis, $\beta \neq 0$ is invariant under C^3 conjugacy fixing p, so the second program's computation in the coordinates Φ of Lemma 3 gives the same conclusion for P.

Appendix C The steps of Proposition 4

The Lean proof follows these steps; constants not given here are explicit in the Lean files.

R1. In the coordinates of Lemma 3, translated by the periodic solution, the streamlines solve $\dot{u} = J\nabla_u H_1(u, z)$ with H_1 analytic on a polydisc times a strip, real, 2π-periodic in z and $O(|u|^2)$. The period map P_1 is conjugate to P by an analytic chart. Analyticity of the flow in complex time and of P_1 is proved from uniform limits of Euler polygons.

R2. With $X(t)$ the fundamental matrix of the linear part, the matrix $Q(t) = X(t)SR(-\nu t)$, for $S \in SL(2, \mathbb{R})$ bringing $DP_1(0)$ to the rotation $R(2\pi\nu)$, is real, 2π-periodic and symplectic, and $u = Q(t)w$ turns H_1 into $\frac{\nu}{2}|w|^2 + O(|w|^3)$. Here ν is not a multiple of $1/q$ for $q \leq 4$, by the certified trace.

R3. Two finite compositions of time-periodic polynomial shears, solving the homological equations with divisors $m + \nu l$ bounded below for $|l| \leq 4$, give $K = \frac{\nu}{2}|w|^2 + \frac{b}{4}|w|^4 + R$ with $|R| \leq C|w|^5$ on a polydisc times a strip, and conjugate the period maps.

R4. The period map of K has twist coefficient a nonzero multiple of b, and the conjugacy invariance of Appendix B gives $b \neq 0$.

R5. With $p = |w|^2/2 = p_0(\omega) + I$ and polar angle θ_1, the time $\theta_2 = t$ and its conjugate J, the Hamiltonian $\omega_2(K + J)$ equals $e(\omega) + \omega_1 I + \omega_2 J + P_\omega$ exactly, with $P_\omega = \omega_2(bI^2 + R)$.

R6. For $\varepsilon \leq \varepsilon_0$, with $r = \varepsilon^{5/4}$, $\alpha = \varepsilon^{9/8}$ and $h = \alpha s^3$, P_ω is real analytic on $D_{r,s} \times O_h$ with $|P_\omega| \leq C_1 \varepsilon^{5/2}$, and every hypothesis of Theorem 9 holds.

R7. The time row of $X_H \circ \Phi = D\Phi \cdot X_N$, averaged over the torus, shows $\varphi_2(\omega) = \omega_2$ and that the θ_2-component of $\Phi - \Phi_0$ is constant, so for $(\sigma, 1) \in \Omega_\alpha$ the section $t = 0$ of the torus is a P-invariant curve. The set $\{\sigma : (\sigma, 1) \in \Omega_\alpha\}$ has measure at least $|b|\varepsilon(1 - 8\alpha) - 16\alpha(2 + |\nu|)$.

Assembly. With $\varepsilon = \kappa^8$ and κ small, the angle part of Φ minus the identity is $\frac{1}{2}$-Lipschitz, so each curve is the radial graph of $g_\sigma = \sqrt{2(p_0 + \Phi^I)}$ over the angle, in the chart given by R1–R3; the bounds on $\varphi - \mathrm{id}$ and $\Phi - \Phi_0$ and on their Lipschitz constants give the Lipschitz label.

Appendix D The two programs

The certificate lists the center and half-width of $\mathcal{X}$ and the claimed ranges of the return time and the trace. The first program (CAPD 6.1.0, at a pinned commit, with machine intervals) integrates the variational equations to order 3 over $\mathcal{X}$ by the C^r-Lohner algorithm [16], finds the crossing with $z = 2\pi$, and checks every claim; the second (Arb through python-flint 0.9.0, 256-bit precision, order-36 Taylor steps of size $1/32$ with a priori enclosures) does the same independently, refining the fixed point in high precision first. As a test, both treat the integrable flow $x' = \cos y$, $y' = \sin x$, $z' = 13/10$ at $(0, \pi/2)$, whose return map is the time-$2\pi/1.3$ map of $\cos x + \sin y$ with exact twist $-(20\pi/13)/4$; both reproduce it (the second to within $8 \cdot 10^{-29}$). Both reject three altered certificates: the center moved by 2^{-19}, the upper trace bound set to -1.9646, and the upper return time set to 3.2352.

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