A counterexample to a lit-only sigma-game conjecture

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Abstract

A move in the sigma-game toggles the neighbors of a vertex; the lit-only game permits such a move only at a lit vertex. The gap for a graph is the largest difference, over starting configurations, between the minimum numbers of lights attainable in these two games. Goldwasser, Wang and Wu conjectured that a connected graph on n vertices with gap greater than $n/2$ must be complete m-partite for $m \equiv 3 \pmod 4$. We refute their Conjecture 5, published in 2011. The complement of the disjoint union of a six-cycle and an isolated vertex has a starting configuration reducible to zero lights by regular moves but to no fewer than four by lit-only moves. Its lit-only orbit has three configuration types. Replacing every vertex by m independent copies gives connected graphs on $7m$ vertices that are not complete multipartite, with a uniform starting configuration having minimum light numbers 0 and $4m$. The counterexample and this family are proved in Lean from the definitions.

The proofs, the Lean formalization, the certificate checkers and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

1 Introduction

We define the gap between the two games, state the conjecture about large gaps, and give a counterexample. A configuration on a finite undirected graph G assigns to each vertex either ON (lit) or OFF (dark). Its weight is the number of lit vertices. A regular move at a vertex toggles all its neighbors. A lit-only move makes the same change only if the played vertex is ON. Loops are allowed in the source convention: the played vertex is its own neighbor exactly when it has a loop. Moves at dark vertices do nothing in the lit-only game and may be omitted.

For a starting configuration x, let $\mathrm{ML}_G(x)$ be the least number of lit vertices attainable by regular moves, and let $\mathrm{ML}^*_G(x)$ be the corresponding lit-only minimum. These minima include the empty sequence of moves. Following Goldwasser, Wang and Wu [1], define

$$\mathcal{D}(G) = \max_x \big(\mathrm{ML}^*_G(x) - \mathrm{ML}_G(x)\big).$$

Both minima in each difference use the same starting configuration.

Theorem 1. *Let G be the complement of the disjoint union of the six-cycle $0,1,2,3,4,5,0$ and the isolated vertex 6. Let x have lit vertices $\{0,1,3,4\}$. Then*

$$\mathrm{ML}_G(x) = 0, \qquad \mathrm{ML}^*_G(x) = 4, \qquad \mathcal{D}(G) \geq 4 > \frac{7}{2}.$$

The graph G is connected and is not complete multipartite.

Theorem 2. *For each integer $m \geq 1$, replace every vertex v of G by an independent fiber $\{(v,i) : 0 \leq i < m\}$, joining all pairs between two fibers exactly when their base vertices are adjacent. Call the resulting graph G_m, and set $x_m(v,i) = x(v)$. Then*

$$\mathrm{ML}_{G_m}(x_m) = 0, \qquad \mathrm{ML}^*_{G_m}(x_m) = 4m, \qquad \mathcal{D}(G_m) \geq 4m > \frac{7m}{2}.$$

Every G_m is connected and is not complete multipartite.

The multiplicity in Theorem 2 is independent of the number of parts in the conjecture below.

A complete m-partite graph is loopless, with its vertex set partitioned into m nonempty parts, two distinct vertices being adjacent exactly when they lie in different parts. Goldwasser, Wang and Wu's Conjecture 5 [1, p. 5] starts, "Let G be a connected graph on n vertices." Its full implication, in mathematical form, is

$$\mathcal{D}(G) > \frac{n}{2} \quad \Longrightarrow \quad \begin{array}{c} G \text{ is a complete } m\text{-partite graph for some positive integer } m, \\ m \equiv 3 \pmod 4. \end{array}$$

Theorem 1 contradicts that implication even for a loopless graph. The conjecture appears in the article published October 31, 2011 [1].

The contribution is the explicit graph and configuration in Theorem 1, together with a short certificate for the difference between their minimum light numbers. Goldwasser, Wang and Wu studied the parameter and posed the question, supplied the complete multipartite examples motivating it, and reported earlier theoretical and experimental investigations [1, Example 4 and pp. 5–6]. Wang and Wu's 2009 preprint [2] studies related differences, including a difference between two graph-wide maxima; that quantity must be distinguished from $\mathcal{D}(G)$. Wu and Xiang restated the present implication in their 2019 survey and described prior experiments on lit-only reachability [4, Conjecture 3.3 and pp. 10–11 of the author manuscript]. Their investigations form the background to the present question. An outside AI model suggested the three-type description and uniform blow-up extension proved here.

Hau-wen Huang describes the lit-only orbits of nondegenerate graphs that are not line graphs and proves that they are 2-lit, with a criterion for being 1-lit [5, Theorems 3.1 and 3.2]. Our loopless seven-vertex witness is degenerate: its adjacency matrix over $\mathbb{F}_2$ has the all-ones vector in its kernel. It therefore lies outside that class [5, Lemma 2.1]. No result, code or data from that work is used here.

Nearby algebraic work includes Wu's analysis of lit-only games on line graphs [3], and Sjöstrand's reachability characterization, first posted in 2023 and published in 2025 [6]. The argument below does not require either theory: its essential lower-bound step is to exclude every lit-only path to a configuration of weight less than four. Exhibiting some paths alone would not do this. A small set closed under every allowed move supplies the exclusion.

Proof overview

Section 2 specifies the adjacency lists and the starting configuration. Two regular moves turn every light off. For the lit-only game, a closed set contains the starting configuration and has no state with fewer than four lights. Induction on the length of a move sequence therefore proves the lower bound, and the starting configuration attains it. The nonedges 01 and 12, together with the edge 02, show that nonadjacency fails transitivity and rule out every complete multipartite graph. For the family, every move from a uniform state projects to a base move and preserves uniformity; every base move lifts. Weights therefore scale by m. Section 3 describes the Lean proof, the certificate's independent checkers, and their scope.

This work has not yet been reviewed by experts.

2 The finite certificate

We give all graph and configuration data needed to prove Theorem 1. Figure 1 shows the graph with the starting lights. Its universal vertex 6 makes the graph connected.

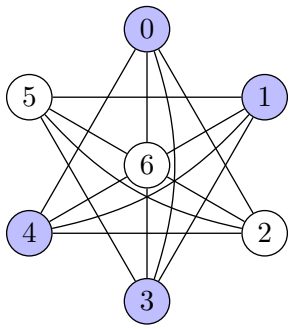

Figure 1: The graph in Theorem 1. Filled vertices are ON in the starting configuration x; open vertices are OFF.

For compact exact data, encode a configuration with lit set $A \subseteq \{0, 1, \ldots, 6\}$ by the integer $\sum_{i \in A} 2^i$. Thus the starting configuration has code $27 = 1 + 2 + 8 + 16$. Write r_v for the code of the neighbor set of vertex v. The adjacency data are

$$(r_0, r_1, r_2, r_3, r_4, r_5, r_6) = (92, 120, 113, 99, 71, 78, 63). \tag{1}$$

For example, $r_0 = 92 = 4 + 8 + 16 + 64$ means that vertex 0 has neighbors $\{2, 3, 4, 6\}$. The rows are symmetric and have no diagonal entries. Among vertices $0, \ldots, 5$, exactly the consecutive cycle pairs are missing; vertex 6 is universal. These are precisely the graph in Theorem 1.

Write $a \oplus b$ for bitwise exclusive or of two nonnegative integer codes. A regular move at v sends z to $z \oplus r_v$. In particular, the regular moves $0, 4$ give

$$27 \xrightarrow{0} 27 \oplus 92 = 71 \xrightarrow{4} 71 \oplus 71 = 0.$$

No configuration has a negative number of lights, so this proves $\mathrm{ML}_G(x) = 0$. The second move in this sequence is unavailable in the lit-only game, because vertex 4 is dark in state 71.

For the lit-only lower bound consider the set of configuration codes

$$S = \{27, 45, 54, 63, 71, 78, 92, 99, 113, 120\}. \tag{2}$$

This set consists exactly of three types, with indices on the outer cycle interpreted modulo six:

(A) The hub is dark and two of the three antipodal pairs $\{0, 3\}, \{1, 4\}, \{2, 5\}$ are lit: three states, each of weight four.

(B) The hub is dark and all six outer vertices are lit: one state, of weight six.

(C) The hub is lit and three consecutive outer vertices are lit: six states, each of weight four.

The initial state is of type (A). Rotations and reflections of the outer cycle fix the hub and preserve adjacency, hence carry moves to moves. Rotations carry each representative through every state of its type. For (A), a rotation by three and the reflection $v \mapsto 1 - v$ act transitively on the four lit vertices of the representative $\{0, 1, 3, 4\}$. For (B), rotations act transitively on its six lit vertices.

For the representative (C) state $\{0,1,2,6\}$, the reflection $v \mapsto 2-v$ exchanges the outer endpoints; the center and hub are separate cases. These symmetries cover all allowed moves by five cases, writing $(z,v) \mapsto z'$ for the successor of state z played at v:

$$(27,0) \mapsto 71, \quad (63,0) \mapsto 99, \quad (71,0) \mapsto 27, \quad (71,1) \mapsto 63, \quad (71,6) \mapsto 120.$$

Their successors have types (C), (C), (A), (B), (C), respectively. Thus every permitted move stays in S; a play at a dark vertex is the identity. Induction over every finite move sequence confines the lit-only orbit to S, giving $\mathrm{ML}^*_G(x) \geq 4$. The initial state has weight four, so equality holds.

In fact every state of S is attainable. The other two type (A) states, 45 and 54, are reached by the legal move lists $(1,5)$ and $(0,2)$. From 27, single moves reach $71, 99, 120, 92$; from 45, moves 0 and 3 reach 113 and 78. These are all six type (C) states. The move list $(0,1)$ reaches the type (B) state 63. Consequently S is exactly the lit-only orbit, and its three-type description proves the claimed minimum. Together with the regular minimum this gives $\mathcal{D}(G) \geq 4 > 7/2$.

Finally, 01 and 12 are nonedges of G, whereas 02 is an edge. In a complete multipartite graph, nonadjacent distinct vertices must lie in the same part. The two nonedges would put $0,1,2$ in one part, in contradiction to the edge 02. Thus G is not complete multipartite. This completes the proof of Theorem 1.

Uniform blow-ups

We prove Theorem 2 by identifying all configurations reachable from the uniform initial state. For a base configuration y, write $L_m y$ for its uniform lift, $(L_m y)(v,i) = y(v)$. If R_v is a regular move at v, then for every copy i,

$$R_{(v,i)}(L_m y) = L_m(R_v y).$$

Indeed, precisely the fibers of the neighbors of v are toggled, each in its entirety. The fiber of v is not toggled, since G is loopless. The played copy is lit exactly when $y(v)$ is lit, so the same identity holds for lit-only moves, including identity moves at dark vertices.

Induction now projects every finite move list to its base-vertex list, preserving order, repetitions, and lit-only legality at every step. Conversely, every base list lifts by playing copy 0 of each vertex; this copy exists because $m \geq 1$. Thus the configurations reachable from $x_m = L_m x$ in either game are exactly the uniform lifts of those reachable from x in that game. Each lit base vertex contributes exactly m lit copies, so $|L_m y| = m|y|$. The two minima are therefore 0 and $4m$, with the empty sequence attaining the latter and the lifted moves $0, 4$ attaining the former. The same-start difference proves $\mathcal{D}(G_m) \geq 4m > 7m/2$.

Every outer copy is adjacent to copy 0 of the hub. Every hub copy is adjacent to copy 0 of vertex 0, which is adjacent to copy 0 of the hub. These paths prove connectedness. The first copies of vertices $0,1,2$ remain distinct and induce the nonedges $01, 12$ and the edge 02. The same part-membership contradiction as before excludes every complete multipartition. This completes Theorem 2.

3 Formalization, computational checks and scope

The Lean project in `lean/` proves Theorem 1 from the game definitions, including the induction over arbitrary finite move sequences. The theorem `LitOnlySigma.counterexample` proves the graph's symmetry and connectivity, the two same-start minima, failure of every complete multipartition, and the negation of Conjecture 5. The new endpoints `LitOnlySigma.three_type_description` and `LitOnlySigma.uniform_blowup` prove the exact three-type orbit and Theorem 2, with every

hypothesis discharged. The projection and lifting lemmas quantify over arbitrary finite move lists and preserve strict lit-only legality. Weight scaling is proved by a bijection with the product of the base lit set and the copy set. The direct refutation is also named `LitOnlySigma.not_conjecture5`. The definitions and their correspondence with the source are given in `lean/Statement.lean` and `lean/FAITHFULNESS.md`.

With elan installed, the proof and its axiom guards are checked by running the following commands from the accompanying folder:

```
cd lean
lake exe cache get
lake build
lake env lean Audit.lean
cd ..
```

The toolchain and dependency manifest pin Lean and Mathlib to version 4.34.0. Finite certificate checks use kernel reduction; the proof imports no search program or external checker. The build checks the expanded endpoint types, the graph and starting-configuration descriptions, and a control showing that deleting state 45 destroys closure. Its guarded axiom reports require exactly `propext`, `Classical.choice` and `Quot.sound`; an additional axiom or proof hole fails the build. The audit command replays the axiom check separately. The trusted base is Lean's kernel and these standard logical foundations, together with the stated identification of the definitions with the source conjecture.

The accompanying checkers verify the finite certificate used in the proof. The file `data/certificate.json` contains the adjacency rows (1), the starting code, the two regular moves, and the closed set (2). The public check is

```
python3 checks/check.py
```

It runs two independent exact checkers and corruption controls, and confirms that the graph illustration and the retained ten-row regression table agree with the certificate. One checker uses integer bitmasks; the other uses explicit sets of vertices and symmetric difference, and additionally enumerates the configurations reachable from the starting state in each game. Neither checker imports search code. Both reject the corrupted certificate obtained by deleting state 45 from (2). The entry point needs Python 3.10 or later and only its standard library. The small finite data can also be checked directly from (1). The ten-row table is retained in `data/closure-table.tex` as a regression artifact; the proof above uses the three configuration types.

The verified finite obligations are the graph's stated adjacency, connectivity and failure to be complete multipartite; the regular path to zero; and containment of every allowed successor of every listed state in the listed set, with its asserted minimum number of lights. The mathematical passage from closure to all finite lit-only move sequences is the induction given above, also proved in Lean. The checkers use exact integer or set operations; there is no numerical approximation.

The result concerns the exhibited starting configuration and its consequence for the strict-threshold implication. The family theorem concerns uniform initial configurations only. The stronger statement $\mathcal{D}(G_m) = 4m$ over arbitrary nonuniform initial configurations remains unproved here. The result asserts neither minimality of the example nor an exact graph-wide value of $\mathcal{D}(G)$, and it does not settle the other light-number or maximum-degree conjectures in the cited papers.

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