The $(3, 4, \infty)$ family of complex six-spheres

I. Periods, moduli and the two ends

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Abstract

We determine the periods, moduli and two limiting regimes of the $(3, 4, \infty)$ family of complex six-spheres. A theta expression for its elliptic period gives the exact parameter disc $0 < |w| < 216$. Integral affine and cusp comparison maps classify the discrete data, parameter identifications and real structures, including the fixed three-spheres. We compute the integral monodromy group, its maximal domain of proper discontinuity and the algebraic dimension of every smooth fibre. Logarithmic direct images and explicit relative differential modules determine the Dolbeault, Bott–Chern, Aeppli and tangent cohomology, all Picard twists and the deformation parameter. Global Dolbeault representatives extend through the multiple fibres and cusp. At the puncture, the corner monodromy obstructs every proper Hausdorff analytic filling after any finite parameter cover; the analogous obstruction holds for the coprime higher-multiplicity sphere families. At the outer boundary, the intrinsic Albanese invariant obstructs continuation, while the dual period lattices determine the local Hausdorff reflection and all additive limits. The arithmetic norm satisfies exact division and height identities; we describe its torsion values, critical points and level curves.

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1 Results and their proofs in outline

1.1 The series and this paper

This series studies the $(3,4,\infty)$ family of torus-fibred complex structures on S^6 (Claude and L. Alpöge, 2026; Engel, 2026), its periods, maps and smooth monodromy, and the higher-dimensional complex manifolds built from it. The table gives the main result of each paper and the papers it uses.

Paper	Main result	Uses
I	Exact periods, moduli, fibre strata and the two ends	—
II	Torus-fibred threefolds with $b_1 = b_2 = 0$: the Campana–Demailly–Peternell argument and the sign theorem for all degenerations	I, IV
III	Maps from the six-spheres and the manifolds they give	I, II, IV, V
IV	Smooth monodromy in Θ_7	I, II, III, V
V	Complex fourfolds on $3\mathbb{HP}^2\#(-\mathbb{HP}^2)$ and a resolved fibre product	I, III, IV

1.2 The main theorems

We use the Claude–Alpöge construction with its specified periodic cusp fan, clockwise finite root generators and affine translations. Its base is $B = \mathbb{P}^1_t$, with special values $0, 1, \infty$, and $B^\circ = B \setminus \{0,1,\infty\}$. The ordered homology lattice and period matrix are

$$\Lambda = \mathbb{Z}\langle \widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta}\rangle, \qquad \Pi_c = \begin{pmatrix} 6\mu & \tau & 1 & 0 \\ \beta_{\mathrm{part}} + c & \mu & 0 & 1 \end{pmatrix}.$$

Write $d = \widehat{\delta}$, and use γ, u, w, δ for the dual integral covectors when they occur in lattice forms. The parameter $w = e^{-\pi i c}$ and its square $u = w^2$ are distinguished by their arguments or by the words "parameter" and "covector." The particular solution is fixed by $(\beta_{\mathrm{part}} + \tau)(\infty) = 0$; μ is the cusp-selected zero of $\wp_\tau$. A statement about $t = 0$ or $t = 1$ uses the specified smooth finite root model. Unless further data are displayed, X_c denotes the standard $(3,4)$ sphere member with $v_1 = \varepsilon = (1,2,-4,0)^{\mathsf{t}}$, $v_2 = -\varepsilon' = (-1,-3,3,0)^{\mathsf{t}}$ and the untwisted cusp collar. We write $e = (0,1)^{\mathsf{t}} = \Pi_c d$ for the invariant vertical vector. The broader admissible translation and higher multiplicity data are stated in Theorems B and F.

Theorem A (The period, its exact domain, and its arithmetic norm)**.** [tag 03JT] The missing period and the normalized norm are

$$\beta_{\mathrm{part}} = \frac{6i}{\pi} \operatorname{Log} \frac{\vartheta_1(\mu,\tau)}{2\sqrt{3}\,\eta_D(\tau)}, \qquad N(P) = \left|\frac{\vartheta_1(\mu,\tau)}{\eta_D(\tau)}\right| e^{-\pi(\Im\mu)^2/\Im\tau}, \quad P = [\mu].$$

The displayed logarithm is the branch fixed by the cusp condition. Putting $L = \log N$ and $D_c = \Im(\beta_{\mathrm{part}} + c) - 6(\Im\mu)^2/\Im\tau$, one has

$$D_{\mathrm{part}} = \frac{6}{\pi} \log \frac{N}{2\sqrt{3}}, \qquad \max N = \sqrt{2} \quad (t = 1), \qquad \max D_{\mathrm{part}} = -\frac{3}{\pi}\log 6.$$

All ordinary fibres and both normalized finite root fibres are compact complex tori exactly on

$$\Im c < H_* := \frac{3}{\pi}\log 6, \qquad 0 < |w| < 216.$$

There is no coarse critical point of L away from $t = 1$; $t = 0$ is a noncritical C^1 point with $L_t(0) = 7/384$. For every angle θ, $L(1 + re^{i\theta})$ decreases strictly from $\log\sqrt{2}$ to $-\infty$ as r increases from zero. Every level below the maximum is one Jordan curve strictly star-shaped about 1, real

analytic except for a possible C^1 point with nonzero tangent at $t=0$ on the level $L=\frac{1}{4}\log 3$. Its bounded complementary component is the strict superlevel set; these sets form nested discs.

In any of the explicitly transported pointed Weierstrass models, for $m\ge 2$ and $mP\neq O$,

$$N(P)^{m^2}=\frac{N(mP)|\Delta_E|^{(m^2-1)/12}}{|\psi_m(P)|}.$$

For torsion P of order n and $m\ge 2$ with $m\equiv\pm 1 \pmod n$, this gives

$$N(P)^{12}=\frac{|\Delta_E|}{|\psi_m(P)|^{12/(m^2-1)}}.$$

In particular the five specified values are

t	$\operatorname{ord}(P)$	$N(P)^{12}$
0	3	27
1	2	64
$32/5$	4	10
$64(47-27\sqrt{3})/11$	6	$17+10\sqrt{3}$
$64(47+27\sqrt{3})/11$	6	$10\sqrt{3}-17$

and the body gives the complete exact division-polynomial lists for orders four through eight. For every number field K and $t\in K\backslash\{0,1\}$, in the arithmetic model $y^2=x^3-3t^3(t-1)x/4+t^4(t-1)^2/4$ with $P=(0,t^2(t-1)/2)$,

$$\sum_{\sigma:K\hookrightarrow\mathbb{C}}\log N(P_{\sigma(t)})=\sum_{v\nmid\infty}n_v\lambda_v(P_t)-[K:\mathbb{Q}]\widehat{h}_O(P_t).$$

The normalization is $\lambda_\sigma(P)=-\log N(P)$, with complex embeddings counted separately. For rational t, the positive algebraic factor $A_t=\exp(\sum_p\lambda_p(P_t))$ gives $N(P_t)=A_te^{-\widehat{h}_O(P_t)}$. At the eleven nonspecial rational CM parameters $t=j/1728$ its values are

j	A_t	j	A_t
8000	$\sqrt{6}$	-3375	2
54000	$2^{2/3}3^{1/4}$	-32768	$\sqrt{3}$
287496	$2^{3/4}$	-884736	1
16581375	2	-12288000	$3^{5/12}$
-884736000	1	-147197952000	1
-262537412640768000	1		

Mechanism. Theta transformation laws fix the additive constant, division identities control the norm, and the local-height product formula fixes its arithmetic normalization. *Literature status.* New period evaluations, norm values and level classification, using the elliptic section of (Claude and L. Alpöge, 2026, Remark 3.12, p. 19). Proof in outline.

Theorem B (All discrete data, the parameter, and real forms)**.** [tag 03JU] For the fixed period solution and cusp fan, take

$$w_1=r_1\varepsilon+k_1d,\qquad w_2=r_2\varepsilon'+k_2d,\qquad 3\nmid r_1,\quad 2\nmid r_2,\quad p=12l_0-4r_1-3r_2,$$

where l_0 is the first integral winding of the full holomorphic cusp collar in the stated positive convention. Two such members are biholomorphic exactly when

$$n=c'-c\in\mathbb{Z},\qquad p'=ap\ (a\in\{1,-1\}),\qquad n\equiv k_2-k_2' \pmod 2.$$

Every biholomorphism is over B, and the map with sign a has fibre-linear part aI_2. The complete invariants are

$$\big(|p|, e^{-\pi i(c+k_2)}\big), \qquad |p| \geq 1, \quad \gcd(|p|, 6) = 1,$$

and every displayed integer occurs. The additional analytic collar part and its second integral winding contribute no further invariant. For sphere members, $|p| = 1$, a fixed datum has period subgroup $2\mathbb{Z}$. At each fixed admissible $u = w^2$ there are exactly two sphere isomorphism classes, exchanged by a positive circuit of u about zero.

For positive k, l with $\gcd(3k, 4l) = 1$, put $m_1 = 3k, m_2 = 4l$. Use Viaclovsky's sphere data $w_j = r_j\varepsilon_j + s_j d$, $\gcd(r_j, m_j) = 1$, $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon'$, and $p = m_1 m_2 l_0 - m_2 r_1 - m_1 r_2 = \pm 1$. Two members are biholomorphic precisely when their multiplicity pairs agree and, for some $a = \pm 1$ and integer $n = c' - c$,

$$p' = ap, \quad r'_j \equiv a r_j \pmod{m_j}, \quad n r'_1 \equiv a s_1 - s'_1 \pmod{k}, \quad n r'_2 \equiv a s_2 - s'_2 \pmod{2l}.$$

The fixed-data period subgroup is $2kl\mathbb{Z}$. At fixed admissible u there are exactly $2kl$ classes for this multiplicity pair, and a parameter circuit permutes them in one cycle.

For the standard family the reduced finite fibres have Albanese nomes

$$q_2 = iw/216, \qquad q_1 = -w^2/48^3 = (27/64)q_2^2.$$

An antiholomorphic diffeomorphism $X_c \to X_{c'}$ exists exactly when $c' - (1 - \bar{c}) \in 2\mathbb{Z}$, equivalently $w' = -\bar{w}$. The members with real structures are precisely $\Re c \in \frac{1}{2} + \mathbb{Z}$. On each,

$$\mathrm{Aut}^{\pm}(X_c) = \mathbb{C}^* \rtimes \mathbb{Z}/2, \qquad z \longmapsto 1/\bar{z},$$

and there are two holomorphic-conjugacy classes of antiholomorphic involutions. For negative imaginary w both fixed manifolds are S^3, with the semifree circle action having the circle $D_2^{\sigma_c}$ as fixed set. For positive imaginary w one class is fixed-point free and the other has fixed manifold S^3 with the free Hopf circle action, up to the stated circle and orientation conventions.

Mechanism. Integral centralizers, finite norm maps and cusp winding determine the parameter equivalence, while the marked reflection and cap-meridian maps determine the real forms. *Literature status.* New classification for the displayed discrete data and real forms; the standard $2\mathbb{Z}$ equivalence is known (Du et al., 2026a, Theorem 1.4, p. 3). Proof in outline.

Theorem C (The monodromy quotient and every fibre's algebraic dimension)**.** [tag 03JV] Let $\psi = 6\gamma \wedge \delta + u \wedge w$ on Λ, and let $P_{d,\mathbb{Z}}$ fix d as a vector in its integral symplectic group. Write $J_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Every member has unique block coordinates

$$g(B, x, t) = \begin{pmatrix} 1 & 0 & 0 \\ 6x & B & 0 \\ t & -x^{\mathrm{t}} J_2 B & 1 \end{pmatrix}, \quad B \in \mathrm{SL}_2(\mathbb{Z}),\ x \in \mathbb{Z}^2,\ t \in \mathbb{Z}.$$

For $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, let $\alpha : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}/12$ be the homomorphism with $\alpha(S) = 9$, $\alpha(T) = 1$. The monodromy group is

$$\Gamma = \{g(B, x, t) : t + 6(x_1 + x_2 + x_1 x_2) - \alpha(B) = 0 \pmod{12}\}.$$

It has index 12 in $P_{d,\mathbb{Z}}$ and index 24 in $P_{\mathbb{Z}}$, the group preserving $\mathbb{Z}d \subset \ker\gamma \subset \Lambda$. In the real vector stabilizer $P_d(\mathbb{R})$, a fundamental set for $\Gamma\backslash P_d(\mathbb{R})$ is given by the classical modular region for τ, the Iwasawa angle $0 \leq \theta < \pi$, $0 \leq x_1, x_2 < 1$ and $0 \leq t < 12$, with the modular lifts $g(S, 0, 9)$, $g(T, 0, 1)$, $g(-I, 0, 6)$ and the Heisenberg face identifications

$$(x_1, x_2, t) \mapsto (x_1 + 1, x_2, t - 6 - 6x_2), \quad (x_1, x_2, t) \mapsto (x_1, x_2 + 1, t - 6 + 6x_1).$$

On $\mathcal{D} = \mathrm{Sp}_4(\mathbb{R})/\,\mathrm{U}(1,1)$ put $q_d(J) = \psi(d, Jd)$. The largest Γ-invariant open domain of proper discontinuity is $\mathcal{O} = \{q_d \neq 0\}$; the same assertion holds for $P_{\mathbb{Z}}$-invariant open domains and its action. The null locus has a dense set of points with infinite monodromy stabilizer. Each sign region has a Hausdorff complex analytic orbifold quotient. In the family chart, $\mathcal{O}_+$ has $\Im\tau > 0$, $D_c < 0$ and $q_d = -6/D_c$. For every admissible c the period map is injective and immersive on B° and extends to a proper holomorphic embedding

$$\mathbb{C} = B \setminus \{\infty\} \hookrightarrow \Gamma\backslash\mathcal{O}_+.$$

For every smooth fibre its Néron–Severi group is the integral kernel in $\mathbb{Z}^6$ of

$$E_{12} - E_{13}\tau + (6E_{23} - E_{14})\mu + E_{24}\beta_c + E_{34}(6\mu^2 - \tau\beta_c) = 0.$$

With its complex orientation, $E^2 = -2\,\mathrm{Pf}(E)$, where $\mathrm{Pf}(E) = E_{12}E_{34} - E_{13}E_{24} + E_{14}E_{23}$. Let $\mathcal{E}_c$ be the image on B° of the union of these zero sets for $E \neq 0$ with $\mathrm{Pf}(E) = 0$, and let $\mathcal{A}_c$ use $\mathrm{Pf}(E) < 0$. They are exactly the elliptic-quotient locus and abelian-surface locus, respectively, and

$$a(F_t) = 2 \ (t \in \mathcal{A}_c), \quad a(F_t) = 1 \ (t \in \mathcal{E}_c \setminus \mathcal{A}_c), \quad a(F_t) = 0 \ (t \notin \mathcal{E}_c \cup \mathcal{A}_c).$$

For each nonzero form E in this integral kernel with zero Pfaffian, put $K_{\mathbb{R}} = \ker(E : \Lambda_{\mathbb{R}} \to \Lambda_{\mathbb{R}}^*)$ and $K = K_{\mathbb{R}} \cap \Lambda$. The maps

$$(\Pi_c K_{\mathbb{R}})/\Pi_c K \ \hookrightarrow F_t \longrightarrow \mathbb{C}^2/(\Pi_c K_{\mathbb{R}} + \Pi_c \Lambda)$$

give every connected elliptic kernel and its quotient, including kernels that mix the two displayed coordinates. The invariant class $\eta = 6\gamma \wedge \delta + u \wedge w$ has square -12; a fibre with $\mathrm{NS} = \mathbb{Z}\eta$ has no compact curves.

Mechanism. The integral congruence kernel determines the parabolic quotient, and the wedge equation with its Hermitian form determines the fibre strata. *Literature status.* New integral monodromy and maximal proper domain, answering (Engel, 2026, Remark 8.4, pp. 23–24). Proof in outline.

Theorem D (Cohomology and the ordinary relative differential modules)**.** [tag `03JW`] Every admissible standard sphere member has the following Dolbeault, Bott–Chern and Aeppli dimensions, with columns $p = 0, 1, 2, 3$ and rows $q = 3, 2, 1, 0$:

$q\backslash p$	0	1	2	3
3	0	0	0	1
2	0	1	2	1
1	1	2	1	0
0	1	0	0	0

$\bar\partial$

$q\backslash p$	0	1	2	3
3	0	0	1	1
2	0	1	2	1
1	0	2	1	0
0	1	0	0	0

BC

$q\backslash p$	0	1	2	3
3	0	0	0	1
2	0	1	2	0
1	1	2	1	0
0	1	1	0	0

A .

The Frölicher sequence satisfies $E_1 \neq E_2 = E_\infty$; the only nonzero d_1 have rank one at $(p, q) = (0, 1), (1, 1), (1, 2), (2, 2)$. The Bott–Chern algebra is $\mathbb{C} \cdot 1 \oplus \mathcal{N}$ with $\mathcal{N}^2 = 0$.

For the ordinary relative differentials the local presentations retain both the degree and the full fibre multiplicity. For $R = \mathbb{C}\{x, y, z\}$ and $t = xy$,

$$\Omega^1_{R/\mathbb{C}\{t\}} = R^3/R(y, x, 0) \simeq (x, y) \oplus R, \quad \Omega^2 \simeq (x, y) \oplus R/(x, y), \quad \Omega^3 \simeq R/(x, y).$$

For $t = xyz$, $\Omega^1 = R^3/R(yz, xz, xy)$ is torsion-free of rank two and not locally free on the double locus. With $I = (xy, xz, yz)$,

$$0 \longrightarrow \frac{Ru \oplus Rv}{(xu,\ y(u - v),\ zv)} \longrightarrow \Omega^2 \xrightarrow{\wedge dt} I \longrightarrow 0, \qquad \Omega^3 = R/I.$$

The left term is exactly the torsion and is annihilated by I. For $t = s^m$ in $R = \mathbb{C}\{s, y_1, y_2\}$, the three modules are

$$R^2 \oplus R/(s^{m-1})\,ds, \qquad R\,dy_1 \wedge dy_2 \oplus (R/(s^{m-1}))^2, \qquad R/(s^{m-1})\,ds \wedge dy_1 \wedge dy_2.$$

Consequently the degree-one torsion of $\Omega^1_{X/B}$ is supported exactly on $S_1 \cup S_2$, while degree two also has torsion along the cusp double locus.

There are specified global forms $G \in \mathcal{A}^{0,1}(X)$ and $Q \in \mathcal{A}^{1,1}(X)$ satisfying

$$\bar\partial G = \bar\partial Q = 0, \qquad [G] = g_\gamma, \qquad i_b^*[Q] = \eta, \qquad \exp(12[G]) = f^*\mathcal{O}_{\mathbb{P}^1}(1).$$

On the specified common refinement of the original atlas they are $G = \bar\partial \sum w_n f_{a,n}$ and $Q = \frac{1}{12}\bar\partial \sum w_n c_{a,n}$, with the locally finite weights, holomorphic scalar cocycle and holomorphic one-form cocycle of (14.13). The elementary derivative recursion specifies every coefficient through both multiple fibres and every cusp stratum. The form Q is complex; a real $\bar\partial$-closed form cannot have this fibre class.

Mechanism. Root characters and logarithmic residues compute the connecting scalar and direct images, exterior quotients by dt give the ordinary modules, and theta factors with cusp Cartier frames give the coefficient cocycles. *Literature status.* New evaluated cohomology and coefficient atlas; the ordinary-module calculation corrects (Claude and L. Alpöge, 2026, Remark 9.21, p. 81). Proof in outline.

Theorem E (Every Picard twist and the deformation parameter)**.** [tag 03JX] Use the primitive coordinate $M_z = \exp(z\mathfrak{g})$ on $\mathrm{Pic}(X) = \mathbb{C}$, so $M_n = \mathcal{O}(n(S_1 - S_2))$ for $n \in \mathbb{Z}$. Let $r_1(n) \in \{0, 1, 2\}$ and $r_2(n) \in \{0, 1, 2, 3\}$ be the residues of n modulo 3 and $3n$ modulo 4, respectively, and set

$$d(n) = \frac{n - 4r_1(n) - 3r_2(n)}{12},$$
$$U(n) = \max\{d(n) + 1, 0\}, \quad V(n) = \max\{-d(n) - 1, 0\},$$
$$\epsilon_n = \begin{cases} 0 & 3 \mid n, \\ 1 & 3 \nmid n. \end{cases}$$

Then

$$\begin{aligned} h^0(T_X \otimes M_n) &= U(n) + U(n-1), \\ h^1(T_X \otimes M_n) &= V(n) + V(n-1) + U(n) + U(n-1) + U(n+2), \\ h^2(T_X \otimes M_n) &= V(n) + V(n-1) + V(n+2) + U(n+6) + \epsilon_n, \\ h^3(T_X \otimes M_n) &= V(n+6). \end{aligned}$$

For $z \notin \mathbb{Z}$, let $\mathcal{R}_z$ be the set of distinct regular base points for which integers n, p, q, r satisfy

$$q\tau + r\mu = p, \qquad z = n - 6q\mu - r\beta_c.$$

It is finite; write $N_z = |\mathcal{R}_z|$. Put $\rho = e^{\pi i/3}$, $\beta_j = \beta_c(z_j)$ at the specified order-three and order-four marked root centers. Define $\ell_1(z) = 1 + \epsilon_n$ at $z = n + k(3\beta_1 + 4 - 2\rho)$, $k \neq 0$, and zero elsewhere; define $\ell_2(z) = 1$ at $z = n + k(2\beta_2 + 3 - 3i)$, $k \neq 0$, $n \equiv 1, 2 \pmod 4$, and zero elsewhere. With $J_z = 2N_z + \ell_1(z) + \ell_2(z)$,

$$(h^0, h^1, h^2, h^3)(T_X \otimes M_z) = (0, J_z, 1 + J_z, 0).$$

The cohomology jump loci are locally discrete in this Picard line, and the generic tuple is $(0, 0, 1, 0)$.

The untwisted tangent tuple is $(1, 1, 1, 0)$, the holomorphic automorphism group of the standard member is $\mathbb{C}^*$, and its Kuranishi germ is a smooth reduced line. For every sphere

member with coprime multiplicities $(3k, 4l)$ and the data of Theorem B, put $z_0 = kl$, $d_0 = 3k+4l$. On the analytic weighted projective root stack $\mathfrak{B}_{k,l} = \mathbb{P}(3k, 4l)$, the vertical root bundle E and the root lift satisfy

$$E = \mathcal{O} \oplus \mathcal{O}(-z_0), \quad R^q \bar{f}_* \mathcal{O}_X = \mathcal{O},\ \mathcal{O} \oplus \mathcal{O}(-z_0),\ \mathcal{O}(-z_0), \quad K_X = \bar{f}^* \mathcal{O}(z_0 - d_0).$$

Its tangent tuple is again $(1, 1, 1, 0)$ and its Kuranishi germ is a smooth reduced line. For the groupoid derivative $\kappa(\Phi) = (\partial_c \Phi)\Phi^{-1}$ the classes of ∂_c and ∂_u are $e \otimes [\gamma]$ and $-(2\pi i u)^{-1} e \otimes [\gamma]$, respectively, and are nonzero. Every stated fixed-data period family is locally miniversal.

Mechanism. Weighted section counts and local Koszul modules compute every Picard twist, while the nonzero Kodaira–Spencer class gives the coordinate on the Kuranishi line. *Literature status.* New all-twist and higher-multiplicity formulas; the standard untwisted deformation results are known (Du et al., 2026a, Theorem 6.12, pp. 81–82; Proposition 6.16, p. 85). Proof in outline.

Theorem F (The puncture and the obstruction to every proper filling)**.** [tag 03JY] For an admissible $(3, 4)$ logarithmic datum put $b = 4l_1 + 3l_2$, $p = 12l_0 - b$, $a = -p$, with $3 \nmid l_1$ and l_2 odd. After $w = r^6$ there is a proper holomorphic partial filling over

$$(\mathbb{P}^1 \times \Delta_r) \setminus \{(\infty, 0)\}$$

with smooth total space and reduced normal-crossing parameter central divisor. Over a regular finite base point the central fibre is a cycle of twelve ruled surfaces, with component cover

$$y^{12} = t^{4l_1}(t-1)^{3l_2}$$

and rotations $4l_1, 3l_2, -b$ modulo 12. On $q = Q^{12m}$ and $r = R^n$, the difference between the cusp-extending and parameter-extending origins has deck differences

$$am\gamma - 6am^2\delta, \qquad an\gamma - 6an(n-1)\delta,$$

so the radial class modulo $\mathbb{Z}\widehat{\delta}$ is $-pn\widehat{\gamma}$ in these fixed conventions.

The punctured family has no proper Hausdorff complex-analytic filling at $w = 0$ after any finite parameter cover. The conclusion permits arbitrary singularities, multiplicities and extra central components, and holds for every admissible datum, including every sphere member. It also holds for every admissible datum of the coprime $(3k, 4l)$ construction. In that case

$$N = 12kl, \quad b = 4l\, l_1 + 3k\, l_2, \quad a = b - Nl_0, \qquad \gcd(l_1, 3k) = \gcd(l_2, 4l) = 1,$$

after $w = r^{N/2}$ the corresponding partial model has N components. On $q = Q^{Nm}$ and $r = R^n$, the two displayed deck differences have $N/2$ in place of 6.

For the untwisted elliptic-surface quotient $X(u)$, whose logarithmic translations are zero, the origins have zero corner winding. That family admits a proper flat Hausdorff analytic filling at $u = 0$, obtained by the two-projective-line-bundle plumbing, proper modification and finite quotient in the body. The particular one-component central model proposed in (Engel, 2026, Conjecture 8.2, p. 22) cannot be the central fibre of a proper flat smoothing to these $X(u)$: its virtual Euler characteristic is zero, whereas $e(X(u)) = 2$.

Mechanism. The intrinsic canonical system recovers the marked base, finite-orbit vanishing kills horizontal meridians, and their constant ratio across the connected compact corner contradicts the nonzero cusp coefficient. *Literature status.* New filling obstruction and untwisted plumbing; the Euler calculation corrects the proposed central model in (Engel, 2026, Conjecture 8.2, p. 22). Proof in outline.

Theorem G (The outer boundary and all marked additive limits)**.** [tag 03JZ] For every w_0 with $|w_0| = 216$ and every connected open neighbourhood U of w_0, there is no flat proper holomorphic map from a reduced complex space to U whose fibre at each $w \in U \cap \{0 < |w| < 216\}$ is biholomorphic to X_w. The intrinsic divisor $2S_2$ and its Albanese nome $iw/216$ retain the invariant $J(iw/216)$, where $J(q) = q^{-1} + 744 + 196884q + \cdots$ has the unit circle as a natural boundary.

Fix c_* with $\Im c_* = H_*$, put $a = \Re c_* + 3/2$, and use the specified order-four root coordinate h at $t = 1$ and its fixed complex-linear fibre coordinates (η, ξ). In the transformed integral period basis the four vectors are

$$(1, 1/2), \quad (\tau, 1/2 + h), \quad (0, 1), \quad (6h, a + B(h) + c - c_*), \qquad B(h) = -3ih^2 + O(h^4).$$

For the total root quotient at fixed $c = c_*$, two central points are nonseparated exactly when

$$\Re(\xi' - \xi) \in G_a = \tfrac{1}{2}\mathbb{Z} + a\mathbb{Z} \quad (a \in \mathbb{Q});$$

every central pair is nonseparated for $a \notin \mathbb{Q}$. Distinct noncentral points and distinct base fibres are separated. The Hausdorff reflection keeps the noncentral complex tori and has central fibre $\mathbb{R}/G_a \simeq S^1$ for rational a, and one point for irrational a. For reduced $a = m/n$, $n > 0$, $G_a = \operatorname{lcm}(2, n)^{-1}\mathbb{Z}$.

For sequences $c_n \to c_*$ with $\Im c_n < H_*$ and $h_n \to 0$, write

$$\delta_n = -D_{c_n}(h_n) > 0, \quad V_n = \delta_n^{-1}(6\Re h_n, 6\Im h_n, \Re(c_n - c_*)),$$

$$L_a = \{(p, q, r) : r \in \mathbb{Z},\ p + r/2 \in \mathbb{Z},\ q + r/2 \in \mathbb{Z},\ ar \in \mathbb{Z}\}.$$

It has rank three for rational a, and equals $\mathbb{Z}^2 \times \{0\}$ for irrational a. Each convergent subsequence of the closed period lattices has limit

$$\Gamma_\infty = \{(\eta, \xi) : p\Re\eta + q\Im\eta + r\Re\xi + \ell(p, q, r)\Im\xi \in \mathbb{Z} \text{ for every } (p, q, r) \in K\},$$

where $K \subset L_a$ is primitive, $K = \{\lambda : \lambda \cdot V_n \text{ has a finite limit}\}$, and $\ell(\lambda) = \lim \lambda \cdot V_n$. For labels outside K the absolute values tend to infinity. Conversely every such primitive K and homomorphism $\ell : K \to \mathbb{R}$ occurs. The resulting pointed additive quotient limits are real tori of every dimension zero through three for rational a, and zero through two for irrational a.

Mechanism. The intrinsic Albanese invariant detects the outer boundary, and the exact dual-period equations with their quadratic cancellation determine nonseparation and every marked closed-group limit. *Literature status.* New boundary obstruction and marked limit formulas, using the annihilator duality of (Cornulier, 2011, Theorem 3.1, p. 2017). Proof in outline.

1.3 The proofs in outline

Theorem A.

1. The pointed maps $(x, y) \mapsto (2x, -2\sqrt{2}y)$ and $(x_0, y_0) \mapsto (-x_0, y_0/(2i))$ transport the chosen section, invariant differential and period lattice. The quotient exponential identifies the period model with the elliptic-surface line-bundle quotient (2.10, 2.11).

2. The two theta transformation factors give the prescribed affine period laws; the cusp product fixes the additive constant and hence the displayed logarithm (3.2).

3. The distribution identity gives the exact division-polynomial norm formula, and $mP = \pm P$ cancels the remaining norm. The stated polynomial roots and values then determine every order-four-through-eight specialization (4.7, 4.8, 4.9).

4. Exact polynomial bounds away from the square elliptic point and the controlled local theta expansion give $\max N = \sqrt{2}$. The determinant identity and its cusp limit then give exactly $\Im c < H_*$ (6.2, 6.3, 6.4, 6.5, 6.6).

5. The normalized critical factor satisfies $|1 - X| < 1$ and $(t-1)L_t = X(X-2)/24$. Its radial sign, the coarse derivative at zero and the cusp limit give the complete critical and level classification (25.6, 19.18).

6. The local multiplication identity evaluates every finite-place contribution on the stated minimal models. The product formula identifies the archimedean term and gives the eleven exact CM factors (24.2, 24.4).

Theorem B.

1. The intrinsic canonical pencil fixes the base. Its period centralizer forces $c' - c \in \mathbb{Z}$ and the fibre-linear part aI_2 (7.2, 7.8).

2. Solving the two finite norm equations gives the parity condition; comparing the cusp collar gives $p' = ap$. The holomorphic correction solves the remaining gluing, so these necessary equations are sufficient (8.1, 8.2, 8.3).

3. The higher root pullbacks multiply the norm images by k, l, yielding exactly the congruences modulo $k, 2l$. Their intersection is $2kl\mathbb{Z}$, and a positive u circuit is the cyclic parameter translation (8.4, 9.4).

4. The invariant-coordinate lattice, including its affine half-period, and the two exact finite theta values give the Albanese nomes (17.10).

5. The explicit reflected period and affine lattice maps give $c' \equiv 1 - \bar{c} \pmod{2\mathbb{Z}}$. Their square and holomorphic conjugacy identify the two involution classes (17.2, 17.3).

6. The fixed-lift congruences and cusp units determine the nonempty pieces. The exact cap-meridian matrices close them to S^3 and determine the semifree or Hopf circle action (17.5, 17.6).

Theorem C.

1. The parabolic multiplication law and the polarization of $x_1 + x_2 + x_1x_2$ prove the character law. The commutator and its conjugates generate its integral kernel, giving the two exact indices (10.2).

2. The modular lifts first reduce the Iwasawa factor, integral Heisenberg translations reduce x, and the center reduces t. Interior uniqueness and the displayed face maps give the fundamental set (17.9).

3. Real parabolic stabilizers are compact on $q_d \neq 0$. Rational parabolic conjugates of the explicit null unipotent give a dense infinite-stabilizer locus, which proves maximality (10.6, 10.7).

4. The modular function $j/1728$ is the period map's left inverse, while the finite root charts extend it at zero and one. Its control on compact inverse images proves properness on the finite t-line (10.8).

5. Wedge the period rows with an integral alternating form to obtain the exact $(1,1)$ equation. Its Hermitian determinant and the complex orientation give $E^2 = -2\,\mathrm{Pf}(E)$ and the quotient/projectivity tests (11.5, 11.7).

Theorem D.

1. Root characters and the cusp normalization compute the logarithmic direct images. The residue connecting scalar is $-iC_4/4 \neq 0$, which fixes the remaining Hodge integer at one (12.11, 12.13).

2. Coherent Leray on the curve, the structure-sheaf inputs and Serre duality give the complete Dolbeault table, uniformly for every admissible parameter (12.4, 12.15).

3. The middle rows and the S^6 de Rham groups force the four rank-one differentials. The double-complex decomposition then gives the Bott–Chern and Aeppli tables; wedge products of the specified representatives give the square-zero multiplication (13.2, 13.5, 13.10).

4. Quotienting by dt gives the degree-one modules. The maps $(a, b, c) \mapsto (xa - zc, yb - zc)$ and $\Omega^2 \xrightarrow{\wedge dt} I$ compute their kernels and torsion exactly; finite étale descent gives the global support assertion (12.17, 12.18).

5. The theta factors with their original finite affine phases and the extending cusp Cartier units define the local twelfth-power lines on the four pieces. The exact determinant divisor makes the displayed holomorphic connection inverse valid on every overlap annulus. Its one-form cocycle and the specified coordinate weights give Q; the scalar cocycle gives the primitive G and its factor twelve normalization (14.2).

Theorem E.

1. The primitive Picard normalization identifies the integer twists, and weighted coarse degrees give their section counts and four dimensions (13.8, 15.1).

2. The integral cocycle modulo the two period rows gives the regular resonance equation and both finite root lattices. At the cusp the normalization complex gives exactly the reduced degree-two skyscraper (15.2, 15.3).

3. The joint order-three coefficient $123ik/(64\pi)$ fixes the local normal form; Smith modules and every stabilizer character give the complete finite stalk lengths. Leray then gives the formula for J_z (15.4, 15.5).

4. The established $\bar{\partial}$-closed form $G_\gamma = G_0/12$ defines the Picard family by $\bar{\partial} - 2\pi i z G_\gamma$. Proper coherent base change and a real nonintegral nonresonant point make every jump locus locally discrete (13.7, 13.8, 15.5).

5. The root-forgetting map $(x, y) \mapsto (x^l, y^k)$ with character $\lambda \mapsto \lambda^{kl}$ pulls back the standard Hodge line. Its weighted direct images and logarithmic tangent sequence give $(1, 1, 1, 0)$ for every stated higher sphere member (8.4, B.2).

6. Differentiating the period groupoid gives the nonzero class $e \otimes [\gamma]$. The induced parameter map forces the one-variable Kuranishi ring to be reduced and smooth. The integral translation sheaf gives the standard automorphism group (B.2, C.4).

Theorem F.

1. The node charts $X_i = W/r^i$, $Y_i = r^{i+1}/W$, $X_i Y_i = r$ and their inverse overlaps give the proper rank-one cycle. The finite affine actions rotate its components freely and produce the stated cover. Subtraction of the two origins fixes both deck differences (18.2).

2. Resolve a proper filling's dominating component. Its relative anticanonical direct image recovers the pencil, and the three labelled critical-value sections recover the marked base by their meromorphic cross-ratio (18.3).

3. Modulo $\widehat{\delta}$ the cross-cocycles vanish. The three unipotent annihilator equations have determinant 6; Stein factorization on a horizontal central component then kills its normal-meridian class (18.4, 18.5).

4. Principalization makes both axes normal crossing. The Stein finite factor is birational over the normal bidisc, so the compact corner fibre is connected. The lifts $Q_o = T_{-mb\widehat{\gamma}}Q$, $R_o = R$ commute; local meridians therefore satisfy the constant-ratio equation (18.6).

5. The connected incidence graph joins a horizontal ratio zero to the strict cusp ratio a. Admissibility gives $\gcd(a, 12) = 1$, producing the contradiction for every proper analytic filling and every finite cover (18.6).

6. For higher multiplicities, $K_X^{-N} = f^*\mathcal{O}(4l + 3k - kl)$ with nonzero degree. The indicated canonical or anticanonical power and its three labelled critical points recover the degree-one base parameter. The same ratio argument and $\gcd(a, N) = 1$ give the full higher conclusion (18.8).

7. With zero logarithmic translations the two line-bundle neck equations have zero corner winding and give the proper flat plumbing after its specified modification and finite quotient. Normalization and the double-locus smoothing formula give virtual Euler zero for Engel's particular one-component proposal, excluding its smoothing to $e = 2$ (21.3, 21.8).

Theorem G.

1. The canonical divisor recovers S_2 intrinsically in a proper extension. Its normalization and Albanese invariant retain $J(iw/216)$, whose natural boundary excludes crossing the outer circle (19.4).

2. The fixed complex-linear and integral basis changes give the displayed four periods. The local theta expansion gives $B(h) = -3ih^2 + O(h^4)$ and the exact inverse transpose gives every integral dual covector (19.5, 19.3).

3. Bounded dual covectors have constant integral labels after subselection; their labels form L_a. The quadratic real term cancels, leaving $\lambda \cdot V_n + o(1)$. The primitive surviving subgroup and its limiting homomorphism give exactly Γ_∞ (19.9).

4. Choose v orthogonal to K and outside the annihilator hyperplanes of $L_a \setminus K$. The vectors $V_n = nv + v_0$, with the positive scales δ_n chosen as in the proof, realize every primitive subgroup and limiting homomorphism. Pontryagin–Chabauty duality converts their dual graphs to the stated real-torus additive quotients (19.9).

5. At fixed boundary parameter, irrational root rays make all noncentral dual labels escape. The surviving rational label is $\operatorname{lcm}(2, n)\Re\xi$; its circle-valued function separates precisely the asserted classes. Collapsing those classes proves the universal Hausdorff-reflection property (19.7).

1.4 Known and new

Theorem	Status and credited inputs
A	New explicit evaluations and sharp norm bounds from the known section (Claude and L. Alpöge, 2026, Remark 3.12, p. 19); the quotient and height inputs are (Engel, 2026, Definition 2.3, p. 5; proof of Lemma 2.1, pp. 3–4), transported by lemma 2.10.
B	New discrete-data and real-form extensions of the standard equivalence (Du et al., 2026a, Theorem 1.4, p. 3); the higher construction is (Viaclovsky, 2026a, Theorem 1.1, p. 1; Section 2.1, pp. 4–6), with $(k, l) = (1, 1)$ giving the standard data.
C	New exact group, quotient and proper-domain answer to (Engel, 2026, Remark 8.4, pp. 23–24); the fibre tests use the integral wedge equation of (Claude and L. Alpöge, 2026, Lemma 9.2, p. 65) with the period row $\beta = \beta_c$.
D	New value of the Hodge integer, full cohomology and coefficient representatives extending (Claude and L. Alpöge, 2026, Theorem 9.1(vi), p. 65; Proposition 9.20, p. 80); the degree-one cusp module corrects its Remark 9.21, p. 81, and specifies the degree dependence of Remark 1.3, p. 5.
E	New Picard-twist and coprime-root calculations extending the known standard tangent, Kuranishi and automorphism results (Du et al., 2026a, Theorems 1.3–1.4, p. 3; Theorem 6.12, pp. 81–82; Proposition 6.16, p. 85; Theorem 7.14, p. 95), under the parameter map of remark A.1.
F	New marked nonfilling obstruction and explicit untwisted filling in the degeneration setting of (Engel, 2026, Remark 8.1, p. 22); virtual Euler number zero excludes the particular central model in its Conjecture 8.2, p. 22.
G	New intrinsic boundary and marked additive-limit calculations; the map $w \mapsto iw/216$ supplies the modular boundary, and (Cornulier, 2011, Theorem 3.1, p. 2017) supplies closed-group duality under the real annihilator map.

1.5 Methodology

This text is AI-produced; no part of it was written by a human. GPT-5.6 Sol (OpenAI), in Codex at reasoning effort ultra, contributed to the working edition of 24 August–9 September 2026. The pinned repository's authorship record also credits GPT-6 Astra (OpenAI), at reasoning effort ultra, from 5 September. GPT-6 Astra (OpenAI), in ChatGPT Pro mode, carried out the research of 12–16 November, the independent re-derivation of 16–18 November, and the writing of 16–20 November 2026. GPT-6 Astra (OpenAI), in ChatGPT Pro mode, also undertook the publication revision and further checks of this paper on 20–21 November 2026, including the continuation on 21 November.

Claude Fable 5.2 (Anthropic), at effort high and at effort max, and Claude Opus 5.5 (Anthropic), at effort max, undertook readings, intakes, checks and recomputations on 12–20 November 2026. Their objections and calculations were compared with the statements, proofs and cited sources, and the resulting corrections were incorporated into the text. The records specify the passages and calculations examined by each model.

For the statements printed in this paper, the record contains 767 examinations of proof steps, 10 independent re-derivations, and 143 computations or formal certificates. It also records 809 readings of the proofs and applications, 194 uses of background results without a fault found, and 1 refutation associated with its corrected statement. Each total counts recorded links from a check to a statement; one calculation used for several statements can contribute several links. Every outcome specifies its scope. Re-reading an edited proof is recorded separately from deriving its steps before comparison. Exact algebra, numerical enclosures and analytic arguments retain their respective scopes, and the records identify the implementations used for independent computational comparisons. No human has checked the mathematics.

The check records and programs are linked from the project repository. The text, programs and check records are dedicated to the public domain under CC0 1.0 Universal.

1.6 How the paper is organized

The marked period and quotient coordinates lead to the exact norm, its admissible domain, and the proper family. The intrinsic classification is followed by discrete logarithmic data, the monodromy quotient and the exceptional fibres. Sections 12–15 compute the cotangent, Bott–Chern and tangent cohomology; the complete coefficient construction precedes its use in the Picard family. The integral circle quotient and the real structures then lead to the puncture and outer-boundary problems. Their local models, component obstructions and topology are developed with the specified marked comparison maps.

The remaining main-body sections treat other multiplicities and finite covers, rational CM heights, the critical level curves, and arithmetic periods. Appendices A–C contain the credited proofs of parameter equivalence, tangent cohomology, local deformations and automorphisms. The final part states the unresolved arithmetic questions. Each arrow in the diagram identifies a particular supplying and receiving body argument in the table below it.

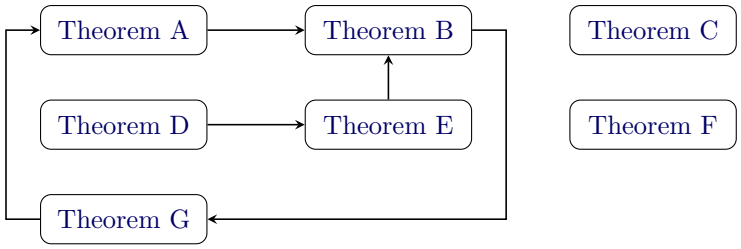

Figure 1: Reuse of the indicated body arguments among the main theorem groups.

Arrow	Supplying and receiving arguments
$A \to B$	I.6.6 $\longrightarrow$ I.17.10. The exact parameter domain is used for the intrinsic Albanese nome.
$E \to B$	I.C.4 $\longrightarrow$ I.17.3. The automorphism group acts in the conjugacy classification of real involutions.
$D \to E$	I.13.2 $\longrightarrow$ I.13.7. The differential and Bott–Chern dimensions determine the Picard curvature statement.
$B \to G$	I.7.2 $\longrightarrow$ I.19.4. The intrinsic multiplicity-four divisor supplies the Albanese continuation obstruction.
$G \to A$	I.19.5 $\longrightarrow$ I.6.3. The half-period derivative identity supplies the theta-heat calculation.

Further results

- Propositions 4.7 to 4.9: division polynomials give the exact torsion loci and their norm values.
- Propositions 24.4 and 24.18: local heights give the rational CM factors, and two-isogeny descent determines the stated rank over $\mathbb{Q}(\sqrt{3})$.
- Proposition 16.2 and theorem 16.7: the integral circle quotient has an explicit cusp replacement and relative uniqueness.
- Proposition 14.2: holomorphic cocycles and a locally finite coordinate partition give global coefficient representatives.
- Propositions 19.18 and 25.6: the theta norm has one maximum and strictly star-shaped lower level curves.

2 The marked family and its coordinates

Definition 2.1 (Base and integral marking)**.** [tag 0101] Let $B = \mathbb{P}^1_t$, $p_1 = 0$, $p_2 = 1$, $p_0 = \infty$, $\Sigma = \{p_0, p_1, p_2\}$, and $B^\circ = B \setminus \Sigma$. The full fibres of the marked family are $3S_1$, $4S_2$ and W; S_1, S_2, W are reduced, and $D = W + S_1 + S_2$. We orient base meridians counterclockwise in holomorphic local coordinates and use clockwise root deck generators $s_j(g_j z) = e^{-2\pi i/m_j} s_j(z)$, where $(m_1, m_2) = (3, 4)$ and $g_1 g_2 g_0 = 1$. The lift of a positive base loop ends at the inverse deck translate, and its endpoint identification applies g_j.

The integral homology lattice and its dual basis are

$$\Lambda = \mathbb{Z}\langle \widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta}\rangle, \qquad (\gamma, u, w, \delta).$$

The period matrix in coordinates (ζ_1, ζ_2) is

$$\Pi_c = \begin{pmatrix} 6\mu & \tau & 1 & 0 \\ \beta_c & \mu & 0 & 1 \end{pmatrix}, \qquad \beta_c = \beta^{\mathrm{part}} + c, \qquad (\beta^{\mathrm{part}} + \tau)(p_0) = 0. \tag{2.1}$$

Its rows are $\sigma_1 = 6\mu\gamma + \tau u + w$ and $\sigma_2 = \beta_c\gamma + \mu u + \delta$. The monodromy matrices are

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ -6 & -1 & -1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -6 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 \end{pmatrix}, \quad A_0 = (A_1 A_2)^{-1}. \tag{2.2}$$

They act on columns in the displayed integral basis. Homology has integral coefficients unless specified, and complex manifolds carry their complex orientations. Reversing the two additive coordinates acts by $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$: the period and Hermitian matrices become $J\Pi_c$ and JHJ^{t}, respectively.

Definition 2.2 (The parameter and the finite translations)**.** [tag PI01] Put

$$D_{\mathrm{part}} = \Im\beta^{\mathrm{part}} - 6(\Im\mu)^2/\Im\tau, \qquad M = \sup D_{\mathrm{part}}, \qquad \mathcal{P} = \{c \in \mathbb{C} : \Im c < -M\}, \qquad w = e^{-\pi i c}. \tag{2.3}$$

The scalar w is the family parameter; in an exterior product, w denotes the marked covector of Definition 2.1. The standard affine translations are $v_1 = (1, 2, -4, 0)^{\mathsf{t}}$ and $v_2 = (-1, -3, 3, 0)^{\mathsf{t}}$, and the cusp collar is untwisted. We write $u_{\mathrm{Engel}} = e^{-2\pi i c} = w^2$ for the multiplicative parameter with the inverse period lift at $c = 0$ as its reference. The line-bundle identification and the effect of changing that reference are proved in Proposition 2.13. We use $\rho = e^{\pi i/3}$; the transformation $\tau \mapsto \tau + 1$ carries the convention $e^{2\pi i/3}$ to ρ.

Proposition 2.3 (Integral changes of marking)**.** [tag 01S2] *In Engel's common integral basis, our marked basis has column matrix*

$$P_{\mathrm{mark}} = \begin{pmatrix} 0 & -2 & -1 & -2 \\ 2 & -1 & 0 & -1 \\ 0 & 2 & 1 & 3 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

His base indices $(0, 1, \infty)$ *correspond to* $(1, 2, 0)$ *here. The matrices satisfy* $T_0 P_{\mathrm{mark}} = P_{\mathrm{mark}} A_1$ *and* $T_1 P_{\mathrm{mark}} = P_{\mathrm{mark}} A_2$*, and his distinguished circle is* $\delta_{\mathrm{E}} = P_{\mathrm{mark}}\widehat{\delta}$*. The positive-root affine vectors are* $a_0 = -P_{\mathrm{mark}} v_1/3$ *and* $a_1 = -P_{\mathrm{mark}} v_2/4$*. For reversed meridians the change of marking is*

$$Q = P_{\mathrm{mark}} S, \quad S = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -6 & 0 & -1 & 0 \\ 6 & -1 & 0 & 0 \\ 2 & 1 & 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & -1 & 0 & -2 \\ 2 & -1 & 0 & -1 \\ 0 & 2 & 1 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and $T_0 Q = Q A_1^{-1}$*,* $T_1 Q = Q A_2^{-1}$*.*

Proof. The matrices in (Engel, 2026, Theorem 5.1, p. 13; equation (6.1), p. 16) in that common basis are

$$T_0 = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} -1 & 1 & -1 & 2 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Multiplication gives all four intertwining identities, $\det P_{\mathrm{mark}} = -1$ and $\det S = 1$. Thus these are integral lattice isomorphisms with the stated actions. The last column of P_{mark} is $(-2,-1,3,0)^{\mathsf{t}}$, the distinguished circle in that basis. Since $A_j v_j = v_j$, inversion of the affine deck generator sends its translation v_j/m_j to $-v_j/m_j$. Applying P_{mark} gives the displayed affine vectors. □

Definition 2.4 (Root bundles and logarithmic differentials)**.** [tag PI02] Let $\rho : \mathcal{B} \to B$ be the root curve of orders $(3,4)$, with tautological root bundles R_1, R_2. Put $R = R_1 R_2^{-1}$ and $\mathcal{O}_{\mathcal{B}}(n) = R^n$. Thus $R_1 = R^4$, $R_2 = R^3$, $\rho^*\mathcal{O}_B(1) = R^{12}$ and $K_{\mathcal{B}} = R^{-7}$. Here $\mathcal{O}_B(n)$ has coarse degree n; its pullback is $\mathcal{O}_{\mathcal{B}}(12n)$. We use

$$\mathcal{L} = \Omega^1_B(\log \Sigma), \qquad \mathcal{A} = \Omega^1_X(\log D), \qquad \mathcal{E} = \mathcal{A}/f^*\mathcal{L}.$$

Fibres in coherent base change are the full scheme fibres.

Proposition 2.5 (The root base and its explicit cohomology)**.** [tag 01SM]

Let $\rho : \mathcal{B} \to B$ be the analytic root curve with orders 3 and 4 at p_1, p_2. Write R_1, R_2 for its tautological root line bundles. Its Picard group is infinite cyclic, with generator $R = R_1 \otimes R_2^{-1}$, and

$$R_1 = R^4, \quad R_2 = R^3, \quad \rho^*\mathcal{O}_B(1) = R^{12}, \quad K_{\mathcal{B}} = R^{-7}.$$

We abbreviate R^n to $\mathcal{O}_{\mathcal{B}}(n)$. Choose sections $x \in H^0(R^3)$ and $y \in H^0(R^4)$ vanishing at p_2 and p_1, respectively. For every integer n, put

$$r_1(n) \equiv n \pmod 3, \quad 0 \le r_1(n) < 3, \qquad r_2(n) \equiv 3n \pmod 4, \quad 0 \le r_2(n) < 4,$$

$$d(n) = \frac{n - 4r_1(n) - 3r_2(n)}{12}.$$

Then

$$\rho_*\mathcal{O}_{\mathcal{B}}(n) = \mathcal{O}_B(d(n)), \qquad H^i(\mathcal{B}, \mathcal{O}(n)) = H^i(B, \mathcal{O}(d(n))). \tag{2.4}$$

Equivalently, $h^0(\mathcal{O}(n))$ is the number of pairs $(a,b) \in \mathbb{Z}^2_{\ge 0}$ with $3a + 4b = n$, and $h^1(\mathcal{O}(n)) = h^0(\mathcal{O}(-n-7))$.

Proof. The Picard group and graded cohomology are those of (Du et al., 2026a, Lemma 6.1, p. 74); the following chart maps specify the coarse invariant direct image used here.

1. On the root charts the group acts on a disc coordinate by a primitive m_jth root of unity. The stabilizer character of any line bundle is removed uniquely by tensoring with $R_1^{-r_1}R_2^{-r_2}$, with the exponents in the displayed ranges. A line bundle with trivial stabilizer characters descends: an equivariant nonzero local frame can be averaged, and its invariant frame gives the descent transition function. It is therefore the pullback of a unique line bundle on B. Consequently
$$\mathrm{Pic}(\mathcal{B}) = \langle H, R_1, R_2 \mid 3R_1 = H,\ 4R_2 = H \rangle = \mathbb{Z}R.$$
The weighted quotient presentation is
$$\mathcal{B} \simeq [(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*], \quad \lambda(x,y) = (\lambda^3 x, \lambda^4 y), \qquad t = \frac{y^3}{y^3 - x^4}.$$
The root sections are $(\mathcal{O}(4), y)$ and $(\mathcal{O}(3), x)$. They give isomorphisms on each root chart, and the same coarse coordinate on their ordinary overlap makes those isomorphisms glue. Additive notation in the Picard presentation means tensor product. The determinant of the coordinate change $t = s^m$ gives $K_{\mathcal{B}} = \rho^*K_B \otimes R_1^2 R_2^3 = R^{-24+8+9}$.

2. An invariant section of $R_j^{r_j}$ on its chart is s^{r_j} times a holomorphic function of $t = s^{m_j}$. It follows, by the local invariant comparison on overlaps, that $\rho_*(\rho^*\mathcal{O}(k)R_1^{r_1}R_2^{r_2}) = \mathcal{O}(k)$. In characteristic zero taking finite-group invariants is exact: the averaging idempotent splits the invariant submodule. Thus there are no higher direct images for ρ, and (2.4) follows. This proves the cohomology comparison with its canonical invariant map. Finally every monomial of degree n is $x^{r_2}y^{r_1}$ times a homogeneous polynomial of degree $d(n)$ in x^4, y^3. This gives the stated count of sections. The negative-degree Čech complex on $x \neq 0$, $y \neq 0$ has cokernel represented by $x^{-a}y^{-b}$ with $a, b \geq 1$ and $3a + 4b = -n$; replacing (a, b) by $(a-1, b-1)$ proves the h^1 formula.

□

Definition 2.6 (Theta, period and height conventions)**.** [tag PI03] The theta variable has periods $(1, \tau)$. With $q = e^{2\pi i\tau}$, define

$$N(z,\tau) = \left|\frac{\vartheta_1(z,\tau)}{\eta(\tau)}\right| e^{-\pi(\Im z)^2/\Im\tau}, \quad \Delta_0 = q\prod_{n\geq 1}(1-q^n)^{24}, \quad t = j(\tau)/1728.$$

Then $t = E_4^3/(1728\Delta_0)$ and $t - 1 = E_6^2/(1728\Delta_0)$. The arithmetic Weierstrass equation is

$$E_t : Y^2 = X^3 - \tfrac{3}{4}t^3(t-1)X + \tfrac{1}{4}t^4(t-1)^2, \qquad P_t = (0, t^2(t-1)/2), \qquad \omega = dX/(2Y).$$

Its discriminant is $\Delta_E = 27t^8(t-1)^3$. The lattice coordinates are $X = \wp(u)$, $Y = \wp'(u)/2$; for $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, $g_2^3 - 27g_3^2 = (2\pi)^{12}\eta(\tau)^{24}$. We use the height of (O), $\widehat{h}_O(P) = \frac{1}{2}\lim n^{-2}h(x(nP))$; the surface self-pairing and rank-one BSD regulator are $2\widehat{h}_O$. The real period is $\Omega = \int_{E(\mathbb{R})}|\omega|$, summed over both real components when there are two. The archimedean local height is $-\log N$. At finite places $|p|_v = p^{-1}$ and $n_v = [K_v : \mathbb{Q}_p]$; complex embeddings are counted separately.

Quasi-periods satisfy $\zeta(z + \omega_j) = \zeta(z) + \eta_j$ and $\omega_1\eta_2 - \omega_2\eta_1 = -2\pi i$; $\eta(u)$ denotes their real-linear extension. In the critical-point calculation we use $\chi = (\tau - i)/(\tau + i)$, $z = k\chi^2$, and $\beta_{\mathrm{cusp}} = \log(5 + 2\sqrt{6})/(2\pi)$.

Lemma 2.7 (The real period determinant and the fibre class)**.** [tag PI04] *Put $y = \Im\tau$, $m = \Im\mu$, $r = \Im\beta_c$, and $D_c = r - 6m^2/y = D_{\mathrm{part}} + \Im c$. The real period determinant in the row order $(\Re\zeta_1, \Im\zeta_1, \Re\zeta_2, \Im\zeta_2)$ is yD_c. For $D_c < 0$, the integral fibre class $\eta_F = u \wedge w + 6\gamma \wedge \delta$ satisfies*

$$\int_{F,\mathrm{or}_{\mathbb{C}}} \eta_F^2 = -12, \qquad \Gamma = \frac{i}{2D_c}\left(d\bar{\zeta}_2 - \frac{m}{y}d\bar{\zeta}_1\right) = \gamma^{0,1}.$$

Moreover $\eta_F \wedge \Gamma \wedge d\zeta_2$ equals the marked volume form $\gamma \wedge u \wedge w \wedge \delta$.

Proof. Writing $a = \Re\mu$, $x = \Re\tau$, $b = \Re\beta_c$, the real matrix is

$$\begin{pmatrix} 6a & x & 1 & 0 \\ 6m & y & 0 & 0 \\ b & a & 0 & 1 \\ r & m & 0 & 0 \end{pmatrix}.$$

Its determinant is $yr - 6m^2 = yD_c$. Also $\eta_F^2 = 12\gamma \wedge u \wedge w \wedge \delta$. The negative determinant reverses the marked orientation relative to the complex orientation, giving the stated integral. The imaginary parts of the two period rows give $\gamma = (\Im d\zeta_2 - (m/y)\Im d\zeta_1)/D_c$; taking its $(0,1)$ part gives Γ. Substitution of the same two rows into the exterior product gives the final identity, whose coefficient is one. □

Definition 2.8 (The Picard coordinate)**.** [tag PI05] Let $\mathfrak{g} = g_\gamma$ be the coherent Leray class with fibre image $\gamma^{0,1}$ and put $M_z = \exp(z\mathfrak{g})$, where exp is the sheaf map $z \mapsto e^{2\pi i z}$. We normalize $\int_B \omega_{FS} = 1$ and $c_{1,BC}(L) = [iF_L/(2\pi)]$; the unnormalized Chern–Ricci class is $2\pi c_{1,BC}(T_X)$.

Remark 2.9 (Special-fibre and Picard coordinates)**.** [tag PI06] Propositions 4.3, 7.6 and 2.13 give

	κ	upper-half-plane period	nome
S_1	$c + \frac{1}{2} - \frac{3i}{2\pi}\log 48$	$-\kappa_1$	$q_1 = -w^2/48^3$
S_2	$c + \frac{3}{2} - \frac{3i}{\pi}\log 6$	$-\kappa_2/2$	$q_2 = iw/216.$

In particular $q_1 = (27/64)q_2^2$ and the full compact-family domain is $0 < |w| < 216$, equivalently $0 < |u_{\mathrm{Engel}}| < 46656$. Proposition 13.8 fixes the collar orientation and gives $\exp(12g_\gamma) = f^*\mathcal{O}_B(-p)$, where $p = 12l_0 - 4l_1 - 3l_2$. For the standard data $p = -1$, $\exp(12\mathfrak{g}) = f^*\mathcal{O}_B(1)$, $\mathcal{O}_X(S_1 - S_2) = \exp(\mathfrak{g})$, and $[G_0] = 12\mathfrak{g}$. The prescribed Chern–Ricci form in Proposition 13.13 is $\pi f^*\omega_{FS}$.

Lemma 2.10 (The exact comparison of the two Weierstrass equations)**.** [tag 0002] *Put $C = t^3(t-1)$ and $D = t^4(t-1)^2$, with $t \neq 0, 1$. The pointed map*

$$\Phi : E_t^{\mathrm{ar}} \longrightarrow E_t^{\mathrm{geom}}, \qquad (x, y) \longmapsto (2x, -2\sqrt{2}y)$$

is an isomorphism from $y^2 = x^3 - 3Cx/4 + D/4$ to $Y^2 = X^3 - 3CX + 2D$. It sends $(0, t^2(t-1)/2)$ to $(0, -\sqrt{2}t^2(t-1))$ and preserves the normalized theta norm under the induced pointed elliptic-curve identification. The discriminants satisfy $\Delta_{\mathrm{geom}} = 64\Delta_{\mathrm{ar}}$.

Proof. 1. Substitution gives $Y^2 = 8y^2 = 8x^3 - 6Cx + 2D = X^3 - 3CX + 2D$. The inverse is $(X, Y) \mapsto (X/2, -Y/(2\sqrt{2}))$ and both maps extend over the unique point at infinity. The displayed section identity follows by the same substitution.

The original equation in (Claude and L. Alpöge, 2026, Remark 3.12, p. 19) is

$$y_0^2 = 4x_0^3 - 3Cx_0 - D, \qquad P_0 = (0, it^2(t-1)).$$

Its exact pointed map to E_t^{ar} is $(x_0, y_0) \mapsto (-x_0, y_0/(2i))$, with inverse $(x, y) \mapsto (-x, 2iy)$; its composite to E_t^{geom} is $(x_0, y_0) \mapsto (-2x_0, i\sqrt{2}y_0)$. These maps carry the displayed sections to one another. With $\omega_0 = dx_0/y_0$, the pullback of $dx/(2y)$ is $-i\omega_0$; transporting the period basis and elliptic logarithm by this scalar therefore gives the same normalized theta norm by the calculation below.

2. Set $a = -\sqrt{2}$. In Weierstrass uniformization the equations are related by $X = a^2x$, $Y = a^3y$ and $\Lambda_{\mathrm{geom}} = a^{-1}\Lambda_{\mathrm{ar}}$. The induced elliptic logarithm map is $z \mapsto a^{-1}z$. Writing $\Lambda_{\mathrm{ar}} = \omega(\mathbb{Z} + \tau\mathbb{Z})$ shows directly that the argument z/ω in the defining theta quotient is unchanged. Thus the norms agree through this map.

3. The discriminant scales by $a^{12} = 64$. The divisor and leading term of the division polynomial give $\psi_m(\Phi(Q)) = a^{m^2-1}\psi_m(Q)$. The two absolute scaling factors in $|\Delta_E|^{(m^2-1)/12}/|\psi_m(Q)|$ cancel exactly.

□

2.1 The elliptic surface and the period quotient

Proposition 2.11 (The period quotient and its marked elliptic point)**.** [tag 01Q7] [tag 01Q9]
Let $E_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, $P=[\mu]$, and $L_\mu = \mathcal{O}_{E_\tau}(P-O)$. The holomorphic map

$$[\xi,\eta] \longmapsto [\xi, e^{2\pi i\eta}]$$

identifies $\mathbb{C}^2/\Pi_c\Lambda$ with the quotient of $L_\mu^\times$ by the lift $A(\xi,W)=(\xi+6\mu, e^{2\pi i\beta_c}W)$. The generator in (Engel, 2026, Definition 2.3, p. 5) is its inverse, with translation $-6P$ and multiplier $e^{-2\pi i\beta_c}$. The section $P=[\mu]$ is a generator of the Mordell–Weil group of the stated pointed Weierstrass surface, and

$$\wp_\tau(\mu) = 0.$$

Under the pointed map in the proof it is one of $P_* = (0, \sqrt{2}t^2(t-1))$ and $-P_*$.

Proof. 1. First quotient by columns two, three and four. Exponentiation kills column four; columns three and two then give

$$(\xi, W) \sim (\xi+1, W) \sim (\xi+\tau, e^{2\pi i\mu}W).$$

For the odd theta function with periods $1,\tau$, the quotient $\theta(\xi-\mu)/\theta(\xi)$ is a meromorphic section of this line bundle: its factors are $1, e^{2\pi i\mu}$ and its divisor is $P-O$. Sending its divisor section to this quotient is the asserted line-bundle isomorphism. Quotienting the remaining first column gives exactly A. On sufficiently small simply connected charts the logarithm of W is a holomorphic inverse; changes of logarithm are column-four periods. Thus the displayed map is a biholomorphism of the two quotient complex spaces, with its inverse explicitly defined by local logarithms. Choosing A^{-1} changes the generator of the same infinite cyclic group and gives precisely Engel's declared inverse-translation convention.

2. The pointed map to the elliptic surface is determined by the branch f_{sc} satisfying

$$f_{\mathrm{sc}}^4 = \frac{12t^3(t-1)}{g_2(\tau)}, \qquad f_{\mathrm{sc}}^6 = -\frac{8t^4(t-1)^2}{g_3(\tau)}, \qquad f_{\mathrm{sc}}/t \longrightarrow i\sqrt{3}/\pi.$$

The identities $g_2^3/(g_2^3-27g_3^2) = t$ make these equations compatible. Their zero and pole orders give a holomorphic branch on the period cover; the indicated cusp sign fixes it. The map $\xi \mapsto (f_{\mathrm{sc}}^2\wp_\tau(\xi), \frac{1}{2}f_{\mathrm{sc}}^3\wp_\tau'(\xi))$ is a pointed isomorphism to $y^2 = x^3 - 3t^3(t-1)x + 2t^4(t-1)^2$. Its factors are $f_{\mathrm{sc}}(g_1 z) = -f_{\mathrm{sc}}/\tau$, $f_{\mathrm{sc}}(g_2 z) = f_{\mathrm{sc}}/\tau$; therefore this isomorphism descends over B°. It pulls $dx/(2y)$ back to $f_{\mathrm{sc}}^{-1}d\xi$.

3. Write $C = t^3(t-1)$ and $D = t^4(t-1)^2$. In (Claude and L. Alpöge, 2026, Remark 3.12, p. 19), the surface

$$y_0^2 = 4x_0^3 - 3Cx_0 - D$$

has Mordell–Weil generator $P_0 = (0, it^2(t-1))$, whose normalized Abel–Jacobi coordinate is $-\mu$. Its Weierstrass scale r satisfies

$$r^4 = 3C/g_2(\tau), \qquad r^6 = D/g_3(\tau), \qquad r^2 = \frac{Dg_2(\tau)}{3Cg_3(\tau)}.$$

Thus $x_0(P_0) = r^2\wp_\tau(-\mu) = 0$. Evenness of $\wp$ gives the asserted zero for the original affine logarithm. Changing the sign of r composes this uniformization with $\xi \mapsto -\xi$, and leaves this zero identity unchanged. The exact pointed map of lemma 2.10,

$$(x_0, y_0) \longmapsto (-2x_0, i\sqrt{2}y_0),$$

sends P_0 to $-P_*$ and $-P_0$ to P_*. Both visible sections therefore generate the Mordell–Weil group of the geometric equation. The preceding f_{sc} map sends $[\mu]$ to one of these two sections, proving the generator claim. The Shioda self-pairing is $\langle P, P\rangle = 2 - 4/3 - 1/2 = 1/6$ (Engel, 2026, proof of Lemma 2.1, pp. 3–4); the two sign choices have the same pairing. The chosen sign in our family is fixed by the cusp calculation in proposition 2.13.

4. The nodal comparison needed for that sign is explicit. Put $v = 1/t$, $X = v^2 x$, $Y = v^3 y$; the geometric equation becomes $Y^2 = X^3 - 3(1-v)X + 2(1-v)^2$. At $v = 0$, the chosen scale gives

$$X = 1 + \frac{12U}{(U-1)^2}, \qquad Y = \frac{12\sqrt{3}\,U(U+1)}{(U-1)^3}, \qquad U = e^{2\pi i\xi}.$$

On the smooth locus away from O its inverse is $U = (Y/(X-1) + \sqrt{3})/(Y/(X-1) - \sqrt{3})$; at O a local parameter is X/Y. The normally convergent multiplicative expansion of $\wp$ gives the corresponding holomorphic map for small v.

□

Proposition 2.12 (The compactness radius)**.** [tag 01Q8]

The exact set of nonzero parameters for which every ordinary fibre and both normalized finite root fibres of Engel's period-reference quotient are compact complex two-tori is

$$0 < |u| < e^{-2\pi M}.$$

This is the parameter domain for the compact family with both smooth finite root models. The condition on ordinary fibres alone at equality is determined by whether a maximizing point of D_{part} is ordinary; the maximizing locus is determined in proposition 25.6. For an independent reference, the radius is $|\lambda_0| e^{-2\pi M}$.

Proof. 1. The function $N(\xi, W) = |W| \exp(2\pi \operatorname{Im}\mu \operatorname{Im}\xi / \operatorname{Im}\tau)$ is invariant under both line-bundle period identifications. The inverse lift multiplies it by $e^{2\pi D_c}$. More directly the real determinant of the four period columns is a nonzero sign times $(\operatorname{Im}\tau) D_c$. Thus the four columns generate a full lattice exactly when $D_c \neq 0$, and then the quotient is a compact torus by a closed fundamental parallelepiped. At $D_c = 0$ their real span has dimension three; a rank-four lattice in $\mathbb{R}^4$ cannot result, so the quotient cannot be a compact complex two-torus. The function D_{part} descends continuously to the connected base with the cusp removed and tends to $-\infty$ at the cusp. Its image is therefore $(-\infty, M]$: the maximum is attained, and the intermediate value theorem fills all smaller values. Consequently D_c is nonzero at every base point exactly when $\operatorname{Im} c < -M$. Since $|u| = e^{2\pi \operatorname{Im} c}$, the assertion follows, with the indicated reference scaling.

□

Proposition 2.13 (The reference linearization)**.** [tag 01QA]

The line-bundle comparison of the period family with Engel's family over B° transports its inverse period lift to a global section of $\mathrm{Hom}(t_{6P}^* M_S, M_S)$ with divisor F_∞. Take this transported section for $c = 0$ as the reference linearization. Then $\lambda_0 = 1$ exactly and $u = e^{-2\pi i c} = w^2$. For another independently fixed reference the precise constant is its scalar ratio with this section. With the cusp branch $e^{2\pi i\mu} \to -(5 + 2\sqrt{6})$ and $f_{\mathrm{sc}}/t \to i\sqrt{3}/\pi$, the section is $P = (0, -\sqrt{2}t^2(t-1))$.

Proof. 1. The zero identity $\wp_\tau(\mu) = 0$, proved in 2.11, identifies P with the signed visible section. With this cusp branch and the specified scale, substitution of $U = -(5 + 2\sqrt{6})$ in the nodal uniformization of 2.11 gives $Y = -\sqrt{2}$, so $P = (0, -\sqrt{2}t^2(t-1))$.

2. To compare the line bundles as families, use their common divisor $P - O$ and rigidify at the zero section. Put $F_\theta(\xi) = \theta(\xi - \mu)/\theta(\xi)$, $r_\theta = \theta(-\mu)/\theta'(0)$, and let s_{P-O} be the canonical meromorphic divisor section. The map

$$[\xi, W] \longmapsto \frac{2r_\theta}{f_{\mathrm{sc}}} W \frac{s_{P-O}(\xi)}{F_\theta(\xi)}$$

is a holomorphic line-bundle isomorphism, including at P, O after cancellation of the equal divisors. At O it sends the chosen frame to dx/y, since $F_\theta \sim r_\theta/\xi$ and $x/y \sim -\xi/f_{\mathrm{sc}}$ with the corresponding canonical divisor frame. Two such maps that agree at O differ by a holomorphic scalar on each compact elliptic fibre and hence agree everywhere. This proves descent under the specified period identifications.

3. Engel (2026, Lemma 2.1, pp. 3–4) identifies $\mathrm{Hom}(t_{6P}^* M_S, M_S)$ with $\pi_S^* \mathcal{O}_B(1)$. The lemma applies because the preceding pointed cubic map and Mordell–Weil calculation identify his (S, O, P) with ours, including the chosen sign. To determine which section is transported, it suffices to compute its order at one generic smooth point of each special fibre, since a section of this pulled-back line is a base section. At a finite centre write $6\mu = p + r\tau + \delta$ with $(p, r) = (4, -2), (3, -3)$ and δ/f_{sc} holomorphic; this divisibility follows by linearizing the affine laws, whose homogeneous orders are two and one. Evaluate at $\xi = f_{\mathrm{sc}} z$ for generic fixed z. The factor $(2r_\theta/f_{\mathrm{sc}})/F_\theta(f_{\mathrm{sc}} z)$ tends to $2z$, a unit. Restoring the lattice shift $p + r\tau$ at the target changes the fibre multiplier to $e^{-2\pi i(\beta_c - r\mu)}$, whose finite limits are $e^{-2\pi i\kappa_1}$ and $e^{-2\pi i\kappa_2}$. The source and target theta ratios remain units. Therefore the base section has order zero at both finite points. At the cusp f_{sc} has a simple pole and r_θ has a nonzero limit; the two factors $2r_\theta/f_{\mathrm{sc}}$ cancel between source and target. The remaining multiplier is $e^{-2\pi i\beta_c} = qe^{-2\pi i(\beta_c + \tau)}$, of exact order one. Thus the transported section has divisor F_∞. Engel (2026, Corollary 2.2, p. 4) identifies it with a nonzero scalar multiple of his reference, since two base sections with the same divisor have constant ratio. Changing β by c multiplies it exactly by $e^{-2\pi ic}$. With our stated period-reference normalization, this scalar is one. For any independently chosen reference ϕ_{ref}, the scalar is instead the precisely defined ratio $\lambda_0 = \phi_{\mathrm{part}}/\phi_{\mathrm{ref}}$, evaluated on any ordinary fibre. Scaling the reference by $a \in \mathbb{C}^*$ changes λ_0 to λ_0/a.

□

3 Normalized periods and the real determinant

3.1 The normalized theta period

Definition 3.1 (Theta functions and the cusp branch)**.** [tag IA21] The odd Jacobi function and the Dedekind function are

$$\vartheta_1(v, \tau) = 2 \sum_{n \geq 0} (-1)^n e^{\pi i \tau (n+1/2)^2} \sin((2n+1)\pi v),$$
$$\eta_D(\tau) = \eta(\tau) = e^{\pi i \tau/12} \prod_{n \geq 1} (1 - e^{2\pi i n \tau}).$$

Choose the cusp lift reached from $\tau = iT$, $T > 1$, on which $\mu = 1/2 - iy(T)$ and $0 < y(T) < T/2$. Put $a = 5 + 2\sqrt{6}$ and $\rho = e^{\pi i/3}$.

Proposition 3.2 (The normalized theta period)**.** [tag 01QH]

For the period section identified in proposition 2.11, the particular solution with $(\beta^{\mathrm{part}} + \tau)(p_0) = 0$ *is*

$$\beta^{\mathrm{part}} = \frac{6i}{\pi} \operatorname{Log} \frac{\vartheta_1(\mu, \tau)}{2\sqrt{3}\, \eta_D(\tau)}. \tag{3.1}$$

Here the logarithm is continued on the simply connected uniformizing half-plane from its cusp expansion. Its special values are

$$\mu_0 = \frac{1}{2} - \frac{i \log a}{2\pi}, \qquad \mu(z_1) = \frac{2-\rho}{3}, \qquad \mu(z_2) = \frac{1-i}{2},$$

$$\beta^{\mathrm{part}}(z_2) = i\left(\frac{3}{2} - \frac{3}{\pi}\log 6\right), \qquad \operatorname{Re}\beta^{\mathrm{part}}(z_1) = -\frac{1}{2}.$$

In particular $e^{2\pi i \mu_0} = -a$.

Proof. 1. Proposition 2.11 identifies the elliptic point represented by μ with one of the sections $(0, \pm\sqrt{2}t^2(t-1))$. The uniformization map $(v,\tau) \mapsto (f^2\wp_\tau(v), f^3\wp'_\tau(v)/2)$ therefore gives $\wp_\tau(\mu) = 0$. No zero of $\wp$ is a lattice point. Thus the numerator of (3.1) is nonzero; the denominator is nonzero as well. A holomorphic logarithm exists on the simply connected cover.

2. The cusp value follows from the normally convergent Fourier expansion:

$$\frac{\wp_\tau(v)}{\pi^2} = \csc^2(\pi v) - \frac{1}{3} + 8\sum_{n\geq 1}\frac{nq^n}{1-q^n}(1-\cos(2\pi n v)), \qquad q = e^{2\pi i\tau}. \tag{3.2}$$

This identity follows by differentiating the logarithm of the Jacobi product twice and using $E_2 = 1 - 24\sum nq^n/(1-q^n)$. Boundedness of the chosen elliptic logarithm at the cusp permits passage to the limit. We obtain $\sin^2(\pi\mu_0) = 3$, equivalently $u^2 + 10u + 1 = 0$, $u = e^{2\pi i\mu_0}$. On the chosen imaginary-axis branch $\operatorname{Im}\mu < 0$, so $u = -a$. On the rectangular elliptic curve with $\tau = iT$, $T > 1$, $\wp$ is real on the segment $1/2 - iy$, $0 \leq y \leq T/2$. Its endpoint values are the largest and middle real roots of its cubic, respectively positive and negative. Its derivative has no zero in the open segment, since the zeros of $\wp'$ are the three nonzero half-periods. There is exactly one zero there; this gives the indicated continuation. Consequently $2\sin(\pi\mu_0) = 2\sqrt{3}$.

3. Put $R = \vartheta_1/\eta_D$. Poisson summation for the defining Gaussian series and its index shift give

$$\begin{aligned} R(v,\tau+1) &= e^{\pi i/6}R(v,\tau), \\ R(v/\tau,-1/\tau) &= -ie^{\pi i v^2/\tau}R(v,\tau), \\ R(1-v,\tau) &= R(v,\tau), \qquad R(v+1,\tau) = -R(v,\tau). \end{aligned}$$

Inserting the transformed arguments gives

$$\begin{aligned} R(\mu\circ g_1, \tau\circ g_1) &= e^{-\pi i/3}e^{\pi i(1-\mu)^2/\tau}R(\mu,\tau), \\ R(\mu\circ g_2, \tau\circ g_2) &= ie^{\pi i\mu^2/\tau}R(\mu,\tau). \end{aligned}$$

Multiplication of the logarithms by $6i/\pi$ gives exactly $2-6(1-\mu)^2/\tau$ and $-3-6\mu^2/\tau$, modulo constants in $12\mathbb{Z}$. These constants vanish: each defect is constant on the connected cover, and evaluation at the corresponding fixed point makes both its two sides zero. The fixed-point equations themselves give $\mu(z_1) = (2-\rho)/3$ and $\mu(z_2) = (1-i)/2$. The Jacobi product has cusp expansion $R(\mu,\tau) = 2q^{1/12}\sin(\pi\mu)(1+O(q))$. Equation (3.1) consequently gives $\beta^{\mathrm{part}} + \tau \to 0$. The difference from the specified particular solution is an invariant holomorphic function extending across all three points of B, hence a constant, and the cusp normalization makes it zero.

4. At $\tau = i$, the half-period translation formula gives $\vartheta_1((1-i)/2, i) = e^{\pi/4}\vartheta_3(0,i)$. The identities $2\eta_D^3 = \vartheta_2\vartheta_3\vartheta_4$, $\vartheta_2(0,i) = \vartheta_4(0,i) = 2^{-1/4}\vartheta_3(0,i)$ give $R = e^{\pi/4}\sqrt{2}$, proving the displayed value of β. At ρ, q is negative real, $\mu = 1/2 - i/(2\sqrt{3})$, and the product for R has phase $\pi/12$, with all its remaining factors positive real. This gives $\operatorname{Re}\beta = -1/2$.

□

3.2 The invariant norm

Proposition 3.3 (The norm and the real determinant)**.** [tag 01QJ] *For the conventions of definitions 2.2 and 2.6,*

$$D_{\mathrm{part}} = \frac{6}{\pi}\log\frac{N(\mu,\tau)}{2\sqrt{3}}, \qquad \mathcal{N}(v,\tau) = N(v,\tau). \tag{3.3}$$

The norm is preserved by integral translations and sign change of v, and by the lattice isomorphism

$$[v] \longmapsto [v/(c\tau+d)], \qquad \tau \longmapsto \frac{a\tau+b}{c\tau+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

At the cusp, $D_{\mathrm{part}} = -\Im\tau + o(1)$.

Proof. The imaginary part of (3.1) is $(6/\pi)\log|\vartheta_1(\mu,\tau)/(2\sqrt{3}\eta_D(\tau))|$. Subtracting $6(\Im\mu)^2/\Im\tau$ gives (3.3). Thus the two notations $\mathcal{N}$ and N denote the identical function of definition 2.6. The theta transformation formulas in proposition 3.2 have exactly the exponential moduli cancelled by this Gaussian; the theta and eta square-root factors cancel in their quotient. The displayed lattice map also sends $\wp_\tau(v) = 0$ to the zero equation on the transformed lattice, since $\wp$ has weight two. Finally $\beta^{\mathrm{part}} + \tau \to 0$ and the bounded cusp limit of μ give the stated asymptotic directly from the real determinant. □

4 Special values, local expansions and torsion

4.1 The normalized Green function and special arguments

Definition 4.1 (The logarithmic norm on a lattice)**.** [tag IA22] Write $\rho = (1+i\sqrt{3})/2$ and $T = \operatorname{Im}\tau$. For the norm of definition 2.6, put

$$U_\tau(z) = \log N(z,\tau) = \log\left|\frac{\vartheta_1(z,\tau)}{\eta(\tau)}\right| - \frac{\pi(\operatorname{Im} z)^2}{T}.$$

The theta variable has periods $1, \tau$; a convention with first period π therefore uses the argument πz. For $\Lambda = \omega(\mathbb{Z}+\tau\mathbb{Z})$ and $z \notin \Lambda$ set $U_\Lambda(z) = U_\tau(z/\omega)$.

Well-definedness. Write $v = z/\omega$. An oriented basis change has

$$\tau' = \frac{a\tau+b}{c\tau+d}, \qquad \omega' = \omega(c\tau+d), \qquad v' = \frac{v}{c\tau+d}, \qquad ad-bc = 1.$$

The theta transformations (National Institute of Standards and Technology, 2026, Equations 20.7.26 and 20.7.30), with the first argument multiplied by π, and the eta law (National Institute of Standards and Technology, 2026, Equations 23.18.5–23.18.6) give the two modular generators; the eta translation law also follows directly from its product. Composition gives

$$\log\left|\frac{\vartheta_1(v',\tau')}{\eta(\tau')}\right| = \log\left|\frac{\vartheta_1(v,\tau)}{\eta(\tau)}\right| - \pi\Im\frac{cv^2}{c\tau+d}.$$

The identity

$$\frac{(\Im v')^2}{\Im\tau'} = \frac{(\Im v)^2}{\Im\tau} - \Im\frac{cv^2}{c\tau+d}$$

cancels this term, so $U_{\tau'}(v') = U_\tau(v)$. This proves independence of the basis of Λ. For a homothety $\lambda \neq 0$, the same normalized argument gives $U_{\lambda\Lambda}(\lambda z) = U_\Lambda(z)$. □

Proposition 4.2 (Distribution under isogenies). [tag 000I]

Let $\Lambda, \Lambda' \subset \mathbb{C}$ be lattices and let $\alpha \neq 0$ satisfy $\alpha\Lambda \subset \Lambda'$. Write $\phi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ for $z \mapsto \alpha z$, $K = \alpha^{-1}\Lambda'/\Lambda$, and $d = [\Lambda' : \alpha\Lambda]$. Then

$$U_{\Lambda'}(\alpha z) = \sum_{v \in K} U_\Lambda(z + v). \tag{4.1}$$

The equality holds away from the zeros and as an equality of locally integrable functions at them. If $\Delta_\Lambda = g_2(\Lambda)^3 - 27 g_3(\Lambda)^2$, then

$$\prod_{0 \neq v \in K} N_\Lambda(v) = |\alpha| \left| \frac{\Delta_{\Lambda'}}{\Delta_\Lambda} \right|^{1/12}.$$

In particular, for every nonzero endomorphism of one elliptic curve, $\prod_{v \in \ker \alpha,\ v \neq 0} N(v) = |\alpha|$.

Proof. 1. The Jacobi product, with $q = e^{2\pi i \tau}$ and $X = e^{2\pi i z}$, gives

$$\vartheta_1(z, \tau)/\eta(\tau) = i q^{1/12} X^{-1/2} (1 - X) \prod_{m \geq 1} (1 - q^m X)(1 - q^m X^{-1}),$$

where a harmless fixed unit changes with the theta convention. For $0 < y < T$, integration of the logarithm over $z = x + iy$, $0 \leq x \leq 1$, gives $-\pi T/6 + \pi y - \pi y^2/T$. Each product factor has mean logarithm zero by the absolutely convergent logarithmic series; the endpoint singularities are integrable. Integrating in y proves $\int U_\Lambda \, d\mu_\Lambda = 0$ for Haar probability measure. Moreover

$$\Delta_{\mathrm{Eucl}} U_\Lambda = 2\pi\delta_0 - \frac{2\pi}{\operatorname{area} \Lambda}, \qquad U_\Lambda(z) = \log|z| + \frac{1}{12} \log|\Delta_\Lambda| + o(1).$$

The second formula follows from $\vartheta_1'(0, \tau) = 2\pi\eta(\tau)^3$ and $\Delta_{\mathbb{Z}+\tau\mathbb{Z}} = (2\pi)^{12}\eta(\tau)^{24}$, then scaling by ω. The two sides of (4.1) have the same logarithmic masses at K and the same constant Laplacian, since $d = |\alpha|^2 \operatorname{area} \Lambda / \operatorname{area} \Lambda'$. Their difference is a smooth harmonic function on the compact torus, hence constant. Both means vanish: translations preserve Haar measure and an isogeny sends Haar probability measure to Haar probability measure. The constant is zero. Taking $z \to 0$ and cancelling the logarithmic singularity gives the product.

□

Proposition 4.3 (Phases and the two periods). [tag 000J]

With the branch continued from the specified cusp, the exact values are

$$\begin{aligned}
\frac{\vartheta_1((2 - \rho)/3, \rho)}{\eta(\rho)} &= e^{\pi i/12}\, 3^{1/4} e^{\pi/(6\sqrt{3})}, \\
\frac{\vartheta_1((1 - i)/2, i)}{\eta(i)} &= \sqrt{2}\, e^{\pi/4}, \\
\beta^{\mathrm{part}}(z_1) &= -\frac{1}{2} + i \left(\frac{1}{\sqrt{3}} - \frac{3}{2\pi} \log 48 \right), \\
\beta^{\mathrm{part}}(z_2) &= i \left(\frac{3}{2} - \frac{3}{\pi} \log 6 \right).
\end{aligned}$$

Consequently $N(z_1) = 3^{1/4}$, $N(z_2) = \sqrt{2}$, $D_{\mathrm{part}}(z_1) = -3 \log 48/(2\pi)$, and $D_{\mathrm{part}}(z_2) = -3 \log 6/\pi$.

Proof. 1. Multiplication by $1 + \rho$ has matrix $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 2 \end{smallmatrix}\right)$, its kernel is $\{0, \mu, -\mu\}$, and $(1 + \rho)\mu = 1$ for $\mu = (2 - \rho)/3$. Proposition 4.2 gives $N(\mu)^2 = \sqrt{3}$. At i, multiplication by $1 + i$ has kernel $\{0, (1 - i)/2\}$ and absolute value $\sqrt{2}$; the same identity gives $N((1 - i)/2, i) = \sqrt{2}$.

2. At ρ, $\mu = 1/2 - i/(2\sqrt{3})$ and $q = -e^{-\pi\sqrt{3}}$. In

$$R(z,\tau) = 2e^{\pi i\tau/6}\sin(\pi z)\prod_{m\geq 1}(1 - 2q^m\cos(2\pi z) + q^{2m}),$$

$\sin(\pi\mu)$ is positive real. The factors are $(1 + q^m e^{\pi/\sqrt{3}})(1 + q^m e^{-\pi/\sqrt{3}})$, both positive: when m is odd the larger subtracted number is at most $e^{-2\pi/\sqrt{3}} < 1$. Thus the phase is exactly $\pi/12$ modulo 2π. There is no winding ambiguity in the continued logarithm. Indeed continue from the cusp along $\tau = 1/2 + iT$, $T \geq \sqrt{3}/2$. Real conjugation of the rhombic lattice keeps the chosen zero on $\mu = 1/2 - iy$, $0 < y < T$. It cannot leave that component: a collision of the two zeros would require $g_3 = 0$, which does not occur on this ray. Indeed, writing $q = -r$, $0 < r < 1/200$, the tail after the first term of the E_6 series has absolute value at most $504 \cdot 32r^2/((1 - r^2)(1 - 8r)) < 1/2$, while its first term is positive. Thus $E_6 > 1/2$. At $y = 0$ the point is a nonzero half-period, so a zero there would force $g_3 = 0$. At $y = T$ it is the lattice point $1 - \tau$, where $\wp$ has a pole; a zero cannot reach that boundary. Each displayed product factor remains positive because $e^{-2\pi T}e^{2\pi y} < 1$. The cusp phase tends to $\pi/12$, so the continued argument has that value. On the imaginary ray $\tau = iT$, $T \geq 1$, the same branch is $\mu = 1/2 - iy$ with $0 < y \leq T/2$; every product factor is positive, so its argument at i is zero. The Gaussian factors are respectively $e^{\pi/(6\sqrt{3})}$ and $e^{\pi/4}$. Substitution in the definition of β^{part} proves all four identities.

□

Corollary 4.4 (Difference of the finite invariant periods)**.** [tag 01QI] *Put* $\kappa_1^{\mathrm{part}} = \beta^{\mathrm{part}}(z_1) + 2\mu(z_1)$ *and* $\kappa_2^{\mathrm{part}} = \beta^{\mathrm{part}}(z_2) + 3\mu(z_2)$. *Then*

$$\kappa_1^{\mathrm{part}} - \kappa_2^{\mathrm{part}} = -1 - \frac{3i}{2\pi}\log\frac{4}{3}, \qquad -\frac{1}{5} < \Im(\kappa_1^{\mathrm{part}} - \kappa_2^{\mathrm{part}}) < -\frac{1}{10}.$$

Proof. Substitution of the exact values in proposition 4.3 gives $\kappa_1^{\mathrm{part}} = \frac{1}{2} - \frac{3i}{2\pi}\log 48$ and $\kappa_2^{\mathrm{part}} = \frac{3}{2} - \frac{3i}{\pi}\log 6$. Their difference is the displayed expression. Integrating $1/(1 + x)$ on $0 \leq x \leq 1/3$ gives $1/4 < \log(4/3) < 1/3$; together with $3 < \pi < 15/4$, these inequalities give the stated strict bounds. □

4.2 Division identities and the entire torsion locus

Remark 4.5 (The arithmetic model in the division identities)**.** [tag IA23] Use the model and point of definition 2.6, written as

$$E_t: \quad y^2 = x^3 + A(t)x + B(t), \qquad A = -\tfrac{3}{4}t^3(t-1), \quad B = \tfrac{1}{4}t^4(t-1)^2, \quad P = (0, t^2(t-1)/2).$$

Its discriminant is $\Delta_E(t) = 27t^8(t-1)^3$. Under $(x,y) \mapsto (\lambda^2 x, \lambda^3 y)$ the period lattice and its logarithm change by $(\Lambda, z) \mapsto (\lambda^{-1}\Lambda, \lambda^{-1}z)$ for the invariant differential. The identity in definition 4.1 therefore preserves N.

Cited result 4.6 (Siegel functions and local heights)**.** [tag 000P]

For $a = (a_1, a_2) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ and $v = a_1\tau + a_2$, define

$$g_a(\tau) = -e^{\pi i a_2(a_1-1)}q^{B_2(a_1)/2}(1 - e^{2\pi iv})\prod_{r\geq 1}(1 - q^r e^{2\pi iv})(1 - q^r e^{-2\pi iv}), \quad B_2(x) = x^2 - x + \tfrac{1}{6}.$$

Then

$$g_a(\tau) = ie^{\pi i a_1 v}\frac{\vartheta_1(v,\tau)}{\eta(\tau)}, \qquad N(v,\tau) = |g_a(\tau)|.$$

When $a \in n^{-1}\mathbb{Z}^2 \setminus \mathbb{Z}^2$, $n \geq 2$, g_a^{12n} is a modular unit for $\Gamma(n)$ with Fourier coefficients in $\mathbb{Q}(\zeta_n)$ in the variable $q^{1/n}$. On every complex elliptic curve the model-independent Néron local function for (O) is

$$\lambda_\infty(Q) = -\log N(Q).$$

Thus $D_{\mathrm{part}} = -(6/\pi)(\lambda_\infty(P) + \log(2\sqrt{3}))$.

Proof. Dividing the defining products gives $g_a = ie^{\pi i a_1 v}\vartheta_1(v,\tau)/\eta(\tau)$. Since $\Im v = a_1 \Im\tau$, its absolute value is exactly $N(v,\tau)$. For a nonintegral rational pair of denominator $n \geq 2$, (Shin, 2015, Remark 2.2, p. 3, and Proposition 2.3, pp. 3–4) gives the modular-unit assertion. Its product has coefficients in $\mathbb{Q}(\zeta_n)$ after taking the indicated power; the exponent $6nB_2(r/n)$ belongs to $n^{-1}\mathbb{Z}$.

By proposition 4.2, $-\log N$ has mean zero, the logarithmic pole at O, and the Laplacian of the normalized Green function of (O). The difference of two functions with these properties is harmonic on the compact torus, and its mean makes the constant zero. In the Weierstrass coordinate the normalization is

$$-\log N(z) - \frac{1}{2}\log|x(z)| \longrightarrow -\frac{1}{12}\log|\Delta|.$$

Under the exact homothety $(\Lambda, z, x, \Delta) \mapsto (c\Lambda, cz, c^{-2}x, c^{-12}\Delta)$, N is unchanged and both sides of the last comparison change by $\log|c|$. This proves the stated model transport and the formula for D_{part}. □

Proposition 4.7 (The full division identity)**.** [tag 000Q]

Let ψ_m be the standard division polynomial of a smooth short Weierstrass model, normalized by $\psi_1 = 1$, $\psi_2 = 2y$. On a general equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, the exact change $Y = y + (a_1x + a_3)/2$, followed by $X = x + (a_2 + a_1^2/4)/3$, gives a short equation and $\psi_2 = 2Y = 2y + a_1x + a_3$. The division functions below are transported by this isomorphism. For $m \geq 2$ and $Q, mQ \neq O$,

$$N(Q)^{m^2} = N(mQ)\frac{|\Delta_E|^{(m^2-1)/12}}{|\psi_m(Q)|}. \tag{4.2}$$

For the displayed family this specializes to

$$N(P)^4 = 3^{3/4}|t-1|^{-1/4}N(2P), \qquad N(P)^9 = 16|t|^{-2/3}N(3P).$$

If P_t has exact order n and $m \geq 2$ satisfies $m \equiv 1$ or $-1 \pmod n$, then

$$N(P_t)^{12} = \frac{|\Delta(t)|}{|\psi_m(P_t)|^{12/(m^2-1)}}. \tag{4.3}$$

The positive value on the right is independent of m.

Proof. 1. Use the complex uniformization $x = \wp_\Lambda(z)$, $y = \wp'_\Lambda(z)/2$. The rational function ψ_m has divisor $\sum_{0 \neq Q \in E[m]}(Q) - (m^2-1)(O)$ and leading Laurent term $(-1)^{m-1}mz^{1-m^2}$. Both the divisor and the leading Laurent coefficient follow from $(-1)^{m-1}\sigma_\Lambda(mz)/\sigma_\Lambda(z)^{m^2}$. Its lattice factors cancel, its zeros are exactly the nonzero m-division points, and its expansion at zero is the displayed one. The addition formula for σ gives the standard division-polynomial recurrence with initial values $1, 2y$, so it is the stated rational function.

2. The function

$$m^2U_\Lambda(z) - U_\Lambda(mz) + \log|\psi_m(Q(z))|$$

is harmonic after its logarithmic singularities cancel. Its Laplacian has zero constant term as well, so it is constant. The expansion in 4.2 and the leading term of ψ_m make that constant $(m^2-1)\log|\Delta_E|/12$. This proves (4.2). Direct substitution of

$$\psi_2(P) = t^2(t-1), \qquad \psi_3(P) = -9t^6(t-1)^2/16$$

gives the two special identities. For $m \equiv \pm 1 \pmod{n}$, $mP = \pm P$ and N is even. Cancelling $N(P)$ in (4.2) gives (4.3). Every right side is the same positive number $N(P)^{12}$, which proves independence of m.

□

Proposition 4.8 (Algebraicity and fields of division values)**.** [tag 000R]

Let $t_0 \in \mathbb{C} \setminus \{0,1\}$ and suppose P_{t_0} has exact order $n \geq 2$. Then t_0 and $N(P_{t_0})$ are algebraic. Put $K_0 = \mathbb{Q}(t_0, \bar{t}_0)$,

$$W = \Delta(t_0)\Delta(\bar{t}_0) > 0, \qquad V = \psi_{n+1}(P_{t_0})\psi_{n+1}(P_{\bar{t}_0}) > 0.$$

With positive real radicals,

$$N(P_{t_0})^{12n} = W^{n/2}V^{-6/(n+2)} \ \in \ K_0(W^{1/2}, V^{1/(n+2)}).$$

More generally, if $m \equiv \pm 1 \pmod{n}$, then

$$\lambda_\infty(P_{t_0}) = \frac{\log|\psi_m(P_{t_0})|}{m^2-1} - \frac{1}{12}\log|\Delta(t_0)|.$$

These formulas determine the height and N^{12} on the entire torsion locus.

Proof. 1. The division polynomial $\psi_n(P_t)$ is a nonzero rational polynomial in t. To check nonvanishing for every n, a hypothetical torsion identity would express the elliptic logarithm as $a\tau + b$ with constant rational a, b. Its bounded cusp limit forces $a = 0$, but that limit has nonzero imaginary part, a contradiction. Thus t_0 is a zero of a nonzero polynomial over $\mathbb{Q}$. Formula (4.3) shows algebraicity of N using only positive rational powers of nonzero algebraic absolute values. Taking $m = n+1$, raising to the nth power, and using $|\Delta| = W^{1/2}$, $|\psi_{n+1}| = V^{1/2}$ gives the exact field statement. The local-height formula is its logarithm.

2. Put $r = N^{12}$. Remove the factors at $0, 1$ from $\psi_n(P_t)$ and divide successively by the corresponding polynomials for proper divisors of n; let F_n be the resulting primitive polynomial. At a point of exact order $d \mid n$, the formal group gives $z([n/d]Q) = (n/d)z(Q) + O(z(Q)^2)$, so the vanishing multiplicities for ψ_d and ψ_n agree. Hence this cancellation leaves exactly the exact-order-n divisor, with its multiplicities. Set $m = n+1$ and $e = m^2 - 1$. Clearing rational denominators in

$$\mathrm{Res}_S\left(F_n(S),\ r^{2e}\{\psi_m(P_T)\psi_m(P_S)\}^{12} - \{\Delta(T)\Delta(S)\}^e\right)$$

gives a polynomial in $\mathbb{Q}[T, r]$ that vanishes at $(t_0, N(P_{t_0})^{12})$, since $S = \bar{t}_0$ is a common root. It is nonzero: at an exact-order-n root no ψ_m vanishes, so the leading coefficient in r of the resultant is nonzero. This is an explicit recurrence-and-resultant prescription for every n, including nonreal torsion parameters.

□

Proposition 4.9 (Five norm values and torsion through order eight)**.** [tag 000S]

The exact special values are

t	$\operatorname{ord}(P)$	$N(P)^{12}$
0	3	27
1	2	64
$32/5$	4	10
$64(47-27\sqrt{3})/11$	6	$17+10\sqrt{3}$
$64(47+27\sqrt{3})/11$	6	$10\sqrt{3}-17.$

At $t=0,1$ the orders refer to the smooth models after the specified base changes. On the smooth base, $F_2=F_3=1$ and

$$\begin{aligned}
F_4 &= 5t-32,\\
F_5 &= 89t^2+4736t-4096,\\
F_6 &= 11t^2-6016t+8192,\\
F_7 &= 881t^4+4745344t^3-12079104t^2\\
&\qquad +24641536t-16777216,\\
F_8 &= 49t^4-2614336t^3-7864320t^2\\
&\qquad +19398656t-8388608.
\end{aligned}$$

All these polynomials are squarefree. At each root of exact order $n=4,5,6,7,8$, the positive number $r=N^{12}$ is respectively determined by

n	*equation for* r
4	$r=64/\lvert t\rvert$
5	$r^5=432^5/((1-t)(5t-32)^4)$
6	$r^2=1728^2/F_5(t)$
7	$r^{35}=2^{192}3^{81}/(\lvert t\rvert^8\lvert t-1\rvert^3\lvert F_6(t)\rvert^{12})$
8	$r^4=1728^4/\lvert F_7(t)\rvert.$

The order-five roots are real and below one; the order-six roots are real and above one. The two complex order-seven roots are included in the formula.

Proof. 1. The division-polynomial recurrence, evaluated at $x=0$, $y=t^2(t-1)/2$, gives

$$\begin{aligned}
\psi_4(P) &= -t^{10}(t-1)^4F_4/32,\\
\psi_5(P) &= t^{16}(t-1)^6F_5/4096,\\
\psi_6(P) &= 9t^{24}(t-1)^9F_6/65536,\\
\psi_7(P) &= -t^{32}(t-1)^{12}F_7/16777216,\\
\psi_8(P) &= -t^{42}(t-1)^{16}F_4F_8/268435456.
\end{aligned}$$

The recurrence below gives these identities over $\mathbb{Q}[t]$. The Euclidean algorithm gives $\gcd(F_n,F_n')=1$ and excludes common factors with the polynomials for proper orders. Thus the listed factors give every point, with its exact order and multiplicity. Applying (4.3) with $m=n-1$ gives every displayed formula. The signs for F_5,F_6 follow directly from their quadratic formula; in particular F_5 is positive at the two roots of F_6. At order six, substituting $t=64(47\mp27\sqrt{3})/11$ gives the exact identities

$$F_5(t)(17\pm10\sqrt{3})^2=1728^2.$$

The positive roots are $17+10\sqrt{3}$ and $10\sqrt{3}-17$; their product is 11. The order-four value follows from $64/(32/5)=10$, and the two central values are 4.3.

2. The division-polynomial recurrence is

$$\psi_{2m+1} = \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3,$$
$$\psi_{2m} = \frac{\psi_m}{\psi_2}(\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2).$$

Its initial values at P are $\psi_0 = 0, \psi_1 = 1, \psi_2 = 2y$, $\psi_3 = -A^2$, $\psi_4 = 4y(-8B^2 - A^3)$. The resultant prescription of proposition 4.8 gives a rational polynomial relation for the nonreal order-seven values.

□

Corollary 4.10 (The small-order torsion parameters)**.** [tag 000T] *The numbers of smooth parameters of exact orders* $2, 3, 4, 5, 6, 7, 8$ *are respectively* $0, 0, 1, 2, 2, 4, 4$*. Thus there are thirteen parameters in total, including the two nonreal conjugate parameters of order seven.*

Proof. The polynomials in proposition 4.9 are squarefree and their roots have the indicated exact orders. Their degrees give the count. □

5 Local expansions of the norm

Definition 5.1 (Local expansion constants and differential conventions)**.** [tag IA03]

Put $G = g_2(i) = \Gamma(1/4)^8/(16\pi^2)$ and $D_* = -3\log 6/\pi$. Our theta convention is $\vartheta_j(z,\tau) = \theta_j(\pi z \mid \tau)$, with lattice $\mathbb{Z} + \tau\mathbb{Z}$. The heat identity and modular transformations are (National Institute of Standards and Technology, 2026, Equations 20.13.1, 20.7.26, 20.7.30 and 23.18.5). The classical differential system, in the notation $P = E_2$, $Q = E_4$, $R = E_6$, is

$$xP_x = (P^2 - Q)/12, \qquad xQ_x = (PQ - R)/3, \qquad xR_x = (PR - Q^2)/2;$$

see (Erban and Van Gorder, 2021, Equations (1.1)–(1.4), p. 1). Here $x = e^{2\pi i\tau}$, so $d/d\tau = 2\pi i x\, d/dx$. For $E = \pi^2 E_2/3$, $g_2 = 4\pi^4 E_4/3$ and $g_3 = 8\pi^6 E_6/27$, this gives the exact normalization

$$E' = \frac{i}{2\pi}(E^2 - g_2/12),$$
$$g_2' = \frac{i}{\pi}(2Eg_2 - 3g_3), \qquad g_3' = \frac{i}{\pi}(3Eg_3 - g_2^2/6).$$

5.1 Differential identities

Proposition 5.2 (The two first-derivative identities)**.** [tag 001L]

On an ordinary modulus chart write $L(\tau) = \log N(\mu(\tau), \tau)$ *and put*

$$a = \frac{\Im\mu}{T}, \qquad W = \frac{\vartheta_1'(\mu,\tau)}{\vartheta_1(\mu,\tau)} + 2\pi i a, \qquad R = \frac{d\mu}{d\tau} - a.$$

Then

$$L_\tau = \frac{W}{8\pi i}(W + 4\pi i R), \qquad \det \frac{d}{d\tau}\begin{pmatrix} \tau & \mu \\ \mu & \beta^{\mathrm{part}}/6 \end{pmatrix} = \frac{(W + 2\pi i R)^2}{4\pi^2}.$$

Moreover

$$R = \frac{g_2}{6\pi i \wp'(\mu)} - \frac{W}{2\pi i}, \qquad L_\tau = \frac{iW(3W\wp'(\mu) - 2g_2)}{24\pi\wp'(\mu)}.$$

Multiplication by $6/\pi$ *gives the critical-point factorization for* D_{part} *in Proposition 19.14.*

Proof. 1. Put $f = \vartheta_1'/\vartheta_1$ and $E = \pi^2 E_2/3$. The theta heat equation and $\eta'/\eta = \pi i E_2/12$ give

$$\partial_\tau \log(\vartheta_1/\eta) = \frac{f_z + f^2 + E}{4\pi i}.$$

At $\wp(\mu) = 0$, the identity $f_z = -\wp - E$ reduces this to $f^2/(4\pi i)$. Thus, along the section,

$$\frac{d}{d\tau} \log(\vartheta_1(\mu,\tau)/\eta) = f\mu' + \frac{f^2}{4\pi i}.$$

Differentiate the Gaussian $-\pi(\Im\mu)^2/T$ using Wirtinger derivatives, retaining the factor $1/2$ in the derivative of the real part of a holomorphic logarithm. Substitution of $f = W - 2\pi i a$ and $\mu' = R + a$ gives the first formula. Since $\beta^{\mathrm{part}}/6 = (i/\pi)\log(\vartheta_1/(2\sqrt{3}\eta))$, its derivative is $if\mu'/\pi + f^2/(4\pi^2)$. Subtracting μ'^2 gives $(f + 2\pi i\mu')^2/(4\pi^2)$, the second formula.

2. Finally differentiate $\wp_\tau(\mu(\tau)) = 0$ by the theta heat equation. After substituting $\wp(\mu) = 0$, $\wp''(\mu) = -g_2/2$, and $f = W - 2\pi i a$, the result is $6\pi i\wp' R = g_2 - 3W\wp'$. This is the formula for R. Inserting it into the first identity gives the last one. The formulas involving $1/\wp'(\mu)$ apply where $\wp'(\mu) \neq 0$. The half-period case is treated in the root coordinate of proposition 5.3.

□

5.2 The square and equianharmonic points

Proposition 5.3 (The fourth-order expansion at the maximum)**.** [tag `000N`]
On either root branch at $\tau = i$, let $h = \mu - (1-\tau)/2$ and $\delta = \tau - i$. Then

$$\begin{aligned}
\tau &= i - \frac{3\pi i}{2}h^2 - \frac{9\pi^2 i}{8}h^4 + O(h^6),\\
h^2 &= \frac{2i}{3\pi}\delta + \frac{1}{3\pi}\delta^2 + O(\delta^3),\\
\mu &= \frac{1-\tau}{2} \pm \left(\frac{2i\delta}{3\pi}\right)^{1/2}\left(1 - \frac{i}{4}\delta + O(\delta^2)\right).
\end{aligned}$$

The square root is taken holomorphically on the chosen root chart. Furthermore

$$\begin{aligned}
\log N &= \tfrac{1}{2}\log 2 - \frac{\pi}{2}|h|^2 + \frac{3\pi^2}{8}|h|^4 - \frac{3\pi^2}{4}|h|^2\,\mathrm{Re}(h^2) - \frac{5G}{192}\,\mathrm{Re}(h^4) + O(|h|^6), \qquad (5.1)\\
D_{\mathrm{part}} &= D_* - \frac{2}{\pi}|\delta| + \frac{1}{\pi}(|\delta|^2 + |\delta|\,\mathrm{Im}\,\delta) + \frac{5G}{72\pi^3}\,\mathrm{Re}(\delta^2) + O(|\delta|^3).
\end{aligned}$$

In particular the leading terms agree with the boundary expansion in Lemma 19.5.

Proof. 1. Write $e(\tau) = \wp_\tau((1-\tau)/2)$ and $E = \pi^2 E_2/3$. The half-period Taylor series is

$$\wp_\tau((1-\tau)/2 + h) = e + (3e^2 - g_2/4)h^2 + (3e^3 - g_2 e/4)h^4 + O(h^6).$$

The exact half-period coefficients used in Lemma 19.5 are $e(i) = 0$, $e'(i) = iG/(6\pi)$. This proves the claimed identity $3e'(i) = (2i/\pi)G/4$. Its closed form follows from $e = \pi^2(\vartheta_2^4 - \vartheta_4^4)/3$ and the heat equation: if $A = \vartheta_2(i)^4 = \vartheta_4(i)^4$, then $G = 4\pi^4 A^2$ and $e'(i) = 2\pi^3 i A^2/3$. For the classical closed value, (National Institute of Standards and Technology, 2026, Equation 23.5.3) gives $e_1 = \Gamma(1/4)^4/(32\pi\omega_1^2)$, $e_2 = 0$ and $e_3 = -e_1$. The lattice identification is $2\omega_1 = 1$, $2\omega_3 = i$; hence $G = -4(e_1e_2 + e_2e_3 + e_3e_1) = 4e_1^2 = \Gamma(1/4)^8/(16\pi^2)$.

2. Ramanujan's identities give $g_2'(i) = 2iG$. Differentiating $e(-1/\tau) = \tau^2 e(\tau)$ twice at i gives $e''(i) = 3ie'(i) = -G/(2\pi)$. Inserting $\delta = A_2h^2 + A_4h^4 + O(h^6)$ into the displayed half-period series therefore gives

$$A_2 = -3\pi i/2, \qquad A_4 = \frac{g_2'A_2/4 - e''A_2^2/2}{e'} = -9\pi^2 i/8.$$

The implicit-function theorem applies because $e'(i) \neq 0$; evenness in h gives the stated remainder. Series inversion gives the formulas for h^2 and μ.

3. Let $H(h,\tau) = \log(\vartheta_3(h,\tau)/\eta(\tau))$. The exact identity from the half-period translation is $\log N = \operatorname{Re} H - \pi(\operatorname{Im} h)^2/\operatorname{Im}\tau$. At $(0,i)$ its relevant Taylor data are

$$H_0 = \tfrac{1}{2}\log 2, \quad H_\tau = 0, \quad [h^2]H = -\pi/2, \quad [h^4]H = G/48,$$
$$\partial_\tau[h^2]H = -\pi i/4 - iG/(16\pi), \qquad H_{\tau\tau} = -G/(24\pi^2).$$

Indeed $\partial_h^2 \log\vartheta_3 = -\wp(h + (1-\tau)/2) - E$; the heat equation at $h = 0$ gives $H_\tau = ie/(4\pi)$, and $E'(i) = \pi i/2 - iG/(24\pi)$. These identities prove every entry. After substitution $\tau = \tau(h)$, the pure h^4 coefficient of H is $-3\pi^2/8 - 5G/192$. Expanding $\operatorname{Im}\tau = 1 - (3\pi/2)\operatorname{Re}(h^2) + O(|h|^4)$ in the Gaussian gives (5.1). Finally

$$|h|^2 = \frac{2}{3\pi}|\delta|\left(1 + \tfrac{1}{2}\operatorname{Im}\delta + O(|\delta|^2)\right)$$

and substitution into the quartic expression, followed by multiplication by $6/\pi$, gives the formula for D_{part}.

□

Proposition 5.4 (The equianharmonic expansion)**.** [tag 0000]

Set $T_0 = \sqrt{3}/2$, $\mu_0 = (2-\rho)/3$, and $p = \wp'_\rho(\mu_0)$. Then $p \neq 0$, $p^2 = -g_3(\rho)$, and, for $\delta = \tau - \rho$ on the order-three root chart,

$$\mu = \frac{2-\tau}{3} + \frac{3p}{16\pi^2}\delta^2 + \frac{3ip}{32\pi^2 T_0}\delta^3 + O(\delta^4),$$
$$t = \frac{ip^2}{\pi^3}\delta^3\left(1 + \frac{3i}{2T_0}\delta + O(\delta^2)\right),$$
$$\log N = \tfrac{1}{4}\log 3 + \frac{7}{192}\operatorname{Re} t - \frac{9\pi}{512T_0|p|^{2/3}}|t|^{4/3} + O(|t|^{5/3}).$$

Thus

$$D_{\text{part}}(t) = -\frac{3}{2\pi}\log 48 + \frac{7}{32\pi}\operatorname{Re} t - \frac{27}{256T_0|p|^{2/3}}|t|^{4/3} + O(|t|^{5/3}).$$

The point $t = 0$ is neither a local maximum nor a local minimum.

Proof. 1. Use the notation of proposition 5.2, with $\ell = \partial_z \log\vartheta_1(\mu,\tau)$ and $Z = W$. The differential identities proved there apply to the zero locus. At ρ the unit endomorphism fixing the nonzero order-three point forces the gradient of U_ρ there to vanish, so $Z = 0$. Explicitly the unit $\rho - 1$ fixes μ_0 modulo the lattice and has no fixed nonzero real covector. Hence $a = \mu' = -1/3$ and $\ell = 2\pi i/3$.

2. Along the zero locus, a further heat-equation differentiation gives

$$\ell' = \frac{i\wp'}{4\pi} + \frac{iEg_2}{6\pi\wp'}.$$

The Ramanujan derivatives at ρ are

$$g_2' = \frac{3ip^2}{\pi}, \quad g_2'' = \frac{5i}{T_0} g_2', \quad \frac{(\wp')'}{\wp'} = \frac{3i}{2T_0}, \quad E = \frac{\pi}{T_0}.$$

They give $\mu'' = 3p/(8\pi^2)$, $\mu''' = 9ip/(16\pi^2 T_0)$ and hence the asserted μ expansion. Expansion of $t = g_2^3/(g_2^3 - 27g_3^2)$ gives the displayed t formula. Since $\partial_\tau a = R/(2iT)$, one has

$$Z = \frac{ip}{4\pi}\delta + O(|\delta|^2), \qquad R = \frac{3p}{8\pi^2}\delta + O(|\delta|^2).$$

The derivative formula now gives the harmonic cubic $\operatorname{Re}(7ip^2\delta^3/(192\pi^3)) = (7/192)\operatorname{Re} t + O(|\delta|^4)$.

3. For the next term, direct differentiation of the Gaussian gives

$$\partial_\tau \partial_{\bar\tau} \log N = -\frac{\pi}{2T}|R|^2 = -\frac{9|p|^2}{128\pi^3 T_0}|\delta|^2 + O(|\delta|^3).$$

After the harmonic cubic has been written as a function of t, the order-three stabilizer permits only a radial term in degree four: the stabilizer map is $\gamma\tau = 1 - 1/\tau$ and the zero branch transforms as $\mu(\gamma\tau) = (1 - \mu(\tau))/\tau$. The theta and eta transformations cited above cancel against the Gaussian factor, so $N(\gamma\tau) = N(\tau)$. Thus in the linearizing coordinate $(\tau - \rho)/(\tau - \bar\rho)$ invariant monomials have exponents whose difference is divisible by three. The coefficient of $|\delta|^4$ is therefore $-9|p|^2/(512\pi^3 T_0)$, since $\partial_\delta \partial_{\bar\delta}|\delta|^4 = 4|\delta|^2$. Converting $|\delta|^4$ to $|t|^{4/3}$ proves the expansion, including the remainder. Its nonzero linear term in $\operatorname{Re} t$ has both signs, which proves the last assertion. In particular superharmonicity at this root point does not make it a local maximum.

□

6 An analytic proof of the elliptic maximum

Remark 6.1 (Modular and arithmetic discriminants)**.** [tag IA24] Use $q = e^{2\pi i\tau}$, $T = \Im\tau$, Δ_0 and t from definition 2.6. Thus $t = E_4^3/(1728\Delta_0)$ and $t - 1 = E_6^2/(1728\Delta_0)$. On the smooth rational model $t \neq 0, 1$, choose λ with

$$\lambda^2 = -\frac{t(t-1)g_2}{3g_3}.$$

The exact uniformization is $(x, y) = (\lambda^2\wp(z), \lambda^3\wp'(z)/2)$, where $\wp$ has lattice $\mathbb{Z} + \tau\mathbb{Z}$. Choose the sign of λ so that the marked zero μ maps to P_t. Indeed $g_2^3/g_3^2 = 27t/(t-1)$ gives

$$A = -\lambda^4 g_2/4, \qquad B = -\lambda^6 g_3/4, \qquad \Delta_E = \lambda^{12}(2\pi)^{12}\Delta_0.$$

The invariant differential pulls back to $\lambda^{-1}dz$, so its period lattice is $\lambda^{-1}(\mathbb{Z} + \tau\mathbb{Z})$.

Proposition 6.2 (Three uniform bounds on the elliptic norm)**.** [tag 001J]

For every $z \in \mathbb{C}$ and every τ in the standard modular fundamental region, we have

restriction	$T \le 2$	$T \le 7/5$	$T \le 11/10$
$N(z, \tau)$	$< 17/10$	$< 3/2$	$< 143/100.$

If $\wp_\tau(z) = 0$ and $T \ge 2$, then $N(z, \tau) < 541/500$.

Proof. 1. Taking absolute values in the theta series and completing the square gives

$$e^{-\pi(\Im z)^2/T}|\vartheta_1(z, \tau)| \le \sum_{n\in\mathbb{Z}} e^{-\pi T(n+1/2+\Im z/T)^2} \le \sum_{n\in\mathbb{Z}} e^{-\pi T n^2}.$$

The last inequality follows from Poisson summation: the Fourier coefficients of the periodized Gaussian are positive, so its maximum is at an integer. The eta product and $n^2 \geq 1+3(n-1)$ for $n \geq 1$ therefore imply, on $a \leq T \leq b$,

$$N(z,\tau) \leq \exp\left(\frac{r_a}{(1-r_a)^2}\right) \max\{F(a), F(b)\}, \qquad F(T) = e^{\pi T/12}\left(1 + \frac{2e^{-\pi T}}{1-e^{-3\pi T}}\right), \quad (6.1)$$

where $r_a = e^{-2\pi a}$. Indeed, $-\sum \log(1-r^n) \leq r/(1-r)^2$. The function F is convex, as follows by expanding its denominator as a geometric series; each summand is an exponential of a real linear function of T. We use $6/7 < \sqrt{3}/2$ and the following rational bounds:

T	$6/7$	1	$11/10$	$7/5$	2
$F(T)$	$< 14211/10000$	$< 14116/10000$	$< 14180/10000$	$< 14782/10000$	$< 16944/10000.$

The eta factors in (6.1) are less than $10047/10000$ for $a = 6/7$ and less than $10019/10000$ for $a = 1$. Multiplying these fractions proves the asserted bounds, using $[6/7, 1]$ and $[1, 2]$ separately for the first one.

2. The scalar estimates follow from $314159/100000 < \pi < 355/113$ and, for $0 \leq x \leq 20$, the Taylor bounds

$$S_{80}(x) \leq e^x \leq S_{80}(x) + \frac{x^{81}}{81!}\frac{1}{1-x/82}, \qquad S_{80}(x) = \sum_{j=0}^{80} \frac{x^j}{j!}. \quad (6.2)$$

Taking reciprocals bounds negative exponentials. The remainder bound follows by comparison of consecutive terms. These inequalities give each displayed rational bound.

3. For the cusp use the Fourier expansion of $\wp$. Choose $|\Im z| \leq T/2$ by translation, put $b_0 = 43/100$ and $r_0 = e^{-4\pi}$. The Fourier series

$$\frac{\wp_\tau(z)}{\pi^2} = \csc^2(\pi z) - \frac{1}{3} + 8\sum_{n\geq 1} \frac{nq^n}{1-q^n}(1-\cos(2\pi n z))$$

shows, if $|\Im z| \geq b_0$, that

$$\left|\frac{\wp_\tau(z)}{\pi^2} + \frac{1}{3}\right| \leq \frac{1}{\sinh^2(\pi b_0)} + 8\left\{\frac{r_0}{(1-r_0)^3} + \frac{1}{2(1-r_0)}\left(\frac{e^{-2\pi}}{(1-e^{-2\pi})^2} + \frac{e^{-6\pi}}{(1-e^{-6\pi})^2}\right)\right\} < \frac{1}{3}.$$

The two terms in the last parentheses bound respectively $r^n e^{2\pi n|\Im z|}$ and $r^n e^{-2\pi n|\Im z|}$. This inequality excludes a zero. Hence a zero has $|\Im z| < b_0$. The Jacobi product, $\log(1+x) \leq x$, and the monotonicity of $\cosh(\pi y)e^{-\pi y^2/T}$ on $0 \leq y \leq b_0$ give

$$N(z,\tau) \leq 2\cosh(\pi b_0) e^{-\pi/3 - \pi b_0^2/2} \exp\left(\frac{2\cosh(2\pi b_0) r_0}{1-r_0} + \frac{r_0^2}{1-r_0^2}\right) < \frac{541}{500}.$$

For the stated monotonicity, the concave function $\tanh(\pi y) - y$ is zero at zero and positive at b_0. For $T \geq 2$ the right side before substituting $T = 2$ decreases: the derivative of $-\pi T/6 - \pi b_0^2/T$ is negative and both product errors decrease. The two strict scalar inequalities follow from (6.2). This proves the cusp assertion on the whole cusp region.

□

Proposition 6.3 (The maximum on an explicit local disc)**.** [tag 001Q]
If $|\tau - i| \le 1/50$ and $\wp_\tau(z) = 0$, then $N(z,\tau) \le \sqrt{2}$, with equality exactly when $\tau = i$.

Proof. 1. Write $\delta = \tau - i$, $h = z - (1-\tau)/2$ and

$$G(h,\tau) = \wp_\tau((1-\tau)/2 + h), \qquad H(h,\tau) = \log \frac{\vartheta_3(h,\tau)}{\eta(\tau)}.$$

We choose the holomorphic logarithm whose value at $(0,i)$ is $\log\sqrt{2}$. The half-period theta identity gives

$$\log N(z,\tau) = \Re H(h,\tau) - \frac{\pi(\Im h)^2}{T}. \tag{6.3}$$

Both G and H are even in h. On the polydisc $|h| \le 1/4$, $|\delta| \le 1/10$, they satisfy

$$|G| < 30, \qquad |H| < 2/3. \tag{6.4}$$

Here are sufficient series estimates, including their margins. Put

$$A_k = \sum_{n\ge 1} n^k e^{-9\pi n^2/10} \cosh(\pi n/2), \quad a_0 = 2A_0, \quad a_1 = 4\pi A_1, \quad a_2 = 8\pi^2 A_2.$$

Then $a_0 < 3/10$, $a_1 < 19/10$, $a_2 < 12$. The first two summands suffice to verify these inequalities: for $n \ge 3$, each of the two exponentials in the hyperbolic cosine has successive-term ratio at most $e^{-29\pi/5}$, and the factor n^k increases this ratio by at most $(4/3)^2$. We use (6.2) for the first terms and this geometric bound for the tail. With $r = e^{-9\pi/5}$, the identities

$$G = -\partial_h^2 \log\vartheta_3 - \frac{\pi^2}{3}E_2, \qquad |E_2| \le 1 + \frac{24r}{(1-r)^3}$$

give

$$|G| \le \frac{a_2}{1-a_0} + \frac{a_1^2}{(1-a_0)^2} + \frac{\pi^2}{3}\left(1 + \frac{24r}{(1-r)^3}\right) < 30.$$

Also

$$|H| \le -\log(1-a_0) + \frac{11\pi}{120} + \frac{r}{(1-r)^2} < \frac{2}{3}.$$

The logarithm is well defined since $|\vartheta_3 - 1| < 3/10$; the displayed eta product fixes its branch. A geometric bound for $-\log(1-x) = \sum_{n\ge1} x^n/n$ verifies the last estimate.

2. Let

$$e(\tau) = \frac{\pi^2}{3}(\vartheta_2(0,\tau)^4 - \vartheta_4(0,\tau)^4), \qquad E(\tau) = \frac{\pi^2}{3}E_2(\tau), \qquad g = \frac{g_2(i)}{4}.$$

On $|\delta| \le 1/50$ the following loose bounds hold:

$$|e'| < 14, \quad |e''| < 60, \quad |g_2'| < 500, \quad |E'| < 2, \quad |E_6''| < 100, \qquad 47 < g < 95/2. \tag{6.5}$$

The scalar Fourier series give, at $T \ge 49/50$, the following upper bounds for $|\vartheta_j|, |\vartheta_j'|, |\vartheta_j''|$:

j	0 derivatives	1 derivative	2 derivatives
2	929/1000	742/1000	670/1000
4	1093/1000	290/1000	911/1000.

They follow by differentiating the absolutely convergent theta series, taking absolute values, retaining two terms, and using the same successive-term bound. Differentiating the fourth

powers proves the first two bounds in (6.5). For $r \le 1/470$, the remaining derivative majorants are

$$|g_2'| \le 640\pi^5 \frac{r(1+11r+11r^2+r^3)}{(1-r)^7} < 500,$$
$$|E'| \le 16\pi^3 \frac{r(1+r)}{(1-r)^5} < 2,$$
$$|E_6''| \le 2016\pi^2 \frac{r(1+r)(1+120r+1191r^2+2416r^3+1191r^4+120r^5+r^6)}{(1-r)^{11}} < 100.$$

These follow by summing $\sum n^k r^n$ and bounding $(1-r^n)^{-1}$ by $(1-r)^{-1}$. Substitution of $r = 1/470$ and $\pi < 355/113$ verifies the rational inequalities. Finally $g = \pi^4 E_4(i)/3$ and $E_4 = 1+240\sum n^3 q^n/(1-q^n)$ give $47 < g < 95/2$. The half-period identity used in Lemma 19.5, obtained by differentiating e and using the theta heat equation, reads

$$e(i) = 0, \qquad e'(i) = id, \qquad d = \frac{2g}{3\pi} > 99/10.$$

3. The differential equation $\wp'' = 6\wp^2 - g_2/2$ determines the first coefficients in the even expansion:

$$G(h,\tau) = e + ah^2 + eah^4 + \mathcal{R}_G, \qquad a = 3e^2 - g_2/4.$$

Consequently, on $|\delta| \le 1/50$,

$$|e| \le 7/25, \quad |a+g| \le \frac{5}{2} + 3(7/25)^2 = \frac{3419}{1250}, \quad |a| \le \frac{62794}{1250}.$$

Cauchy's estimate from (6.4) yields

$$|\mathcal{R}_G| \le \frac{30(4|h|)^6}{1-16|h|^2}.$$

On $|h| = 1/10$ the term ah^2 has modulus at least $55331/125000 > 0.4426$, whereas the sum of the other terms is at most

$$\frac{7}{25} + \frac{7}{25}\frac{62794}{1250}\frac{1}{10000} + \frac{30(4/25)^3}{1-4/25} < 0.4277.$$

Rouché's theorem puts two zeros, with multiplicity, in $|h| < 1/10$. The disc injects into the torus and $\wp$ has exactly two zeros on the torus, so it contains both of them.

4. Write $G = e + h^2 Q$. The preceding estimates give, for $|h| \le 1/10$,

$$|Q+g| \le \frac{3419}{1250} + \frac{7}{25}\frac{62794}{1250}\frac{1}{100} + \frac{30(4/25)^3}{(1-4/25)/100} < \frac{88}{5}.$$

Taylor's formula and $|e''| < 60$ give $|e| \ge (99/10 - 3/5)|\delta| = (93/10)|\delta|$. At either zero, therefore,

$$|\delta| < 7|h|^2, \tag{6.6}$$

because $(95/2 + 88/5)/(93/10) = 7$.

5. Expand $H = H_0(\tau) + H_2(\tau)h^2 + H_4(\tau)h^4 + \mathcal{R}_H$. The heat equation and the identity for G give

$$H_0' = \frac{ie}{4\pi}, \quad H_2 = -\frac{e+E}{2}, \quad H_4 = -\frac{a}{12}.$$

Thus $H_0(i) = \log\sqrt{2}$, $H_0'(i) = 0$, $|H_0''| < 6/5$, $H_2(i) = -\pi/2$, $|H_2'| < 8$, and $|H_4| < 21/5$. Cauchy's estimate gives

$$|\mathcal{R}_H| \le \frac{(2/3)(4|h|)^6}{1-16|h|^2} \le \frac{2048}{63}|h|^4 \quad (|h| \le 1/10).$$

Combining (6.3) and (6.6), we obtain

$$\begin{aligned}\log N - \log\sqrt{2} &\le -\frac{\pi}{2}\Re(h^2) - \pi(\Im h)^2 + C|h|^4 \\ &= -\frac{\pi}{2}|h|^2 + C|h|^4, \\ C &= \frac{147}{5} + 56 + \frac{21}{5} + \frac{2048}{63} + \frac{1100}{49} < 150.\end{aligned}$$

The final term of C bounds the change of $1/T$ from 1: $7\pi/(1-1/50) < 1100/49$. Since $150|h|^2 < 3/2 < \pi/2$, the result is strict unless $h = 0$. Equation (6.6) then gives $\delta = 0$, where equality holds by proposition 4.3.

□

Proposition 6.4 (Comparison of the two local discs)**.** [tag 001R]

Every complex number t_0 with $|t_0 - 1| \le 1/500$ has a representative τ with $|\tau - i| < 1/50$ and $j(\tau) = 1728t_0$. Consequently every zero of $\wp$ for that modulus satisfies $N \le \sqrt{2}$, with equality only for $t_0 = 1$.

Proof. 1. At i, $E_6 = 0$ and the Ramanujan differential identity gives $|E_6'(i)| = \pi E_4(i)^2 > 13/2$, since $E_4(i) > 29/20$. On $|\delta| = 1/50$, (6.5) gives

$$|E_6(\tau)| > \frac{13}{100} - \frac{1}{50} = \frac{11}{100}.$$

On the whole disc, $|q| < 1/470$ and $|\Delta_0| \le |q|\exp(24|q|/(1-|q|)) < 1/400$. It follows on the boundary that

$$|t-1| = \frac{|E_6|^2}{1728|\Delta_0|} > \frac{121}{43200} > \frac{1}{500}.$$

Rouché applied to $E_6'(i)\delta$ shows that E_6 has exactly one zero in the disc. Hence $t-1$ has two zeros there, with multiplicity, and another application of Rouché shows that $t - t_0$ has two for each $|t_0 - 1| \le 1/500$. Thus the stated t-disc is covered. Elliptic curves over $\mathbb{C}$ with the same j are isomorphic. In terms of lattices the isomorphism is a homothety; it carries the two zeros of $\wp$ to the two zeros of the scaled $\wp$ and preserves N. Proposition 6.3 applies to either zero.

□

Proposition 6.5 (The first three normalized duplication steps)**.** [tag 001S]

Put $u = t - 1$. For $n = 2^k$, $k = 1, 2, 3$, and $nP \ne O$,

$$\left(\frac{N(P)}{\sqrt{2}}\right)^{n^2} = \frac{3^{3/4}}{2n}\,\frac{N(nP)|u|^{-1/4}}{|A_k(u)|}, \tag{6.7}$$

where

$$\begin{aligned}&A_1(u) = 1, \qquad A_2(u) = 1 - 5u/27, \\ &A_3(u) = 1 - \frac{217}{27}u - \frac{20486}{729}u^2 + \frac{10910}{19683}u^3 + \frac{161383}{177147}u^4 - \frac{245}{14348907}u^5.\end{aligned}$$

On $|u| \le 1/28$ we have $|A_3(u)| \ge 1 - 10|u|$. For every $k \ge 1$, define

$$E_k(t) = \frac{|\Delta_E|^{(4^k-1)/12}|t-1|^{1/4}}{2^{4^k/2}|\psi_{2^k}(P)|}.$$

Then $(N(P)/\sqrt{2})^{4^k} = E_k(t)N(2^kP)|t-1|^{-1/4}$, and the removable limiting value is

$$\lim_{t\to 1} E_k(t) = \frac{3^{3/4}}{2^{k+1}} = 2^{1-k}\frac{3^{3/4}}{4}.$$

Proof. 1. Substitute the exact division polynomials from proposition 4.7 in (4.2). Equivalently, remove $t^{2(n^2-1)/3}(t-1)^{n^2/4}$ from $\psi_n(P)$ and divide the resulting polynomial by its value at $t = 1$. For $n = 2, 4, 8$ this gives the displayed polynomials and the constant $3^{3/4}/(2n)$. These are rational polynomial operations using the recurrence in proposition 4.9. The general identity is proposition 4.7. For its limit write $\mu = (1-\tau)/2 + h$ as in 5.3. Since $n = 2^k$ is even, $n\mu$ is congruent to nh modulo the period lattice. Hence

$$N(nP) = 2\pi|\eta(i)|^2 n|h|(1+o(1)), \qquad t - 1 = \tfrac{9}{4}\pi^4 E_4(i)h^4(1+o(1)).$$

The identity $E_4(i)^3 = 1728\eta(i)^{24}$ then gives $N(nP)|t-1|^{-1/4} \to 2n/3^{3/4}$. Since the left side of the general ladder tends to one, this proves the stated limit for every k. The triangle inequality gives $|A_3(u)| \ge 1 - |u|M$, where

$$M \le \frac{81}{10} + \frac{29}{28} + \frac{1}{28^2} + \frac{1}{28^3} + \frac{1}{28^4} < 10.$$

Here we bounded respectively the five absolute coefficients by $81/10, 29, 1, 1, 1$. The bound holds on the whole disc.

□

Theorem 6.6 (Exact norm maximum and parameter discs)**.** [tag 01QL] *At every zero of $\wp_\tau$,*

$$N \le \sqrt{2},$$

with equality precisely in the modular orbit of $\tau = i$. For the cusp normalization $\beta^{\mathrm{part}} + \tau \to 0$ this gives

$$M = -\frac{3}{\pi}\log 6, \qquad \arg\max_{B\setminus\{p_0\}} D_{\mathrm{part}} = \{1\}.$$

The real period determinant is nonzero throughout the base exactly when $\Im c < -M$. Consequently the parameter domains are

$$0 < |w| < 216, \qquad 0 < |u_{\mathrm{Engel}}| < 46656.$$

Changing the reference to $u_{\lambda_0} = \lambda_0 w^2$ multiplies the latter radius by $|\lambda_0|$.

Proof. 1. Reduce to the standard modular region. The portion $T \ge 2$ is covered by proposition 6.2. Suppose $T \le 2$ and put $r = |t-1|$. For $r \ge 9/10$, the step $n = 2$ in (6.7) and $N(2P) < 17/10$ give strict inequality, because

$$\frac{27(17/10)^4}{256} < \frac{9}{10}.$$

There is no 2-torsion point of the section on the smooth base.

2. If $r \le 9/10$, then $T < 7/5$. Indeed, $T \ge 7/5$ would give $|q| < 1/6500$ and

$$|E_4| > 24/25, \qquad |\Delta_0| < 1/6400, \qquad |t| > \frac{6400(24/25)^3}{1728} > 19/10,$$

contrary to $|t| \le 19/10$. The estimates follow from

$$\sum_{n\ge1} \sigma_3(n) r^n \le \frac{r(1+4r+r^2)}{(1-r)^5}, \qquad |\Delta_0| \le r \exp \frac{24r}{1-r}.$$

For $1/28 \le r \le 9/10$, the $n = 4$ step and $N(4P) < 3/2$ now apply. The function $(1-5r/27)r^{1/4}$ is increasing up to $r = 27/25$. At its left endpoint its fourth power is

$$\frac{1}{28}\left(\frac{751}{756}\right)^4 > \frac{2187}{65536} = \left(\frac{3^{3/4}}{8}\frac{3}{2}\right)^4.$$

This proves strict inequality on the whole annulus. The lower bound for A_2 is positive there.

3. If $r \le 1/28$, then $T < 11/10$. For otherwise $|q| < 1/1000$, and the Lambert series with

$$\sum_{n\ge1} \sigma_5(n) r^n \le \frac{r(1+26r+66r^2+26r^3+r^4)}{(1-r)^7}$$

gives $|E_6| > 9/20$, $|\Delta_0| < 1/900$, and hence $|t-1| > 900(9/20)^2/1728 > 1/28$. On $1/500 \le r \le 1/28$, the $n = 8$ step, $N(8P) < 143/100$, and $|A_3| \ge 1 - 10r$ prove the claim. Indeed $(1-10r)r^{1/4}$ has a single critical point in this interval, a maximum at $r = 1/50$, so its minimum is at an endpoint. At both endpoints its fourth power exceeds $9/5000$, whereas

$$\left(\frac{3^{3/4}}{16}\frac{143}{100}\right)^4 < \frac{7}{4000} < \frac{9}{5000}.$$

Again the displayed positive bound excludes a division-polynomial zero. The remaining disc $r \le 1/500$ is covered by proposition 6.4. The smooth limiting fibre at $t = 0$ has $N = 3^{1/4} < \sqrt{2}$; the formula also extends there by continuity. The only equality occurs at $t = 1$.

4. The norm identity (3.3) gives $M = (6/\pi)\log(\sqrt{2}/(2\sqrt{3})) = -3\log 6/\pi$. The cusp asymptotic in proposition 3.3 and continuity show that $D_{\mathrm{part}} + \Im c$ has a zero for $\Im c \ge -M$: use the maximum at $t = 1$ and a path to the cusp. For $\Im c < -M$ it is strictly negative everywhere. By lemma 2.7, this is exactly nonvanishing of the real period determinant throughout the base. Since $|w| = e^{\pi \Im c}$ and $u_{\mathrm{Engel}} = w^2$, the radii are $e^{-\pi M} = 6^3$ and $e^{-2\pi M} = 6^6$. The reference change is the displayed multiplication by λ_0.

□

7 The family and its intrinsic classification

Definition 7.1 (Affine covariance)**.** [tag PI08] Use the standard data of Definition 2.2 and write $\varepsilon = v_1$, $\varepsilon' = -v_2$, $\epsilon_j = \gamma(v_j)$, so $(\epsilon_1, \epsilon_2) = (1, -1)$. Put

$$d = \widehat{\delta}, \qquad e = (0,1)^{\mathsf{t}} = \Pi_c d, \qquad E = d \otimes \gamma, \qquad E^2 = 0, \quad \Pi_c E = e\gamma. \tag{7.1}$$

The fibre covariance is $R_g(z)\Pi_c(z) = \Pi_c(gz)A_g$, where

$$R_{g_1} = \begin{pmatrix} -1/\tau & 0 \\ (1-\mu)/\tau & 1 \end{pmatrix}, \qquad R_{g_2} = \begin{pmatrix} 1/\tau & 0 \\ -\mu/\tau & 1 \end{pmatrix}, \qquad R_{g_0} = I_2. \tag{7.2}$$

In the cusp coordinate s, the generator satisfies $s(g_0 z) = s(z) - 1$. The matrices satisfy $R_g e = e$ and $A_j E = E A_j = E$.

7.1 Canonical sections and the finite covers

Proposition 7.2 (The canonical bundle and anticanonical pencil)**.** [tag 01PZ]

For the standard marked $(3, 4, \infty)$ family, including every admissible choice of its affine translations and cusp collar with the fixed fan, the evaluation map

$$f^*(f_*\omega_{X/B}) \longrightarrow \omega_{X/B}$$

has zero divisor exactly $2S_2$. Consequently

$$\omega_X \simeq f^*\mathcal{O}_B(-p_2) \otimes \mathcal{O}_X(2S_2) \simeq \mathcal{O}_X(-2S_2), \qquad \omega_X^{-2} \simeq f^*\mathcal{O}_B(p_2). \tag{7.3}$$

For every integer k the direct image of the vertical divisor is

$$f_*\mathcal{O}_X(kS_2) = \mathcal{O}_B(\lfloor k/4 \rfloor p_2) \tag{7.4}$$

as subsheaves of the meromorphic-function sheaf, using pullback to identify their sections. In particular, for each integer $m \geq 0$,

$$h^0(X, \omega_X^{-m}) = \lfloor m/2 \rfloor + 1, \qquad h^0(X, \omega_X^{m}) = 0 \quad (m > 0). \tag{7.5}$$

There is a graded algebra isomorphism

$$\mathbb{C}[U, V] \xrightarrow{\sim} \bigoplus_{m \geq 0} H^0(X, \omega_X^{-m}), \qquad \deg U = 1, \quad \deg V = 2. \tag{7.6}$$

The degree-two complete linear system has no base points and its map is the map f, followed by a specified projective-coordinate map of B. The degree-one system consists of the single divisor $2S_2$. Every biholomorphism between two such threefolds induces the identity on B.

Proof. 1. (Claude and L. Alpöge, 2026, Lemma 9.10(ii), p. 70, and Proposition 9.11, pp. 70–71) give the evaluation divisor $2S_2$ and $f_*\omega_{X/B} \simeq \mathcal{O}_B(p_0)$ for these same marked fillings. Use the explicit base identifications

$$h = (t-1)^{-1}, \quad \operatorname{div}(h) = p_0 - p_2, \qquad \eta_B = dt/(t-1)^2, \quad \operatorname{div}(\eta_B) = -2p_2.$$

These local canonical transitions hold for every admissible affine translation and cusp collar with the fixed fan. A translation $\zeta \mapsto \zeta + a(s)$ contributes only ds terms to its differential, which vanish after wedging with the base differential. A cusp multiplier $x_j \mapsto h_j(t_c)x_j$ adds $(h_j'/h_j)dt_c$ to dx_j/x_j, with the same cancellation. Thus the determinant characters and evaluation divisor are unchanged. Tensoring evaluation with $f^*\eta_B$, and multiplying the base line by f^*h, gives (7.3), since $f^*p_2 = 4S_2$.

2. Connected compact fibres give $f_*\mathcal{O}_X = \mathcal{O}_B$ by (Claude and L. Alpöge, 2026, Proposition 9.13, pp. 74–76). A section of $\mathcal{O}_X(kS_2)$, viewed as a meromorphic function, descends away from p_2 to a function $b(t)$. On the finite cover $t-1 = s_2^4$, its pullback is invariant under $s_2 \mapsto \zeta_2 s_2$, so its Laurent exponents are multiples of four. The exact order condition is

$$4\,\mathrm{ord}_{p_2} b + k \geq 0 \quad \Longleftrightarrow \quad \mathrm{ord}_{p_2} b \geq -\lfloor k/4 \rfloor.$$

Conversely every such b pulls back to an allowed section. This is (7.4) as an equality of subsheaves, and applying it to $k = 2m, -2m$ gives (7.5).

3. Fix the isomorphism $\omega_X^{-1} \simeq \mathcal{O}_X(2S_2)$ from the first step. Let u correspond to its canonical divisor section and choose $\ell \in H^0(B, \mathcal{O}_B(p_2))$ with $\ell(p_2) \neq 0$. Let v correspond to $f^*\ell$, so $u^2 = f^*\sigma_{p_2}$ under the induced degree-two map. The map $U \mapsto u, V \mapsto v$ sends the degree-m basis $U^{m-2j}V^j$, $0 \leq j \leq \lfloor m/2 \rfloor$, to the pullback basis from (7.4), multiplied by u when m is odd. It is therefore the graded isomorphism (7.6). Replacing ℓ by $a\ell + b\sigma_{p_2}$ gives $v \mapsto av + bu^2$; multiplying the chosen line isomorphism by $c \neq 0$ gives $(u, v) \mapsto (c^{-1}u, c^{-2}v)$. The base-point-free pair (u^2, v) induces exactly

$$x \longmapsto [\sigma_{p_2}(f(x)) : \ell(f(x))].$$

For $\ell = A/(t-1) + B$, $A \neq 0$, the specified base map is $t \mapsto [t-1 : A + B(t-1)]$. Degree one has the single divisor $2S_2$.

4. A biholomorphism preserves the anticanonical graded ring and its complete degree-two system, so it induces a projective automorphism of B. The reduced singular cusp fibre and the finite fibres of multiplicities three and four distinguish the three values: the pullback identity for a fibre divisor preserves its multiplicity, and smoothness of its reduction is preserved under the induced fibre isomorphism. Hence each of $0, 1, \infty$ is fixed. A projective automorphism fixing these three points is the identity.

□

Remark 7.3 (Divisor classes in the Picard coordinate)**.** [tag PI09]

In the additive coordinate of definition 2.8, the identification in (13.6) gives

$$\mathcal{O}_X(S_1) = L_4, \qquad \mathcal{O}_X(S_2) = L_3, \qquad \omega_X = L_{-6}, \qquad \omega_X^{-2} = L_{12}.$$

Proposition 7.4 (Normalized root covers and the finite fibres)**.** [tag 01Q0]

The normalized base change $N_j \times_{D_j} \Delta_j \to \Delta_j$, $t_j = s_j^{m_j}$, is the original smooth torus family J_j'. The maps $E_1 \times F_1 \to S_1$ and $E_2 \times F_2 \to S_2$ described below are finite étale covers of degrees 9 and 8.

Proof. 1. On J_j', the deck generator is $(s, \zeta) \mapsto (\zeta_j s, R_j(s)\zeta + \Pi_c(g_j s)v_j/m_j)$. It has exact order m_j. Its rth power changes the invariant circle coordinate γ by $r\gamma(v_j)/m_j$; this is nonintegral for $0 < r < m_j$, so the action is free. The quotient map is therefore a local biholomorphism. Its local base function is s^{m_j} and its fibre multiplicity is exactly m_j. The map

$$J_j' \longrightarrow N_j \times_{D_j} \Delta_j, \qquad y \longmapsto (q_j(y), s_j(y))$$

is finite, is an isomorphism over $s \neq 0$, and has normal source. Locally at the central fibre the base-change ring has equation $z^{m_j} - s^{m_j} = 0$. Its normalization is the disjoint union of branches $z = \zeta_j^r s$. The displayed map gives those branches in the covering charts. It is thus the normalization everywhere, with its specified root coordinate.

2. Put $\rho = e^{\pi i/3}$, $\mu_1 = (2-\rho)/3$, $\mu_2 = (1-i)/2$, $\kappa_1 = \beta_c(z_1) + 2\mu_1$, and $\kappa_2 = \beta_c(z_2) + 3\mu_2$. The fixed-point transformation laws for μ give the two displayed values directly. We have $\operatorname{Im}\kappa_j = D_c(z_j) < 0$. Define

$$E_j = \mathbb{C}/(\mathbb{Z} + \kappa_j\mathbb{Z}), \quad F_1 = \mathbb{C}/(\mathbb{Z}(\rho-2) - 3\mathbb{Z}), \quad F_2 = \mathbb{C}/(\mathbb{Z}(i-1) - 2\mathbb{Z}).$$

Let $e_1 = (0,1,-2,0)^t, f_1 = (0,0,-3,1)^t, e_2 = (0,1,-1,0)^t, f_2 = (0,0,-2,1)^t$. Direct solution of the fixed equations and inspection of the columns give

$$\Lambda^{A_1} = \mathbb{Z}\varepsilon + \mathbb{Z}d, \quad \Lambda^{A_2} = \mathbb{Z}\varepsilon' + \mathbb{Z}d, \quad (A_j - I)\Lambda = \mathbb{Z}e_j + \mathbb{Z}f_j.$$

The image lattices are saturated: they are the integer solutions of $\gamma = 0, 2u + w + 3\delta = 0$ and $\gamma = 0, u + w + 2\delta = 0$, respectively. The determinants of the fixed-plus-moving basis matrices are $-3, -2$.

3. The complex-linear maps $L_1(\xi,\eta) = (\eta, \xi - \eta/3)$ and $L_2(\xi,\eta) = (\eta, \xi - \eta/2)$ carry the product periods to those fixed and moving sublattices. They induce isogenies to the original central tori, with kernel generated respectively by $(2/3, \rho)$ and $(1/2, i)$, of exact orders three and two. For example $\widehat{u} = 2d/3 + e_1 - 2f_1/3 = d/2 + e_2 - f_2/2$ gives both kernel representatives.

4. The affine deck maps lift to

$$(\xi,\eta) \longmapsto (\xi + \kappa_1/3, (\rho-1)\eta), \qquad (\xi,\eta) \longmapsto (\xi - \kappa_2/4, -i\eta).$$

Substitution in L_j gives exactly $R_jL_j + \Pi_c v_j/m_j$. They commute with their kernel translations: the moving differences are periods, since $(\rho-2)\rho = -1 - \rho = -(\rho-2) - 3$ can equally be checked in the lattice basis, and $(-i-1)i = 1 - i = -(i-1)$. More directly these assertions follow by the two-dimensional period matrices of the indicated rotations. A nonidentity element of either resulting product group translates the E_j coordinate by $2a/3 + b\kappa_1/3$ or $a/2 - b\kappa_2/4$, respectively. Real independence of $1, \kappa_j$ proves that this translation vanishes only for the identity element. Hence the groups $C_3 \times C_3$ and $C_2 \times C_4$ act freely and have quotient precisely S_j.

□

7.2 The proper holomorphic family and coherent descent

Proposition 7.5 (Coherent even shifts and the quotient family)**.** [tag 01Q2]

For each $n \in 2\mathbb{Z}$ there is a biholomorphism $F_{n,c} : X_c \to X_{c+n}$, holomorphic jointly in c, with $F_{2h,c+2k}F_{2k,c} = F_{2(h+k),c}$. The quotient family over $0 < |w| < e^{-\pi M}$, $w = e^{-\pi i c}$, is proper and holomorphic, with pairwise non-biholomorphic fibres.

Before the quotient, every disc Q with $\overline{Q} \subset \mathcal{P}$ has the proper holomorphic submersion and proper base map

$$\pi : \mathcal{X}_Q \longrightarrow Q, \qquad F : \mathcal{X}_Q \longrightarrow B \times Q, \qquad \pi = \mathrm{pr}_Q \circ F. \tag{7.7}$$

Their restrictions to c are X_c and f_c, and their three distinguished values are the constant sections $0, 1, \infty$.

Proof. 1. *The holomorphic family.* Let Q be a disc with $\overline{Q} \subset \mathcal{P}$. On an ordinary lifted base chart use the period matrix (2.1), jointly holomorphic in (z,c). Its real determinant is $\Im\tau\, D_c$, and its inverse is uniformly bounded on compact base-parameter sets. Thus the lattice action is free and properly discontinuous there. A fixed closed real lattice parallelotope gives compact surjective representatives over each compact subset of the base chart times Q. The

quotient is holomorphic and its projection to Q is a submersion. The covariance convention of definition 7.1 is

$$R_g(z)\Pi(z,c) = \Pi(gz,c)A_g, \qquad M_g = A_g^{-1}, \qquad \Pi(gz,c) = R_g(z)\Pi(z,c)M_g.$$

Consequently the base deck quotient is jointly holomorphic; its proper discontinuity follows from that of the lifted base action.

At a finite root chart the family action is

$$(z,c,\zeta) \longmapsto \left(g_j z, c, R_{g_j}(z)\zeta + \Pi(g_j z,c)\frac{v_j}{m_j} \right), \qquad (m_1,m_2) = (3,4). \tag{7.8}$$

In the real marking it is $(z,c,u) \mapsto (g_j z, c, A_j u + v_j/m_j)$. The invariant-covector freeness test in proposition 7.4 is independent of c, so these quotients are local biholomorphisms and submersions to Q. The same parallelotope proves properness over the finite base disc times Q. Their punctured connectors are

$$\frac{\log s_j(z)}{2\pi i}\Pi(z,c)v_j. \tag{7.9}$$

A logarithm change adds a period, so this is a jointly holomorphic torus section. Writing $\sigma_j = (\log s_j)/(2\pi i)\,\Pi(z,c)v_j$, the relations $A_j v_j = v_j$ and $\log s_j(g_j z) = \log s_j(z) - 2\pi i/m_j$ give

$$R_{g_j}(z)\sigma_j(z,c) - \sigma_j(g_j z,c) = \Pi(g_j z,c)v_j/m_j.$$

Thus translation by σ_j intertwines the finite affine action with the middle linear action, with the specified root coordinate.

2. *Uniform cusp charts and properness.* On the fixed unimodular fan write the full cusp period matrix as

$$C(t_c,c) = \begin{pmatrix} 6\mu(t_c) & h(t_c) \\ b^{\mathrm{part}}(t_c) - h(t_c) + c & \mu(t_c) \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{7.10}$$

Choose a closed cusp disc $|t_c| \le r_1$ on which these functions are holomorphic, and set

$$K_Q = \max_{|t_c|\le r_1,\ c\in\overline{Q}} \| -2\pi\Im C(t_c,c)\|, \qquad 0<\epsilon<\min(r_1,1), \quad |\log\epsilon| > 2K_Q. \tag{7.11}$$

The lattice acts on the toric family times Q by

$$(x,t_c,c) \longmapsto (e^{2\pi i C(t_c,c)\lambda} t_c^{B_0\lambda} x, t_c, c), \qquad \lambda\in\mathbb{Z}^2. \tag{7.12}$$

The fan translation and nowhere-zero torus multiplier extend this map across every toric stratum. With Euclidean operator norm, $B_0^{\mathsf{t}}B_0 = I$, and

$$B_{t_c,c} = B_0 + \frac{-2\pi\Im C(t_c,c)}{\log|t_c|}, \quad \|B_{t_c,c} - B_0\| < \tfrac{1}{2}, \quad \tfrac{1}{2}\|\lambda\| < \|B_{t_c,c}\lambda\| \le \tfrac{3}{2}\|\lambda\| \quad (\lambda\neq 0). \tag{7.13}$$

These inequalities exclude stabilizers on the dense torus. For a height-one triangle, use its three dual unimodular cone coordinates z_0,z_1,z_2, with $t_c = z_0z_1z_2$. In the open chart $|z_i|<2$, $|t_c|<\epsilon$, the normalized logarithmic barycentric coefficients have sum one and are bounded below by $-\log 2/|\log\epsilon|$. Thus the logarithmic positions in each fixed chart lie in a bounded set, uniformly in c. For a fixed pair of charts, (7.13) bounds the finitely many λ which can translate one into the other. At the central fibre a nonzero $B_0\lambda$ moves every bounded fan cell. These charts cover the entire toric space, including its boundary. On the dense torus the duality identity is

$$\log|z_i| = \log|t_c|\,\theta_i^{\sigma}(y),$$

where θ_i^σ are the barycentric coordinates of the height-one triangle. A triangle containing y therefore gives $|z_i| \le 1$. For a boundary point in the orbit of the cone over a cell P, the coordinates belonging to vertices of P vanish. The remaining characters are dual to the image cones of triangles containing P in the quotient lattice by $\operatorname{span}(P,1)$. Those cones cover the quotient real space: lift a vector, add a sufficiently small positive multiple of it to a relative interior point of $(P,1)$, and rescale to height one. Local finiteness places that point in a triangle containing P. Applied to the negative logarithm of the orbit coordinates, this gives the same closed unit-polydisc bound. Thus the open radius-two charts cover the boundary as well. Every compact set has a finite subcover. The chart-pair estimate applies to an intersection meeting the boundary because that intersection is open and meets the dense torus. It therefore proves freeness and proper discontinuity on the entire toric family.

For properness, fix $\epsilon_1 < \epsilon$ and let K_0 be the finite union of closed unit polydiscs for height-one triangles meeting the closed radius-$\sqrt{2}$ ball, intersected with $|t_c| \le \epsilon_1$. For $t_c \neq 0$, choose λ with $\|B_{t_c,c}^{-1} y + \lambda\|_\infty \le 1/2$. The translated logarithmic position has norm at most $3/(2\sqrt{2}) < \sqrt{2}$. A triangle containing it has nonnegative barycentric coordinates of sum one, so each corresponding cone coordinate has modulus at most one. This gives a representative in K_0. At $t_c = 0$, translate the height-one component and use the six closed toric bidiscs covering $E_0 \simeq \mathrm{d}P_6$. The multiplier $\exp(2\pi i C(0,c) B_0 v)$ is included in this translation; the bidiscs cover the resulting whole compact component. For each compact $K \subset Q$, the compact set $K_0 \times K$ therefore maps onto the full inverse image of $\{|t_c| \le \epsilon_1\} \times K$. This proves properness. The quotient charts are open subsets of a smooth toric variety times Q, so projection to Q is a submersion.

3. Exponentiating the logarithmic coordinates gives the cusp connector with period identity $Z = sB_0 + C(t_c, c)$. Glue it and (7.9) to the middle family. The distinguished base discs are disjoint. Points with distinct (t,c) values separate on the base; points with the same value lie in one Hausdorff local piece. The resulting total space is therefore Hausdorff and smooth. The local proper maps give a proper map F to $B \times Q$: cover a compact subset by finitely many relatively compact sets inside the displayed base pieces. Compactness of B proves properness of $\pi = \mathrm{pr}_Q F$. The fibres and overlaps are the specified X_c and f_c. These local families agree on smaller cusp radii and glue over $\mathcal{P}$.

4. *Local maps.* Let $n \in 2\mathbb{Z}$. Define the following rational columns, integral cocycles, and holomorphic translation vectors:

$$\begin{aligned} r_1 &= -\frac{n}{3}(\widehat{u} + \widehat{w}), \quad \ell_1 = -n\widehat{w}, \quad b_1(z) = \Pi_c(z) r_1 = -\frac{n}{3}(\tau(z) + 1, \mu(z))^{\mathsf{t}}, \\ r_2 &= \frac{n}{4}(\widehat{u} + \widehat{w}), \qquad \ell_2 = \frac{n}{2}\widehat{u}, \qquad b_2(z) = \Pi_c(z) r_2 = \frac{n}{4}(\tau(z) + 1, \mu(z))^{\mathsf{t}}. \end{aligned} \tag{7.14}$$

Direct multiplication of the complete matrices in (2.2) gives

$$(I - A_j) r_j = \frac{n\epsilon_j}{m_j}\widehat{\delta} + \ell_j. \tag{7.15}$$

For $j = 1$ the four entries are $(0, 0, -n, n/3)$, and for $j = 2$ they are $(0, n/2, 0, -n/4)$. Since both r_j and $n\epsilon_j\widehat{\delta}/m_j + \ell_j$ lie in $\ker\gamma$, the period covariance gives

$$b_j(g_j z) - R_{g_j}(z) b_j(z) = \Pi_{c+n}(g_j z)\left(\frac{n\epsilon_j}{m_j}\widehat{\delta} + \ell_j\right).$$

This is the deck-equivariance identity for translation by b_j between the finite fillings at c and $c + n$. Its overlap with the fixed middle identification is as follows.

Keep the marked linearizing coordinate, radius, and phase $s_j(g_j z) = e^{-2\pi i/m_j} s_j(z)$. On a logarithm branch put

$$w_j(z) = \frac{\log s_j(z)}{2\pi i}, \qquad a_j(z) = b_j(z) + n\epsilon_j w_j(z) e. \tag{7.16}$$

Changing the logarithm adds $n\epsilon_j \widehat{\delta}$ under the period map, so a_j is branch independent as a torus section. For the specified negative generator, $w_j(g_j z) = w_j(z) - 1/m_j$. Equations (7.2) and (7.15) consequently give the exact equality

$$a_j(g_j z) = R_{g_j}(z) a_j(z) + \Pi_c(g_j z)\ell_j. \tag{7.17}$$

In fact the marked finite overlap maps are translations by $\sigma_{j,c} = w_j \Pi_c v_j$, and

$$\sigma_{j,c+n} - \sigma_{j,c} = n\epsilon_j w_j e. \tag{7.18}$$

Thus a_j is exactly the middle translation induced by the local translation b_j, with the marked gluing maps retained.

At the cusp retain $s = \tau - h$, $e^{2\pi i s} = t_c$, $s(g_0 z) = s(z) - 1$, and the complete period matrix

$$Z_c = sB_0 + C_c(t_c), \quad B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C_c = \begin{pmatrix} 6\mu & h \\ b^{\mathrm{part}} - h + c & \mu \end{pmatrix}. \tag{7.19}$$

For $\lambda = (\lambda_1, \lambda_2)^{\mathsf{t}} \in \mathbb{Z}^2$, $C_{c+n}\lambda - C_c\lambda = (0, n\lambda_1)^{\mathsf{t}}$. Therefore

$$\exp(2\pi i C_{c+n}\lambda) t_c^{B_0\lambda} = \exp(2\pi i C_c \lambda) t_c^{B_0\lambda}. \tag{7.20}$$

The entire cusp actions, including their nonconstant holomorphic units, agree under the integer-shift identification.

The affine cocycle convention in (7.17) has multiplication law

$$\ell_{gh} = \ell_g + A_g \ell_h. \tag{7.21}$$

Since $g_0 = (g_1 g_2)^{-1}$, it gives

$$\begin{aligned} \ell_1 + A_1\ell_2 &= -\frac{3n}{2}\widehat{w} + \frac{n}{2}\widehat{\delta}, \\ \ell_0 = -A_{g_0}(\ell_1 + A_1\ell_2) &= \frac{3n}{2}\widehat{w} - \frac{n}{2}\widehat{\delta}. \end{aligned} \tag{7.22}$$

Here A_{g_0} fixes both toric lattice vectors, as is also immediate from $A_{g_0} = \left(\begin{smallmatrix} I & 0 \\ B_0 & I \end{smallmatrix}\right)$. Put

$$q_n = (-3n/2, n/2)^{\mathsf{t}} \in \mathbb{Z}^2, \qquad a_0(z) = s(z) q_n. \tag{7.23}$$

Then $a_0(g_0 z) - a_0(z) = -q_n = \Pi_c(g_0 z)\ell_0$, exactly as required by (7.17). Exponentiating gives the map $(x, t_c) \mapsto (t_c^{q_n} x, t_c)$ on the dense cusp torus. It extends holomorphically and invertibly to the marked infinite fan: the integral map

$$(y, y_3) \longmapsto (y + y_3 q_n, y_3)$$

translates each height-one triangle by the integer vector q_n and therefore permutes the cones, with inverse given by $-q_n$. It commutes with the cusp quotient action in (7.20). This proves extension across every cusp stratum.

5. *The global additive correction.* Let $\widetilde{B}^\circ$ be the universal cover of the three-punctured base. The fundamental group of B° is free on the marked two negative peripheral generators. Pull back the marked Π_c, R to this cover. Assign ℓ_1, ℓ_2 from (7.14) to its free generators and extend by (7.21). Freeness makes this an integral cocycle; its third peripheral value is precisely (7.22). The maps

$$(z, y) \longmapsto (gz, R_g(z)y + \Pi_c(gz)\ell_g) \tag{7.24}$$

form an affine action. To check the action law, covariance gives $R_g(hz)\Pi_c(hz) = \Pi_c(ghz)A_g$; substituting this equality in the composition gives exactly (7.21). The quotient is an affine holomorphic bundle over B°, since the deck action on the base is free and properly discontinuous and one may use evenly covered base opens as affine charts.

On a finite punctured disc, (7.17) says that a_j is a section of this affine bundle. On the cusp punctured disc, (7.22)–(7.23) say the same for a_0. Define a sheaf $\mathscr{H}_n$ on B as follows. Away from the special points it is the sheaf of sections of (7.24). Near p_j, a section is required to have the form $a_j + \eta_j$, where $\eta_j : \Delta_j \to \mathbb{C}^2$ is holomorphic and obeys $\eta_j(g_j z) = R_{g_j}(z)\eta_j(z)$. Near the cusp it is required to have the form $a_0 + \eta_0$, with $\eta_0(t_c)$ holomorphic on the full cusp disc. These are compatible definitions on the punctured overlaps by the equal affine cocycles just proved. They also prove local nonemptiness, including at all three special points.

The differences of these local sections form a vector sheaf $\mathscr{V}$. Its ordinary sections have covariance $\eta(gz) = R_g\eta(z)$; at the two finite points and the cusp its extensions are precisely the holomorphic extensions specified above. Projection to the first coordinate gives an exact sequence on the marked compact base

$$0 \longrightarrow \mathcal{O}_B e \longrightarrow \mathscr{V} \longrightarrow \mathcal{L} \longrightarrow 0, \qquad \mathcal{L} \simeq \mathcal{O}_B(-p_0). \tag{7.25}$$

The homogeneous first-coordinate line is determined by its transition factors. Indeed the first diagonal entries in (7.2) are its factors $-1/\tau$ and $1/\tau$, and at the cusp both extensions require ordinary holomorphicity. The marked function

$$\Theta = \frac{(E_4 \circ \tau)^2 (E_6 \circ \tau)^{1/2}}{\Delta_{\mathrm{mod}} \circ \tau}$$

has those exact factors, orders two and one in s_1, s_2, and a simple pole in t_c. Dividing by Θ identifies the ordinary and finite stalks with $\mathcal{O}_B$, and the cusp stalk with $\mathfrak{m}_{p_0}$. This recovers the displayed $\mathcal{L}$ with its fixed cusp condition. For the finite stalks in (7.25), use the full matrix R_g. Given an equivariant first component $u(z)$, begin with the vector $(u(z), 0)^{\mathsf{t}}$ and average it under the finite group action on vector-valued holomorphic functions. The resulting vector is equivariant, and its first component is still u. Thus projection is surjective there. Its kernel is exactly the invariant second-component functions, which are holomorphic functions of $s_j^{m_j} = t_j$, multiplied by e. The ordinary and cusp stalks are immediate. This also proves that $\mathscr{V}$ is locally free of rank two: lift a local frame of $\mathcal{L}$ and join it to e.

Both $H^1(B, \mathcal{O}_B)$ and $H^1(B, \mathcal{O}_B(-p_0))$ vanish: on the two standard affine charts of $\mathbb{P}^1$, split a holomorphic Laurent series on their overlap into its nonnegative and negative powers. For $\mathcal{O}_B$ these are the two coboundary contributions, and for $\mathcal{O}_B(-p_0)$ its transition factor shifts the second contribution by one, still leaving no missing exponent. The positive part converges on the full first affine chart and the transformed negative part on the full second affine chart. This proves the two vanishings, equivalently the usual additive Cousin calculation on these charts. The exact sequence (7.25) now gives

$$H^1(B, \mathscr{V}) = 0. \tag{7.26}$$

Choose local sections h_α of $\mathscr{H}_n$, taking the displayed a_j, a_0 on sufficiently small special discs and affine charts elsewhere. Their differences $d_{\alpha\beta} = h_\beta - h_\alpha$ are a Čech 1-cocycle with values

in $\mathscr{V}$. Project it to $\mathcal{L}$ and solve that additive cocycle, using its just proved H^1 vanishing. Lift the resulting local $\mathcal{L}$ sections to $\mathscr{V}$ by the local lifting construction in (7.25). Subtracting their coboundary from d leaves a scalar $\mathcal{O}_B e$ cocycle. Solve that scalar additive cocycle by its H^1 vanishing. This produces sections v_α with $d_{\alpha\beta} = v_\beta - v_\alpha$. The local sections $h_\alpha - v_\alpha$ agree and define a global section $a \in \mathscr{H}_n(B)$. It gives holomorphic vectors η_j, η_0 with

$$a = a_j + \eta_j \quad (j = 1, 2), \qquad a = a_0 + \eta_0 \quad \text{at the cusp.} \tag{7.27}$$

Any choices made in the additive solutions give maps with the same asserted properties. Their differences as sections of this affine bundle lie in $H^0(B, \mathscr{V}) = \mathbb{C}e$: this follows from (7.25), $H^0(B, \mathcal{L}) = 0$, and $H^0(B, \mathcal{O}_B) = \mathbb{C}$.

Projecting a modulo $\Pi_c\Lambda$ gives a holomorphic section of the middle torus family because every ℓ_g in (7.24) is integral. The three local maps are therefore

$$\begin{aligned} F_{\mathrm{mid}}[z, \zeta] &= [z, \zeta + a(z)], \\ F_j[z, \zeta] &= [z, \zeta + b_j(z) + \eta_j(z)] \quad (j = 1, 2), \\ F_0[x, t_c] &= [\exp(2\pi i \eta_0(t_c)) t_c^{q_n} x, t_c]. \end{aligned} \tag{7.28}$$

The first descends by (7.24) and equality of the integer-shift lattices. The finite maps descend by (7.15), the finite deck-equivariance identity above, and the exact covariance of η_j. The cusp map descends by (7.20); its holomorphic-unit factor extends by torus multiplication on every toric chart, and its integer factor extends by the fan map in (7.23). Each map is a biholomorphism, with inverse given by the negative translation or the inverse torus multiplier and fan map.

On a finite overlap, write the marked gluing as $h_{j,c}(\zeta) = \zeta + \sigma_{j,c}$. Then (7.18), (7.27), and (7.28) give the equality of torus maps

$$h_{j,c+n} F_j = F_{\mathrm{mid}} h_{j,c}. \tag{7.29}$$

Both sides add $\sigma_{j,c} + a = \sigma_{j,c+n} + b_j + \eta_j$. On the cusp overlap the marked map is exponentiation of ζ; (7.23) and (7.27) give $\exp(2\pi i a) = t_c^{q_n} \exp(2\pi i \eta_0)$, exactly the third line of (7.28). Thus that overlap also agrees, including all nonconstant period units. The maps and their inverses glue to a global biholomorphism $X_c \to X_{c+n}$ over B.

The choices can be made coherently for every parameter and every even shift. Both initial cocycle columns in (7.14) have $\gamma = 0$. Because every A_g preserves γ, (7.21) shows that $\gamma(\ell_g) = 0$ for every word. Therefore the affine term $\Pi_c(gz)\ell_g$ in (7.24) is independent of c. The displayed a_j, a_0, and the vector sheaf $\mathscr{V}$ are also independent of c. Perform the additive construction once for $n = 2$, keeping one resulting section $a^{(2)}$. For $n = 2k$ choose $a^{(2k)} = k a^{(2)}$ and multiply its local correcting vectors by k. All their affine cocycles and local formulas are then exactly those for $n = 2k$. The resulting maps satisfy the full group law

$$F_{2h,c+2k} \circ F_{2k,c} = F_{2(h+k),c}, \qquad F_{0,c} = \mathrm{id}_{X_c}. \tag{7.30}$$

This follows by adding the translation vectors in the first two lines of (7.28), and by multiplying their units and adding the integer exponents in its third line. The formulas are jointly holomorphic in c, since their corrections are independent of it.

6. The local families constructed above are proper and holomorphic. The coherent $2\mathbb{Z}$ action is properly discontinuous because its action on $\mathcal{P}$ is. Local inverse logarithms for $w = e^{-\pi i c}$ identify its quotient with the proper local families just proved. Proposition A.2 gives pairwise non-biholomorphic fibres.

□

7.3 Albanese invariants and the derivative of the deformation

Proposition 7.6 (Albanese curves and parameter recovery)**.** [tag 01Q3]
With $\kappa_j(c) = \kappa_j^{\mathrm{part}} + c$, the Albanese elliptic curves of the reduced finite fibres are

$$\mathrm{Alb}(S_1) = \mathbb{C}/(\mathbb{Z} + \kappa_1(c)\mathbb{Z}), \qquad \mathrm{Alb}(S_2) = \mathbb{C}/(\mathbb{Z} + \tfrac{1}{2}\kappa_2(c)\mathbb{Z}).$$

Writing $J(q) = q^{-1} + 744 + 196884q + \cdots$, their invariants are

$$j_1(w) = J(e^{-2\pi i\kappa_1^{\mathrm{part}}} w^2), \qquad j_2(w) = J(e^{-\pi i\kappa_2^{\mathrm{part}}} w).$$

These are convergent formulas because the displayed nome moduli are below one. Outside the explicitly defined countable exceptional set in the proof, equality of both Albanese invariants forces $c' - c \in 2\mathbb{Z}$. Thus the generic Albanese invariants recover the parameter modulo its exact period subgroup.

Proof. 1. In the product covers of 7.4 the invariant holomorphic one-forms are exactly the multiples of $d\xi$: the moving factor is multiplied by a primitive third or fourth root. Integration thus identifies the Albanese with the quotient of the fixed line by the fixed-coordinate projections of every deck translation. These are, respectively,

$$\mathbb{Z} + \kappa_1\mathbb{Z} + \tfrac{2}{3}\mathbb{Z} + \tfrac{\kappa_1}{3}\mathbb{Z} = \tfrac{1}{3}\mathbb{Z} + \tfrac{\kappa_1}{3}\mathbb{Z}, \quad \mathbb{Z} + \kappa_2\mathbb{Z} + \tfrac{1}{2}\mathbb{Z} + \tfrac{\kappa_2}{4}\mathbb{Z} = \tfrac{1}{2}\mathbb{Z} + \tfrac{\kappa_2}{4}\mathbb{Z}.$$

Multiplication by three and two gives the stated maps of complex tori. The two displayed lattices include both the product-isogeny kernels and the affine quotient translations. Using the positive-imaginary representatives $-\kappa_1$ and $-\kappa_2/2$ in the standard modular j-series gives the formulas. At sufficiently small nome J is injective: $1/J(q) = q + O(q^2)$ has nonzero derivative at zero. Thus j_2 alone distinguishes parameters within a sufficiently small punctured subdisc when both parameters lie there. For the full generic assertion put $x = -\kappa_2(c)/2$ and $d_* = \kappa_1^{\mathrm{part}} - \kappa_2^{\mathrm{part}}$, so $-\kappa_1(c) = 2x - d_*$. The definitions $\kappa_1^{\mathrm{part}} = \beta^{\mathrm{part}}(z_1) + 2\mu_1$ and $\kappa_2^{\mathrm{part}} = \beta^{\mathrm{part}}(z_2) + 3\mu_2$, together with proposition 4.3, give

$$\operatorname{Im} d_* = -\frac{3}{2\pi}\log 48 + \frac{3}{\pi}\log 6 = -\frac{3}{2\pi}\log\frac{4}{3} < 0.$$

Equality of the two elliptic invariants at c, c' gives matrices $M, N \in \mathrm{PSL}_2(\mathbb{Z})$ with $x' = M(x)$ and

$$N(2x - d_*) = 2M(x) - d_*.$$

Define the exceptional set to be the $c \in \mathcal{P}$ obtained from roots of these equations for all pairs for which the two rational functions are not identical. Each equation has at most two roots after clearing denominators, so this is a countable, explicitly specified set. For a pair giving an identity, let $C(x) = 2x - d_*$. Then $N = CMC^{-1}$. The real matrix M preserves both the real circle $\mathbb{R} \cup \{\infty\}$ and $C^{-1}(\mathbb{R} \cup \{\infty\})$, the parallel line with imaginary part $\operatorname{Im} d_*/2$ together with infinity. Their unique intersection is infinity, so M fixes infinity. An element of $\mathrm{PSL}_2(\mathbb{Z})$ fixing infinity is $x \mapsto x + b$, $b \in \mathbb{Z}$, and then $N(y) = y + 2b$. It follows that $c' - c = -2b$. This proves parameter recovery from the two Albanese invariants outside the specified exceptional set. For biholomorphic total spaces, proposition A.2 gives the parameter congruence at every point.

□

Remark 7.7 (Period variation and norm comparisons)**.** [tag 01Q5]

1. On a fixed ordinary fibre, the three last periods remain fixed and span $\mathbb{C}^2$, whereas the first changes by δe. A marked complex-linear identification fixing those three periods must be the identity, so the marked period map is nonconstant. The set of integral markings of one torus is countable. A holomorphic nonconstant map from a disc has uncountable image; hence the unmarked fibre family is locally non-isotrivial. 7.2 transfers this to total spaces. This is the period argument of (Viaclovsky, 2026a, Proposition 4.7, pp. 14–15). A nonconstant classifying map gives positive dimension of its local target. It does not by itself determine the derivative at the base point, as the explicit holomorphic map $z \mapsto z^2$ has nonconstant image and derivative zero at zero. A.3 computes the derivative here.

2. The comparison in (Yang, 2026, Lemmas 6.6–6.7 and 6.9, pp. 31–32; Remark 6.8, p. 32; Proposition 6.15, pp. 37–38) uses an integral commuting lattice map, an affine fibre lift, normalization of the root base change, and a norm-lattice defect. His Proposition 6.15 obtains $6/r$ from the first-column scaling and its order-six norm image. Here the first-column scaling is one and the two computed norm images impose parity, yielding two. These matrix substitutions identify the commuting maps, finite norm equations and collar corrections with the ones just displayed. The nonzero absolute Kodaira–Spencer conclusion in (Yang, 2026, Theorem 6.18, p. 39) is obtained for the present family by the explicit computation and intrinsic-pencil transfer in proposition A.3.

7.4 The standard isomorphism classes

Theorem 7.8 (Classification of the standard family)**.** [tag 01Q6]

Fix the original cusp-regular period solution, the clockwise peripheral marking, the prescribed periodic triangular cusp fan, the affine vectors $(\varepsilon, -\varepsilon')$, and the untwisted cusp collar. For every $c, c' \in \mathcal{P}$, the constructed compact complex threefolds satisfy

$$X_c \simeq X_{c'} \quad \Longleftrightarrow \quad c' - c \in 2\mathbb{Z}.$$

Every biholomorphism is over the identity of the original $\mathbb{P}^1$. The proper holomorphic family of their isomorphism classes has parameter $0 < |w| < e^{-\pi M}$ and nonzero Kodaira–Spencer class at every point.

Proof. For the standard twist data, the classification of (Du et al., 2026a, Theorem 1.4, p. 3), with the exact parameter identification of remark A.1, is re-derived in theorem C.5. It gives $X_c \simeq X_{c'}$ exactly when $c - c' \in 2\mathbb{Z}$. The explicit maps on the regular, finite, and cusp charts are those of proposition 7.5, displayed in (7.28) and satisfying (7.30).

As a complex parameter space for the isomorphism classes represented by this marked family, the quotient $\mathcal{P}/(2\mathbb{Z})$ has the explicit coordinate

$$w = \exp(-\pi i c), \qquad 0 < |w| < \exp(-\pi M). \tag{7.31}$$

Indeed $|w| = \exp(\pi \operatorname{Im} c)$, and two complex numbers have the same value of this exponential exactly when they differ by an even integer. Every point in the displayed punctured disc has a logarithm giving a $c \in \mathcal{P}$. The exponential has nonzero derivative, so these local inverse logarithms give the claimed quotient complex structure. The integer-period quotient seen by the middle family instead has coordinate w^2; it is covered two-to-one by (7.31). Its two lifts are distinguished by the exact order-two summand in (A.7). The resulting parameter space classifies the members of the marked family. Proposition 7.5 proves properness of this quotient family and its joint holomorphic dependence. Its Kodaira–Spencer class is nonzero by Proposition A.3. □

Remark 7.9 (A rank-one nodal model)**.** [tag IP09] The family of theorems B.2 and 7.8 uses the original period system with multiplicities $(3, 4)$. Its arbitrary affine and cusp data are classified in section 8.1, and its higher multiplicities in section 8.2. For the elliptic surface $y^2 = x^3 + tx + 1$ of (Viaclovsky, 2026b, Section 2, pp. 3–5; Proposition 3.1, pp. 6–7; Proposition 6.1, pp. 17–18), fix the nodal value p_1 with $4p_1^3 + 27 = 0$. In that paper's equation (3.1), put $\gamma_0 = \gamma(0)$, $\beta_0 = \beta(0)$ and $c_0 = \lambda\delta(0)$, with $0 < |c_0| < 1$. The normalization of this rank-one filling is

$$E' = \mathbb{C}^*/\langle c_0\rangle, \qquad R = \mathbb{P}_{E'}(\mathcal{O}_{E'} \oplus L),$$
$$R = (\mathbb{P}^1 \times \mathbb{C}^*)/\langle (z, v) \mapsto (\gamma_0 z, c_0 v)\rangle,$$

and its boundary sections are identified by $v \mapsto \beta_0 v$. The invariant nodal volume form in equation (6.2) of that paper trivializes the ambient canonical line near this fibre. The branch-normal map therefore has quotient $\nu_* K_R$ by the local coefficient map in (B.7). For the ruling $\pi : R \to E'$, one has $\pi_*\mathcal{O}_R = \mathcal{O}_{E'}$ and $R^1\pi_*\mathcal{O}_R = 0$; Leray and Serre duality give $h^\bullet(K_R) = (0, 1, 1)$. The corresponding summand of the connecting homomorphism is

$$H^1(R, K_R) = \mathbb{C} \longrightarrow H^2(T_X(-\log D)),$$

where D is the union of that model's three reduced nodal fibres. Its defining boundary-translation Čech cocycle uses the full holomorphic units in equation (3.1), their central values (γ_0, c_0, β_0) and the vertical frames. The integral comparison equations use that model's four monodromy matrices and their collar defects. These are the explicit local and global inputs for this model; the displayed cohomology calculation evaluates the source of the connecting map.

8 Discrete data and the geometric parameter

8.1 All affine vectors and cusp sections

Proposition 8.1 (Classification with arbitrary affine and cusp data)**.** [tag 01QB]

Let

$$w_1 = r_1\varepsilon + k_1 d, \qquad w_2 = r_2\varepsilon' + k_2 d, \qquad 3 \nmid r_1, \quad 2 \nmid r_2,$$

and use an arbitrary holomorphic cusp collar section σ in the construction of (Claude and L. Alpöge, 2026, Section 6.4, pp. 39–40). Write $l_0 = \gamma(c(\sigma))$ in its positive cusp-winding convention, and set $p = 12l_0 - 4r_1 - 3r_2$. Let primes denote a second such datum with the same period solution and cusp fan. Then

$$X(c; w_1, w_2, \sigma) \simeq X(c'; w_1', w_2', \sigma')$$

holds exactly for the following arithmetic conditions:

$$n = c' - c \in \mathbb{Z}, \qquad p' = ap \text{ for some } a \in \{1, -1\}, \qquad n \equiv k_2 - k_2' \pmod 2.$$

All maps are over B. A map with the indicated a has fibre-linear part aI_2. In particular the complete invariants are

$$(|p|,\ e^{-\pi i(c+k_2)}), \qquad |p| \geq 1, \quad \gcd(|p|, 6) = 1.$$

Every value of this positive integer occurs. The full collar section enters the construction, but its additional analytic part and its second integral winding contribute no further invariant.

Proof. 1. The determinant calculation in 7.2 was independent of translations, so the base is still intrinsic. The centralizer computation of proposition A.2 and its period equation (A.3) give

$c' = c + n$ and fibre-linear part aI_2. Put $S_n = I + nE$ and compare all target periods in source coordinates. On the finite root cover the exact deck equation is

$$B_j(g_j z) - R_j B_j(z) = \Pi_c(g_j z)\left(\frac{d_j}{m_j} + \ell_j\right), \quad d_j = S_n w_j' - a w_j, \quad N_j \ell_j = -d_j. \tag{8.1}$$

Necessity follows by the normalized-base-change lift already proved. Conversely the norm equation has a rational solution r_j^* of $(I - A_j) r_j^* = d_j/m_j + \ell_j$, since $\ker N_j$ over $\mathbb{Q}$ equals the image of $I - A_j$. We can choose $\gamma(r_j^*) = 0$: the columns $\widehat{u}, \widehat{w}$ already span that image over $\mathbb{Q}$. Then $B_j = \Pi_c r_j^*$ is holomorphic on the whole root disc and solves the exact equation. Thus the norm condition supplies a biholomorphism on the complete root disc.

2. The computed norm images translate (8.1) into

$$r_1' \equiv a r_1 \pmod 3, \quad r_2' \equiv a r_2 \pmod 4, \quad a k_2 - k_2' - n r_2' \equiv 0 \pmod 2. \tag{8.2}$$

The integer k_1 is arbitrary. Since r_2' is odd, the last condition is the displayed parity. On the finite collar the middle translation is $A_j = B_j + (\log s_j)/(2\pi i)\,\Pi_c d_j$ and has integral defect ℓ_j. Its γ defect is $\alpha_j = (a r_j - r_j')/m_j$.

3. A holomorphic isomorphism between cusp neighbourhoods with fibre-linear part aI_2 has, after lifting near a generic smooth central point to the toric covers, the form

$$(x, t_c) \longmapsto (t_c^q u(t_c) x^a, t_c), \qquad q \in \mathbb{Z}^2, \quad u(0) \in (\mathbb{C}^*)^2. \tag{8.3}$$

Here the assertion concerns this local lift near a generic central point. To prove it, take a source chart near the generic point of its component and a target chart near its image. The étale toric quotient permits a local lift over a simply connected neighbourhood. The torus characters in a chart at the height-one ray v are $x_i = t_c^{v_i} X_i$, with X_i invertible. On $t_c \neq 0$ the affine fibre equation gives x_i'/x_i^a independent of the X coordinates. Its expression in the two charts is therefore $t_c^{v_i' - a v_i}$ times a holomorphic unit of t_c, including zero. This proves (8.3). Taking logarithms shows that its winding is in $\Lambda_{\mathrm{tor}} = \mathbb{Z}\widehat{w} + \mathbb{Z}d$. Changing a lift adds $(I - A_0)\Lambda = \Lambda_{\mathrm{tor}}$, so the vanishing of its class in $\bar{\Lambda} = \Lambda/\Lambda_{\mathrm{tor}}$ is intrinsic. Thus every extending translation has the stated integral winding and holomorphic unit factor.

4. The map from the cusp piece to the middle family is $T_{-\sigma} G_0^{-1}$. Consequently the middle translation induced by (8.3) is

$$A_0 = sq + a\sigma - \sigma' + \eta_0,$$

with η_0 holomorphic on the cusp disc after a logarithm of the unit. For the clockwise generator $s \mapsto s - 1$, the class of its defect in $\bar{\Lambda}$ is $-ac(\sigma) + c(\sigma')$. On the other hand every middle integral cocycle satisfies $\ell_0 = -A_0(\ell_1 + A_1 \ell_2)$, whose γ coordinate is $-\alpha_1 - \alpha_2$. Equality of these coordinates is exactly

$$l_0' - a l_0 = -(a r_1 - r_1')/3 - (a r_2 - r_2')/4,$$

or $p' = ap$. This proves the remaining necessity. Notice that $p \equiv -r_1 \pmod 3$ and $p \equiv r_2 \pmod 2$, so p is nonzero and coprime to six.

5. For sufficiency, $p' = ap$ implies the first two congruences of (8.2) by reduction modulo three and four. The parity supplies the remaining one. Choose integral solutions ℓ_1, ℓ_2 of (8.1). Their γ coordinates already have the required values. Adding any element of $\ker N_1 = (I - A_1)\Lambda$ preserves (8.1). In particular adding an arbitrary integer multiple of $e_1 = (0, 1, -2, 0)^t$ changes the $\widehat{u}$ coordinate of ℓ_0 by an arbitrary integer while leaving its γ coordinate fixed. We can therefore arrange $[\ell_0] = -ac(\sigma) + c(\sigma')$ in the entire rank-two group $\bar{\Lambda}$. Choose lifts of σ, σ' on the cusp half-plane. Their remaining defect differs from ℓ_0 by a toric lattice vector; an integral q makes $sq + a\sigma - \sigma'$ have exactly defect ℓ_0.

6. The finite translations and this cusp reference now define a locally nonempty affine torsor with linear difference sheaf $\mathscr{V}$ defined by (7.25) in the proof of proposition 7.5. The specified collar reference contains the full terms $a\sigma - \sigma'$, including any essential singularities of their chosen lifts at the puncture; differences of allowed extensions are holomorphic η_0. The proof of $H^1(B, \mathscr{V}) = 0$ gives a global middle translation. The finite maps descend by (8.1), and the cusp map is $x \mapsto t_c^q u(t_c) x^a$. The fan is preserved by all integer translations and by $y \mapsto -y$; exponentials of the integer period shift agree. Hence this cusp map and its inverse extend over every stratum. The finite overlap is the equality defining A_j, and the cusp overlap is the equality defining A_0. Thus the maps glue globally.

7. Finally parity is equivalent to equality of the two displayed exponentials. Passing from p to $-p$ is effected by $a = -1$ and the same proof; for fixed sign the congruences have just been shown sufficient. Every positive integer coprime to six is obtained by choosing $r_1 \in \{1, 2\}$ and odd r_2 in the appropriate residue classes modulo three and four, and then taking $l_0 = (p + 4r_1 + 3r_2)/12$. The construction in the preceding steps absorbs the analytic part and the second winding of the collar.

□

Proposition 8.2 (The two classes over a fixed linearization parameter)**.** [tag 01QC]

For multiplicities $(3, 4)$ *and* $|p| = 1$*, every admissible discrete member is biholomorphic to one member of 7.8, uniquely modulo* $2\mathbb{Z}$*. At fixed nonzero admissible Engel parameter* u *there are exactly two isomorphism classes, exchanged by a positive circuit of* u *about zero. In the convention that Engel's root generator is the inverse of our clockwise one and his circle class is* d*, the identity of the middle quotient sends*

$$X_{c+1} \longleftrightarrow X(u; a_0 - d/3,\ a_1 + d/4).$$

Reversing both declared root generators reverses these two displayed signs.

Proof. 1. Take the canonical target data $(r_1', r_2', k_1', k_2', l_0') = (1, -1, 0, 0, 0)$, for which $p' = -1$. Choose a so that $p' = ap$, and an integer $n \equiv k_2 \pmod 2$. 8.1 supplies the isomorphism to X_{c+n}. The uniqueness is 7.8. At fixed $u = e^{-2\pi i c}$, its logarithms differ by integers; the invariant $e^{-\pi i(c+k_2)}$ takes exactly the two square roots of u. Both occur, already as X_c, X_{c+1}, and A.2 distinguishes them.

2. Put $P_{\mathrm{E}} = P_{\mathrm{mark}}$, the integral isomorphism of proposition 2.3, and retain Engel's $Cf_4 = e_1 + e_4$. Besides the intertwining identities in that proposition, multiplication gives $P_{\mathrm{E}} d = \delta_0$, $\psi P_{\mathrm{E}} = -\gamma$, $P_{\mathrm{E}}\varepsilon = -e_4$ and $P_{\mathrm{E}}\varepsilon' = \delta_0 - Cf_4$. They also agree with the complex periods: at z_1 the complex-linear map $(\zeta_1, \zeta_2) \mapsto (-\zeta_1/(\rho + 1), \zeta_2 + \zeta_1/3)$ carries $\Pi_c P_{\mathrm{E}}^{-1}$ to Engel's columns $(1, 0), (\rho - 1, 0), ((1 + \rho)/3, 1/3), (0, -\kappa_1)$. At z_2 his Cf_4 has preimage $d - \varepsilon'$ and fixed period $1 - \kappa_2$. Thus the comparison fixes the positive circle class and the local complex structures, in addition to their monodromy matrices.

3. The inverse-generator convention of proposition 2.3 gives $a_{j-1} = -P_{\mathrm{E}} v_j / m_j$ in Engel's lattice marking (and its period image in the torus). Integer shift gives $\Pi_{c+1} v_j = \Pi_c(v_j + \gamma(v_j) d)$, with $\gamma(v_1) = 1, \gamma(v_2) = -1$, proving the two signs. A positive u circuit changes $\log u$ by $2\pi i$ and hence changes c by -1, exchanging the square roots. This gives the claimed permutation with its orientation.

□

Proposition 8.3 (The double cover of the linearization parameter)**.** [tag 01QD]

Fix the sphere logarithmic data in (Engel, 2026, Remark 8.1, p. 22). The resulting family has a proper holomorphic realization on every simply connected admissible u-open set and globally over the punctured w-disc of proposition 7.5, with $u = w^2$. The order-two class of d in $\Lambda^{A_2}/N_2\Lambda$ obstructs descent of this family to the punctured u-disc.

Proof. Local inverse logarithms of u give the proper local families of proposition 7.5; its coherent even shifts give their global family over the punctured w-disc. Descent through $u = w^2$ would furnish an isomorphism between the two pulled-back fibres at w and $-w$, namely X_c and X_{c+1}. Proposition A.2 excludes that isomorphism by the nonzero class of d in $\Lambda^{A_2}/N_2\Lambda$. This class has exact order two on the normalized finite cover. The same obstruction applies to all sphere logarithmic data by proposition 8.1, because r_2 is odd. □

8.2 Higher multiplicities

Proposition 8.4 (Classification for higher multiplicities)**.** [tag 01QE]

Let k, l be positive integers with $\gcd(3k, 4l) = 1$, and put $m_1 = 3k, m_2 = 4l$. Consider Viaclovsky's sphere members expressed in the same clockwise marking, with $w_j = r_j\varepsilon_j + s_j d$, $\gcd(r_j, m_j) = 1$, cusp winding l_0, and $p = m_1m_2l_0 - m_2r_1 - m_1r_2 = \pm 1$, where $\varepsilon_1 = \varepsilon, \varepsilon_2 = \varepsilon'$. Two such members are biholomorphic precisely when their multiplicity pair agrees and, for some $a \in \{1, -1\}$ and integer $n = c' - c$,

$$p' = ap, \quad r'_j \equiv ar_j \pmod{m_j}, \quad nr'_1 \equiv as_1 - s'_1 \pmod{k}, \quad nr'_2 \equiv as_2 - s'_2 \pmod{2l}.$$

The congruences on r'_j already follow from $p' = ap$. For fixed discrete vectors and cusp collar the identification subgroup is $2kl\mathbb{Z}$. For fixed admissible u and this multiplicity pair, all the stated sphere translation data give exactly $2kl$ isomorphism classes, permuted in one cycle by a circuit of u about zero.

Proof. 1. The base is intrinsic for these sphere members. A biholomorphism $F : X \to X'$ gives $g = f'F$. Since $H^2(S^6, \mathbb{R}) = 0$, the closed form $g^*\omega_{\mathrm{FS}}$ is exact. Restrict to a regular torus T and wedge with a flat Kähler form on T. Stokes' theorem gives integral zero, while the integrand is nonnegative and strictly positive wherever $dg|_T \neq 0$. Thus g is constant on each regular fibre. The image of (f, g) is a compact analytic curve whose projection to B has degree one; its normalization gives a holomorphic $h : B \to B$ with $g = hf$. Apply the same construction to F^{-1} to see that h is an automorphism. The three homological local monodromy orders are three, four and infinity, independently of k, l; their intertwining maps under F therefore fix each special point. Hence $h = 1$. The nonreduced fibre multiplicities now force equal k, l. This is the base-preservation argument of (Viaclovsky, 2026a, Proposition 4.7, pp. 14–15), with the displayed maps.

2. The root model of degree m_1 is the pullback of the degree-three model by $s \mapsto s^k$, and likewise at the other point. Its linear action remains A_1 or A_2, while its root generator has order m_j. The primitive invariant coordinate r_j/m_j proves freeness of every nonidentity power. The normalization and deck-commutation arguments of 7.4 and A.2 apply to these root covers, preserving the chosen generator. Their norms are

$$N_{3k} = kN_1, \qquad N_{4l} = lN_2,$$

so the fixed-coordinate norm images are $3k\mathbb{Z}\varepsilon + k\mathbb{Z}d$ and $4l\mathbb{Z}\varepsilon' + 2l\mathbb{Z}d$. The norm equation (8.1), with m_j replaced by these covering degrees, gives exactly the displayed congruences. On ordinary fibres the affine lift has simultaneous rational centralizer $\mathbb{Q}I + \mathbb{Q}E$, as in proposition A.2. Comparison of the constant period columns gives fibre-linear part aI, integer

shift n and lattice map aS_n^{-1}, where $S_n = I + nE$. Functorial normalization of the root base change makes the lift commute with the prescribed deck generator; summing its defect gives the norm equation. The explicit base norms are

$$N_3 v = 3\gamma(v)\varepsilon + (2v_u + v_w + 3v_\delta)d, \qquad N_4 v = 4\gamma(v)\varepsilon' + (2v_u + 2v_w + 4v_\delta)d,$$

where $N_3 = N_1$ and $N_4 = N_2$ in the notation above. At the cusp the character ratio of any extending map is t_c^q times a holomorphic unit, by proposition 8.1; its winding vanishes in the quotient by the toric directions. The middle cocycle then forces $p' = ap$.

3. The integer p records the primitive sphere relation in this marking. The common torus coinvariants are detected by γ, since $(A_1 - I)\Lambda + (A_2 - I)\Lambda = \ker\gamma$ by their explicit saturated bases. The finite meridian relations are $x^{m_1} = c^{r_1}$, $y^{m_2} = c^{r_2}$, and the cusp relation is $xy = c^{l_0}$ with c central. Eliminating y gives an abelian relation matrix with determinant $-p$ and primitive first elementary divisor because $\gcd(m_1, m_2) = 1$. Thus the sphere data have $p = \pm 1$. Reducing $p' = ap$ modulo m_1, m_2 gives the two congruences for r_j.

4. For the local sufficiency maps put $\alpha = a$, $h = n$, $S_h = I + hE$ and $d_j = S_h w_j' - \alpha w_j$. Suppose $p' = \alpha p$ and $hr_1' \equiv \alpha s_1 - s_1' \pmod{k}$, $hr_2' \equiv \alpha s_2 - s_2' \pmod{2l}$. Reduction of $p' = \alpha p$ modulo m_1, m_2 gives $r_j' \equiv \alpha r_j \pmod{m_j}$. We may choose integral solutions of $N_{m_j}\ell_j = -d_j$ explicitly:

$$\ell_1 = \left(\frac{\alpha r_1 - r_1'}{m_1}, 0, -\frac{s_1' + hr_1' - \alpha s_1}{k}, 0\right)^t,$$
$$\ell_2 = \left(\frac{\alpha r_2 - r_2'}{m_2}, -\frac{s_2' + hr_2' - \alpha s_2}{2l}, 0, 0\right)^t.$$

Put $y_j = d_j/m_j + \ell_j$; these vectors belong to the rational kernel of N_{m_j} and have first coordinate zero. The following rational vectors solve $(I - A_j)b_j = y_j$:

$$b_1 = (0, (2y_{1u} + y_{1w})/3, (y_{1w} - y_{1u})/3, 0)^t,$$
$$b_2 = (0, (y_{2u} - y_{2w})/2, (y_{2u} + y_{2w})/2, 0)^t.$$

The last coordinates of the equations follow from the displayed norm-kernel equations. The maps

$$(s, \zeta) \longmapsto (s, \alpha\zeta + \Pi_c(s)b_j)$$

are holomorphic maps of the normalized torus families over the degree-m_j root discs. Their inverses are holomorphic, and period covariance and $(I - A_j)b_j = d_j/m_j + \ell_j$ give exact commutation with the prescribed affine deck generators, modulo the integral period ℓ_j. They therefore descend to biholomorphisms of the finite fillings, including the full multiple fibres.

5. On the punctured root collar the corresponding middle translation is

$$\mathcal{A}_j = \Pi_c b_j + \frac{\log s_j}{2\pi i}\Pi_c d_j.$$

Its integral defect is ℓ_j. The middle cocycle relation fixes $\ell_0 = -A_0(\ell_1 + A_1\ell_2)$. In the quotient $\overline{\Lambda} = \Lambda/(\mathbb{Z}\widehat{w} + \mathbb{Z}\widehat{\delta})$, the first-coordinate condition is exactly $p' = \alpha p$. Adding an integral multiple of $(0, 1, -2, 0)^t \in \ker N_{m_1}$ to ℓ_1 changes the other quotient coordinate by a generator, without changing this first coordinate. We can consequently arrange

$$[\ell_0] = -\alpha c(\sigma) + c(\sigma') \quad \text{in } \overline{\Lambda}.$$

An integral toric vector then supplies $q \in \mathbb{Z}^2$ such that the cusp reference $sq + \alpha\sigma - \sigma'$ has precisely defect ℓ_0. Here $s = (2\pi i)^{-1}\log t_c$ is the cusp logarithm; the maps of cusp toric covers are

$$(x, t_c) \longmapsto (t_c^q u(t_c)x^\alpha, t_c), \qquad u(0) \in (\mathbb{C}^*)^2.$$

Integer translations of the cocharacter lattice, its inversion, and multiplication by a holomorphic unit preserve the specified hexagonal fan and all its strata. These maps and their inverses extend holomorphically across the three double curves and both triple points.

6. Differences of allowed finite extensions are the equivariant holomorphic vertical vectors on the complete root discs. The extra subgroup of order k at the first point acts trivially on the vertical vector coefficients and rotates s; its invariant holomorphic coefficients are exactly those in s^k. The remaining order-three equivariance is the original one. The argument at the second point uses s^l and order four. These invariant-coefficient maps identify the coarse difference sheaf with $\mathcal{V}_B$ for multiplicities $(3,4)$. Its first-coordinate quotient is $\mathcal{O}_B(-p_0)$: the modular frame has vanishing orders $2k, l$ on the two enlarged roots and a simple cusp pole. Its fixed second-coordinate subline is $\mathcal{O}_B$. Finite averaging provides local lifts, hence

$$0 \longrightarrow \mathcal{O}_B \longrightarrow \mathcal{V}_B \longrightarrow \mathcal{O}_B(-p_0) \longrightarrow 0.$$

Both end terms have vanishing H^1 on $\mathbb{P}^1$, so $H^1(B, \mathcal{V}_B) = 0$. The permitted local translations form a torsor under this sheaf: on each overlap their defects agree, so their difference is a single-valued section of the indicated difference sheaf. Vanishing of its Čech class gives a global middle translation. Its overlap equalities are exactly the two finite formulas and the cusp formula just displayed. Consequently the local biholomorphisms glue, with their inverses, over B.

7. For identical data, $p \neq 0$ excludes $a = -1$. The congruences reduce to $k \mid nr_1$ and $2l \mid nr_2$. The two primitivity assumptions imply $k \mid n$ and $2l \mid n$. Since $\gcd(k, 2l) = 1$, their least positive common multiple is $2kl$. For different sphere translation data, r_1' is invertible modulo k, r_2' modulo $2l$, and the Chinese remainder theorem gives exactly one integer shift class modulo $2kl$. The values of s_1 modulo k and s_2 modulo $2l$ all occur without affecting freeness or p. Thus there are exactly $2kl$ classes at fixed u. The change $c \mapsto c - 1$ on a positive u circuit acts by a generator of this cyclic residue group.

□

9 Classification in elliptic-surface coordinates

9.1 Higher logarithmic transformations

Definition 9.1 (Higher multiplicities and source coordinates)**.** [tag IP10] Put $m = 3k$, $n = 4l$ with $\gcd(m, n) = 1$. Viaclovsky's indices $(0, 1)$ correspond to the finite indices $(1, 2)$ of definition 2.1. Lattice coordinates are integral homology coordinates, and matrices act on columns.

Proposition 9.2 (The integral dictionary)**.** [tag 032R]

Let (δ, a, b, d) be the marking in (Viaclovsky, 2026a, Section 2, equations (2.1)–(2.2), p. 3, and (2.7), (2.11)–(2.12), p. 4), and put $u_0 = d + 2a + 2b$, $u_1 = d + 3a + 3b$. The homomorphism from the marking of definition 2.1 to this marking is

$$P_V = \begin{pmatrix} -2 & 1 & 0 & 1 \\ -6 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \det P_V = -1.$$

It intertwines the two finite monodromies and satisfies

$$P_V \widehat{\delta} = \delta, \qquad P_V \varepsilon = -u_0, \qquad P_V \varepsilon' = \delta - u_1.$$

With the counterclockwise deck generators of that paper, its data $m_0 = 3k$, $m_1 = 4l$ and $a_i = [v_i/m_i]$ with $v_i = \rho_i\delta + N_i u_i$ correspond to

$$w_1 = N_0\varepsilon - \rho_0\widehat{\delta}, \qquad w_2 = N_1\varepsilon' - (\rho_1 + N_1)\widehat{\delta}.$$

For the section-preserving cusp comparison used there, $l_0 = 0$ and $p = -(nN_0 + mN_1)$. Thus its sphere condition is $b := nN_0 + mN_1 = \pm 1$. The brackets in a_i mean the torsion point in the smooth root torus, using its section-preserving origin and the transported period vector. Replacing v_i by $v_i + m_i\lambda$, $\lambda \in \Lambda$, leaves the class $[v_i/m_i]$ unchanged, while its chosen lift fixes the integral collar defect.

Proof. 1. Use $P_E = P_{\mathrm{mark}}$ from proposition 2.3. In Engel's marking, the four columns of the change to Viaclovsky's basis are

$$Q_E = \begin{pmatrix} -2 & 1 & 0 & -2 \\ -1 & 0 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Direct integral multiplication gives $P_V = Q_E^{-1}P_E$ and $V_0P_V = P_VA_1$, $V_1P_V = P_VA_2$, where V_0, V_1 are the matrices in that paper's Lemma 2.1. Multiplication by the three specified invariant vectors gives the displayed identities. Inverting a clockwise affine generator replaces its invariant translation w_j/m_j by $-w_j/m_j$. Consequently the counterclockwise translation vector is $-P_Vw_j$, which gives the two formulas, including the term N_1 in the second coefficient. Viaclovsky's reference section extends through the smooth locus of the cusp; its comparison therefore has zero class in the quotient lattice of toric directions. This is precisely $l_0 = 0$. Substitution into the definition of p proves its formula.

□

Definition 9.3 (The residual cusp defect)**.** [tag IP11] For a general cusp collar σ, write $c(\sigma)$ for its residual translation defect in $\overline{\Lambda} = \Lambda/(\mathbb{Z}\widehat{w} + \mathbb{Z}\widehat{\delta})$, using the convention $\sigma(g_0z) - R_0\sigma(z) = \Pi_c(g_0z)\ell(\sigma)$ and $c(\sigma) = [\ell(\sigma)]$. Set $l_0 = \gamma(c(\sigma))$. In the marked coordinates $w_j = r_j\varepsilon_j + s_j\widehat{\delta}$, the integer is $p = mnl_0 - nr_1 - mr_2$. The section-preserving source data of 9.2 have $c(\sigma) = 0$.

Theorem 9.4 (Classification in Viaclovsky's coordinates)**.** [tag 032S]

Use a local logarithm c of the transported linearization, $u = e^{-2\pi ic}$, and the same reference section and cusp comparison on both sides. Two sphere members with data $(m, \rho_0, N_0; n, \rho_1, N_1; c)$ and primed data are biholomorphic if and only if their multiplicities agree and there are $\alpha \in \{1, -1\}$ and $h = c' - c \in \mathbb{Z}$ such that

$$b' = \alpha b, \qquad hN_0' \equiv \rho_0' - \alpha\rho_0 \pmod{k}, \qquad hN_1' \equiv \rho_1' - \alpha\rho_1 \pmod{2l}.$$

Every such biholomorphism covers the identity of B. For fixed discrete data the period identification subgroup is $2kl\mathbb{Z}$. At fixed u all sphere translation data give exactly $2kl$ classes. A positive u-circuit acts on them by a single cycle.

Proof. Apply proposition 8.4 through the integral isomorphism P_V of proposition 9.2. Its affine and cusp coordinates are

$$r_1 = N_0, \quad s_1 = -\rho_0, \qquad r_2 = N_1, \quad s_2 = -(\rho_1 + N_1), \qquad p = -b.$$

Thus $p' = \alpha p$ is $b' = \alpha b$, and the first norm congruence is $hN_0' \equiv \rho_0' - \alpha\rho_0 \pmod{k}$. The second becomes

$$hN_1' \equiv \rho_1' - \alpha\rho_1 + N_1' - \alpha N_1 \pmod{2l}.$$

The relation $b' = \alpha b$, reduced modulo $4l$, gives $N_1' \equiv \alpha N_1 \pmod{4l}$, since $\gcd(3k, 4l) = 1$. It removes the last two terms and yields the stated congruence. The base-preservation assertion, identification subgroup and number of classes transfer through this same integral map. It preserves $u = e^{-2\pi i c}$, so a positive u-circuit is still $c \mapsto c - 1$ and induces the single cycle proved in proposition 8.4. □

Corollary 9.5 (Holomorphic dependence of the comparisons)**.** [tag 032T]

For fixed discrete data in 9.4 and a fixed admissible shift h, the constructed comparisons are holomorphic isomorphisms of families over a sufficiently small parameter disc. In particular, for sphere data with multiplicities $(3, 4)$ the identification with the standard family uses the parameter

$$W = e^{-\pi i(c+s_2)} = (-1)^{s_2} w.$$

The smooth monodromy conjugacy class around the positive parameter circle is unchanged by this identification of twist data.

Proof. 1. The rational vectors b_j, integral defects ℓ_j and toric exponents in the proof of proposition 8.4, transported through proposition 9.2, are independent of c. All period matrices, root comparisons, and cusp unit factors are holomorphic in c. On a small Stein parameter disc D, the correction sheaf is $\mathcal{V}_B \boxtimes \mathcal{O}_D$. Its first cohomology on $B \times D$ is zero: project to D, use the same two-chart complex and $H^1(B, \mathcal{V}_B) = 0$, and then Stein acyclicity. Thus the affine correction can be chosen holomorphically in c. For $(k, l) = (1, 1)$ the only nonvacuous finite congruence is $h \equiv s_2 - s_2' \pmod 2$, because the admissible r_2' is odd. Taking $s_2' = 0$ gives the displayed parameter identification. Two choices of these family comparisons differ fibrewise by an automorphism of the standard member. The automorphism calculation in theorem C.4 identifies that group with $\mathbb{C}^*$; it is connected. On overlapping parameter arcs the transitions are therefore isotopic to the identity. Composing around a circle changes the resulting diffeomorphism only by such an isotopy and by conjugation at the chosen base fibre. Multiplication of the parameter by $(-1)^{s_2}$ preserves the positive orientation.

□

10 The monodromy group and its period domain

Definition 10.1 (The period domain and proper discontinuity)**.** [tag IP01] We use the clockwise root deck generators and lattice basis of Definition 2.1. A group acts properly discontinuously when, for every compact subset K of its space, only finitely many group elements g satisfy $gK \cap K \neq \varnothing$. This definition permits finite stabilizers. We write $e = \widehat{\delta}$, $E = e \otimes \gamma$, and

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -6 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the alternating form is $\psi = u \wedge w + 6\gamma \wedge \delta$, an integral alternating form of type $(1, 6)$ on Λ. We use Ω for its matrix and retain J for a real complex structure. The period domain $\mathcal{D}$ consists of the real endomorphisms J of $\Lambda_{\mathbb{R}}$ satisfying $J^2 = -I$, $J^{\mathrm{t}}\Omega J = \Omega$, and $\operatorname{sign}(\Omega J) = (2, 2)$. It is $\mathrm{Sp}_4(\mathbb{R})/U(1, 1)$, with its complex structure as an open subset of the Lagrangian Grassmannian. An integral symplectic matrix acts by $J \mapsto gJg^{-1}$.

10.1 An exact arithmetic description

Theorem 10.2 (The integral monodromy group)**.** [tag 01R9]

Let $P_{e,\mathbb{Z}}$ be the integral symplectic group fixing e as a vector, and let $P_{\mathbb{Z}}$ be the integral symplectic group preserving $\mathbb{Z}e \subset \ker\gamma \subset \Lambda$. Every element of $P_{e,\mathbb{Z}}$ has a unique expression

$$g(B,x,t) = \begin{pmatrix} 1 & 0 & 0 \\ 6x & B & 0 \\ t & -x^{\mathrm{t}}J_2B & 1 \end{pmatrix}, \qquad B \in \mathrm{SL}_2(\mathbb{Z}), \quad x \in \mathbb{Z}^2, \quad t \in \mathbb{Z}, \tag{10.1}$$

where the block sizes are $1,2,1$. Put

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

There is a homomorphism $\alpha : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}/12$ determined by $\alpha(S) = 9$ and $\alpha(T) = 1$; in particular $\alpha(a) = 4$. With $Q(x) = x_1 + x_2 + x_1x_2$, the monodromy group is

$$\Gamma = \{g(B,x,t) : t + 6Q(x) - \alpha(B) = 0 \text{ in } \mathbb{Z}/12\}\,. \tag{10.2}$$

Consequently $[P_{e,\mathbb{Z}} : \Gamma] = 12$ and $P_{\mathbb{Z}} = \{\pm I\} \times P_{e,\mathbb{Z}}$, so $[P_{\mathbb{Z}} : \Gamma] = 24$. Its image in the Levi quotient is exactly $\mathrm{SL}_2(\mathbb{Z})$, with scalar on e equal to 1.

Proof. 1. Use the matrices A_1, A_2 of (2.2). Multiplication gives $A_1^3 = A_2^4 = I$ and $A_j^{\mathrm{t}}\Omega A_j = \Omega$. In particular $A_1 = g(a,(1,-1)^{\mathrm{t}},-2)$ and $A_2 = g(S,(0,-1)^{\mathrm{t}},3)$. Their images a, S generate $\mathrm{SL}_2(\mathbb{Z})$ because $Sa = T$ and S,T generate that group by (Conrad, n.d., Theorem 1.1, p. 1).

2. A matrix fixing e and preserving Ω preserves $e^{\perp} = \ker\gamma$. Its first row is $(1,0,0,0)$, its last column is e, its middle block B is symplectic on $e^{\perp}/\mathbb{R}e$, and its remaining middle column, say v, satisfies $6r = -v^{\mathrm{t}}J_2B$. Integrality of r and unimodularity of B give $v = 6x$ with $x \in \mathbb{Z}^2$. The lower left entry t varies freely in $\mathbb{Z}$. This proves (10.1). Direct multiplication gives

$$g(B,x,t)g(C,y,s) = g(BC, x + By, t + s - 6x^{\mathrm{t}}J_2By). \tag{10.3}$$

3. Since $a = -ST$, its image in $\mathrm{PSL}_2(\mathbb{Z})$ is the image of the generator $R = ST$ in (Conrad, n.d., Theorem C.1, pp. 18–19). That theorem gives $\mathrm{PSL}_2(\mathbb{Z}) = \langle \bar{a}, \bar{S} \mid \bar{a}^3 = \bar{S}^2 = 1\rangle$. Lifting through the central kernel $\{I, -I\}$ gives

$$\mathrm{SL}_2(\mathbb{Z}) = \langle a, S \mid a^3 = S^4 = 1,\ [a, S^2] = 1\rangle. \tag{10.4}$$

   Indeed the group on the right modulo its central element S^2 is the cited free product, and S^2 maps to the nonidentity matrix $-I$. Its kernel in the displayed matrix group is consequently trivial. The values $4, 9$ respect all these relations, proving the assertion about α; $T = Sa$ has value 1.

4. The quadratic function Q modulo 2 is zero at $(0,0)$ and one at all three other points of $\mathbb{F}_2^2$. Hence $Q(Bx) = Q(x) \pmod 2$ for every $B \in \mathrm{SL}_2(\mathbb{Z})$. Also

$$Q(x+y) - Q(x) - Q(y) \equiv x^{\mathrm{t}}J_2y \pmod 2.$$

   Equation (10.3) now shows that $h(g(B,x,t)) = t + 6Q(x) \pmod{12}$ is a homomorphism. The homomorphism $\chi = h - \alpha \circ B$ is surjective, since $\chi(g(I,0,1)) = 1$. Its values on A_1, A_2 are zero.

5. We prove $\Gamma = \ker \chi$. Let $N_{\mathbb{Z}}$ be the kernel of the map to the middle block. The kernel of $\langle g_1, g_2 \mid g_1^3 = g_2^4 = 1\rangle \to \mathrm{SL}_2(\mathbb{Z})$ is normally generated by $[g_1, g_2^2]$, by (10.4). Its image in Γ is therefore normally generated by

$$K = [A_1, A_2^2] = g(I, (0,-1)^{\mathsf{t}}, 6).$$

Put

$$U = A_2 K A_2^{-1} = g(I, (1,0)^{\mathsf{t}}, -6), \quad V = K^{-1} = g(I, (0,1)^{\mathsf{t}}, -6), \quad Z = I + 12E.$$

Equation (10.3) gives $[U, V] = Z^{-1}$. Thus U, V, Z generate

$$H = \{g(I, x, t) : t \in 6\mathbb{Z}, \quad t/6 \equiv Q(x) \pmod 2\}. \tag{10.5}$$

More explicitly, $U^r V^s Z^k = g(I, (r,s)^{\mathsf{t}}, -6(r+s+rs) + 12k)$, so these expressions give every member of the set. Conjugation by $g(B, x, t)$ sends $g(I, y, s)$ to $g(I, By, s - 12x^{\mathsf{t}} J_2 By)$; the invariance of Q modulo 2 proves that H is normal in $P_{e,\mathbb{Z}}$. The normal closure of K is contained in H and contains U, V, Z, so it is exactly H. We have proved $\Gamma \cap N_{\mathbb{Z}} = H$. Both Γ and $\ker \chi$ surject onto the same Levi group and have intersection H with $N_{\mathbb{Z}}$. Given $g \in \ker \chi$, choose $\gamma \in \Gamma$ with the same middle block. Then $g\gamma^{-1} \in H$, and $g \in \Gamma$. This proves (10.2) and the index 12.

6. Finally an integral automorphism preserving the primitive line $\mathbb{Z}e$ sends e to e or $-e$. Multiplication by the central matrix $-I$ converts the latter case to the former. The intersection of $\{\pm I\}$ with $P_{e,\mathbb{Z}}$ is trivial. This proves the direct product and the index 24.

□

Proposition 10.3 (The center and the Zariski closure)**.** [tag 01RA]

The group in (10.2) has center $\langle Z\rangle \cong \mathbb{Z}$ and is the following explicit central extension of $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ by $\mathbb{Z}$. Choose the representative $a(B) \in \{0, \dots, 11\}$ of $\alpha(B)$. Its underlying set consists of triples (B, x, k), with product

$$(B, x, k)(C, y, l) = (BC, x + By, k + l + \kappa((B, x), (C, y))),$$

where

$$\begin{aligned} 12\kappa((B, x), (C, y)) = a(B) + a(C) - a(BC) - 6Q(x) - 6Q(y) \\ - 6x^{\mathsf{t}} J_2 By + 6Q(x + By). \end{aligned}$$

Its real and rational Zariski closures are respectively the full real and rational symplectic stabilizers of the vector e. They have dimension 6, with three-dimensional Heisenberg unipotent radical and Levi quotient SL_2.

Proof. 1. The numerator defining κ is divisible by 12, by the two congruences proved above. The bijection

$$(B, x, k) \longmapsto g(B, x, a(B) - 6Q(x) + 12k)$$

intertwines the displayed multiplication with matrix multiplication by (10.3). This proves associativity and identifies the central extension with the displayed cocycle. The kernel of the homomorphism to $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is $\langle Z\rangle$. That affine group has trivial center: commuting with all translations forces its linear part to be I, and commuting with S, T forces its translation part to vanish. Thus the center of Γ is exactly the stated kernel.

2. The logarithms $U-I, V-I, Z-I$ are linearly independent elements of the three-dimensional Heisenberg Lie algebra. The square of each is zero. A polynomial vanishing on all integral powers $I+n(U-I)$ vanishes on the whole one-parameter line; the same holds for V and Z. Hence the Zariski closure contains the full unipotent radical. Its Levi image contains S, T; the powers of T and of STS^{-1} similarly supply the two opposite unipotent subgroups of SL_2. Their Lie algebras and their bracket span $\mathfrak{sl}_2$. They generate the connected algebraic group SL_2. A subgroup containing the entire unipotent kernel and surjecting onto this Levi quotient is the entire vector stabilizer. Its dimension is $3+3=6$. This reasoning uses equations over $\mathbb{Q}$, so proves the rational assertion as well as the real assertion.

□

10.2 Properness and the maximal open set

Definition 10.4 (The sign domains)**.** [tag IP02] Put

$$q_e : \mathcal{D} \longrightarrow \mathbb{R}, \qquad q_e(J) = \psi(e, Je), \qquad \mathcal{O} = \{J : q_e(J) \neq 0\}.$$

The function q_e is invariant under $P_{\mathbb{Z}}$: replacing e by $-e$ changes both of its arguments. Write $\mathcal{O}_+$ and $\mathcal{O}_-$ for its two sign regions.

Proposition 10.5 (An infinite stabilizer)**.** [tag 01RB]

The following point of $\mathcal{D}$ has an infinite Γ-stabilizer:

$$J_0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -6 \\ 1 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 72 & 0 \\ 0 & 0 & 1 & 0 \\ -12 & 0 & 0 & 1 \end{pmatrix} \in \Gamma.$$

Every power of R fixes J_0, and all these powers are distinct.

Proof. 1. Direct multiplication gives $J_0^2 = -I$ and $J_0^{\mathrm{t}} \Omega J_0 = \Omega$. Moreover

$$\Omega J_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 6 & 0 \end{pmatrix}$$

has eigenvalues $1, -1, 6, -6$, so $J_0 \in \mathcal{D}$ and $q_e(J_0) = 0$. The matrix R equals $g(T^{72}, 0, -12)$ and belongs to (10.2), because $\alpha(T^{72}) = 0 \pmod{12}$. Its commutator with J_0 is zero. Writing $R = I + N$ gives $N \neq 0$ and $N^2 = 0$, so $R^m = I + mN$ and the powers are distinct. The identity $N^2 = 0$ also gives $(I+mN)(I+nN) = I + (m+n)N$. The compact singleton $\{J_0\}$ meets every one of its translates by R^m, so the action on $\mathcal{D}$ is not properly discontinuous.

□

Theorem 10.6 (The maximal domain of proper discontinuity)**.** [tag 01RC]

The set $\mathcal{O}$ is the largest Γ-invariant open subset of $\mathcal{D}$ on which the action is properly discontinuous. It is also the largest such open subset for $P_{\mathbb{Z}}$. Each sign region is a complex manifold in the analytic topology, and its quotient by either discrete group is a Hausdorff complex analytic orbifold.

Proof. 1. Let $P_{\mathbb{R}}$ be the real symplectic group preserving the real line $\mathbb{R}e$. For a fixed J, the form $h_J(v,w) = \psi(v, Jw) + i\psi(v,w)$ is a nondegenerate Hermitian form of signature $(1,1)$ on the complex vector space $(\Lambda_{\mathbb{R}}, J)$, with the convention conjugate linear in the first entry. When $q_e(J) \neq 0$, the complex line $\mathbb{C}e$ is nondegenerate and its Hermitian orthogonal complement is

a complex line of the opposite sign. Normalize e by a positive real scalar and choose a unit vector in this complement. Comparing these two Hermitian bases for two complex structures with the same sign gives a real symplectic map preserving $\mathbb{R}e$ and conjugating one complex structure to the other. Thus $P_{\mathbb{R}}$ is transitive on each sign region.

2. An element of the stabilizer of J inside $P_{\mathbb{R}}$ is complex linear and sends e to re for a real nonzero r. Preservation of $q_e(J)$ gives $r^2 = 1$. On the complementary complex line it acts by an element of $U(1)$. The stabilizer is consequently the compact group $\{\pm 1\} \times U(1)$. Each sign region is therefore a homogeneous space of $P_{\mathbb{R}}$ with compact stabilizer L. Local sections of $P_{\mathbb{R}} \to P_{\mathbb{R}}/L$, and a finite cover of a compact subset K, give a compact subset C of $P_{\mathbb{R}}$ mapping onto K. Every element carrying a point of K back into K belongs to CLC^{-1}, which is compact. This proves properness. Restricting a proper action to a closed discrete subgroup gives the required finite-intersection condition. Both Γ and $P_{\mathbb{Z}}$ are closed discrete matrix groups. The two sign regions are disjoint invariant open-and-closed subsets of $\mathcal{O}$, so the same finite-intersection condition holds on their union. Proper actions by discrete groups have finite stabilizers and Hausdorff quotients; their holomorphic local finite-group quotient charts supply the asserted analytic orbifolds.

3. We prove maximality by exhibiting a dense set of points with infinite stabilizer in the zero locus of q_e. The real vector stabilizer $P_{e,\mathbb{R}}$ acts transitively on $\mathcal{D}_0 = \{J : q_e(J) = 0\}$. In the Hermitian space just described, e is a nonzero null vector. Choose v with $h_J(e, v) = 1$, and replace v by $v - \frac{1}{2}h_J(v, v)e$. This preserves $h_J(e, v) = 1$ and makes v null. The ordered complex basis (e, v) has Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The complex-linear map between two such bases fixes e and preserves h, hence its imaginary part ψ. As a real map it belongs to $P_{e,\mathbb{R}}$ and conjugates the two complex structures.

4. The subgroup $P_{e,\mathbb{Q}}$ is dense in $P_{e,\mathbb{R}}$. For the unipotent coordinates in (10.1) this is density of $\mathbb{Q}^3$ in $\mathbb{R}^3$. For SL_2 one may approximate a, b, c by rational numbers in the chart $a \neq 0$ and set $d = (1 + bc)/a$; translates of that chart cover the group. Hence

$$\{rJ_0r^{-1} : r \in P_{e,\mathbb{Q}}\}$$

is dense in $\mathcal{D}_0$. Put $R_0 = g(T^6, 0, -1)$, so $R_0^{12} = R$ and $(R_0 - I)^2 = 0$. At rJ_0r^{-1} the nonzero rational nilpotent $L = r(R_0 - I)r^{-1}$ commutes with the complex structure and satisfies $L^2 = 0$. Choose a positive integer d clearing its denominators. The matrix $I + dL$ and its inverse $I - dL$ are integral, symplectic, and fix e; this follows either by conjugating the identity $(I + s(R_0 - I))^{\mathrm{t}}\Omega(I + s(R_0 - I)) = \Omega$ or by expanding it. The twelfth power of $I + dL$ lies in Γ, because $P_{e,\mathbb{Z}}/\Gamma \cong \mathbb{Z}/12$. It has infinite order and fixes rJ_0r^{-1}. Thus points with infinite Γ-stabilizer are dense in $\mathcal{D}_0$.

5. Every open neighborhood in $\mathcal{D}$ of any point of $\mathcal{D}_0$ therefore contains a point with infinite stabilizer. An open set on which the action is proper cannot meet $\mathcal{D}_0$. This proves the largest-open-set assertion for Γ. The same points have infinite $P_{\mathbb{Z}}$-stabilizer, so it proves that assertion for the full parabolic as well.

□

Proposition 10.7 (A global chart for positive periods)**.** [tag 01RD]

The positive sign region admits the global period chart

$$\mathcal{O}_+ = \left\{(\tau, \mu, \beta) \in \mathbb{C}^3 : \operatorname{Im}\tau > 0, \quad \operatorname{Im}\beta - \frac{6(\operatorname{Im}\mu)^2}{\operatorname{Im}\tau} < 0\right\},$$

represented by the period matrix (2.1), with β_c replaced by the coordinate β. Put $D = \Im\beta - 6(\Im\mu)^2/\Im\tau$. In this chart,

$$q_e(J) = -\frac{6}{D}. \tag{10.6}$$

Every admissible period $\Pi_c(z)$ belongs to $\mathcal{O}_+$, for every admissible c. Thus this single maximal proper-action open set contains all the period images in question.

Proof. 1. Let $J \in \mathcal{O}_+$. The two vectors $\widehat{w}, e$ are complex-linearly independent in $(\Lambda_{\mathbb{R}}, J)$. Otherwise $\widehat{w} = re + sJe$ for real r, s. Pairing with e gives $0 = sq_e(J)$; hence $s = 0$, contradicting the real linear independence of $\widehat{w}, e$. There is consequently a unique complex-linear isomorphism $(\Lambda_{\mathbb{R}}, J) \to \mathbb{C}^2$ sending these two vectors to $(1, 0)$ and $(0, 1)$. The corresponding period matrix has right block I. The equation $\Pi\Omega^{-1}\Pi^{\mathrm{t}} = 0$ says that its upper left entry is six times its lower middle entry, giving exactly the displayed matrix. This construction is the ordinary holomorphic Grassmannian chart, so its coordinates and inverse are holomorphic.

2. In the real symplectic basis $(\widehat{u}, \widehat{\gamma}/6, \widehat{w}, e)$ the normalized symmetric period matrix is $\left(\begin{smallmatrix} \tau & \mu \\ \mu & \beta/6 \end{smallmatrix}\right)$. Its imaginary part has signature $(1, 1)$, equivalently $6(\operatorname{Im}\mu)^2 - \operatorname{Im}\tau \operatorname{Im}\beta > 0$. Solving $\Pi(Je) = i\Pi e = (0, i)$ in real coefficients gives

$$\gamma(Je) = \frac{\operatorname{Im}\tau}{\operatorname{Im}\tau \operatorname{Im}\beta - 6(\operatorname{Im}\mu)^2}.$$

Since $\psi(e, v) = -6\gamma(v)$, positivity of q_e now gives $\operatorname{Im}\tau > 0$ and $D < 0$, and proves (10.6). Conversely these inequalities make the imaginary period matrix invertible with the stated signature and give $q_e > 0$. The admissibility inequality of Definition 2.2 is $D_c < 0$ everywhere, so the displayed comparison map places every admissible period in $\mathcal{O}_+$.

□

10.3 The curve in the proper quotient

Theorem 10.8 (The embedded period curve)**.** [tag 01RE]

For every admissible c, the marked period map on the covering of B° is a holomorphic immersion into $\mathcal{O}_+$. The induced map

$$p_c : B^\circ \longrightarrow \Gamma\backslash\mathcal{O}_+$$

is injective and a holomorphic immersion. It extends over $t = 0, 1$ to a proper holomorphic embedding

$$\bar{p}_c : \mathbb{C} = B \setminus \{\infty\} \hookrightarrow \Gamma\backslash\mathcal{O}_+.$$

The closure of $p_c(B^\circ)$ in this quotient is exactly $\bar{p}_c(\mathbb{C})$.

Proof. 1. For $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, changing the marking by $g(B, x, t)$ replaces Π by $\Pi g(B, x, t)^{-1}$ and then renormalizes its right block. On the quotient by the complex line of e, the two periods are the row $(\tau, 1)$. Right multiplication by B^{-1} sends it to $(d\tau - c, a - b\tau)$. Hence

$$\tau \longmapsto \frac{d\tau - c}{a - b\tau}. \tag{10.7}$$

The denominator never vanishes in the upper half plane. The function $j(\tau)$ is therefore a holomorphic invariant of this action and descends to a holomorphic function j_Γ on the proper quotient. Here j is the elliptic invariant, normalized by $j(E) = 1728\, 4A^3/(4A^3 + 27B^2)$ for $y^2 = x^3 + Ax + B$; changing a lattice basis preserves the isomorphism of elliptic curves and hence this function.

2. The elliptic curve whose period is τ has the Engel equation of Lemma 2.10,

$$y^2 = x^3 - 3t^3(t - 1)x + 2t^4(t - 1)^2.$$

Its discriminant is $1728t^8(t-1)^3$ and direct cancellation gives

$$j(\tau) = 1728t \qquad (t \in B^\circ). \tag{10.8}$$

In a local base coordinate t, the derivative of the composite $j \circ \tau$ equals 1728. Thus the derivative of the marked period map is nonzero. The same identity after passage to the quotient shows both injectivity and injectivity of the tangent map of p_c: $j_\Gamma/1728$ is a holomorphic left inverse.

3. At the two elliptic points, the period functions are holomorphic in the cyclic covering coordinates s_1, s_2, with $t = s_1^3$ and $t - 1 = s_2^4$ in the root coordinates of Definition 2.1. Their values are nondegenerate because $D_c < 0$ there. The maps from these discs to $\mathcal{O}_+$ are equivariant for A_1, A_2. Composing with the quotient makes them invariant, so they descend to holomorphic maps from the base discs. This is the finite-quotient comparison map: a holomorphic invariant germ on a disc is a convergent series in $s_j^{m_j}$, which proves descent on local function rings. The extensions agree with p_c off their centers and therefore with each other. Identity (10.8) extends by holomorphy, so the resulting $\bar{p}_c$ still has holomorphic left inverse $j_\Gamma/1728$.

4. The quotient is Hausdorff by 10.6. For a compact subset K of it, $\bar{p}_c^{-1}(K)$ is closed in $\mathbb{C}$ and lies in the compact set $j_\Gamma(K)/1728$. Thus $\bar{p}_c$ is proper. A holomorphic section of a holomorphic map is locally a closed immersion: in local embeddings into polydiscs, its coordinate along the base is t and the remaining coordinates are its holomorphic component functions, whose graph defines the image. Applying this to the left inverse shows that $\bar{p}_c$ is a holomorphic embedding; properness makes its image closed. Since $\mathbb{C} \setminus \{0, 1\}$ is dense in $\mathbb{C}$, its image has the asserted closure.

□

Proposition 10.9 (Finite limits and the cusp limit)**.** [tag 01RF]

At the two finite punctures the period map has the holomorphic finite-cover limits

$$\Pi_1 = \begin{pmatrix} 2(2-\rho) & \rho & 1 & 0 \\ \beta_c(z_1) & (2-\rho)/3 & 0 & 1 \end{pmatrix}, \qquad \Pi_2 = \begin{pmatrix} 3(1-i) & i & 1 & 0 \\ \beta_c(z_2) & (1-i)/2 & 0 & 1 \end{pmatrix}, \quad \rho = e^{\pi i/3}.$$

Their monodromies are the given order-three and order-four matrices. At the cusp the curve leaves every compact subset of $\Gamma\backslash\mathcal{O}_+$. On the prescribed cusp branch its Hodge plane in the compact dual tends to $\mathbb{C}\gamma + \mathbb{C}u$. The latter plane has complex dimension two in common with its conjugate and therefore does not represent a point of $\mathcal{D}$.

Proof. 1. The finite values of τ, μ follow by evaluating their transformation laws at the fixed points: $\tau(z_1) = \rho$, $\mu(z_1) = (2-\rho)/3$, $\tau(z_2) = i$, $\mu(z_2) = (1-i)/2$. Substitution in Π gives the two matrices. The finite covering comparison and its descent were proved in 10.8. The unipotent cusp transformation is $\tau \mapsto \tau - 1$, $\mu \mapsto \mu$, $\beta \mapsto \beta + 1$. On the cusp branch, $\operatorname{Im}\tau \to +\infty$, $\mu \to \mu_0$, and $\beta_c + \tau \to b_0 + c$, from the holomorphic cusp expansions. Consequently

$$D_c = -\operatorname{Im}\tau + O(1), \qquad q_e(\Pi_c) \longrightarrow 0^+, \qquad j_\Gamma(p_c(t)) = 1728t \longrightarrow \infty.$$

The last equality alone excludes a convergent subnet in any compact subset of the proper quotient; the first two identities give the position relative to the maximal proper-action open set. The rows of Π_c, regarded as covectors, are $6\mu\gamma + \tau u + w$ and $\beta_c\gamma + \mu u + \delta$. Dividing each by τ gives limits u and $-\gamma$. Their exterior product has a nonzero limit, so their span converges in the Grassmannian to the stated plane. It is spanned by real covectors, hence equals its conjugate. A complex structure has $H^{1,0} \cap H^{0,1} = 0$; here the intersection is the entire two-dimensional plane.

□

Proposition 10.10 (The determinant of the period derivative)**.** [tag 01RG]

Let $Z = \left(\begin{smallmatrix} 6\mu & \tau \\ \beta & \mu \end{smallmatrix}\right)$ be the left block of (2.1). Its derivative has generically nonzero determinant, with a zero at z_1:

$$\det \dot{Z} = 6\dot{\mu}^2 - \dot{\tau}\dot{\beta}, \qquad \det \dot{Z}(z_1) = 0.$$

The period map is an immersion at every point of B° by Theorem 10.8; its differential and $\det \dot{Z}$ are related by the polynomial map $\dot{Z} \mapsto \det \dot{Z}$.

Proof. 1. In the cusp coordinate s, the expansion $Z = s\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) + C(q)$, $q = e^{2\pi i s}$, gives $\det(dZ/ds) = 1 + O(q)$, so the determinant is not the zero function. At z_1 the fixed-point transformation law of μ gives $\dot{\mu} = -\dot{\tau}/3$. One may obtain this by differentiating $\mu \circ g_1 = (1-\mu)/\tau$, using $g_1'(z_1) = -\rho$ and $\rho^2 - \rho + 1 = 0$. The law $\beta \circ g_1 = \beta + 2 - 6(1-\mu)^2/\tau$ then gives $\dot{\beta} = 2\dot{\tau}/3$. Substitution yields $6\dot{\tau}^2/9 - 2\dot{\tau}^2/3 = 0$. The derivative of a map from a curve to the three-dimensional period chart is a vector of three coordinate derivatives. Identity (10.8) proves that this vector is nonzero over B°; the determinant tests the rank of the associated two-by-two Kodaira–Spencer matrix. Thus the coordinate derivative is nonzero on B°, while its associated matrix has the displayed determinant.

□

11 Exceptional fibres, Néron–Severi classes, and compact subspaces

11.1 Elliptic quotients and the torsion locus

Proposition 11.1 (The distinguished elliptic projection)**.** [tag 01QR]

The specified linear projection $\mathrm{pr}_1 : \mathbb{C}^2 \to \mathbb{C}$ induces a holomorphic group homomorphism $F_t \to E_t = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ exactly on $6\mu = A\tau + B$, $A, B \in \mathbb{Z}$. It is then surjective with connected kernel $\mathbb{C}/(\mathbb{Z} + (A\mu - \beta)\mathbb{Z})$. On the smooth base the two values of t are precisely

$$t_\pm = \frac{64(47 \pm 27\sqrt{3})}{11}.$$

Proof. 1. The images of the four lattice generators under the linear projection are $6\mu, \tau, 1, 0$. This proves descent in both directions. The integral basis change replacing $\widehat{\gamma}$ with $\widehat{\gamma} - A\widehat{u} - B\widehat{w}$ changes its period to $(0, \beta - A\mu)$; the remaining vertical generator is $(0, 1)$. The other two periods map onto the entire target lattice, so the kernel is connected. Its second modulus has imaginary part $A \operatorname{Im}\mu - \operatorname{Im}\beta = -D > 0$.

2. The geometric model and its marked section are transported by $\Phi^{-1}(X, Y) = (X/2, -Y/(2\sqrt{2}))$ to the arithmetic model below. The inverse map is $\Phi(x, y) = (2x, -2\sqrt{2}y)$; substitution in both cubic equations proves the isomorphism and its effect on the marked point (lemma 2.10). Thus the arithmetic equation is

$$Y^2 = X^3 - \frac{3}{4}t^3(t-1)X + \frac{1}{4}t^4(t-1)^2, \qquad P = (0, t^2(t-1)/2).$$

Its sixth division polynomial at P is

$$\psi_6(P) = \frac{9}{65536}t^{24}(t-1)^9(11t^2 - 6016t + 8192).$$

For a smooth elliptic fibre, $\psi_n(P) = 0$ is exactly the condition $nP = O$, since the multiplication rational map has these and only these poles in its affine coordinate. The two roots displayed

above are simple and avoid $0, 1$. Chord-and-tangent addition gives the rational function

$$\begin{aligned} x(6P) &= \frac{(89t^2 + 4736t - 4096)Q(t)}{1296(11t^2 - 6016t + 8192)^2}, \\ Q(t) &= 881t^4 + 4745344t^3 - 12079104t^2 \\ &\quad + 24641536t - 16777216. \end{aligned}$$

The division-polynomial recurrence is given in proposition 4.9.

□

Proposition 11.2 (The torsion locus and its degrees)**.** [tag 01QS]

The locus on B° where $P(t)$ is torsion is countable and dense. Every fibre over this locus is an abelian surface. The divisor on the smooth base cut out by $6kP = O$ has degree $3k^2 - 1$, including local intersection multiplicity, for every positive integer k.

Proof. 1. On a simply connected open subset write uniquely $\mu = a_0\tau + b_0$, a_0, b_0 real. The section is torsion precisely when $(a_0, b_0) \in \mathbb{Q}^2$. In a local complex parameter z, direct differentiation gives

$$\det \frac{\partial(a_0, b_0)}{\partial(\operatorname{Re} z, \operatorname{Im} z)} = -\frac{|\mu'(z) - a_0\tau'(z)|^2}{\operatorname{Im} \tau}.$$

This determinant vanishes on no open set. On an open set of vanishing, the two real differential equations imply $da_0 = db_0 = 0$, hence $\mu = a_0\tau + b_0$ with constant real coefficients. Analytic continuation would give this identity on the cover. The cusp law $(\tau, \mu) \circ g_0 = (\tau - 1, \mu)$ then forces $a_0 = 0$, and the g_2-law makes a constant μ impossible. Thus every open subset contains a point where this real map is a local diffeomorphism, and its image contains rational pairs. This proves density. For each integer triple (n, A, B), $n \neq 0$, the function $n\mu - A\tau - B$ is holomorphic and not identically zero by the same argument. Its zeros are discrete; a countable union proves countability.

2. For $N\mu = A\tau + B$, $A, B, N \in \mathbb{Z}$, $N > 0$, use the linear coordinate change $\zeta_2' = \zeta_2 - (A/N)\zeta_1$. The four integral vectors

$$N\widehat{u} - B\widehat{\delta}, \quad N\widehat{w} + A\widehat{\delta}, \quad N\widehat{\gamma} - 6A\widehat{u} - 6B\widehat{w}, \quad \widehat{\delta}$$

span a sublattice of index N^3. Its first two periods lie on $\zeta_2' = 0$, and its last two lie on $\zeta_1 = 0$. Both intersections are genuine rank-two lattices because $D < 0$. Their product quotient is therefore a product of elliptic curves, and inclusion of this sublattice induces a finite surjective holomorphic homomorphism onto F_t. A positive integral form on the product can be summed over its finite translation kernel and multiplied by the index to descend an integral positive form to F_t. The Riemann polarization criterion proves projectivity.

3. For the degree statement, (Engel, 2026, Lemma 2.1, pp. 3–4, with the height computation on p. 4) gives $\langle P, P\rangle = 1/6$, $\chi(\mathcal{O}_S) = 1$, the component groups at the two reducible fibres have orders three and two, and the local corrections for P are $4/3$ and $1/2$. The application to $6kP$ is as follows. The corrected divisor class of a section is its class minus O and a fibre multiple, plus the unique rational combination of nonidentity fibre components that makes it perpendicular to those components. On restriction to the generic fibre this assignment is additive because the divisor class $[Q] - [O]$ is the group law in Pic^0. The perpendicular correction makes its self-intersection the negative of the height, so the height is quadratic. The element $6kP$ is in the identity component at both reducible fibres; its correction terms vanish. Hence

$$6k^2 = (6k)^2\langle P, P\rangle = 2 + 2((6kP) \cdot O),$$

which gives the formula. The rational functions for $6P$ show that it has finite affine coordinates at $0, 1$, so it is a nonzero point of each additive identity component; multiplication by k remains nonzero in $\mathbb{C}$. At infinity its multiplicative coordinate is $(-a)^{6k} = a^{6k} \neq 1$. Thus all intersections counted by the height occur on B°, so their total degree is $3k^2 - 1$.

□

Theorem 11.3 (A second elliptic quotient)**.** [tag 01QT]

At $t = 32/5$ there is an admissible parameter for which $L(\zeta_1, \zeta_2) = 3\zeta_1 + 4\zeta_2$ induces a surjective holomorphic group homomorphism $F_t \to E_t$ with connected elliptic kernel, while $P(t)$ has exact order four. The distinguished projection of 11.1 descends exactly at the order-dividing-six locus specified there.

Proof. 1. Set $t = 32/5$. The exact fourth division polynomial is

$$\psi_4(P) = -\frac{1}{32} t^{10}(t-1)^4(5t-32),$$

whereas $\psi_2(P) = t^2(t-1) \neq 0$. Thus P has exact order four. On the rectangular branch write $\tau = iT$, $T > 1$, with $j(iT) = 1728 \cdot 32/5$. The chosen zero has real part $1/2$, negative imaginary part of absolute value less than $T/2$, and order four; therefore $\mu = (2 - \tau)/4$. Choose the concrete parameter

$$c = \left(\frac{9}{8} - \frac{101}{4}\right)\tau - \frac{9}{4} - \beta^{\mathrm{part}}(t).$$

It is globally admissible. Indeed the positive-real Jacobi product at this (μ, τ) gives $\operatorname{Im}\beta^{\mathrm{part}} \geq -T - (3/\pi)\log 3$. Consequently $\operatorname{Im} c \leq -185T/8 + (3/\pi)\log 3 < -22 < -M$, by 3.3. Thus the displayed parameter is admissible.

2. The holomorphic linear map $L : \mathbb{C}^2 \to \mathbb{C}$, $L(\zeta) = 3\zeta_1 + 4\zeta_2$, takes the four periods to

$$-101\tau, \qquad 2\tau + 2, \qquad 3, \qquad 4.$$

These generate exactly $\mathbb{Z}\tau + \mathbb{Z}$: the last two generate 1, and the first two then generate τ, since $\gcd(101, 2) = 1$. Thus L descends to a surjective holomorphic group homomorphism $F_{32/5} \to E_{32/5}$ with connected one-dimensional kernel. The kernel is compact as a closed subset of a compact torus, hence an elliptic curve. Nevertheless $6P = 2P \neq O$.

□

Definition 11.4 (The Hermitian form of an integral class)**.** [tag IP03]

At a smooth fibre the real vector-space isomorphism $\Pi : \mathbb{R}^4 \to \mathbb{C}^2$ transfers an alternating form E to $\widetilde{E}(v, w) = E(\Pi^{-1}v, \Pi^{-1}w)$. For a J-invariant $\widetilde{E}$, put

$$H_E(v, w) = \widetilde{E}(v, Jw) - i\widetilde{E}(v, w).$$

Here H_E is complex-linear in the first argument. Direct substitution gives $H_E(Jv, w) = iH_E(v, w)$ and $H_E(w, v) = \overline{H_E(v, w)}$. Its real Chern form is $-\operatorname{Im} H_E = \widetilde{E}$, since $(i/2)(H_E(v, w) - H_E(w, v)) = \widetilde{E}(v, w)$.

Proposition 11.5 (The Néron–Severi lattice)**.** [tag 01QU]

For every smooth fibre and every admissible c, its Néron–Severi lattice is the following explicit subgroup of $\mathbb{Z}^6$:

$$\left\{E :\ E_{12} - E_{13}\tau + (6E_{23} - E_{14})\mu + E_{24}\beta + E_{34}(6\mu^2 - \tau\beta) = 0\right\}. \tag{11.1}$$

The complex orientation has intersection form $E^2 = -2\,\mathrm{Pf}(E)$, where $\mathrm{Pf}(E) = E_{12}E_{34} - E_{13}E_{24} + E_{14}E_{23}$. In particular the invariant class $\eta = 6\gamma \wedge \delta + u \wedge w$ has $\int_{F_t} \eta^2 = -12$. Its Hermitian form is indefinite. The fibres with lattice exactly $\mathbb{Z}\eta$ have no compact curves and algebraic dimension zero.

Proof. 1. On the universal cover the holomorphic two-form is the wedge of the two rows of Π. Its six coefficients in the integral coframe are $6\mu^2-\tau\beta, -\beta, 6\mu, -\mu, \tau, 1$. An integral real two-form is of type $(1,1)$ exactly when its wedge with this form vanishes: its $(0,2)$ part is then zero, and reality also annihilates its $(2,0)$ part. Expanding the wedge gives (11.1). The exponential sequence identifies these integral $(1,1)$ classes with Chern classes of holomorphic line bundles, proving the lattice assertion.

2. The ordered real period matrix in lemma 2.7 has determinant $TD<0$. Thus the integral orientation $(\widehat{\gamma},\widehat{u},\widehat{w},\widehat{\delta})$ is opposite to the complex orientation. Algebraically $E\wedge E = 2\,\mathrm{Pf}(E)\gamma\wedge u\wedge w\wedge\delta$, which proves the sign in the displayed intersection formula. In particular (Claude and L. Alpöge, 2026, Lemma 9.4, pp. 66–67) gives the coefficient identity $\eta\wedge\eta = 12\gamma\wedge u\wedge w\wedge\delta$. The real period isomorphism above takes this integral orientation to the negative of the complex orientation, yielding $\int_{F_t}\eta^2=-12$.

3. For the Hermitian form H_E of Definition 11.4, direct substitution of the four periods gives $H_\eta = -2G^{-1}$, where

$$G=\begin{pmatrix} 2T & 2U \\ 2U & \operatorname{Im}\beta/3 \end{pmatrix}, \qquad \det G = 2TD/3<0.$$

Thus H_η has one positive and one negative eigenvalue, and every nonzero multiple remains indefinite. The integration current of an effective divisor, averaged under translations of the compact torus, is a nonzero semipositive constant $(1,1)$-form in its Chern class. It cannot be an indefinite multiple of η. Thus no curve exists when $\mathrm{NS}=\mathbb{Z}\eta$. A nonconstant meromorphic function would have a nonempty polar divisor (after subtracting a constant if necessary), so the algebraic dimension is zero.

□

Proposition 11.6 (Generic and projective fibres)**.** [tag 01QV]

For a fixed c, all but countably many smooth fibres have $\mathrm{NS}=\mathbb{Z}\eta$. Conversely, for every fixed smooth t, there are admissible parameters tending to the cusp of the parameter disc for which F_t is an abelian surface, irrespective of the order of $P(t)$.

Proof. 1. By (Claude and L. Alpöge, 2026, Lemma 9.3, p. 66), the functions $1,\tau,\mu,\beta^{\mathrm{part}},6\mu^2-\tau\beta^{\mathrm{part}}$ are linearly independent. For $\beta=\beta^{\mathrm{part}}+c$, the relation $A+B\tau+C\mu+D\beta+E(6\mu^2-\tau\beta)=0$ has coefficient vector

$$(A+Dc,\ B-Ec,\ C,\ D,\ E)$$

in that ordered list. This coefficient change is invertible, with inverse $(A',B',C',D',E')\mapsto(A'-D'c,B'+E'c,C',D',E')$. Thus every integral class outside $\mathbb{Z}\eta$ has a nonidentically-zero holomorphic left side in (11.1); its zeros are discrete. Taking the countable union over $\mathbb{Z}^6$ proves the assertion for each fixed c, as in the application (Claude and L. Alpöge, 2026, Proposition 9.5, p. 67).

2. For the second assertion put $\beta_n=-3n^2\tau+12n\mu$, $c_n=\beta_n-\beta^{\mathrm{part}}(t)$. Its imaginary part tends to minus infinity, so the already proved finite bound on M makes every sufficiently large n admissible. The integral alternating matrix with nonzero upper entries

$$E_{13}=-3n^2,\quad E_{14}=6n,\quad E_{23}=-n,\quad E_{24}=1$$

satisfies (11.1) exactly at β_n, and $\mathrm{Pf}(E)=-3n^2<0$. Its complex-oriented square is positive. A nondegenerate Hermitian form in complex dimension two with positive determinant is definite. Choosing its sign gives a positive integral Riemann form, so F_t is projective. This supplies an explicit family of projective fibres even at any chosen non-torsion value of the section.

□

Theorem 11.7 (Elliptic quotients and algebraic dimension)**.** [tag 01QW]

Fix any admissible c. For each integral alternating matrix E, let $Z_E(c)$ be the zero set on the marking cover of the explicit holomorphic function in (11.1), with $\beta = \beta^{\mathrm{part}} + c$ given by (3.1). Let π be the cover map to B°, and define

$$\mathcal{E}_c = \pi\left(\bigcup_{E\neq 0,\ \mathrm{Pf}(E)=0} Z_E(c)\right), \qquad \mathcal{A}_c = \pi\left(\bigcup_{\mathrm{Pf}(E)<0} Z_E(c)\right).$$

These give the exact loci of all elliptic quotients and all abelian surfaces, respectively. The algebraic dimension of every smooth fibre is

$$a(F_t) = \begin{cases} 2, & t \in \mathcal{A}_c, \\ 1, & t \in \mathcal{E}_c \setminus \mathcal{A}_c, \\ 0, & t \notin \mathcal{E}_c \cup \mathcal{A}_c. \end{cases}$$

Together with (11.1), these classify every smooth fibre by countable unions of explicit analytic zero sets. The saturated null lattices in the proof determine all connected elliptic kernels and their quotient maps.

Proof. 1. We apply the Appell–Humbert correspondence and Riemann projectivity criterion (Schnell, 2019, Theorem 4.1, p. 1, and Theorem 4.4, p. 3). Equation (11.1) gives J-invariance, so Definition 11.4 identifies the integral class E with the real Chern form $-\operatorname{Im} H_E = \tilde{E}$. The Appell–Humbert integrality condition is therefore exactly the integrality of E. The complex-oriented square computed in 11.5 shows that $\mathrm{Pf}(E) < 0$ is equivalent to definiteness of this Hermitian form. Its sign can be chosen positive. The cited projectivity criterion then gives precisely the asserted abelian locus, in both directions.

2. For nonzero E with Pfaffian zero, its real rank is two. Its nullspace $K_{\mathbb{R}} \subset \mathbb{R}^4$ is a rational plane because it is the kernel of an integral matrix. J-invariance makes $\Pi K_{\mathbb{R}}$ a complex line, and $K = K_{\mathbb{R}} \cap \mathbb{Z}^4$ is saturated. The maps

$$(\Pi K_{\mathbb{R}})/\Pi K \ \hookrightarrow F_t \ \longrightarrow\ \mathbb{C}^2/(\Pi K_{\mathbb{R}} + \Pi\mathbb{Z}^4)$$

are holomorphic group morphisms; the first is a closed embedded elliptic curve and the second is its quotient onto an elliptic curve, with connected kernel. A basis of the saturated K extends to an integral basis of $\mathbb{Z}^4$. The other two periods project to real-linearly independent vectors in the one-dimensional complex quotient, because Π is a real isomorphism. They therefore form a discrete lattice there. Compactness follows by taking the image of its compact real fundamental parallelogram. Conversely any elliptic quotient pulls an integral positive area form back to a nonzero rank-two integral $(1,1)$ form. This proves the characterization of $\mathcal{E}_c$. Equivalently all its kernels can be found by saturated rank-two lattices with a basis v, w satisfying $\det_{\mathbb{C}}(\Pi v, \Pi w) = 0$ and real independence. This is a second exact integer test, with explicit inclusion and quotient morphisms.

3. For the algebraic-dimension assertion, an effective divisor on a torus has a semipositive Hermitian form: averaging its integration current under translations gives that constant form. A nonzero semipositive form has rank one or two. Rank two gives the abelian case. Rank one gives the elliptic quotient just constructed. In the absence of a definite form, there cannot be two different such kernel subtori. Indeed addition from their product onto F_t would be a surjective holomorphic homomorphism with finite kernel, and the product polarization would descend after multiplication by an integer, giving a definite form. Thus there is a single kernel K_t in this case. Every effective divisor is a sum of its translates: its intersection number with K_t is zero, whereas a curve mapping nonconstantly to the elliptic quotient has strictly positive

intersection with a general quotient fibre. Every meromorphic function therefore factors through this quotient. One can check the last statement on a fibre avoiding its polar divisor, where it is a holomorphic function on a compact elliptic curve and hence constant; analytic continuation then gives the factorization. The quotient supplies nonconstant meromorphic functions, so the algebraic dimension is exactly one. If neither locus occurs, a nonconstant meromorphic function would have a polar divisor of one of the two forbidden ranks. This proves dimension zero. Finally monodromy acts by integral determinant-one changes of basis, preserving (11.1) and its Pfaffian. Thus the stated unions descend as the stated subsets of B°.

□

Proposition 11.8 (An explicit locus of algebraic dimension one)**.** [tag 01QX]

There is an explicit nonempty locus of admissible members with ordinary fibres of algebraic dimension one. On a sufficiently small punctured cusp neighbourhood set

$$c(z) = -6\tau(z) - 12\mu(z) - \beta^{\mathrm{part}}(z).$$

Outside a countable set of z, the fibre of $X_{c(z)}$ over $\pi(z)$ has Néron–Severi lattice $\mathbb{Z}\eta + \mathbb{Z}E$, where the nonzero upper entries of E are $E_{13} = -6$, $E_{14} = -6$, $E_{23} = 1$, $E_{24} = 1$. Its algebraic dimension is one. The section $P(\pi(z))$ can simultaneously be required to be non-torsion.

Proof. 1. At the specified parameter $\beta = -6\tau - 12\mu$. Both E and η satisfy (11.1), and exact multiplication gives $\mathrm{Pf}(a\eta + bE) = 6a^2$. For any further integral class F, substitution in its period equation gives a linear relation among $1, \tau, \mu, (\mu+\tau)^2$. These four functions are linearly independent: applying g_0 to a relation $A + B\tau + C\mu + D(\mu+\tau)^2 = 0$ and subtracting yields $-B + D(1 - 2(\mu+\tau)) = 0$. A further cusp difference forces $D = 0$, and then $B = 0$; nonconstancy of μ forces $A = C = 0$. Comparison of coefficients now gives $F_{34} = F_{12} = 0$, $F_{13} = -6F_{24}$, and $6F_{23} - F_{14} = 12F_{24}$, exactly $F \in \mathbb{Z}\eta + \mathbb{Z}E$. Every other integral class therefore has a discrete zero set along this period locus. Excluding their countable union gives the asserted exact Néron–Severi lattice. It contains the nonzero isotropic E and no matrix of negative Pfaffian; 11.7 gives algebraic dimension one. Finally $c = -5\tau - 12\mu - (\beta^{\mathrm{part}} + \tau)$ has imaginary part tending to minus infinity at the cusp. A sufficiently small cusp neighbourhood is therefore admissible by 3.3. The torsion locus of P is countable by 11.2, so its exclusion still leaves such points in every open subset of that neighbourhood.

□

11.2 Compact curves and surfaces

Proposition 11.9 (Vertical compact subspaces)**.** [tag 01QY]

For a standard-twist member X_c with underlying manifold S^6, every compact irreducible complex curve lies in one fibre of f, and every compact irreducible complex surface is one of the reduced fibres.

Proof. 1. The given diffeomorphism with S^6 implies $H^2(X_c; \mathbb{Z}) = 0$. For a compact irreducible curve C, let $\nu : \widetilde{C} \to C \hookrightarrow X_c$ be its normalization map. The holomorphic map $f\nu$ has $\deg(f\nu)^*\mathcal{O}_{\mathbb{P}^1}(1) = 0$, since its Chern class is pulled back from the zero class on X_c. A nonconstant holomorphic map of compact Riemann surfaces has positive degree: a regular value has a positive number of preimages, all with positive local multiplicity. Therefore $f\nu$, and hence $f|_C$, is constant.

2. An irreducible surface Z in the smooth threefold is an effective Cartier divisor. If its image were all of B, a general smooth fibre would meet it in a nonempty effective divisor D whose line bundle is $\mathcal{O}_{X_c}(Z)|_{F_t}$. Its integral Chern class is zero. Choose a flat Kähler form ω on the torus F_t. The equality $\int_{F_t} c_1(D) \wedge \omega = \int_D \omega$ has left side zero and right side positive, a

contradiction. Thus Z lies in a single fibre. Each reduced fibre is irreducible of dimension two, so containment forces equality.

□

Proposition 11.10 (Curves in the special fibres)**.** [tag 01QZ]

There is no rational curve in $X_c \setminus W_c$. There are elliptic curves outside all three special fibres, in particular in each of the two smooth fibres $F_{t_\pm}$. All curves in W_c are precisely the images of irreducible algebraic curves in dP_6 *under its finite normalization map. Distinct curves have the same image only for the three paired boundary lines. All curves in a bielliptic fibre S_j are images of irreducible curves in its finite product cover.*

Proof. 1. By 11.9 every curve is vertical. A holomorphic map from $\mathbb{P}^1$ into a torus lifts to its vector-space universal cover, since $\mathbb{P}^1$ is simply connected, and its coordinate functions are constant by compactness. Thus a smooth torus fibre contains no rational curve. A map from $\mathbb{P}^1$ into a bielliptic surface lifts through its finite unramified torus cover by the same covering criterion; the lifted map is constant. These arguments prove the first assertion, including possibly singular rational images. 11.1 gives an embedded connected elliptic kernel in each $F_{t_\pm}$, proving the second.

2. For the third assertion let $\eta : \mathrm{dP}_6 \to W_c$ be the finite normalization morphism. Each component of the inverse image of a compact curve is compact and one-dimensional and maps onto that curve, because finite maps have zero-dimensional fibres. It is algebraic in the projective surface dP_6. Conversely its image is a compact analytic curve by properness. Away from the conductor, η is a biholomorphism, so two distinct curves can have the same image only when both lie in the conductor; its identifications are exactly the three opposite pairs computed in 11.16. This is also an explicit algebraic parametrization: the curves of dP_6 are the three exceptional lines and strict transforms of irreducible plane curves under the blow-up of the three coordinate points. Their homogeneous equations and their three multiplicities specify the strict transforms. The finite-product cover statement for S_j follows from the identical proper-image and finite-preimage argument applied to its unramified morphism $E_j \times F_j \to S_j$; curves in the product are algebraic.

□

Theorem 11.11 (All compact curves)**.** [tag 01R0]

The following descriptions give all compact irreducible curves of X_c. In W_c and S_j they are exactly the images described in 11.10. In a smooth fibre of algebraic dimension zero there are none. In a smooth fibre of algebraic dimension one they are exactly the translates of its unique elliptic kernel K_t. In an abelian fibre they are exactly the translates of elliptic subtori, together with the irreducible zero divisors of the positive Appell–Humbert theta sections described below. These descriptions, 11.7, and 11.9 constitute a classification including every smooth exceptional fibre and every compact irreducible surface.

Proof. 1. All curves are vertical by 11.9, so the list of fibre types is exhaustive. The dimension-zero and dimension-one assertions, including uniqueness, were proved in 11.7. Let now $F_t = \mathbb{C}^2/\Pi\Lambda$ be abelian. For every positive definite Hermitian form H whose imaginary part takes integer values on $\Pi\Lambda$, and every function $\alpha : \Pi\Lambda \to U(1)$ satisfying

$$\alpha(\lambda+\nu) = \alpha(\lambda)\alpha(\nu)e^{\pi i \operatorname{Im} H(\lambda,\nu)},$$

take the nonzero entire solutions $\theta : \mathbb{C}^2 \to \mathbb{C}$ of

$$\theta(\zeta+\lambda) = \alpha(\lambda)e^{\pi H(\zeta,\lambda)+\pi H(\lambda,\lambda)/2}\theta(\zeta). \tag{11.2}$$

The cocycle identity follows by multiplying the displayed factors; its integral imaginary cross-term has exponential one. It defines a holomorphic line bundle by quotienting $\mathbb{C}^2 \times \mathbb{C}$. The invariant zero set of θ descends to a compact divisor on F_t. Retain precisely those zero divisors that are reduced and irreducible. This is an exact parametrization, with all periods and multiplier equations given; multiplying θ by a nonzero scalar does not change its divisor.

2. The correspondence (Schnell, 2019, Theorem 4.1, p. 1, and equation (4.3), p. 2) proves that every holomorphic line bundle and every section on this quotient has this form. For the cohomology matrix the exact comparison is $H_E = \widetilde{E}(\,\cdot\,, J\,\cdot\,) - i\widetilde{E}$ from Definition 11.4; we choose the sign of E making H_E positive. Its imaginary part is $-\widetilde{E}$, which gives the integral semicharacter equation displayed above. The effective line-bundle criterion (Schnell, 2019, pp. 2–3) shows that a divisor class with positive Hermitian rank two gives exactly these positive theta sections. A divisor of Hermitian rank one is pulled back from its elliptic quotient and, when irreducible and reduced, is a single fibre. Thus it is a translate of an elliptic subtorus. The null lattice is computed by the explicit integer kernel in 11.7. No other rank is possible for an effective nonempty divisor, by positivity of its averaged current. This proves exhaustiveness on the abelian fibre.

3. The same equations give the curves in the product covers of S_j: compute them on its two-elliptic-curve lattice and take images under the specified finite morphism. Conversely each curve downstairs is the image of a component of its finite inverse image. In W_c the analogous procedure uses the homogeneous polynomial parametrization of the normalization in 11.10, with the three boundary pairings already identified. The maps used throughout are finite or closed embeddings, so they preserve compactness and give one-dimensional images.

□

11.3 The rank-three period quotient

Proposition 11.12 (Rank-three period quotients)**.** [tag 01QM]

At every point with $D = 0$, $T > 0$, the real-linear map $\Pi : \mathbb{R}^4 \to \mathbb{C}^2$ has rank three. Write $\mu = a_0\tau + b_0$, $a_0, b_0 \in \mathbb{R}$, and put

$$k = (1, -6a_0, -6b_0, 6a_0 \operatorname{Re}\mu - \operatorname{Re}\beta).$$

Its real kernel is $\mathbb{R}k$. The subgroup $\Pi\mathbb{Z}^4$ is discrete exactly when $k \in \mathbb{Q}^4$. In that case it is a rank-three lattice in a real hyperplane and its quotient in $\mathbb{C}^2$ is a noncompact complex Lie group. In every other case the topological quotient $\mathbb{C}^2/\Pi\mathbb{Z}^4$ is non-Hausdorff.

Proof. 1. The real matrix, with output order $(\operatorname{Re}\zeta_1, \operatorname{Im}\zeta_1, \operatorname{Re}\zeta_2, \operatorname{Im}\zeta_2)$, is

$$\begin{pmatrix} 6\operatorname{Re}\mu & \operatorname{Re}\tau & 1 & 0 \\ 6\operatorname{Im}\mu & T & 0 & 0 \\ \operatorname{Re}\beta & \operatorname{Re}\mu & 0 & 1 \\ \operatorname{Im}\beta & \operatorname{Im}\mu & 0 & 0 \end{pmatrix}.$$

Its last three columns are real-linearly independent because $T > 0$. Its determinant is TD, so at zero determinant the rank is exactly three. Substitution gives the displayed kernel, including its signs. If the image subgroup is discrete, it is a lattice of rank at most three in its real span. The homomorphism $\mathbb{Z}^4 \to \Pi\mathbb{Z}^4$ therefore has a nonzero kernel, forcing the displayed kernel vector with first coordinate one to be rational. Conversely a rational line has a primitive integral generator extendable to a basis of $\mathbb{Z}^4$. The remaining three basis images are a real basis of the image hyperplane and hence generate a discrete lattice. Its quotient leaves one unbounded real direction. Finally a non-discrete finitely generated subgroup of this

real hyperplane cannot be closed: a closed subgroup is a product of a real vector group and a discrete group, and a nonzero real vector group is uncountable. A quotient topological group is Hausdorff exactly when its identity subgroup is closed.

□

Proposition 11.13 (The quotient at the parameter boundary)**.** [tag 01QN]

The unique degenerate base point at the admissibility boundary is $t = 1$. The boundary constants $c = x + i(3/\pi)\log 6$ give

$$\ker_{\mathbb{R}} \Pi = \mathbb{R}(1, 3, -3, -x - 3/2).$$

For rational $x + 3/2 = m/n$ in lowest terms, the quotient is a finite holomorphic quotient of $E_i \times \mathbb{C}^*$, with a covering of degree four. For irrational $x + 3/2$, the closure of the period subgroup is generated by the real line $\mathbb{R}(0, 1)$ and the two periods $(i, (1-i)/2)$, $(1, 0)$. Its Hausdorff quotient is a real manifold homeomorphic to $\mathbb{R} \times (S^1)^2$.

Proof. 1. 3.2 gives $\operatorname{Im} \beta = 3/2$ at this value of c. The vector $\widehat{\gamma} + 3\widehat{u} - 3\widehat{w}$ maps to $(0, x + 3/2)$, while $\widehat{\delta}$ maps to $(0, 1)$. For $x + 3/2 = m/n$ in lowest terms, change coordinates by $(\zeta_1, \zeta_2) \mapsto (\zeta_1, \zeta_2 + \zeta_1/2)$. The vectors $2\widehat{u} - \widehat{\delta}$ and $2\widehat{w} - \widehat{\delta}$ then have periods $(2i, 0)$ and $(2, 0)$. Bézout's identity applied to $\widehat{\gamma} + 3\widehat{u} - 3\widehat{w}$ and $\widehat{\delta}$ gives the period $(0, 1/n)$. These three periods generate a sublattice of index four in the rank-three image: its two remaining generators are represented by $(i, 1/2)$ and $(1, 1/2)$, each of order two modulo this sublattice. The product identification is the explicit map $([\zeta_1], [\zeta_2]) \mapsto ([\zeta_1/2], e^{2\pi i n \zeta_2})$ from $\mathbb{C}^2$ modulo these three periods to $E_i \times \mathbb{C}^*$. Lattice inclusion gives the claimed degree-four map. For irrational $x + 3/2$, its additive subgroup with 1 is dense in the indicated real line. The other two displayed periods are independent modulo that line. This proves the closure and the quotient description. By 6.6 this is the only degenerate torus cover at the admissibility boundary; every other finite-base period matrix retains real rank four.

□

Remark 11.14 (The local boundary obstruction)**.** [tag 01QO]

The only degenerate base fibre at the parameter boundary lies over $t = 1$. By 11.13, its rank-three period quotient is noncompact for rational $x + 3/2$ and non-Hausdorff for irrational $x + 3/2$; neither quotient can be a fibre of a proper family of complex spaces. Proposition 11.15 proves that the elliptic Albanese model also has no holomorphic Weierstrass continuation across this boundary. The obstruction to a general proper holomorphic extension is proved in lemma 19.3 and theorem 19.4 by recovering the intrinsic divisor.

Proposition 11.15 (The natural boundary of the Albanese invariant)**.** [tag 01QP]

On its half-plane of definition, the j-invariant of $\operatorname{Alb}(S_2)$ has no meromorphic continuation across any point of the line $\operatorname{Im} c = (3/\pi)\log 6$. In particular the rank-three quotient at $t = 1$ cannot be replaced by a Weierstrass family of elliptic Albanese curves whose coefficients are holomorphic in the same parameter across that line.

Proof. 1. The explicit Albanese lattice in proposition 7.6 is $\mathbb{Z} + (\kappa_2/2)\mathbb{Z}$, with $\kappa_2 = c + 3/2 - i(3/\pi)\log 6$ by 3.2. A positive-imaginary modulus is $z = -\kappa_2/2$, so the invariant is $j(z)$. At a reduced rational $r = a/b$ on the real boundary, choose an integral determinant-one modular matrix with bottom row $(b, -a)$. Its value at $r + iy$ has imaginary part $1/(b^2 y)$. The convergent Fourier expansion $j = q^{-1} + 744 + O(q)$ therefore gives growth of order $\exp(2\pi/(b^2 y))$ as $y \downarrow 0$. A meromorphic function at r is bounded by a fixed power of $1/y$, so continuation there is impossible. A meromorphic continuation near an irrational real point would also continue the same function near some rational real point in that neighbourhood, which has just been excluded. The affine change from c to z transfers this obstruction to the stated line.

A Weierstrass family with holomorphic coefficients has a meromorphic j-invariant, expressed as the quotient of its cubic invariants, so it cannot supply the indicated continuation. This obstruction applies to continuation of the displayed elliptic Albanese family.

□

11.4 Classification of the non-normal cusp surface

Theorem 11.16 (Gluing classification of the cusp surface)**.** [tag 01QQ]

For the period normalization of proposition 3.2, the three opposite-side identification constants of W_c, in the characters specified below, are

$$(g_1, g_2, g_3) = (-a^{-1}, a^6, -a^{-7}e^{-2\pi ic}).$$

Two such compact complex surfaces are biholomorphic exactly when $c' - c \in \mathbb{Z}$. The vertical $\mathbb{C}^*$-action inherited from translation in the $\widehat{\delta}$-direction fixes exactly the double curve represented by the pair of edges of direction e_2.

Proof. 1. Keep the lattice of toric cocharacters $\mathbb{Z}^2$, with $e_1 = \widehat{w}$, $e_2 = \widehat{\delta}$, and cusp logarithm $s = (2\pi i)^{-1} \log t_c$. The cusp period matrix is $[sB_0 + C(t_c) \mid I]$, where

$$B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad C(0) = \begin{pmatrix} 6\mu_0 & h_0 \\ b_0 - h_0 & \mu_0 \end{pmatrix}, \quad b_0 = c.$$

For $v = e_1, e_2, e_1 - e_2$, the gluing map from the side of direction v to its opposite is the restriction of the toric action with lattice label $B_0 v$. A character row r with $rv = 0$ pulls back by this map to itself multiplied by $\exp(2\pi i\, rC(0)B_0 v)$: the remaining power of t_c is $rB_0^2 v = -rv = 0$. Taking respectively $r = (0,1), (1,0), (1,1)$ gives exponents $-\mu_0, 6\mu_0, -7\mu_0 - c$. This proves the three constants, including the cancellation of h_0. These are morphisms of the whole projective boundary lines, since multiplication by a nonzero constant extends over both endpoints. The restrictions agree at both triple points, as the ambient toric action supplies the same vertex identifications.

2. A biholomorphism of the two surfaces lifts uniquely to a biholomorphism of their normalizations: integral closure of the reduced local rings is functorial under an isomorphism. It preserves the inverse image of the singular locus, the hexagon on dP_6. Its permutation of the six components preserves their cyclic incidence, hence lies in the order-twelve dihedral group. After composition with the corresponding toric lattice automorphism, it fixes each boundary component. Each torus character has its divisor supported on those components, so its pullback divided by itself has divisor zero on the compact normalization and is constant. Such an automorphism is therefore multiplication by an element of $(\mathbb{C}^*)^2$. These torus multipliers cancel in the opposite-side comparison because the same character is used on both sides.

3. The remaining six effective actions on the triple, with the central lattice automorphism $-I$ acting trivially, are described by the equations ${g'_{k_j}}^{\epsilon_j} = g_j$, where

$$((k_1,\epsilon_1),(k_2,\epsilon_2),(k_3,\epsilon_3)) \in \begin{cases} ((1,+),(2,+),(3,+)), & ((1,-),(3,+),(2,+)), \\ ((2,-),(3,-),(1,+)), & ((2,+),(1,+),(3,-)), \\ ((3,-),(2,-),(1,-)), & ((3,+),(1,-),(2,-)). \end{cases}$$

Enumerate the 36 ordered pairs of ray images of an integral basis, retain those with determinant ± 1, and require preservation of the six rays. This gives the 12 fan automorphisms. For $Av_j = \alpha v_k$ and $r_k A = \beta r_j$, the transported equation is $(g'_k)^{\alpha\beta} = g_j$, yielding the six displayed actions. Every nonidentity action requires one of $g_1^{-1} = g_1$, $g_2^{-1} = g_2$, or equality of a positive

real number $g_2^{\pm 1}$ with a negative real number $g_1^{\pm 1}$. All fail for $a > 1$. Therefore $g_3' = g_3$, precisely $e^{-2\pi i c'} = e^{-2\pi i c}$. Conversely equality of the three constants gives the identity gluing morphism on the normalization and hence a biholomorphism of the quotients. This proves the unmarked assertion.

4. For the action of the subtorus with cocharacter e_2, a one-dimensional toric orbit on the normalization is pointwise fixed exactly when its stabilizer line contains e_2. Among the six rays this occurs exactly for $\pm e_2$. Their images form one double curve; all zero-dimensional orbits map to its two endpoints. No dense two-dimensional orbit is fixed. The quotient action has the same fixed set: a continuous orbit cannot jump between the finitely many branches of a normalization fibre.

□

12 Cotangent direct images and the period scalar

12.1 Root-atlas cohomology and logarithmic bundles

Proposition 12.1 (Root-atlas base change and relative duality)**.** [tag 033I]

Let k, l be positive integers with $\gcd(3k, 4l) = 1$. For the root lift $\bar{f} : X \to \mathfrak{B}_{k,l} = \mathbb{P}(3k, 4l)$ of the construction in proposition 8.4, the direct images $R^q \bar{f}_* \mathcal{O}_X$, for $q = 0, 1, 2$, are locally free of ranks $1, 2, 1$. The canonical geometric-point maps

$$(R^q \bar{f}_* \mathcal{O}_X) \otimes k(b) \longrightarrow H^q(X_b, \mathcal{O}_{X_b})$$

are isomorphisms, and the trace induces the canonical isomorphism

$$(R^2 \bar{f}_* \mathcal{O}_X)^\vee \simeq \bar{f}_* \omega_{X/\mathfrak{B}_{k,l}}.$$

Both comparisons commute with the analytic root-atlas groupoid. Here the geometric fibre at a finite root point is the smooth torus on the normalized root chart.

Proof. The fibre-divisor identities $f^* p_1 = 3kS_1$ and $f^* p_2 = 4lS_2$ define the root lift. On a full root chart its atlas pullback is

$$[J_j'/\mu_{m_j}] \times_{[\Delta_{s_j}/\mu_{m_j}]} \Delta_{s_j} = J_j', \qquad (m_1, m_2) = (3k, 4l).$$

The affine quotient action is free and the family $J_j' \to \Delta_{s_j}$ is the smooth proper torus family. The atlas map is étale. At the cusp the local charts are $q = z_0 z_1 z_2$, with the two-factor and smooth charts on the other strata. They have smooth total space and pure two-dimensional Cartier fibres. A base parameter is a non-zero-divisor, so the atlas family is flat. Properness is properness of the compact torus and cusp fillings. These local comparisons make the atlas families proper flat Gorenstein morphisms of relative dimension two.

A smooth two-torus has structure-sheaf dimensions $(1, 2, 1)$. Let $\nu : P \to W$ be the normalization, $P = \mathrm{dP}_6$, let C_h, C_v, C_d be the three double-curve normalizations, and let P_+, P_- be the two triple points. Each C_e is $\mathbb{P}^1$. Write $\rho_e^0, \rho_e^1 : C_e \to P$ for its two paired boundary maps and $j_e = \nu\rho_e^0 = \nu\rho_e^1$. The sheaf sequence is

$$0 \longrightarrow \mathcal{O}_W \longrightarrow \nu_* \mathcal{O}_P \xrightarrow{\delta^0} \bigoplus_{e=h,v,d} j_{e*} \mathcal{O}_{C_e} \xrightarrow{\delta^1} \mathbb{C}_{P_+} \oplus \mathbb{C}_{P_-} \longrightarrow 0, \tag{12.1}$$

The differentials in this complex are

$$\begin{aligned} \delta^0(s)_e &= (\rho_e^1)^* s - (\rho_e^0)^* s, \\ \delta^1(t) &= \big((t_h - t_v + t_d)(P_+), (t_h - t_v + t_d)(P_-)\big). \end{aligned} \tag{12.2}$$

Its exactness is local: for $xyz=0$ it is the alternating restriction complex of the three coordinate planes, their pairwise lines and their common point. Decomposing convergent power series by the variables occurring in each monomial gives the augmented simplex complex, which is exact. The one- and two-branch cases are the same calculation with fewer planes. This verifies exactness for the boundary maps, including the torus multipliers in their gluing.

All positive-degree structure-sheaf cohomology of P, C_e vanishes. Consequently (12.1) computes cohomology by

$$\mathbb{C} \xrightarrow{0} \mathbb{C}^3 \xrightarrow{\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}} \mathbb{C}^2.$$

It gives $(h^0,h^1,h^2)(\mathcal{O}_W)=(1,2,1)$. The branch-intersection resolution is the augmented semisimplicial sheaf algebra with its restriction face maps; its local exactness makes the augmentation a multiplicative quasi-isomorphism. Its Alexander–Whitney product therefore computes the sheaf-cohomology cup product. In the ordered triangles $[0,e_1,e_2]$ and $[e_1,e_2,e_1+e_2]$ it is

$$\alpha \smile \beta = (\alpha_h\beta_d, \alpha_d\beta_h), \qquad [\alpha]\smile[\beta] = \alpha_h\beta_d - \alpha_d\beta_h \quad \text{in } \mathbb{C}^2/\mathbb{C}(1,1).$$

The cocycles $(1,1,0)$ and $(0,1,1)$ pair to one. Thus $\bigwedge^2 H^1(\mathcal{O}_W)\to H^2(\mathcal{O}_W)$ is an isomorphism. Thus the three dimensions are constant on every atlas chart, including its centre.

The constant-cohomology theorem (Bănică and Stănăşilă, 1976, Chapter III, Theorem 4.12(ii), pp. 134–135) now gives local freeness, and the canonical analytic base-change maps are those of (Bănică and Stănăşilă, 1976, Chapter III, Theorem 4.10, pp. 132–133). The hypotheses are properness, flatness of the structure sheaf, a reduced smooth base, and the constant fibre dimensions just computed. The constant-dimension formulation also appears in (Grauert, 1960, Section 7, Satz 5, pp. 63–64), with (Grauert, 1963, pp. 35–36). On a double atlas overlap, both pullbacks identify with the direct image of the same pulled-back family. Functoriality makes these identifications satisfy the triple-overlap cocycle condition, so they descend with their stabilizer representations.

For an atlas family $p:Y\to U$, proper analytic duality gives

$$Rp_*\omega_{Y/U}[2] \simeq R\mathcal{H}om_U(Rp_*\mathcal{O}_Y,\mathcal{O}_U).$$

This is the canonical trace identity of (Ramis et al., 1971, introduction, equation (1), p. 261; Section 1, pp. 264–265; Proposition 2.1, p. 266). The smooth-space dualizing complex is $K_Y[3]$ and the relative Gorenstein complex is $\omega_{Y/U}[2]$; see (Ramis and Ruget, 1970, Section 3, Theorem 1, p. 81; Proposition 2 and the complete-intersection calculation, p. 91). The locally free cohomology sheaves already obtained have no positive sheaf Ext against $\mathcal{O}_U$. Degree -2 of the trace identity is therefore $p_*\omega_{Y/U}\simeq (R^2p_*\mathcal{O}_Y)^\vee$. The trace commutes with the local biholomorphisms on the atlas and has the same cocycle compatibility. It consequently descends to the asserted isomorphism on $\mathfrak{B}_{k,l}$. □

Proposition 12.2 (Vertical and logarithmic tangent bundles)**.** [tag 01SN]

There is a canonical representable proper flat holomorphic map $\bar f: X\to\mathcal{B}$ with $f=\rho\bar f$. If

$$\mathcal{T}=T_X(-\log W), \qquad \mathcal{V}=\ker(\mathcal{T}\to \bar f^{*}T_{\mathcal{B}}(-p_0)),$$

then, for rank-two vector bundles E,G on $\mathcal{B}$,

$$\mathcal{V}=\bar f^{*}E, \quad E=\mathcal{O}\oplus\mathcal{O}(-1), \qquad G=R^1\bar f_*\mathcal{O}_X=\mathcal{O}\oplus\mathcal{O}(-1), \tag{12.3}$$

$$R^0\bar f_*\mathcal{O}_X=\mathcal{O}, \qquad R^2\bar f_*\mathcal{O}_X=\mathcal{O}(-1). \tag{12.4}$$

and the following is an exact sequence of locally free sheaves on X:

$$0\longrightarrow \bar f^{*}E\longrightarrow\mathcal{T}\longrightarrow \bar f^{*}\mathcal{O}(-5)\longrightarrow 0. \tag{12.5}$$

The trivial summands in E,G are generated by the vertical field ξ and the constant integral cocycle γ, respectively.

Proof. These bundles and maps are the standard-data calculation of (Du et al., 2026a, Lemma 6.2, pp. 74–75; Lemma 6.7, p. 78; Proposition 6.6, p. 77; Proposition 6.10, pp. 79–80).

1. The fibre-divisor root objects and their proper flat atlas comparison are those of proposition 12.1 with $k = l = 1$.

2. On a smooth torus chart a vertical field is a holomorphic linear combination of the translation fields. Its coefficients are constant on each compact fibre and therefore depend only on the base. The vertical derivative of an affine transition is its linear part R_g; the translating part, including the logarithmic regluing, has vertical derivative the identity. At the cusp the two fields $x_1\partial_{x_1}, x_2\partial_{x_2}$ extend as logarithmic vector fields in every toric chart. The deck maps multiply each x_i by a base-dependent unit and a power of the base parameter. Their vertical logarithmic derivative fixes both coefficient directions. The exponential overlap sends ∂_{ζ_i} to $2\pi i x_i \partial_{x_i}$. These transitions depend only on the base and define E; they prove the bundle identity $\mathcal{V} = \bar{f}^* E$.

3. At $t = xy$, the kernel of df is freely generated by $x\partial_x - y\partial_y$ and ∂_z. At $t = xyz$ it is generated by $x\partial_x - y\partial_y$ and $x\partial_x - z\partial_z$. Indeed the equation $yza + xzb + xyc = 0$ in the regular local ring forces $x \mid a, y \mid b, z \mid c$; after division the only relation is that the three coefficients sum to zero. The logarithmic map sends $\sum a_i z_i \partial_{z_i}$ to $(\sum a_i) t\partial_t$ and is surjective. At a finite root point it is the ordinary relative tangent sequence of the smooth map to the root coordinate s. These comparisons prove (12.5), since $T_{\mathcal{B}}(-p_0) = \mathcal{O}(7 - 12)$.

4. The invariant vector e_2 defines a subbundle $\mathcal{O}\xi \subset E$. It remains nonzero as a vector of the logarithmic vertical bundle at the cusp, even where its image in the ordinary tangent fibre is zero. The quotient line has automorphy factors $-1/\tau$ and $1/\tau$ under the order-three and order-four generators. The modular discriminant has weight twelve and no zero in the upper half-plane. Therefore its reciprocal is a meromorphic section of the twelfth tensor power of this quotient line. On the cusp chart $\tau = (\log q)/(2\pi i) + h(q)$ with h holomorphic, so the product expansion of the discriminant is q times a holomorphic unit. There are no other zeros or poles, including on the finite root charts. The quotient line thus has twelfth power $\mathcal{O}(-p_0) = \mathcal{O}(-12)$, and the torsion-free Picard group in 2.5 makes it $\mathcal{O}(-1)$. Its extension by $\mathcal{O}$ splits because $H^1(\mathcal{B}, \mathcal{O}(1)) = 0$.

5. Proposition 12.1 gives local freeness, geometric-fibre comparison and trace duality on the root atlas. The determinant of (12.5), using $W = \bar{f}^* p_0$, gives

$$\omega_{X/\mathcal{B}} = \bar{f}^*(\det E)^{-1}. \tag{12.6}$$

   Relative Serre duality, applied on that same atlas and descended by its trace pairing, gives $R^2\bar{f}_*\mathcal{O}_X = \det E = \mathcal{O}(-1)$. Cup product is an isomorphism $\det G \to R^2\bar{f}_*\mathcal{O}_X$, since it is so on every geometric fibre by the preceding calculation.

6. The class γ is nowhere zero in G. On a smooth marked fibre, $H^1(\mathcal{O})$ is the quotient of constant lattice cocycles by the two period rows. If γ were a combination of those rows, evaluation on $\widehat{w}, \widehat{\delta}$ would make both coefficients zero, contradicting $\gamma(\widehat{\gamma}) = 1$. The same argument applies on each finite root cover, and affine translations act trivially on these classes. At the cusp γ kills $\Lambda_{\mathrm{tor}} = \mathbb{Z}\langle \widehat{w}, \widehat{\delta} \rangle$. Local sections of the free toric quotient with transitions $s_i = \Psi_{\lambda_{ij}} s_j$ give the holomorphic constant Čech cocycle $\gamma(\lambda_{ij})$ on the whole neighbourhood. For $B_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, the edges $e_1, e_2, e_2 - e_1$ have deck vectors $(0,1), (-1,0), (-1,-1)$, so its central values are $(0,-1,-1)$. Their cup product with $(1,0,-1)$ is one by (12.1) and its displayed product. On the punctured overlap the exponential map kills exactly Λ_{tor}, giving the same full-lattice cocycle. The Leray edge map over the Stein cusp disc is an isomorphism, since its positive coherent cohomology vanishes. This gives a holomorphic nonvanishing extension of $[\gamma]$. Thus

$\mathcal{O}\gamma \subset G$ is a subbundle; its quotient is $\mathcal{O}(-1)$ by the determinant just computed. Once again $H^1(\mathcal{O}(1)) = 0$ splits the extension.

□

Definition 12.3 (Logarithmic cotangent notation)**.** [tag PI17]

For an admissible marked member, set

$$\omega_f = K_X \otimes f^* K_B^{-1}, \qquad \mathcal{L} = \Omega^1_B(\log \Sigma) = \mathcal{O}_B(1), \qquad \mathcal{A} = \Omega^1_X(\log D), \qquad \mathcal{E} = \mathcal{A}/f^*\mathcal{L}.$$

All coherent cohomology groups have complex coefficients. The symbols γ, u in $R^1 f_* \mathcal{O}_X$ denote the images of the marked integral classes under the constant-to-coherent cohomology map. Representatives in the Dolbeault resolution are specified separately by their displayed forms.

Cited result 12.4 (Structure-sheaf direct images)**.** [tag 01RQ]

This is the structure-sheaf calculation of Claude and Alpöge: (Claude and L. Alpöge, 2026, Lemma 9.12, pp. 72–74; Proposition 9.13, pp. 74–76, and the base-change clarification on p. 92). The direct images $R^q f_* \mathcal{O}_X$, for $q = 0, 1, 2$, are $\mathcal{O}$, $G = \mathcal{O}\gamma \oplus \mathcal{O}(-1)$, and $\mathcal{O}(-1)$. The inclusion $\mathcal{O}\gamma \subset G$ is a subbundle. At the cusp, (γ, u) is a holomorphic frame, and cup product identifies $\bigwedge^2 G$ with $R^2 f_* \mathcal{O}_X$. On the normalized root family, the cohomology pullback by the cyclic generator has characters $(1, \zeta^{-a})$, where $(m, a) = (3, 2)$ or $(4, 1)$ and $\zeta = e^{-2\pi i/m}$. These assertions concern full scheme fibres, including $3S_1$ and $4S_2$. At the cusp, the natural map $H^1(W; \mathbb{C}) \to H^1(W, \mathcal{O}_W)$ is an isomorphism. In degree two the comparison is surjective and is injective on $\ker[H^2(W; \mathbb{C}) \to H^2(\mathrm{dP}_6; \mathbb{C})]$. The cup product $\bigwedge^2 H^1(W, \mathcal{O}_W) \to H^2(W, \mathcal{O}_W)$ is an isomorphism. Every base-change map $(R^q f_* \mathcal{O}_X) \otimes k(b) \to H^q(X_b, \mathcal{O}_{X_b})$ is an isomorphism, including at the nonreduced fibres.

Proposition 12.5 (The root-to-coarse comparison maps)**.** [tag PI10] *Write $G_B = R^1 f_* \mathcal{O}_X$ and $G_{\mathcal{B}} = R^1 \bar{f}_* \mathcal{O}_X$. Then*

$$G_B = \rho_* G_{\mathcal{B}}.$$

On a root disc $t = s^m$, take a frame (e_1, e_2) in which the covariant action is $(1, \zeta^a)$, where $(m, a) = (3, 2)$ or $(4, 1)$ and $\zeta = e^{-2\pi i/m}$. The invariant frame is $(e_1, s^a e_2)$, and the adjunction map is

$$\rho^* G_B \longrightarrow G_{\mathcal{B}}, \qquad (u, v) \longmapsto (u, s^a v).$$

It is the map induced by normalization after the root base change. The covariant action on cohomology is $(g^{-1})^* : G_{\mathcal{B},s} \to G_{\mathcal{B},gs}$; the pullback $g^* : G_{\mathcal{B},gs} \to G_{\mathcal{B},s}$ has the inverse eigenvalues $(1, \zeta^{-a})$.

For the relative logarithmic cotangent bundle of definition 2.4, the evaluation pairing gives

$$\mathcal{E} = \bar{f}^*(E^\vee),$$

where E is the vertical root bundle in proposition 12.2.

Proof. The factorization $f = \rho \bar{f}$ and exactness of finite-group invariants give $R^q f_* \mathcal{O}_X = \rho_* R^q \bar{f}_* \mathcal{O}_X$. In the chosen root frame, invariance of $h(s) e_2$ is exactly $h(\zeta s) = \zeta^a h(s)$. Its convergent series is therefore $s^a H(s^m)$. This gives both the invariant frame and the displayed adjunction matrix. The normalization map agrees with this map on the punctured disc by its marked torus identification, hence on the whole disc by holomorphic continuation. At $s = 0$ its second component is zero; this specifies the comparison with the normalized torus fibre while the coarse base-change theorem uses the full scheme fibre. Inverting the covariant action gives the displayed pullback eigenvalues.

On an ordinary chart the pairing is $\langle d\zeta_i, \partial_{\zeta_j} \rangle = \delta_{ij}$. At the cusp the exponential map sends ∂_{ζ_i} to $2\pi i x_i \partial_{x_i}$ and $d\zeta_i$ modulo the base differential to $(2\pi i)^{-1} d \log x_i$, preserving that pairing.

On a finite root chart, $dt/t = m\,ds/s$ removes exactly the base logarithmic direction from the quotient defining $\mathcal{E}$. Thus these local perfect pairings agree under the transition maps and give $\mathcal{E} = \bar{f}^*(E^\vee)$. □

Proposition 12.6 (Local freeness and the determinant line)**.** [tag 01RR]

Both $\mathcal{A}$ and $\mathcal{E}$ are locally free, of ranks three and two. The vertical vector $\xi = \partial/\partial\zeta_2$ is nowhere zero as a section of $\mathcal{E}^\vee$. There is an exact sequence of holomorphic vector bundles

$$0 \longrightarrow N \longrightarrow \mathcal{E} \xrightarrow{\langle\cdot,\xi\rangle} \mathcal{O}_X \longrightarrow 0, \qquad N = \det \mathcal{E} = \omega_f(-2S_1 - 3S_2). \tag{12.7}$$

Proof. 1. At a smooth point of a reduced cusp branch the assertion is the usual coordinate calculation. At a double point write $t_c = xy$ with third coordinate z; then $\mathcal{A} = R\,d\log x \oplus R\,d\log y \oplus R\,dz$ and its base subline is $R(d\log x + d\log y)$. At either triple point write $t_c = xyz$; then $\mathcal{A}$ has the three logarithmic coordinate basis vectors and its base subline is their sum. In each case that subline is a direct summand. These calculations apply on every one of the three double curves, including their endpoints.

2. At S_j we use the specified free cyclic quotient $q : J_j' \to N_j$, which is finite and étale on total spaces, and $t_j = s^{m_j}$ on the base. On J_j' the logarithmic conormal is $m_j\,ds/s$, and its coefficient is a unit over $\mathbb{C}$. It is again a direct summand of the logarithmic cotangent bundle. The comparison $q^*\mathcal{A} = \Omega^1_{J_j'}(\log(s=0))$ identifies the quotient bundles and their exact sequence; faithful flat descent proves the claim downstairs.

3. On the marked torus family ξ is the constant second vertical vector. The linear monodromies fix it, and translations preserve it. On a cusp toric chart it is a nonzero constant vector in the relative logarithmic tangent lattice of rank two. A toric coordinate change $x_i \mapsto c_i(t_c)t_c^{b_i}x_i$ changes $d\log x_i$ by a multiple of $d\log t_c$, so this constant relative vector descends and is nowhere zero in that bundle, including where its image in the ordinary tangent fibre vanishes. Hence evaluation is surjective and its kernel is the determinant line. Taking determinants of $0 \to f^*\mathcal{L} \to \mathcal{A} \to \mathcal{E} \to 0$ gives

$$\det \mathcal{E} = K_X(D) \otimes f^*\mathcal{L}^{-1} = \omega_f(D - f^*\Sigma) = \omega_f(-2S_1 - 3S_2),$$

which proves (12.7).

□

Proposition 12.7 (The logarithmic direct images by vertical twisting and duality)**.** [tag 01RS]

One has, as holomorphic vector bundles on B,

q	0	1	2
$R^q f_*\mathcal{E}$	$\mathcal{O} \oplus \mathcal{O}(-1)$	$\mathcal{O}^2 \oplus \mathcal{O}(-1)^2$	$\mathcal{O} \oplus \mathcal{O}(-1).$

They commute with base change to every full scheme fibre.

Proof. 1. Put $H = 2S_1 + 3S_2$. Sections of the line bundle $\mathcal{O}_X(H)$ on the cover at p_j allow poles of order at most $m_j - 1$ in s. A fixed character has invariant exponent zero and gains no pole, whereas the variable character ζ^{-a} has least allowed invariant exponent $a - m$. Thus the inclusion $R^1 f_*\mathcal{O} \to R^1 f_*\mathcal{O}(H)$ is the identity on $\mathcal{O}\gamma$ and changes $s^a b_1$ to $s^{a-m}b_1 = t_j^{-1}s^a b_1$ on its quotient. It makes exactly one upper modification at each finite point. The same computation applies to R^2. No modification is made at the cusp or a regular fibre. The result is

$$f_*\mathcal{O}(H) = \mathcal{O}, \qquad R^1 f_*\mathcal{O}(H) = \mathcal{O} \oplus \mathcal{O}(1), \qquad R^2 f_*\mathcal{O}(H) = \mathcal{O}(1). \tag{12.8}$$

Here the middle extension splits because $H^1(\mathcal{O}(-1)) = 0$. The local invariant modules are free. Tensoring them with $\mathbb{C}\{t_j\}/(t_j)$ gives invariants of $s^{-(m_j-1)}\mathbb{C}\{s\}/(s^m)$, with the indicated

surviving character basis vectors. The Reynolds idempotent is linear over the base, so this calculation also proves the comparison with the nonreduced fibre. At the cusp the line bundle is trivial. Cohomology and base change, applied successively from the top degree, proves the claimed local freeness and comparison for (12.8).

2. The relative duality map we use is the cup product followed by the analytic trace, giving a morphism of bundles
$$R^q f_*(\omega_f \otimes \mathcal{O}(-H)) \longrightarrow (R^{2-q} f_* \mathcal{O}(H))^\vee.$$
Every fibre is a compact Gorenstein surface: it is the Cartier fibre in a smooth threefold. Serre duality on that scheme makes the map an isomorphism on its residue field, also for $m_j S_j$. The constant dimensions just computed imply local freeness on both sides, so Nakayama's lemma makes the bundle map an isomorphism. Therefore
$$(f_*N, R^1 f_* N, R^2 f_* N) = (\mathcal{O}(-1), \mathcal{O} \oplus \mathcal{O}(-1), \mathcal{O}). \tag{12.9}$$

3. The connecting maps of (12.7) are zero on a regular fibre: its relative cotangent is a trivial bundle and evaluation on a constant nonzero vector splits. Source and target direct images in (12.9) and cited result 12.4 are vector bundles, so these holomorphic connecting maps, zero on a dense open set, are zero everywhere. We obtain extensions in each degree of the listed $R^q N$ by $R^q \mathcal{O}$. Every resulting pair of line-bundle degrees is zero or minus one; their extension groups are $H^1(\mathcal{O}(-1)), H^1(\mathcal{O}), H^1(\mathcal{O}(1))$, all zero. These extensions give exactly the three asserted bundles. Applying the already proved fibre comparisons to the short exact sequences proves base change for $R^q \mathcal{E}$ as well.

□

12.2 The elliptic logarithm and the period determinant

Definition 12.8 (The finite local coordinates)**.** [tag PI07] At the order-four point choose $t-1 = s^4$ so that on the positive $t > 1$ arc toward the cusp, $\tau = iy$ with $y > 1$, and
$$\tau = i + \frac{i}{\pi\sqrt{E_4(i)}} s^2 + O(s^4), \qquad \operatorname{Im} \mu_s(0) > 0. \tag{12.10}$$

Replacing s by $i^j s$ changes $\mu_s(0)$ by i^{-j}. For any other local coordinate z the derivative is transported by $\mu_z = \mu_s s_z$. At the order-three point use $r^3 = t$, with $r > 0$ on the arc $0 < t < 1$ along which τ travels from ρ to i on the unit circle.

Proposition 12.9 (The elliptic logarithm at the finite points)**.** [tag 01RW] *Let $[\mu]$ be the generator identified in Proposition 2.11 with one of the two sections $P_* = (0, \sqrt{2}t^2(t-1))$ and $-P_*$ of $y^2 = x^3 - 3t^3(t-1)x + 2t^4(t-1)^2$. Along the branch in (12.10), put $x_0 = 1 - 1/t$. Then*
$$\mu = \frac{1-\tau}{2} + \frac{i\sqrt{6}}{3\pi} \frac{x_0^{1/4}\, {}_3F_2(1/3, 2/3, 1; 3/4, 5/4; x_0)}{{}_2F_1(1/12, 5/12; 1; 1-x_0)}. \tag{12.11}$$

The difference $\mu - (2-\tau)/3$ has exact order two at the order-three point in the coordinate $r^3 = t$.

Proof. The pointed Weierstrass isomorphism in Proposition 2.11 and the section calculation in Proposition 2.11 give $\wp(\mu, \tau) = 0$. In (Duke and Imamoğlu, 2008, Theorem 1, equation (5), author preprint p. 2), write z_0 for the displayed zero. The exact affine map $\mu = 1 - z_0$ preserves the zero identity, since $\wp(1 - z_0, \tau) = \wp(z_0, \tau)$, and gives (12.11). At $t = 1 + s^4$ its initial value is $(1-i)/2$. The coefficient of s is
$$\frac{i\sqrt{6}}{3\pi\, {}_2F_1(1/12, 5/12; 1; 1)} = \frac{i\sqrt{6}}{3\pi E_4(i)^{1/4}}.$$

It has positive imaginary part and therefore selects the branch of (12.10). Continuation gives the chosen bounded lift at the cusp.

The chord-and-tangent formulas applied to P_* give

$$x(2P_*) = \frac{9t^2}{8}, \qquad y(2P_*) = \frac{t^2(32-5t)}{16\sqrt{2}}, \qquad x(3P_*) = \frac{16}{81}(t-1)(5t-32).$$

The last expression tends to $512/81$ at $t = 0$. Let $F = f_{\mathrm{sc}}$ be the elliptic scale of Proposition 2.11; it has exact order two in r. For $\nu = \mu - (2-\tau)/3$, the point $3P$ has local logarithm 3ν modulo its period, and $x(3P) = F^2\wp(3\nu,\tau)$. The expansion $\wp(3\nu,\tau) = (3\nu)^{-2} + O(\nu^2)$ therefore proves that ν has exact order two. Its leading coefficient is $\pm 3F_2/(16\sqrt{2})$, where $F_2 = \lim_{r\to 0} F/r^2$; the sign is fixed by the same logarithm branch. □

Proposition 12.10 (The complete raw period determinant)**.** [tag 01RU]

On the marked upper half-plane,

$$6\mu_z^2 - \tau_z\beta_z = \tau_z^2\frac{E_4(\tau)^2}{E_6(\tau)}. \tag{12.12}$$

The right side extends holomorphically through the apparent poles. Its exact orders are two at z_1 and zero at z_2; it has no other zero away from their translates. The logarithmic connecting map $\kappa_0 : f_*\mathcal{E} \to \mathcal{L}\otimes G$ has divisor $\operatorname{divisor}(\det\kappa_0) = p_1 + p_2$.

Proof. 1. Consider the holomorphic quadratic differential $\mathscr{Q} = 6(d\mu)^2 - d\tau\, d\beta$. Direct differentiation of the two affine transformation laws in (Claude and L. Alpöge, 2026, Definition 3.1, p. 14) gives

$$g_j^*\mathscr{Q} = \tau^{-2}\mathscr{Q} \quad (j = 1, 2).$$

In particular g_2^2 fixes τ and sends

$$\mu \longmapsto 1 - \tau - \mu, \qquad \beta \longmapsto \beta + 6\tau + 12\mu - 6,$$

which leaves $\mathscr{Q}$ exactly unchanged. The kernel of $\Delta = \mathbb{Z}/3 * \mathbb{Z}/4 \to \mathbb{Z}/3 * \mathbb{Z}/2 = \mathrm{PSL}_2(\mathbb{Z})$ is the normal subgroup generated by g_2^2. The squared multiplier of $\mathscr{Q}$ is the ordinary modular multiplier $(c\tau + d)^{-2}$, so it is one on every kernel element. Two generic points of the marked cover with the same τ have the same t and differ by an element of Δ; its modular image fixes a generic point, hence is the identity. It follows that $k = \mathscr{Q}/(d\tau)^2$ is a single-valued meromorphic function of τ, transforming as a modular form of weight two.

2. At z_1, write a linearizing parameter r with $g_1 r = \zeta r$, $\zeta = e^{-2\pi i/3}$. If $\mathscr{Q} = K_r dr^2$ has first nonzero coefficient of order n, its transformation law at the fixed point reads $\zeta^{n+2} = \rho^{-2} = \zeta$. Thus $n \equiv 2 \pmod 3$, and K_r vanishes to order at least two. The map τ is unramified there by (Claude and L. Alpöge, 2026, Theorem 3.4(i), p. 14; Proposition 3.6, pp. 16–17). At z_2, $\tau - i$ has order two and K_s is holomorphic; therefore k has at most a simple pole at i. Away from these points τ is unramified. At the cusp μ and $\beta + \tau$ are holomorphic in q, so $k = 1 + O(q)$.

3. The modular function kE_6/E_4^2 is therefore holomorphic at ρ, at i, and at the cusp, as well as everywhere else. The usual zero statements for E_4, E_6 (simple at ρ, i, respectively) are the modular facts used in (Claude and L. Alpöge, 2026, Section 3.1, pp. 15–17). This function descends to the compact modular curve $X(1) = \mathbb{P}^1$, so it is constant. Its cusp value is one. This proves (12.12) and the exact two local orders.

4. For the coarse lattices, choose cover eigenframes e_0, e_1 of $\mathcal{E}$ with characters $1, \zeta^a$, and b_0, b_1 of G with characters $1, \zeta^{-a}$. Their invariant bases are $e_0, s^{m-a}e_1$ and $b_0, s^a b_1$. The logarithmic

base frame is $\ell = dt_j/t_j = m\,ds/s$. Changing determinants from these frames to the marked ones gives

$$\det \kappa_0 = \epsilon(t_j) m^{-2} s^{m+2-2a} K_s, \qquad \epsilon(0) \neq 0.$$

This equality follows by taking the two factors s/m from replacing ds by ℓ, the factor s^{m-a} in the source, and the factor s^{-a} in the target. Thus at $(m,a) = (3,2)$ the total order in s is $1+2=3$, and at $(4,1)$ it is $4+0=4$. Both coarse zeros are simple. At the cusp, the frame change has no elementary divisor and $dZ/d\tau$ tends to $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, with determinant one. Formula (12.12) excludes any further zero, proving the asserted divisor.

□

Proposition 12.11 (The finite period coefficients)**.** [tag 01RV]

In the coordinate (12.10)*,*

$$\mu_s(0) = i\,C_4, \qquad C_4 = \frac{\sqrt{2/3}}{\pi E_4(i)^{1/4}} = \frac{4\sqrt{\pi}}{3^{3/4}\Gamma(1/4)^2} > 0. \tag{12.13}$$

For $r^3 = t$, with $r > 0$ on the arc $0 < t < 1$ and τ travelling from ρ to i along the unit circle, set

$$\lambda = \frac{3e^{5\pi i/6}}{2\pi E_6(\rho)^{1/3}}.$$

Then $\tau = \rho + \lambda r + O(r^2)$ and

$$K_r = -\frac{3i\lambda}{2\pi} r^2 + O(r^3).$$

The exact expressions have the rational enclosures

$$\begin{aligned} &0.2366093140807852 < C_4 < 0.2366093140807853, \\ &0.0801022110875382 < \operatorname{Re} K_2 < 0.0801022110875383, \\ &0.1387410994022233 < \operatorname{Im} K_2 < 0.1387410994022234. \end{aligned}$$

Proof. 1. Ramanujan's differential relations give $E_6'(i) = -\pi i E_4(i)^2$ and $E_4'(\rho) = -(2\pi i/3)E_6(\rho)$. Using $j = 1728E_4^3/(E_4^3 - E_6^2)$, we obtain

$$j(\tau) - 1728 = -1728\pi^2 E_4(i)(\tau - i)^2 + O((\tau - i)^3).$$

Since $j = 1728(1+s^4)$, the chosen branch is precisely (12.10). Formula (12.12) now gives

$$K_s(0) = -\frac{4}{\pi^2\sqrt{E_4(i)}} = 6\mu_s(0)^2,$$

because $\tau_s(0) = 0$. This proves nonvanishing and the first formula in (12.13); the sign is fixed by the stated coordinate convention. Similarly $j(\tau) = -512i\pi^3 E_6(\rho)(\tau-\rho)^3 + \cdots$. This gives the stated λ, and substitution into (12.12) gives the coefficient $-3i\lambda/(2\pi)$.

2. The real period also has the integral expression

$$\Omega_R = \frac{\sqrt{2}}{3^{1/4}} \int_0^{\pi/2} (1 - \tfrac{1}{2}\sin^2\theta)^{-1/2}\, d\theta, \qquad C_4 = \frac{\sqrt{2}}{3\Omega_R}.$$

It follows by the elliptic-integral substitution for the three roots $\sqrt{3}, 0, -\sqrt{3}$ of $Y^2 = X^3 - 3X$ and the differential $dX/(2Y)$. The beta-integral evaluation gives the gamma expression in (12.13).

For the rational enclosures, evaluate the divisor-sum series of $E_4(i)$ and $E_6(\rho)$ by interval arithmetic with $N = 100$ terms. The elementary inequalities $\sigma_3(n) \le n^4$, $\sigma_5(n) \le n^6$, $e^{-2\pi} < 1/500$ and $e^{-\pi\sqrt{3}} < 1/200$ give respective tail bounds

$$250(N+1)^4 500^{-N-1}, \qquad 1024(N+1)^6 200^{-N-1} \quad (N = 100).$$

The ratios of successive majorant terms are less than $1/480$ and $1/180$, respectively, which proves these tail estimates. Outward rounding in the finite sums, together with these tail bounds, gives the three stated enclosures. The preceding modular calculation proves the exact nonvanishing and the local orders.

□

Lemma 12.12 (Residues and the kernel of the logarithmic connecting map)**.** [tag PI00] *For every admissible parameter, the logarithmic connecting map $\kappa_1 : R^1 f_* \mathcal{E} \to \mathcal{L} \otimes R^2 f_* \mathcal{O} = \mathcal{O}$ is surjective. Put $\mathcal{K} = \ker \kappa_1$ and $r = \operatorname{rank} H^0(\kappa_1)$. The residue maps give*

$$f_* \Omega^1_X = \Omega^1_B, \qquad R^1 f_* \Omega^1_X = \mathcal{K}, \qquad R^2 f_* \Omega^1_X = R^2 f_* \mathcal{E}.$$

Thus the first and third bundles are isomorphic to $\mathcal{O}(-2)$ and $\mathcal{O} \oplus \mathcal{O}(-1)$, respectively; the latter splitting is proposition 12.7. Moreover,

$$h^0(\mathcal{K}) = 2 - r, \qquad h^1(\mathcal{K}) = 1 - r, \qquad h^{1,2} = 2 - r, \qquad h^{1,1} = 3 - r.$$

Proof. 1. Let $d_{ij}(s)$ be the cover Kodaira–Spencer coefficients in the eigenframes of proposition 12.10 after removing ds. The invariant coarse matrix and its cup-product row have the forms

$$[\kappa_0] = \begin{pmatrix} A & t_j B \\ C & D_0 \end{pmatrix}, \qquad [\kappa_1] = (-C, A, -D_0, B),$$

in the ordered basis $e_0 b_0, s^a e_0 b_1, s^{m-a} e_1 b_0, e_1 b_1$. Specifically $A = s d_{00}/m$, $B = s^{1-a} d_{01}/m$, $C = s^{1-a} d_{10}/m$, $D_0 = s^{m+1-2a} d_{11}/m$. These are holomorphic invariant functions; the row follows by wedging the connecting class with b_0, b_1. If the row had no unit at a finite point, every entry would vanish modulo t_j and $\det \kappa_0$ would vanish to order at least two, contradicting proposition 12.10. Away from the finite points, including the cusp, κ_0 is invertible, which gives a unit in the cup-product row. Thus κ_1 is surjective.

2. By proposition 12.10, $\operatorname{coker} \kappa_0 = \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2}$. The logarithmic conormal sequence and proposition 12.7 give

$$f_* \mathcal{A} = \mathcal{O}(1), \quad 0 \to T \to R^1 f_* \mathcal{A} \to \mathcal{K} \to 0, \quad R^2 f_* \mathcal{A} = \mathcal{O} \oplus \mathcal{O}(-1), \qquad T = \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2}.$$

Let $\nu : \widetilde{D} = \mathrm{dP}_6 \sqcup S_1 \sqcup S_2 \to D$ be normalization. The local bases in proposition 12.6 give the exact residue sequence

$$0 \to \Omega^1_X \to \mathcal{A} \to \nu_* \mathcal{O}_{\widetilde{D}} \to 0.$$

Its three residue direct images are $\mathbb{C}_{p_0} \oplus \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2}$, $\mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2}$, and zero: the normalization of W is rational, and a bielliptic surface has $(h^0, h^1, h^2)(\mathcal{O}) = (1, 1, 0)$. In degree zero the map from $\mathcal{O}(1)$ is local evaluation with the nonzero multipliers $1, 3, 4$. Its kernel is $\mathcal{O}(-2)$. Under the base logarithmic inclusion this kernel is Ω^1_B, and its map to $f_* \Omega^1_X$ is pullback.

3. At p_j, in the invariant frame $(b_0, s^a b_1)$ of G, residue of $(dt_j/t_j)(x b_0 + y s^a b_1)$ in $H^1(S_j, \mathcal{O})$ is $m_j x [b_0]$. The residue factors through the base logarithmic line on the covering chart. Naturality of the connecting map gives $\operatorname{res}_1 \kappa_0 = 0$. The determinant of κ_0 has a simple zero, so its reduction has rank one, and its image is exactly the one-dimensional kernel $\mathbb{C} s^a b_1$ of this residue. Hence $\operatorname{coker} \kappa_0 \to H^1(S_j, \mathcal{O})$ is an isomorphism. Both torsion summands cancel, giving the stated canonical identifications of $R^1 f_* \Omega^1_X$ and $R^2 f_* \Omega^1_X$.

4. The long exact cohomology sequence of

$$0 \to \mathcal{K} \to \mathcal{O}^2 \oplus \mathcal{O}(-1)^2 \xrightarrow{\kappa_1} \mathcal{O} \to 0$$

gives $h^0(\mathcal{K}) = 2 - r$ and $h^1(\mathcal{K}) = 1 - r$. Leray on the curve B and the preceding direct images then give

$$h^{1,2} = h^1(\mathcal{K}) + h^0(\mathcal{O} \oplus \mathcal{O}(-1)) = 2 - r, \qquad h^{1,1} = h^1(\mathcal{O}(-2)) + h^0(\mathcal{K}) = 3 - r.$$

□

Proposition 12.13 (The global rank and the period scalar)**.** [tag 01RX]

The map $\kappa_1 : R^1 f_* \mathcal{E} \to \mathcal{L} \otimes R^2 f_* \mathcal{O} = \mathcal{O}$ is surjective, and $H^0(\kappa_1) : \mathbb{C}^2 \to \mathbb{C}$ has rank one. Let $h = a\, d\zeta_1 + d\zeta_2$ be the unique global section of $f_* \mathcal{E}$ with $h(\xi) = 1$. In an invariant target frame normalized as below,

$$\kappa_1(h\gamma)(p_2) = -\tfrac{1}{4}\mu_s(0) = -\tfrac{i}{4}C_4.$$

The rank-one condition is equivalent to $a\tau_z + \mu_z \not\equiv 0$. The same direct-image exact sequence expresses the two Hodge numbers in terms of the global rank r as $h^{1,2} = 2 - r$ and $h^{1,1} = 3 - r$; here $r = 1$.

Proof. 1. Sheaf surjectivity is proved in lemma 12.12. The lift h exists and is unique because $f_* \mathcal{E}$ is an extension of $\mathcal{O}$ by $\mathcal{O}(-1)$. Its invariance under g_2 gives $a(g_2 z) = \tau a + \mu$, hence $a(z_2) = 1/2$. In the quotient frame (γ, u) of G the connecting map is

$$\kappa_0(h) = \frac{s}{4}((6a\mu_s + \beta_s)\gamma + (a\tau_s + \mu_s)u)\ell.$$

Choose the variable eigenframe b_1 by Reynolds averaging so that $\gamma \wedge b_1 = v(s)\gamma \wedge u$, $v(0) = 1$. Then $s\, v(s)(\gamma \wedge u)\ell$ is the invariant target frame. Cup product with γ gives at the central point $-(a\tau_s + \mu_s)/4 = -\mu_s/4$, proving rank one.

2. To check the stated equivalence, there is a second global section

$$q_E = \sigma_1 \otimes u + 6\sigma_2 \otimes \gamma \quad \text{of } R^1 f_* \mathcal{E}.$$

It is the negative of the relative $(1,1)$ image of the invariant flat class $\eta = u \wedge w + 6\gamma \wedge \delta$. It extends at the cusp because the relative logarithmic forms and γ, u extend there. At a finite point it is an invariant holomorphic tensor on the smooth cover, hence belongs to the four explicit invariant generators of lemma 12.12. Its cup-product row is zero, since the two terms are $6\mu_z$ and $-6\mu_z$. It is independent of $h\gamma$ on every regular fibre (its $\sigma_1 \otimes u$ coefficient is nonzero), so these two sections form a basis of the two-dimensional global section space. Consequently the global map has rank one precisely when its value on $h\gamma$, equivalently $a\tau_z + \mu_z$, is not identically zero. The Hodge-number formulas follow from lemma 12.12, and the nonzero scalar above gives $r = 1$.

□

12.3 Absolute cotangent direct images

Proposition 12.14 (All absolute cotangent direct images)**.** [tag 01RT]

For every admissible c, the six direct images are

q	$R^q f_* \Omega^1_X$	$R^q f_* \Omega^2_X$
0	$\mathcal{O}(-2)$	$\mathcal{O}(-2) \oplus \mathcal{O}(-1)$
1	$\mathcal{O} \oplus \mathcal{O}(-1)^2$	$\mathcal{O}(-2) \oplus \mathcal{O}(-1)^2$
2	$\mathcal{O} \oplus \mathcal{O}(-1)$	$\mathcal{O}$.

(12.14)

In particular every listed sheaf is torsion-free.

Proof. 1. By lemma 12.12, the first-column direct images are $\mathcal{O}(-2)$, $\mathcal{K} = \ker \kappa_1$, and $\mathcal{O} \oplus \mathcal{O}(-1)$. By proposition 12.13, the global map $H^0(\kappa_1)$ has rank one. Hence the bundle map $\mathcal{O}^2 \oplus \mathcal{O}(-1)^2 \to \mathcal{O}$ has a nonzero constant component on a trivial summand. Eliminating all other components by bundle automorphisms gives $\mathcal{K} = \mathcal{O} \oplus \mathcal{O}(-1)^2$.

2. Finally $\Omega^2_X = (\Omega^1_X)^\vee \otimes K_X$. Relative duality, with the locally free direct images now established, gives
$$R^q f_* \Omega^2_X = \left(R^{2-q} f_* \Omega^1_X\right)^\vee \otimes K_B.$$
Substitution of $K_B = \mathcal{O}(-2)$ gives every entry of the second column.

□

12.4 The Hodge table and its Leray generators

Proposition 12.15 (The complete Hodge table for every admissible parameter)**.** [tag 01RY]

For every $\operatorname{Im} c < -M$*, the Hodge numbers are*

$q \backslash p$	0	1	2	3
3	0	0	0	1
2	0	1	2	1
1	1	2	1	0
0	1	0	0	0.

The vanishing of holomorphic forms and the Hodge-diamond pattern with one undetermined integer are due to Claude and L. Alpöge (2026, Proposition 9.20, p. 80; Theorem 9.1(vi), p. 65). The nonzero direct-image scalar in proposition 12.13 determines that integer.

Proof. 1. Since B is a curve, coherent Leray has no differential and gives for $F = \Omega^p_X$ the exact sequences $0 \to H^1(B, R^{q-1} f_* F) \to H^q(X, F) \to H^0(B, R^q f_* F) \to 0$. Substitution of (12.14), with $h^0(\mathcal{O}) = h^1(\mathcal{O}(-2)) = 1$ and all cohomology of $\mathcal{O}(-1)$ zero, gives columns $p = 1, 2$ directly. Cited result 12.4 gives column zero, and Serre duality gives column three. Replacing β by $\beta + c$ leaves every derivative, finite eigencharacter, invariant lattice and logarithmic residue used above unchanged. The toric gluing units change, but their relative logarithmic derivatives still differ only by base logarithmic forms. Hence the same computations apply at each c in the stated open parameter domain.

□

Proposition 12.16 (Associated-graded generators and a Dolbeault lift)**.** [tag 01RZ]

The Leray associated-graded spaces for the four middle groups are

	$E_2^{1,q-1}$	$E_2^{0,q}$
$H^{1,1}$	$H^1(\mathcal{O}(-2)) = \mathbb{C}\alpha$	$\mathbb{C}\eta_{\mathrm{rel}}$
$H^{1,2}$	0	$\mathbb{C}\upsilon$
$H^{2,1}$	$\mathbb{C}\upsilon^\vee$	0
$H^{2,2}$	$\mathbb{C}\eta_{\mathrm{rel}}^\vee$	$\mathbb{C}\alpha^\vee$.

Here α is the Čech class of dt/t on the two standard affine base charts, $\eta_{\mathrm{rel}} = -q_E$ is the invariant relative class from 12.13, and υ is the image of the trace-dual of $1 \in f_*\mathcal{O}(H)$ under $R^2f_*N \to R^2f_*\mathcal{E}$ and the isomorphism $R^2f_*\Omega^1_X = R^2f_*\mathcal{E}$. The superscript $\vee$ denotes the bases dual under the relative trace and the base Serre pairing. There exists a Dolbeault class $Q \in H^1(X, \Omega^1_X)$ whose restriction to every regular fibre is $\eta = u \wedge w + 6\gamma \wedge \delta$.

Proof. 1. The isomorphism $f_*\Omega^1_X = \Omega^1_B$ in the residue sequence is the pullback map, so α gives its stated Leray class; it can be represented by a nonzero normalization of $f^*\omega_{\mathrm{FS}}$ in Dolbeault cohomology. The section η_{rel} belongs to the canonical kernel identification $R^1f_*\Omega^1_X = \ker \kappa_1$ and spans its global sections. Leray is surjective onto this space, so any preimage defines Q with the asserted fibre restriction. Its preimages form the affine space $Q + \mathbb{C}\alpha$. The injection $R^2f_*N = \mathcal{O} \to R^2f_*\mathcal{E}$ from (12.7) spans its global sections, since the quotient is $\mathcal{O}(-1)$. This specifies υ through the displayed injection. Duality used in 12.14 gives the remaining entries and their specified dual pairings. Choosing a lift of η_{rel} and of $\alpha^\vee$ gives bases of the whole middle cohomology groups. The named trace pairings determine the associated-graded bases; the choices of preimages determine the corresponding Leray splittings.

□

12.5 Ordinary relative differentials and the correction at the cusp

Proposition 12.17 (Relative differentials in the three local models)**.** [tag 01S0]

Let $R = \mathbb{C}\{x, y, z\}$. For $t = xy$,

$$\Omega^1_{R/\mathbb{C}\{t\}} = R^3/R(y, x, 0) \simeq (x, y) \oplus R,$$

and

$$\Omega^2_{R/\mathbb{C}\{t\}} \simeq (x, y) \oplus R/(x, y), \qquad \Omega^3_{R/\mathbb{C}\{t\}} \simeq R/(x, y).$$

For $t = xyz$, $\Omega^1 = R^3/R(yz, xz, xy)$ is torsion-free of rank two and is not free at the triple point. If $I = (xy, xz, yz)$, there is an exact sequence

$$0 \to T \to \Omega^2 \xrightarrow{\wedge dt} I \to 0, \qquad T = \frac{Ru \oplus Rv}{(xu,\ y(u - v),\ zv)}, \quad IT = 0,$$

and $\Omega^3 = R/I$. Finally, for $t = s^m$ in $R = \mathbb{C}\{s, y_1, y_2\}$,

$$\begin{aligned} \Omega^1 &= R^2 \oplus R/(s^{m-1})\, ds, \\ \Omega^2 &= R\, dy_1 \wedge dy_2 \oplus \left(R/(s^{m-1})\right)^2, \\ \Omega^3 &= R/(s^{m-1})\, ds \wedge dy_1 \wedge dy_2. \end{aligned}$$

Proof. 1. The degree-one presentation is the quotient by dt; higher degrees are its exterior powers. At a double point the transverse quotient $R^2/R(y, x)$ maps isomorphically to (x, y) by $(a, b) \mapsto xa - yb$. Its kernel is exactly $R(y, x)$: the equality $xa = yb$ gives $a = yh$ and $b = xh$ by relative primality. In degree two, with basis $A = dx \wedge dy, B = dx \wedge dz, C = dy \wedge dz$, the relations are $xA = yA = 0$, $yB + xC = 0$. This gives the displayed torsion summand and the ideal (x, y); wedging once more gives $R/(x, y)$.

2. At the triple point the map from ordinary to logarithmic relative one-forms sends (a, b, c) to $(xa - zc, yb - zc)$. Its kernel is precisely $R(yz, xz, xy)$: the two equations imply $c = xyh$, $a = yzh$, $b = xzh$ by relative primality of the coordinates. Thus it injects into R^2, proving torsion-freeness. Its fibre at the maximal ideal has dimension three although its rank is two, so it is not free. The same minimal-generator argument works at the double locus.

3. In degree two, wedging with dt is the row $(xy, -xz, yz)$ in the basis (A, B, C). Its syzygies are generated freely by $u = (z, y, 0)$ and $v = (0, y, x)$. To check generation, an equation $xyA - xzB + yzC = 0$ implies $x \mid C$, say $C = xC_1$; then $yA = z(B - yC_1)$ implies $A = zA_1$, $B = y(A_1 + C_1)$. Thus the syzygy is $A_1u + C_1v$. The three relations from $dt \wedge (dx, dy, dz)$ are $-xu, y(u - v), zv$, proving the stated presentation of the kernel. The ideal I kills it by those relations. Conversely the quotient I is torsion-free, so this kernel is exactly the torsion of Ω^2. The presentation immediately gives $\Omega^3 = R/I$. For $t = s^m$ the sole degree-one relation is $ms^{m-1}ds = 0$; m is a unit and taking exterior powers gives the three formulas.

□

Proposition 12.18 (The support of relative torsion)**.** [tag 01S1]

The torsion of $\Omega^1_{X/B}$ is supported precisely on $S_1 \cup S_2$. Its cusp modules inject into the free logarithmic relative modules, and their minimal number of generators exceeds their rank at the double locus. In degree two, the torsion also has support on the cusp double locus. This corrects (Claude and L. Alpöge, 2026, Remark 9.21, p. 81), where degree-one torsion is asserted on $W \cup S_1 \cup S_2$; the support statement in (Claude and L. Alpöge, 2026, Remark 1.3, p. 5) therefore depends on the degree.

Proof. 1. The double and triple charts in proposition 12.17 exhaust the singular strata of W. Their injections into free logarithmic modules prove absence of degree-one cusp torsion. Their degree-two presentations have a nonzero torsion generator at every general point of a double curve. At a multiple fibre the finite étale comparison

$$q^*\Omega^1_{N_j/B} = \Omega^1_{J'_j/B}$$

is an isomorphism; its summand $R/(s^{m_j-1})\,ds$ is nonzero for $m_j = 3, 4$. Faithful flatness descends both the torsion and its support. There are no other critical fibres, and ordinary relative differentials are locally free on their complement. Thus the degree-one support is exactly $S_1 \cup S_2$, whereas degree two also has cusp torsion, as asserted.

□

13 Frölicher, Bott–Chern, Aeppli, and Hermitian metrics

13.1 The differentials and their dependence on the Hodge number

Definition 13.1 (Cohomological conventions)**.** [tag PI13] For the standard marked threefold, put $h = h^{1,2}(X)$. Proposition 12.15 gives $h = 1$. We write the dimensions below as functions of h in the common Hodge pattern

$q\backslash p$	0	1	2	3
3	0	0	0	1
2	0	h	$h+1$	1
1	1	$h+1$	h	0
0	1	0	0	0.

(13.1)

The numerical tables of this family are obtained by the evaluation map $\mathbb{Z}[h] \to \mathbb{Z}$, $h \mapsto 1$. We use $d = \partial + \bar\partial$ on complex-valued smooth forms and the complex orientation for integration.

Proposition 13.2 (The complete Frölicher differential in terms of h)**.** [tag 01SA]

For a compact complex threefold with (13.1) and the complex de Rham cohomology of S^6, the Frölicher spectral sequence satisfies $E_1 \neq E_2 = E_\infty$. Its only nonzero d_1 ranks are

$$\operatorname{rank} d_1^{0,1} = 1, \quad \operatorname{rank} d_1^{1,1} = h, \quad \operatorname{rank} d_1^{1,2} = h, \quad \operatorname{rank} d_1^{2,2} = 1.$$

In particular its two middle rows are exact.

Proof. 1. Here $E_1^{p,q} = H^q(X, \Omega_X^p)$ and $d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ are complex-linear. For $r = 2$, any source in the two middle rows has target in one of $(2,0), (3,0), (3,1)$ or outside the table, except the source $(0,2)$, which is zero. Sources in row $q = 3$ outside $(3,3)$ are zero, and $(3,3)$ has no target. For $r = 3$ the only possible targets inside the table have sources $(0,2)$ or $(0,3)$, both zero. For $r > 3$ there is no arrow inside the table. A zero term remains zero on subsequent pages. Hence $E_2 = E_\infty$.

2. The filtration on $H^k_{\mathrm{dR}}(X;\mathbb{C})$ has no nonzero associated graded term for $1 \le k \le 5$. Every middle term of E_2 therefore vanishes. Consequently

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^{h+1} \longrightarrow \mathbb{C}^h \longrightarrow 0, \qquad 0 \longrightarrow \mathbb{C}^h \longrightarrow \mathbb{C}^{h+1} \longrightarrow \mathbb{C} \longrightarrow 0$$

are exact, giving the stated ranks. The first rank is one, so E_1 cannot equal E_2.

□

Proposition 13.3 (A basis on which all first differentials are specified)**.** [tag 01SB]

Choose a nonzero $g \in H^{0,1}_{\bar\partial}(X)$, a basis $q_1, \dots, q_h$ of a complement to $\mathbb{C}d_1 g$ in $H^{1,1}_{\bar\partial}(X)$, a basis $v_1, \dots, v_h$ of $H^{1,2}_{\bar\partial}(X)$, and a class e complementary to the image of $d_1 : H^{1,2} \to H^{2,2}$. With $a = d_1 g$, $c_j = d_1 q_j$, $d_j = d_1 v_j$, and $t = d_1 e$, the first differential is

$$g \mapsto a \mapsto 0, \quad q_j \mapsto c_j \mapsto 0, \qquad v_j \mapsto d_j \mapsto 0, \quad e \mapsto t \mapsto 0.$$

These are bases in the corresponding bidegrees. If $h = 1$ and a class q restricts to the invariant fibre class η described below, we may take $q_1 = q$, $v_1 = gq$, and $e = q^2$, after scaling g if necessary. Propositions 12.14 and 12.16 supply these data for every member of the marked family.

Proof. 1. The first assertions follow from the injectivity and surjectivity proved in 13.2. For the last assertion write $\eta = u \wedge w + 6\gamma \wedge \delta$ in the marked integral cohomology of a regular fibre. Its complex-oriented square has integral -12. Its product with the $(0,1)$ image of γ is nonzero: the explicit period calculation in 13.14 gives its nonzero pairing with $d\zeta_2$. Thus gq and q^2 have nonzero fibre restrictions. The images under d_1 restrict to zero on the fibre, because the fibre is a complex torus and its invariant representatives are d-closed. Therefore q complements $d_1 g$, gq spans the one-dimensional $H^{1,2}$, and q^2 complements $d_1 H^{1,2}$. The lift in Proposition 12.16 maps to η under the Leray restriction, and hence supplies the class q used here.

□

13.2 A computation from the two definitions

Definition 13.4 (Bott–Chern and Aeppli cohomology)**.** [tag PI14]

On the double complex $A^{p,q}(X)$ of smooth forms, set

$$H^{p,q}_{BC}(X) = \frac{\ker \partial \cap \ker \bar\partial \subset A^{p,q}(X)}{\operatorname{im}(\partial\bar\partial : A^{p-1,q-1} \to A^{p,q})}, \quad H^{p,q}_{A}(X) = \frac{\ker(\partial\bar\partial : A^{p,q} \to A^{p+1,q+1})}{\partial A^{p-1,q} + \bar\partial A^{p,q-1}}.$$

Complex conjugation sends bidegree (p,q) to (q,p).

Proposition 13.5 (Bott–Chern and Aeppli tables as functions of h)**.** [tag 01SC]

The complete dimension tables are

$q\backslash p$	0	1	2	3
3	0	0	1	1
2	0	h	$2h$	1
1	0	2	h	0
0	1	0	0	0

BC

$q\backslash p$	0	1	2	3
3	0	0	0	1
2	0	h	2	0
1	1	$2h$	h	0
0	1	1	0	0.

A (13.2)

At the value $h = 1$ of Proposition 12.15, the totals are $(1,0,2,2,2,2,1)$ and $(1,2,2,2,2,0,1)$, respectively.

Proof. 1. Apply (Stelzig, 2021, Theorem 3, p. 7; Proposition 6, p. 16; Lemma 8, p. 17) to the bounded-bidegree complex of smooth forms. The theorem permits arbitrary cardinal dimensions. Its odd zigzags contribute to total cohomology; the sphere Betti groups and the single extreme bidegrees therefore leave only dots at $(0,0)$ and $(3,3)$. By proposition 13.2 and conjugation, both spectral sequences have no differential after d_1, so every remaining zigzag is an arrow. The horizontal arrows are exactly

source	target	multiplicity
$(0,1)$	$(1,1)$	1
$(1,1)$	$(2,1)$	h
$(1,2)$	$(2,2)$	h
$(2,2)$	$(3,2)$	1.

(13.3)

The vertical arrows are their reflections in $p = q$.

2. Under the cited decomposition, an arrow contributes its target to Bott–Chern cohomology and its source to Aeppli cohomology; a dot contributes to both and a square to neither. The induced isomorphisms in Lemma 8 respect Definition 13.4. Counting the displayed arrows and their reflections gives every entry of (13.2). Kernels, images and these quotients commute with direct sums, including the possible infinite sum of squares.

□

Proposition 13.6 (Comparison with McHugh and Angella)**.** [tag 01SD]

The specialization of McHugh's Theorem 6.3, Proposition 6.4, equation (7), and Section 7.1 to (13.1) agrees with 13.5. The corresponding relations and Figure 5 in Angella's Section 4 also agree through the displayed specialization formulas.

Proof. 1. McHugh's quotient $K^{p,q}$ is zero in intermediate total degrees on every arrow and square just used: a closed corner of an arrow is the image of the incoming differential and a closed corner of a square is in both images. Thus $k^{p,q} = 0$ for $0 < p+q < 6$. Also $E_2^{3,1} = 0$ by 13.2 and $H^{1,1}(X;\mathbb{R}) = 0$ because $H^2(X;\mathbb{R}) = 0$. Reading the formulas in McHugh (2019, arXiv v4, Theorem 6.3, pp. 20–23; Proposition 6.4 and (7), p. 23; §7.1, pp. 24–25) therefore gives

$$h^{1,1}_{BC} = 2h^{0,1} = 2, \qquad h^{1,2}_{BC} = h^{1,2} = h, \qquad h^{2,2}_{BC} = -1 + (h+1) + h = 2h.$$

The remaining entries in his table are precisely the boundary entries of (13.2). In Angella (2018, arXiv v2, §4, pp. 13–15; Figure 5, p. 16), the relation

$$h^{2,2}_{BC} = 2h^{2,1}_{BC} - 2h^{0,1} + 2$$

gives $2h$, and the boundary values reduce to those above because $h^{2,0} = h^{0,2} = 0$.

□

13.3 The algebra, line bundles, and metric consequences

Proposition 13.7 (The real Bott–Chern Chern-class map and a global G)**.** [tag 01SE]

The Picard-group identification and fibrewise injectivity below are due to Claude and Alpöge (Claude and L. Alpöge, 2026, Theorem 10.5(b), proof Steps 1 and 4, p. 88). Let G be a $\bar\partial$-closed smooth $(0,1)$ representative of a nonzero $g \in H^1(X,\mathcal{O}_X)$. Set

$$\mathsf{A} = [\partial G]_{BC}, \qquad \mathsf{B} = [\bar\partial \bar G]_{BC}.$$

These classes form a complex basis of $H^{1,1}_{BC}(X)$. With the exponential $z \mapsto e^{2\pi i z}$ and the Chern convention $c_{1,BC} = [iF/(2\pi)]$, the map of real Lie groups is

$$\mathrm{Pic}(X) = \{\exp(zg) : z \in \mathbb{C}\} \longrightarrow H^{1,1}_{BC}(X), \qquad \exp(zg) \longmapsto z\mathsf{A} + \bar z \mathsf{B}. \tag{13.4}$$

It is injective and its image is the real subspace fixed by conjugation.

There is an operator-explicit global representative. Normalize the Fubini–Study form by $\int_{\mathbb{P}^1} \omega_{FS} = 1$, put $\theta = f^*\omega_{FS}$, choose any smooth Riemannian metric on X, and let $\mathcal{G}_d$ be its Green operator for the real de Rham Laplacian. Then

$$\alpha = d^*\mathcal{G}_d\theta, \qquad G_0 = \alpha^{0,1} \tag{13.5}$$

satisfy $d\alpha = \theta$, $\bar\partial G_0 = 0$, $[G_0] \neq 0$, and $\exp([G_0]) = f^*\mathcal{O}_{\mathbb{P}^1}(1)$. Restriction $H^1(X,\mathcal{O}_X) \to H^1(X_b,\mathcal{O}_{X_b})$ is injective for every scheme-theoretic fibre; on a smooth fibre its image is $\mathbb{C}\gamma^{0,1}$.

Proof. 1. By 13.2, the Dolbeault image of A is $d_1 g \neq 0$. The class $\mathsf{B} = \overline{\mathsf{A}}$ is nonzero, whereas its Dolbeault image is zero. Thus these two classes are independent; the dimension calculation of 13.5 makes them a basis. In this basis conjugation sends (a,b) to $(\bar b, \bar a)$, so its fixed real subspace consists precisely of $(z,\bar z)$.

2. The exponential sequence and $H^1(X;\mathbb{Z}) = H^2(X;\mathbb{Z}) = 0$ give the cited identification $\mathrm{Pic}(X) = H^1(X,\mathcal{O}_X) = \mathbb{C}g$. On a smooth trivial line with unit frame s, put $\bar\partial_z s = -2\pi i z G \otimes s$. Integrability follows from $\bar\partial G = 0$. On a cover with $G = \bar\partial\phi_i$, the holomorphic frames $e_i = e^{2\pi i z\phi_i}s$ satisfy $e_j = e^{2\pi i z(\phi_j - \phi_i)}e_i$; hence this is exactly the line represented by the exponential of the Čech class zg. Its unitary Chern connection and curvature are

$$\nabla_z = d - 2\pi i(zG + \bar z\bar G), \qquad F_z = -2\pi i(z\partial G + \bar z\bar\partial\bar G).$$

Multiplication by $i/(2\pi)$ proves (13.4). The two independent basis vectors prove injectivity, additivity as real Lie groups and the assertion about the image.

3. The closed real form θ is exact since $H^2(X;\mathbb{R}) = 0$. It has zero harmonic projection. De Rham Hodge decomposition, with $d\mathcal{G}_d = \mathcal{G}_d d$, therefore gives $dd^*\mathcal{G}_d\theta = \theta$. The Green operator and d^* preserve reality, so α is real and $d\alpha = \theta$. The $(0,2)$ component yields $\bar\partial G_0 = 0$. To prove nonvanishing, an equation $G_0 = \bar\partial u$ would give

$$\theta = \partial\bar\partial(u - \bar u) = 2i\partial\bar\partial\,\mathrm{Im}\,u.$$

Semipositivity of θ would make $\mathrm{Im}\,u$ plurisubharmonic. The maximum principle on compact connected X would make it constant, contradicting the nonzero pullback of ω_{FS} on the regular locus of f.

4. Since $\alpha = G_0 + \bar G_0$, the Chern formula already proved sends $\exp([G_0])$ to $[\theta]_{BC}$, which is also $c_{1,BC}(f^*\mathcal{O}(1))$. Its injectivity proves the stated line-bundle equality, with exactly the declared $2\pi i$ exponential and curvature conventions.

5. For the last assertion, the exact sequence of a full scheme fibre is

$$0 \longrightarrow f^*\mathcal{O}(-1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_b} \longrightarrow 0.$$

The cited direct images in 12.4 give $f_*f^*\mathcal{O}(-1) = \mathcal{O}(-1)$ and $R^1f_*f^*\mathcal{O}(-1) = \mathcal{O}(-1)\oplus\mathcal{O}(-2)$. Both terms bounding $H^1(X, f^*\mathcal{O}(-1))$ in the degree-one Leray edge sequence vanish, so this group vanishes. Restriction on $H^1(\mathcal{O})$ is injective. On a smooth fibre, the same edge map is the evaluation of $H^0(B, G) = \mathbb{C}\gamma^{0,1}$ in G_b. This identifies its image as claimed and reproduces exactly the application of (Claude and L. Alpöge, 2026, Theorem 10.5(b), Step 4, p. 88).

□

Proposition 13.8 (The factor 12 with a fixed collar orientation)**.** [tag 01SF]

Use the marked cohomology class γ dual to $\hat{\gamma}$, the analytic peripheral generators and the lift relation of (Claude and L. Alpöge, 2026, §7.5, pp. 54–57, especially (7.10), p. 55), and put $p = 12l_0 - 4l_1 - 3l_2 = \pm 1$. Let g_γ be the unique coherent Leray class whose fibre image is $\gamma^{0,1}$. Then

$$\exp(12g_\gamma) = f^*\mathcal{O}_{\mathbb{P}^1}(-p). \tag{13.6}$$

For the standard data $(l_0, l_1, l_2) = (0, 1, -1)$, this is $\exp(12g_\gamma) = f^*\mathcal{O}(1)$. For the opposite value $p = 1$, the exponent representing $f^*\mathcal{O}(1)$ is $-12g_\gamma$.

Proof. 1. A degree-one integral class $k\gamma$ on a regular fibre extends over the cyclic filling of order m_j precisely when the affine generator $\hat{g}_j$ can be assigned an integer value n_j satisfying $m_jn_j = k\gamma(v_j) = kl_j$. Admissibility gives $\gcd(l_j, m_j) = 1$, so this is precisely $m_j \mid k$. At the cusp γ extends, as follows from its specialization to $H^1(W; \mathbb{Z})$ in (Claude and L. Alpöge, 2026, Theorem B.1, pp. 101–105; Proposition 7.12, pp. 51–53). The global monodromy invariants have rank one generated by γ. It follows, by intersection of these three local images in the marked local system, that $H^0(B, R^1f_*\mathbb{Z}) = 12\mathbb{Z}\gamma$. This is the degree-one calculation of (Claude and L. Alpöge, 2026, Proposition 7.26, pp. 61–62). The closed embedding $F_{\mathrm{red}} \hookrightarrow F$ is a homeomorphism on underlying spaces and therefore induces the isomorphism on integral topological cohomology used here. The coherent restriction map has target $H^1(F, \mathcal{O}_F)$.

2. Let $U = \bigsqcup_i D_i$ be small discs around the special points and let V be their pair-of-pants complement with collars. Lift 12γ to integral classes Θ_i on $f^{-1}(D_i)$ and Θ_V on $f^{-1}(V)$. Existence over V follows from $H^2(V; \mathbb{Z}) = 0$ and the low-degree Leray sequence; over each disc it follows from the local images just computed. Write $a_j = \Theta_V(\hat{\rho}_j)$. On the two finite analytic peripheral generators the differences $\Theta_i - \Theta_V$ have periods

$$n_1 = 4l_1 - a_1, \qquad n_2 = 3l_2 - a_2.$$

At the cusp, the specified lift relation $\widetilde{a}_0 = t_\mu\hat{\rho}_2^{-1}\hat{\rho}_1^{-1}$, with $\gamma(\mu) = l_0$, and the null-homotopy of $\widetilde{a}_0$ in the cusp filling give $n_0 = -12l_0 + a_1 + a_2$. Hence

$$n_0 + n_1 + n_2 = -p. \tag{13.7}$$

These boundary evaluations use the orientations in (Claude and L. Alpöge, 2026, Proposition 7.27, pp. 62–63).

3. Represent the lifts by real closed one-forms. On a collar their difference is the pullback of a real closed base form b_i with clockwise period n_i, modulo an exact form. This follows from the degree-one Leray sequence over the annulus, since the difference has zero fibre restriction and the annulus has no second cohomology. Subtract exact terms on smaller collars, extending their primitives with cutoffs, so the equality of forms holds wherever a partition of unity

changes. For a real base cutoff ρ_i equal to one toward D_i and zero toward V, the resulting glued real one-form α on X has

$$d\alpha = f^*\vartheta, \qquad \vartheta = \sum_i d\rho_i \wedge b_i.$$

The form ϑ is a smooth real base two-form; thus $\bar\partial\alpha^{0,1} = 0$. Its fibre restriction is the $(0,1)$ image of 12γ, so the coherent Leray edge isomorphism $H^1(X,\mathcal{O}_X) \simeq H^0(B, R^1 f_*\mathcal{O}_X)$ identifies $[\alpha^{0,1}] = 12g_\gamma$.

4. In a positively oriented polar coordinate (r,φ) on D_i, $b_i = -n_i\, d\varphi/(2\pi)$ modulo an exact form. Because ρ_i decreases from one to zero as r increases,

$$\int_{D_i\cap V} d\rho_i \wedge b_i = -n_i \int_0^1 d\rho_i = n_i.$$

The exact part does not change this integral, since $d\rho_i$ is supported inside the collar. Therefore $\int_B \vartheta = -p$ by (13.7). 13.7 identifies the Bott–Chern class of $\exp(12g_\gamma)$ with $[f^*\vartheta]$. On $\mathbb{P}^1$ two real $(1,1)$ forms of the same integral differ by $i\partial\bar\partial$ of a real function (solve the ordinary scalar Laplace equation). Hence this class is the Chern class of $f^*\mathcal{O}(-p)$, and injectivity in 13.7 proves (13.6).

□

Definition 13.9 (The nine positive-degree Bott–Chern classes)**.** [tag PI15]

For $h = 1$, choose $\bar\partial$-closed representatives G, Q of g, q as in Proposition 13.3, and choose a smooth top-degree form ν with $\int_X \nu = 1$. Define

class	representative	bidegree
A	∂G	$(1,1)$
B	$\bar\partial\bar G$	$(1,1)$
C	∂Q	$(2,1)$
$\overline{\mathsf{C}}$	$\bar\partial\bar Q$	$(1,2)$
D	$\partial(G\wedge Q)$	$(2,2)$
$\overline{\mathsf{D}}$	$\bar\partial(\bar G\wedge\bar Q)$	$(2,2)$
E	$\partial(Q^2)$	$(3,2)$
$\overline{\mathsf{E}}$	$\bar\partial(\bar Q^2)$	$(2,3)$
T	ν	$(3,3)$.

(13.8)

Proposition 13.10 (The complete Bott–Chern algebra, for every h)**.** [tag 01SG]

The positive-total-degree ideal $N \subset H^{\bullet,\bullet}_{BC}(X)$ satisfies

$$H^{\bullet,\bullet}_{BC}(X) = \mathbb{C}\cdot 1 \oplus N, \qquad N^2 = 0.$$

Its nonzero bidegrees and dimensions are those of 13.5. Thus the square-zero conclusion is independent of the evaluated value of h. For $h = 1$, the nine classes specified in (13.8), together with the unit, give bases of every nonzero Bott–Chern group.

Proof. 1. Take global $\bar\partial$-closed representatives G, Q_j, V_j, E of the classes g, q_j, v_j, e of 13.3. By the arrow computation in 13.5, bases for the positive Bott–Chern groups consist of

$$\partial G,\ \bar\partial\bar G; \qquad \partial Q_j,\ \bar\partial\bar Q_j; \qquad \partial V_j,\ \bar\partial\bar V_j; \qquad \partial E,\ \bar\partial\bar E; \qquad \nu,$$

where the pairs have bidegrees $(1,1), (2,1), (1,2), (2,2), (3,2), (2,3)$, respectively, and $\int_X \nu = 1$. In particular the natural map $H^{3,2}_{BC} \to H^{3,2}_{\bar\partial}$ is an isomorphism: on its unique horizontal arrow the closed target maps nontrivially to column cohomology.

2. Write $a = \partial G$, $b = \bar\partial \bar G$ and $c_j = \partial Q_j$. The graded Leibniz rule gives

$$\partial\bar\partial(G \wedge \bar G) = -a \wedge b, \quad \partial\bar\partial(G \wedge \bar Q_j) = -a \wedge \bar c_j, \quad \partial\bar\partial(\bar G \wedge Q_j) = b \wedge c_j.$$

Consequently $\mathsf{AB} = 0$, $\mathsf{A}\overline{\mathsf{C}_j} = 0$, and $\mathsf{BC}_j = 0$.

3. Let $f^*\mathcal{O}(1) = \exp(\lambda g)$, where $\lambda \neq 0$. Its curvature may be chosen to be the pullback of a two-form on B, whose square vanishes pointwise. By 13.7, $\lambda^2\mathsf{A}^2 + \bar\lambda^2\mathsf{B}^2 = 0$. The Dolbeault image of B^2 is zero. Hence $[a^2]_{\bar\partial} = 0$. But $a^2 = \partial(G \wedge a)$, and 13.2 says that $d_1 : H^{1,2} \to H^{2,2}$ is injective. Thus $[G \wedge a]_{\bar\partial} = 0$. Choose a smooth $(1,1)$ form Φ satisfying $G \wedge a = \bar\partial\Phi$. Applying ∂ proves $a^2 = \partial\bar\partial\Phi$. Conjugation gives $\mathsf{B}^2 = 0$ as well.

4. To treat AC_j, pair its Dolbeault image with the generator $g \in H^{0,1}_{\bar\partial}$. Since $\bar\partial c_j = 0$ and $G \wedge a = \bar\partial\Phi$,

$$\int_X a \wedge c_j \wedge G = -\int_X \bar\partial(\Phi \wedge c_j) = 0.$$

Dolbeault–Serre duality and the isomorphism $H^{3,2}_{BC} \to H^{3,2}_{\bar\partial}$ give $\mathsf{AC}_j = 0$. Complex conjugation gives $\mathsf{B}\overline{\mathsf{C}_j} = 0$.

5. Every remaining product whose bidegree lies in $[0,3]^2$ lands in $(3,3)$: it is a product of a $(1,1)$ class with a $(2,2)$ class, or of a $(2,1)$ class with a $(1,2)$ class. Each non-top generator displayed above is d-exact, since its chosen primitive is $\bar\partial$-closed, or is the conjugate of such a form. The products therefore have integral zero. Integration is an isomorphism $H^{3,3}_{BC} \to \mathbb{C}$ by 13.5 and the class of ν, so all these products vanish. All other products exceed the complex dimension in at least one bidegree. This exhausts the possibilities.

□

Proposition 13.11 (Non-Kähler degrees and comparison maps)**.** [tag 01SH]

For $k = 0, \dots, 6$ the non-Kähler degrees are

$$(\Delta^k) = (0, 2, 2+2h, 4h, 2+2h, 2, 0). \tag{13.9}$$

If $h \geq 1$, the two maps $\bigoplus_{p+q=k} H^{p,q}_{BC} \to H^k_{\mathrm{dR}}(X;\mathbb{C}) \to \bigoplus_{p+q=k} H^{p,q}_A$ are both isomorphisms exactly for $k = 0, 6$. For the weaker assertion that a d-exact pure d-closed form of degree k is $\partial\bar\partial$-exact, the valid degrees are $k = 0, 1, 6$, and also $k = 3, 4$ if $h = 0$.

Proof. 1. The totals from 13.5 are

$$(h^k_{BC}) = (1, 0, 2, 2h, 2h, 2, 1), \qquad (h^k_A) = (1, 2, 2h, 2h, 2, 0, 1).$$

Subtracting twice the Betti numbers gives (13.9). On the two endpoint dots both comparison maps are isomorphisms; all intermediate de Rham groups vanish. The first assertion follows by inspecting when both outside groups vanish. The weaker assertion is exactly injectivity of the first comparison map. In intermediate degree this is equivalent to vanishing of the Bott–Chern group alone, which gives the stated list. In degree zero there is no nonzero exact form, and in degree six integration identifies Bott–Chern and de Rham cohomology. In degree one the first comparison map is $0 \to 0$, while the second is $0 \to \mathbb{C}^2$; the formula gives $\Delta^1 = 2$.

□

Proposition 13.12 (Obstructions from exact positive forms). [tag 01SI]

The threefold X admits no balanced, strongly Gauduchon, Kähler, locally conformally Kähler, or Hermitian-symplectic metric.

Proof. 1. Let $\theta = f^*\omega_{FS}$. It is a nonzero semipositive exact real $(1,1)$ form. If ω were balanced, the real four-form ω^2 would be closed and $\int_X \theta \wedge \omega^2 > 0$, because the integrand is positive on the regular locus. Exactness of θ and Stokes' theorem give zero, a contradiction.

2. More generally, if ω were strongly Gauduchon, write $\partial\omega^2 = \bar\partial u$ with $u \in A^{3,1}$. The real four-form $\Omega = \omega^2 - u - \bar u$ is closed. Bidegrees give $\theta \wedge \Omega = \theta \wedge \omega^2$, so the same argument applies. Kähler metrics are balanced. If $d\omega = \ell \wedge \omega$ with a closed real Lee form ℓ, then $b_1 = 0$ gives $\ell = dv$, and $d(e^{-v}\omega) = 0$, excluding locally conformally Kähler metrics. Finally any symplectic two-form on X is exact because $b_2 = 0$. Its cube has nonzero integral for its symplectic orientation, whereas an exact cube has zero integral. Thus X is not symplectic and in particular admits no Hermitian-symplectic metric.

□

Cited result 13.13 (Gauduchon metrics with specified Chern–Ricci form). [tag 01SJ]

There exists a Gauduchon metric on X. More precisely, for every Gauduchon metric ω_0 there is another Gauduchon metric ω with $[\omega^2]_A = [\omega_0^2]_A$ and

$$\mathrm{Ric}^{\mathrm{Ch}}(\omega) = \pi f^*\omega_{FS}. \tag{13.10}$$

This Ricci form is semipositive and has rank one at regular points of f. No Chern–Ricci-flat Hermitian metric exists on X.

Proof. Gauduchon's conformal theorem (Gauduchon, 1985, Theorem 2, pp. 132–133) supplies a Gauduchon metric on the compact complex threefold. Fix any such ω_0. By proposition 7.2 and $4S_2 = f^*(p_2)$, $K_X^{\otimes 2} \simeq f^*\mathcal{O}_B(-1)$. For $\theta = f^*\omega_{FS}$, with $\int_{\mathbb{P}^1}\omega_{FS} = 1$, this gives

$$c_{1,BC}(T_X) = \tfrac{1}{2}[\theta]_{BC}.$$

The convention $\mathrm{Ric}^{\mathrm{Ch}}(\omega) = -i\partial\bar\partial \log\det(g)$ represents $2\pi c_{1,BC}(T_X)$. Thus $\pi\theta$ is the prescribed representative of that class. Theorem 1.3 of (Székelyhidi et al., 2017, arXiv v2, p. 3; conventions on p. 2) gives a Gauduchon metric with this Chern–Ricci form and $[\omega^2]_A = [\omega_0^2]_A$.

The pullback θ is semipositive and has rank one where f is a submersion. In particular $\int_X \theta\wedge\omega_0^2 > 0$. Every $\partial\bar\partial$-exact $(1,1)$ form pairs to zero with the Gauduchon square, by integration by parts. Consequently $[\theta]_{BC} \neq 0$, and a zero Chern–Ricci form would contradict the displayed Chern-class identity. □

Proposition 13.14 (The fibre obstruction to SKT and astheno-Kähler metrics). [tag 01SK]

The threefold X admits no pluriclosed (SKT) metric and no astheno-Kähler metric. The geometric input is the value $h = 1$ and the global Dolbeault class q in 12.16, whose restriction to a regular fibre is $\eta = u \wedge w + 6\gamma \wedge \delta$. More generally, it is enough that the real image of $H_A^{1,1}(X;\mathbb{R}) \to H_A^{1,1}(F;\mathbb{R})$ be contained in $\mathbb{R}\eta$ for one regular fibre F.

Proof. 1. Put $y = \operatorname{Im}\tau > 0$, $m = \operatorname{Im}\mu$, and $\Delta = \operatorname{Im}\beta - 6m^2/y < 0$. For the marked period rows $\sigma_1 = 6\mu\gamma + \tau u + w$ and $\sigma_2 = \beta\gamma + \mu u + \delta$ we have $\eta = u \wedge \sigma_1 + 6\gamma \wedge \sigma_2$. This proves that the real class η has type $(1,1)$. The determinant of the four real period columns, with rows ordered as $\operatorname{Re}\zeta_1, \operatorname{Im}\zeta_1, \operatorname{Re}\zeta_2, \operatorname{Im}\zeta_2$, is $y\Delta < 0$. Since $\eta^2 = 12\gamma \wedge u \wedge w \wedge \delta$,

$$\int_{F,\mathrm{or}_{\mathbb{C}}} \eta^2 = -12. \tag{13.11}$$

Equivalently the invariant representative of η has Hermitian coefficient matrix

$$\frac{1}{y\Delta}\begin{pmatrix}\operatorname{Im}\beta & -6m\\ -6m & 6y\end{pmatrix}, \qquad \det = \frac{6}{y\Delta} < 0,$$

so it has one positive and one negative eigenvalue.

2. The explicit invariant fibre representative of $\gamma^{0,1}$ is

$$\Gamma = \frac{i}{2\Delta}\left(d\bar{\zeta}_2 - \frac{m}{y}d\bar{\zeta}_1\right). \tag{13.12}$$

Its invariant $(1,1)$ representative is

$$Q_F = \frac{1}{2iy\Delta}(\operatorname{Im}\beta\, d\zeta_1\wedge d\bar{\zeta}_1 - 6m\, d\zeta_1\wedge d\bar{\zeta}_2 - 6m\, d\zeta_2\wedge d\bar{\zeta}_1 + 6y\, d\zeta_2\wedge d\bar{\zeta}_2). \tag{13.13}$$

Direct exterior multiplication gives

$$Q_F\wedge\Gamma = \frac{1}{4y\Delta}d\zeta_1\wedge d\bar{\zeta}_1\wedge d\bar{\zeta}_2, \qquad \int_F Q_F\wedge\Gamma\wedge d\zeta_2 = -1.$$

This also proves the nonvanishing used in 13.3.

3. When $h=1$, 13.5 identifies $H^{1,1}_A(X)$ with the sum of the one horizontal source and its vertical conjugate. A representative Q of q is nonzero in the horizontal source, since its fibre restriction is nonzero whereas $d_1 g$ restricts to zero. Thus $[Q]_A, [\bar{Q}]_A$ form a basis. Their restrictions to F both equal the real class η. The real image of the restriction map is therefore $\mathbb{R}\eta$.

4. If ω were a pluriclosed positive real $(1,1)$ form, its restriction to F would be pluriclosed and positive. Average this restriction under translations of the compact complex torus F. The average ω_{av} is invariant, closed, and strictly positive. Translation preserves the Aeppli class: on a torus the $\partial\bar{\partial}$-lemma identifies Aeppli cohomology with the invariant (p,q) forms, so the average is the unique invariant representative of that class. This identification can also be checked Fourier mode by Fourier mode: each nonzero mode is a Koszul square for two independent type components of its real nonzero covector and contributes zero Aeppli cohomology. Thus $[\omega_{\mathrm{av}}] = r\eta$ for some real r. But positivity gives $\int_F \omega_{\mathrm{av}}^2 > 0$, whereas (13.11) gives $-12r^2 \le 0$, a contradiction. In dimension three the astheno-Kähler equation $\partial\bar{\partial}\omega^{n-2}=0$ is exactly the SKT equation.

□

14 Global coefficient representatives

14.1 A coefficient atlas for the two generators

Remark 14.1 (Sources and coefficient conventions)**.** [tag PI16] The Green representative of proposition 13.7 and the harmonic normalization of a lift of proposition 12.16 give operator descriptions. The following construction specifies a holomorphic Čech cocycle and every smooth coefficient in its Dolbeault representative. It uses the original chart maps through all three special fibres. The period matrix, two affine finite fillings, logarithmic collar maps and untranslated cusp fan are the original construction (Claude and L. Alpöge, 2026, Sections 3–6, pp. 13–40). The same local pieces in the two versions of Du et al. (2026a,b) are identified by the parameter translation $a = c+\delta$ of remark A.1; in particular $\Omega_a = \Pi_c$, using the same ordered lattice, vectors and fan. The scalar cocycle below is precisely $f = -\eta_{\mathrm{DGKX}}$ on every displayed generating arrow in (Du et al., 2026a, Section 6.4, scalar cocycle and Lemma 6.14 with proof, p. 84; conventions

and generating arrows, pp. 82–83). The minus sign is the stated line-coordinate convention. The exponential quotient comparison with Engel's elliptic-surface presentation is the map already given in proposition 2.11.

Conversion of a holomorphic cocycle to a Dolbeault form by smooth weights is the usual Čech–Dolbeault construction (Abate et al., 2013, Theorem 3.1 and proof, p. 10). On two charts its formula applied to $(0, 0, c_{01}/12)$ is $-\bar\partial w_0 \wedge c_{01}/12 = \bar\partial(w_1 c_{01})/12$, exactly the convention used here; replace $c_{01}/12$ by f_{01} for G.

Proposition 14.2 (A specified global coefficient atlas)**.** [tag 01SL] *Use the standard marked family of proposition 7.5, with vectors $v_1 = (1, 2, -4, 0)^t$, $v_2 = (-1, -3, 3, 0)^t$, the untranslated original cusp fan, and any $c \in \mathcal{P} = \{\operatorname{Im} c < 3\log(6)/\pi\}$. There are specified smooth forms*

$$G \in \mathcal{A}^{0,1}(X), \qquad Q \in \mathcal{A}^{1,1}(X), \qquad \bar\partial G = \bar\partial Q = 0,$$

whose classes satisfy

$$[G] = g_\gamma, \qquad i_b^*[Q]_{\bar\partial} = \eta, \qquad \exp(12[G]) = f^*\mathcal{O}_{\mathbb{P}^1}(1).$$

Here i_b is any regular fibre inclusion and $\eta = u \wedge w + 6\gamma \wedge \delta$ in the original integral marking. The formulas (14.13) and (14.14), with the elementary recursion (14.12), give their coefficients on a specified countable refinement of the original atlas. Every sum is locally finite, also at the multiple fibres, cusp double curves and both cusp triple points. The form Q is a complex representative; no real $\bar\partial$-closed representative can have this fibre class.

Proof.

1. *The chart domains and their inverse maps.* Use the ordinary, finite and cusp covering charts of propositions 7.4 and 7.5, with their specified affine, toric and exponential transition maps. More explicitly, write $t_1 = t$, $t_2 = t - 1$, $q = 1/t$, and retain the fixed disjoint special discs of those constructions. After restricting their charts by inclusion, write their radii as r_1, r_2, ϵ, with $r_1, r_2 < 1/4$, and use

$$V_j = \{|t_j| < r_j\}, \quad V_0 = \{|q| < \epsilon\}, \quad V_* = B \setminus (\{|t_1| \le r_1/2\} \cup \{|t_2| \le r_2/2\} \cup \{|q| \le \epsilon/2\}).$$

Thus only $V_* \cap V_j$ are nonempty pairwise overlaps between distinct pieces. All three are annuli. The finite coordinate domains are the original torus covers over $|s_j|^{m_j} < r_j$, with $t_j = s_j^{m_j}$ and $(m_1, m_2) = (3, 4)$; their maps to X are the original free affine quotient maps. The middle domains are the ordinary lifted base opens over V_*, times $\mathbb{C}^2$.

For the cusp, retain the full holomorphic matrix $C(q)$ and a closed disc $|q| \le r_0$ on which it is holomorphic. The original radius satisfies

$$0 < \epsilon < \min(r_0, 1), \qquad |\log\epsilon| > 2 \max_{|q|\le r_0} \| - 2\pi \operatorname{Im} C(q)\|.$$

These are the chart and properness bounds in Proposition 7.5. For every triangle σ of the original periodic triangulation, use

$$\mathcal{U}_\sigma(2, \epsilon) = \{(z_0, z_1, z_2) : |z_i| < 2,\ |z_0 z_1 z_2| < \epsilon\}, \qquad q = z_0 z_1 z_2.$$

The toric characters are the basis dual to the three original height-one ray generators. Their integral exponent matrix is unimodular, so its inverse gives every inverse toric overlap. The toric orbit calculation in that proof shows that these domains cover the cusp space before quotient, including its boundary.

All these maps are local biholomorphisms to X. Their overlaps are fibre products, retaining every deck branch. A lattice map $\zeta \mapsto \zeta + \Pi\lambda$ has inverse $\zeta \mapsto \zeta - \Pi\lambda$. A finite affine map $(s, \zeta) \mapsto (\omega s, R(s)\zeta + \Pi(\omega s)v/m)$ has inverse

$$s = \omega^{-1}s', \qquad \zeta = R(s)^{-1}(\zeta' - \Pi(s')v/m).$$

The cusp period map has inverse $x \mapsto q^{-B_0 n}\exp(-2\pi i C(q)n)x$. The finite collar inverse subtracts its full translating section, and the cusp collar inverse uses the branches $\zeta = (2\pi i)^{-1}\log x$, whose differences are the integral periods. These maps specify the refinement and its local inverse branches inside the given X.

2. *The regular class and its sign.* Put

$$\Pi = [Z \mid I_2], \quad Z = \begin{pmatrix} 6\mu & \tau \\ \beta & \mu \end{pmatrix}, \quad T = \begin{pmatrix} \tau & 6\mu \\ \mu & \beta \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 6 \\ 1 & 0 \end{pmatrix}, \quad D = \operatorname{diag}(1, 6).$$

The exact real marking is

$$q_B = (r_u, r_\gamma)^t, \quad p_B = (r_w, r_\delta)^t, \qquad \zeta = p_B + Tq_B,$$

with inverse $q_B = (\operatorname{Im} T)^{-1}\operatorname{Im}\zeta$, $p_B = \operatorname{Re}\zeta - (\operatorname{Re} T)q_B$. For $\lambda = (n_\gamma, n_u, m_w, m_\delta)$, write $n = (n_\gamma, n_u)^t$ and define

$$a_\lambda(\zeta) = \exp\left(2\pi i\left[n^t K\zeta + \tfrac{1}{2}n^t KZn\right]\right). \tag{14.1}$$

Symmetry of KZ makes its logarithmic composition defect the integer $n^t Km'$. Thus these factors satisfy the multiplicative cocycle identity.

Let $Y = \operatorname{Im} T$, $S = DY^{-1}$ and $\varphi = 2\pi q_B^t DY q_B$. The metric coefficient e^φ obeys $h(\zeta+\Pi\lambda)|a_\lambda|^2 = h(\zeta)$. With $F = \bar\partial\partial\log h$ and $c_1 = iF/(2\pi)$ its Chern form is

$$Q_{\mathrm{sf}} = -\frac{i}{2\pi}\partial\bar\partial\varphi = -\frac{i}{2}(d\zeta - dTq_B)^t S \wedge \overline{(d\zeta - dTq_B)}. \tag{14.2}$$

Expanding under the exact real inverse gives $dq_B^t D \wedge dp_B = dr_u \wedge dr_w + 6dr_\gamma \wedge dr_\delta$. In particular the fibre class is η, with its original sign. If $y = \operatorname{Im}\tau$, $m = \operatorname{Im}\mu$, $r = \operatorname{Im}\beta$ and $\Delta = r - 6m^2/y$, then

$$S = \frac{1}{y\Delta}\begin{pmatrix} r & -6m \\ -6m & 6y \end{pmatrix},$$

which gives the invariant fibre form (13.13).

3. *The line on the middle piece.* Use the original matrices A_1, A_2 in section 7. Their period transformation laws and complex-linear maps are

j	τ_j	μ_j	β_j
1	$1 - 1/\tau$	$(1-\mu)/\tau$	$\beta - 6(\mu-1)^2/\tau + 2$
2	$-1/\tau$	$1 + \mu/\tau$	$\beta - 6\mu^2/\tau - 3,$

$$R_1 = \begin{pmatrix} -1/\tau & 0 \\ (1-\mu)/\tau & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1/\tau & 0 \\ -\mu/\tau & 1 \end{pmatrix}, \qquad R_j\Pi = \Pi_j A_j.$$

For both generators put $F_j(\zeta) = -\zeta_1^2/(2\tau)$. The differences

$$\frac{\log a_{A_j\lambda}(R_j\zeta) - \log a_\lambda(\zeta)}{2\pi i} - F_j(\zeta + \Pi\lambda) + F_j(\zeta)$$

on the four original lattice generators are $(24, 0, 1/2, 0)$ and $(-9, 0, 0, 0)$. Their exponential ratios are characters, so the four values determine the full lattice comparison. Twelfth powers kill every defect. The line $L_* = L_\eta^{12}$ has the linear lifts

$$(\zeta, v) \longmapsto (R_j\zeta, e^{2\pi i\, 12F_j(\zeta)}v).$$

The Gaussian sums vanish on the order-three and order-four relations. Applying generator two and then generator one gives $(\tau, \mu, \beta, \zeta) \mapsto (\tau + 1, \mu, \beta - 1, \zeta)$ and Gaussian sum zero. These identities give the middle line on the specified quotient, with trivial fibre multiplier for that cusp logarithm turn.

4. *Both finite affine lifts, including their phases.* Let $b_j = v_j/m_j$. Use the same polynomial logarithm in (14.1) to define a_b for this rational vector. Since $12b_j$ is integral, the affine map

$$g_j(s, \zeta) = (e^{-2\pi i/m_j}s, R_j\zeta + \Pi_j b_j)$$

has the following line lift:

$$k_j(s, \zeta) = \exp\left(2\pi i\left[12F_j(\zeta) + 12\frac{\log a_{b_j}(R_j\zeta)}{2\pi i} + \kappa_j\right]\right), \quad \kappa_1 = \frac{32}{3}, \quad \kappa_2 = \frac{81}{8}. \tag{14.3}$$

The factor a_{b_j} uses the target period Z_j. Direct composition gives $g_j^{m_j}\zeta = \zeta + \Pi v_j$. Before adding κ_j, the logarithmic difference from $a_{v_j}^{12}$ is -32 for $j = 1$ and $-81/2$ for $j = 2$. The stated phases cancel these differences exactly. Hence the power of (14.3) is the lattice action, and it defines a line L_j on the original free finite quotient. The functions are holomorphic through $s = 0$: the original periods are holomorphic there, $\tau \neq 0$, and every displayed exponential is a unit. Thus this is a line over the full multiple-fibre neighbourhood. Its regular fibre Chern class is 12η.

5. *Cusp Cartier frames and all extending units.* Write the original full period matrix as

$$Z = sB_0 + C(q), \quad s = \frac{\log q}{2\pi i}, \quad B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The period $n \in \mathbb{Z}^2$ acts by $x \mapsto q^{B_0 n}\exp(2\pi i C(q)n)x$. Use the factor

$$A_n = x^{K^t n}q^{h(n)}\exp(\pi i n^t KC(q)n), \qquad h(n) = -3n_\gamma^2 + \frac{n_u(n_u - 1)}{2}. \tag{14.4}$$

Both $KB_0 = \mathrm{diag}(-6, 1)$ and $KC(q)$ are symmetric. Consequently multiplication proves the exact cocycle in (14.4). Its q exponent is integral. Its ratio to (14.1), on a fixed punctured fibre, is $q^{-n_u/2}$, independent of x; its fibre Chern class is therefore η.

Interpolate $L(i, j) = i(i-1)/2 - 3j^2$ on the prescribed triangulation. Both triangles in the square with lower corner (i, j) have

$$m_{ij} = (i, -6j - 3), \quad l_{ij} = -i(i + 1)/2 + 3j(j + 1), \quad f_{ij} = x^{m_{ij}}q^{l_{ij}}.$$

These are Cartier frames. The successive ratios $f_{i+1,j}/f_{ij}$ and $f_{i,j+1}/f_{ij}$ are x_1q^{-i-1} and $x_2^{-6}q^{6(j+1)}$; their exponents vanish on the corresponding common face, so they are units on the toric overlap. The two triangles inside a square have ratio one. In the line coordinate $w = z/f_{ij}$ the overlap multiplier is the reciprocal frame ratio.

For $v = B_0 n$ the exact identity $L(u + v) - L(u) = (K^t n) \cdot u + h(n)$ cancels every x and q power in the period action on w. The remaining factor is

$$U_{n,ij}(q) = \exp\left(\pi i n^t KC(q)n - 2\pi i\, m_{i+v_1,j+v_2}^t C(q)n\right). \tag{14.5}$$

It is a holomorphic unit at zero. Its cocycle follows either from these exact cancellations or by direct multiplication. Thus it descends on the original free properly discontinuous toric quotient. Let L_0 be its twelfth power. On the punctured cusp the exact ratio $L_0 \otimes L_*^{-1}$ has factors q^{-6n_u}.

6. *The inverse used on every annulus.* For a line with normalized lattice factors 1 on the two real periods and $e^{2\pi i n_T^t l(t)}$ on the columns of T, the form $A = a(t)^t d\zeta$ is a holomorphic connection when

$$T'^t a = -2\pi i l', \qquad (T'^t)^{-1} = \frac{1}{\tau'\beta' - 6\mu'^2}\begin{pmatrix} \beta' & -\mu' \\ -6\mu' & \tau' \end{pmatrix}. \tag{14.6}$$

Indeed translation by Tn_T adds $a^t T' n_T\, dt = -d\log(e^{2\pi i n_T^t l})$.

Every use of this inverse is on one of the three annuli above. By proposition 12.10, $6\mu_z^2 - \tau_z\beta_z = \tau_z^2 E_4(\tau)^2/E_6(\tau)$, with the apparent poles removed, has order two at the order-three point, order zero at the order-four point, and no further zeros on the regular locus. A local change of base coordinate multiplies the determinant by the square of a nonzero derivative. Thus $\det T'$ is nonzero throughout the three annular domains of the inverse. In the cusp variable its leading term is $1/(4\pi^2 q^2)$, in agreement with the exact logarithmic period matrix.

7. *The cusp overlap connection.* For $F_0 = L_0 \otimes L_*^{-1}$, use $l = (-6\log q/(2\pi i), 0)^t$. Its connection is

$$A_0 = a_0^t d\zeta, \qquad a_0 = \frac{6}{q}(T'^t)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{6}{q(\tau'\beta' - 6\mu'^2)}\begin{pmatrix} \beta' \\ -6\mu' \end{pmatrix}. \tag{14.7}$$

It is single-valued: a logarithm turn adds $\mathrm{diag}(1,-1)$ to T and changes neither T' nor ζ. In the cusp frame f_{ij}^{12} the form is $A_0 + 12d\log f_{ij}$. This transformation retains the full units (14.5).

8. *The finite overlap connections and all logarithm turns.* The collar map, as in proposition 7.5, is

$$H(s,\zeta) = (s, \zeta + \sigma), \qquad \sigma = h\Pi v_j, \qquad h = \frac{\log s}{2\pi i}.$$

The identities $h(g_j s) = h(s) - 1/m_j$ and $A_j v_j = v_j$ give the exact square $H g_j^{\mathrm{aff}} = g_j^{\mathrm{lin}} H$. For $F = H^* L_* \otimes L_j^{-1}$ its lattice factors are $e^{2\pi i\, 12 n_T^t D\sigma}$. Put $v_q = (v_u, v_\gamma)^t$ and $S = v_q^t D\Pi v_j$. Then

$$A = a^t d\zeta - 12\pi i\, d(h^2 S), \qquad a = -24\pi i (T'^t)^{-1} D\sigma'. \tag{14.8}$$

The last differential is a base one-form. Equation (14.6) proves lattice descent. For an integer logarithm turn $h \mapsto h + k$, the frame changes by $a_{kv_j}(\zeta+\sigma)^{12}$. The vertical change in (14.8) is $-24\pi i k D v_q$. The base change is exactly

$$-12\pi i\, d((2kh + k^2)S) = -d\log a_{kv_j}(\zeta+\sigma)^{12} + 24\pi i k v_q^t D\, d\zeta.$$

Thus the complete connection descends across every logarithm turn, with the displayed base term.

For the remaining finite cyclic descent its line multiplier is

$$k_F = \frac{e^{2\pi i\, 12 F_j(\zeta+\sigma)}}{k_j(s,\zeta)}.$$

Let $k_F^{(r)}$ be the product over the first r affine iterates. The finite sum

$$A^{\mathrm{inv}} = \frac{1}{m_j}\sum_{r=0}^{m_j-1}\left((g_j^r)^* A + d\log k_F^{(r)}\right) \tag{14.9}$$

is a connection on the same line. The exact power relation in (14.3) makes cyclic permutation of the summands valid, so it descends to the original annulus. For $F_j = L_j \otimes L_*^{-1}$ use the dual connection $-A^{\mathrm{inv}}$. This is the tensor-dual connection, with all transition factors inverted and the displayed sign. These holomorphic connections need not be flat: da can supply a $(2,0)$ curvature term.

9. *The holomorphic one-form cocycle.* Choose the specified line frames in each same-piece chart. If the line-coordinate map is $v_b = g_{ba} v_a$, put

$$c_{ab} = \frac{i}{2\pi} d \log g_{ba} \quad \text{within a piece}, \qquad c_{ab} = -\frac{i}{2\pi} A_{F_j,(a,b)} \quad (a \text{ middle},\ b \text{ in piece } j), \tag{14.10}$$

and $c_{ba} = -c_{ab}$. On a triple with two middle charts, the ratio frame of F_j changes by the inverse middle transition, and its connection changes by $+d \log g_*$. On a triple with two special charts it changes by the special transition, and its connection changes by $-d \log g_j$. These signs give $c_{ab} + c_{bc} = c_{ac}$ in both cases. The special pieces do not intersect one another. This proves the cocycle identity on every component of every overlap.

The result is the holomorphic one-form cocycle c_{ab}. Its construction uses the connection transformation law; the connections may have the $(2,0)$ curvature dA displayed above.

10. *A completely specified smooth partition.* In every original uniformizing domain enumerate all rational closed Euclidean balls $B(c_n, R_n)$ compactly contained in that domain. Use a fixed enumeration of the original charts by piece, cone label and finite deck word. Order the balls by integer height bounding the chart index and all coordinate numerators and denominators, then lexicographically, removing repeats. Each height block is finite. The half-radius balls cover every chart.

Set

$$E(t) = \begin{cases} e^{-1/t} & t > 0, \\ 0 & t \le 0, \end{cases} \quad \chi(t) = \frac{E(t)}{E(t) + E(1-t)}, \quad b_n(u) = \chi\!\left(\frac{R_n^2 - \|u - c_n\|^2}{3R_n^2/4} \right).$$

Extend b_n by zero outside its ball. It is one on the half-radius ball and is flat on its support boundary. For the étale map π_n, define

$$\begin{aligned} S_n(x) &= \sum_{u \in \pi_n^{-1}(x)} b_n(u), \qquad a_n(x) = \chi(2S_n(x)), \\ P_1 &= 1, \qquad P_{n+1} = P_n(1 - a_n), \\ R(s) &= \chi(2s)/s \quad (s > 0), \qquad R(0) = 0, \\ w_n(u) &= P_n(\pi_n u) R(S_n(\pi_n u)) b_n(u). \end{aligned} \tag{14.11}$$

The function R is smooth at zero because its numerator is flat. Each trace S_n is locally a finite smooth sum: its source support is compact and the map is étale, hence has discrete fibres. Every point has a lift in an open half-radius ball. For some N one therefore has $S_N \ge 1$ on a neighbourhood, after shrinking that neighbourhood if necessary. There $a_N = 1$, so all subsequent weights vanish. Consequently the weights are locally finite. The finite telescoping product gives

$$\sum_n \sum_{u \in \pi_n^{-1}(x)} w_n(u) = 1.$$

This construction also handles original chart maps that are not globally injective; it retains all their local inverse branches.

Every derivative in the final coefficients is given by the following finite recursion. In a local

inverse branch write $\mathcal{D} = \partial/\partial\bar{z}_l$ and $v_n = (R_n^2 - \|u - c_n\|^2)/(3R_n^2/4)$. Then

$$\begin{aligned}
\mathcal{D}b_n &= -\frac{4}{3R_n^2}\chi'(v_n)\sum_j (u_j - c_{n,j})\mathcal{D}\bar{u}_j, \\
\mathcal{D}S_n &= \sum_{u\in\pi_n^{-1}(x)} \mathcal{D}b_n, & \mathcal{D}a_n &= 2\chi'(2S_n)\mathcal{D}S_n, \\
\mathcal{D}P_1 &= 0, & \mathcal{D}P_{n+1} &= (1-a_n)\mathcal{D}P_n - P_n\mathcal{D}a_n, \\
R'(s) &= \frac{2s\chi'(2s) - \chi(2s)}{s^2} \quad (s>0), & R'(0) &= 0, \\
\mathcal{D}w_n &= R(S_n)b_n\mathcal{D}P_n + P_nR'(S_n)b_n\mathcal{D}S_n + P_nR(S_n)\mathcal{D}b_n.
\end{aligned} \tag{14.12}$$

The derivatives $\mathcal{D}\bar{u}_j$ are the conjugate Jacobians of the explicit inverse chart maps in step 1. Finally

$$\chi'(t) = \frac{E'(t)E(1-t) + E(t)E'(1-t)}{(E(t)+E(1-t))^2}, \qquad E'(t) = \begin{cases} e^{-1/t}/t^2 & t>0, \\ 0 & t\le 0. \end{cases}$$

Together with the inverse-chart Jacobians, this evaluates every weight derivative in the coefficient formulas.

11. *The primitive scalar cocycle.* For a complex parameter z, use lattice factors $e^{-2\pi i z\gamma(\lambda)}$ and middle linear factors one. The finite affine factor is $e^{-2\pi i z\gamma(v_j)/m_j}$; its m_jth power is the period factor of v_j. The cusp period factor is $e^{-2\pi i z n_\gamma}$, which extends in every toric chart. The finite collar comparison is $e^{-2\pi i z\gamma(v_j)h(s)}$. Its changes under $h \mapsto h - 1/m_j$ and $h \mapsto h + k$ are exactly those required by the affine generator and the period kv_j.

    Write every resulting transition as $e^{2\pi i z f_{ab}}$. Its coefficients are respectively $-\gamma(\lambda)$, 0, $-\gamma(v_j)/m_j$, $-n_\gamma$ and $-\gamma(v_j)h(s)$ on those arrows; pure toric frame changes have coefficient zero. Composition and inversion use addition and negation. Since the identities hold for every complex z, $f_{ab} + f_{bc} = f_{ac}$ on each overlap component.

12. *The evaluated global forms and their normalization.* In a chart a, with the indicated component of each fibre-product overlap, put

$$H_a = \sum_n\sum_u w_n(u)f_{a,n}(x,u), \quad B_a = \sum_n\sum_u w_n(u)c_{a,n}(x,u), \quad G = \bar{\partial}H_a, \quad Q = \frac{1}{12}\bar{\partial}B_a. \tag{14.13}$$

    The cocycles and sum of weights give $H_a - H_b = f_{ab}$ and $B_a - B_b = c_{ab}$. Their right sides are holomorphic. Therefore these formulas agree on every overlap and define global $\bar{\partial}$-closed forms. Writing $c_{a,n} = \sum_k c_{a,n,k}dz_k$, their coefficients are

$$G_{\bar{l}} = \sum_n\sum_u f_{a,n}(x,u)\frac{\partial w_n(u)}{\partial\bar{z}_l}, \qquad Q_{k\bar{l}} = -\frac{1}{12}\sum_n\sum_u c_{a,n,k}(x,u)\frac{\partial w_n(u)}{\partial\bar{z}_l}. \tag{14.14}$$

    Every coefficient is a locally finite sum of the specified period, Cartier, connection and elementary cutoff expressions. The minus sign in the second formula is from moving $d\bar{z}_l$ past dz_k.

    On a regular fibre, the one-form cocycle is that of the line with Chern class 12η, with the Chern sign fixed in step 2. Consequently $i_b^*[Q] = \eta$. This is a statement about the exact Dolbeault quotient; the resulting coefficient form need not be pointwise translation invariant on the fibre. For the scalar cocycle, $f_{ab} = -\Delta r_\gamma$, so its fibre image is $\bar{\partial}r_\gamma$, explicitly

$$\frac{i}{2\Delta}\left(d\bar{\zeta}_2 - \frac{\operatorname{Im}\mu}{\operatorname{Im}\tau}d\bar{\zeta}_1\right).$$

This is (13.12); injectivity in proposition 13.7 gives $[G] = g_\gamma$. At $z = 1$ the scalar line has the finite root characters and collar frames of $\mathcal{O}(S_1 - S_2)$. Its twelfth power is

$$\mathcal{O}(12S_1 - 12S_2) = f^*\mathcal{O}(4p_1 - 3p_2) \simeq f^*\mathcal{O}(1),$$

where the last map is induced by multiplication by $t^4/(t-1)^3$: its divisor is $4p_1 - 3p_2 - p_0$, so it identifies $\mathcal{O}(4p_1 - 3p_2)$ with the hyperplane line $\mathcal{O}(p_0) = \mathcal{O}(1)$ in the original base coordinate. This agrees with the original sign and factor in proposition 13.8.

13. *The fibre square of a closed real form.* Suppose a real $(1,1)$ form with the stated fibre class were $\bar\partial$ closed. Reality would give $\partial Q = 0$, hence $dQ = 0$. Since $H^2(X;\mathbb{R}) = 0$, write $Q = dA$. On a regular compact fibre, Dolbeault equality with η and Stokes' theorem give $\int_F (i^*Q)^2 = \int_F \eta^2 = -12$. But $(i^*Q)^2 = d(i^*A \wedge i^*Q)$ has integral zero. This contradicts the exact fibre square and proves the last assertion.

□

14.2 Comparison with the Green and harmonic representatives

Proposition 14.3 (Green and harmonic representatives)**.** [tag 03LI]

The real restriction map

$$r_b : H_A^{1,1}(X;\mathbb{R}) \longrightarrow H_A^{1,1}(F_b;\mathbb{R})$$

has image $\mathbb{R}\eta$ by 12.16 and 13.14. Its source has real dimension two and its target has real dimension four. The map is induced by pullback of $\partial\bar\partial$-closed forms followed by the quotient by ∂- and $\bar\partial$-exact forms on F_b. Thus it specifies the class of each restricted representative through that exact quotient map.

The formula (13.5) supplies a global G after a metric choice. For the standard primitive marking, $G_\gamma = G_0/12$ represents g_γ by 13.8. There is a precise operator construction of Q_H as well. Fix a Hermitian metric on X and a regular fibre F, identify η with its real invariant form on F, and choose an orthonormal basis e_1, e_2 of the Dolbeault-harmonic $(1,1)$ forms for $\Delta_{\bar\partial}$. Define

$$\ell(e) = -\frac{1}{12}\int_F i_F^* e \wedge \eta, \qquad Q_H = \frac{\overline{\ell(e_1)}e_1 + \overline{\ell(e_2)}e_2}{|\ell(e_1)|^2 + |\ell(e_2)|^2}.$$

The denominator is positive by 12.16 and $\int_F \eta^2 = -12$. The expression is independent of the orthonormal basis, is $\bar\partial$-closed, and has fibre Dolbeault class η. Indeed the image of fibre restriction on $H^{1,1}_{\bar\partial}(X)$ is $\mathbb{C}\eta$, with kernel $\mathbb{C}d_1 g$; the displayed functional measures exactly the coefficient of η, and the formula makes that coefficient one.

The exact comparison on cohomology is

$$[G] = [G_0]/12 = g_\gamma, \qquad [Q_H] - [Q] \in \mathbb{C}d_1 g.$$

The second inclusion follows from the restriction kernel in proposition 12.16. Thus the two formulas choose lifts of the same fibre class; their difference is the specified one-dimensional Leray ambiguity. The harmonic normalization remains basis independent for the fixed metric and regular fibre.

Proof. 1. The operator formulas have the following normalization. The Green formula and $\exp([G_0]) = f^*\mathcal{O}(1)$ were proved in 13.7; the primitive marking in 13.8 then gives $[G_0]/12 = g_\gamma$.

2. Restriction of a Dolbeault-harmonic representative is a closed Dolbeault form on the regular fibre. By 12.16, its class is $c\eta$. Integration against the closed $(1,1)$ form η annihilates Dolbeault-exact changes by Stokes' theorem. Since $\int_F \eta^2 = -12$, the displayed functional gives $\ell(e) = c$. The image is $\mathbb{C}\eta$, hence ℓ is nonzero on the two-dimensional harmonic space and its squared norm $|\ell(e_1)|^2 + |\ell(e_2)|^2$ is positive.

3. For another orthonormal basis $e'_j = \sum_k U_{jk} e_k$ with U unitary, put $l_k = \ell(e_k)$. Then $l' = Ul$ and
$$\sum_j \overline{l'_j} e'_j = \sum_{j,k,r} \overline{U_{jk}}\,\overline{l_k} U_{jr} e_r = \sum_k \overline{l_k} e_k, \qquad \sum_j |l'_j|^2 = \sum_k |l_k|^2.$$
Thus Q_H is independent of the orthonormal basis. It is a linear combination of harmonic forms and $\ell(Q_H) = 1$, so its fibre class is precisely η. These computations establish the operator representatives and their restrictions. The evaluated coefficient construction above provides another representative with the same fibre normalization.

□

14.3 Changes of the specified coefficient gauge

Remark 14.4 (Fibre restriction of the coefficient representative)**.** [tag 03CT]

The formulas (14.13) and (14.14) choose global smooth forms through all three special fibres. Their restriction statement uses the pullback map followed by the Dolbeault quotient on a regular fibre. In particular, its translation-invariant value is the class represented in (13.13). The specified coefficient form on that fibre can be related to the invariant representative by a Dolbeault coboundary. The cocycle and weight formulas below describe the changes produced by these choices.

Remark 14.5 (Changes of coefficient gauge)**.** [tag 03CS]

Use the same holomorphic cocycles c_{ab}, f_{ab} on a common refinement, and two locally finite weight systems w_n, w'_n with sum one. In a chart a set
$$K_a = \sum_n \sum_u (w'_n - w_n) c_{a,n}, \qquad U_a = \sum_n \sum_u (w'_n - w_n) f_{a,n}.$$
The cocycle identities and $\sum(w'_n - w_n) = 0$ give $K_a - K_b = 0$ and $U_a - U_b = 0$. They therefore define a global smooth $(1,0)$ form K and a global smooth function U, with
$$Q' - Q = \tfrac{1}{12}\bar\partial K, \qquad G' - G = \bar\partial U.$$
For a holomorphic change of cocycles $c'_{ab} = c_{ab} + r_a - r_b$ and $f'_{ab} = f_{ab} + u_a - u_b$, where r_a is a holomorphic one-form and u_a a holomorphic function, the same weights give
$$B'_a - B_a = r_a - \sum_n \sum_u w_n r_n, \qquad H'_a - H_a = u_a - \sum_n \sum_u w_n u_n.$$
The two sums on the right are globally defined smooth forms of the indicated degrees. Consequently
$$Q' - Q = -\tfrac{1}{12}\bar\partial \sum_n \sum_u w_n r_n, \qquad G' - G = -\bar\partial \sum_n \sum_u w_n u_n.$$
These formulas identify the Dolbeault coboundaries under the specified gauge changes and preserve all class and product calculations. A change of coherent-Leray lift of the fibre class is measured separately by $\mathbb{C}d_1 g$, as in the operator comparison above.

15 Twisted tangent cohomology

15.1 All integral twists and the generic nonintegral twist

Proposition 15.1 (Integral Picard twists, including their torsion)**.** [tag 01S0]

For $M_n = \mathcal{O}_X(n(S_1 - S_2))$, define $u(n) = \max\{d(n) + 1, 0\}$, $v(n) = \max\{-d(n) - 1, 0\}$ and $\epsilon_n = 0$ if $3 \mid n$, $\epsilon_n = 1$ otherwise. Then

$$\begin{aligned} f_*(T_X \otimes M_n) &= \mathcal{O}(d(n)) \oplus \mathcal{O}(d(n-1)), \\ R^1 f_*(T_X \otimes M_n) &= \mathcal{O}(d(n)) \oplus \mathcal{O}(d(n-1)) \oplus \mathcal{O}(d(n+2)), \\ R^2 f_*(T_X \otimes M_n) &= \mathcal{O}(d(n+6)) \oplus \mathbb{C}_{p_1}^{\epsilon_n}. \end{aligned} \tag{15.1}$$

In particular its cusp stalk is free of rank one. The four global dimensions, for every integer n, are

$$\begin{aligned} h^0 &= u(n) + u(n-1), \\ h^1 &= v(n) + v(n-1) + u(n) + u(n-1) + u(n+2), \\ h^2 &= v(n) + v(n-1) + v(n+2) + u(n+6) + \epsilon_n, \\ h^3 &= v(n+6). \end{aligned} \tag{15.2}$$

Proof. 1. The defining roots give $\bar{f}^* R = \mathcal{O}_X(S_1 - S_2)$, so the projection formula applies to (B.8), with its coherent torsion summand. Tensoring (B.5) changes its generator character from -1 to $n - 1$. There is exactly one invariant coefficient among exponents $0, 1$ if $n - 1$ is 0 or 1 modulo three, and none if it is 2. Multiplication by the coarse parameter $t = s^3$ is zero on this module. Its invariant direct image is therefore exactly the displayed skyscraper, of length ϵ_n, not a higher-order thickening. The two other special places contribute no torsion. Formula (2.4) proves (15.1), and the two-column Leray sequence proves (15.2).

□

Proposition 15.2 (The cusp length and the generic twist)**.** [tag 01SP]

Use the coordinate $M_z = \exp(z\mathfrak{g})$ of 13.8. For every $z \notin \mathbb{Z}$ the cusp contribution is exactly

$$(R^0 f_*(T_X \otimes M_z))_{p_0} = 0, \quad (R^1 f_*(T_X \otimes M_z))_{p_0} = 0, \quad (R^2 f_*(T_X \otimes M_z))_{p_0} = \mathbb{C}\{q\}/(q). \tag{15.3}$$

Outside the countable exceptional set explicitly defined in 15.3, these are the only nonzero direct-image stalks. Hence the generic twist has $(h^0, h^1, h^2, h^3) = (0, 0, 1, 0)$, and the cusp nilpotence exponent r equals one there.

Proof. 1. The map $H^1(W, \mathbb{C}) \to H^1(W, \mathcal{O}_W)$ is an isomorphism by the normalization complex in 12.2. The restriction of M_z has trivial pullback to P and descent character $(\exp(2\pi i z), 1)$ on the two generators of the dual torus. The normalization complex with these coefficients is the cellular cochain complex of that rank-one local system. It may be computed on the product cell structure as

$$\mathbb{C} \xrightarrow{(a-1,0)^t} \mathbb{C}^2 \xrightarrow{(0,a-1)} \mathbb{C}, \qquad a = \exp(2\pi i z).$$

For $z \notin \mathbb{Z}$ this is acyclic. All normalization components remain rational with trivial pulled-back twist, so this normalization resolution computes $H^q(W, M_z|_W)$ through the displayed cellular complex. Analytic semicontinuity and base change make $R^q \bar{f}_* M_z$ zero on a neighbourhood of p_0 for all q. The projection formula and (12.5) imply the same vanishing for $\mathcal{T} \otimes M_z$. Tensoring (B.7) remains exact, and $\nu^* M_z = \mathcal{O}_P$ because P is rational and the topological first Chern class of M_z is zero. Thus the degree-two direct image is exactly the reduced skyscraper $H^2(P, K_P) \otimes \mathbb{C}_{p_0}$. The reduced support gives annihilator (q) and proves (15.3).

2. If the twist is nontrivial on every smooth fibre and on both finite root-cover central tori, all cohomology of its fibre restriction vanishes at those places. Indeed a nontrivial topologically trivial line bundle on a complex torus has zero cohomology: in a unitary flat presentation its Dolbeault Laplacian has no zero Fourier mode. Tensoring the tangent extension by that line preserves acyclicity. Base change on the root charts and exact invariants then give zero direct images away from p_0. 15.3 proves that the excluded twists form a countable set. Leray now gives the asserted four dimensions.

□

Proposition 15.3 (The exceptional nonintegral twists)**.** [tag 01SQ]

On a regular marked fibre, M_z is trivial if and only if there are integers n, p, q, r such that

$$q\tau(t) + r\mu(t) = p, \qquad z = n - 6q\mu(t) - r\beta_c(t). \tag{15.4}$$

For $z \notin \mathbb{Z}$ these resonances occur in a countable set of parameter values z; every resonant z is nonreal. On the finite root covers the exceptional lattices are

$$\Lambda_1 = \mathbb{Z} + (3\beta_c(z_1) + 4 - 2\rho)\mathbb{Z}, \qquad \Lambda_2 = \mathbb{Z} + (2\beta_c(z_2) + 3 - 3i)\mathbb{Z}. \tag{15.5}$$

Take the union of $\mathbb{Z}$, these two lattices, and all values from (15.4) over regular fibres; this is the countable exceptional set used in 15.2. The set is larger than $\mathbb{Z}$: the regular value $t = 32/5$ supplies nonintegral twists with $h^1(X, T_X \otimes M_z) \geq 2$.

Proof. 1. The exponential sequence on a torus identifies trivial line bundles with integral constant cocycles modulo the two holomorphic period rows. Write $z\gamma = (n, p, q, r) + a(6\mu, \tau, 1, 0) + b(\beta_c, \mu, 0, 1)$. The third and fourth coordinates give $a = -q, b = -r$; the first two give exactly (15.4). If $r = 0$, the positivity of $\operatorname{Im}\tau$ forces $q = p = 0$, leaving $z = n$. Otherwise the imaginary parts of (15.4) give

$$\operatorname{Im}\mu = -\frac{q}{r}\operatorname{Im}\tau, \qquad \operatorname{Im} z = -rD \neq 0. \tag{15.6}$$

For fixed (p, q, r) with $(q, r) \neq (0, 0)$, the first equation of (15.4) has discrete zeros, since $1, \tau, \mu$ are linearly independent. A discrete subset of this second-countable Riemann surface is countable. There are countably many integer tuples and markings, so the union of the resulting z is countable.

2. At z_1, insert $\tau = \rho$, $\mu = (2 - \rho)/3$ in the first equation. Its solutions have $(q, r, p) = (k, 3k, 2k)$. At z_2 they have $(q, r, p) = (k, 2k, k)$. The second equation gives (15.5). These are discrete lattices because their nonintegral generators have imaginary parts $3D$ and $2D$.

3. For the explicit regular resonance, use the section $P = (0, \sqrt{2}t^2(t-1))$ of the stated elliptic surface. The chord-and-tangent law gives

$$x(2P) = 9t^2/8, \qquad y(2P) = t^2(32 - 5t)/(16\sqrt{2}).$$

At $t = 32/5$, the discriminant $1728t^8(t-1)^3$ is nonzero, P is not two-torsion, and $2P$ is a nonzero two-torsion point. Thus P has exact order four. The period identification in lemma 2.10 supplies a nonzero relation $q\tau + 4\mu = p$ with integers p, q. Equation (15.4), with $r = 4$, defines a twist M_z for which the regular torus restriction is trivial; (15.6) shows that it is nonintegral. The period derivative there has nonzero determinant, so the tangent restriction has $h^0(T_X|_F) = 2$. The degree-zero base-change exact sequence gives

$$H^0(F, (T_X \otimes M_z)|_F) \simeq (R^1 f_*(T_X \otimes M_z))_t[t],$$

because $f_*(T_X \otimes M_z) = 0$ for a twist nontrivial on the general fibre. Its right side has dimension two. The degree-one direct image is a finite-support sheaf, so its global sections

have dimension at least two, proving the bound on h^1. The integers p, q are the marked elliptic-logarithm coordinates of this order-four point; the displayed period relation determines the corresponding resonance lattice.

□

Proposition 15.4 (Local complexes and every noncusp torsion length)**.** [tag 01SR]
Let $z \notin \mathbb{Z}$. At a regular resonance t_* in (15.4), with $A = \mathbb{C}\{t - t_*\}$, the two stalks are

$$(R^1 f_*(T_X \otimes M_z))_{t_*} = (A/(t - t_*))^2, \qquad (R^2 f_*(T_X \otimes M_z))_{t_*} = A/((t - t_*)^2). \tag{15.7}$$

At p_1, write $z = n + k(3\beta_1 + 4 - 2\rho)$, $k \neq 0$, when this is possible. Both coarse stalks are killed by t and have length $1 + \epsilon_n$, with ϵ_n as in 15.1. At p_2, write $z = n + k(2\beta_2 + 3 - 3i)$, $k \neq 0$, when possible. Both coarse stalks are killed by $t - 1$ and have length one for $n \equiv 1, 2 \pmod 4$ and length zero for $n \equiv 0, 3 \pmod 4$. If the relevant lattice representation does not exist, both stalks at that finite fibre are zero.

Proof. 1. On a simply connected disc of smooth torus fibres, including a finite root cover, retain the real marking and the holomorphic period matrix. Translation by $\Pi(s)v$, for fixed $v \in \mathbb{R}^4$, is holomorphic on the total family. After subtracting the integer cocycle of a resonance, the line bundle has constant lattice multipliers. Multiplying the lifted real translation by its constant factor of automorphy makes the lattice act as the identity and gives a genuine compact real torus action on the line bundle. The same action acts on the tangent bundle by differentiation. It commutes with the full Dolbeault differential.

2. Averaging over this compact torus projects the relative Dolbeault complex onto its invariant part. Choose the lift so that the central trivial line has trivial character. On every nonzero Fourier character, the relative Dolbeault symbol is exterior multiplication by a nonzero covector; contraction by a dual covector is a contracting homotopy. After shrinking the disc, these nonzero covectors have a uniform positive lower bound for bounded Fourier frequency and grow linearly for unbounded frequency, because the real period matrices and their inverses stay bounded. The homotopies therefore preserve smooth Fourier series, with all derivatives, and are uniform on compact subsets of the disc. The tangent extension is triangular with respect to its vertical and horizontal parts; its nilpotent off-diagonal term is treated by the finite homological perturbation series, so every nonzero Fourier complex is still contractible. These homotopies exhibit averaging as a quasi-isomorphism.

3. The chain-level comparison uses the following frames. Write the normalized period matrix as $[Z(s), I]$ and use the real coordinates $\zeta = Z(s)x + y$, with $x, y \in \mathbb{R}^2/\mathbb{Z}^2$. Set

$$G_j = (dx_j)^{0,1}, \qquad e_i = \partial_{\zeta_i}, \qquad h = \partial_s + \sum_i (Z'(s)x)_i e_i.$$

These are invariant under the real translations. Since each dx_j is closed on the total real product, $\bar\partial G_j = 0$. The forms G_j frame the relative invariant $(0,1)$ forms because $\operatorname{Im} Z$ is invertible, and

$$\bar\partial h = \sum_{i,j} Z'_{ij}(s) e_i G_j, \qquad (dy)^{0,1} = -Z(s)(dx)^{0,1}.$$

For the smooth line frame defined by its flat character, the last identity gives $\bar\partial \ell = 2\pi i(a_1 G_1 + a_2 G_2)\ell$, with a_1, a_2 in (15.9); there is no additional base term. All displayed coefficients are holomorphic in s. The full invariant complex is thus the ordinary base Dolbeault resolution of the following finite complex of holomorphic bundles:

$$C^j = (V \oplus L) \otimes \bigwedge^j G, \quad 0 \le j \le 2, \qquad d(v \otimes \eta) = v \otimes a \wedge \eta,$$

$$d(h\otimes\eta)=h\otimes a\wedge\eta+\kappa\wedge\eta. \tag{15.8}$$

Here V is the rank-two vertical bundle, L the horizontal line, $G=R^1\mathcal{O}$ on the root chart, and $\kappa\in V\otimes G\otimes L^*$ is the tangent-extension cocycle. The matrix is obtained by differentiating the period translations. In the raw period and constant-cocycle frames it is

$$\kappa=\begin{pmatrix}6\mu' & \tau'\\ \beta_c' & \mu'\end{pmatrix},\qquad a=(z-n+6q\mu+r\beta_c,-p+q\tau+r\mu). \tag{15.9}$$

Let $\lambda=2\pi i$. The Dolbeault exponential convention gives the differential with Picard vector λa and tangent term κ. Its explicit chain isomorphism to (15.8) is, in degree j,

$$F^j|_{V\otimes\bigwedge^j G}=\lambda^{-j}\,\mathrm{id},\qquad F^j|_{L\otimes\bigwedge^j G}=\lambda^{-j-1}\,\mathrm{id}\,.$$

Direct substitution gives $F^{j+1}d_{\lambda a,\kappa}=d_{a,\kappa}F^j$ on both summands. The inverse multiplies them by λ^j and λ^{j+1}, respectively. These constant maps also commute with the finite cyclic action. The displayed formula also follows directly by subtracting the holomorphic period rows from the constant lattice cocycle. Together these formulas identify the chain map and its tangent extension. For the nonzero Fourier characters, use the filtration by the number of base $(0,1)$ factors: the relative contractions make the associated graded complexes acyclic, and the two-step filtration then proves acyclicity of the full complexes. The finite cyclic group normalizes the real translation group, so averaging and this comparison are equivariant; exact invariants compute the coarse multiple-fibre stalks.

4. At a regular resonance, $a'=\kappa^t(q,r)^t$ and $(q,r)\neq(0,0)$. The determinant of κ is nonzero there by proposition 12.10. Therefore a has a simple zero. Choose its nonzero leading direction as the second frame of G and use invertibility of κ to choose the two vertical frames. Holomorphic invertible changes reduce (15.8) to

$$a=(0,s),\qquad \kappa=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{15.10}$$

The second cohomology is generated by $h\gamma\delta$ and is killed by s^2; the relations eliminate the two vertical generators. The first cohomology is generated by $e_1\delta, e_2\delta$, each killed by s. These are equalities of modules over $\mathbb{C}\{s\}$ and prove (15.7).

5. There is an additional joint coefficient to check at the order-three point. Put $t=s^3$, $\nu=\mu-(2-\tau)/3$ and $b=\beta_c+4\mu+(2/3)\tau-b(0)$. The order-three transformation laws give $\nu=\nu_2s^2+O(s^3)$ and $b=b_3s^3+O(s^4)$. The elliptic calculation in 12.9 gives, with elliptic scale $f(s)=Fs^2+O(s^3)$,

$$\nu_2^2=9F^2/512.$$

Use the original Weierstrass normalization. Since $x=f^2\wp$, $y=f^3\wp'/2$, one has $f^4g_2=12t^3(t-1)$ and $f^6g_3=-8t^4(t-1)^2$. The identities $g_2=4\pi^4E_4/3$, $g_3=8\pi^6E_6/27$ and $E_4'(\rho)=-2\pi iE_6(\rho)/3$ give

$$E_6(\rho)=-\frac{27}{\pi^6F^6},\qquad \tau(s)=\rho+\tau_1s+O(s^2),\quad \tau_1=\frac{i\pi F^2}{2}.$$

The period determinant identity in proposition 12.10 gives

$$K(s)=6\mu'^2-\tau'\beta_c'=K_2s^2+O(s^3),\quad K_2=\frac{3F^2}{4}.$$

Equivalently $K=6\nu'^2-\tau'b'$, so $K_2=24\nu_2^2-3\tau_1b_3$ and $b_3=7i/(32\pi)$.

6. For a nonintegral resonance the relation is $(q,r,p)=\pm(k,3k,2k)$, $k\neq 0$; the sign only changes the following nonzero scalar. After a constant change of the two cocycle frames, its Picard section is $3k(\nu,b)$, and the transpose period derivative becomes

$$\begin{pmatrix} \tau' & \nu' \\ 6\nu' & b' \end{pmatrix}.$$

Eliminating the unit τ' leaves the joint transverse component $3k(b-6\nu\nu'/\tau')$. Its coefficient of s^3 is

$$-\frac{k(K_2+12\nu_2^2)}{\tau_1}=\frac{123ik}{64\pi}\neq 0. \tag{15.11}$$

This nonzero joint coefficient determines the two transverse orders in the following Smith presentation.

7. Choose the second cocycle frame to be the Picard section divided by s^2. Equivariance and (15.11) give a matrix with a unit in its lower-right entry, first-order off-diagonal terms with nonzero coefficients, and a second-order upper-left term. Subtracting a holomorphic multiple of the second vertical row removes the last term; the nonzero determinant coefficient and (15.11) keep both off-diagonal coefficients units. Rescaling the equivariant frames therefore reduces (15.8) to

$$a=(0,s^2),\qquad \kappa=\begin{pmatrix} 0 & s \\ s & 1 \end{pmatrix}. \tag{15.12}$$

Its cohomology is

$$H^1=\mathbb{C}\{s\}/(s^2)\oplus\mathbb{C}\{s\}/(s^3),\qquad H^2=\mathbb{C}\{s\}/(s^2)\oplus\mathbb{C}\{s\}/(s^3). \tag{15.13}$$

For example in degree one put $u=e_1\delta$, $v=e_2\gamma+sh\delta$; the remaining cycle $e_2\delta$ is eliminated by the boundary $su+e_2\delta+sv$. The two surviving relations are $s^2u=0, s^3v=0$. In degree two eliminate $e_2\gamma\delta$; the remaining generators $e_1\gamma\delta, h\gamma\delta$ have annihilators s^2, s^3 respectively.

8. A frame has invariant exponent w when its coefficient satisfies $g(\zeta s)=\zeta^w g(s)$; its pullback-frame weight is the negative of w. For the trivialized twist in (15.5), these exponents are n at p_1 and $3n$ at p_2. This agrees with M_n in 15.1 and follows by evaluating its integer cocycle on v_1, v_2. At p_1 the two generators in H^1 of (15.13) have invariant exponents $n+2, n+2$, and the two generators in H^2 have exponents $n+2, n$. The length-three summand has one invariant coefficient for every n; the length-two summand has one precisely when $n\not\equiv 0 \pmod 3$. The coarse parameter is s^3, so all these invariants are killed by it. This proves the asserted order-three stalks with the full nilpotent layers included.

9. At p_2 the determinant is a unit. The Picard section has a simple zero, so (15.10) applies on the order-four cover. The invariant exponents of the two degree-one generators are $3n+1, 3n+2$; the cyclic degree-two module of length two has generator exponent $3n+2$. In both degrees its invariant dimension is one exactly for $n\equiv 1,2 \pmod 4$. The coarse parameter s^4 kills the modules. This proves the stated order-four calculation.

□

Proposition 15.5 (The full twisted cohomology and its jump loci)**.** [tag 01SS]

For $z\notin\mathbb{Z}$, let $\mathcal{R}_z$ be the set of regular base points satisfying (15.4), counted as points, not as marked integer tuples. It is finite. Put $N_z=|\mathcal{R}_z|$. Define $\ell_1(z)=0$ off Λ_1, and $\ell_1(z)=1+\epsilon_n$ if $z=n+k(3\beta_1+4-2\rho)$, $k\neq 0$. Define $\ell_2(z)=1$ if $z=n+k(2\beta_2+3-3i)$, $k\neq 0$, with $n\equiv 1,2 \pmod 4$, and $\ell_2(z)=0$ otherwise. Then, setting $J_z=2N_z+\ell_1(z)+\ell_2(z)$,

$$\boxed{(h^0,h^1,h^2,h^3)(T_X\otimes M_z)=(0,J_z,1+J_z,0).} \tag{15.14}$$

More precisely, with $\mathfrak{m}_t$ the maximal ideal at t,

$$R^1 f_*(T_X \otimes M_z) = \bigoplus_{t \in \mathcal{R}_z} \mathbb{C}_t^2 \oplus \mathbb{C}_{p_1}^{\ell_1(z)} \oplus \mathbb{C}_{p_2}^{\ell_2(z)},$$

$$R^2 f_*(T_X \otimes M_z) = \mathbb{C}_{p_0} \oplus \bigoplus_{t \in \mathcal{R}_z} \mathcal{O}_{B,t}/\mathfrak{m}_t^2 \oplus \mathbb{C}_{p_1}^{\ell_1(z)} \oplus \mathbb{C}_{p_2}^{\ell_2(z)}. \tag{15.15}$$

Together with 15.1 these formulas cover every element of $\mathrm{Pic}(X) = \mathbb{C}$. *The cohomology jump loci are locally discrete subsets of this complex line.*

Proof. 1. The support assertions and all stalks were proved in 15.2–15.4. The direct image in degree zero is zero for a nonintegral twist: it has generic rank zero and is torsion-free, since multiplication by a nonzero base function is injective on the bundle on the smooth integral total space. The support of either higher coherent direct image is a proper closed analytic subset of the compact curve, and hence is finite. This proves finiteness of $\mathcal{R}_z$ as well as the absence of an accumulation at the special values. If two integer tuples at a marked fibre represented the same trivialized line, their difference would be an integral real class in $H^{1,0}$, which is zero. Thus there is no multiplicity from duplicate tuples. Formula (15.15) follows by the decomposition of a finite-support sheaf into its local modules. Its global sections have dimension equal to its length and its H^1 is zero, so Leray gives (15.14).

2. The $\bar\partial$-closed representative $G_\gamma = G_0/12$ of propositions 13.7 and 13.8 defines a holomorphic line bundle on $X \times \mathbb{C}$ by the integrable operators $\bar\partial - 2\pi i z G_\gamma$. Proper coherent base change for the projection to $\mathbb{C}$ makes each cohomology-dimension threshold an analytic subset. A real noninteger z has no resonance by (15.6) and (15.5), so the generic values are attained. All possible exceptional values were shown to form a countable set. Thus each threshold exceeding a generic dimension is a proper analytic subset of a complex curve and is locally discrete. The jump set is the union of the four thresholds $h^0 \geq 1$, $h^1 \geq 1$, $h^2 \geq 2$ and $h^3 \geq 1$, hence is locally discrete. The order-four congruences $n \equiv 0, 3 \pmod 4$ have zero contribution in (15.14).

3. The jump loci can also be indexed algebraically on the elliptic surface. If a regular fibre has torsion section P of exact order r_t, choose its marked relation $q_t\tau + r_t\mu = p_t$. Its full resonance lattice is

$$\mathbb{Z} + (6q_t\mu(t) + r_t\beta_c(t))\mathbb{Z}.$$

   Every regular resonance arises this way. The underlying values t are exactly the non-special roots of the division polynomials $\psi_N(P)$, $N \geq 2$. The nonintegral lattice points contribute two to J_z for each distinct such fibre. The finite-point lattices have the congruence weights already stated. The division-polynomial recursion gives the first nonconstant regular factors $5t - 32$, $89t^2 + 4736t - 4096$ and $11t^2 - 6016t + 8192$. The displayed lattice formula sends every such root to its resonance values in the fixed Picard coordinate.

4. For the index, $c_1(X) = c_2(X) = c_1(M) = 0$ in integral cohomology, so the degree-six Riemann–Roch integrand for the twisted tangent bundle is $c_3/2$. Its integral is one. Both (15.2) and (15.14) have alternating sum one.

□

Proposition 15.6 (Local separation and Kodaira dimension)**.** [tag 0335]

The multiplicity pair $(3k, 4l)$ is constant on every sufficiently small deformation of one of the preceding members. Distinct pairs therefore determine disjoint open loci among these isomorphism classes. Moreover

$$\kappa(X) = \begin{cases} -\infty, & kl < 3k + 4l, \\ 1, & kl > 3k + 4l. \end{cases}$$

Equality $kl = 3k + 4l$ never occurs under $\gcd(3k, 4l) = 1$.

Proof. 1. Local miniversality identifies every small deformation with the fixed-data period family; 9.4 makes its multiplicity pair an intrinsic biholomorphic invariant. The canonical formula of B.2 and connected fibres identify its pluricanonical sections with those of $\mathcal{O}(r(kl - 3k - 4l))$ on the root curve. Negative weight has no sections. Positive weight has linearly growing section counts along multiples of $12kl$, giving Kodaira dimension one. Equality would imply $k \mid 4l$; coprimality forces $k = 1$, and then $l = 3 + 4l$ is impossible.

□

Proposition 15.7 (Separation in irreducible analytic families)**.** [tag 033B]

Suppose a proper holomorphic submersion over a connected irreducible reduced complex base has two fibres of the type in B.2. Then their multiplicity pairs agree. For these members the asymptotic number

$$s(X) = \lim_{r\to\infty} \frac{h^0(X, K_X^r) - h^0(X, K_X^{-r})}{r} = \frac{1}{12} - \frac{1}{3k} - \frac{1}{4l}$$

recovers the ordered multiplicity pair uniquely.

Proof. 1. For coprime m, n, the number of nonnegative solutions of $mi+nj = N$ is $N/(mn)+O(1)$ as $N \to \infty$: there is one permitted residue class of j modulo m, and its representatives lie between 0 and N/n. The canonical formula in B.2 proves the displayed limit, with a negative value when the anticanonical counts grow. Write $1/12 - s(X) = a/b$ in lowest positive terms. Since $\gcd(m + n, mn) = 1$, we have $a = m + n$, $b = mn$. Thus m, n are the two roots of $T^2 - aT + b$. Their order is determined because only m is divisible by three and only n is divisible by four.

2. For every fixed integer r, apply the finite-free-complex theorem of (Bănică and Stănăşilă, 1976, Chapter III, Theorem 4.1, p. 124) to the relative canonical power, which is an invertible flat sheaf for a holomorphic submersion. Its zeroth cohomology dimension has analytic jumping loci. On an irreducible base any value constant on a nonempty open set is its generic value: a proper analytic jumping locus has empty interior. Local miniversality in B.2 makes these dimensions constant near either of the two given fibres, since all nearby fibres have the same multiplicities and their canonical formula is independent of the period parameter. Consequently the two fibres have equal dimensions for every positive and negative r. Their limits $s(X)$ are equal, and the preceding arithmetic recovers the same pair.

□

16 The integral circle quotient and its cusp replacement

Definition 16.1 (Circle-quotient conventions)**.** [tag IP06] We write g, u, w, d for the ordered homology basis $(\widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta})$ in this section, and γ for its first dual covector. The cusp coordinate is $q = 1/t$. Positive target orientations are the complex orientation on the base followed by the positive γ-circle orientation. The complex orientation on X fixes the source orientation.

Proposition 16.2 (The primitive integral quotient and its finite fillings)**.** [tag 02PA]

Use the monodromy matrices of (2.2) *and the standard affine vectors of Definition 2.2. Put*

$$M_0 = (A_1A_2)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

The common invariant vectors and covectors are $\mathbb{Z}d$ and $\mathbb{Z}\gamma$, respectively. The smallest invariant primitive sublattice containing the cusp vanishing lattice is $L = \mathbb{Z}\langle u, w, d\rangle = \ker\gamma$. The formula

$$[z, \Pi(z)x] \longmapsto ([z], \gamma(x) \bmod \mathbb{Z}) \tag{16.1}$$

defines a real-analytic fibrewise group epimorphism $J \longrightarrow B^\circ \times S^1$ with connected kernel T_L. At the finite fillings it descends to a smooth proper T^3-bundle over a Seifert solid torus, of multiplicities 3 *and* 4 *and translation residues* 1 *and* -1*.*

Proof. 1. For the matrices of (2.2), direct integral multiplication gives $A_1^3 = A_2^4 = A_1A_2M_0 = I$ and $A_jv_j = v_j$. Solving the two fixed-vector equations gives $(0, 0, 0, d_0)$. For a row vector (a, b, c, e), the first fixed-covector equation gives $b = 2c, e = 3c$ and the second gives $b = c, e = 2c$; therefore $b = c = e = 0$. Also

$$(M_0 - I)(a, b, c, e) = (0, 0, b, -a), \qquad A_1w = u - w.$$

These identities prove the assertions about L and the cusp lattice $K = \mathbb{Z}\langle w, d\rangle$. Their primitivity follows from the displayed bases. The flat real marking identifies the deck action on a fibre with $x \mapsto A_jx$. Since $\gamma A_j = \gamma$, (16.1) descends. Its kernel is connected because γ is primitive.

2. On a root disc the affine generator is

$$(s, x) \longmapsto (e^{-2\pi i/m}s, A_jx + v_j/m), \qquad (m, \kappa) = (3, 1), (4, -1).$$

It induces $(s, \theta) \mapsto (e^{-2\pi i/m}s, \theta + \kappa/m)$ on the character quotient. This action is free since $\gcd(\kappa, m) = 1$. Writing $a = e^{2\pi i\theta}$ and choosing r with $r\kappa \equiv 1 \pmod m$, the invariant coordinates sa^r and a^m identify its quotient with $D^2 \times S^1$. The base coordinate becomes $(sa^r)^m(a^m)^{-r}$, and hence has multiplicity m at its central circle. Before the finite quotient the map is a trivial T_L-bundle. The free actions on source and target are equivariant for that map, so it descends locally to a smooth T_L-bundle. Properness follows from compactness of T_L. The logarithmic translation used to glue the root chart to J changes the character by its γ-component. Applying γ to the gluing translation therefore gives the target gluing, proving agreement on every overlap.

□

Remark 16.3 (Comparison of the quotient characters)**.** [tag IP08]

Engel's primitive invariant covector ψ (Engel, 2026, Lemma 3.8, p. 11) is related to the present character by

$$P_E = P_{\mathrm{mark}}, \qquad \psi P_E = -\gamma, \tag{16.2}$$

where P_{mark} is the integral isomorphism of Proposition 2.3. Thus the quotient characters agree after the target-circle map $\theta \mapsto -\theta$. Inverting the root generators gives the translation vectors $a_{j-1} = -P_Ev_j/m_j$, as in Proposition 8.2.

Proposition 16.4 (Pointwise cusp limits and the fixed double curve)**.** [tag 02PB]

The literal character has no continuous extension at either triple point or at generic points of two of the three double curves. It has the constant limit 0 *on the smooth locus of W and on the third double curve away from the triple points. In the chart $q = z_0z_1z_2$, $x_1 = z_1$, $x_2 = z_2$, the extendible double curve is $z_0 = z_1 = 0$, $z_2 \neq 0$. The fixed curve of the $\mathbb{C}^*$-action in direction $d = e_2$ is instead $z_0 = z_2 = 0$, $z_1 \neq 0$, and is one of the nonextendible curves. Consequently the extendible double curve and the fixed double curve are distinct.*

Proof. 1. Put $q = e^{-2\pi T}$ on a positive radial ray and use the exact cusp periods

$$Z_T = iTB_0 + C(q), \qquad B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where C is bounded and holomorphic at 0. For a fixed $a \in \mathbb{R}^2$ the representative $x_T(a) = \exp(2\pi i Z_T a)$ has character $a_1 \bmod \mathbb{Z}$ and logarithmic position

$$\frac{\log|x_T(a)|}{\log|q|} = B_0 a + T^{-1} \operatorname{Im} C(q) a. \tag{16.3}$$

The choices $a = (-1/4, 1/4)$ and $a' = (-1/2, 1/4)$ have logarithmic positions tending to $(1/4, 1/4)$ and $(1/4, 1/2)$. Both are interior to the triangle with vertices $0, e_1, e_2$, so all three chart coordinates tend to 0. Their character values are $3/4$ and $1/2$. This proves nonextension at the first triple point. The other triangle type has vertices $e_1, e_2, e_1 + e_2$. Its coordinates can be taken as $x_1x_2/q, q/x_2, q/x_1$. Positions $(3/4, 3/4)$ and $(3/4, 1/2)$ lie in its interior, give distinct values of $-y_2$ modulo one, and converge to its origin. Translation by the lattice preserves triangle type; these are precisely the two triple points of W.

2. For arbitrary convergent paths in the first chart, inversion of the exact logarithmic period matrix gives

$$\theta_\gamma = -\frac{\log|z_2|}{\log|q|} + o(1) \pmod{\mathbb{Z}}. \tag{16.4}$$

Here the error tends to zero whenever the logarithmic ratios stay bounded; they do stay bounded near a fixed point of a normal-crossing stratum because $q = z_0z_1z_2$ and the three coordinates are bounded. If $z_0, z_1 \to 0$ and $z_2 \to c \neq 0$, the right side tends to zero. If $z_0, z_2 \to 0$ and $z_1 = c \neq 0$, the paths $z_2 = q^a, z_0 = q^{1-a}/c$, $0 < a < 1$, have limits $-a \pmod 1$ at the same double point. The same construction with $z_0 = c$ proves nonextension on $z_1 = z_2 = 0$. At a smooth point of a component the ratio in (16.4) tends to 0 or 1, giving zero modulo one. Integer translates of these charts give the stated assertion on all of W.

3. The torus action in direction $e_2 = d$ is

$$\lambda \cdot (z_0, z_1, z_2) = (\lambda^{-1}z_0, z_1, \lambda z_2).$$

Its fixed curve is $z_0 = z_2 = 0$, while the character extends along $z_0 = z_1 = 0$ by the preceding limit formula. The displayed action and limit formula give these two loci in the same toric chart.

□

Lemma 16.5 (The cusp neighbourhood and its circle classes)**.** [tag IP07]

For the cusp neighbourhood N_0 and the vanishing lattice $K = \mathbb{Z}\langle w, d\rangle$,

$$\pi_1(N_0) = \overline{\Lambda} = \Lambda/K. \tag{16.5}$$

The punctured-cusp map to this group kills K and the base meridian. Both long-period covectors extend in integral $H^1(N_0; \mathbb{Z})$. The literal character on an ordinary fibre has no extension to all of $X \simeq S^6$.

Proof. The toric covering has rays $(v, 1)$ for all $v \in \mathbb{Z}^2$. Meridians about its toric divisors kill the whole cocharacter lattice: $(0, 1)$ is a ray and $(e_i, 1) - (0, 1) = (e_i, 0)$. Hence the toric cover is simply connected. Its free period deck group is $\overline{\Lambda} = \Lambda/K$, proving the displayed isomorphism. The base meridian bounds the disc given by a point of the dense orbit on a central component. Both long-period covectors kill these relations and therefore extend in integral H^1. An extension of θ_γ to all of $X \simeq S^6$ would give a nonzero homomorphism $H_1(F) \to \mathbb{Z}$ factoring through $H_1(X) = 0$, which is impossible. □

Proposition 16.6 (The filled three-dimensional target)**.** [tag 02PC]

For the original data the quotient over the complement of the cusp disc is the exterior of the $(3,4)$ *torus knot, with the specified Seifert structure. For general admissible* (l_0,l_1,l_2) *the filled quotient is the lens space determined by the following meridian slopes in one oriented integral torus basis* (ρ_1,h)*:*

$$\mu_1=(3,-l_1),\qquad \mu_2=(-4,4l_0-l_2),\qquad \det(\mu_1,\mu_2)=p=12l_0-4l_1-3l_2. \tag{16.6}$$

In the lens-space convention that the second meridian is $-q_0\mu_1+p\lambda_1$*, where* $\det(\mu_1,\lambda_1)=1$*, the lens parameter satisfies* $3q_0\equiv 4 \pmod p$*. In particular the target depends only on the signed integer* p*, and is a three-sphere exactly when* $|p|=1$*.*

Proof. 1. Let P be the three-holed sphere, with oriented boundary meridians $\rho_0+\rho_1+\rho_2=0$ and positive fibre h. The two finite meridian relations are $3\rho_1-l_1h=0$ and $4\rho_2-l_2h=0$; the cusp solid-torus meridian is ρ_0+l_0h. Eliminating ρ_0 gives $\rho_2=l_0h-\rho_1$ on the annulus joining the finite solid tori, which gives exactly (16.6). Since both slopes are primitive, this is a genus-one Heegaard splitting. Choose $a,b\in\mathbb{Z}$ with $al_1+3b=1$ and put $\lambda_1=(a,b)$. Direct multiplication gives

$$\mu_2=-q_0\mu_1+p\lambda_1,\quad q_0=a(4l_0-l_2)+4b,\quad 3q_0-ap=4. \tag{16.7}$$

This proves the claimed lens parameter in a convention fixed by the displayed oriented meridians. The fundamental group of this gluing is $\mathbb{Z}/|p|$. Admissibility implies $\gcd(p,6)=1$, so the modular inverse of 3 used above exists.

2. For $(l_0,l_1,l_2)=(0,1,-1)$ the two exceptional-fibre data are $(3,1),(4,-1)$. They are realized by the circle action
$$s\cdot(z_1,z_2)=(e^{4is}z_1,e^{3is}z_2)\quad\text{on } S^3\subset\mathbb{C}^2.$$
Its invariant map is $[z_1^3:z_2^4]$; the local circle translation residues are $4^{-1}=1 \pmod 3$ and $3^{-1}=-1 \pmod 4$. Removing a regular orbit gives the same two Seifert solid tori and their product annulus as above. A regular orbit is the $(3,4)$ torus knot under the coordinate exchange $(z_1,z_2)\mapsto(z_2,z_1)$ when the opposite ordering is used. The displayed orbit map and meridian slopes identify the manifold with its peripheral structure.

□

16.1 The cusp replacement and its relative uniqueness

Theorem 16.7 (The cusp replacement and its relative homotopy class)**.** [tag 02PD]

For every admissible member, there is a smooth map $e_{\mathrm{q}}:X\to Q$ *over* B *whose restriction outside an arbitrarily small cusp neighbourhood is the literal circle quotient. For fixed quotient gluing, its homotopy class is independent of the cusp replacement, provided the replacement maps the cusp neighbourhood into its designated solid torus and is kept fixed on its outer collar. In particular there is a well-defined class for the original member* $X\simeq S^6$*.*

Proof. 1. On the cusp boundary use the product coordinates (q,θ_γ) after the specified collar shear. By (16.5), the boundary circle class extends to N_0. Since $S^1=K(\mathbb{Z},1)$, it has a continuous representative $\theta':N_0\to S^1$. Its restriction to the boundary is homotopic to the given one: the two maps induce the same homomorphism on π_1 of the boundary mapping torus. The homotopy-extension property for a manifold with boundary changes θ' on a collar so that it agrees exactly with θ_γ there. Relative smooth approximation gives a smooth representative, still equal on a smaller collar. Then (f,θ') is the required map into $D^2\times S^1$, and agrees with the finite-filling maps on the overlap. Both components of (f,θ') are smooth on N_0, so the product map is smooth at the critical points of f as well.

2. The inclusion $\partial N_0 \to N_0$ surjects on π_1, as (16.5) explicitly shows. It follows that $H^1(N_0;\mathbb{Z}) \to H^1(\partial N_0;\mathbb{Z})$ is injective. Since both spaces are connected, the long exact sequence gives $H^1(N_0,\partial N_0;\mathbb{Z}) = 0$. Two circle replacements agreeing on the boundary therefore have quotient whose relative H^1 class is zero, hence have a relative homotopy. Keeping the base coordinate fixed gives the required homotopy of maps into the solid torus. Different choices of the collar size are compared on a common smaller collar.

□

17 Real structures and their fixed three-spheres

Definition 17.1 (Real-period data)**.** [tag IP04] Use the lattice, period matrix and clockwise root generators of Definition 2.1, and the affine vectors of Definition 2.2. In this section write $d = \widehat{\delta}$, $g = \widehat{\gamma}$, $u = \widehat{u}$ and $w_\Lambda = \widehat{w}$. An unadorned w in a lattice expression denotes w_Λ; the scalar parameter remains $w = e^{-\pi i c}$. Put

$$e = \Pi_c d = (0,1)^{\mathsf{t}}, \qquad E = d \otimes \gamma, \qquad M_0 = (A_1 A_2)^{-1}, \qquad D_c = D_{\mathrm{part}} + \Im c < 0.$$

The circle $\rho(s)$ is translation by se, $s \in \mathbb{R}/\mathbb{Z}$. In cusp coordinates $q = 1/t$ and $s = (2\pi i)^{-1}\log q$, define the bounded matrix $C_c(q)$ by $Z_c = sB_0 + C_c(q)$, where $B_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Thus $\Pi_c g = (6\mu, -s + C_{c,21})^{\mathsf{t}}$. The cusp double curves D_1, D_2, D_{12} belong to the edge directions $e_1, e_2, e_1 - e_2$. The action ρ fixes D_2, while the literal character has a limit along D_1, as located by the explicit toric coordinates in Proposition 16.4.

Proposition 17.2 (Conjugate members and antiholomorphic automorphisms)**.** [tag 01QG]

Use the normalization $(\beta^{\mathrm{part}} + \tau)(p_0) = 0$. There is an antiholomorphic diffeomorphism $X_c \to X_{c'}$ exactly when

$$c' - (1 - \bar{c}) \in 2\mathbb{Z}, \qquad \textit{equivalently } w' = -\bar{w}.$$

Consequently $\overline{X_c} \simeq X_{1-\bar{c}}$. The members admitting antiholomorphic self-maps are exactly those with $\operatorname{Re} c \in \frac{1}{2} + \mathbb{Z}$, equivalently $w = -\bar{w}$. On each such member the group of holomorphic and antiholomorphic diffeomorphisms is

$$\mathrm{Aut}^{\pm}(X_c) \simeq \mathbb{C}^* \rtimes \mathbb{Z}/2, \qquad z \longmapsto 1/\bar{z}$$

for the action of the second factor. Its antiholomorphic maps form the nonidentity coset. For all other members the full group is the holomorphic automorphism group of Theorem C.4.

Proof. 1. Lift conjugation of the real orbifold base to the antiholomorphic involution ι of its universal cover whose cusp branch has $\tau(\iota z) = -\overline{\tau(z)}$. A lift exists by the universal covering property; it extends at the finite orbifold points, whose local equations are $s_j^{m_j} = t_j$. Choose the usual paths from a real point $t > 1$, so reflection fixes the second centre and sends the first chosen lift to $g_2^{-1} z_1$. The induced involution of the deck group is

$$\phi(g_1) = g_2^{-1} g_1^{-1} g_2, \qquad \phi(g_2) = g_2^{-1}, \qquad \phi(g_0) = g_0^{-1}.$$

These identities also follow by reflecting the lifted peripheral paths; their product and their squares agree with $g_1 g_2 g_0 = 1$. The cusp branch fixes the lift uniquely.

2. Denote the affine period transformations by G_1, G_2 on the triple (τ, μ, β) and put $K(\tau,\mu,\beta) = (-\bar{\tau}, 1 - \bar{\mu}, -\bar{\beta})$. Direct rational substitution gives

$$KG_1K = G_2^{-1} G_1^{-1} G_2, \qquad KG_2K = G_2^{-1}.$$

For example $G_2^{-1}(\tau,\mu,\beta)=(-1/\tau,(1-\mu)/\tau,\beta+3-6(1-\mu)^2/\tau)$ and $G_1^{-1}(\tau,\mu,\beta)=(1/(1-\tau),1-\mu/(1-\tau),\beta-2+6\mu^2/(1-\tau))$. These formulas verify every constant in the two equalities. The reflected μ is cusp bounded and has the same affine laws, so their difference is a global section of $\mathcal{O}_B(-p_0)$ and vanishes. After this equality, the difference between the reflected and original β^{part} is a holomorphic invariant base function, hence constant; its cusp value is zero under the stated normalization. Therefore

$$\mu(\iota z)=1-\bar{\mu}(z),\qquad \beta^{\mathrm{part}}(\iota z)=-\overline{\beta^{\mathrm{part}}(z)}.$$

The original τ identity follows from its j equation and its specified cusp branch. In particular $D_{-\bar{c}}(\iota z)=D_c(z)$, so this comparison stays in the exact admissible domain.

3. Let

$$H=\begin{pmatrix}1&0&0&0\\0&1&0&0\\-6&0&-1&0\\0&-1&0&-1\end{pmatrix}. \tag{17.1}$$

Then $H^2=I$, $HA_2H=A_2^{-1}$, $HA_1H=A_2^{-1}A_1^{-1}A_2$, and

$$\Pi_{-\bar{c}}(\iota z)H=-\overline{\Pi_c(z)}.$$

Thus $(z,\zeta)\mapsto(\iota z,-\bar{\zeta})$ is a well-defined antiholomorphic torus map. It also descends under the middle deck action. Indeed covariance and the displayed lattice conjugacies imply $R_{\phi(g)}(\iota z)=\overline{R_g(z)}$ by multiplying the full real-invertible period matrices. Thus the reflection intertwines the middle deck actions.

4. At the first finite centre compose this map with the target deck identification g_2 to return the centre $g_2^{-1}z_1$ to z_1. Its lattice map becomes A_2H and it conjugates the clockwise generator to the inverse generator. At the second centre its lattice map is H with the same reversal. Consequently the target affine vectors are exactly

$$w_1'=-A_2H\varepsilon=-\varepsilon+d,\qquad w_2'=-H(-\varepsilon')=\varepsilon'-3d. \tag{17.2}$$

The minus signs arise because the inverse of a generator with invariant translation w_j'/m_j has translation $-w_j'/m_j$. This calculation verifies the full affine actions on the root covers. For the overlaps choose the root coordinates of the real local base coordinates t and $t-1$ compatible with these local reflections. This choice has an explicit comparison with any original root coordinate: their ratio is a holomorphic unit, so changing it changes the collar by $(2\pi i)^{-1}\log h\,\Pi w_j$, a translation extending on the entire root family. Its inverse gives a holomorphic isomorphism of the two glued descriptions. Thus the extension property and final classification transfer to the original coordinates. Then $\log s_j'=\overline{\log s_j}$ on paired branches. Under $\zeta\mapsto-\bar{\zeta}$, conjugation of $1/(2\pi i)$ supplies the compensating sign, and the identity of the two logarithmic collar translations follows exactly. Two roots of the same fixed real coordinate differ by an m_jth root of unity; this changes the comparison only by a constant period-coordinate translation, and the additive correction of 7.5 glues those constants. The preceding holomorphic-unit comparison handles a different original local base coordinate. Its inverse has the same construction.

5. At the cusp choose $s=(2\pi i)^{-1}\log t_c$, so $s\mapsto-\bar{s}$. The full holomorphic unit matrix satisfies

$$C_{-\bar{c}}(\bar{t}_c)=-\overline{C_c(t_c)}+\begin{pmatrix}6&0\\0&1\end{pmatrix}.$$

The integral extra matrix has trivial exponential. Hence $(x,t_c)\mapsto(\bar{x},\bar{t}_c)$ intertwines the full cusp deck groups, with the same integral index λ. The toric fan is defined over $\mathbb{Z}$, so this

map extends antiholomorphically on every toric chart and stratum. It preserves the untwisted cusp collar. We have therefore constructed an antiholomorphic map to the target datum with parameter $-\bar{c}$, $r_1' = -1, k_1' = 1, r_2' = 1, k_2' = -3, l_0' = 0$, whose p' is $+1$.

6. Apply 8.1 to normalize this target back to the canonical $p = -1$ datum. Take $a = -1$ and the odd shift $n = 1$. This gives a holomorphic map to $X_{1-\bar{c}}$. Every other antiholomorphic map, composed with the inverse of this one, is a holomorphic comparison between canonical members. 7.8 gives exactly the stated congruence. Exponentiating it gives $w' = -\bar{w}$.

7. For a real member, choose any resulting antiholomorphic self-map J. Its derivative on ordinary fibre coordinates is +complex conjugation: the initial reflection has derivative −complex conjugation, while the normalization just used has linear part $-I_2$. Hence it conjugates translation by $b \in \mathbb{C}/\mathbb{Z}$ to translation by $\bar{b}$. Its square is a holomorphic translation b_0. Commuting J with J^2 shows $b_0 = \bar{b}_0$ in $\mathbb{C}/\mathbb{Z}$. This equality forces the imaginary part of a representative of b_0 to vanish, so take $b_0 \in \mathbb{R}$. Replacing J by the composition of translation by $-b_0/2$ with J changes its square to $b_0 - b_0/2 - b_0/2 = 0$. We have an antiholomorphic involution. Under $b \mapsto e^{2\pi i b}$, its conjugation action is $z \mapsto 1/\bar{z}$. This proves the split group and specifies its antiholomorphic coset.

□

Proposition 17.3 (Two conjugacy classes of real structures)**.** [tag 02PE]

For a real member the antiholomorphic involutions have exactly two conjugacy classes under holomorphic automorphisms. For any fixed involutive splitting σ they are represented by σ and $\rho(1/2)\sigma$. Every nonempty fixed set is a closed connected orientable totally real three-manifold with the mod-two homology of S^3. The circle ρ preserves each such fixed set and acts semifreely on it.

Proof. 1. Proposition 17.2 identifies the full holomorphic group with $\mathbb{C}^*$ and gives the conjugation action $z \mapsto 1/\bar{z}$. Thus the square of $z\sigma$ is $z/\bar{z}$, and it is an involution exactly for $z \in \mathbb{R}^*$. Conjugation by $y \in \mathbb{C}^*$ changes this coefficient to $|y|^2 z$, giving exactly the two sign classes.

2. At a fixed point an antiholomorphic involution has a real three-dimensional fixed tangent space, and its complexification is the full tangent space. An invariant metric and its exponential map identify its fixed locus locally with this real subspace. It is therefore a closed totally real three-manifold. The smooth $\mathbb{Z}/2$ action admits an equivariant triangulation by (Illman, 1972, Theorem 3.1, p. 48); compactness and finite dimension give the finite-dimensional G-CW hypothesis. Smith's theorem, in the form of (Knudson, 2004, Theorem 4.1(1), p. 10), applies to the smooth finite action on the sphere and gives the stated mod-two homology. A nonempty three-dimensional fixed set is consequently connected, and its first mod-two cohomology vanishes; its orientation class w_1 therefore vanishes. On the unit circle $1/\bar{z} = z$, so σ commutes with ρ. Its stabilizers are inherited from the semifree action on X and are therefore either trivial or the whole circle.

□

Definition 17.4 (Representatives of the real involutions)**.** [tag IP05]

Put

$$L_F = (2\pi i)^{-1} \log F, \qquad L_G = (2\pi i)^{-1} \log G, \qquad F^3 = t, \quad G^4 = t - 1. \tag{17.3}$$

The roots are positive over $t > 1$. We continue from that interval through the upper half t-plane. Their logarithms are taken on the orbifold uniformizing cover. Write ι for its reflection lift. Every point of $-i\mathbb{R}_{>0}$ has a representative $c = 1/2 + iy$, and every point of $i\mathbb{R}_{>0}$ has a representative

$c = -1/2 + iy$, with y in the admissible range. Set

	α	$S(t)$
$c = 1/2 + iy$	$1/3$	$2L_F - 4L_G - 1/6$
$c = -1/2 + iy$	$1/2$	$-2L_G + 1/4$

$$b_c(z) = \alpha\Pi_c(z)g + S(t(z))e. \tag{17.4}$$

Theorem 17.5 (Explicit real involutions and all fixed fibres)**.** [tag 02PF]

On the ordinary period charts the formulas

$$\sigma_c[z,\zeta] = [\iota z, \bar{\zeta} + b_c(\iota z)], \qquad \sigma_c^- = \rho(1/2)\sigma_c \tag{17.5}$$

extend to antiholomorphic involutions of the compact threefold. With these representatives their complete fixed-set data are as follows. Here $2T^2$ means the disjoint union of two real two-tori; every displayed circle in a double curve avoids its two triple-point endpoints.

w	*involution*	*image in* $\mathbb{RP}^1$	*on* S_1	*on* S_2	*on* W
$-ir$, $r > 0$	σ_c	*all* $\mathbb{RP}^1$	$2T^2$	$2T^2$	$T^2 \sqcup S^1_{D_2} \sqcup S^1_{D_{12}}$
$-ir$, $r > 0$	σ_c^-	$[1,\infty]$	$\varnothing$	T^2	$S^1_{D_1} \sqcup S^1_{D_2}$
ir, $r > 0$	σ_c	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
ir, $r > 0$	σ_c^-	$[\infty,1]$ *through* 0	$2T^2$	T^2	$S^1_{D_1} \sqcup S^1_{D_{12}}$

The first interval is the closure of $t > 1$; the second is the closure of $t < 1$. On every open real arc contained in the indicated image the real regular fibre is $2T^2$. No triple point is real for these structures.

Proof. 1. Start with the reflection $(z,\zeta) \mapsto (\iota z, -\bar{\zeta})$ and the integral matrix H of (17.1). Proposition 17.2 proves its period compatibility and gives the target affine datum (17.2). The normalization from this target at parameter $c_0 = -\bar{c}$ to the canonical datum at $c_0 + 1$ has linear part $-I$ and translation

$$b_{c_0}^{\mathrm{norm}} = \tfrac{1}{3}\Pi_{c_0}g + (2L_F - 4L_G)e. \tag{17.6}$$

Its period defects at the two finite generators are $(0,-2,2,0)^t$ and $(0,0,2,0)^t$. To check them, the logarithmic increments are $-2/3$ and 1, and one multiplies the two displayed matrices A_j by $g/3$. The finite difference vectors in the complex translation space are $2e$ and $-4e$, so subtracting their logarithmic collars leaves holomorphic translations on both full root discs. At the cusp $2L_F - 4L_G = s/3 - (2\pi i)^{-1}\log(1-q)$ and $\Pi g = (6\mu, -s + C_{21})^t$. The logarithmic terms cancel and give the holomorphic toric unit $(2\mu, C_{21}/3 - (2\pi i)^{-1}\log(1-q))$. Thus (17.6) extends over all three fillings, with its inverse.

2. For $c = 1/2 + iy$ compose this normalization with the initial reflection and then with the constant translation $e/6$. This gives exactly the first line of (17.4). For $c = -1/2 + iy$, additionally close the parameter shift by the following explicit even-shift formula with $h = -2$,

$$b_{h,c_0} = -\frac{h}{12}\Pi_{c_0}g + \left(h(L_F - L_G) - \frac{h^2}{24}\right)e.$$

A final constant translation $-e/4$ gives the second line of (17.4). The displayed translation represents the even shift of Proposition 7.5. Its finite-chart corrections are holomorphic. For its cusp expression, put $\ell = L_F - L_G = -s/12 + \kappa_0(q)$ and $C_{c,21}(q) = c + C_{0,21}(q)$, and set $D_0(q) = C_{0,21}(q) - 12\kappa_0(q)$. The explicit even-shift translation becomes

$$b_{h,c} = \left(-h\mu/2, -h(c + D_0)/12 - h^2/24\right)^t.$$

The logarithmic terms have cancelled exactly. Exponentiating both coordinates gives holomorphic nonzero torus multipliers on the whole cusp disc; multiplication by these units extends over every toric stratum. Compatibility with the period deck group is the integral identity $\exp(2\pi i h\lambda_1) = 1$ for $h \in 2\mathbb{Z}$ and $\lambda_1 \in \mathbb{Z}$. The finite-chart defects are checked by the same displayed A_j and root logarithms used for the normalization.

3. Over $t > 1$ we have $\Re\mu = 1/2$, $\Re\tau = \Re\beta^{\mathrm{part}} = 0$ and the logarithms in (17.3) are purely imaginary. The two squares are respectively the integral period translations $2\widehat{w}$ and $3\widehat{w}$. The square is a holomorphic automorphism and is consequently the identity everywhere. This proves the involution claim.

4. Put $a = \Re c$ and $R = -H + 2aE$. In the chosen branch the reflection identifications on $t > 1$, $0 < t < 1$, $t < 0$ are respectively $1, g_2, g_0^{-1}$. They follow by reflecting the chosen upper-half-plane paths; equivalently the root arguments are $(0, 0)$, $(0, \pi/4)$ and $(\pi/3, \pi/4)$. The three fibre involution matrices are

$$R_> = R, \qquad R_m = A_2^{-1}R, \qquad R_< = M_0 R. \tag{17.7}$$

In marked real coordinates let $r_c(t) = \Pi_c(t)^{-1}b_c(t)$, where Π_c^{-1} is the real inverse. The fixed tori are exactly

$$\{x \in \mathbb{R}^4/\Lambda : (I - R_\bullet)x = r_c(t) + \epsilon d/2 \pmod{\Lambda}\}, \qquad \epsilon = 0, 1. \tag{17.8}$$

This specifies the cosets, including their parameter dependence. Their identity subgroups are $T_{\langle w,d\rangle}$ on both outer arcs and $T_{\langle u+w,d\rangle}$ on the middle arc. When nonempty there are two cosets. Relative to any one solution their offsets are $(g + u)/2$ on the right arc and $g/2$ on the other two arcs.

5. Existence of fixed points is decided by integral affine congruences. An affine involution $x \mapsto R_\bullet x + \beta$ has a fixed point if and only if its square translation $(I + R_\bullet)\beta$ belongs to $(I + R_\bullet)\Lambda$. In (17.4) the square translation is

$$(I + R_\bullet)\alpha g + 2\Re S\, d.$$

For $c = 1/2 + iy$ the values $2\Re S$ on the three arcs are $-1/3, -4/3, -2/3$; for $c = -1/2 + iy$ they are $1/2, 0, 0$. The resulting integral square vectors are respectively $(2w, 0, 2w)$ and $(3w, d, 3w)$. Adding $d/2$ to the involution adds d to each square vector. Taking column Hermite normal forms of $I + R_\bullet$ gives exactly the nonempty entries in the table. All three $I - R_\bullet$ have nonzero Smith factors $1, 2$. This proves the coset counts and offsets. The Hermite and Smith computations are listed in table 4. The left-arc lift g_0^{-1} follows from the reflected upper-half-plane path above.

6. At $t = 0$ the root degree is odd. Conjugation by the degree-three deck group acts transitively on the three reflection lifts; the centralizer of a reflection in that deck group is trivial. Thus the quotient has exactly the same two real tori as the adjacent nonempty real regular fibres. In the empty cases there are none. This also follows by solving the limiting affine equations on the full multiplicity-three root quotient.

7. At $t = 1$ write the affine deck map as $G(x) = A_2 x + v_2/4$. The real root involution has matrix $R_>$ and translation

$$b_- = (1/3, 0, 0, -1/6)^t, \qquad b_+ = (1/2, 0, 0, 1/4)^t$$

for the two parameter signs. The real locus of the quotient is the quotient of the fixed sets of $G^k h$, $0 \le k < 4$. Its even and odd lift parities are identified internally by G, while G^2

preserves each parity. Direct integer congruences show that the nonempty lift parities are: both for negative imaginary w and $\epsilon = 0$; only even for negative imaginary w and $\epsilon = 1$; neither for positive imaginary w and $\epsilon = 0$; only odd for positive imaginary w and $\epsilon = 1$. Every nonempty lift has two tori, exchanged by G^2. Each parity therefore contributes one torus downstairs. The congruence enumeration uses denominator 24: translation denominators divide 12, and the nonzero Smith factors are $1, 2$, so every component has such a representative. The component permutation is included in the exact enumeration listed in table 4. The normal real line has fourth power of sign positive for the even parity and negative for the odd parity. This establishes the two one-sided endpoints at $t = 1$ recorded in the table.

8. Put $A = C_{12}(0)$; the cusp normalization gives

$$\Re\mu(0) = 1/2, \quad A \in i\mathbb{R}, \quad C_{21}(0) = c - A, \quad C_{11}(0) = 6\mu(0), \quad C_{22}(0) = \mu(0).$$

In a toric lift the involution is $x \mapsto U/\bar{x}$, with

	$b_1(0)$	$b_2(0)$
$w = -ir$	$2\mu(0)$	$C_{21}(0)/3 - 1/6$
$w = ir$	$3\mu(0)$	$C_{21}(0)/2 + 1/4$

$$U_j = e^{2\pi i b_j(0)}. \tag{17.9}$$

Thus $U_2 > 0$ in both rows, whereas $U_1 > 0$ in the first and $U_1 < 0$ in the second. Replacing σ_c by σ_c^- negates U_2. The smooth central orbit has a real torus precisely when both units are positive, with equations $|x_j|^2 = U_j$.

9. For an edge v, set $n = (-v_2, v_1)^t$. The period translate bringing the inverted edge back is $B_0 n = v$. Its curve coordinate $z = x^n$ has involution

$$z \longmapsto K_v/\bar{z}, \qquad K_v = U^n e^{2\pi i n^t C(0) n}. \tag{17.10}$$

The signs for (D_1, D_2, D_{12}) are $(-,+,+)$ for negative w and $(-,-,-)$ for positive w. Negating U_2 reverses the first and third signs, but not the second. A positive sign gives the single circle $|z|^2 = K_v$; a negative sign gives no fixed point. Central inversion exchanges the two triangle classes of the periodic fan, so it exchanges the two triple points. These calculations prove every cusp entry. Finally a nonempty fixed three-manifold cannot be confined to the special fibres: a holomorphic function constant on an open piece of a totally real submanifold is locally constant. This also proves that the positive-w representative with no regular real fibres is entirely fixed-point free.

□

Theorem 17.6 (The real three-manifolds)**.** [tag 02PG]

Every nonempty fixed set in Theorem 17.5 is diffeomorphic to S^3. Its fundamental group is trivial and its integral homology is $(\mathbb{Z}, 0, 0, \mathbb{Z})$. For negative imaginary w, the restricted circle action is the standard semifree action on S^3 with fixed circle $D_2^{\sigma_c}$. For positive imaginary w, the nonempty real form has a free circle action isomorphic, up to circle and orientation conventions, to the Hopf action.

Proof. 1. For negative imaginary w both fixed manifolds meet D_2 in one circle. Their circle action is semifree, so the orbit space is a compact orientable surface with boundary. The local quotient at a fixed circle is the quotient of $S^1 \times \mathbb{C}$ by scalar rotation on $\mathbb{C}$, hence is a boundary collar. Over the surface interior the principal circle bundle is trivial because the surface has boundary and $H^2 = 0$. Changes of boundary trivialization extend over the attached normal discs by rotation. Thus the three-manifold is

$$(B \times S^1) \cup_{\partial B \times S^1} (\partial B \times D^2).$$

If B has genus g_B and b_B boundary components, van Kampen or its surface handle decomposition gives first homology $\mathbb{Z}^{2g_B+b_B-1}$. Smith's theorem makes this rank zero. Since $b_B > 0$, we have $g_B = 0, b_B = 1$. The displayed manifold is then $\partial(D^2 \times D^2) = S^3$, with the indicated circle action. This derives the smooth recognition from an explicit surface decomposition.

2. For positive imaginary w the nonempty real manifold is free under ρ. The two real tori on $t < 1$ join at the single real torus over 1. At 0 the odd root comparison is a torus product. At infinity they are capped by the real neighbourhoods of D_1 and D_{12}. These are solid tori: the primitive circle in the edge direction collapses and the remaining real circle is their core. In the outer real-fibre lattice $\mathbb{Z}\langle w, d\rangle$ the two cap meridians are w and $w - d$.

3. Put $B_1 = A_2^{-1}A_1^{-1}A_2$ and $C = B_1^2$. The two real-arc involutions satisfy $R_< R_m = B_1$, $R_m B_1 R_m = B_1^{-1}$ and $B_1^3 = I$. Moving between the real root lifts requires $B_1^k R_m B_1^{-k} = B_1 R_m$, so $2k \equiv 1 \pmod 3$ and $k = 2$. Thus the odd-root transport is C. It satisfies $R_< = C R_m C^{-1}$, and $C(u + w) = -w + d$. The exchange across the multiplicity-four real torus is A_2^2; after this odd-root transport it is $CA_2^2C^{-1}$, whose restriction to $\langle w, d\rangle$ is $w \mapsto -w$, $d \mapsto d$. Consequently the meridians, compared on one boundary torus, have determinant of absolute value one. The resulting genus-one Heegaard splitting is S^3: primitive meridians intersecting once can be taken to the two coordinate circles by an integral torus automorphism, giving the usual two solid tori of S^3. The free circle d then has Euler number of absolute value one, so the displayed bundle over S^2 is the Hopf bundle. This proves the fundamental-group and integral-homology assertions.

□

Proposition 17.7 (Restrictions of the circle quotient)**.** [tag 02PH]

The map e_{q} of Theorem 16.7 has a ρ-invariant cusp replacement. Its restriction to every nonempty real form is null-homotopic. On an ordinary real fibre its image consists of the two quotient-circle values

$$\theta = \tfrac{1}{2}\gamma(r_c(t)), \qquad \theta = \tfrac{1}{2}\gamma(r_c(t)) + \tfrac{1}{2} \pmod{\mathbb{Z}}, \quad \gamma(r_c(t)) = \alpha + \frac{\Im S(t)}{D_c(t)}, \tag{17.11}$$

with α, S from (17.4). The circle action is along each real two-torus and is annihilated by e_{q}.

Proof. 1. The literal quotient is circle invariant. In a cusp extension $\theta' : N_0 \to S^1$, each orbit circle has degree zero: its class is d, which vanishes in $\pi_1(N_0)$. The quotients $\theta'(\rho(s)x)/\theta'(x)$ therefore have a continuous real lift vanishing at $s = 0$ and periodic in s. Averaging this real lift over $s \in \mathbb{R}/\mathbb{Z}$ changes θ' to a circle-invariant map. The cocycle identity proves invariance, and on an invariant outer collar the lift is zero, so the replacement agrees exactly with the prescribed map. The same construction is smooth. Thus $e_{\mathrm{q}}|_{\mathrm{Fix}\,\sigma}$ factors through its circle orbit space, which is D^2 or S^2 by the preceding proof. Both have only two-dimensional homotopy type and S^3 is two-connected, so this restriction is null-homotopic. The first row of $I - R_\bullet$ is $(2, 0, 0, 0)$, giving the two values in (17.11). The real inverse of the period matrix sends Se to first coordinate $\Im S/D_c$; this follows by subtracting $(\Im\mu/\Im\tau)$ times its first imaginary-coordinate equation from the second. Both real-fibre identity subgroups have zero γ-coordinate, proving the remaining assertions.

□

Proposition 17.8 (The coherent closing translation)**.** [tag 0322]

Fix the standard twists and the root and cusp markings of Proposition 7.5. There is a coherent choice of the even-shift maps $\widetilde{F}_{h,c} : X_c \to X_{c+h}$, $h \in 2\mathbb{Z}$, with the following explicit ordinary-chart translation. Choose global roots $F^3 = t$, $G^4 = t - 1$ on the uniformizing cover and

put $\ell = (2\pi i)^{-1}\log(F/G) \pmod{\mathbb{Z}}$, $e = (0,1)^t$. *Then the translation is*

$$b_{h,c} = -\frac{h}{12}\Pi_c\widehat{\gamma} + \left(h\ell - \frac{h^2}{24}\right)e. \tag{17.12}$$

For $h = 12$, *it is the scalar multiplication*

$$K(t) = \frac{t^4}{(t-1)^3} \tag{17.13}$$

on the ordinary $\mathbb{C}^*$*-coordinate, after removing an integral period. On the finite root charts its scalar representatives are* $K(t)s_1^{-12}$ *and* $K(t)s_2^{12}$, *respectively, both units. In a cusp coordinate* $q = t_c$, *the scalar representative is* $q^{-1}H(q)$, *with* H *a holomorphic nonvanishing function. The same map has a holomorphic* $(\mathbb{C}^*)^2$*-unit representative on the entire cusp filling. These two representatives differ by exactly the* $\widehat{\gamma}$ *cusp deck transformation.*

For six positive w*-loops the closing map is* $\widetilde{F}_{12,c_0-12}$*; thus the stated cusp exponent is* -1 *with* $w = e^{-\pi ic}$ *and positive loops in the* w*-plane. For parallel transport* $P : X_{c_0} \to X_{c_0-12}$ *along this lifted path, the resulting six-loop diffeomorphism is* $\widetilde{F}_{12,c_0-12} \circ P$.

Proof. The even-shift translation and its quadratic correction are those of Du et al. (2026a, Proposition 7.8, pp. 91–92, and Proposition C.2, p. 102). Their representatives in the fixed root and cusp charts are as follows. The global roots exist because the zero orders of t and $t-1$ on the simply connected uniformizing surface are divisible by three and four. The local characters and their coprime orders give

$$\ell(g_1z) - \ell(z) = -\frac{1}{3}, \qquad \ell(g_2z) - \ell(z) = \frac{1}{4}, \qquad \ell(g_0z) - \ell(z) = \frac{1}{12} \pmod{\mathbb{Z}}.$$

A change of logarithm adds an integral $h\widehat{\delta}$ period, so (17.12) is a well-defined torus translation. Since $\Pi_{c+h} = \Pi_c(I + hE)$, the two ordinary lattices agree; the source-to-target lattice map of the identity linear part is $I - hE$.

For compatible logarithm lifts, covariance gives the affine defects

$$b_{h,c}(g_jz) - R_j(z)b_{h,c}(z) = \Pi_c(g_jz)\eta_j, \qquad \eta_1 = \frac{h}{2}(\widehat{u} - \widehat{w} - \widehat{\delta}), \quad \eta_2 = \frac{h}{2}(-\widehat{w} + \widehat{\delta}).$$

Their γ-coordinates vanish, so the same right sides use Π_{c+h}. The vectors are integral for even h. At the cusp the defect is zero: $(A_0 - I)\widehat{\gamma} = -\widehat{\delta}$ cancels the increment of ℓ. These equations prove ordinary and cusp descent with the stated orientation of the peripheral loops.

For the finite comparison put $w_j = (2\pi i)^{-1}\log s_j$. The functions $F/(s_1G)$ and Fs_2/G are invariant holomorphic units on the respective full root discs. Choose holomorphic logarithms κ_j of them. Then

$$\ell = \epsilon_j w_j + \kappa_j \pmod{\mathbb{Z}}, \qquad \epsilon_1 = 1, \quad \epsilon_2 = -1.$$

The exact finite-to-ordinary comparison satisfies $\sigma_{j,c+h} - \sigma_{j,c} = h\epsilon_j w_j e$. Consequently the required finite-chart translation is

$$b^{(j)}_{h,c} = -\frac{h}{12}\Pi_c\widehat{\gamma} + \left(h\kappa_j - \frac{h^2}{24}\right)e.$$

It is holomorphic on the full root disc and satisfies

$$b^{(j)}_{h,c}(g_jz) - R_j(z)b^{(j)}_{h,c}(z) - \frac{h\epsilon_j}{m_j}e = \Pi_c(g_jz)\eta_j.$$

This is precisely equivariance for the source and target affine finite actions, including the changed period column. The exact identity $\sigma_{j,c+h} + b^{(j)}_{h,c} = \sigma_{j,c} + b_{h,c}$ proves overlap agreement.

At the cusp let $q = e^{2\pi i s}$ and $C_{c,21} = \beta^{\mathrm{part}} + c + s$, which is holomorphic at $q = 0$. The function

$$\widehat{H} = e^{2\pi i s/12} F/G \quad \text{satisfies} \quad \widehat{H}^{12} = q\frac{t^4}{(t-1)^3}.$$

The right side is a unit because t has a simple pole at the cusp. The character formula makes $\widehat{H}$ single-valued; it therefore extends as a nonvanishing holomorphic function. Write $\ell = -s/12 + \kappa_0(q)$, with κ_0 holomorphic. Since $\Pi_c\widehat{\gamma} = (6\mu, -s + C_{c,21})^t$, the logarithmic terms cancel in (17.12):

$$\bar{b}_{h,c}(q) = \left(-\frac{h}{2}\mu(q),\, -\frac{h}{12}C_{c,21}(q) + h\kappa_0(q) - \frac{h^2}{24}\right)^t.$$

Exponentiating gives a holomorphic unit in $(\mathbb{C}^*)^2$. On every toric chart its action multiplies a character coordinate by an integral character of this unit, so it and its inverse extend across every double curve and both triple points. The source and target cusp lattice multipliers differ by $(1, e^{2\pi i h\lambda_1}) = (1, 1)$, for $\lambda \in \mathbb{Z}^2$. They are therefore identical. The unit multiplication commutes with their toric shears and descends. Exponentiation of the displayed formula gives exactly the ordinary translation on the punctured cusp.

The uncorrected translations have composition defect $-hke/12$. The constants in (17.12) cancel it, since $(h+k)^2 - h^2 - k^2 = 2hk$. Hence

$$\widetilde{F}_{k,c+h}\widetilde{F}_{h,c} = \widetilde{F}_{h+k,c}, \qquad \widetilde{F}_{0,c} = \mathrm{id}.$$

Composing this even-shift map with the inverse of the fixed choice in Proposition 7.5 gives an automorphism over the identity of the base. The full automorphism theorem, Theorem C.4, identifies it with a scalar in $\mathbb{C}^*$. A path from that scalar to 1 supplies an isotopy, so the two closing choices define the same smooth monodromy class.

For $h = 12$, remove the period $-\Pi_c\widehat{\gamma}$ from (17.12). The remaining translation is $(12\ell - 6)e$. Its exponential is $(F/G)^{12} = K(t)$, proving (17.13). The finite corrections divide it by s_1^{12} and multiply it by s_2^{12}. With the original uniformizers, these are units because t/s_1^3 and $(t-1)/s_2^4$ are units.

Put $H(q) = qK(t(q)) = e^{24\pi i\kappa_0(q)}$. The cusp unit for $h = 12$ is exactly

$$U_{12,c}(q) = \left(e^{-12\pi i\mu(q)},\, e^{-2\pi i C_{c,21}(q)}H(q)\right).$$

The $\widehat{\gamma}$ deck transformation has multiplier

$$\left(e^{12\pi i\mu(q)},\, e^{2\pi i C_{c,21}(q)}q^{-1}\right),$$

because $B_0(1,0)^t = (0,-1)^t$. Multiplying these two expressions gives $(1, q^{-1}H) = (1, K)$. This proves the claimed comparison, including its sign, on the entire cusp quotient. The extending map is represented by $U_{12,c}$, and the displayed deck multiplier identifies it with the scalar representative on the punctured chart.

Finally $w = e^{-\pi i c}$ makes six positive loops the path $c(r) = c_0 - 12r$; its endpoint is closed by the positive shift $h = 12$. For parallel transport P along that path the closed diffeomorphism is therefore $\widetilde{F}_{12,c_0-12} \circ P$. The overlap difference between the scalar logarithm and the two-coordinate unit logarithm is exactly $\Pi_c\widehat{\gamma}$. Scaling both logarithms by a real number a changes this difference to $a\Pi_c\widehat{\gamma}$. For $0 < a < 1$, this vector is not a period: the real period map is an isomorphism and $a\widehat{\gamma} \notin \Lambda$. Thus the separately scaled chart maps fail the overlap equation at precisely this fractional period. □

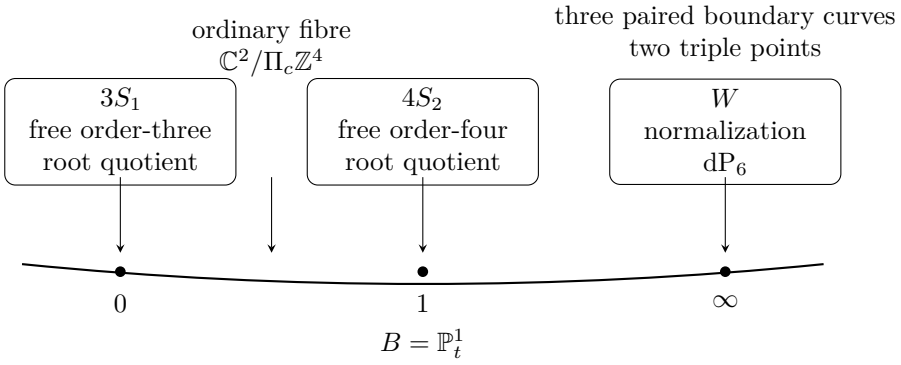

Figure 2: The fixed $(3, 4, \infty)$ base data. The arrows denote the restriction of f to each indicated fibre, with its scheme multiplicity. The ordinary period quotient, finite affine quotients and the paired hexagon are the three local models in the common conventions; their overlap maps are fixed there.

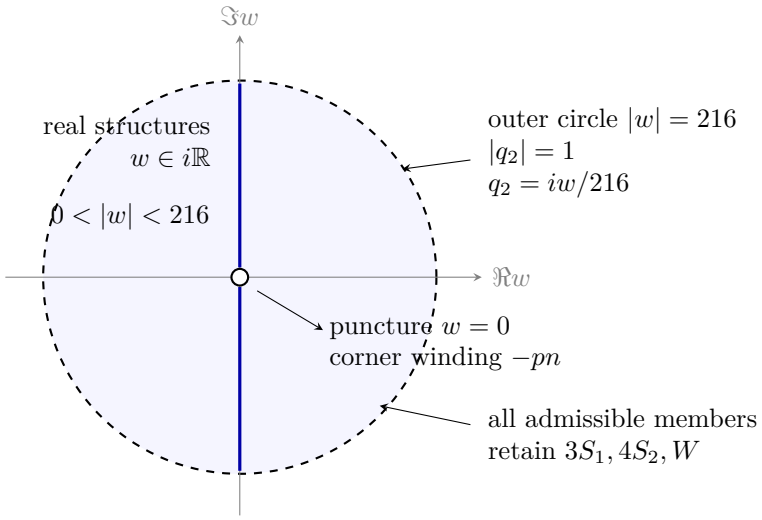

Figure 3: The exact parameter disc for the standard fixed data. The coordinate map is $w = e^{-\pi i c}$ and the coarser coordinate map is $u = w^2$. The imaginary diameter is the fixed locus of $w \mapsto -\bar{w}$, which is the reflected-family map. The two boundary obstructions use the meridian comparison at the puncture and the intrinsic Albanese nome at the outer circle. The special fibres are over $t = 0, 1, \infty$ in every member, as indicated in Figure 2.

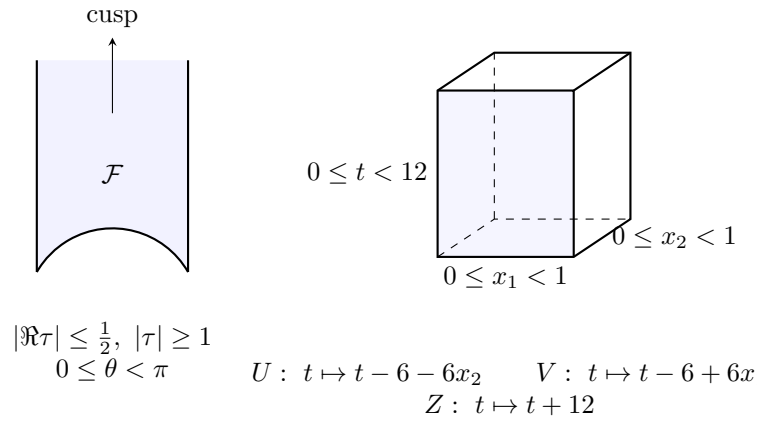

Figure 4: A fundamental set for the index-twelve monodromy kernel in the real parabolic stabilizer of d. The base is a modular fundamental set with its rotation coordinate; the fibre is the indicated Heisenberg parallelepiped. The displayed face maps include the central shears. The representative map and the boundary identifications are proved in 17.9; the index twelve occurs in the central fibre coordinate.

Proposition 17.9 (The fundamental set of the monodromy kernel)**.** [tag 0001] *In the real coordinates $g(B,x,t)$ of 10.2, a measurable fundamental set for $\Gamma\backslash P_d(\mathbb{R})$ consists of the matrices*

$$g(B,x,t), \qquad B = n(\Re\tau)a(\Im\tau)k(\theta), \quad \tau \in \mathcal{F}, \quad 0 \le \theta < \pi, \quad x_1, x_2 \in [0,1), \quad t \in [0,12).$$

Here $n(u) = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$, $a(v) = \operatorname{diag}(\sqrt{v}, 1/\sqrt{v})$, and $k(\theta)$ is the rotation matrix. Paired modular edges are assigned to one side, and their lower-dimensional stabilizer identifications use the corresponding lifts in Γ: for $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, we use $\widetilde{S} = g(S,0,9)$, $\widetilde{T} = g(T,0,1)$ and $\widetilde{-I} = g(-I,0,6)$. Their maps on the three coordinates are $(B,x,t) \mapsto (AB, Ax, t + \alpha(A))$, with $\alpha(S) = 9$, $\alpha(T) = 1$ and $\alpha(-I) = 6$. The Heisenberg face identifications are the maps printed in Figure 4.

Proof. 1. The projection $g(B,x,t) \mapsto B$ maps Γ onto $\mathrm{SL}_2(\mathbb{Z})$ by 10.2. For any $B \in \mathrm{SL}_2(\mathbb{R})$, the usual modular reduction gives an integral matrix A with $AB = n(u)a(v)k(\theta)$ in the stated set. Choose a lift $g(A,y,s) \in \Gamma$ and multiply on the left. For the displayed edge generators, the character formula $\chi(g(A,0,t)) = t - \alpha(A) \pmod{12}$ proves that the three specified lifts lie in Γ. Substitution in the group law gives each displayed coordinate map. This is the specified map from a general parabolic element to one over the modular set.

2. Two choices of lift over the same A differ by $\Gamma \cap H$. The group calculation in 10.2 identifies this kernel with $\langle U, V, Z\rangle$, where $U = g(I,(1,0),-6)$, $V = g(I,(0,1),-6)$ and $Z = g(I,0,12)$.

3. The multiplication law sends (x,t) under U to $(x + (1,0), t - 6 - 6x_2)$, and under V to $(x + (0,1), t - 6 + 6x_1)$. Powers of these maps first put x_1 and x_2 in $[0,1)$. A power of Z then puts t in $[0,12)$. All three are left translations in Γ, so they retain the original coset.

4. Within the interiors of these intervals equality of two reduced representatives forces their kernel element to have zero x-translation and then zero central translation. Modular uniqueness gives the same conclusion in the base interior. The usual edge and elliptic identifications, lifted by the specified surjection, account for the measure-zero boundary. This proves the fundamental-set assertion and all displayed Heisenberg identifications.

□

17.1 The Albanese coordinates and real structures

Proposition 17.10 (Exact nomes and continuation radius)**.** [tag 000L]
For $w = e^{-\pi i c}$, the fixed branches give

$$\kappa_1 = c + \tfrac{1}{2} - \frac{3i}{2\pi}\log 48, \qquad q_1 = -w^2/48^3,$$
$$\kappa_2 = c + \tfrac{3}{2} - \frac{3i}{\pi}\log 6, \qquad q_2 = iw/216, \qquad q_1 = (27/64)q_2^2.$$

The family is parameterized by $0 < |q_2| < 1$ and exists precisely in that domain. Writing $J(q) = q^{-1} + 744 + \cdots$, one has

$$j(\mathrm{Alb}\, S_1) = J(-w^2/48^3).$$

This function is holomorphic on $0 < |w| < 48^{3/2}$, has a pole of order two at zero, and has a natural boundary at $|w| = 48^{3/2}$. In particular it continues holomorphically through every point of the family boundary $|w| = 216$.

Proof. 1. The precise Albanese lattices in proposition 7.6 are verified by projecting all translations on the product covers. At the order-three fibre the projection is $(1/3)(\mathbb{Z}+\kappa_1\mathbb{Z})$; multiplication by three gives the first stated lattice. At the order-four fibre it is $\frac{1}{2}\mathbb{Z}+\frac{1}{4}\kappa_2\mathbb{Z}$; multiplication by two gives $\mathbb{Z}+(\kappa_2/2)\mathbb{Z}$. Thus the positive-imaginary period representatives are $-\kappa_1$ and $-\kappa_2/2$. Adding $2\mu_1 = 1 - i/\sqrt{3}$ and $3\mu_2 = 3/2 - 3i/2$ to 4.3 proves the two κ identities, including their real parts. Exponentiation gives the signs in q_1, q_2; the ratio is $216^2/48^3 = 27/64$.

2. The identity $q_2 = iw/216$ in Proposition 19.2 has factor $1/2$ precisely because of the Albanese projection just computed. Its natural-boundary proof uses modular invariance at rational boundary points. The existence criterion and exact $2\mathbb{Z}$ identification are theorems 6.6 and 7.8. The latter is also (Du et al., 2026a, Theorem 1.4, p. 3) under the explicit parameter map of remark A.1: with $\delta = \beta^{\mathrm{part}} - \beta_0$ and $a = c + \delta$, one has $\Omega_a = \Pi_c$ and $a' - a = c' - c$. That comparison also identifies the finite affine twists and the cusp collar.

3. The Fourier expansion for J converges for $0 < |q| < 1$ and has a simple pole at $q = 0$. The inverse image under $q = -w^2/48^3$ is the claimed punctured disc and the pole has order two. The inequality $216 < 48^{3/2}$ is equivalent to $27 < 64$. At every nonzero point on the larger boundary the map to q is locally biholomorphic. A continuation there would continue J through the unit circle, contradicting the rational-cusp argument proved in Proposition 19.2.

□

Corollary 17.11 (The real locus)**.** [tag 000M] *An antiholomorphic equivalence $X_c \to X_{c'}$ exists exactly when $c' - (1 - \bar{c}) \in 2\mathbb{Z}$. Consequently X_c admits an antiholomorphic involution exactly when $q_2 \in \mathbb{R} \setminus \{0\}$.*

Proof. The equivalence criterion and the involution are proposition 17.2. Its real locus is $w = -\bar{w}$. Under the exact coordinate change $q_2 = iw/216$, this is $q_2 = \bar{q}_2$; the parameter domain excludes zero. □

Remark 17.12 (Affine corrections for the real comparison)**.** [tag IA20] The normalization in proposition 17.2 has linear fibre map $-I_2$, parameter shift 1, and lattice map $-I + E$, where $d = (0, 0, 0, 1)^t$ and $E = d(1, 0, 0, 0)$. For the canonical vectors $v_1 = \varepsilon$, $v_2 = -\varepsilon'$ and the reflected vectors $v_1' = -\varepsilon + d$, $v_2' = \varepsilon' - 3d$, its integral correcting cocycles and rational translations can be taken as

$$\ell_1 = (0, 0, -2, 0)^t, \quad \ell_2 = (0, 2, 0, 0)^t, \qquad r_1 = (0, -2/3, -2/3, 0)^t, \quad r_2 = (0, 1, 1, 0)^t.$$

Direct substitution in the matrices of proposition 8.1 gives

$$N_1\ell_1 = -2d, \qquad N_2\ell_2 = 4d, \qquad (I - A_j)r_j = \frac{v_j - (-I + E)v_j'}{m_j} + \ell_j.$$

The cusp cocycle is

$$\ell_0 = -A_0(\ell_1 + A_1\ell_2) = (0, 0, 4, -2)^t.$$

These are the explicit corrections for the root-disc and cusp extensions in propositions 8.1 and 17.2.

18 The uniform corner obstruction

18.1 Period transport and parameter descent

Theorem 18.1 (Parameter descent and cusp transport)**.** [tag 02R9]

For an admissible $(3, 4)$ datum with integral twists (l_0, l_1, l_2), where $3 \nmid l_1$ and l_2 is odd, use $w = r^6$, $c = -12v$, $r = e^{2\pi i v}$ and $q = e^{2\pi i s}$, and put

$$b = 4l_1 + 3l_2, \qquad p = 12l_0 - 4l_1 - 3l_2, \qquad a = b - 12l_0 = -p.$$

On the logarithmic cover of the cusp bidisc the period columns are

$$\gamma = \begin{pmatrix} 6\mu \\ C - s - 12v \end{pmatrix}, \quad u = \begin{pmatrix} s + h \\ \mu \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where h, μ, C are holomorphic at $q = 0$. Their positive-loop column comparisons are

$$K_q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad K_r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -12 & 0 & 0 & 1 \end{pmatrix}.$$

The parameter descent on the ordinary Jacobian chart is

$$R : (t, v, \zeta) \longmapsto (t, v + 1, \zeta + H(t)\delta), \qquad H(t) = -\frac{4l_1 \log t + 3l_2 \log(t - 1)}{2\pi i}. \tag{18.1}$$

In the cusp collar it extends across $q = 0$ and has coefficient a on $s\delta$.

Proof. 1. We substitute $c = -12v$ in the period matrix (2.1) and use its cusp expansions $\tau = s + h$, $\beta^{\mathrm{part}} = C - s$. Direct subtraction gives the two displayed matrices. On fixed vector coordinates the lattice conjugation matrices are K_q^{-1} and K_r^{-1}. In the characters Z, W, addition of u and γ gives respectively

$$(Z, W) \mapsto (qe^{2\pi i h}Z, e^{-2\pi i \mu}W), \qquad (Z, W) \mapsto (e^{12\pi i \mu}Z, qr^{12}e^{-2\pi i C}W).$$

These are holomorphic character identities, including the signs of both meridians.

2. The collar section carrying the primitive $\widehat{\gamma}$ class has lift

$$z_\gamma(s, v) = \begin{pmatrix} 6s\mu \\ -s(s + 1)/2 + s(C - 12v) \end{pmatrix}. \tag{18.2}$$

It satisfies $z_\gamma(s+1, v) - z_\gamma(s, v) = \gamma(s+1, v)$ and $z_\gamma(s, v+1) - z_\gamma(s, v) = -12s\delta$. We therefore use the cusp coordinate $\eta = \zeta + l_0 z_\gamma$, retaining the collar integer.

3. At finite point j, the logarithmic gluing is translation by $(\log z_j)/(2\pi i)\Pi v_j$, with $t_j = z_j^{m_j}$ and $\widehat{\gamma}$ coefficient of v_j equal to l_j. Changing v by one changes this gluing by $-12l_j \log z_j/(2\pi i)\delta$. This is $-4l_1 \log t/(2\pi i)\delta$ at 0 and $-3l_2 \log(t-1)/(2\pi i)\delta$ at 1. The complementary summand of H is holomorphic on each disc, so its translation extends. The finite affine action changes by $-12l_j/m_j$ times the integral period δ, and is unchanged on the torus. Thus (18.1) is the descent compatible with both finite caps. Branch changes of H are integral δ periods.

4. At infinity,
$$H = bs + U(q), \qquad U(q) = -\frac{3l_2 \log(1-q)}{2\pi i}.$$
In η the descent is translation by $(as+U)\delta$. Adding the lattice period $a\gamma$ rewrites this vector as $(6a\mu, U + aC - 12av)$. It is holomorphic at $q = 0$ for each v; its exponentials act by units on every toric chart. This proves the extension and its collar coefficient. The displayed formulas verify the collar coefficient by direct substitution.

□

18.2 The partial filling and its origins

Theorem 18.2 (Partial filling and extending origins)**.** [tag 02RD] [tag 02RC]

There is a proper holomorphic partial filling over $(\mathbb{P}^1 \times \Delta_r) \setminus \{(\infty, 0)\}$ with smooth total space and reduced normal-crossing parameter central divisor. Over a regular finite base point its central fibre is a cycle of twelve ruled surfaces. Its component rotations at $0, 1, \infty$ are $4l_1, 3l_2, -b$ modulo 12, with component cover
$$y^{12} = t^{4l_1}(t-1)^{3l_2}.$$

On $q = Q^{12}$, extending origins have lifts
$$o = \begin{pmatrix} -b\mu s/2 \\ bs^2/24 - bCs/12 + l_0 s/2 + v(bs+U) \end{pmatrix},$$
$$\sigma = -l_0 z_\gamma + \begin{pmatrix} 6a\mu v \\ vU + aCv - 6av^2 + 6av \end{pmatrix}.$$

The section σ extends across the cusp for $r \neq 0$, and o extends across the parameter central fibre of the partial filling. Their difference is
$$F = \sigma - o = a\begin{pmatrix} \mu(s+12v)/2 \\ C(s+12v)/12 - (s+12v)^2/24 + 6v \end{pmatrix}. \tag{18.3}$$

On $q = Q^{12m}$, $r = R^n$, with m, n positive integers,
$$\Delta_Q F = am\gamma - 6am^2\delta, \tag{18.4}$$
$$\Delta_R F = an\gamma - 6an(n-1)\delta. \tag{18.5}$$

The radial class modulo $\mathbb{Z}\widehat{\delta}$ is $-pn\widehat{\gamma}$.

Proof. 1. On a simply connected finite-base chart subtract a local origin with v-difference $H\delta$, quotient by δ, and put $W = e^{-2\pi i \eta_2}$, $z = \eta_1$. The remaining lattice transformations are
$$(z, W) \mapsto (z+1, W),$$
$$(z, W) \mapsto (z+\tau, e^{-2\pi i\mu} W),$$
$$(z, W) \mapsto (z + 6\mu, d(t) r^{12} W),$$

with d a nonzero holomorphic unit. The first two give the complement of the zero section of a degree-zero line bundle over $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Use the infinite chain of node charts

$$X_i = W/r^i, \quad Y_i = r^{i+1}/W, \quad X_iY_i = r \qquad (i \in \mathbb{Z}). \tag{18.6}$$

Adjacent charts are identified on $Y_i \neq 0$ by $X_{i+1} = Y_i^{-1}$, $Y_{i+1} = rY_i$. The charts and their inverse overlaps give an infinite projective-line chain at $r = 0$. Unit multiplication on W multiplies X_i, Y_i inversely, so all elliptic line-bundle transitions extend. The third period shifts the index by twelve, translates the elliptic coordinate by 6μ, and multiplies by d. Its quotient has the stated twelve-cycle.

2. The total node chart is smooth by eliminating $r = X_iY_i$, and its central divisor is reduced NC. On a compact smaller finite-base chart the elliptic lattices have uniformly bounded compact fundamental parallelograms. Choose middle annuli and node discs using upper and lower bounds for d and the finitely many transition units. Twelve consecutive indices give a compact fundamental cover over a closed smaller parameter disc. The overlap formulas have closed graphs on that cover, and the positive index translation separates nonadjacent charts. Thus the quotient is Hausdorff and proper. At $r \neq 0$ it is exactly the original torus quotient.

3. At root point j, addition of v_j/m_j multiplies W by a factor containing r^{12l_j/m_j} and rotates the cycle by $12l_j/m_j$. Every nonidentity power moves the cycle, because $3 \nmid l_1$ and l_2 is odd. The full finite action is free on the central divisor, including its intersection strata. Its quotient is smooth and preserves the reduced NC parameter divisor. Logarithmic gluing translations differ on overlaps by holomorphic units and integral chain shifts, which extend by the same formulas. The rotations give the cyclic cover; the third is their negative sum and equals p modulo 12.

4. The lift descent identities are

$$\begin{aligned} \Delta_{s,12}o &= -b\gamma + 6(b+l_0)\delta, & \Delta_{v,1}o &= H\delta, \\ \Delta_{s,12}\sigma &= -12l_0\gamma + 78l_0\delta, & \Delta_{v,1}\sigma &= H\delta + a\gamma. \end{aligned}$$

Every additional term is a lattice period. Both lifts therefore define holomorphic sections of the descended torus torsor. In the cusp chart $\sigma + l_0 z_\gamma$ is holomorphic and bounded in q and lies in the smooth toric open orbit at $q = 0$. It extends there. Subtracting o makes the vector coordinate invariant under the parameter deck. The zero coordinate extends in the node charts (18.6). This constructs the required second origin.

5. Subtracting gives (18.3). Integral shifts of s leave μ, C, U unchanged; expansion of the square gives (18.4)–(18.5). For $(l_0, l_1, l_2) = (0, 1, -1)$, $p = -1$, $a = 1$. Its radial class is $+n\widehat{\gamma}$. For every admissible twist $p \neq 0$, so the radial class is nonzero in the free quotient lattice.

6. We glue this finite-base filling to the prescribed cusp filling over $r \neq 0$ by the translation $\sigma - o$. The displayed origin formulas identify the overlap with the original family. Properness and the Hausdorff property are local on the displayed target, so the glued map has both. The local equations prove flatness over the parameter disc.

□

18.3 Extending the marked base map

Theorem 18.3 (Extension of the marked base map)**.** [tag 02RE] [tag 0314]

For every admissible $(3,4)$ *twist, the marked map* $f : X_c \to \mathbb{P}^1$ *extends meromorphically over every proper Hausdorff analytic filling of a finite parameter cover. For a smooth given total space, its graph is a proper projective modification carrying the marked map. In general, a smooth proper modification carries a holomorphic marked map to* $\mathbb{P}^1$*, and hence a proper map to* $\mathbb{P}^1 \times \Delta$*.*

Proof. 1. Take the reduced irreducible component containing the punctured family. It is closed and proper; discarding other central components preserves that family. Resolve it isomorphically over the smooth punctured family. We use analytic resolution (Bierstone and Milman, 1997, Theorem 1.6 and the following noncompact extension, p. 215): reduced complex analytic spaces admit proper modifications by locally finite sequences of smooth-centre blowups with smooth result. It can be chosen unchanged over the nonsingular locus. Properness makes the neighborhoods of the central fibre relatively compact, so the stated analytic hypotheses hold. Write $g : \mathcal{Z} \to \Delta_s$ for the smooth result. The following argument also applies directly to a smooth given total space.

2. We shrink the disc whenever necessary. Put $L = K^{-2}_{\mathcal{Z}/\Delta_s}$. The relative canonical bundle here is the line bundle $K_{\mathcal{Z}} \otimes g^* K^{-1}_{\Delta_s}$; its restriction to a nonzero fibre is that fibre's canonical bundle. The precise proposition 7.2, proposition 7.4 input is $K^{-2}_{X_s} \simeq f_s^* \mathcal{O}_{\mathbb{P}^1}(1)$, with connected fibres and no further sections. Therefore $h^0(X_s, L|_{X_s}) = 2$, and its complete linear system is the given map f_s, up to a projective change of coordinate. The three critical values are intrinsically distinguished by their fibres: multiplicity three, multiplicity four, and reduced normal crossings, respectively.

3. Grauert's proper direct-image theorem (Grauert, 1960, §6, Hauptsatz I, p. 59) makes $\mathcal{F} = g_* L$ coherent. It is torsion-free: multiplication by a nonzero base germ is injective on L, because g is nonconstant on the smooth connected total space, and hence is injective on its direct image. A coherent torsion-free sheaf on the smooth disc is locally free. Its rank is two. On the punctured disc, constant $h^0 = 2$ and the analytic base-change theorem of Proposition 12.1 identify its fibres with the complete spaces of sections. Choose a frame (σ_0, σ_1) of $\mathcal{F}$. Evaluation gives two global holomorphic sections of L on $\mathcal{Z}$, and thus a meromorphic map $F_0 = [\sigma_0 : \sigma_1]$ to $\mathbb{P}^1$.

4. Its graph is analytic and proper by the following local construction. In a local trivialization of L, write the sections as a, b. Remove their common divisorial factor in the regular local ring of the smooth space $\mathcal{Z}$. The equation $va - ub = 0$ in $\mathcal{Z} \times \mathbb{P}^1_{[u:v]}$ then has a unique component dominating $\mathcal{Z}$; it is the closure of the graph where $(a, b) \neq (0, 0)$. These local components glue. Denote the resulting reduced graph by $\mathcal{G}$. The projection $\mathcal{G} \to \mathcal{Z}$ is proper and projective, and is an isomorphism over Δ_s^*. The map

$$H : \mathcal{G} \longrightarrow \mathbb{P}^1 \times \Delta_s, \qquad H = (F_0, g),$$

   is proper. Locally $\mathcal{G}$ is a hypersurface in the smooth space $\mathcal{Z} \times \mathbb{P}^1$. The vanishing of the maximal Jacobian minors for its equation and the two target coordinates defines the closed analytic locus C where H is not a submersion; singular points of $\mathcal{G}$ are included.

5. The image $H(C)$ is a closed analytic subset of the target. One may deduce this precise proper-image assertion from the same direct-image theorem: the coherent sheaf $H_* \mathcal{O}_C$ has support exactly $H(C)$, since the constant section 1 is nonzero at every nonempty proper fibre. Over Δ_s^* this image consists exactly of three distinct critical-value sections. It cannot have a two-dimensional component, since such an analytic component would contain an open set of the connected two-dimensional target and hence meet the punctured region in dimension

two. Remove any component contained in $\mathbb{P}^1 \times \{0\}$. The remaining reduced analytic curve is proper with zero-dimensional fibres over the disc, and is therefore finite over it.

6. Each of the three labelled sections on the punctured disc belongs to a degree-one branch of that finite curve. Indeed their labels are intrinsic and cannot be permuted by continuation. The normalization of a degree-d branch over a punctured disc has cyclic monodromy of length d; a single-valued labelled section therefore has $d = 1$. The normalization extends it holomorphically to a map from the full disc to $\mathbb{P}^1$. Let these three maps be $c_1(s), c_2(s), c_0(s)$, in the multiplicity-three, multiplicity-four and cusp order. They may collide at $s = 0$.

7. For $s \neq 0$ the unique projective transformation sending (c_1, c_2, c_0) to $(0, 1, \infty)$ has meromorphic coefficients at 0. In a common affine chart its value at z is

$$A_s(z) = \frac{(z - c_1(s))(c_2(s) - c_0(s))}{(z - c_0(s))(c_2(s) - c_1(s))};$$

homogeneous brackets give the same formula if a value is infinite. The denominators are not identically zero, since the three values are distinct on the punctured disc. Thus $A_s \circ F_0$ is meromorphic on $\mathcal{Z}$. On every nonzero fibre it agrees with the marked f_s, because both are the complete bi-anticanonical map and have the same three labelled values. Thus the two maps agree over the punctured disc.

8. Finally the graph of $A_s \circ F_0$ is constructed by the same local two-section argument after clearing powers of the base coordinate in its meromorphic coefficients. Its projection to $\mathcal{Z}$ is the asserted proper projective modification, and its second projection is the required holomorphic marked map. Resolve this reduced graph isomorphically over the smooth punctured family by the same analytic resolution theorem. Composing its holomorphic marked map with the resolution gives the asserted smooth proper modification.

□

Theorem 18.4 (Vanishing of finite monodromy orbits)**.** [tag 02RF]

The global monodromy on $\Lambda_{\mathbb{Q}}/\mathbb{Q}\widehat{\delta}$ has no nonzero vector with finite orbit.

Proof. 1. In coordinates (g, u, w), a cusp unipotent has nilpotent part $N = E_{32}$. The two finite homology generators are

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ -6 & -1 & -1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -6 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 \end{pmatrix}.$$

A vector with finite orbit is fixed by a positive power of every conjugate of that unipotent. Since $(I + N)^j = I + jN$ in characteristic zero, it belongs to $\ker N$, $\ker(NA_1)$ and $\ker(NA_2)$. The equations are $u = 0$, $6g + w = 0$, $-w = 0$, forcing zero. The coefficient matrix of these three equations has determinant 6.

□

Theorem 18.5 (Vanishing of horizontal meridians)**.** [tag 02RG]

In a smooth resolved proper filling carrying the marked base map, every parameter-central component dominating $\mathbb{P}^1$ has zero normal-meridian lattice class modulo $\mathbb{Z}\widehat{\delta}$, relative to the ordinary parameter origin.

Proof. 1. Put $B^\circ = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. After $r = R^n$, the quotient fundamental group of the punctured family over B° by the normal subgroup generated by $\widehat{\delta}$ is

$$((\Lambda/\mathbb{Z}\widehat{\delta}) \rtimes \pi_1(B^\circ)) \times \mathbb{Z}_R. \tag{18.7}$$

The finite Jacobian linear changes are independent of c and fix $\widehat{\delta}$. The R descent is the δ translation H, all its branch changes are integral δ periods, and its lattice conjugation differs from the identity by a δ multiple. The finite gluing changes under $c \mapsto c - 12n$ also lie in $\mathbb{Z}\delta$. Thus every cross-cocycle disappears in exactly the quotient (18.7); this proves the presentation from the chart action.

2. Let D dominate B. Its positive normal circle has parameter degree $e = \operatorname{ord}_D(R) > 0$, and in (18.7) is (λ, e) with trivial base-loop coordinate. Changing a basing path by a fibre loop changes λ by $(I - T_R^e)$ of that loop, which vanishes modulo δ. Transport along D therefore acts on λ only by base monodromy.

3. Resolve the reduced compact component D by a proper map $\nu : \widetilde{D} \to D$ which is an isomorphism over D_{reg}, using the analytic resolution theorem in 18.3. Apply Stein factorization to $f|_D \circ \nu$:

$$\widetilde{D} \xrightarrow{p} C \xrightarrow{h} B, \qquad f|_D \circ \nu = h \circ p.$$

   Here p has connected fibres, $p_*\mathcal{O}_{\widetilde{D}} = \mathcal{O}_C$, and h is finite, by (Movasati, 2024, Theorem 8.5, pp. 80–81). The curve C is normal: a locally meromorphic function integral over $\mathcal{O}_C$ pulls back to a holomorphic function on the normal space $\widetilde{D}$ and descends by the displayed direct-image equality. Thus C is a smooth compact curve.

   Let $\Sigma \subset B$ consist of $0, 1, \infty$, the branch values of h, and the h-images of the critical values of p. This set is finite. Put

$$B' = B \setminus \Sigma, \qquad C' = h^{-1}(B'), \qquad V = p^{-1}(C').$$

   Then $p : V \to C'$ is a proper submersion with connected fibres, and $h : C' \to B'$ is a finite unramified cover. Path lifting followed by a path in a fibre gives $\pi_1(V) \twoheadrightarrow \pi_1(C')$. Consequently the image of $\pi_1(V)$ in $\pi_1(B')$ has index $\deg h$.

   Let E range over the other irreducible components of the reduced parameter-central divisor, and set

$$A = V \cap \nu^{-1}\left(D_{\mathrm{sing}} \cup \bigcup_{E \neq D} (D \cap E) \right), \qquad U = V \setminus A.$$

   This is a proper complex analytic subset A of the smooth manifold V. General position moves every based loop in V off A, so $\pi_1(U) \to \pi_1(V)$ is surjective. The inclusion $B' \hookrightarrow B^\circ$ is also surjective on fundamental groups; therefore the image of $\pi_1(U)$ in $\pi_1(B^\circ)$ has index at most $\deg h$.

   The restriction of ν identifies U with its image in D_{reg} away from the other central components and over B'. Composing these loops with ν thus transports the normal circle in the original smooth total space. Its complex orientation, lattice class λ and parameter degree $e = \operatorname{ord}_D(R)$ are those of step 2. Hence a finite-index subgroup of the base group transports the original normal meridian.

4. The orbit of λ is finite. 18.4 kills it rationally, and the quotient lattice is free, so it vanishes integrally. This includes every multiplicity e. After a local finite root in the base, each horizontal component inherits this zero class from its global preimage.

□

18.4 The commuting-meridian obstruction

Theorem 18.6 (Nonexistence of proper fillings)**.** [tag 0313] [tag 030D] [tag 031A] [tag 02RH]

For every admissible $(3,4)$ *twist, the punctured family has no proper Hausdorff complex-analytic filling at* $w=0$ *after any finite parameter cover. This includes every sphere member and permits arbitrary singularities, multiplicities and extra central components.*

Proof. 1. A proper filling pulls back properly to a further cover $w=R^{6n}$. By 18.3 we resolve its dominating component and marked-map graph. Near infinity take the finite base change $q=Q^{12m}$, normalize the dominating component, and resolve it. Principalize the inverse image of the two axes using (Bierstone and Milman, 1997, Theorem 1.10, p. 216, with the analytic category in (0.1)(2), p. 208). That theorem turns a coherent ideal on a smooth analytic space into a locally monomial ideal after proper smooth-centre blowups. The axis ideal is coherent and the relevant inverse image is compact over a smaller closed bidisc, so its hypotheses hold. The smooth torus family over $Q^*\times R^*$ is unchanged.

2. The proper map to the bidisc has connected fibres. Its Stein finite factor is generically one sheet because its general fibre is a connected torus. A finite birational map to the normal bidisc is an isomorphism: its finite local algebra is integral over, and contained in the fraction field of, the integrally closed local ring of that bidisc. In particular the corner fibre is connected.

3. In the axis complement, the original logarithmic deck lifts have

$$A_Q = I - 12mE_{32} + 12mE_{41}, \qquad A_R = I + 12nE_{41}, \qquad [Q,R] = -12mnb\widehat{\delta}.$$

The affine relation is the difference $H(s+12m)-H(s)=12mb$ through n parameter turns. The lifts

$$Q_o = T_{-mb\widehat{\gamma}}Q, \qquad R_o = R$$

commute. They give a specific semidirect product $\Lambda\rtimes\mathbb{Z}^2$; central δ corrections do not affect the following coefficients.

4. For every irreducible axis component D_i, put

$$\alpha_i = \operatorname{ord}_{D_i}(Q), \quad \beta_i = \operatorname{ord}_{D_i}(R), \quad \kappa_i = \text{the } \widehat{\gamma} \text{ coefficient of its lattice meridian.}$$

Its positive meridian is $(\lambda_i,\alpha_i,\beta_i)$, with nonnegative orders and positive sum. Conjugation changes its lattice vector by a vector killed by the first-coordinate covector; thus κ_i is independent of its basing path in the complement. Meeting components have commuting meridians, represented by coordinate circles in a smooth NC chart. The $\widehat{\delta}$ coordinate of their commutator gives

$$(m\alpha_i+n\beta_i)\kappa_j = (m\alpha_j+n\beta_j)\kappa_i. \tag{18.8}$$

Both nilpotents have square zero and their products vanish, so there are no higher terms. The E_{32} contribution lies in the $\widehat{w}$ coordinate; E_{41} gives exactly (18.8). The ratio $\kappa_i/(m\alpha_i+n\beta_i)$ is therefore constant along every intersection edge. Its denominator is positive for every axis component, including vertical exceptional components.

5. A parameter-central component dominating the original base exists, since the proper map onto $\mathbb{P}^1\times\Delta$ is surjective and finitely many vertical components cannot cover its central base. Its strict transform meets the corner by properness. Its ratio is zero by 18.5.

6. The strict cusp component met by σ also meets the corner. Its meridian in the smooth open orbit is the loop traced by the extending section. 18.2 gives $\widehat{\gamma}$ coefficient $-12ml_0$ in the original coordinates. Relative to Q_o this becomes $-12ml_0+mb=ma$. Its orders are $(1,0)$, so its ratio is a. Further ramification multiplies the order and coefficient by the same positive factor, and proper transforms preserve a smooth open part of the branch.

7. The compact connected corner fibre is covered by its intersections with finitely many axis components. The incidence graph of this closed cover is connected: a partition with no edge would partition the fibre into disjoint nonempty closed subsets, both also open. A chain therefore joins the horizontal central component to the strict cusp component. Equation (18.8) forces $a = 0$.

8. Admissibility gives $3 \nmid l_1$ and l_2 odd, whence p is coprime to 12 and nonzero. Since $a = -p$, this is a contradiction. Each comparison was proper and preserved the punctured family, so it applies to the original filling and every finite parameter cover.

□

Theorem 18.7 (The logarithmic datum $(0, 2, -1)$)**.** [tag 02RI]

For $(l_0, l_1, l_2) = (0, 2, -1)$ we have $p = -5$, $a = b = 5$, component cover $y^{12} = t^8/(t-1)^3$, and rotations $8, -3, -5$ modulo 12. The descent and origins are

$$H = 5s + U, \qquad U = \frac{3\log(1-q)}{2\pi i},$$
$$o = \begin{pmatrix} -5\mu s/2 \\ 5s^2/24 - 5Cs/12 + v(5s+U) \end{pmatrix},$$
$$\sigma = \begin{pmatrix} 30\mu v \\ vU + 5Cv - 30v^2 + 30v \end{pmatrix}, \qquad F = 5F_0,$$

where F_0 is the vector in (18.3) without a. Its two differences are $5m\gamma - 30m^2\delta$ and $5n\gamma - 30n(n-1)\delta$. No finite parameter cover admits a proper filling.

Proof. 1. Substitution in 18.1–18.2 gives every displayed formula. The lattice and character laws remain those of 18.1 and the charts those of (18.6). The corner has $[Q, R] = -60mn\widehat{\delta}$ and $Q_o = T_{-5m\widehat{\gamma}}Q$. Its horizontal ratio is zero and strict cusp ratio five. Thus 18.6 gives the contradiction for this member itself.

□

18.5 Higher coprime multiplicities

Theorem 18.8 (The obstruction for coprime multiplicities)**.** [tag 02RJ]

Every admissible logarithmic datum in the coprime $(3k, 4l)$ family of Viaclovsky's logarithmic construction has no proper Hausdorff analytic filling at parameter zero after any finite cover.

Proof. 1. Put $m_1 = 3k$, $m_2 = 4l$, $N = m_1m_2 = 12kl$, with $\gcd(m_1, m_2) = 1$. Let l_j be the $\widehat{\gamma}$ coefficient of the invariant finite twist. Admissibility is $\gcd(l_j, m_j) = 1$. Retain l_0 and define $b = m_2l_1 + m_1l_2$, $a = b - Nl_0$. On $w = r^{N/2}$, $c = -Nv$, and the gamma period is $(6\mu, C - s - Nv)$. The descent is

$$H = -\frac{m_2l_1\log t + m_1l_2\log(t-1)}{2\pi i} = bs + U, \qquad U = -\frac{m_1l_2\log(1-q)}{2\pi i}.$$

The finite actions are unchanged by $c \mapsto c - N$, since N/m_j is integral. The collar coefficient is a; adding $a\gamma$ gives a unit action on cusp charts. The rank-one filling has N components with rotations $m_2l_1, m_1l_2, -b$. Admissibility makes the finite actions free on the cycle.

2. On $q = Q^N$ the origins are

$$o = \begin{pmatrix} -6b\mu s/N \\ bs^2/(2N) - bCs/N + l_0s/2 + v(bs+U) \end{pmatrix},$$
$$\sigma = -l_0z_\gamma + \begin{pmatrix} 6a\mu v \\ vU + aCv - Nav^2/2 + Nav/2 \end{pmatrix},$$

with $z_\gamma = (6s\mu, -s(s+1)/2 + s(C - Nv))$. Their difference is

$$aF_N = a\begin{pmatrix} 6\mu(s+Nv)/N \\ C(s+Nv)/N - (s+Nv)^2/(2N) + Nv/2 \end{pmatrix}.$$

On $q = Q^{Nm}$, $r = R^n$,

$$\begin{aligned} \Delta_Q(aF_N) &= am\gamma - \tfrac{N}{2}am^2\delta, \\ \Delta_R(aF_N) &= an\gamma - \tfrac{N}{2}an(n-1)\delta. \end{aligned}$$

All coefficients are integral because $12 \mid N$. Expanding the two quadratic expressions gives the origin subtraction, the four descent identities and both covered differences.

3. Pulling the order-three and order-four tangent root charts through k and l powers changes their vertical determinant orders from $2, 1$ to $2k, l$. The vertical determinant is therefore $f^*\mathcal{O}(-1) \otimes \mathcal{O}(2kS_1 + lS_2)$. The determinant of the image of df is $f^*\mathcal{O}(2) \otimes \mathcal{O}(-(3k-1)S_1 - (4l-1)S_2)$; the cusp singular ideal has codimension two. Hence

$$\begin{aligned} K_X^{-1} &= f^*\mathcal{O}(1) \otimes \mathcal{O}((1-k)S_1 + (1-3l)S_2), \\ K_X^{-N} &= f^*\mathcal{O}(d), \qquad\qquad d = 4l + 3k - kl. \end{aligned}$$

The integer d is nonzero. Otherwise $k \mid 4l$, while $\gcd(k, 4l) = 1$, so $k = 1$, giving $d = 3l + 3 > 0$.

4. Choose K^{-N} when $d > 0$ and K^N when $d < 0$; its restriction is $f^*\mathcal{O}(|d|)$. On a resolved proper filling the corresponding relative direct image is coherent torsion free of rank $|d| + 1$. Evaluation and graph resolution give a proper pluricanonical image. Over the field K of meromorphic parameter germs, its image is a smooth rational normal curve. More explicitly, the kernels of multiplication on the direct-image modules, tensored with K, give its quadratic equations; on the punctured disc these are the Veronese equations and generate its ideal. The three labelled critical-value curves give distinct K-rational points. The coprime multiplicities are unequal, and the third point is the cusp.

5. Over an algebraic closure of K there is a unique rational parameter taking those points to $0, 1, \infty$. Uniqueness makes it invariant under every field automorphism over K, so its coefficients belong to K. They are meromorphic germs with finite principal parts. Composing and resolving the graph yields the marked base map required in 18.3. This explicitly recovers a degree-one base parameter.

6. Modulo δ, the global lattice and finite-orbit calculation are unchanged. The cross-cocycles still lie in $\mathbb{Z}\delta$, so 18.5 applies. At the corner,

$$A_Q = I - NmE_{32} + NmE_{41}, \quad A_R = I + NnE_{41}, \quad [Q, R] = -Nmnb\widehat{\delta}.$$

The commuting lifts $Q_o = T_{-mb\widehat{\gamma}}Q$, $R_o = R$ give (18.8). Connectedness forces equality of the horizontal ratio zero and cusp ratio a. But $a \equiv m_2 l_1 \pmod{m_1}$ and $a \equiv m_1 l_2 \pmod{m_2}$, so $\gcd(a, N) = 1$. This contradiction proves nonexistence. Proper pullback to the further cover preserves the punctured family and gives the implication back to every finite parameter cover.

□

19 The boundary of the parameter disc

Definition 19.1 (Boundary coordinates and the Hausdorff reflection)**.** [tag PT08] For the standard twists put $H_* = (3/\pi)\log 6$, fix $c_* = x + iH_*$ and set $a = x + 3/2$. All topological group quotients carry their quotient topology. A Hausdorff reflection is a continuous quotient universal for continuous maps to Hausdorff spaces.

19.1 The intrinsic obstruction to extension

Proposition 19.2 (The natural boundary in the multiplicative parameter)**.** [tag 032A]

For $0 < |w| < 216$, the reduced multiplicity-four surface has

$$j(\operatorname{Alb} S_2(w)) = J(iw/216), \qquad J(q) = q^{-1} + 744 + 196884q + \cdots .$$

This holomorphic function has no meromorphic continuation through any point of $|w| = 216$.

Proof. 1. The fixed-coordinate periods on the product cover, including the isogeny kernel and the affine deck translation, generate $\frac{1}{2}\mathbb{Z} + \frac{\kappa_2}{4}\mathbb{Z}$. Multiplication by two therefore identifies the Albanese lattice with $\mathbb{Z} + (\kappa_2/2)\mathbb{Z}$. This verifies the precise step of proposition 7.6 used here; taking only the invariant eigenspace would omit the half-period. The values in proposition 3.2, theorem 6.6 and $\mu(z_2) = (1-i)/2$ give

$$\kappa_2 = c + \frac{3}{2} - iH_*, \qquad e^{-\pi i\kappa_2} = e^{-3\pi i/2}e^{-\pi H_*}e^{-\pi ic} = \frac{iw}{216}.$$

The positive-imaginary representative is $-\kappa_2/2$, proving the identity with the stated Fourier normalization.

2. Proposition 11.15 proves the natural boundary in the c-coordinate for this exact Albanese lattice. The parameter map is $w = e^{-\pi ic}$, with nonzero derivative at every finite c. A local inverse logarithm at a point of $|w| = 216$ maps that circle to $\operatorname{Im} c = \log 216/\pi = H_*$. A meromorphic continuation in w would pull back to the excluded continuation in c.

□

Lemma 19.3 (The intrinsic anticanonical surface family)**.** [tag 032B]

Let V be a connected Riemann surface and $\pi : \mathcal{X} \to V$ a flat proper holomorphic map of reduced complex spaces. Suppose that over a nonempty open subset V_0, the fibres are smooth compact threefolds whose unique anticanonical divisor is $2S_v$, where S_v is a smooth connected surface with $b_1(S_v) = 2$. Then there is an open subset $V^{\mathrm{good}} \subset V$, with at most countable complement, on which the reduced support of the relative anticanonical divisor is a smooth proper surface family. Its Albanese elliptic invariant is holomorphic.

Proof. 1. Flatness and the smoothness of the fibres over V_0 imply that π is a submersion there: the local analytic smoothness criterion applies because the base is smooth. The relative nonsmooth locus is a closed analytic subset, by the Jacobian criterion. Its proper image is analytic by the coherent direct-image argument in Theorem 18.3, applied to the structure sheaf of this proper analytic locus. It misses V_0, hence is discrete. We remove it and work over the connected remaining curve V'. Removing a discrete subset from a connected Riemann surface does not disconnect it. Thus $\mathcal{X}' \to V'$ is a smooth proper map.

2. Write $L = K^{-1}_{\mathcal{X}'/V'}$. Grauert's direct-image theorem (Grauert, 1960, §6, Hauptsatz I, p. 59) makes $F = \pi_*L$ and $R^1\pi_*L$ coherent. His erratum (Grauert, 1963, item 2, p. 36) corrects the assertion in §7, Satz 2 that the locus where the comparison map fails to be bijective is always closed. For the present smooth curve, the following exact sequence computes the canonical

base-change map and its torsion obstruction. For a local base parameter t vanishing at v, smoothness and invertibility of L give an exact sequence $0 \to L \xrightarrow{t} L \to i_* L_v \to 0$, where $i : X_v \hookrightarrow \mathcal{X}'$ is the closed embedding. Its direct-image sequence gives, at v,

$$0 \longrightarrow F_v/tF_v \longrightarrow H^0(X_v, L_v) \longrightarrow (R^1\pi_* L)_v[t] \longrightarrow 0.$$

The last term denotes elements annihilated by t. The support of the torsion subsheaf of the coherent module $R^1\pi_* L$ is closed discrete, by the structure theorem for finitely generated modules over the local discrete valuation rings of the smooth base curve. Outside that support the canonical base-change map is an isomorphism. Also F is torsion-free: multiplication by a nonzero base function is injective on sections of L. It is therefore locally free. Its rank is one by choosing an interior point outside the discrete torsion support. Locally a frame gives a relative section σ of L, nowhere identically zero on a fibre. Its divisor is independent of the frame. Let D be its reduced support. On V_0, the divisor is $2S_v$ and D is exactly S_v.

3. The map $D \to V'$ is proper, since D is closed in a proper space. There is no vertical divisor component: a vertical divisor in the smooth total space would contain a whole irreducible fibre, contrary to the nonzero restriction of σ. Every horizontal component has proper image all of the connected base. Since the interior fibre is irreducible, there is exactly one horizontal component, and its multiplicity in $\operatorname{div}\sigma$ is two. Thus $\operatorname{div}\sigma = 2D$ on this base. In the smooth ambient space D is Cartier. It is flat over the curve because its local rings are torsion-free over the base discrete valuation ring. Restriction of the Cartier identity to an interior fibre gives $2D_v = 2S_v$, whence $D_v = S_v$ with its reduced structure. Flatness and the smoothness of these fibres make $D \to V'$ smooth over V_0. Its analytic nonsmooth locus therefore misses V_0; the proper mapping theorem once more makes its image discrete. After deleting that image, $D \to V^{\mathrm{good}}$ is a smooth proper family of connected surfaces. Each deletion is closed discrete in the then-current open curve. Their finite union is at most countable, and the final good locus is open.

4. Ehresmann's theorem identifies the smooth fibres along every path; in particular their first Betti number is two. (Buchdahl, 1999, Theorem 11, p. 297) then gives a Kähler metric on every fibre. Hence $h^{1,0} = 1$, every holomorphic one-form is closed, and the integral period map has a rank-two lattice image. Constant $h^{1,0}$, properness, and analytic base change supply a local relative holomorphic one-form. Locally on a simply connected base disc, a relative one-form and a basis of the free part of H_1 give two holomorphic periods. Their ratio lies in one half-plane after orienting the basis. To see holomorphicity directly, choose local primitives of the closed relative form on simply connected fibre charts; their differences are holomorphic functions of the base, and the finite sum around a transported cycle is its period. The modular invariant of the ratio is independent of all choices, so these local invariants glue to a holomorphic function on V^{good}.

□

Theorem 19.4 (Nonextension across the parameter boundary)**.** [tag 032C]

For every w_0 with $|w_0| = 216$ and every connected open neighbourhood U of w_0, there is no flat proper holomorphic map $\mathcal{X} \to U$ from a reduced complex space whose fibres over $U \cap \{0 < |w| < 216\}$ are biholomorphic, fibre by fibre, to X_w. In particular an extension isomorphic to the given family over that intersection does not exist.

Proof. 1. We may restrict a putative extension to a small disc meeting the circle, with closure away from zero. This is a holomorphic base restriction; properness and flatness are preserved. The calculation of the canonical evaluation in proposition 7.2 gives $K_{X_w} = \mathcal{O}(-2S_2(w))$ and $h^0(K_{X_w}^{-1}) = 1$. Its sole divisor is $2S_2(w)$, so it is intrinsic under every fibrewise biholomorphism. The surface has $b_1 = 2$: on its finite product cover the rotation kills the moving factor in

invariant first cohomology and fixes the elliptic factor. These are the exact proposition 7.2, proposition 7.4 and proposition 7.6 steps required in Lemma 19.3.

2. The at most countable exceptional set in that lemma cannot contain an arc of the circle. Choose a boundary point outside it and then a smaller disc on which the reduced divisor is a smooth proper surface family. Its holomorphic Albanese invariant agrees, pointwise on the inner part, with $J(iw/216)$. The pointwise agreement gives an analytic continuation through that point, contrary to Proposition 19.2.

□

19.2 The local quotient and its Hausdorff reflection

Lemma 19.5 (The quadratic normal form at the boundary)**.** [tag 032D]

Near the order-four root centre there is a holomorphic coordinate

$$h = \mu - (1-\tau)/2$$

with nonzero derivative in s_2. In the fixed complex-linear fibre coordinates $\eta = \zeta_1$, $\xi = \zeta_2 + \zeta_1/2$, and the integral ordered basis $(\widehat{w}, \widehat{u}, \widehat{\delta}, \widehat{\gamma} + 3\widehat{u} - 3\widehat{w})$, the periods at c_* are

$$(1, \tfrac{1}{2}), \quad (\tau, \tfrac{1}{2} + h), \quad (0, 1), \quad (6h, a + B(h)), \tag{19.1}$$

where

$$\tau = i - \frac{3\pi i}{2}h^2 + O(h^4), \qquad B(h) = -3ih^2 + O(h^4), \qquad D_{c_*} = -3|h|^2 + O(|h|^4).$$

For c near c_*, only the last ξ-period changes, by $c - c_*$.

Proof. 1. The basis change has determinant of absolute value one. Substitution into the original period matrix gives (19.1) exactly, with $B = \beta + 6h + 3(1-\tau)/2 - a$. At the centre $B(0) = 0$. Put $G(h,\tau) = \wp_\tau((1-\tau)/2 + h)$ and $H(h,\tau) = \log(\vartheta_3(h,\tau)/\eta_D(\tau))$. The half-period identities and theta heat equation give

$$G(0,i) = 0, \quad G_{h^2}(0,i) = -g_2(i)/4, \quad G_\tau(0,i) = \frac{ig_2(i)}{6\pi},$$

$$H(0,i) = \log\sqrt{2}, \qquad H_\tau(0,i) = 0, \qquad H_{h^2}(0,i) = -\pi/2.$$

Here G_{h^2} is the power-series coefficient. For the derivative constant, write $A = \vartheta_2(0,i)^4 = \vartheta_4(0,i)^4$. Then $g_2(i) = 4\pi^4 A^2$, while differentiating $e_2 = (\pi^2/3)(\vartheta_2^4 - \vartheta_4^4)$ with the heat identities gives $e_2'(i) = 2\pi^3 i A^2/3$. The logarithmic derivatives used in this differentiation are

$$\frac{\partial_\tau \vartheta_2}{\vartheta_2} = \frac{\pi i}{12}(E_2 + \vartheta_3^4 + \vartheta_4^4), \qquad \frac{\partial_\tau \vartheta_4}{\vartheta_4} = \frac{\pi i}{12}(E_2 - \vartheta_2^4 - \vartheta_3^4).$$

They follow by applying the theta heat equation to the half-period identities. In particular $G_\tau(0,i) \neq 0$. The holomorphic implicit-function theorem and evenness of G give $\tau = \tau(h)$, even in h, with the displayed quadratic coefficient. Since $\tau - i$ has order two in the root coordinate, h has order one.

2. The identity verified in proposition 6.3 is

$$\log \mathcal{N} = \operatorname{Re} H - \frac{\pi(\operatorname{Im} h)^2}{\operatorname{Im} \tau}.$$

The displayed coefficients, $\tau - i = O(h^2)$, and evenness give $\log \mathcal{N} - \log\sqrt{2} = -(\pi/2)|h|^2 + O(|h|^4)$. Multiplication by $6/\pi$, using the boundary normalization, gives the formula for D_{c_*}. Directly from the definition,

$$D_{c_*} = \operatorname{Im} B - 6(\operatorname{Im} h)^2/\operatorname{Im}\tau.$$

Consequently $\operatorname{Im} B = -3\operatorname{Re}(h^2) + O(|h|^4)$. Since B is holomorphic and $B(0) = 0$, its Taylor coefficients force $B = -3ih^2 + O(h^4)$.

□

Proposition 19.6 (Nonseparation beyond the rank-drop line)**.** [tag 032E]

At the boundary value $a = 0$, the central points with $(\eta, \xi) = (0, 0)$ and $(0, -3i)$ cannot be separated in the root-family quotient. Their difference lies outside the sum of $\mathbb{C}(2, -1)^t$ and the central period group.

Proof. 1. Choose $h_n = 1/n$, which defines root points by Lemma 19.5, and take the integral lattice vector

$$\lambda_n = n^2(\widehat{\gamma} + 3\widehat{u} - 3\widehat{w}) - 6n\widehat{w} + 3n\widehat{\delta}.$$

Its period is exactly $(0, n^2B(1/n))$, and this tends to $(0, -3i)$. Thus the orbit of $(h_n, 0)$ also has a representative converging to $(0, 0, -3i)$. The two central quotient points are nonseparated. In these coordinates the line $\mathbb{C}(2, -1)^t$ is $\{\xi = 0\}$, and every central period has real ξ-coordinate; therefore their difference has nonzero imaginary second coordinate modulo that line and the central periods. The analytic error is $n^2O(n^{-4}) \to 0$, so the counterexample is for the family.

□

Theorem 19.7 (The Hausdorff reflection of the boundary family)**.** [tag 032F]

Let $\mathcal{T} = (\Delta_h \times \mathbb{C}^2)/\Lambda$ at the fixed boundary parameter c_*, after shrinking Δ_h. Two central points represented by (η, ξ) and (η', ξ') are nonseparated precisely when

$$\operatorname{Re}(\xi' - \xi) \in G_a := \tfrac{1}{2}\mathbb{Z} + a\mathbb{Z} \quad (a \in \mathbb{Q}),$$

and every pair is nonseparated when $a \notin \mathbb{Q}$. There is no nonseparation between distinct base fibres, or between distinct points of a noncentral fibre. The Hausdorff reflection $\mathcal{T}^{\mathrm{H}} \to \Delta_h$ has the original complex tori over $h \neq 0$, and its central fibre is

$$\mathbb{R}/G_a \simeq S^1 \quad (a \in \mathbb{Q}), \qquad \{*\} \quad (a \notin \mathbb{Q}).$$

If $a = m/n$ is in lowest terms, $n > 0$, then $G_a = \operatorname{lcm}(2, n)^{-1}\mathbb{Z}$.

Proof. 1. The limiting-group formula in 19.9 also applies to the fixed boundary family with $c = c_*$, $h \neq 0$. Put $d_0 = \operatorname{lcm}(2, n)$ in the rational case. In the dual lattice, the fixed integral covector whose central limit is $d_0 \operatorname{Re}\xi$ has $(p, q, r) = (0, 0, d_0)$. The exact dual formula (19.3) below shows that it extends continuously, as a real-linear functional ℓ_h on $\mathbb{C}^2$, to that limit: its last coefficient is $O(h^2)$, its other moving coefficients are $O(h)$. Therefore

$$(h, z) \longmapsto \exp(2\pi i \ell_h(z))$$

is continuous, invariant under every period, and restricts at the centre to $\exp(2\pi i d_0 \operatorname{Re}\xi)$. Points with different indicated central classes are separated by this map to the circle.

2. For the converse take a ray $h = \epsilon\omega$, $\epsilon \downarrow 0$, whose real and imaginary parts have irrational ratio. More precisely choose ω so that $p\operatorname{Re}\omega + q\operatorname{Im}\omega \neq 0$ for every nonzero pair (p, q) in the countable half-integral lattice in 19.9. The asymptotic dual formula shows that all such covectors escape to infinity. For rational a, the surviving dual group is exactly $\mathbb{Z}d_0 \operatorname{Re}\xi$; for

irrational a, it is zero. Pontryagin–Chabauty duality (Cornulier, 2011, Theorem 3.1, p. 2017; proof, p. 2018) now gives the period-group limits

$$\{z : d_0 \operatorname{Re} \xi(z) \in \mathbb{Z}\}, \qquad \mathbb{C}^2,$$

respectively. Every element of the indicated limit is the limit of period vectors. Such a sequence supplies intersecting images of every two prescribed central neighbourhoods, proving nonseparation. This includes the quadratic-scale differences of 19.6, not only first-order differences.

3. Collapse exactly these central classes and leave every other fibre unchanged. The resulting quotient is Hausdorff. The base coordinate separates different fibres. Within a noncentral fibre, continuous functions supported over a compact subdisc away from zero separate distinct points, because there the original action is properly discontinuous and gives a smooth torus bundle. In the rational central fibre the displayed circle-valued function separates different classes. These functions descend through this quotient, so it is Hausdorff. Every continuous map from $\mathcal{T}$ to a Hausdorff space identifies nonseparated pairs and therefore factors uniquely and continuously through this quotient. This proves the universal property, including its topology. The elementary greatest common divisor calculation gives the stated value of G_a. □

Proposition 19.8 (The finite boundary quotient and its leaves)**.** [tag 032G]

At the boundary, the construction over $B \setminus \{1\}$ is a complex manifold, since proposition 3.2, theorem 6.6 makes every remaining finite period map invertible and the cusp completion is unchanged. The order-four deck action descends continuously to $\mathcal{T}^{\mathrm{H}}$. On its rational central circle it is translation by $-a/4$; on the irrational central point it is trivial. Its finite quotient is Hausdorff with central fibre a circle or a point, respectively. Neither quotient is a complex space over the base compatible with the given holomorphic family off the centre. The linear foliation by $\mathbb{C}(2,-1)^t$ on the original central torus cover has compact elliptic leaves. For rational a the leaves are closed; for irrational a the closure of a leaf is the set with its fixed value of $\operatorname{Im} \xi$.

Proof. 1. The deck transformation is a biholomorphism of the original quotient, so the universal property extends it uniquely to a homeomorphism of the Hausdorff reflection. The action on the fixed product coordinate at the centre is $\xi \mapsto \xi - \kappa_2/4 = \xi - a/4$. A finite group quotient of a Hausdorff space is Hausdorff: finite orbits are separated by intersecting finitely many translated open neighbourhoods. A finite rotation quotient of a circle is a circle.

2. On a closed small base disc every noncentral torus point has a representative in the image of $[0,1]^4$ under its period matrix. These representatives belong to a single compact subset of $\mathbb{C}^2$. At the centre their images meet every class of the circle or the point. The image of that compact parameterized parallelepiped is therefore the full inverse image of the closed disc in the reflection. The reflected map is proper; the same holds after the finite quotient. If it were holomorphic on a compatible complex space, the central fibre would be analytic. A circle cannot be a complex analytic space: a positive-dimensional analytic germ has real dimension at least two, and a compact zero-dimensional analytic space is finite. A central point is also impossible here: upper semicontinuity of fibre dimension, or the local dimension inequality at a point in the closure of the three-dimensional smooth locus, forces a fibre dimension of at least two for a holomorphic map to a curve. Thus this Hausdorff reflection cannot be glued as a complex-space filling to the remaining constructed manifold.

3. For the leaf assertion, the moving line is $\{\xi = 0\}$. At the centre the projection of the period group to the ξ-line is $G_a \subset \mathbb{R}$. Its intersection with the moving line is a rank-two lattice in $\mathbb{C}_\eta$: the periods are $p + qi$ with $(p+q)/2 + ka + \ell = 0$ for integers p, q, k, ℓ. For irrational a this is

the index-two lattice $p+q\equiv 0 \pmod 2$; for rational a it is either that lattice or $\mathbb{Z}[i]$. Both give intrinsic elliptic leaf manifolds isomorphic to E_i. In the irrational case their maps into the non-Hausdorff central quotient need not be topological embeddings; "leaf" here means the image with this intrinsic immersed topology. When G_a is discrete, the saturated leaf is closed. When it is dense in $\mathbb{R}$, its saturation has closure $\mathbb{C}_\eta \times (\operatorname{Re}\xi + \mathbb{R})$, with $\operatorname{Im}\xi$ fixed. The natural map from this central quotient to the central fibre of $\mathcal{T}^{\mathrm{H}}$ is $[(\eta,\xi)] \mapsto \operatorname{Re}\xi \bmod G_a$ for rational a, and the constant map for irrational a, as in Theorem 19.7. It is well defined because the real second coordinate of every central period belongs to G_a.

□

19.3 All local limiting period groups

Theorem 19.9 (All marked limiting period groups)**.** [tag 032H]

Let $c_n \to c_*$ with $\operatorname{Im} c_n < H_*$, and let $h_n \to 0$. Set

$$\delta_n = -D_{c_n}(h_n) > 0, \qquad V_n = \left(\frac{6\operatorname{Re} h_n}{\delta_n}, \frac{6\operatorname{Im} h_n}{\delta_n}, \frac{\operatorname{Re}(c_n - c_*)}{\delta_n} \right).$$

Define the discrete group

$$L_a = \{(p,q,r) \in \mathbb{R}^3 : r \in \mathbb{Z},\ p + r/2 \in \mathbb{Z},\ q + r/2 \in \mathbb{Z},\ ar \in \mathbb{Z}\}.$$

It has rank three for rational a, and is $\mathbb{Z}^2 \times \{0\}$ for irrational a. Every convergent subsequence of the closed lattices $\Gamma_n = \Pi_{c_n}(h_n)\mathbb{Z}^4 \subset \mathbb{C}^2$ has the following form. There is a primitive subgroup $K \subset L_a$ and a homomorphism $\ell : K \to \mathbb{R}$ such that

$$\Gamma_\infty = \{(\eta,\xi) : p\operatorname{Re}\eta + q\operatorname{Im}\eta + r\operatorname{Re}\xi + \ell(p,q,r)\operatorname{Im}\xi \in \mathbb{Z} \text{ for every } (p,q,r) \in K\}. \tag{19.2}$$

Here

$$K = \{\lambda \in L_a : \lambda \cdot V_n \text{ has a finite limit}\}, \qquad \ell(\lambda) = \lim_n \lambda \cdot V_n,$$

and for $\lambda \notin K$ the absolute value tends to infinity. Conversely every primitive $K \subset L_a$ and every such ℓ occur for sequences with $\operatorname{Im} c_n < H_*$. Thus the possible pointed additive quotient limits are real tori of any dimension from zero to three for rational a, and from zero to two for irrational a, with the marked groups given explicitly by (19.2).

Proof. 1. We use the fixed complex-linear coordinates of 19.5. Write $h = X + iY$, $\tau = R + iT$, and $z = B(h) + c - c_*$. The real period matrix, with the basis of 19.5, is

$$A = \begin{pmatrix} 1 & R & 0 & 6X \\ 0 & T & 0 & 6Y \\ 1/2 & 1/2 + X & 1 & a + \operatorname{Re} z \\ 0 & Y & 0 & \operatorname{Im} z \end{pmatrix}, \qquad \det A = TD_c.$$

For an integral dual label (m_w, m_u, m_d, m_g), put $r = m_d$, $p = m_w - r/2$, $q = m_u - r/2$. The real-linear covector $A^{-t}(m_w, m_u, m_d, m_g)^t$ is

$$\left(p, \frac{q - pR - rX - tY}{T}, r, t\right), \quad t = \frac{m_g - ra - 6pX - 6Y(q - pR - rX)/T - r\operatorname{Re} z}{D_c}. \tag{19.3}$$

Multiplication by A^t proves the formula.

2. A bounded sequence of dual vectors has bounded integral labels, since A_n is bounded. After subselection its label is constant. Its limit must satisfy $m_g = ra$, because A_0^t has that relation.

The remaining labels are exactly L_a. Conversely consider a fixed label from L_a. Lemma 19.5 gives

$$T = 1 + O(|h|^2), \quad R = O(|h|^2), \quad \operatorname{Re} B = 6XY + O(|h|^4),$$
$$\delta_n = -\operatorname{Im}(c_n - c_*) + 3|h_n|^2 + O(|h_n|^4).$$

In particular $\delta_n \geq C|h_n|^2$ for a fixed $C > 0$ after shrinking the disc. The numerator in (19.3), with $m_g = ra$, is

$$-6pX - 6qY - r\operatorname{Re}(c - c_*) + O(|h|^3)(|p| + |q| + |r|).$$

The cancellation of the quadratic real term is essential. Division by $D_c = -\delta_n$ makes the error $O(|h_n|)$. Thus

$$t_n = (p, q, r) \cdot V_n + o(1).$$

Whenever this is bounded, the other three coefficients converge to (p, q, r). This proves that the dual limit, if it exists, is the discrete graph $\{(p, q, r, \ell(p, q, r)) : (p, q, r) \in K\}$. There can be no further elements from varying labels, because a bounded sequence of labels lies in a finite set. A convergent dual group cannot contain two different limits with the same label: their difference would have label zero, whose sole preimage under every A_n^t is zero. Nor can a surviving label escape along a subsequence, since an element of a Chabauty limit must be approximable along the whole convergent sequence. Consequently every label either has the indicated finite limit or escapes to infinity. Addition and integer division show that K is a subgroup and is primitive.

3. By (Cornulier, 2011, Theorem 3.1, p. 2017), the map from closed subgroups of a finite-dimensional real vector space to their Pontryagin annihilators is a homeomorphism. Applying it to the dual graph gives exactly (19.2). This theorem's hypotheses are met by the real vector space underlying $\mathbb{C}^2$ and by the closed full lattices Γ_n.

4. To prove realizability, extend ℓ linearly to a covector represented by a vector $v_0 \in \mathbb{R}^3$. Since K is primitive, $L_a \cap \operatorname{span}_{\mathbb{R}} K = K$. Choose v orthogonal to that span and not orthogonal to any member of $L_a \setminus K$; the excluded proper hyperplanes are countably many, and cannot cover the orthogonal space. If $K = L_a$, take $v = 0$. Put $V_n = nv + v_0 = (A_n', B_n', C_n')$. Choose positive $d_n \to 0$ with $d_n(1 + \|V_n\|^2) \to 0$, and define

$$h_n = d_n(A_n' + iB_n')/6, \quad x_n = d_n C_n', \quad \epsilon_n = d_n + D_{c_*}(h_n) > 0, \qquad c_n = c_* + x_n - i\epsilon_n.$$

The inequality follows from $D_{c_*}(h) = -3|h|^2 + O(|h|^4)$. These choices give $-D_{c_n}(h_n) = d_n$ exactly, so their vectors are the prescribed V_n. On K the scalar products converge to ℓ, and off K they escape. This constructs every claimed limit inside the admissible parameter domain.

5. Finally the dual graph is a discrete free group of rank $\operatorname{rank} K$. Its annihilator quotient is a compact connected real torus of that dimension. In the fixed Euclidean metric, $d_n([z], [z']) = \operatorname{dist}(z - z', \Gamma_n)$. Chabauty convergence gives local uniform convergence of these functions: minimizers relevant on a bounded set can be restricted to a bounded set, and the two defining Chabauty approximation properties then give both inequalities. This identifies the stated pointed quotient limits. Their group structures and quotient metrics are the ones displayed above.

□

Corollary 19.10 (Path dependence of the limiting torus)**.** [tag 032I]

There is no path-independent limiting group at the boundary even when $c_n \to c_*$ from the admissible side. At $h_n = 0$ and $c_n = c_* - i\epsilon_n$, all of L_a survives, giving a three-dimensional real torus if a is rational and a two-dimensional real torus if a is irrational. Other admissible paths give a point. The fixed-boundary Hausdorff reflection of 19.7 therefore cannot be identified with all pointed fibre limits.

Proof. 1. The first sequence has $V_n = 0$, hence $K = L_a$ and $\ell = 0$. For the second, take $K = 0$ in the explicit realization in 19.9. The limits have different real dimensions. Away from $t = 1$, the periods stay invertible and the local complex families converge to their ordinary value at c_*; the ambiguity is at the specified root centre. Theorem 19.4 excludes a flat proper holomorphic extension across the parameter boundary.

□

19.4 The supercritical locus

Proposition 19.11 (The small supercritical level curve)**.** [tag 032J]

For $\operatorname{Im} c = H_* + \epsilon$, with $0 < \epsilon \ll 1$, the zero locus $D_c = 0$ is one real-analytic simple closed curve surrounding $t = 1$. In the coordinate of 19.5,

$$|h|^2 = \epsilon/3 + O(\epsilon^2).$$

In the fixed coarse coordinate it has the leading shape

$$|t - 1| = \frac{\pi^4 E_4(i)}{4}\epsilon^2(1 + O(\epsilon)).$$

The error is uniform along the curve. The region containing $t = 1$ has $D_c > 0$; the region containing the cusp has $D_c < 0$.

Proof. 1. The Hessian of $D_{c_*}(h) = -3|h|^2 + O(|h|^4)$ is negative at zero. Its radial derivative is $-6|h| + O(|h|^3)$, uniformly in angle. Consequently each sufficiently small positive level of $-D_{c_*}$ has exactly one radius in each direction and is a real-analytic simple closed curve, with the displayed expansion. The order-four deck action preserves it, acts freely on it, and the coarse map t is a four-fold covering of its image. Since the punctured root quotient is a punctured disc, that image is again a simple closed curve. The strict global maximum in Theorem 6.6 gives a uniform gap on the complement of any fixed root neighbourhood; the cusp tends to minus infinity. Thus no other zero-locus component exists for small enough ϵ.

2. Ramanujan's differential identities give $j''(i) = -2\pi^2 j(i)E_4(i)$: indeed $j' = -2\pi i j E_6/E_4$ and $E_6'(i) = -\pi i E_4(i)^2$. Since $j = 1728t$, Lemma 19.5 gives

$$t - 1 = \frac{9\pi^4 E_4(i)}{4}h^4 + O(h^6).$$

Insertion of the radial expansion proves the coarse-coordinate formula, including its order of error. The sign statement follows from the strict radial decrease near zero and from the cusp sign.

□

Proposition 19.12 (The maximal open locus and the negative region)**.** [tag 032K]

For any height $H = \operatorname{Im} c > H_*$, the zero locus is the well-defined modular-norm level set

$$Z_H = \left\{ t : \mathcal{N}(\mu(t), \tau(t)) = 2\sqrt{3}\, e^{-\pi H/6} \right\}, \qquad \wp_{\tau(t)}(\mu(t)) = 0.$$

It is compact away from the cusp, and is a real-analytic set in the finite root charts. At every regular point it is a smooth real curve. The construction gives a complex manifold precisely over $B \setminus Z_H$, retaining the original Mumford fibre at the cusp. The open sublevel set $\{D_c < 0\}$, with the cusp adjoined, is connected. At every point of Z_H, the period quotient is the rank-three quotient of proposition 11.12, proposition 11.13; its exact Hausdorffization is described below.

Proof. 1. The formula for D in proposition 3.2, theorem 6.6 gives the equation, and the theta transformation law shows that it is independent of the modular and period representatives. Compactness away from the cusp follows from $D_{\mathrm{part}} \to -\infty$. The analytic and regular-level assertions follow in each root chart from the real-analytic implicit function theorem. On the complement, the real period matrix is invertible. On compact subsets of that complement its inverse is bounded, which proves uniform discreteness and proper discontinuity of the lattice action. The quotient is therefore a complex manifold with compact torus fibres. The finite deck action is free for the same invariant-coordinate reason as proposition 7.2, proposition 7.4: a nontrivial power translates that circle by a noninteger fraction, regardless of the sign of D. Near the cusp $D < 0$, so its existing holomorphic Mumford completion and its overlaps remain available. At a zero with irrational kernel line, the fibre quotient is already non-Hausdorff. At a zero with rational kernel choose a primitive integral kernel vector. Its holomorphic period vanishes at that root point but is not identically zero nearby. Its first nonzero Taylor coefficient, multiplied by arbitrarily large integer multiples and approached along suitable root directions, yields nonseparated central points. Thus no neighbourhood of a zero in the naive quotient is a Hausdorff complex manifold. This justifies the asserted maximal open base locus.

2. For connectedness, in any holomorphic coordinate z with $T = \operatorname{Im} \tau > 0$, differentiation gives

$$\partial_z \partial_{\bar{z}} D_{\mathrm{part}} = -\frac{3}{T} \left| \mu'(z) - \frac{\operatorname{Im} \mu}{T} \tau'(z) \right|^2 \leq 0.$$

Thus the function is superharmonic on the root charts. This property and the minimum principle descend under the finite quotient; one can equally apply the principle on a root cover of a putative minimum. A component of the strict sublevel set not containing the cusp would have compact closure with boundary at level $-H$, and would attain a smaller interior minimum. The strong minimum principle excludes this. A sufficiently small cusp neighbourhood is contained in the sublevel set, so all components reaching the cusp are the same one. Hence the sublevel set is connected.

□

Proposition 19.13 (Closed period groups at a rank-three fibre)**.** [tag 032L]

At a zero of D_c, let $A : \mathbb{R}^4 \to \mathbb{C}^2$ be the real period map and $\ker A = \mathbb{R}k$, with the kernel vector stated in proposition 11.12, proposition 11.13. Define

$$L_k = \{ m \in \mathbb{Z}^4 : m \cdot k = 0 \}, \qquad r = \operatorname{rank} L_k, \qquad V_k = \bigcap_{m \in L_k} \ker m.$$

Then

$$\overline{A\mathbb{Z}^4} = A(\mathbb{Z}^4 + V_k), \qquad \mathbb{C}^2 / \overline{A\mathbb{Z}^4} \simeq \mathbb{R} \times (S^1)^r$$

as real topological groups. The original quotient is a complex Lie group precisely when k is rational, in which case $r = 3$. Otherwise it is non-Hausdorff. Every Chabauty limit of nearby full lattices contains $\overline{A\mathbb{Z}^4}$; the quotient topology at the fixed zero and a limit of full lattices need not coincide.

Proof. 1. The last three period columns are real-linearly independent, so the rank is exactly three, as in proposition 11.12, proposition 11.13. The annihilator characters of $A\mathbb{Z}^4$ are the real covectors λ with $A^t\lambda \in \mathbb{Z}^4$. Their integral labels belong exactly to L_k. The smallest rational subspace containing k is V_k; the closure of the image of $\mathbb{Z}^4$ in $\mathbb{R}^4/\mathbb{R}k$ is its image after adding V_k. For an elementary verification, the two groups have the same annihilator characters, and characters separate closed subgroups of a real vector space by putting such a subgroup into the form $\mathbb{R}^d \times \mathbb{Z}^e$. This proves the closure formula. Inside the three-dimensional image hyperplane its connected part has dimension $3 - r$, and its discrete quotient has rank r. The one transverse real direction gives the displayed group.

2. The image is discrete exactly when the kernel line has a primitive integral generator: extend such a generator to an integral basis for the converse; for necessity, a discrete subgroup of a real three-space has rank at most three, forcing a nonzero integral kernel. A nondiscrete finitely generated subgroup cannot be closed, since a nondiscrete closed subgroup contains an uncountable vector group. Finally each fixed integral period converges to its limiting period; a Chabauty limit is closed and therefore contains their closure.

□

Proposition 19.14 (The derivative along the moving Weierstrass zero)**.** [tag 032M]

On an ordinary modulus chart where $P = \wp'_\tau(\mu) \neq 0$, put

$$L = \partial_\mu \log \vartheta_1(\mu, \tau), \qquad Z = L + 2\pi i \frac{\operatorname{Im} \mu}{\operatorname{Im} \tau}.$$

Then the exact real-analytic derivative is

$$\partial_\tau D_{\mathrm{part}} = \frac{iZ}{4\pi^2 P}(3ZP - 2g_2).$$

Away also from the ramification of j, the critical-point equation on the coarse base is therefore $Z = 0$ or $3ZP = 2g_2$.

Proof. 1. Put $E = \pi^2 E_2/3$. The theta heat equation, $\wp = -\partial_\mu L - E$, the differential equation $\wp'' = 6\wp^2 - g_2/2$, and $4\pi i E' = -2E^2 + g_2/6$ give, with μ held fixed,

$$\partial_\tau \wp = \frac{2\wp^2 - 2E\wp + L\wp' - g_2/3}{2\pi i}.$$

Indeed differentiating the heat equation twice in μ gives $(\wp'' - 2(\wp + E)^2 + 2L\wp')/(4\pi i) - E'$, which simplifies to the displayed formula. Implicit differentiation of $\wp(\mu) = 0$ therefore yields $\mu' = iL/(2\pi) - ig_2/(6\pi P)$. At a zero of $\wp$, the same heat equation and the eta derivative give

$$\frac{d}{d\tau} \log(\vartheta_1(\mu, \tau)/\eta_D(\tau)) = L\mu' + \frac{L^2}{4\pi i}.$$

Differentiating $D = \operatorname{Im} \beta - 6(\operatorname{Im} \mu)^2/\operatorname{Im} \tau$, and writing $b = \operatorname{Im} \mu/\operatorname{Im} \tau$, now gives

$$\partial_\tau D = \frac{3}{\pi} Z(\mu' - b) - \frac{3i}{4\pi^2} Z^2, \qquad \mu' - b = \frac{iZ}{2\pi} - \frac{ig_2}{6\pi P}.$$

Substitution proves the formula. Where $dt/d\tau \neq 0$, the chain rule transfers vanishing of this derivative to the coarse base.

□

Proposition 19.15 (The equianharmonic crossing)**.** [tag 032N]

At the equianharmonic point $\tau = \rho = e^{\pi i/3}$, with the original period representative $\mu = (2-\rho)/3$, one has

$$\mathcal{N}(\mu,\rho) = 3^{1/4}, \qquad D_{\mathrm{part}}(z_1) = -\frac{3}{2\pi}\log 48.$$

Consequently the zero locus passes through the coarse point $t = 0$ exactly at the height

$$H_0 = \frac{3}{2\pi}\log 48 = H_* + \frac{3}{2\pi}\log\frac{4}{3}.$$

It has $D_c(0) < 0$ below this height and $D_c(0) > 0$ above it. In the fixed coarse coordinate the function is C^1 at zero and

$$D_{\mathrm{part}}(t) = D_{\mathrm{part}}(0) + \frac{7}{32\pi}\operatorname{Re} t + O(|t|^{4/3}), \qquad \partial_t D_{\mathrm{part}}(t) = \frac{7}{64\pi} + O(|t|^{1/3}).$$

Thus at $H = H_0$ the level set is a regular C^1 arc through zero, tangent to the imaginary axis.

Proof. 1. For a fixed modulus write $T = \operatorname{Im}\tau$ and

$$u(z) = \log\mathcal{N}(z,\tau) = \log\left|\frac{\vartheta_1(z,\tau)}{\eta_D(\tau)}\right| - \frac{\pi(\operatorname{Im} z)^2}{T}.$$

The theta transformation law makes u periodic for $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. It is locally integrable, with $u(z) = \log|z| + c_\tau + o(1)$ at zero, and has no other singularity on the torus. In the Jacobi product, with $Q = e^{2\pi i\tau}$ and $X = e^{2\pi iz}$, the quotient is

$$\frac{\vartheta_1(z,\tau)}{\eta_D(\tau)} = iQ^{1/12}X^{-1/2}(1-X)\prod_{m\geq 1}(1-Q^mX)(1-Q^mX^{-1}),$$

up to a constant of absolute value one. For $z = x + iy$, $0 < y < T$, every product factor has horizontal mean logarithm zero, by the power series for $\log(1-ae^{2\pi ix})$, $|a| < 1$. The series converges uniformly on closed substrips. Thus

$$\int_0^1 u(x+iy)\,dx = -\frac{\pi T}{6} + \pi y - \frac{\pi y^2}{T}.$$

The logarithmic singularities at the strip endpoints are integrable; alternatively one integrates first on closed substrips and passes to the limit. Integration in y gives $\int_{\mathbb{C}/\Lambda} u\,dA/T = 0$. A tilted fundamental parallelogram has the same horizontal averages, by period one. This proves the zero-mean normalization directly.

2. For $\tau = \rho$ put $\alpha = 1 + \rho$. Since $\rho^2 = \rho - 1$, multiplication by α on $(1,\rho)$ has integral matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}, \qquad \det = 3.$$

Moreover $\alpha\mu = 1$, $3\mu = 2 - \rho \in \Lambda$, and $\mu \notin \Lambda$. Its kernel on the torus is therefore $K = \{0, \mu, -\mu\}$, and $|\alpha|^2 = 3$. The distributional Euclidean Laplacian is

$$\Delta u = 2\pi\delta_0 - \frac{2\pi}{T}.$$

Indeed the holomorphic theta factor contributes its simple zero, and the Gaussian contributes the displayed constant. Pulling back by the covering $z \mapsto \alpha z$ shows that $u(\alpha z)$ and $\sum_{v\in K} u(z+v)$ have the same Laplacian: both have one logarithmic mass at every point of K and constant term $-6\pi/T$. Their difference is consequently a periodic harmonic function. The logarithmic

singularities cancel locally, so the removable-singularity theorem gives an ordinary harmonic function on the compact torus. It is constant. Multiplication by α and translation preserve normalized Haar measure. The mean just computed therefore makes that constant zero, proving the exact distribution identity

$$\mathcal{N}(\alpha z, \rho) = \prod_{v \in K} \mathcal{N}(z + v, \rho).$$

As $z \to 0$, divide by $\mathcal{N}(z, \rho)$. The left ratio tends to $|\alpha| = \sqrt{3}$. Since $\mathcal{N}$ is even, the other two factors tend to $\mathcal{N}(\mu, \rho)^2$. Positivity now gives $\mathcal{N}(\mu, \rho) = 3^{1/4}$.

3. Finally the exact proposition 3.2, theorem 6.6 normalization is

$$D_{\text{part}} = \frac{6}{\pi} \log \frac{\mathcal{N}(\mu, \tau)}{2\sqrt{3}}.$$

Substitution gives $-3 \log 48/(2\pi)$. The boundary height is $H_* = 3 \log 6/\pi$, so the stated difference is $3 \log(4/3)/(2\pi) > 0$. Addition of $\operatorname{Im} c$ proves the crossing height and sign assertions.

4. For regularity in the coarse coordinate, multiplication by ρ^2 is an isometry of the torus and preserves the zero-mean Green norm just used. Also $(\rho^2 - 1)\mu = \rho - 1 \in \Lambda$. Hence its nontrivial rotation fixes μ, and the real gradient of $\log \mathcal{N}$ vanishes there. In the notation of 19.14 this says $Z(\rho) = 0$. Write $P_0 = \wp'_\rho(\mu) \neq 0$, $E = \pi^2 E_2/3$, and $b = \operatorname{Im} \mu/T$. The identities of 19.14 give $\mu' = b$ at ρ, since $g_2(\rho) = 0$ and $L = -2\pi i b$. The heat equation, with μ held fixed, gives

$$L_\tau = \frac{-P - 2EL}{4\pi i}, \qquad L_\mu = -E.$$

When differentiating Z along $\mu(\tau)$, its two terms involving Eb cancel and the derivative of b vanishes at ρ. Thus

$$\partial_\tau Z(\rho) = \frac{iP_0}{4\pi}, \qquad \partial_{\bar\tau} Z(\rho) = 0.$$

Ramanujan's identity and the Weierstrass equation yield

$$g_2'(\rho) = -\frac{3i}{\pi} g_3(\rho) = \frac{3i}{\pi} P_0^2.$$

For $v = \tau - \rho$, it follows that

$$3ZP - 2g_2 = -\frac{21iP_0^2}{4\pi} v + O(|v|^2), \qquad \partial_\tau D_{\text{part}} = \frac{21iP_0^2}{64\pi^4} v^2 + O(|v|^3).$$

Here all remainders are real analytic in $v, \bar v$. The exact normalization $t = j/1728$ gives

$$t = -\frac{ig_3(\rho)}{\pi^3} v^3 + O(v^4) = \frac{iP_0^2}{\pi^3} v^3 + O(v^4).$$

For example this follows immediately from $t = E_4^3/(E_4^3 - E_6^2)$, $E_4'(\rho) = -(2\pi i/3)E_6(\rho)$, and $g_3 = 8\pi^6 E_6/27$. Dividing the derivative expansions on $v \neq 0$ proves $\partial_t D_{\text{part}} = 7/(64\pi) + O(|t|^{1/3})$. The conjugate derivative has the conjugate expansion. The resulting real Taylor expansion extends the coarse function as a C^1 function with nonzero real gradient, giving the asserted local arc by the implicit function theorem. In particular this is not a singular level at $t = 0$, despite the vanishing modulus derivative at the order-three ramification point.

□

Remark 19.16 (The Green-function pullback along the moving zero)**.** [tag 0320] For a fixed modulus τ, let

$$G_\tau(z) = -\frac{1}{2\pi}\log\mathcal{N}(z,\tau), \qquad Z(z,\tau) = \partial_z \log\vartheta_1(z,\tau) + 2\pi i\frac{\operatorname{Im} z}{\operatorname{Im}\tau}.$$

The normalization proved in Proposition 19.15 gives the zero-mean Green function, and differentiation gives

$$\partial_z G_\tau(z) = -\frac{Z(z,\tau)}{4\pi}.$$

This is the critical equation of Lin and Wang (2010, equation (1.1), p. 912, and Lemma 2.3, p. 919). The section of the universal elliptic curve used here is $\iota(\tau) = (\mu(\tau),\tau)$, with $\wp_\tau(\mu(\tau)) = 0$, and its exact pullback is

$$D_{\mathrm{part}}(\tau) = -12G_\tau(\mu(\tau)) - \frac{6}{\pi}\log(2\sqrt{3}).$$

Along this section, Proposition 19.14 computes the full derivative, including the variation of τ:

$$\partial_\tau D_{\mathrm{part}} = \frac{iZ}{4\pi^2 P}(3ZP - 2g_2), \qquad P = \wp'_\tau(\mu).$$

Thus the two factors controlling the level curves on ordinary charts are the pullback of the fixed-modulus critical factor Z and the additional factor $3ZP-2g_2$. Propositions 19.17 and 19.18 supply their nonvanishing and boundary estimates; Proposition 19.15 gives the coarse equianharmonic crossing.

Proposition 19.17 (Nonvanishing near the elliptic points and cusp)**.** [tag 032P]

The function D_{part} has no critical point on the punctured root neighbourhood $0 < |\tau - i| \le 3/20000$, no critical point on the coarse neighbourhood represented by $|\tau - \rho| \le 1/50$, and no critical point where $\Im\tau \ge 2$. At ρ the two factors Z and $F = 3ZP - 2g_2$ of 19.14 vanish, but the coarse gradient is nonzero, as in 19.15.

Proof. 1. The two norms agree by their exact sigma–theta formula in 4.2. Hence, with $L = \log\mathcal{N}$,

$$D_{\mathrm{part}} = \frac{6}{\pi}(L - \log(2\sqrt{3})).$$

Theorem 25.6 proves that the coarse differential is nonzero away from $t = 1$, including at $t = 0$, where $L_t(0) = 7/384$. It gives $(D_{\mathrm{part}})_t(0) = 7/(64\pi)$. Its two exact root-critical orbits are i and ρ, with the factor vanishing specified in 19.14, 19.15.

2. The root-to-coarse map is unramified off those elliptic orbits. The stated small disc about i contains no other elliptic representative, and removing its centre therefore removes its only critical point. The coarse disc about ρ has the nonzero differential just calculated. Finally

$$\Im\frac{ai+b}{ci+d} = \frac{1}{c^2+d^2} \le 1, \qquad \Im\frac{a\rho+b}{c\rho+d} = \frac{\Im\rho}{c^2+cd+d^2} \le 1.$$

Thus no elliptic-orbit critical point lies at height at least two. These facts prove every claimed domain of nonvanishing.

□

Proposition 19.18 (The supercritical level classification)**.** [tag 032Q]

For every $H > H_*$, the degeneracy locus Z_H is exactly one embedded C^1 Jordan curve in the finite t-plane. It is real analytic unless $H = H_0 = 3\log 48/(2\pi)$; at that height it is real analytic away from $t = 0$ and has the regular crossing described in 19.15. There are no singular coarse levels at any supercritical height. The bounded complementary disc contains $t = 1$ and has $D_c > 0$; the complementary disc in $\mathbb{P}^1$, containing the cusp, has $D_c < 0$. The point $t = 0$ is in the latter disc for $H < H_0$, on the curve for $H = H_0$, and in the former disc for $H > H_0$.

The bounded discs are strictly nested as H increases, shrink to $\{1\}$ as $H \downarrow H_*$, and exhaust the finite plane as $H \to \infty$. Their small-height boundary shape is the expansion in 19.11. The maximal open base locus of the complex manifold construction consists of these two complementary discs. Approaching their common boundary, the covolume of the marked root torus lattice is

$$\operatorname{covol}(\Pi_c\mathbb{Z}^4) = \operatorname{Im}\tau\,|D_c| \longrightarrow 0.$$

At the boundary the period map has rank three and the exact quotient and Hausdorff reflection are those of 19.13.

Proof. 1. Use the same exact affine relation and write

$$f = -D_{\mathrm{part}}, \qquad Z_H = \{L = \log(2\sqrt{3}) - \pi H/6\}.$$

The maximum $L(1) = \log\sqrt{2}$ corresponds exactly to $H_* = 3\log 6/\pi$. Thus $H > H_*$ corresponds to a level strictly below that maximum. Theorem 25.6 proves that each such level is one strictly star-shaped C^1 Jordan curve about 1, real analytic away from the possible point $t = 0$. Its nonzero coarse derivative there proves regularity also on that exceptional level.

2. Since $L(0) = \frac{1}{4}\log 3$, its height is $H_0 = 3\log 48/(2\pi)$. The exact equation $D_c = H - f$ gives the two signs, the position of 0, and the bounded component $\{f < H\}$ containing 1. Strict radial monotonicity proves strict nesting. The unique minimum $f(1) = H_*$ and properness of f off 1 prove shrinking as $H \downarrow H_*$ and exhaustion of every compact subset as $H \to \infty$. The expansion at the minimum is 19.11.

3. In the ordered real coordinates and marked lattice basis of 19.13, direct expansion of the period determinant gives

$$\det_{\mathbb{R}} \Pi_c = \Im\tau\,\Im\beta_c - 6(\Im\mu)^2 = \Im\tau\, D_c.$$

Its absolute value is the covolume and tends to zero at the level curve. The same matrix retains three independent columns there, so its rank is exactly three. Apply the exact quotient and Hausdorff-reflection computation of 19.13. The local proper-action criterion of 19.12 gives complex manifolds on both complementary regions and excludes every point of their common boundary, including its discrete fixed-fibre quotients. This proves the maximality and all the local degeneration assertions.

□

20 The limit at the puncture

20.1 The untwisted pinching and its local smoothings

Definition 20.1 (The elliptic surface and the pinching datum)**.** [tag PT01] S is the smooth minimal rational elliptic surface with Weierstrass equation

$$y^2 = x^3 - 3t^3(t-1)x + 2t^4(t-1)^2,$$

$\pi : S \to B$ is its elliptic fibration, P is the section used in the period dictionary, $T = t_{6P} : S \to S$, and $L = M_S = \mathcal{O}_S(P - O)$. We use the convention that $\mathbb{P}_S(\mathcal{O} \oplus L)$ contains the total space of L, with fibre coordinate a, zero section s_0, and infinity section s_∞. The two normal bundles are respectively L and L^{-1}. We write $R = \mathbb{P}_S(\mathcal{O} \oplus L)$.

Proposition 20.2 (The pinching and its smoothing line)**.** [tag 01R1]

The quotient

$$Q = R/\{s_0(x) \sim s_\infty(Tx) : x \in S\}$$

has a compact, reduced complex-space structure. Its normalization is the finite holomorphic map $\nu : R \to Q$. Its singular locus is a copy of S, and its local analytic equation along that locus is $ab = 0$, with two further coordinates on S. The induced map $Q \to B$ is proper. Its sheaf of first-order local smoothing parameters is

$$\mathcal{T}^1_Q = \mathcal{E}xt^1_{\mathcal{O}_Q}(\Omega_Q, \mathcal{O}_Q) \simeq L \otimes T^*L^{-1} \simeq \pi^*\mathcal{O}_B(1)$$

on the double locus.

Proof. 1. The two sections of R are disjoint closed submanifolds. The equivalence relation consists of their two graphs and the diagonal; it is closed in $R \times R$. A quotient of a compact Hausdorff space by a closed equivalence relation is Hausdorff: for distinct equivalence classes, compactness gives disjoint saturated neighbourhoods by taking finite subcovers of the complements of the relation. Thus the displayed quotient is compact and Hausdorff.

2. For a coordinate neighbourhood $U \subset S$ and frames of $L|_U$ and $L^{-1}|_{T(U)}$, the two tubular coordinate charts are (x, a) at $s_0(U)$ and (Tx, b) at $s_\infty(T(U))$. We give their identification the ring

$$\{(f(x,a), g(x,b)) : f(x,0) = g(x,0)\} = \mathbb{C}\{x_1, x_2, a, b\}/(ab).$$

Changes of frame are holomorphic and invertible on each branch and have the same change of x on their intersection. They therefore glue these hypersurface charts to the inherited charts off the sections. This constructs the asserted complex space. The direct sum of the two regular branch rings is the integral closure: an integral element in the total ring of fractions is integral on each smooth branch, and conversely the two branch idempotents are integral. The resulting map ν is proper with at most two points in each fibre, and its local ring extension is finite. The map to B is well defined because $\pi T = \pi$; its properness follows from compactness of Q.

3. The conormal presentation for the hypersurface $ab = 0$ gives $\mathcal{E}xt^1(\Omega, \mathcal{O}) = \mathcal{O}/(a, b)$. A change of normal frames $a' = g(x)a$, $b' = h(x)b$ changes the equation and its smoothing parameter by gh. Consequently these local modules glue as $N_{s_0/R} \otimes T^*N_{s_\infty/R} = L \otimes T^*L^{-1}$. For this particular surface, Engel's Lemma 2.1 states $\mathrm{Hom}(T^*L, L) = \pi^*\mathcal{O}_B(1)$ (Engel, 2026, Lemma 2.1, pp. 3–4). Its identification applies here with $T = t_{6P}$ and $L = \mathcal{O}(P - O)$. One can recover the degree in that lemma directly: the bundle has multidegree zero on every fibre, since T preserves every component; hence it is $\pi^*\mathcal{O}_B(n)$. Its degree on O is

$$n = (O - P) \cdot 6P + (P - O) \cdot O = (2 - 2) + (0 - (-1)) = 1.$$

The section intersections used here are the explicit computations in the proof of that lemma. This derives the displayed sheaf in the normal-bundle convention of our pushout.

□

Definition 20.3 (The smoothing section)**.** [tag PT06] Set $J = L \otimes T^*L^{-1}$. Proposition 20.2 identifies J with $\pi^*\mathcal{O}_B(1)$. We fix the section $\phi \in H^0(S, J)$ with divisor S_∞ from (Engel, 2026, Corollary 2.2, p. 4). Thus ϕ_x is the bundle map $L_{Tx} \to L_x$ used in the plumbing equation.

Proposition 20.4 (The nodal plumbing and its small modification)**.** [tag 01R3]

The first simultaneous local plumbing at the nodal point of S_∞ has equation

$$xy = uzv, \qquad q = zv, \tag{20.1}$$

after a holomorphic change of one normal coordinate, where q is a local coordinate on B at infinity. This hypersurface is flat over the u-disc. For each $u \neq 0$, its three-dimensional fibre has an ordinary double point at $x = y = z = v = 0$. The projective modification defined by the blow-up of (x, z) resolves that point for $u \neq 0$. Its two strict-transform charts are

$$x = zr, \quad ry = uv; \qquad z = xa, \quad y = uav. \tag{20.2}$$

The modified family remains flat over u. At $u = 0$, the normalization of its central fibre contains the blow-up of one branch of (20.1) along the smooth curve $x = z = 0$; hence this particular modification changes the normalization of the unmodified pinching.

Proof. 1. The surface S is smooth and its nodal elliptic fibre has local equation $q = zv$ at its node. The section ϕ has a simple zero along that fibre. Thus $\phi = zv$ times an invertible holomorphic function. Absorbing the function in one normal coordinate gives (20.1). The convergent power-series ring $\mathbb{C}\{x, y, z, v, u\}$ is factorial, and u is coprime to $xy - uzv$. Hence $uh \in (xy - uzv)$ implies $h \in (xy - uzv)$. The quotient has no u-torsion. At a nonzero parameter the identical argument uses $u - u_0$. This proves flatness over the smooth parameter curve. For fixed $u \neq 0$ the four first derivatives vanish only at the origin, and the quadratic form $xy - uzv$ is nondegenerate. This proves the ordinary-double-point assertion by its defining equation.

2. The two standard affine charts of the blow-up give (20.2), after removing the common factors z and x, respectively. For $u \neq 0$ the first chart solves for v and the second for y, so both are smooth. The map is an isomorphism away from the origin in the fixed u-fibre: where the ideal fails to be visibly principal, the hypersurface equation and one of y, v make it principal. The fibre over the origin is $\mathbb{P}^1$. The modification is projective over (20.1), since it is a closed subspace of its product with $\mathbb{P}^1$. The first strict-transform equation is coprime to $u - u_0$ in its convergent power-series ring for every u_0; the second chart is smooth with free parameter u. Thus flatness over u persists.

3. At $u = 0$ the first chart has $ry = 0$. The branch $r = 0$ maps isomorphically to $x = 0$ in the original central fibre. The branch $y = 0$, together with the second chart, is the ordinary blow-up of the branch $y = 0$ in coordinates (x, z, v) along (x, z). This description follows from the two affine substitutions defining that blow-up and identifies the normalizations. For $u \neq 0$, the base coordinate in the first chart is $q = zry/u$ and in the second it is $q = xav$. Thus the modification also exhibits the triple-coordinate strata that the cusp filling requires.

□

Proposition 20.5 (The unsubdivided cusp fan)**.** [tag 02ZH]

For $k, l \in \mathbb{Z}$, let $P_{k,l}$ be the cone defined by $h_q, h_u \geq 0$, $kh_q \leq y_1 \leq (k+1)h_q$ and $l(h_q + h_u) \leq y_2 \leq (l+1)(h_q + h_u)$. These cones and their faces form a fan. Its two period automorphisms on the cocharacter lattice are

$$G_1(y_1, y_2, h_q, h_u) = (y_1 + h_q, y_2, h_q, h_u), \qquad G_2(y_1, y_2, h_q, h_u) = (y_1, y_2 + h_q + h_u, h_q, h_u).$$

The affine chart of $P_{0,0}$ is

$$U_{0,0} = \operatorname{Spec} \mathbb{C}[x, y, z, v, u]/(xy - uzv), \qquad q = zv.$$

In particular the unsubdivided chart is not a smooth simultaneous model of the cusp.

Proof. 1. At fixed $h_q, h_u \geq 0$, the cone $P_{k,l}$ cuts out the rectangle

$$kh_q \leq y_1 \leq (k+1)h_q, \qquad l(h_q + h_u) \leq y_2 \leq (l+1)(h_q + h_u).$$

When $h_q = 0$ the first interval is a point. These rectangles tile their support, and their intersections are common faces, including this collapsed boundary. This proves the fan assertion. The displayed integral transformations preserve the fan and have integral inverses.

2. For a character (a, b, c, d), the six inequalities of the dual cone are $c, a+c, b+c, a+b+c, d, b+d \geq 0$. The characters

$$x = (0, 1, 0, 0), \quad y = (0, -1, 1, 1), \quad z = (1, 0, 0, 0), \quad v = (-1, 0, 1, 0), \quad u = (0, 0, 0, 1)$$

generate its integral monoid. Indeed if $b \geq 0$ we subtract bx; if $b < 0$ we subtract $(-b)y$. The remaining character has second coordinate zero and is a nonnegative integral combination of z, v, u. The sole relation is $x + y = u + z + v$, and the character of the first base coordinate is $z + v$. This proves the coordinate-ring assertion, including normality.

□

Theorem 20.6 (The compact nodal plumbing)**.** [tag 02ZI] [tag 01R2]

There is a reduced complex space $\mathcal{X}^\flat$ and a proper flat holomorphic map

$$(f, \varpi) : \mathcal{X}^\flat \longrightarrow B \times \Delta_u$$

whose central threefold is Q. For $u \neq 0$ its fibre is the contraction of Engel's chosen $X(u)$ at one rational curve of the Mumford fibre. This curve is the double curve corresponding to the diagonal of the square in the chosen A_2 triangulation; in the coordinates of the marked cusp fan its direction is $e_1 - e_2$. Its normal bundle in $X(u)$ is $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. Its image is the unique ordinary double point of $\mathcal{X}^\flat_u$.

Over $S^\circ = S \setminus S_\infty$ the total plumbing space is smooth. Its restriction over every relatively compact disc $U \Subset B \setminus \{\infty\}$ is proper and flat over $U \times \Delta_\epsilon$ for sufficiently small $\epsilon > 0$. For $u \neq 0$ it is the untwisted quotient by $u\phi$ covering T^{-1}. This includes discs containing 0 or 1, before the logarithmic transformations.

Proof. 1. On the whole compact surface S, choose a Hermitian metric on L, use the dual metric on L^{-1}, and choose $0 < r < 1/4$. For sufficiently small $\epsilon > 0$ we have $|u|\|\phi_x\| < r^2$ for all $x \in S$ and $|u| < \epsilon$. Take the core

$$C = \{(x, a, u) : a \in L_x,\ r < \|a\| < r^{-1},\ |u| < \epsilon\}$$

and the node space

$$N = \{(x, a, b, u) : a \in L_x,\ b \in L_{Tx}^{-1},\ a \otimes b = u\phi_x,\ \|a\|, \|b\| < 2r,\ |u| < \epsilon\}. \tag{20.3}$$

On $r < \|a\| < 2r$ identify a point of N with $(x, a, u) \in C$. On $r < \|b\| < 2r$ identify it with $(Tx, b^{-1}, u) \in C$. Both identifications are biholomorphisms between open complex spaces; in the first collar $b = u\phi_x/a$, and in the second $a = u\phi_x b^{-1}$. Their domains are disjoint because $\|a\|\|b\| < r^2$. Changes of frames multiply the equation by units and give the stated bundle-valued equation, so no choice of frame enters the gluing.

2. This gluing is Hausdorff. At either end of a collar, exactly one of its two charts contains the limiting circle: the core omits $\|a\| = r$, and the node chart omits $\|a\| = 2r$; the infinity collar has the identical property in b. Thus the graph of the identification has no unpaired pair of limiting points. The two collars in each chart are separated, so there are no additional chains of identifications. These observations give disjoint saturated neighbourhoods of any two inequivalent points. Properness is equally concrete. Over a compact subset of $B \times \Delta_\epsilon$, the closed core $3r/2 \le \|a\| \le 2/(3r)$ together with the closed node subspace $\|a\|, \|b\| \le 7r/4$ is compact and maps onto the glued space. To see surjectivity, a point outside the smaller core lies on a node collar, and a node point outside the smaller node subspace lies on the smaller core. Compactness of S supplies the uniform bound used above. This proves properness of the joint map to (t, u).

3. The total node rings are hypersurface rings over the regular two-dimensional local rings of S. They are Cohen–Macaulay. Every fibre of the joint map has pure dimension two: first fix t, and then impose $xy = u\phi$ on the two line coordinates over the one-dimensional fibre of S. Consequently $t - t_0, u - u_0$ is a regular sequence in every total local ring. The regular-sequence flatness criterion over the smooth two-dimensional base proves flatness. On S°, the derivative of $ab - u\phi_x$ with respect to u is $-\phi_x \neq 0$, so the total node space is smooth. On the core the total space is an open subset of $L \times \Delta_u$ and is smooth; joint flatness follows from that of $S \to B$. This includes the nonreduced Kodaira fibres at $0, 1$, since the argument uses smoothness of S. At $u = 0$ the node equation is $ab = 0$, and the two collar maps recover precisely the specified pinching of the two sections of R.

4. For $u \neq 0$ and $t \neq \infty$, the equation gives $a_{Tx} = a_x/(u\phi_x)$. Hence the resulting quotient is exactly $L^\times/\langle u\phi\rangle$, where the contracting generator covers T^{-1}. Every orbit meets a core or a collar, by the uniform contraction estimate; the displayed comparison identifies exactly its orbit relation. It is therefore a biholomorphism.

5. At the nodal point of S_∞, write $q = zv$. Absorbing a unit in one line coordinate gives $xy = uzv$. At a smooth point of S_∞ the equation is $xy = uq$. For $u \neq 0$ the latter threefold is smooth; the former has one nondegenerate quadratic singularity at the origin. There are no other singularities of a nonzero u fibre.

6. In a Tate covering of a neighbourhood of the I_1 fibre use the torus coordinates (Z, W) of Engel (2026, Proposition 2.7, p. 6). The two periods act as

$$(Z, W, q, u) \longmapsto (c_1 qZ, c_2 W, q, u), \qquad (Z, W, q, u) \longmapsto (c_3 Z, c_4' uqW, q, u),$$

with holomorphic units c_i. Their monomial parts are G_1, G_2. The chart indexed by (k, l) has $z = Zq^{-k}$, $v = q^{k+1}/Z$, $x = W(qu)^{-l}$, $y = qu/x$, with the unit changes given by the line frames. It is the chart of Proposition 20.5. The Tate cover unfolds the node of S_∞ into its infinite chain, and unfolding the two plumbing collars gives the additional integer index l. The foregoing collar identifications are exactly the two deck identifications. Thus, on an invariant neighbourhood over a sufficiently small bidisc, the fan space maps by a holomorphic covering to the plumbing, with deck group $\mathbb{Z}^2$. Each core or node coordinate neighbourhood has disjoint lifts indexed by the two unfolded integers; this proves freeness and proper discontinuity, including the ρ_l orbits. The first period stabilizes such a ray but on its orbit translates the Tate coordinate by $c_1 q$, with $q \neq 0$ and $|c_1 q| < 1$; it does not fix a

point. Properness of its quotient was proved by the compact core and node representatives. This proves the analytic comparison of the fan quotient with the compact plumbing.

7. Subdividing the square at $h_u = 0$ by the chosen diagonal gives the small resolution of $xy = uzv$ for $u \neq 0$. Put $Y = y/u$, so the equation is $xY = zv$. The resolution associated with the diagonal $[e_1, e_2]$ has charts

$$(v, Y, a) \longmapsto (av, uY, aY, v), \qquad (x, z, b) \longmapsto (x, ubz, z, bx)$$

in the coordinates (x, y, z, v). On their overlap, $b = a^{-1}$ and $(x, z) = a(v, Y)$. The character $a = x/v$ has valuation $y_1 + y_2 - h_q$, so these are exactly the two triangles separated by $[e_1, e_2]$. The exceptional curve is the zero section in the two normal coordinates. Their transition is multiplication by a, giving $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. There is one diagonal edge modulo the periods, so there is exactly one such curve. Removing that edge contracts the paired opposite (-1)-curves in the normalization dP_6 to the square model $\mathbb{P}^1 \times \mathbb{P}^1$. This identifies the contracted curve in the selected toric model. The fixed double curve of Proposition 16.4 has direction e_2 in the same marked lattice; the two primitive directions satisfy $\det(e_1 - e_2, e_2) = 1$.

□

Proposition 20.7 (The integral topology of the pinching)**.** [tag 01R4]

The compact pinching Q of Proposition 20.2 has

$$\pi_1(Q) = \mathbb{Z}, \qquad (H_i(Q; \mathbb{Z}))_{i=0}^{6} = (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}^8, \mathbb{Z}^7, \mathbb{Z}^{10}, 0, \mathbb{Z}), \qquad \chi(Q) = 12.$$

Proof. 1. The two section embeddings are cofibrations: a tubular neighbourhood retracts onto each section, and the two neighbourhoods are disjoint. The ordinary pushout is therefore homotopy equivalent to the space formed from R by attaching $S \times [0, 1]$ along s_0 and $s_\infty T$. This comparison follows by collapsing the intervals in the mapping cylinder; the tubular retractions construct its homotopy inverse. Both S and R are simply connected. Van Kampen applied to this mapping-cylinder model gives one free loop and no additional generator or relation, so $\pi_1(Q) = \mathbb{Z}$. The same model has the exact sequence

$$H_i(S) \xrightarrow{d_i = s_{0*} - s_{\infty *} T_*} H_i(R) \longrightarrow H_i(Q) \longrightarrow H_{i-1}(S). \tag{20.4}$$

2. Put $F = [S_t]$, $m = P - O$, and $v_0 = [6P] - [O] - 3F \in H_2(S; \mathbb{Z})$. The intersection computations in (Engel, 2026, Lemma 2.1, pp. 3–4) give

$$\begin{gathered} F^2 = Fv_0 = Fm = 0, \quad v_0^2 = -6, \quad mv_0 = -1, \quad m^2 = -2, \\ OF = 1, \quad Ov_0 = 0, \quad Om = 1. \end{gathered} \tag{20.5}$$

The translation preserves all components of the two reducible Kodaira fibres. Their root classes span a negative definite space of rank seven. Its orthogonal complement is generated over $\mathbb{Q}$ by F, O, v_0. Since $T_* O = O + v_0 + 3F$, preservation of the intersection form gives, on this complement and hence on the entire lattice,

$$T_* a = a + (aF)v_0 - (av_0 - 3aF)F. \tag{20.6}$$

Its value on v_0 is $v_0 + 6F$: it has no O-component and has zero intersections with the rank-seven root space, so it is $\lambda v_0 + aF$. The already established identity $T^* m = m - F$ gives $mT_* v_0 = (m - F)v_0 = -1$, forcing $\lambda = 1$. Orthogonality to $T_* O$ then forces $a = 6$. These values on F, O, v_0 and the root space verify (20.6) on a rational basis, and both sides preserve the integral lattice. In particular $T^* m = m - F$, in agreement with 20.2.

3. The rational elliptic surface is the blow-up of $\mathbb{P}^2$ at nine points, so its homology in even degrees is $\mathbb{Z}, \mathbb{Z}^{10}, \mathbb{Z}$ and vanishes in odd degrees; its intersection lattice is unimodular. The projective-bundle theorem follows here by local trivialization and the fibre hyperplane class: $H_2(R) = H_2(S) \oplus \mathbb{Z}f$ and $H_4(R) = \mathbb{Z} \oplus H_2(S)$, with f the projective-line fibre. The hyperplane class that vanishes on s_0 restricts to $-m$ on s_∞. Therefore

$$d_2(a) = (1 - T_*)a + (mT_*a)f = -(aF)v_0 + (av_0 - 3aF)F + (am - aF)f. \tag{20.7}$$

The span of F, v_0, m is primitive. To check the only nontrivial part of this assertion, split off the unimodular plane spanned by F, O; $m - F$ and v_0 lie in its negative definite orthogonal complement. Their Gram determinant is $\det \left(\begin{smallmatrix} -2 & -1 \\ -1 & -6 \end{smallmatrix}\right) = 11$. The square of an index of a larger integral lattice would divide eleven, so the index is one. The span of F, v_0 is also primitive: pairing with O, m gives determinant -1. Unimodularity consequently makes $a \mapsto (aF, av_0, am)$ surjective onto $\mathbb{Z}^3$. The coefficient matrix in (20.7) has determinant -1. Thus d_2 has primitive image $\mathbb{Z}v_0 \oplus \mathbb{Z}F \oplus \mathbb{Z}f$, rank three, kernel $\mathbb{Z}^7$, and cokernel $\mathbb{Z}^8$.

4. The map d_4 sends $[S]$ to $s_0[S] - s_\infty[S]$, whose Poincaré dual is $\pi_R^* m$. This is primitive and nonzero since $mO = 1$. Thus d_4 is injective with cokernel $\mathbb{Z}^{10}$. The map d_0 is zero. Substitution in (20.4) proves every asserted homology group. The Euler characteristic is independently obtained by stratification: $\chi(R) - \chi(S) = 24 - 12 = 12$.

□

Remark 20.8 (The local modification and the compact central fibre)**.** [tag 01R5] The blow-up charts of Proposition 20.4 resolve the ordinary double point of $xy = uzv$ for $u \neq 0$ and change one branch of the central normalization by the stated blow-up along (x, z). The compact pinching Q has the virtual Euler obstruction of Theorem 21.8. The periodic modifications in Theorem 21.2 give the two smooth total spaces with central normalizations obtained by the explicitly described blow-up and small resolution.

20.2 Finite bielliptic limits and the smooth monodromy

Proposition 20.9 (Semistable limits of the bielliptic surfaces)**.** [tag 01R6]

After the auxiliary base change $w = s^6$, each of the two families of reduced *special surfaces $S_j(w)$ has a proper flat semistable filling by explicit finite free quotients. The central surface for $j = 1$ is a cycle of four ruled surfaces over an elliptic curve; that for $j = 2$ is a cycle of three. They have*

	H_0	H_1	H_2	H_3	H_4	χ
$j = 1$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/3$	$\mathbb{Z}^5$	$\mathbb{Z}^9$	$\mathbb{Z}^4$	0
$j = 2$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}^4$	$\mathbb{Z}^7$	$\mathbb{Z}^3$	0

with integral coefficients. Their fundamental groups are $\mathbb{Z}^2 \rtimes_{R_3} \mathbb{Z}$ and $\mathbb{Z}^2 \rtimes_{R_4} \mathbb{Z}$, respectively, where

$$R_3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad R_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proof. 1. The following product presentations specify the comparison with the marked central surfaces. Let $\rho = e^{\pi i/3}$,

$$E_j = \mathbb{C}/(\mathbb{Z} + \kappa_j\mathbb{Z}), \quad F_1 = \mathbb{C}/(\mathbb{Z}(\rho - 2) + \mathbb{Z}(-3)), \quad F_2 = \mathbb{C}/(\mathbb{Z}(i - 1) + \mathbb{Z}(-2)),$$

where $\kappa_1 = \beta_c(z_1) + 2\mu(z_1)$ and $\kappa_2 = \beta_c(z_2) + 3\mu(z_2)$ have negative imaginary parts. The holomorphic maps to the central tori are induced by $L_1(\xi, \eta) = (\eta, \xi - \eta/3)$ and $L_2(\xi, \eta) = (\eta, \xi - \eta/2)$. Their kernels are generated by

$$k_1 = (2/3, \rho), \qquad k_2 = (1/2, i).$$

The residual affine generators lift to

$$h_1(\xi,\eta) = (\xi + \kappa_1/3, (\rho-1)\eta), \qquad h_2(\xi,\eta) = (\xi - \kappa_2/4, -i\eta). \tag{20.8}$$

These formulas are verified by substituting the four product periods in L_j; the lattice indices are three and two. Moreover $(\rho-1)\rho-\rho$ and $(-i)i-i$ are periods of the displayed F_j, so the generators commute. They give free groups $G_1 = C_3 \times C_3$ and $G_2 = C_2 \times C_4$. Indeed, equality of the first coordinates of $k_j^a h_j^b(\xi,\eta)$ and (ξ,η) forces $b = 0$ from the coefficient of the nonreal period κ_j, and then $a = 0$ from the real period. Thus the composite quotient maps are exactly the degree-nine and degree-eight covers of S_j, with the stated signs.

2. Use the multiplicative coordinate $v = e^{-2\pi i\xi}$ on E_j. Its Tate parameter is

$$q_j = e^{-2\pi i\kappa_j} = A_j w^2 = A_j s^{12}, \qquad A_j = e^{-2\pi i\kappa_j^{\mathrm{part}}} \neq 0.$$

Here A_j denotes the displayed scalar constant within this proof. Choose specifically $A_j^{1/12} = \exp(-2\pi i\kappa_j^{\mathrm{part}}/12)$ and put $r_j = A_j^{1/12} s$. This choice preserves the fractional period translations in (20.8) exactly. The regular Tate model for $q_j = r_j^{12}$ has a central twelve-gon. An infinite chain has node charts $x_i y_i = r_j$; on a branch overlap one uses $x_{i+1} = y_i^{-1}$ and $y_{i+1} = r_j y_i$. The coordinate $r_j^i x_i$ on a general fibre is unchanged on overlaps. The shift $i \mapsto i+12$ acts on that coordinate by r_j^{12}. Quotienting a tubular neighbourhood of the chain by this shift gives the proper twelve-gon family, and on $r_j \neq 0$ it is $\mathbb{C}^*/q_j^{\mathbb{Z}}$. Compactness follows from twelve projective components and a compact annular fundamental domain on a general fibre; the node charts give flatness and smoothness of its total space.

3. Multiplication by $q_1^{1/3} = r_1^4$ shifts the component index by four. Multiplication by $q_2^{-1/4} = r_2^{-3}$ shifts it by minus three. Hence the maps h_j in (20.8) extend, together with their rotations on F_j. The maps k_j multiply v by roots of unity and translate F_j, and extend to every projective component. Their relations hold on the dense set $s \neq 0$ and consequently on all charts. The action is free on the central fibre as well. A nontrivial power of h_j moves every component and every node. A nontrivial power of k_j translates F_j by a nonzero torsion point, including above the nodes where the multiplicative action alone fixes a point. The finite free quotient is therefore a smooth total space, proper over s, with a reduced normal-crossing central surface. Flatness descends along the finite étale quotient. The component orbits have cardinality three and four, leaving four and three components. Quotienting an individual $\mathbb{P}^1 \times F_j$ by k_j is a ruled surface over $F_j/\langle\rho\rangle$ or $F_j/\langle i\rangle$, respectively. Its two distinguished sections have the same integral homology class: the corresponding line bundle has finite order and degree zero on the elliptic base.

4. For the topology, unwrap the component cycle into an infinite chain and then pass to the universal cover $\mathbb{C}$ of its elliptic base. The resulting union of simply connected ruled pieces, with simply connected intersections and a tree as incidence graph, is simply connected by van Kampen. Its deck group is generated by the enlarged lattice of translations on F_j and one step round the component cycle. The latter conjugates the lattice by the order-three or order-four rotation. This gives the asserted semidirect products; inverses of the displayed rotation matrices give isomorphic groups by reversing the cycle generator.

5. For homology, the normalization has $n = 4$ or 3 components, each with ranks $(1,2,2,2,1)$, and n elliptic double curves, each with ranks $(1,2,1)$. There are no triple points. The mapping-cylinder sequence for the normalization and the double curves applies integrally. In degree one the incidence matrix is a twisted cycle: its cokernel is $\operatorname{coker}(R_j - I)$ and its kernel is zero, because R_j has no eigenvalue one. The degree-zero incidence matrix contributes an additional $\mathbb{Z}$ to H_1. In degree two the two section classes on a component agree, so the incidence matrix

is the ordinary cycle matrix on their classes. It has kernel $\mathbb{Z}$ and primitive image of rank $n-1$. Thus $H_2 = \mathbb{Z}^{n+1}$, $H_3 = \mathbb{Z}^{2n+1}$, and $H_4 = \mathbb{Z}^n$. Finally $\operatorname{coker}(R_3 - I) = \mathbb{Z}/3$ and $\operatorname{coker}(R_4 - I) = \mathbb{Z}/2$.

□

21 The untwisted limit

21.1 Two minimal toroidal fillings and further modifications

Proposition 21.1 (The two minimal toroidal refinements)**.** [tag 02ZJ]

Among subdivisions with no new rays, the fan in 20.5 has exactly two smooth period-invariant subdivisions restricting to the chosen A_2 triangulation at $h_u = 0$. These are also all subdivisions which are flat over the original (q, u) bidisc, have reduced u-central divisor, and retain that boundary triangulation. Without the joint-flatness restriction there are infinitely many smooth invariant subdivisions retaining that triangulation and having reduced u-central divisor.

Proof. 1. Write the six rays of $P_{0,0}$ as $A = (0,0,1,0)$, $B = (1,0,1,0)$, $C = (0,1,1,0)$, $D = (1,1,1,0)$, $E = (0,0,0,1)$, $F = (0,1,0,1)$. The return's A_2 diagonal is BC. A triangular prism has six triangulations without new vertices. With this bottom diagonal the three possibilities are

maximal tetrahedra	left diagonal	right diagonal
$ABCE$, $BCDE$, $CDEF$	CE	DE
$ABCE$, $BCDF$, $BCEF$	CE	BF
$ABCF$, $ABEF$, $BCDF$	AF	BF.

Periodicity identifies the left and right faces by $A \mapsto B$, $C \mapsto D$, $E \mapsto E$, $F \mapsto F$; it excludes exactly the middle row. Every listed maximal cone has determinant ± 1. There are no missed triangulations: the prism has normalized volume three, all its nondegenerate four-vertex tetrahedra have normalized volume one, and enumeration of their triples with pairwise face intersections gives exactly these six possibilities before fixing the bottom. To test each pairwise intersection, write both tetrahedra by their rational barycentric inequalities and enumerate the vertices of the combined system. The intersection is their common face exactly when each such vertex belongs to that face.

2. For the stronger assertion, a new ray with $h_q, h_u > 0$ would give a divisor mapping to the origin of the two-dimensional base. Its fibre dimension would be three, contradicting joint flatness. A ray with $h_u = 0$ is excluded by the retained boundary triangulation. On $h_q = 0$, the fan support has $y_1 = 0$. Reducedness requires the primitive ray to have $h_u = 1$, so it is already one of the ρ_l. Thus there are no new rays under the stated three restrictions. Conversely the two triangulations have all heights zero or one, and every cone maps onto a face of the quadrant with saturated image. Their local monomial maps are flat and their u divisors reduced.

3. If joint flatness is omitted, star-subdivide the orbit of the maximal cone $ABCE$ by its interior ray $v_1 = A + B + C + E = (1,1,3,1)$. Distinct translates have disjoint interiors and distinct closed torus orbits. These are simultaneous ordinary blow-ups at disjoint points, so the operation is unambiguously period-invariant. Repeat in the cone $ABCv_{m-1}$, where $v_0 = E$. The new ray is $v_m = v_{m-1} + A + B + C = (m, m, 3m, 1)$. Replacing any basis vector by their sum preserves determinant one, and every new ray has $h_u = 1$. The boundary triangulations are unchanged. On the compact quotient this blows up one smooth central point away from the conductor at each step. There are therefore infinitely many invariant smooth refinements, all reduced over $u = 0$, with increasing numbers of rays modulo the period group. They are not jointly flat.

□

Theorem 21.2 (The two minimal smooth fillings)**.** [tag 02ZK]

The two minimal subdivisions of 21.1 give proper flat holomorphic families $\mathcal{X}^+, \mathcal{X}^- \to \Delta_u$ with smooth total spaces. Their joint maps to $B \times \Delta_u$ are flat. Over Δ_u^* they are isomorphic as families to the specified untwisted family $X(u)$. Each central fibre $Q^\pm$ is reduced, irreducible, and has normal crossings, with smooth double locus S and no triple points.

Its normalization $R^\pm$ is obtained by blowing up R along S_∞ in, respectively, one of its two sections and then taking the small resolution at the unique ordinary double point of that blow-up prescribed by the fixed bottom diagonal. The two proper transforms of the sections are still copies of S, glued by T. Their normal-bundle product is trivial; in particular $\mathcal{T}^1_{Q^\pm} \simeq \mathcal{O}_S$.

Proof. 1. Apply either fan subdivision over the cusp neighbourhood in Theorem 20.6, and use the identity modification elsewhere. Toric modifications are proper, and a subdivision is an isomorphism off the strata it subdivides. Its restriction over $q \neq 0$ is the identity, so this gluing has no additional transition choice. The period actions preserve the subdivisions, and the covering description in 20.6 descends the proper modifications to the quotient. All maximal cones are regular. Every local ring of the resulting fourfold is therefore regular. The coordinate u is the product of those affine coordinates whose rays have $h_u = 1$; there are at most two such rays in a maximal cone. This proves reduced normal crossings and the absence of triple points for the u-central fibre. The regular-sequence argument, or the saturated face images in 21.1, proves joint flatness. Properness follows from properness of the toric modification and of $\mathcal{X}^\flat$.

2. At a node of S_∞, write the original normalization as coordinates (X, z, v), with section $X = 0$ and centre (X, zv). Blowing up that ideal has charts

$$X = zvA; \qquad zv = XB.$$

The first is smooth, and the second has one conifold at its origin. A small resolution of the latter is, for one of the bottom-diagonal conventions,

$$X = za, \quad v = aB; \qquad z = Xd, \quad B = dv.$$

Interchanging z, v gives the other diagonal convention. These are holomorphic charts of a proper small modification: they are the two usual charts of the blow-up of the appropriate rank-one ideal on the conifold. The strict transform of $X = 0$ is $A = 0$ in the first chart and never passes through the conifold. It is therefore unchanged as a surface. The other section is disjoint from the centre and is unchanged as well. Over a smooth point of S_∞ the exceptional fibre is one $\mathbb{P}^1$; over its node it is a chain of two $\mathbb{P}^1$'s, one being the exceptional curve of the small resolution. This identifies the normalization morphism, including its order of operations.

3. For example the corresponding total fourfold modification of $xy = uzv$ can be written by first taking $x = ua$, $ay = zv$ and then resolving the conifold in (a, y, z, v), together with the other chart $u = xb$, $y = bzv$. Its three smooth affine charts map to $(x, y, z, v, u; q)$ by

$$\begin{aligned}(u, z, \lambda, y) &\longmapsto (uz\lambda, y, z, \lambda y, u; z\lambda y),\\ (u, a, \nu, v) &\longmapsto (ua, \nu v, a\nu, v, u; a\nu v),\\ (x, b, z, v) &\longmapsto (x, bzv, z, v, xb; zv).\end{aligned}$$

The first two are the conifold-resolution charts $a = z\lambda$, $v = \lambda y$ and $z = a\nu$, $y = \nu v$. Substitution gives $xy = uzv$ and $q = zv$ in every chart. Exchanging the chosen zero and infinity branches gives the other model.

4. The normal bundle of the section containing the centre changes by tensoring with $\mathcal{O}_S(-S_\infty)$; the second normal bundle is unchanged. Since the original product is J, the new product is $J(-S_\infty) = \mathcal{O}_S$. The local formula for $\mathcal{T}^1$ is the product of the two normal bundles, with T used on the infinity branch. This proves d-semistability in this precise sense. The ρ_l form one orbit, so the central fibre is irreducible and its conductor is exactly the two stated sections. Over $u \neq 0$ both subdivisions restrict to the fixed bottom triangulation, giving precisely Engel's Mumford filling, simultaneously in u.

5. The birational comparison $\mathcal{X}^+ \dashrightarrow \mathcal{X}^-$ is the identity over the punctured u disc and off the exceptional loci over $(q, u) = (0, 0)$. Locally it replaces the two compatible side-face diagonals by the other two. All its rays lie in the Gorenstein hyperplane $h_q + h_u = 1$, so both small modifications are crepant over $\mathcal{X}^\flat$. Over a smooth point of the nodal curve this is the ordinary Atiyah flop times a curve; at the node its extension is exactly the displayed pair of prism subdivisions. This specifies the birational map.

□

21.2 An explicit two-bundle plumbing

Theorem 21.3 (A proper filling of the untwisted family)**.** [tag 02RA]

The untwisted family $X(u)$ has a proper flat Hausdorff analytic filling at $u = 0$. Its corner winding is zero. The filling is obtained from two projective-line bundles by explicit plumbing, a proper modification and a finite quotient.

Proof. 1. Let S be the rational elliptic surface of Definition 20.1, $M = \mathcal{O}_S(P - O)$, $h = t_{6P}$, and $L = \mathrm{Hom}(h^*M, M) = \mathcal{O}_S(F_\infty)$, with section ϕ vanishing on F_∞. The absence of logarithmic twists gives $b = a = 0$ in the corner character calculation. Thus the origins agree and their winding is zero.

2. Take two copies Y_0, Y_1 of $\mathbb{P}_S(\mathcal{O} \oplus M)$. The finite coordinate z_i lies in M, its inverse w_i in M^{-1}. The first neck, over $p \in S$, joins zero in Y_0 to infinity in Y_1 over $h(p)$ by $z_0 w_1 = r\phi_p$. The second joins zero in Y_1 to infinity in Y_0 over the same point by $z_1 w_0 = r$. These equations are typed in $M \otimes h^*M^{-1} = L$ and $M \otimes M^{-1} = \mathcal{O}$.

3. Choose a Hermitian metric and $0 < \epsilon < 1/4$. Use middle annuli $\epsilon/2 < |z_i| < 2/\epsilon$ in Y_i, times the parameter disc, and neck hypersurfaces with $|z|, |w| < \epsilon$. Glue their ends on $\epsilon/2 < |z| < \epsilon$ or $\epsilon/2 < |w| < \epsilon$ by coordinate projection, using h at the first infinity end. In holomorphic frames these are holomorphic maps, and they commute with bundle transitions because the product lines are exactly the ones displayed above. Choose ρ so that $\rho \max(1, \|\phi\|) < \epsilon^2/16$. The product equation makes the two ends of a neck disjoint; the two ends in each middle chart are disjoint as well.

4. For $|r| \leq \rho' < \rho$, a compact fundamental cover consists of closed middle annuli $3\epsilon/4 \leq |z_i| \leq 4/(3\epsilon)$ and closed neck pieces $|z|, |w| \leq 3\epsilon/4$. Every omitted neck end is represented in a middle annulus. The gluing relation on this finite union of compact bundles is the union of diagonals and closed graphs of the coordinate identifications. No additional graph accumulates, because the two neck ends are disjoint. The relation is closed, so its quotient is compact and Hausdorff. The open charts give the analytic structure. This proves properness over the smaller disc and the global Hausdorff property.

5. Over $S \setminus F_\infty$, traversing the two necks gives

$$(p, z_0) \longmapsto \left(h(p), \frac{z_0}{r^2 \phi_p} \right).$$

The inverse covers h^{-1} and is multiplied by $u\phi$, with $u = r^2$. It is the precise generator of Engel's quotient on that open set.

6. At a smooth point of F_∞ the first neck is $zw = rq$; at its node it is $zw = rxy$, with $q = xy$. The ideal of one globally distinguished central component is locally (z, r). Because the two global components are distinct, this is a global coherent ideal. Blow it up. At the node its two charts are

$$\begin{aligned} r = za, \quad w = axy; \qquad & (z, a, x, y) \text{ are smooth coordinates,} \\ z = rb, \quad bw = xy; \qquad & (r, b, w, x, y) \text{ with the displayed relation.} \end{aligned}$$

The remaining singularity is an ordinary threefold node times the parameter disc. Its singular section does not meet the first chart. Resolve (b, y) in the second chart. Its two charts are

$$\begin{aligned} b = yk, \quad x = kw; \qquad & (r, y, k, w), \\ y = bj, \quad w = jx; \qquad & (r, b, j, x). \end{aligned}$$

They are smooth, with transition $j = k^{-1}$ and $(b, x) = k(y, w)$. Off the singular section the ideal is invertible, so this proper small resolution glues to the identity. The analogous blowup charts at smooth points of F_∞ are already smooth. The resulting total space is smooth, and its parameter central divisor is reduced NC: in the first chart $r = za$, and in both small-resolution charts r is a coordinate. Parameter torsion freeness proves flatness.

7. For fixed $r \neq 0$ the first blowup is an isomorphism. Put $W = rw$ and $u = r^2$; the first neck becomes $zW = uxy$. On its dense torus use (x, z, q), with $y = q/x$ and $W = uq/z$. The four affine monomials have exponents

$$(1, 0, 0), \quad (-1, 0, 1), \quad (0, 1, 0), \quad (0, -1, 1).$$

The dual cone is $0 \leq \alpha, \beta \leq \gamma$, the cone over the unit square at height one. The resolution of (z, y), equal to that of (b, y) when r is a unit, divides it along $\alpha + \beta = \gamma$: the chart character $k = z/(ry)$ has valuation $\alpha + \beta - \gamma$. Its two cones, over the triangles with common edge $[e_1, e_2]$, are unimodular. The exact comparison with the coordinates (X, Y, Z, V) in the node $XY = uZV$ of Theorem 20.6 is

$$(X, Y, Z, V) = (z, rw, x, y).$$

In its first resolution chart the parameters are $(V, Y/u, A) = (y, w/r, rk)$, and in its second they are $(X, Z, B) = (rb, x, j/r)$. These give respectively

$$(X, Y, Z, V) = (rky, rw, kw, y), \qquad (X, Y, Z, V) = (rb, rjx, x, bj),$$

and $AB = kj = 1$ on the overlap. Thus both chart maps and their transition agree with the selected resolution of Theorem 20.6. In the two charts, $q = kyw$ and $q = bxj$, respectively. The Tate and linearization periods shift the two marked lattice directions by one; their unit factors act holomorphically on these charts. This is the chosen marked triangular filling supplied by Engel's general construction in (Engel, 2026, Proposition 2.8, pp. 6–7). Together with the quotient comparison, the displayed map identifies the entire compact generic fibre with $X(u)$.

8. The involution $r \mapsto -r$ fixes the Y_0 fibre coordinate and negates the Y_1 coordinate. It preserves both equations and the global component ideal. In the nodal blowup chart it acts as $(b, w, x, y, r) \mapsto (-b, -w, x, y, -r)$ and preserves (b, y). Its lifts on the two resolution charts are

$$(r, y, k, w) \longmapsto (-r, y, -k, -w), \qquad (r, b, j, x) \longmapsto (-r, -b, -j, x).$$

The transition $j = k^{-1}$ intertwines these maps. Its finite quotient is proper and Hausdorff over $u = r^2$. Its invariant local rings are torsion free over the discrete valuation ring $\mathbb{C}\{u\}$ and hence flat. Away from zero the involution exchanges the two roots of u, while $W = rw$ and the Y_0 coordinate are invariant. The displayed marked comparison therefore identifies the quotient fibre with one copy of $X(u)$. This is a filling in the original parameter.

□

Proposition 21.4 (A projective-space component in a point-star refinement)**.** [tag 02ZL] *The first maximal-cone star of Proposition 21.1 contributes a central normalization component $\mathbb{P}^3$. There is no proper bimeromorphic holomorphic map from this component to R.*

Proof. The star is the ordinary blow-up of a point in a smooth four-dimensional chart, with exceptional divisor $\mathbb{P}^3$. The projective-bundle basis gives $b_2(R) = b_2(S)+1 = 11$, while $b_2(\mathbb{P}^3) = 1$. A proper bimeromorphic map $f : \mathbb{P}^3 \to R$ has topological degree one, and the projection formula gives

$$\int_{\mathbb{P}^3} f^*a\, f^*b = \int_R ab \qquad (a \in H^2(R;\mathbb{Q}),\ b \in H^4(R;\mathbb{Q})).$$

Poincaré duality would make f^* injective in degree two, contradicting these ranks. □

Theorem 21.5 (Exhaustive finite fan data)**.** [tag 02ZM]

The following finite-data procedure parametrizes all smooth period-invariant subdivisions of the cusp fan after $u = r^n$, retaining the prescribed bottom triangulation and having finitely many cones modulo the periods. The minimal subdivisions for $n = 1, 2, 12$ are described in Proposition 21.1 and Theorem 22.3. Each output defines a proper flat family over Δ_r with smooth total space, whose punctured restriction is the given untwisted family pulled back by $u = r^n$.

For a bound $H \geq 1$, retain the four bottom rays A, B, C, D and consider all primitive vectors

$$\begin{aligned} v = (a, b, k, \ell), \qquad & \ell \geq 1, \quad k \geq 0, \quad k + \ell \leq H, \\ & 0 \leq a \leq k, \qquad 0 \leq b \leq k + n\ell. \end{aligned} \tag{21.1}$$

Take any collection of four-element subsets of this finite set satisfying the following exact tests:

1. *Each four-column determinant has absolute value one.*
2. *After dividing each ray by $k + \ell$, the resulting tetrahedra intersect pairwise in common faces, and their volumes sum to $(n + 2)/6$.*
3. *Their bottom boundary is the two triangles ABC, BCD. Their boundary subdivisions on $a = 0$ and $a = k$ agree under G_1, and those on $b = 0$ and $b = k + n\ell$ agree under G_2.*

All these are finite rational-arithmetic tests. Extend a passing collection by the two periods, and discard duplicates as H increases. This is exhaustive. For a reduced central divisor, retain only $\ell = 1$ in (21.1).

The central prime components are indexed by period orbits of the selected rays with $\ell > 0$, and the multiplicity of the component indexed by v is exactly ℓ. The normalization charts of that component are the star fan of v in $N/\mathbb{Z}v$, with the inherited bundle units and the stabilizer action. For $k > 0$ the stabilizer is trivial and this normalization is a smooth complete toric threefold. All multiple loci, including possible triple and quadruple loci, are specified by the positive-ℓ rays belonging to each cone. In the reduced case $\mathcal{T}^1 \simeq \mathcal{O}_{\operatorname{Sing} D}$, so every output is d-semistable.

Proof. 1. Write the period shears as

$$\begin{aligned} G_1(a, b, k, \ell) &= (a + k, b, k, \ell), \\ G_2(a, b, k, \ell) &= (a, b + k + n\ell, k, \ell). \end{aligned}$$

One closed fundamental cone is

$$P_n = \{k, \ell \geq 0,\ 0 \leq a \leq k,\ 0 \leq b \leq k + n\ell\}.$$

Its bottom is the original square; its top is the edge from $(0, 0, 0, 1)$ to $(0, n, 0, 1)$. The slice $k + \ell = 1$, in coordinates (a, b, k), is the compact polytope

$$0 \leq k \leq 1, \qquad 0 \leq a \leq k, \qquad 0 \leq b \leq n - (n-1)k.$$

Its volume is $\int_0^1 k(n - (n-1)k)\, dk = (n+2)/6$. A tetrahedron with ray columns v_i has slice volume

$$\frac{|\det(v_1, v_2, v_3, v_4)|}{6 \prod_{i=1}^4 (k_i + \ell_i)}. \tag{21.2}$$

This follows by applying the determinant-one coordinate change from (a, b, k, ℓ) to $(a, b, k, k+\ell)$ and then normalizing the last coordinate to one.

2. A common-face test is also completely explicit: write each tetrahedron in barycentric halfspaces, intersect the two finite lists of halfspaces, and enumerate the possible vertices by solving triples of bounding equations. The intersection is a common face precisely when all its vertices are common tetrahedron vertices. Since the candidate tetrahedra lie in P_n, pairwise face intersections and the volume equality imply that they cover the slice. Otherwise their finite closed union would omit a nonempty relatively open set, which has positive volume. The boundary tests make the translates agree on every shared face. This proves that every passing collection gives the claimed invariant fan.

3. Conversely, intersect any invariant smooth refinement with P_n. A smooth maximal cone has four primitive generators with determinant ± 1, and its slice is one of the stated tetrahedra. Finitely many cone orbits give finitely many rays in this prism, hence some maximum height H. No additional bottom rays are permitted by the boundary condition; every other ray has the form (21.1). The resulting collection passes all three tests. Thus the search at each height is finite, and the procedure is exhaustive.

4. Properness of the quotient forces the finite-orbit condition. Each full-dimensional cone has a closed torus orbit consisting of one point with $q = r = 0$. These points form a closed discrete set: an affine chart of a full-dimensional cone contains exactly its own closed torus orbit. After the properly discontinuous period quotient the points are indexed by cone orbits and remain closed and discrete. Infinitely many such orbits would therefore give an infinite closed discrete subset of a compact central fibre, which is impossible.

5. The fundamental orbit representatives of mixed rays are

$$(a, b, k, \ell), \quad k, \ell > 0, \quad 0 \leq a < k, \quad 0 \leq b < k + n\ell, \quad \gcd(a, b, k, \ell) = 1. \tag{21.3}$$

For pure top rays they are $(0, b, 0, \ell)$ with $0 \leq b < n\ell$ and $\gcd(b, \ell) = 1$. A smooth top subdivision necessarily includes each $(0, j, 0, 1)$, $0 \leq j \leq n$. Indeed an omitted such vector would lie inside a regular two-cone; writing it in that cone's integral basis would express its height one as the sum of two positive integral heights, which is impossible. In the reduced search there are exactly n pure top orbits and $\sum_{k=1}^{H-1} k(k+n)$ available mixed orbits.

6. Enumerating the primitive rays in (21.1) and retaining the four-element subsets whose column

determinants have absolute value one gives the following finite counts:

n	H	reduced	rays in P_n	unimodular four-subsets
1	1	yes	6	12
1	2	yes	12	267
1	2	no	13	357
2	2	yes	15	544
2	2	no	17	784
12	1	yes	17	100

The table counts rays and unimodular four-subsets. The common-face, volume and boundary tests above select the complete collections. At height one these are the two and $n+1$ fans of propositions 21.1 and 22.3, respectively.

7. For existence of the corresponding compact family, start with the proper nodal plumbing of 20.6 and base-change it by $u = r^n$. On the cusp cover apply the proper toric modification defined by the refinement. Its support is unchanged, which is the toric properness criterion (de Cataldo et al., 2018, Section 2, p.6), and each original affine chart is replaced by a finite fan. On $q \neq 0$ use the subdivision of the top two-dimensional fan to modify the chart $ab = r^n\varphi$ fibrewise over S. Changes of trivialization multiply a and b by the given nonvanishing line-bundle units; a torus action extends to every toric fan chart, so these modifications glue. The same top fan occurs on the overlap with the cusp construction. The core outside the plumbing charts is unchanged. This defines a proper modification of the base-changed compact nodal family.

8. The period maps lift because the fan is invariant. Their lifted action is free: a fixed point would project to a fixed point in the nodal covering. It is properly discontinuous: compact sets project to compact sets, and any translating element producing an intersection upstairs produces one downstairs, where 20.6 gives finiteness. Thus the quotient exists as a Hausdorff complex space, and the equivariant proper modification descends to a proper modification of the compact nodal quotient. In a unimodular chart its coordinates z_i satisfy

$$r = \text{unit} \cdot \prod_{i=1}^{4} z_i^{\ell_i}, \qquad q = \text{unit} \cdot \prod_{i=1}^{4} z_i^{k_i}. \tag{21.4}$$

The total space is smooth. The nonconstant coordinate r is a nonzerodivisor, so the morphism to the disc is flat. Its central multiplicities are the indicated ℓ_i; it is reduced exactly when all selected positive heights are one. The bottom fan gives the specified smooth Mumford model over $r \neq 0$, including its original period identification. Thus the punctured-family assertion holds simultaneously in the parameter.

9. The normalization charts are as follows. For a maximal cone with basis (v, v_2, v_3, v_4), let (m_1, m_2, m_3, m_4) be the dual integral basis. The chart of the normalized component is $z_1 = 0$ with coordinates $z_i = \chi^{m_i}$, $i = 2, 3, 4$. On overlaps the transition matrix is the product of the two integer basis matrices, and the original bundle units are raised to those same integer exponents. This is precisely the star fan in $N/\mathbb{Z}v$. When $k > 0$, (21.3) shows that the period stabilizer of the ray is trivial. The ray lies in the interior of the full support, so its star is complete. Its normalized component is therefore the asserted complete smooth toric threefold, lying over the cusp. When $k = 0$, the stabilizer is generated by G_1. Away from S_∞ the resulting component is the ruled bundle $\mathbb{P}_S(\mathcal{O} \oplus L^\ell)$: a character on its open ruled fibre is W^ℓ/r^b, which transforms by the ℓth power of the transition of L. The stated star charts describe its replacement over S_∞. The gluing maps are the original period maps on those same integer characters.

10. Intersections are read from common cones, with separate local branches retained before the period quotient. In a chart (21.4), the number of branches of the reduced central divisor is the number of positive ℓ_i. When all these heights are one, the hypersurface calculation gives $\mathcal{T}^1_D = \mathcal{O}_D(D)|_{\operatorname{Sing} D}$. The divisor is globally principal, cut out by r, so this line on the singular locus is trivial. Finally, any two output fans have an explicit common refinement by all intersections of their cones. Its normal toric modification gives a proper common model, and the resulting birational map is the identity on the fixed lattice and on the punctured family.

□

Proposition 21.6 (The normalization and modification algorithm)**.** [tag 02ZN]

For every finite fan datum Σ of Theorem 21.5, the following algorithm describes the normalized components and their comparison with a reference model.

The star charts and integer transition matrices of 21.5 give every normalized component, its gluing maps, and all multiple loci directly. There is also a terminating finite-data algorithm that expresses its comparison with a reference model as an explicit sequence of blow-ups and blow-downs along specified smooth strata. At each blow-up the algorithm gives the normalized strict transforms, the exceptional projective bundle, its multiplicity, and every new intersection. The algorithm uses both blow-ups and blow-downs.

For a component of multiplicity ℓ, the local nonreduced central equation is the monomial in (21.4). More generally, in a unimodular chart with central equation $F = \prod_i z_i^{\ell_i}$, the cotangent-complex first-order deformation sheaf is

$$\mathcal{T}^1_D = \mathcal{O}_D \Big/ \left(\ell_i z_i^{\ell_i - 1} \prod_{j \neq i} z_j^{\ell_j} : \ell_i > 0 \right). \tag{21.5}$$

Here $\mathcal{T}^1_D$ means $\mathcal{E}xt^1(\mathbb{L}_D, \mathcal{O}_D)$; in the reduced case it also equals $\mathcal{E}xt^1(\Omega_D, \mathcal{O}_D)$. The reduced case is the trivial line on the singular scheme, as in 21.5. For a nonreduced centre the displayed Jacobian quotient is the appropriate deformation sheaf; the stated d-semistability conclusion applies to the reduced case.

Proof. 1. Proposition 21.4 proves that the point-star component $\mathbb{P}^3$ has no proper bimeromorphic map to R. In that example the full normalization is $\operatorname{Bl}_p R^+ \sqcup \mathbb{P}^3$: the first component is the strict transform, and the two meet along the exceptional plane. These are the ordinary point-blow-up charts of the smooth total fourfold.

2. Retain all marking data from 21.5: the lattice, the two period matrices, the original line-bundle units, and the common unsubdivided fan. Strictify the finite quotient cone complex by the decreasing-dimensional barycentric stars used in 21.17. Record these operations and do the same for the reference model. Form their common intersection complex. A regular common subdivision can be found either by toric desingularization or by enumerating rational subdivisions with increasing primitive-ray bounds and testing their determinants and common faces. A regular subdivision exists, so the latter enumeration terminates.

3. On a finite strict regular complex there are finitely many possible ordinary star blow-ups: choose a cone of dimension at least two and insert the sum of its primitive basis rays. There are also finitely many possible inverse moves: choose a ray and test whether its star is exactly the star created by such a subdivision of a regular cone. Both tests are exact integer calculations. Require that every intermediate complex remain a subdivision of the marked original complex. Record face identifications; equality of two marked complexes is a finite rational comparison, using the period orbit representatives in (21.3) and their pure top and pure bottom analogues.

4. Enumerate this locally finite graph of moves in increasing path length, discarding repeated marked complexes. Applied separately to the two proper subdivision morphisms from the common regular complex, the finite-complex weak-factorization theorem used in 21.17 guarantees a path, all of whose intermediates are over the corresponding target. Thus the search terminates. Concatenate the two paths and the recorded strictification paths. This gives a fully specified finite sequence once the input fan is given. Temporary pure bottom subdivisions may occur; they are allowed in a presentation of the birational comparison, and are removed before the prescribed endpoint is reached.

5. If a cone has primitive basis vectors $v_1, \dots, v_c$, its centre is the closed stratum

$$C = \{z_1 = \cdots = z_c = 0\}$$

in the marked toroidal chart. Strictness makes its global closure smooth; the period translates defining one centre are disjoint. Blow-up chart i has coordinates

$$z_i = t_i, \qquad z_j = t_i w_j \quad (1 \le j \le c,\ j \ne i), \qquad z_j \quad (j > c).$$

The changes between charts are these integral monomials, with the original units raised to the same powers. The exceptional divisor is $\mathbb{P}_C(N_{C/Y})$, where Y is the smooth intermediate total space. If C is central, its central multiplicity is

$$\ell_E = \sum_{j=1}^{c} \ell_j.$$

If all ℓ_j vanish, it is a horizontal exceptional divisor and is not a new central component.

6. For an old normalized component, take the inverse image of this centre on its marked branch and the ordinary strict transform in these charts. It is the blow-up along that smooth inverse-image centre; when that centre is Cartier the blow-up is the identity. Its intersections with the new exceptional component are the appropriate coordinate-hyperplane projective subbundles. Intersections of several old branches are given by imposing the corresponding coordinate equations simultaneously. Thus the same chart prescription records double, triple and quadruple loci, their normal lines, and their gluing maps. An inverse star is the inverse contraction of this recorded configuration. This specifies the centres and their order.

7. The algorithm starts with the explicit reference normalizations of 21.2 and the n-cycle reference in 21.17. It is therefore a uniform geometric presentation of every output by specified blow-ups, blow-downs and projective bundles. Independently, the direct star charts avoid running this search when only a normalization chart or an intersection is required. The mixed-ray components are complete smooth toric threefolds; a pure top component $(0, b, 0, \ell)$ is bimeromorphic to $\mathbb{P}_S(\mathcal{O} \oplus L^{\ell})$.

8. Finally D is globally the principal Cartier divisor of the parameter r in a smooth total space. Its two-term cotangent complex is $[\mathcal{O}_D \xrightarrow{dF} \Omega_Y|_D]$, so its first-order deformation module is the cokernel of the gradient. Equivalently, a lifted hypersurface equation $F + \epsilon G$ is changed by an infinitesimal coordinate change precisely by a Jacobian term. This argument applies to nonreduced monomials as well; an ordinary conormal sequence need not be left-exact there. Absorbing the local unit, or reducing its derivative modulo F, gives (21.5). A global parameter makes the normal line trivial, so these Jacobian quotients glue by the canonical hypersurface presentation. In the reduced case the Jacobian ideal defines the singular scheme, giving the asserted trivial deformation line there.

□

21.3 The virtual Euler obstruction and the conjecture

Lemma 21.7 (Analytic specialization of the virtual Euler number)**.** [tag 02Z0]

Let Z be a compact reduced local-complete-intersection complex space of pure dimension n. In any proper flat holomorphic family over a disc with central fibre Z and smooth nearby fibres, their Euler number is

$$e_{\mathrm{vir}}(Z) = \langle c_n(T_Z^{\mathrm{vir}}), [Z] \rangle.$$

Proof. 1. Flat deformation preserves the local-complete-intersection property relative to the disc near Z. Locally this follows by lifting an embedding and its regular sequence of equations; Nakayama's lemma and flatness show that their lifts generate the ideal and remain a regular sequence. Properness allows one to shrink the disc so that these neighbourhoods cover the total space. The relative cotangent complex is consequently perfect of amplitude $[-1, 0]$. Its dual defines a topological complex K-class V on that total space; local finite vector-bundle complexes define the class by descent, or by stabilization and partitions of unity; the analytic-to-topological comparison is given in (Levy, 2008, Section 3.1, Lemmas 3.1–3.2, pp. 588–590). Its restriction to a smooth fibre is its tangent bundle, and its restriction to Z is its intrinsic virtual tangent class.

2. We compare the fundamental cycles on a smooth proper modification, so that the ambient-manifold hypothesis of the triangulation theorem is satisfied. Analytic resolution, in the complex-analytic scope of (Bierstone and Milman, 1997, Analytic category (0.1)(2), p. 208; Theorem 1.6 and its noncompact extension, p. 215; Remark 1.16, p. 221), gives

$$\rho : \widetilde{\mathcal{X}} \longrightarrow \mathcal{X}$$

with smooth total space, isomorphic over the smooth locus. Only a neighbourhood of the compact central fibre is needed; the locally finite construction is finite there after shrinking the base. The total space is reduced: a nilpotent local section lies in every power of the base parameter by reducedness of the central fibre and flatness, and hence vanishes by Krull intersection. Thus the stated resolution applies to the dominating space.

3. Write the divisor of the pulled-back parameter as $\sum_j m_j D_j$. The inverse image of a short radial segment $[0, \epsilon]$ is a compact semianalytic subset of the smooth real analytic manifold $\widetilde{\mathcal{X}}$. We triangulate it, compatibly with its two endpoint fibres and the singular loci of the D_j, by (Łojasiewicz, 1964, Section 3, Theorems 1–2, pp. 463–464). Over the interior of the segment its top-dimensional part is an oriented real $(2n+1)$-manifold. Near a general point of D_j the parameter is uz^{m_j}, with u a holomorphic unit; choosing a local root of u exhibits m_j radial half-sheets, each with the complex boundary orientation. The sum of their oriented top simplices therefore has boundary

$$[\mathcal{X}_\epsilon] - \sum_j m_j [D_j].$$

The singular and crossing sets have real dimension at most $2n-2$ and contribute no boundary simplex of dimension $2n$.

4. At every smooth point of the reduced central fibre the original map is a submersion, by flatness and smoothness of that fibre germ. The modification is consequently an isomorphism there and the strict-transform multiplicity is one. Every other D_j maps into $\operatorname{Sing} Z$, of complex dimension at most $n-1$. Its fundamental class has zero pushforward in dimension $2n$: its map factors through that lower-dimensional analytic set. Thus pushing the displayed chain forward gives exactly $[\mathcal{X}_\epsilon] - [Z]$. This proves the required homology identity in the original total space. Pairing the globally defined $c_n(V)$ with these homologous cycles gives the equality of

Chern numbers. Gauss–Bonnet on the smooth compact complex fibre gives its Euler number. This proves the analytic assertion directly. The proper-scheme counterpart is (Fantechi and Göttsche, 2010, Theorem 5.8 and Remark 5.9, p. 102).

□

Theorem 21.8 (Virtual Euler obstruction to the prescribed pinching)**.** [tag 02RB] [tag 02ZP]

For the pinching Q of Proposition 20.2, one has $e_{\mathrm{vir}}(Q) = 0$, whereas $e(X(u)) = 2$. Every smooth proper-flat smoothing of Q therefore has Euler number zero. In particular there is no proper flat holomorphic filling of Engel's untwisted family with central fibre exactly Q. This disproves the central-fibre assertion in (Engel, 2026, Conjecture 8.2, p. 22). Theorem 20.6 constructs a nodal family with central fibre Q; Theorem 21.2 constructs the two smooth total spaces with central fibres $Q^{\pm}$.

Proof. 1. Let $D_0, D_\infty \subset R$ be the conductor sections and $\nu : R \to Q$ the normalization. On either branch of the hypersurface $xy = 0$, the derivative comparison with its normalization gives in $K^0(R)$, where pullback of the perfect tangent class means derived pullback,

$$\nu^* T_Q^{\mathrm{vir}} = T_R - i_{0*}(J|_{D_0}) - i_{\infty *}(J|_{D_\infty}).$$

Indeed in a smooth local ambient space the normal line A of the branch and the normal line of the whole hypersurface differ by D: the latter is $A(D)$. Their difference in K-theory is $-[i_*(A(D)|_D)]$, and $A(D)|_D$ is exactly the product of the two branch normals, namely $\mathcal{T}_Q^1$. These local comparisons are intrinsic and agree on overlaps. On the infinity branch J means its T^{-1}-pullback from the double locus, canonically the same line bundle because it comes from B. Thus J extends from both sections by the pullback from B, so $[i_* J] = [J] - [J(-D)]$.

2. Put $j = c_1(J) = \pi_R^* F$, $d_i = [D_i]$. Since the sections are disjoint, $d_0 d_\infty = 0$, and since F is a fibre, $j^2 = 0$. Consequently

$$c(\nu^* T_Q^{\mathrm{vir}}) = c(T_R) \frac{(1 + j - d_0)(1 + j - d_\infty)}{(1 + j)^2} = c(T_R)\{1 - d_0 - d_\infty + j(d_0 + d_\infty)\}.$$

The degree-three part has integral

$$e(R) - \sum_i \int_{D_i} c_2(T_R|_{D_i}) + \sum_i \int_{D_i} j\, c_1(T_R|_{D_i}).$$

We have $e(R) = 2e(S) = 24$. On a section with normal class n_i, $c_2(T_R|_{D_i}) = c_2(S) + c_1(S) n_i$ and $c_1(T_R|_{D_i}) = c_1(S) + n_i$. The two normal classes on the unglued normalization are $m, -m$, where $m = c_1(L)$. Their sum is zero; $c_1(S) = F$ and $F^2 = 0$. The displayed number is therefore $24 - 24 + 0 = 0$. Because $\nu_*[R] = [Q]$, this is the intrinsic virtual number on Q. In the projective-bundle presentation, the same calculation uses $H(H + m) = 0$, $d_0 = H + m$, $d_\infty = H$.

3. The untwisted quotient away from its cusp has Euler number zero: scalar multiplication supplies a free circle action, including the two additive places before logarithmic transformation. Indeed a circle scalar identified with an iterate of $u\phi$ must be the identity, since every positive iterate strictly contracts the Hermitian norm and every negative iterate expands it. Compact fibres with a free circle action have Euler number zero. The chosen Mumford fibre has two zero-dimensional torus orbits and all other torus strata have Euler number zero, so $e(X(u)) = 2$. Lemma 21.7 therefore excludes such a smoothing of Q.

□

Proposition 21.9 (An Euler-zero smoothing of the pinching)**.** [tag PT02] *The pinching Q admits a proper flat smoothing with smooth fibres of Euler number zero, obtained by choosing the zero of its smoothing section on a smooth elliptic fibre of S.*

Proof. Replace ϕ in the global construction of 20.6 by a section of J vanishing on a smooth elliptic fibre of S. The compact collar construction still applies. At its zero divisor the fixed nonzero-parameter equation is $xy = uz$ with $dz \neq 0$, so every nearby threefold is smooth. The double-locus nearby Euler function is one on that elliptic curve and zero off it; its integral is zero. Thus this proper flat smoothing has Euler number zero, whereas the fibres of Engel's family have Euler number two by Theorem 21.8. □

Proposition 21.10 (Infinitely many smooth fillings)**.** [tag 02ZQ]

There are infinitely many fillings of the same punctured untwisted family with smooth total space, reduced normal-crossings central divisor and joint flatness over $B \times \Delta$. The two-model classification in Proposition 21.1 concerns the minimal toroidal refinements.

Proof. 1. The following centres preserve joint flatness. The section $2P$ is disjoint from P and O. In the Weierstrass equation its coordinates are

$$x(2P) = \frac{9t^2}{8}, \qquad y(2P) = \frac{\sqrt{2}\,t^2(5t-32)}{32}.$$

At 0 it meets the nonidentity component different from that met by P; at 1 the displayed coordinates are finite and different from those of P, O; at infinity the minimal coordinates are finite and different as well. This also checks directly the zero intersection numbers $P \cdot 2P = O \cdot 2P = 0$. The rational section of $\mathcal{O}(P - O)$ therefore restricts to a nowhere-vanishing holomorphic section on $2P \simeq B$. Its graph is a compact curve $C \subset L^{\times} \subset R^{\pm}$ disjoint from both conductor sections and from the modification centres. It gives a curve in the smooth locus of $Q^{\pm}$ mapping isomorphically to B.

2. Blow up C in the smooth total fourfold $\mathcal{X}^{\pm}$. Locally the coordinates are (t, u, a, b) with $C = (u, a, b) = 0$. The blow-up charts replace u by u itself or by aU or bU. Thus the new central divisor is reduced normal crossings, and the total space is smooth. All fibres over (t, u) still have pure dimension two; the regular base criterion proves joint flatness. The morphism is proper and is an isomorphism for $u \neq 0$. It adds a $\mathbb{P}^2$-bundle over C, hence a new central irreducible component. To justify repetition, $N_{C/Q^{\pm}}$ is an extension of $\mathcal{O}_B(-1)$ by $\mathcal{O}_B$ and splits, and $H^1(N_{C/Q^{\pm}}) = 0$. The normal sequence in the total family therefore splits off the trivial parameter direction. Its section in the exceptional $\mathbb{P}^2$-bundle is disjoint from the intersection with the strict central transform and has the same normal sequence. We may repeat indefinitely. Their differing component counts preclude isomorphism as fillings.

□

21.4 Integral topology and nearby cycles

Proposition 21.11 (The topology of the two modified pinchings)**.** [tag 02ZR]

With integral coefficients, the spaces constructed above have

	b_0	b_1	b_2	b_3	b_4	b_5	b_6	χ
R	1	0	11	0	11	0	1	24
$R^{\pm}$	1	0	12	0	12	0	1	26
Q	1	1	8	7	10	0	1	12
$Q^{\pm}$	1	1	9	7	11	0	1	14

All these homology groups are torsion free. The fundamental groups of $R, R^{\pm}$ are trivial and those of $Q, Q^{\pm}$ are $\mathbb{Z}$. The dual complex of either $Q^{\pm}$ is a circle.

Proof. 1. Proposition 20.7 gives the integral groups and fundamental groups of R, Q, together with the normalization maps $d_i = s_{0*} - s_{\infty *}T_*$. We use exactly its lattice $\mathcal{L} = H_2(S;\mathbb{Z})$, $F = [S_t]$, $m = P - O$, and set $v = v_0 = [6P] - [O] - 3F$. In these coordinates its map $a \mapsto (aF, av, am)$ is onto $\mathbb{Z}^3$, its d_2 has primitive image of rank three, and its d_4 is primitive and injective.

2. Now let $h : R^+ \to R$ be the modification constructed in 21.2. Its fibres are points off the nodal curve $C = S_\infty$, one $\mathbb{P}^1$ over C_{reg}, and a chain of two $\mathbb{P}^1$'s over the node. The chart calculation identifies the sheaf $R^2h_*\mathbb{Z}$ with $\eta_*\mathbb{Z}_{\mathbb{P}^1}$, where $\eta : \mathbb{P}^1 \to C$ is the normalization. If (a, b) are the cohomological degrees on the two exceptional curves, generization to the two branches is a and $a + b$: on one branch the generic exceptional fibre specializes to the first curve, on the other it specializes to their sum. The matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ is unimodular. These generizations follow by substituting $X = za$, $v = aB$ in the blow-up charts, or by taking the two limits of their exceptional fibres. Proper local neighbourhoods retract onto these fibres by the positive toric contraction, so the fibre calculation computes the higher direct images. There is no other positive-degree direct image.

3. The Leray spectral sequence has only the row $H^*(R, \mathbb{Z})$ and the row $H^{*-2}(\mathbb{P}^1, \mathbb{Z})$; both are free and in even degrees. It degenerates integrally for degree reasons, giving the stated groups of R^+. The same local neighbourhood retractions and Van Kampen show that h induces an isomorphism on fundamental groups, so R^+ is simply connected. The case R^- interchanges the branches.

4. For Q^+ use the normalization mapping cylinder once again. The new degree-two difference map is the old d_2 with one extra coordinate aF: the exceptional divisor restricts to F on the blown-up section and to zero on the other. It has the same primitive rank-three image and the same kernel; its cokernel rank increases by one. The degree-four difference remains primitive, since its pushforward to R already was primitive, and its cokernel likewise increases by one. This proves the groups of Q^+. Interchanging the sections changes the extra coordinate by a sign and does not change any group. The dual complex has one vertex and one self-edge, with no higher cells, and Van Kampen again gives $\mathbb{Z}$.

5. In the modification h, $C_{\mathrm{reg}} \simeq \mathbb{C}^*$ contributes zero and the exceptional chain above the node increases Euler number by $3 - 1 = 2$. Thus $\chi(R^\pm) = 26$ and $\chi(Q^\pm) = 26 - 12 = 14$.

□

Proposition 21.12 (Integral homology and monodromy of the smooth fibre)**.** [tag 02ZS]

The untwisted smooth fibre has torsion-free integral homology with Betti numbers

$$(b_0, b_1, b_2, b_3, b_4, b_5, b_6) = (1, 1, 8, 14, 8, 1, 1).$$

The positive-u monodromy is the identity outside degree three. In degree three there is an integral exact sequence

$$0 \longrightarrow \mathcal{L}/\langle F, v, m\rangle \longrightarrow H_3(X(u);\mathbb{Z}) \longrightarrow K \longrightarrow 0, \qquad K = \{a \in \mathcal{L} : aF = av = am = 0\},$$

in which $T_u - I$ is zero on the left term and, on the quotient, is the map

$$K \longrightarrow \mathcal{L}/\langle F, v, m\rangle, \qquad a \longmapsto a \bmod \langle F, v, m\rangle.$$

The sign convention orients the neck circle by increasing $\arg x$ *in $xy = u$; the boundary convention for a transported chain is chosen so that a positive u loop adds this circle times its boundary cycle. With these conventions $(T_u - I)^2 = 0$, its rank is six, and its nonzero Smith factors are* $1, 1, 1, 1, 1, 11$.

Proof. 1. We use the semistable model 21.2. Remove disjoint open tubular neighbourhoods of the two conductor sections from R^+, and denote the remaining compact manifold by Z. Its two boundary circle bundles are identified by the plumbing; let E denote the first. The Euler class of $E \to S$ is $n = m - F$. It is primitive. The Thom sequence for (R^+, Z) gives

$$H_5(Z) = \mathbb{Z}, \quad H_4(Z) = 0, \quad H_3(Z) = \mathcal{L}/\langle F, m\rangle \simeq \mathbb{Z}^8, \quad H_2(Z) \simeq \mathbb{Z}^{10}, \quad H_1(Z) = 0.$$

If H is the projective-bundle class restricting to zero on s_0 and to $-m$ on s_∞, and A is the exceptional divisor class, the two restriction maps on $H^2(R^+)$ are

$$(\alpha, h, a) \longmapsto \alpha + aF, \qquad (\alpha, h, a) \longmapsto \alpha - hm.$$

Their combined image is primitive of rank twelve, and its cokernel is $\mathcal{L}/\langle F, m\rangle$. In degree two, intersection with the two sections is onto $\mathbb{Z}^2$: a generic projective fibre meets both once, and an exceptional projective fibre meets just the section containing the centre once. In degree six the map is the diagonal $\mathbb{Z} \to \mathbb{Z}^2$. These are precisely the Thom maps giving the asserted groups.

2. The Gysin sequence of the primitive circle bundle gives

$$H_2(E) = n^\perp \subset \mathcal{L}, \qquad H_3(E) = \mathcal{L}/\mathbb{Z}n, \qquad H_1(E) = H_4(E) = 0, \quad H_5(E) = \mathbb{Z}.$$

The smooth fibre is obtained from Z by the boundary mapping cylinder. In its Mayer–Vietoris sequence the degree-two difference map is the normalization difference d_2^+ restricted to $n^\perp$. In the coefficient coordinates $(s, t, h) = (aF, av, am)$ the constraint is $h = s$, so this map has primitive rank two and kernel K. The degree-three difference map is, modulo F, m, $(T_* - I)a = (aF)v$, hence has primitive image $\mathbb{Z}v$. These statements use the surjectivity of the three intersection coordinates proved in 21.11. The degree-five difference is zero: the plumbing reverses the normal circle orientation, and the two boundary fundamental classes sum to zero in Z. The degree-zero difference is zero as well. Exactness now gives the claimed groups and the displayed degree-three extension. Poincaré duality, or the same integral exact sequence, confirms the absence of torsion in complementary degrees.

3. We also compute monodromy directly from this decomposition. In the neck $xy = u$, a positive circuit of u rotates one boundary normal coordinate once relative to the other. A cycle in Z is unchanged. A three-cycle whose cut-open boundary represents $a \in K$ acquires the tube obtained by sweeping a once around that normal circle. The Gysin description sends this tube to the class of a in $H_3(E) = \mathcal{L}/\mathbb{Z}n$, and then to its class in $\mathcal{L}/\langle F, m, v\rangle$. This is exactly the map in the statement. The same description in the other degrees is zero, using the groups and difference maps just computed. A tube has no cut-open boundary, which proves $(T_u - I)^2 = 0$.

4. Finally $K \cap \langle F, m, v\rangle = \mathbb{Z}F$, since the Gram matrix of $m - F, v$ has determinant eleven. Hence the rank is six. In the unimodular E_8 summand, the primitive rank-two lattice spanned by $m - F, v$ has discriminant eleven; its orthogonal complement maps into the dual quotient with cokernel of order eleven. Thus the nonzero Smith factors are five ones and eleven.

□

Proposition 21.13 (The pointwise nearby Euler function)**.** [tag 02ZT]

For $\mathcal{X}^\flat \to \Delta_u$ the nearby-fibre Euler function on the double locus of Q is zero off S_∞ and one at every point of S_∞, including its node. Its integral is one. The smooth part of S_∞ has Euler number zero, so its integrated contribution is zero. For either semistable family $\mathcal{X}^\pm$ the nearby Euler function is one off the double locus and zero on it, giving $\chi(X(u)) = 26 - 2 \cdot 12 = 2$.

Proof. 1. Off the zero divisor of ϕ, the transverse nearby fibre of $xy = u\phi$ is an annulus and has Euler number zero. At a smooth zero, the local equation is $xy = uz$; for fixed $u \neq 0$ one solves for z, and the small local nearby space is contractible. Its Euler number is one. At the node it is the quadratic cone $xy = uzv$, also contractible, with Euler number one. The smooth part of the nodal elliptic curve is $\mathbb{C}^*$ and has Euler number zero; the node contributes one. The core $Q \setminus S$ is a $\mathbb{C}^*$-bundle and contributes zero. This proves the first count.

2. In the resolved family the local map on the double locus is simply $ab = u$ times two transverse coordinates, including above the old node. Its nearby annulus has Euler number zero everywhere on that locus. Away from it, the family is a submersion and the local value is one. Integrating gives the second count. If the nodal threefold in the first family is smoothed rather than small-resolved, its ordinary-double-point Milnor fibre is homotopy equivalent to S^3 and has Euler number zero; the replacement changes 1 to 0, whereas the small resolution changes 1 to 2. These agree with the intrinsic virtual number in 21.8.

□

Proposition 21.14 (Non-Kähler models and the monodromy weight filtration)**.** [tag 02ZU]

The smooth fibres of these untwisted models have $b_1 = 1$ and admit no Kähler metric. Moreover the exceptional small-resolution curve in $R^{\pm}$ is null-homologous, so that normalization is not Kähler either. The rational monodromy weight filtration, defined algebraically from $N = T_u - I$ and centred at degree three, has graded ranks $6, 2, 6$ in weights $2, 3, 4$.

Proof. 1. The odd first Betti number excludes a Kähler metric. In the generization matrix of 21.11 a global section of $R^2 h_* \mathbb{Z}$ has equal branch values, hence degree zero on the small exceptional curve. All classes pulled back from R also have degree zero on it. The Leray calculation exhausts $H^2(R^{\pm})$, so the curve has zero homology class; it cannot have positive area for a Kähler form. Finally $N^2 = 0$, rank $N = 6$, and $\dim H_3 = 14$, so the centred filtration is $W_2 = \operatorname{im} N$, $W_3 = \ker N$, $W_4 = H_3$. Its graded dimensions are the asserted ones.

□

Proposition 21.15 (An infinite explicit subclass)**.** [tag 02ZW]

Let D_m be the central fibre after the m maximal-cone star subdivisions in 21.1, starting with Q^+, where $m \geq 0$. Then

$$\pi_1(D_m) = \mathbb{Z}, \qquad H_*(D_m; \mathbb{Z}) \text{ is free,}$$
$$(b_0, \ldots, b_6) = (1, 1, 9 + m, 7, 11 + m, 0, 1 + m).$$

and $\chi(D_m) = 14 + 3m$. For $m \geq 1$ its normalization is the disjoint union of $\operatorname{Bl}_p R^+$, $m - 1$ copies of $\operatorname{Bl}_{p_i} \mathbb{P}^3$, and a final $\mathbb{P}^3$. The original two sections are still glued by T. The new components form a chain attached to the original component. Consecutive components meet in one $\mathbb{P}^2$, with no triple intersections. The dual complex is the original circle with a chain of m edges attached. The nearby Euler function is one off the double locus and zero on it; its integral remains two. The nearby integral monodromy remains exactly that of 21.12.

Proof. 1. The torus orbit of the cone $ABCE$ is a point on a single local central branch, because precisely one of its four rays has positive u-height. Hence it is a smooth point of the central fibre, away from the conductor. A star at the sum of the four basis rays is the ordinary blow-up of that point in the smooth fourfold. The central divisor is the union of its strict transform and the exceptional $\mathbb{P}^3$, both with multiplicity one. Their intersection is the exceptional $\mathbb{P}^2$ in the blow-up of the central threefold, identified with a hyperplane in $\mathbb{P}^3$. Their two normal lines there are $\mathcal{O}(-1)$ and $\mathcal{O}(1)$.

2. The next distinguished cone $ABCv_1$ has only the new central ray v_1. Its point lies on the new $\mathbb{P}^3$ away from its attaching hyperplane. Repeating therefore blows up that component at a point away from the preceding double locus and attaches the next $\mathbb{P}^3$. The incoming hyperplane and outgoing exceptional plane are disjoint in each middle $\mathrm{Bl}_{p_i}\mathbb{P}^3$. This proves the complete normalization and gluing description, the absence of triples and the dual complex.

3. For the integral calculation, blowing up a smooth point of a compact complex threefold adds one free generator in each of H_2 and H_4 and changes no other homology. This also holds when the threefold has singularities away from that point: remove a small ball and use the Mayer–Vietoris sequence with boundary S^5; the replacement retracts to $\mathbb{P}^2$. Now attach $\mathbb{P}^3$ along that $\mathbb{P}^2$. In degrees $0, 2, 4$ the inclusion into $\mathbb{P}^3$ is an integral isomorphism, so the Mayer–Vietoris boundary map is injective and primitive. It contributes no torsion or odd homology. The new component contributes one free H_6 generator, since $H_6(\mathbb{P}^2) = 0$. Each step therefore adds precisely one to b_2, b_4, b_6. Van Kampen gives no change of fundamental group: the blow-up ball replacement, the attaching plane and the new projective component are simply connected. Induction from 21.11 proves the asserted groups.

4. Independently, the strict-transform blow-up adds two to the central Euler number, the new $\mathbb{P}^3$ adds four, and its intersection $\mathbb{P}^2$ subtracts three. Thus each step adds three, agreeing with the homology computation. The double locus is the disjoint union of the original S and m planes, of Euler number $12 + 3m$. Since the local equation is $u = x$ off this locus and $u = xy$ on it, the nearby Euler integral is $(14 + 3m) - (12 + 3m) = 2$. All modifications are the identity over $u \neq 0$, so they do not alter the marked punctured family or its homological monodromy.

□

Theorem 21.16 (Euler number and fundamental group for every toroidal output)**.** [tag 02ZX]

Let Σ be any output of 21.5 for $u = r^n$, and let D_Σ denote the underlying reduced central space. Let $k(\Sigma)$ be the number of selected pure top ray orbits, and let $t_4(\Sigma)$ be the number of maximal four-cones in one fundamental prism. Then

$$\chi(D_\Sigma) = 11k(\Sigma) + t_4(\Sigma), \qquad \pi_1(D_\Sigma) = \mathbb{Z}, \qquad H_1(D_\Sigma; \mathbb{Z}) = \mathbb{Z}.$$

These assertions include nonreduced central divisors. At a point with precisely one local central branch of multiplicity ℓ, the nearby Euler function is ℓ; at a point with at least two local branches it is zero. Its integral is always two. For every reduced output the pure top count is $k(\Sigma) = n$, so $\chi(D_\Sigma) = 11n + t_4(\Sigma)$. The full integral homology is given by Theorem 21.17.

Proof. 1. Over $B \setminus \{\infty\}$ the top fan gives a cycle of $k(\Sigma)$ ruled bundles over $S \setminus S_\infty$, glued along their zero and infinity sections with the specified translations. There are no triple intersections there. Its Euler number is therefore

$$2k(\Sigma)\chi(S \setminus S_\infty) - k(\Sigma)\chi(S \setminus S_\infty) = 11k(\Sigma),$$

because $\chi(S) = 12$ and $\chi(S_\infty) = 1$. This computation concerns the support of the divisor, so the multiplicities do not enter.

2. At $q = r = 0$ the strata are the torus orbits of cones whose generators collectively have positive q-height and positive r-height. Each such cone has trivial period stabilizer. Indeed, an element $G_1^a G_2^b$ preserving its finite set of primitive rays has a power fixing them all. A ray of positive q-height then forces $a = 0$; any ray with positive q- or r-height forces $b = 0$. Consequently a stratum of cone dimension d is $(\mathbb{C}^*)^{4-d}$. The induced gluing units only identify different period translates of the stratum. Its compactly supported Euler number is zero unless $d = 4$, in which case it is one. Every maximal cone has both kinds of height, and each has one

representative in the interior of the fundamental prism. Thus the cusp contributes exactly $t_4(\Sigma)$. Additivity proves the displayed central Euler formula.

3. The local total space is smooth, and its parameter is $r = z_1^{\ell_1} \cdots z_s^{\ell_s}$ times a holomorphic unit, with $\ell_i > 0$. Its Milnor fibre has the homotopy type of a finite disjoint union of $(S^1)^{s-1}$. For $s = 1$ it consists of ℓ_1 contractible pieces; for $s \geq 2$ its Euler number is zero. This proves the pointwise assertion. Off the cusp the one-branch parts are $\mathbb{C}^*$-bundles over the appropriate parts of $S \setminus S_\infty$ and have Euler number zero; the double parts have nearby Euler value zero. At the cusp, positive-dimensional torus strata again contribute zero. A maximal cone with exactly one positive-r ray must have three bottom rays. Its bottom face is consequently one of the two prescribed triangles ABC and BCD. Each triangle belongs to exactly one such maximal cone. Its determinant is the r-height ℓ of the fourth ray, because the bottom triangle is unimodular. Smoothness forces $\ell = 1$. Exactly these two points contribute one each. Thus the nearby Euler integral is two for every output, even when some other components have multiplicity greater than one.

4. For the fundamental group, first choose the reference model obtained by base-changing $\mathcal{X}^+$ by $u = r^n$ and resolving the surface equation $xy = r^n$ along its smooth conductor S. Its normalized central components are R^+ and $n - 1$ projective-line bundles over S. All are simply connected, and their double surfaces are copies of the simply connected S. The gluing graph is an n-cycle, interpreted as the self-gluing circle when $n = 1$. Van Kampen therefore gives fundamental group $\mathbb{Z}$ for this reference centre.

5. Any two fans of 21.5 have the explicit proper common normal modification formed by intersecting their cones. Resolve this analytic space to a smooth common model; analytic resolution, including the noncompact version near the compact central set, is supplied by (Bierstone and Milman, 1997, Theorem 1.6 and the following noncompact extension, p. 215).

6. A proper bimeromorphic map $F : Y \to X$ between smooth complex manifolds induces an isomorphism of fundamental groups. To see this directly, choose an analytic $A \subset X$ of complex codimension at least two such that F is an isomorphism outside A. General position gives $\pi_1(X \setminus A) \simeq \pi_1(X)$, and gives surjectivity of $\pi_1(Y \setminus F^{-1}A) \to \pi_1(Y)$. Under the common-open identification, composition with F_* is the displayed isomorphism to $\pi_1(X)$. Thus the first map is also injective, and F_* is an isomorphism. Apply this argument to both maps from the smooth common model, after restricting to a sufficiently small parameter disc.

7. Finally, each smooth total space retracts to the support of its central divisor, including in the nonreduced case. The gradient proof in 22.14 applies with the explicit bound for $\rho = |r|^2$

$$\|\nabla \rho\| \geq c\rho^{1-1/(2L)}, \qquad L = \ell_1 + \cdots + \ell_s.$$

   On the complement of the divisor it follows from the weighted arithmetic–geometric mean inequality applied to $\sum_i \ell_i^2/|z_i|^2$. A holomorphic unit can be absorbed into one coordinate locally. Compactness of the central set gives a uniform exponent below one, and the resulting finite remaining path-length bound gives the continuous gradient retraction. Hence all the centres and their smooth total neighbourhoods have the same fundamental group as the reference centre. Abelianization gives the stated H_1.

8. For the nonreduced conductor-star family with added pure top rays $(0, 1, 0, j)$, $2 \leq j \leq m + 1$, the pure-top ray-orbit count increases by m and the maximal-cone-orbit count increases by m. The formula therefore gives $\chi(D_{\Sigma_m}) = 14 + 12m$ for the member Σ_m with these m added rays. Equivalently, each conductor blow-up adds the Euler number 12 of its centre.

□

Theorem 21.17 (Integral homology of every toroidal output)**.** [tag 02ZY]

Let D_Σ be the reduced central space of an output of 21.5 for $u = r^n$. Write k for its number of pure top ray orbits, m for its number of mixed ray orbits, and t_4 for its number of maximal four-cone orbits. Then every integral homology group is free, and

$$(b_0, b_1, b_2, b_3, b_4, b_5, b_6) = (1, 1, 8 + k + m, 7, \\ 9k + t_4 - 2m - 1, 0, k + m).$$

This includes nonreduced central divisors: the homology is that of their underlying support. The marked nearby fibre is $X(r^n)$, and its integral monodromy is $I + nN$ in degree three, where N is the map in 21.12; it is the identity in every other degree. In particular the six nonzero Smith factors of this degree-three monodromy minus the identity are $n, n, n, n, n, 11n$.

Proof. 1. We first compute a reference centre by its normalization mapping cylinder. Base-change $\mathcal{X}^+$ by $u = r^n$ and resolve $xy = r^n$ along its smooth conductor S. Put $N_0 = \mathcal{O}_S(m - F)$, using the lattice notation of 21.11. The normalization consists of R^+ and $n - 1$ copies of $\mathbb{P}_S(\mathcal{O} \oplus N_0)$, in a cycle. The latter assertion follows directly on the surface-resolution charts: if x has transition function g, the coordinates x/r^i have the same transition function, while their complementary coordinates have transition function g^{-1}. Only the closing identification is T.

2. In the integral mapping-cylinder sequence the degree-two map has domain $\mathcal{L}^n$ and target of rank $11n + 1$. Project each intermediate ruled component to S. The resulting $n - 1$ difference blocks are integral identity matrices. Eliminate them by integral row and column operations. The remaining map from $\mathcal{L}$ is d_2^+ from 21.11, together with $n - 1$ coordinates $a \mapsto a(m - F)$. Those coordinates are integral combinations of the three primitive coordinates of d_2^+. Thus the original degree-two map has primitive image of rank $10(n - 1) + 3$, kernel $\mathbb{Z}^7$, and cokernel $\mathbb{Z}^{n+8}$.

3. In degree four, the same elimination uses the $n - 1$ identity blocks on the fundamental classes of the bases. The remaining map from $\mathbb{Z}$ contains d_4^+, which is primitive by 21.11; the other entries are the section differences $m - F$. Hence the degree-four map is injective with primitive image of rank n, and cokernel of rank $10n + 1$. In degree zero it is the incidence map of an n-cycle. In degree six the normalization contributes one independent fundamental class per component. Consequently the reference centre has free integral homology and Betti tuple
$$(1, 1, n + 8, 7, 10n + 1, 0, n).$$

4. On each total space take the toroidal boundary
$$E = \big(\{r = 0\} \cup f^{-1}(\infty)\big)_{\mathrm{red}}.$$
In the cusp charts it is the toric boundary, and outside the cusp its nontrivial toroidal charts are the plumbing charts along S. The unit transition functions preserve monomial ideals. 21.5 gives a finite cone complex modulo the two periods, with the possible self-identifications of its faces explicitly retained.

5. Make this complex strict by barycentric star subdivisions, in decreasing cone dimension. At the moment a stratum is blown up, all of its deeper self-intersections have already been separated. Its closure is therefore smooth, and different translates in the chosen orbit are disjoint. The star at the sum of the primitive basis generators is the ordinary blow-up of that closure. Repeating over the finitely many cone orbits gives smooth strict toroidal embeddings. This is the geometric barycentric subdivision used in (Abramovich et al., 2002, Section 5.6, pp.565–566); its analytic realization by the same monomial ideals is described in (Abramovich et al., 2002, Section 5.5, pp. 564–565). Pure q-strata are included in this preparatory step and in the comparison of the punctured r-families.

6. Intersect the two subdivided fans, and take a regular subdivision, to obtain their explicit common strict toroidal modification. Apply toroidal weak factorization to each of its two subdivision maps. The needed theorem is the finite polyhedral-complex factorization by nonsingular star subdivisions and inverse subdivisions, (Abramovich et al., 2002, Theorem 1.6.1, p. 543; Theorem 2.7.1, pp. 551–552). The corrected form for a morphism states that its centres are smooth irreducible closed strata and that all intermediate complexes map to the target (Matsuki, 2000, Section 2, p.2).

7. The complexes are finite, so the cited combinatorial factorization applies. The analytic subdivision–modification correspondence of (Abramovich et al., 2002, Section 5.5, pp. 564–565) realizes each of its finitely many stars by the indicated monomial blow-up. The correspondence respects the face identifications and the bundle units, so the local analytic blow-ups glue. All intermediate morphisms are proper over the target of the subdivision map, hence over a sufficiently small r-disc. They remain smooth, and both parameters remain monomials times units. This gives the required finite chain of analytic blow-ups and blow-downs between the total neighbourhoods.

8. A nontrivial centre has codimension at least two. For a cone having a positive q-height ray, its period stabilizer is trivial: a power of a stabilizing element fixes each of its finitely many primitive rays; the positive q-height first forces the exponent of G_1 to vanish and then the exponent of G_2 to vanish. The same argument holds for pure bottom cones. The normalization of the closure of their torus stratum is the toric variety of its star. If the closed stratum is smooth, this finite birational normalization map is an isomorphism.

9. For a mixed cone the centre is contained in $q = r = 0$; its star is complete, and it is a compact smooth toric point, curve, or surface. It has free integral cohomology in even degrees only. For a pure top centre, the cone must have dimension two. Its stabilizer is precisely $\langle G_1 \rangle$, and the chart quotient and the plumbing identification show that its dense open part is the corresponding open part of S. The compact smooth centre is therefore a surface birational to the rational surface S. Equivalently, its proper map to the unmodified top stratum is a birational map of smooth surfaces. It is a smooth rational surface and again has free even cohomology. It need not remain isomorphic to S after earlier mixed modifications.

10. For a pure bottom centre the cone has dimension two or three. Its star gives a smooth toric surface or curve proper over the r-disc. The induced character r is primitive on its quotient lattice, since it was primitive before quotienting by a smooth cone on which it vanishes. Its generic fibre is therefore connected, and is respectively a complete toric $\mathbb{P}^1$ or a point. In the surface case the ordered two-dimensional fan gives a chain of rational curves as the central support, possibly with positive multiplicities. In the curve case the support is one point. Monomial gradient retraction, as in 21.16, shows that the ordinary integral cohomology of the centre is free and even, with $H^0 = \mathbb{Z}$ and $H^4 = 0$. These centres are not compact; this is why they must be distinguished from the first two types.

11. For any closed smooth complex centre $C \subset Y$ of complex codimension c, compact or not, the integral blow-up splitting is

$$H^j(\mathrm{Bl}_C\, Y; \mathbb{Z}) \simeq H^j(Y; \mathbb{Z}) \oplus \bigoplus_{a=1}^{c-1} H^{j-2a}(C; \mathbb{Z}).$$

    Tubular excision identifies the relative groups by the integral Thom class; the projective-bundle theorem for the exceptional divisor supplies the powers of its hyperplane class. Their exceptional Gysin maps, together with pullback from Y, give the displayed map. The relation that restriction of the exceptional class is minus the hyperplane class makes the relative matrix triangular with unit diagonal. Proper blow-down pushforward, defined by Borel–Moore

duality, splits the pullback summand. This proves the splitting also for the noncompact centres just listed.

12. It follows that all odd cohomology groups and all torsion are unchanged along the factorization. Each connected centre adds one to b_2 and one to the number of prime boundary components. Each compact centre adds one to b_6 and one to the number of central prime components: in degree six the only new group is its top cohomology. A pure bottom centre adds neither, because its H^4 (and, for a curve, its H^2) vanishes. The reverse operations subtract the same contributions.

13. Consequently b_2 minus the number of all boundary ray orbits, and b_6 minus the number of central ray orbits, are invariants of this comparison. Both endpoints have exactly one pure bottom ray orbit. The reference has n central components and $b_2 = n + 8$, so the arbitrary output has
$$b_2 = 8 + k + m, \qquad b_6 = k + m, \qquad (b_1, b_3, b_5) = (1, 7, 0).$$
All groups are free by the integral splitting in both directions. The total neighbourhoods retract onto their central supports by 21.16, and those compact analytic supports have finite triangulations. The universal coefficient theorem therefore transfers the same freeness and ranks to integral homology. Finally the independently computed torus-stratum Euler number $11k + t_4$ of 21.16 gives $b_4 = 9k + t_4 - 2m - 1$.

14. Every output agrees with the marked family $X(r^n)$ off $r = 0$. A positive r-loop makes n positive u-loops. Since $N^2 = 0$, its monodromy is $(I + N)^n = I + nN$. Multiplication of the already computed integral Smith matrix by n gives the stated factors.

□

Proposition 21.18 (Integral cohomology under central blowups)**.** [tag 02ZZ]

The homology formula of 21.17 can also be computed on the central spaces, by their normalization resolutions and the strata of each recorded star. More precisely, let $Y' \to Y$ be one ordinary blow-up in the algorithm of 21.6, with closed smooth centre C of complex codimension c. Let D, D' be the reduced central supports and $C_0 = C \cap D$. In the derived category of integral sheaves on D there is an isomorphism
$$R(\pi_D)_*\mathbb{Z}_{D'} \simeq \mathbb{Z}_D \oplus \bigoplus_{a=1}^{c-1} (i_0)_*\mathbb{Z}_{C_0}[-2a], \tag{21.6}$$
where $\pi_D : D' \to D$ and $i_0 : C_0 \hookrightarrow D$ are the induced continuous maps. In particular
$$H^j(D'; \mathbb{Z}) \simeq H^j(D; \mathbb{Z}) \oplus \bigoplus_{a=1}^{c-1} H^{j-2a}(C_0; \mathbb{Z}). \tag{21.7}$$
The maps in this splitting are integral, with unit diagonal on the exceptional-fibre basis. Thus it also determines the primitive boundary maps and the absence of torsion in the normalization mapping-cylinder computation along the entire finite algorithm.

Proof. 1. To construct the sheaf maps, let $E = \mathbb{P}_C(N_{C/Y})$ be the exceptional divisor, let $p : E \to C$, and put $\xi = c_1(\mathcal{O}_E(1))$. Pullback from Y gives the degree-zero summand. For $1 \le a \le c - 1$, pullback by p, multiplication by ξ^{a-1}, and the exceptional Gysin morphism give a derived map
$$i_*\mathbb{Z}_C[-2a] \longrightarrow R\pi_*\mathbb{Z}_{Y'}, \qquad i : C \hookrightarrow Y.$$
The Gysin map exists integrally for a closed complex divisor, by its oriented tubular Thom class; neither the centre nor the ambient space must be compact. At a point of C, proper

base change identifies the stalk of the target with $H^*(\mathbb{P}^{c-1};\mathbb{Z})$. Restriction of the exceptional normal line is $\mathcal{O}_E(-1)$, so the displayed maps send the generators to

$$1, -\xi, -\xi^2, \dots, -\xi^{c-1}.$$

This is an integral basis. Off C the map is the identity. It is therefore a stalkwise quasi-isomorphism

$$\mathbb{Z}_Y \oplus \bigoplus_{a=1}^{c-1} i_*\mathbb{Z}_C[-2a] \xrightarrow{\simeq} R\pi_*\mathbb{Z}_{Y'}.$$

2. Now restrict this sheaf morphism to D. The central support upstairs is the whole inverse image $\pi^{-1}D$, including when the centre is horizontal. Proper base change applies to this Cartesian square. The inverse-image functor for integral sheaves is exact, and the inverse image of $i_*\mathbb{Z}_C$ is $(i_0)_*\mathbb{Z}_{C_0}$. This proves (21.6). Taking derived global sections proves (21.7). This calculation concerns the stratified map of central spaces.

3. To compare this splitting with the normalization mapping cylinders, resolve $\mathbb{Z}_{D'}$ by the alternating ordered-local-branch complex: its first term is the constant sheaf on the normalization, its next term is on the normalized double-branch space, and so on. Its augmentation is an integral quasi-isomorphism, because at a point with s branches it is the augmented simplex complex on those branches. Use alternating integral cochains, or equivalently the orientation local system of each unordered branch set. Proper pushforward of this resolution computes the left side of (21.6). The same resolution for D, together with the centre complex, computes its right side.

4. The local matrices can be read directly from the blow-up charts in 21.6. If the centre lies on k central branches among its c defining coordinates, the exceptional fibre is $\mathbb{P}^{c-1}$. The new exceptional branch supplies that whole projective space; the strict transform of old branch i supplies its coordinate hyperplane H_i. Intersections supply the corresponding coordinate projective subspaces. In each even cohomological degree the normalization differential is the alternating restriction matrix with entries $0, 1, -1$. The cover consisting of the whole projective space and its selected hyperplanes has the elementary simplex contraction, leaving one generator ξ^a in each degree $2a$, $0 \le a < c$. For a horizontal centre, every central branch meeting its generic central stratum supplies the whole exceptional fibre; the same simplex contraction applies to these copies. Higher boundary strata are treated by the same coordinate restrictions. The global Gysin maps just constructed glue the surviving classes with unit coefficient.

5. Starting from the explicitly computed reference normalization matrices in 21.17, (21.6) therefore computes the normalization complex along every recorded blow-up or inverse blow-up. A compact centre has $C_0 = C$. A horizontal toric surface centre retracts to its rational-chain C_0; a horizontal curve centre has C_0 a point. Their cohomology is free and even. Consequently the central computation preserves odd cohomology and torsion, adds one to b_2 for every new boundary ray, and adds one to b_6 exactly for a new central ray. It gives the same $(b_1, b_3, b_5) = (1, 7, 0)$, $b_2 = 8 + k + m$ and $b_6 = k + m$ as 21.17.

6. Independently, stratification of π_D into $D \setminus C_0$ and C_0, with fibres a point and $\mathbb{P}^{c-1}$, gives

$$\chi(D') - \chi(D) = (c-1)\chi(C_0).$$

The absolute torus-stratum count of 21.16 is $\chi(D) = 11k + t_4$. Together with the central group calculation this gives again $b_4 = 9k + t_4 - 2m - 1$. Thus the two computations agree integrally and in Euler number.

□

Remark 21.19 (Topology determined by the fan data)**.** [tag 02ZV] For an output of Theorem 21.5, Theorem 21.16 computes the Euler number, fundamental group and nearby Euler function, and Theorem 21.17 gives all integral homology groups and the marked monodromy. Proposition 21.18 identifies the integral maps in the normalization calculation through its direct-image splitting and unit exceptional-fibre diagonals. All these calculations use the same marked cone data.

22 Local limits and the global component obstruction

22.1 The finite root neighbourhoods

Proposition 22.1 (Finite-root plumbing and ruled central fibres)**.** [tag 0300]

The marked root family admits a proper flat holomorphic filling over $\Delta_{s_j} \times \Delta_r$, where $u = r^{12}$, with smooth total space and reduced normal-crossings r-central divisor. Its central surface over an ordinary value of s_j is a cycle of twelve ruled surfaces over the smooth elliptic curve E'_{j,s_j}. Their normalizations are projective bundles associated with the degree-zero line bundle represented by $P_{s_j} - O_{s_j}$, with the appropriate pullbacks by translations. Whenever $P_s - O_s$ is non-torsion, no finite isogeny of the elliptic base makes these ruled normalizations into products.

Proof. 1. On the root disc the elliptic family $E'_j \to \Delta_{s_j}$ is smooth. In the additive period coordinates (ξ, ζ_2), put $W = \exp(-2\pi i \zeta_2)$. Quotienting by $\ker \gamma \subset \Lambda$ gives a holomorphic $\mathbb{C}^*$-bundle over this elliptic family. The period $\widehat{\gamma}$ acts by

$$(\xi, W) \longmapsto (\xi + 6\mu(s), a(s)uW), \qquad a(s) = \exp(-2\pi i \beta^{\mathrm{part}}(s)).$$

This is the bundle and translation in proposition 2.11, proposition 2.13, with inverse fibre coordinate; the period identities check its transition functions directly. The factor a is a holomorphic unit on the entire root disc. Apply the compact-core plumbing of 20.6 relatively over a closed smaller root disc. Here the smoothing section never vanishes, so the only local equation at the conductor is $xy = u$ after a change of frame. This gives a smooth total space before the base change.

2. After $u = r^{12}$, the local equation is $xy = r^{12}$, times the two smooth coordinates on E'_j. Resolve its A_{11} surface singularity by its minimal toric resolution. In lattice coordinates this inserts each primitive height-one ray between the two boundary rays, at unit intervals. The construction is canonical for changes $x \mapsto ax$, $y \mapsto a^{-1}y$ by a holomorphic unit, so these local resolutions glue. Its affine charts have the equation $r = XY$, or $r = X$ at a smooth point of the central divisor, times the elliptic and root coordinates. Thus the total space is smooth, the divisor is reduced with normal crossings, and the joint map to (s_j, r) is flat. The resolution is proper, and the plumbing was proper, so the joint map is proper. The comparison over $r \neq 0$ is the original period quotient, since the resolution is an isomorphism there. There are twelve components in the cycle because one application of $\widehat{\gamma}$ shifts the unfolded component index by twelve.

3. The ruled normalization of a component is obtained by adding the zero and infinity sections to the indicated $\mathbb{C}^*$-bundle. For a degree-zero line bundle L_0 on an elliptic curve, $\mathbb{P}(\mathcal{O} \oplus L_0)$ is a product over that curve only if L_0 is trivial. Indeed an isomorphism would give two independent nowhere vanishing quotient sections of a common twist of $\mathcal{O} \oplus L_0$; degree zero forces that twist and L_0 to be trivial. Equivalently, the two distinguished disjoint sections have normal bundles L_0, L_0^{-1}. A finite isogeny can trivialize L_0 only when L_0 is torsion, because the norm of its pullback is a positive tensor power of L_0.

4. The section P of the rational elliptic surface is not torsion, as is checked by its positive Mordell–Weil height in (Engel, 2026, proof of Lemma 2.1, pp. 3–4). If its restriction to the

root disc were identically killed by a positive integer, analytic continuation on the smooth part of $S \to B$ would give the same identity globally. Each nonzero torsion equation therefore has a discrete zero set. Outside a countable union of such sets, $P_s - O_s$ is non-torsion. For each such s, the ruled normalization remains nontrivial after every finite isogeny of the elliptic factor, by the norm argument above. The torsion isogenies at $s_j = 0$ are those in proposition 20.9.

□

Theorem 22.2 (Finite deck quotients and central cycles)**.** [tag 0301]

The twisted finite deck generators extend to the models of 22.1. On the twelve component labels their actions are $i \mapsto i+4$ at $j=1$ and $i \mapsto i-3$ at $j=2$, with labels in $\mathbb{Z}/12$. Every nonidentity group element acts freely, including at every node of the central cycle. The quotient is a smooth complex fourfold, proper and flat over $(t,r) = (s_j^{m_j}, r)$, with a reduced normal-crossings r-central divisor. Its fibre divisor over $t = p_j$ has multiplicity exactly m_j.

The reduction over $s_j = r = 0$ is the central surface of proposition 20.9: a cycle of four ruled surfaces at $j=1$, and a cycle of three at $j=2$. In the root presentation the normal character to $s_j = 0$ is $e^{-2\pi i/m_j}$ and the character on r is trivial.

Proof. 1. Write primes for evaluation at $g_j s$, put $\zeta_j = e^{-2\pi i/m_j}$, and set

$$R_1 = \begin{pmatrix} -1/\tau & 0 \\ (1-\mu)/\tau & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1/\tau & 0 \\ -\mu/\tau & 1 \end{pmatrix}.$$

The identities $R_j \Pi_c = \Pi_c' A_j$ and $A_j v_j = v_j$ follow by substitution of the displayed matrices. In the coordinates (s, ξ, W, r) the two affine deck maps become

$$\begin{aligned} s' &= \zeta_1 s, \qquad \xi' = -\xi/\tau + (6\mu' + 2\tau' - 4)/3, \\ W' &= r^4 U_1(s,\xi) W, \\ U_1 &= \exp\left[-2\pi i \left(\frac{1-\mu}{\tau}\xi + \frac{\beta^{\mathrm{part}\prime} + 2\mu'}{3}\right)\right], \end{aligned}$$

and

$$\begin{aligned} s' &= \zeta_2 s, \qquad \xi' = \xi/\tau + (-6\mu' - 3\tau' + 3)/4, \\ W' &= r^{-3} U_2(s,\xi) W, \\ U_2 &= \exp\left[-2\pi i \left(-\frac{\mu}{\tau}\xi - \frac{\beta^{\mathrm{part}\prime} + 3\mu'}{4}\right)\right]. \end{aligned}$$

Each U_j is a nowhere vanishing holomorphic function on the additive elliptic cover. Period covariance gives the descent rule for U_j and therefore the required bundle isomorphism after quotient by $\ker \gamma$. Multiplication by a unit preserves every toric chart. Multiplication by r^a sends the chart of component index i to the chart of index $i+a$. Therefore the displayed maps and their inverses extend holomorphically to the resolved infinite chain and then to its twelve-component quotient. The identities $g_1^3 = g_2^4 = 1$ on the quotient hold on the dense punctured family, and hence everywhere.

2. A nonidentity power has component shift $4, 8$ in the first case and $-3, -6, -9$ in the second. None is zero modulo twelve. The same shifts act on the edge labels of the cycle, so no node can be fixed either. At $s \neq 0$ the rotation of s already excludes fixed points. The finite quotient map is consequently a holomorphic local isomorphism. Its source is smooth, so its target is smooth. Properness follows from the proper root model and the finite base map. Flatness can be checked after the surjective local-isomorphism cover: there s^{m_j} and r are a regular sequence, and their fibres have pure dimension two. The local central equation

remains $r = XY$ or $r = X$. Finally $t = s^{m_j}$ shows that the Cartier divisor over p_j is exactly m_j times its reduction, including where this divisor meets a central node. This also proves the asserted normal characters.

3. At $s = 0$ use the explicitly marked isogenies

$$L_1(\xi, \eta) = (\eta, \xi - \eta/3), \qquad L_2(\xi, \eta) = (\eta, \xi - \eta/2).$$

Their kernel generators are $(2/3, \rho)$ and $(1/2, i)$, and the finite deck maps lift to

$$(\xi, \eta) \mapsto (\xi + \kappa_1/3, (\rho - 1)\eta), \qquad (\xi, \eta) \mapsto (\xi - \kappa_2/4, -i\eta).$$

In the Tate coordinate $v = e^{-2\pi i\xi}$ these are the shifts $4, -3$ just computed. The kernel generators act by nonzero translations on the smooth elliptic factor F_j, hence freely even at the two fixed points of each Tate component. Every other element with nonzero deck power has a nonzero component shift. The full product groups are therefore free on the product semistable models. Their component orbits have cardinalities three and four, giving precisely four and three ruled components. The original lattice isogenies intertwine the maps on the dense tori and their unit-interval fans, so the quotient identifications extend across all nodes. These quotient identifications give the comparison with proposition 20.9.

4. For the punctured family obtained by descending the c-family, the holomorphic correction $\eta_j(s)$ in theorem 7.8 causes no additional local obstruction here. At the shift $c \mapsto c+12$, the reference translations $-4\Pi_c(\widehat{u} + \widehat{w})$ and $3\Pi_c(\widehat{u} + \widehat{w})$ are periods. The remaining correction is equivariant and independent of c. Translation by $-c\eta_j(s)/12$ on the additive cover conjugates that descent to the period descent. It commutes with the finite deck action by equivariance. Thus the constructed finite models fill the local pullback by $w = r^6$ after a specified holomorphic identification over $r \neq 0$.

□

22.2 The base-changed cusp fan

Proposition 22.3 (Period-invariant fans after base change)**.** [tag 0302]

For the plain base change $u = r^n$ of the untwisted cusp model, the smooth period-invariant refinements which retain the prescribed bottom triangulation, are jointly flat over (q, r), and have reduced r-central divisor are indexed by $k = 0, \dots, n$. In a fundamental prism write its four bottom rays as A, B, C, D as in 21.1, and put $\rho_i = (0, i, 0, 1)$ for $0 \leq i \leq n$. The maximal cones for the index k are

$$ABC\rho_k, \quad BCD\rho_k, \quad AB\rho_i\rho_{i+1}\ (i < k), \quad CD\rho_i\rho_{i+1}\ (i \geq k).$$

There are thirteen such models for $n = 12$ and three for $n = 2$. Each has smooth total space and a reduced normal-crossings central divisor with n components and no triple locus. Its component labels over the punctured q-disc are globally $\mathbb{Z}/n$; in particular their monodromy around $q = 0$ is the identity.

Proof. 1. The map of cocharacter lattices for the base change is

$$(y_1, y_2, h_q, h_r) \longmapsto (y_1, y_2, h_q, nh_r).$$

The old top ray indexed by l pulls back to $(0, nl, 0, 1)$. Smoothness on the two-dimensional top face forces insertion of every primitive ray ρ_{nl+i}, $0 < i < n$: a skipped interval of length $d > 1$ would retain lattice index d on that face. Joint flatness excludes any new ray with both base heights positive, and the fixed bottom excludes new rays of $h_r = 0$. Reducedness permits only top rays of $h_r = 1$. These are consequently all the rays.

2. On the left face $AC\rho_0\rho_n$, a triangulation with these collinear top vertices has a unique triangle containing the bottom edge AC. Write its third vertex as ρ_a. The triangles on either side of that bridge are then forced. The analogous right face has bridge $BD\rho_b$. The first period takes A, C to B, D and fixes every top ray, so periodicity requires $a = b = k$. The fixed bottom triangles ABC and BCD must then cone to ρ_k. The remaining successive top intervals force exactly the displayed cones, leaving no choice. Conversely those cones have determinant ± 1 and common-face intersections. Their normalized volumes sum to $n + 2$, the volume of the prism, so they cover it. For $n = 2$ and $n = 12$, the determinant, intersection and volume calculations use exact rational arithmetic.

3. The period actions are

$$G_1 : (y_1, y_2) \mapsto (y_1 + h_q, y_2), \qquad G_2 : (y_1, y_2) \mapsto (y_1, y_2 + h_q + nh_r).$$

   Thus the choices propagate consistently from the fundamental prism. The proper toric subdivisions descend by the covering construction of 20.6. Every ray has pure q-height or pure r-height, so in a regular affine chart q and r are monomials in disjoint sets of coordinates. This proves joint flatness. All nonzero r-heights equal one, and there are at most two top rays in any maximal cone; this proves reduced normal crossings and absence of triple points. Properness follows by descent from the proper toric modification of the base-changed proper cusp model.

4. The central components and all their intersection maps are specified by the stars of the ρ_i. Modulo the periods their indices are $i \bmod n$; adjacent indices meet in the conductor elliptic surface. Away from the cusp their normalizations are the ruled bundles of the expanded plumbing. At the unique bridge class the star has both bottom triangles, and gives the nodal-curve blow-up and small resolution described in 21.2, on the corresponding ruled bundle. This star description also fixes the gluing when the bridge is on an inserted component. The endpoint choices $k = 0, n$ are the canonical resolutions of the base changes of the two original models. Intermediate values are related to them by the successive side flops.

5. Finally, G_1 fixes each index and G_2 changes it by n. The units in the cusp periods are single-valued holomorphic functions of q and do not change a ray label. The toric charts themselves are defined over q, not a root of q. Hence these labels remain global on the punctured q-disc, proving the monodromy assertion. Further smooth periodic subdivisions with mixed rays give infinitely many models flat over r alone, as in 21.1; they are excluded by the stated joint-flatness condition.

□

Corollary 22.4 (Base-changed cusp refinements)**.** [tag 0304] [tag 0303]

For $n = 2$ and $n = 12$, 21.5–21.6 give all finite-orbit smooth invariant plain-cusp refinements, their proper flat families, normalization charts, finite blow-up and blow-down presentations, multiplicities and all multiple loci. 21.16–21.18 give their complete integral topology and nearby Euler function. Reducedness is exactly the restriction $\ell = 1$ on every positive-height ray.

Proof. 1. The base maps $u = w^2$ and $u = r^{12}$ induce multiplication by 2 and 12 on the integral parameter characteristic. Insert $n = 2$ and $n = 12$ respectively into the periodic lattice action and fan construction of 21.5. Its converse recovers the invariant subdivision from each such plain-cusp model, so the parametrization includes every stated output.

2. For each primitive ray v, the local parameter is a monomial whose exponent on its divisor is its integral height $\ell(v)$. Thus the central divisor is reduced exactly when every such height equals one. Smoothness, flatness, properness, the normalization charts and all intersection maps are those proved in 21.5; 21.6 gives the finite blow-up and blow-down presentation.

Finally substitute the resulting orbit counts and maps into 21.16, 21.17, 21.18 for the claimed topology.

□

22.3 Component monodromy and trace obstructions

Theorem 22.5 (Peripheral component obstruction)**.** [tag 0305]

The finite models of 22.1 and any of the plain $u = r^{12}$ cusp models of 22.2 cannot be glued to a proper complex space over $B \times \Delta_r$ with the prescribed central fibres on all three base neighbourhoods. The obstruction is the peripheral component monodromy of these prescribed finite and plain cusp models.

Proof. 1. If a further plain cusp refinement changes the top fan, its central fibre over an ordinary $q \neq 0$ has additional components or different multiplicities and already cannot agree with the twelve-cycle model of 22.1 on the overlap. We may therefore assume its top fan is the standard reduced twelve-cycle. Additional mixed-height rays lie over $q = 0$ and do not change the following argument over the punctured base.

2. Suppose such a gluing exists and put $B^\circ = B \setminus \{0, 1, \infty\}$. Normalize its reduced central fibre over B°. In each of the prescribed plumbing charts this is a proper smooth family whose fibres are the disjoint union of twelve connected ruled surfaces. The charts and their transition isomorphisms therefore give a locally constant twelve-element set of connected components. Equivalently, the Stein factor of this normalization is an unramified degree-twelve cover of B°: the algebra of fibrewise holomorphic functions is locally $\mathcal{O}^{12}$, since each proper connected ruled surface has only constant holomorphic functions. This description constructs the component cover over B°.

3. The double curves give the adjacency graph of a twelve-cycle. Thus its transition permutations lie in the dihedral group D_{12}, the automorphism group of that graph. We record a peripheral permutation by its deck-identification map. Holonomy defined by lifting a clockwise path and returning its endpoint to the initial root slice uses the inverse permutations throughout; the relation and the following contradiction are unchanged. At 0 the monodromy is a rotation by four steps; at 1 it is a rotation by minus three steps, by the explicit root deck maps in 22.2. Relabelling a cycle can reverse either orientation, but changes these only to ± 4 and ± 3. At the cusp the monodromy is the identity by 22.3.

4. The peripheral relation in $\pi_1(B^\circ)$ is $g_1 g_2 g_0 = 1$ with the consistently chosen clockwise loops. In any common labelling, the product of the two finite rotations would therefore have to be the identity. Its possible rotation numbers are

$$4 + 3, \quad 4 - 3, \quad -4 + 3, \quad -4 - 3 \equiv 7, 1, 11, 5 \pmod{12}.$$

None is zero. This contradicts the peripheral relation.

□

Proposition 22.6 (The marked genus-three component cover)**.** [tag 0306]

Let $B^\circ = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and use the clockwise deck-identification convention of 22.5. The peripheral rotations $4, -3, -1$ in $\mathbb{Z}/12$ define a connected degree-twelve cover of B°. Its compact normalization is the smooth genus-three curve obtained from

$$y^{12} = \frac{t^4}{(t-1)^3}.$$

The ramification indices over $0, 1, \infty$ are $3, 4, 12$, respectively. The peripheral relation determines the last rotation from the first two, so this is the marked cyclic component cover used in the partial filling of 22.41.

Proof. 1. The presentation $g_1 g_2 g_0 = 1$ sends the third peripheral generator to $-4 + 3 = -1$. The first two images generate $\mathbb{Z}/12$, so the resulting action on the twelve sheets is transitive. This constructs the connected cover by its monodromy representation.

2. The rational function $t^4/(t-1)^3$ has orders $4, -3, -1$ at the three punctures. Its twelfth-root equation has the same marked deck representation. The covering-space identification is the map which sends a chosen twelfth root at one base point to its chosen sheet and continues both along the same paths. Its equality of monodromy makes that map single-valued; its local analytic root coordinates make it holomorphic. Normalization extends this identification to the compact covers.

3. Locally $y^{12} = z^a u(z)$ has $\gcd(12, a)$ branches of ramification index $12/\gcd(12, a)$, after absorbing the holomorphic unit in a local twelfth root. Thus the three contributions to ramification are $4(3-1)$, $3(4-1)$, and $12-1$. Their sum is 28. Riemann–Hurwitz gives $2 - 2g = 2 \cdot 12 - 28 = -4$, so $g = 3$. Reversing the deck convention is realized by $y \mapsto y^{-1}$, with the reciprocal defining equation; the compact curve and the ramification calculation are unchanged.

□

Proposition 22.7 (Cusp-root multiplicities)**.** [tag 0307]

Let $q = s^d$, $u = r^n$, and suppose a cyclic deck generator has translation $d^{-1}(aG_1 + bG_2)$ in the additive notation for the two periods. Assume its r-component shift is integral. Then its r-component shift is $nb/d \bmod n$, whereas the multiplicity of every generic quotient q-component is a positive multiple of

$$\frac{d}{\gcd(d, a, b)}.$$

A nonzero r-component shift therefore forces a nonreduced q-fibre. In particular a shift $-1 \bmod 12$ in this class of constructions forces multiplicity at least twelve.

Proof. 1. In the coordinates of 20.6 the two periods have q-valuation vectors $(1, 0)$ and $(0, 1)$; their r-valuation vectors are $(0, 0)$ and $(0, n)$. After $q = s^d$, the period lattice on the q-central component indices is $d\mathbb{Z}^2$. A regular toroidal resolution has indices in $(\mathbb{Z}/d)^2$. The deck generator translates them by (a, b). The order of that translation is $d/\gcd(d, a, b)$, so the subgroup preserving a component has order $h = \gcd(d, a, b)$. Its subgroup fixing a generic point has some order h_0 dividing h; unit factors may make h_0 smaller. That isotropy acts on the normal coordinate s by a primitive h_0th root of unity. The quotient normal coordinate is s^{h_0}, and $q = (s^{h_0})^{d/h_0}$ has multiplicity d/h_0, a positive multiple of d/h. Unit factors cannot decrease this lower bound.

2. Reducedness would require $h_0 = d$, and therefore $h = d$, so both a, b are divisible by d. Then nb/d is zero modulo n. If nb/d is relatively prime to n, its order in $\mathbb{Z}/n$ is n, which divides the order d/h of the full component translation; this gives the final lower bound.

□

Theorem 22.8 (Trace on the split elliptic normalization)**.** [tag 0308]

Let C be the genus-three component cover in 22.6, and let $\mathscr{E} \to C$ be the relatively minimal elliptic surface obtained by pulling back $S \to B$. Write σ for the order-twelve deck transformation,

with its natural lift to $\mathscr{E}$. Then

$$\mathrm{Tr}_\sigma \mathscr{E}(\mathbb{C}(C)) := \left\{ \sum_{i=0}^{11} \sigma^i Q : Q \in \mathscr{E}(\mathbb{C}(C)) \right\} = 12\mathbb{Z}P.$$

In particular there is no section Q for which $(t_Q \circ \sigma)^{12} = t_{6P}$. For a normalization over C ruled over this split elliptic surface, the twelfth iterate of an adjacent conductor map $t_Q \circ \sigma$ is therefore t_{12kP} for some $k \in \mathbb{Z}$.

Proof. 1. There is a convenient model of the cover and its elliptic surface:

$$C:\ b^3 = a^4 + 1, \qquad t = b^3, \quad y = b/a, \qquad \mathscr{E}:\ Y^2 = X^3 - 3bX + 2a^2.$$

Indeed $y^{12} = t^4/(t-1)^3$. The substitution from the original Weierstrass coordinates is $x = a^2b^4X$, $y_{\text{old}} = a^3b^6Y$. The affine discriminant of the new equation is 1728; it has no singular fibre on the affine curve. The curve C has one point at infinity, with pole orders $3, 4$ for a, b. Put $z = a/b$ and $c = z^3a = z^4b$ there. Then z is a uniformizer and

$$c^3(1-c) = z^{12}, \qquad c(0) = 1.$$

In the integral Weierstrass coordinates $X_0 = z^2X$, $Y_0 = z^3Y$, the equation is

$$Y_0^2 = X_0^3 - 3cX_0 + 2c^2.$$

Its discriminant has order twelve and its c_4 is a unit. The unique singular fibre of the minimal regular surface is therefore I_{12}. The fundamental line bundle is $\mathcal{O}_C(\infty)$, so $\chi(\mathcal{O}_\mathscr{E}) = 1$. This is a projective elliptic surface.

2. Let $\omega = e^{2\pi i/3}$ and $\lambda = e^{2\pi i/12}$. The deck lift is

$$\sigma(a, b, X, Y) = (ia, \omega b, \lambda^2 X, \lambda^3 Y).$$

It fixes the section $P = (0, -\sqrt{2}a)$. Two further sections are

$$R = \left(\frac{a^2}{b-1}, \frac{\sqrt{3}a}{b-1} \right), \qquad R' = \sigma R = \left(\frac{a^2}{b-\omega}, \frac{\sqrt{3}\omega a}{b-\omega} \right).$$

Direct substitution verifies their equations. The three orbit points $R, \sigma R, \sigma^2 R$ lie on the line

$$Y = \frac{\sqrt{3}b}{a} X - \sqrt{3}a.$$

Thus $R + \sigma R + \sigma^2 R = O$ by the elliptic chord law, and $\sigma^3 R = R$.

3. The only pole of R on the affine curve is at $(a, b) = (0, 1)$. There a is a uniformizer and $b - 1 = a^4/3 + O(a^8)$, so $X = 3a^{-2} + O(a^2)$ and $Y = 3\sqrt{3}a^{-3} + O(a)$. Consequently $R \cdot O = 1$. The same computation at $(0, \omega)$ gives $R' \cdot O = 1$, while $P \cdot O = 0$. At the other two points with $a = 0$, P and R meet simply: the fibre is smooth at $(X, Y) = (0, 0)$, and the difference of their Y coordinates has a nonzero linear term in a. Thus $P \cdot R = 2$, and likewise $P \cdot R' = 2$. The sections R, R' meet simply at $(a, b) = (0, \omega^2)$. They cannot meet elsewhere on the affine curve, since their X coordinates are distinct for $a \neq 0$.

4. To check infinity, for the section $\sigma^j R$ put $k = \omega^j$. Its integral coordinates are

$$X_0 = \frac{c^2}{c - kz^4}, \qquad Y_0 = \frac{\sqrt{3}kcz^4}{c - kz^4}.$$

At the node use analytic coordinates

$$U = Y_0 - (X_0 - 1)\sqrt{X_0 + 2}, \qquad V = Y_0 + (X_0 - 1)\sqrt{X_0 + 2}.$$

They satisfy

$$UV = (1 - c)\{3X_0 - 2(c + 1)\},$$

whose right side is a unit times z^{12}. The section expansions are

$$U = -\frac{\sqrt{3}k^2}{6}z^8 + O(z^{12}), \qquad V = 2\sqrt{3}kz^4 + O(z^8).$$

Hence R, R' both meet component 4 of the resolved I_{12} cycle, with distinct coordinates on that component. Reversing the cycle numbering replaces both indices by 8 and has no effect on the argument. The section P meets the identity component, at the smooth point $(X_0, Y_0) = (0, -\sqrt{2})$. Thus no further intersections occur at infinity, and $R \cdot R' = 1$.

5. The height pairing formula and its positivity are those of (Shioda, 1990, Theorems 8.4 and 8.6, table (8.16), pp. 227–229). For an I_{12} fibre its correction is $\mathrm{contr}(i, j) = \min(i, j) - ij/12$. The preceding intersections give, in the ordered basis P, R, R',

$$G = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4/3 & -2/3 \\ 0 & -2/3 & 4/3 \end{pmatrix}, \qquad \det G = 8/3.$$

In particular these are three independent sections. The Leray sequence and $R^1\pi_*\mathcal{O}_{\mathscr{E}} = \mathcal{O}_C(-\infty)$ show that its irregularity equals $g(C) = 3$; the right-hand sheaf has no global section. This also agrees with the Picard-variety statement in (Shioda, 1990, Theorem 4.1, pp. 219–220). The equalities $\chi = 1$ and $e(\mathscr{E}) = 12$ give $p_g = 3$, $b_2 = 22$ and $h^{1,1} = 16$. The zero section, a fibre and the eleven nonidentity I_{12} components generate a rank-thirteen trivial lattice. The identification of sections with the Néron–Severi quotient (Shioda, 1990, Theorem 1.3, p. 214), together with $\rho \le h^{1,1}$, now shows that the Mordell–Weil rank is exactly three and $\rho = 16$.

6. There is no nonzero torsion section. Indeed its height would be zero, so for some $0 \le j < 12$ the height formula would give

$$0 = 2 + 2(Q \cdot O) - \frac{j(12 - j)}{12}.$$

The correction is at most three and $Q \cdot O$ is a nonnegative integer. This forces $Q \cdot O = 0$ and $j(12 - j) = 24$, which has no integral solution.

7. Let $L = \mathbb{Z}P + \mathbb{Z}R + \mathbb{Z}R'$. The integral Néron–Severi sublattice generated by the trivial lattice and these three sections has determinant -32. This follows either by eliminating the trivial lattice with the displayed height matrix, or from the integral intersection matrix specified by the following entries. All its entries have just been computed: sections have square -1, their mutual intersections are $2, 2, 1$, and R, R' meet precisely component 4. If $d = [\mathscr{E}(\mathbb{C}(C)) : L]$, the section–Néron–Severi quotient identification gives the same index for this sublattice in $\mathrm{NS}(\mathscr{E})$. Its integral determinant implies $d^2 \mid 32$, hence $d \in \{1, 2, 4\}$. In particular $4Q \in L$ for every section.

8. In the rational basis P, R, R' the deck action and its trace are

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad \sum_{i=0}^{11} M^i = \mathrm{diag}(12, 0, 0).$$

Suppose first that the trace of Q is $6P$. By the index bound we may write

$$Q = \frac{1}{2}P + \frac{1}{4}(uR + vR'), \qquad u, v \in \mathbb{Z}.$$

Let $j \in \mathbb{Z}/12$ be its component index at infinity. Since both R, R' meet component four, the height formula modulo integers gives

$$\frac{2u - v}{6} \equiv \frac{j}{3} \pmod{\mathbb{Z}}, \qquad \frac{-u + 2v}{6} \equiv \frac{j}{3} \pmod{\mathbb{Z}}.$$

Equivalently $2u - v - 2j$ and $-u + 2v - 2j$ are divisible by six. Reducing modulo two shows that both u, v are even. Write $u = 2u'$, $v = 2v'$. The component map is a group homomorphism, so

$$2j \equiv 4(u' + v') \pmod{12}.$$

Therefore j is even. The geometric height formula then lies in $\frac{1}{3}\mathbb{Z}$. In contrast the height from G is

$$\langle Q, Q \rangle = \frac{1}{2} + \frac{u'^2 - u'v' + v'^2}{3},$$

which lies in the disjoint coset $\frac{1}{2} + \frac{1}{3}\mathbb{Z}$. This is a contradiction.

9. The same argument determines the full trace image. If $Q = \alpha P + \beta R + \gamma R'$ in the rational span, pairing with the narrow section P gives $2\alpha = \langle Q, P \rangle \in \mathbb{Z}$. Every odd half-integral value of α leads to the identical congruences and height contradiction, with $1/2$ replaced by an odd square divided by two. Thus $\alpha \in \mathbb{Z}$ and the trace is in $12\mathbb{Z}P$. The section P realizes the generator.

   Finally the identity $(t_Q \circ \sigma)^{12} = t_{\mathrm{Tr}_\sigma Q}$ proves the claimed obstruction.

□

Corollary 22.9 (Integral Mordell–Weil lattice)**.** [tag 0309]

In the notation of 22.8,

$$\mathscr{E}(\mathbb{C}(C)) = \mathbb{Z}P \oplus \mathbb{Z}R \oplus \mathbb{Z}R', \qquad \det \mathrm{NS}(\mathscr{E}) = -32.$$

Proof. 1. The affine coordinate ring of C is $\mathbb{C}[a, b]/(b^3 - a^4 - 1)$, with unique representatives of degree at most two in b. The pole orders $3i + 4j$ of $a^i b^j$, $0 \le j \le 2$, are distinct, so cancellation cannot create functions with pole order one or two at infinity. Consequently

$$H^0(C, \mathcal{O}_C(2\infty)) = \mathbb{C}, \qquad H^0(C, \mathcal{O}_C(3\infty)) = \mathbb{C} \oplus \mathbb{C}a.$$

A section disjoint from O therefore has $X = x_0$, $Y = y_0 + y_1 a$ for constants x_0, y_0, y_1. Substitution in the Weierstrass equation forces $x_0 = y_0 = 0$ and $y_1^2 = 2$. Thus the only nonzero sections with intersection zero with O are $P, -P$. Every other nonzero section has intersection at least one with O, and hence height at least $4 - 3 = 1$.

2. 22.8 proves that every section has integral coefficient along P in the rational height decomposition. The remaining lattice can have index only one, two or four over $\mathbb{Z}R + \mathbb{Z}R'$. Any proper such overlattice contains a nonzero element of order two modulo this sublattice. After subtracting an integral combination of R, R' it is one of $R/2$, $R'/2$, $(R + R')/2$. Each has height $1/3$, contradicting the preceding lower bound. Thus the index is one. The determinant already computed in 22.8 is the full Néron–Severi determinant.

□

Proposition 22.10 (Unramified torsors with prescribed transgression)**.** [tag 030A]

Put $K = \mathbb{C}(t)$, $L = \mathbb{C}(C)$, $G = \mathrm{Gal}(L/K) = \langle\sigma\rangle$, and let E/K be the generic elliptic curve of $S \to B$. There is a nonzero class

$$\xi \in \mathrm{Sha}(E_L/L)^G \quad \textit{whose transgression in} \quad H^2(G, E(L)) = \mathbb{Z}P/12\mathbb{Z}P \quad \textit{is } 6P.$$

Here Sha *denotes the subgroup of genus-one torsors trivial over the henselian local fields of all points of C. Such a class has a relatively minimal projective genus-one surface $V \to C$, locally isomorphic to $\mathscr{E} \to C$, and an affine lift $A : V \to V$ of σ with*

$$A^{12} = t_{6P}.$$

Thus (V, A) is a genus-one normalization base with the specified affine twelfth power.

Proof. 1. We use Galois cohomology with its continuous cochains. The five-term Hochschild–Serre sequence gives

$$H^1(K, E) \longrightarrow H^1(L, E)^G \xrightarrow{d_2} H^2(G, E(L)) \longrightarrow H^2(K, E).$$

The transgression sequence is (Hochschild and Serre, 1953, Chapter III, Section 4, Theorem 2 and the following remark, pp.129–130); its continuous form follows by passing to finite Galois quotients for the finitely many values of each continuous cochain. For this complex function field, $H^2(K, E) = 0$, as in (Cossec et al., 2024, Lemma 4.7.8, p. 445). One can also see this from $\mathrm{cd}\, K = 1$: the torsion submodule of $E(\overline{K})$ has no second cohomology, and its quotient is a rational vector space, whose positive continuous cohomology vanishes by averaging on finite quotients.

2. Cyclic group cohomology gives $H^2(G, E(L)) = E(K)/\operatorname{Tr}_\sigma E(L)$. By 22.8 this is $\mathbb{Z}/12$, with generator the class of P. Hence there is a G-invariant torsor class $\xi_0 \in H^1(L, E)$ with $d_2\xi_0 = 6P$.

3. At a place v of B choose a point w of C above it and its decomposition subgroup G_w. The only nontrivial decomposition groups have orders $3, 4, 12$, at $0, 1, \infty$ respectively. The component groups of the original fibres are $\mathbb{Z}/3$, $\mathbb{Z}/2$ and zero. Thus $6P$ belongs to the identity component of the local Néron group at every ramified place. Multiplication by the ramification index is surjective on that identity group over the henselian local field: on the residue group it is surjective on $\mathbb{G}_a(\mathbb{C})$ or $\mathbb{G}_m(\mathbb{C})$, and its derivative is invertible; Hensel's lemma lifts a division point. Choose such a division point over K_v^h. Viewed over L_w^h, it is invariant, so its local norm is $6P$. The restriction of the transgression class to $H^2(G_w, E(L_w^h))$ is therefore zero. For unramified places the decomposition group is trivial, so the same assertion is automatic.

4. Naturality and exactness of the local five-term sequences now show that $(\xi_0)_w$ is the restriction of some $\eta_v \in H^1(K_v^h, E)$. Only finitely many of these classes need be nonzero. The original elliptic surface is nontrivial, since $j = 1728t$. Ogg–Shafarevich localization is therefore surjective:

$$H^1(K, E) \longrightarrow \bigoplus_{v \in B} H^1(K_v^h, E) \longrightarrow 0.$$

This is the combination of exact sequence (4.7.4) and Theorem 4.7.9 in (Cossec et al., 2024, p. 445); the same surjectivity and its geometric fibre-twist interpretation are given in (Xiao, 1991, Section 4, equation (4), p. 21). Take a global η with these local values and put $\xi = \xi_0 - \mathrm{res}_{L/K}\,\eta$. It is locally trivial at every point of C, remains G-invariant, and has $d_2\xi = 6P$. It is nonzero because its transgression is nonzero.

5. The standard interpretation of the Weil–Châtelet group represents ξ by a smooth projective genus-one curve over L; taking its relatively minimal regular model over C gives V. Local triviality identifies each henselian model with that of E_L. Over $\mathbb{C}$ the corresponding étale identifications yield holomorphic identifications over small base discs. Equivalently, V is obtained by gluing copies of $\mathscr{E}$ over these discs by translations by local sections. These translations extend over the regular I_{12} model. Thus V is locally isomorphic to $\mathscr{E}$, has smooth total surface, no multiple fibres, and the same single I_{12} fibre. Properness and flatness hold locally over the base and hence globally. The local translation gluing specifies V as in (Xiao, 1991, Theorem 3.4, pp. 14–15; Section 4, p. 21).

6. The G-invariance of ξ gives an affine isomorphism of its generic fibre covering σ. It extends to the relatively minimal regular surface: in this particular situation the extension can be checked on a common resolution. If an exceptional (-1)-curve for one projection is not contracted by the other, its image is a vertical rational curve and has self-intersection at least -1, since blowing down only increases self-intersection. On either locally trivial model all vertical rational curves are components of the I_{12} fibre, of square -2. Thus every such (-1)-curve is contracted by both projections and can be removed from the common resolution. Repeating this proves that the map is an isomorphism. The same argument gives the local comparison with the Jacobian model used above. Transporting the base by σ changes none of these facts. Write A_0 for this lift. Its twelfth power is translation by a section $N \in E(L)^G = \mathbb{Z}P$. Choosing the powers $1, A_0, \dots, A_0^{11}$ as lifts of G gives the cyclic factor set $N\lfloor (i+j)/12 \rfloor$. Its cohomology class is precisely the descent obstruction $d_2\xi$, so $N = (6+12m)P$ for an integer m. Replacing A_0 by $A = t_{-mP} \circ A_0$ gives $A^{12} = t_{6P}$.

7. To form the threefold filling of 18.6, the datum (V, A) must be supplemented by its normal line bundles, cusp charts and specified punctured-family identification.

□

Proposition 22.11 (Auxiliary torsor plumbing with cusp multiplicity twelve)**.** [tag 030B]

For the torsor and affine lift (V, A) of 22.10, there is a line bundle M on V representing the narrow Jacobian section P, with

$$M^2 = -2, \qquad M \otimes (A^*M)^{-1} \simeq \pi^*\mathcal{O}_C(\infty).$$

These data give a proper flat nodal plumbing $\mathcal{Z}^\flat \to \Delta_h$ and two uniform smooth-total-space semistable modifications $\mathcal{Z}^\pm \to \Delta_h$, together with holomorphic maps to B. For $h \neq 0$, the nodal fibre has twelve ordinary double points and Euler number twelve; each small-resolved fibre has Euler number twenty-four. The cusp fibre of the latter map to B has multiplicity twelve. Consequently this auxiliary construction does not fill the prescribed S^6 family. No proper birational modification over B can turn its multiple cusp into a reduced cusp.

Proof. 1. The section P determines a relative Picard class of multidegree zero on every fibre component of $V \to C$. It lifts to a line bundle M because $\mathrm{Br}(C) = 0$; this is the relative Picard consequence of (Cossec et al., 2024, Proposition 0.9.2, p. 143; Section 4.2, p. 370). In local trivializations of the torsor, translation acts trivially on the relative degree-zero Picard class. Thus the local Poincaré lines glue up to base lines, with precisely that Brauer obstruction. In particular we may and do retain multidegree zero on all twelve fibre components.

2. Choose a finite multisection cover of degree d which trivializes the torsor generically, and resolve its base change. There the pulled-back M becomes the narrow Poincaré line for the pulled-back P, up to a base line. Its square is $-2d$, by height base change (Shioda, 1990, Proposition 8.12, equation (8.25), p. 231). Base lines do not change the square of a fibre-degree-zero line. The intersection pullback formula therefore gives $M^2 = -2$.

3. Since $\sigma P = P$, the line $J = M \otimes (A^* M)^{-1}$ is trivial on every fibre, including the I_{12} fibre. Grauert base change and the evaluation map give $J = \pi^* J_C$ for a line on C. The line $M \otimes (t_{6P}^* M)^{-1}$ has base degree twelve. To check the sign and degree, pass to the same trivializing cover. On a Jacobian, the degree of the restriction of $\mathcal{O}(P - O)$ to a section Q minus its degree on O is $-\langle P, Q \rangle$, by the height formula and the narrowness of P. For $Q = 6P$ this is $-12d$. Descending the degree factor d proves the assertion. Iterating A^* twelve times now gives $12 \deg J_C = 12$, so $\deg J_C = 1$.

4. The forms da/b^2, $a\,da/b^2$ and da/b form a basis of $H^0(C, K_C)$. Their σ-eigenvalues are $\lambda^7, \lambda^{10}, \lambda^{11}$. Thus the characteristic polynomial on $H^1(C, \mathbb{Z})$ is

$$p(x) = \Phi_{12}(x)\Phi_6(x) = x^6 - x^5 + x^3 - x + 1, \qquad p(x) + (1 - x)(x^5 + x^2 + x) = 1.$$

   It follows that $1 - \sigma^*$ is an automorphism of $\mathrm{Pic}^0(C)$, with inverse $(\sigma^*)^5 + (\sigma^*)^2 + \sigma^*$. Twisting M by a base line changes J_C by $N \otimes (\sigma^* N)^{-1}$. The displayed inverse therefore supplies a base twist making $J_C = \mathcal{O}_C(\infty)$. Let ϕ be its section with divisor the reduced I_{12} fibre F_∞.

5. Apply the plumbing maps of 20.6 to the data (V, M, A). Set $R_V = \mathbb{P}_V(\mathcal{O} \oplus M)$, and identify its zero and infinity sections by $s_0(x) \sim s_\infty(Ax)$. On the node neighbourhood over V use

$$a \in M_x, \quad b \in M_{Ax}^{-1}, \qquad a \otimes b = h\phi_x.$$

   The a collar is glued to the annular core at (x, a); the b collar is glued at (Ax, b^{-1}). Choose the Hermitian metric and collar radii as in 20.6. Compactness of V and the bounded norm of ϕ give disjoint collars, Hausdorffness and compact representatives by exactly the same inequalities. These are holomorphic gluing maps of fourfolds over Δ_h. Their local hypersurface rings are flat over h, and the compact representatives prove properness. The map to B is well defined because $t \circ \sigma = t$.

6. At a smooth point of F_∞ the equation is $ab = hz_C$; at one of its twelve nodes, $z_C = uv$ and it is $ab = huv$. For $h \neq 0$ the latter has one ordinary double point. These twelve centres are distinct even if A rotates the fibre components. A centre has $a = b = 0$ and is in neither annular collar; no overlap identifies it with the centre indexed by Ax. Thus the double locus is parametrized by V. The two uniform prism subdivisions of 21.2 extend over these charts: the affine action preserves the orientation of the I_{12} cycle, and all remaining changes are units. They give proper small modifications, smooth total spaces and reduced normal-crossings h-central fibres. Equivalently their normalizations are the blow-up of R_V along F_∞ in one section, followed by the twelve chosen small resolutions. The normal product becomes $\pi^* \mathcal{O}_C(\infty)(-F_\infty) = \mathcal{O}_V$.

7. The nearby function of the nodal plumbing is one on F_∞ and zero on its complement in the conductor. Since $e(F_\infty) = 12$, its integral is twelve. Each small resolution replaces a point by $\mathbb{P}^1$ and adds one, giving twenty-four. Independently $e(R_V) = 24$; above each of the twelve nodes the modified normalization replaces a point by a chain of two projective lines, of Euler number three. The open parts of the fibre are copies of $\mathbb{C}^*$ and contribute zero. The modified normalization therefore has Euler number 48, its pinching has Euler number 36, and the semistable nearby count is $48 - 2e(V) = 24$.

8. Finally $z = a/b$ is a coordinate on C near infinity and $q_B = 1/t = z^{12}/c^3$, with c a unit. At a generic point of the auxiliary cusp fibre this gives multiplicity twelve. In fact the map q_B is locally a twelfth power throughout the smooth nearby threefold; a holomorphic root of the unit can be chosen locally. On any common resolution of a proper birational comparison over B, every divisor over that fibre consequently has q_B-valuation divisible by twelve. A reduced fibre on another smooth model would have a strict transform of valuation one, a contradiction.

□

Proposition 22.12 (Hodge theory of analytic unramified torsors)**.** [tag 030C]

Let $V \to C$ be any compact smooth analytic genus-one surface locally isomorphic, as a torsor over every base disc, to the Jacobian surface $\mathscr{E} \to C$ of 22.8. Then $b_1(V) = 6$, and V is Kähler. For every holomorphic line bundle M on V, the projective bundle $\mathbb{P}_V(\mathcal{O} \oplus M)$ is Kähler. Every smooth compact threefold bimeromorphic to such a bundle satisfies the $\partial\bar{\partial}$-lemma.

Consequently none of these projective bundles or smooth compact bimeromorphic models can be the non-$\partial\bar{\partial}$ normalization component required in 22.16.

Proof. 1. Write $p : \mathscr{E} \to C$ for the projective Jacobian. The calculation in 22.8 gives $R^1p_*\mathcal{O}_{\mathscr{E}} = \mathcal{O}_C(-\infty)$, and $g(C) = 3$. The coherent Leray sequence therefore gives $h^1(\mathscr{E}, \mathcal{O}_{\mathscr{E}}) = 3$. Projectivity and Hodge decomposition imply $b_1(\mathscr{E}) = 6$. The topological Leray sequence with rational coefficients is

$$0 \longrightarrow H^1(C;\mathbb{Q}) \longrightarrow H^1(\mathscr{E};\mathbb{Q}) \longrightarrow H^0(C, R^1p_*\mathbb{Q}) \longrightarrow H^2(C;\mathbb{Q}) \longrightarrow H^2(\mathscr{E};\mathbb{Q}).$$

The last arrow is injective because p has a section. The first arrow is an isomorphism of six-dimensional vector spaces. Exactness thus implies $H^0(C, R^1p_*\mathbb{Q}) = 0$.

2. The local torsor identifications differ by translations. On a smooth elliptic fibre translations act trivially on H^1. On the I_{12} fibre they preserve the orientation of the component polygon and therefore act trivially on its one-dimensional H^1 as well. These identifications commute with the local specialization maps. Hence, for $\pi : V \to C$, they glue to $R^1\pi_*\mathbb{Q} \simeq R^1p_*\mathbb{Q}$ on the whole curve, including infinity. The same Leray edge sequence for V now gives $H^1(V;\mathbb{Q}) \simeq H^1(C;\mathbb{Q})$. Thus $b_1(V) = 6$ is even. Buchdahl's theorem (Buchdahl, 1999, Theorem 11, p.297) proves that V is Kähler.

3. Choose a Hermitian metric on $\mathcal{O} \oplus M$. The induced metric on the tautological relatively ample line of its projectivization has a closed real $(1,1)$ curvature form θ positive in the vertical direction. If ω is a Kähler form on V, compactness implies that $\theta + A\pi^*\omega$ is positive in every direction for a sufficiently large constant A. It is closed, so it is a Kähler form on the projective bundle. A smooth compact bimeromorphic model is therefore in Fujiki class $\mathcal{C}$ and satisfies the $\partial\bar{\partial}$-lemma; the domination statement is recalled, with its precise reference to the $\partial\bar{\partial}$ theorem, in (Chen, 2026, Section 2.1, p.8). Apply 22.16 for the final obstruction.

□

22.4 Normal-crossing limits and coherent cohomology

Remark 22.13 (The two logarithms in the finite connector)**.** [tag PT09] The finite-chart comparison of proposition 7.5 uses the connector

$$\frac{\log s_j}{2\pi i}\Pi_c v_j, \qquad c = -\frac{12}{2\pi i}\log r.$$

The identity $\Pi_c v_j = \Pi_0 v_j + c\gamma(v_j)e$ separates its vertical term as

$$\frac{\log s_j}{2\pi i}\Pi_0 v_j - \frac{12\gamma(v_j)}{(2\pi i)^2}\log s_j \log r\, e.$$

This is the exact overlap map on the punctured parameter charts. Proposition 22.5 computes its component transport, and Theorem 18.6 gives the resulting obstruction for a proper filling over the full parameter disc.

Theorem 22.14 (Fundamental-group obstruction). [tag 0300]

Let $f : \mathcal{Y} \to \Delta$ be a proper holomorphic map from a smooth complex manifold, with smooth connected fibres over Δ^* and reduced normal-crossings central fibre D. There is a continuous specialization map from a nearby fibre Y_t to D which induces a surjection

$$\pi_1(Y_t) \longrightarrow \pi_1(D).$$

Hence the central fibre of any such semistable filling of S^6 is simply connected. In particular it cannot be a simple pinching of a connected normalization along one pair of disjoint nonempty conductor sections, whether its elliptic normalization base is split or is a genus-one torsor.

Proof. 1. The surjectivity assertion also follows from (Nakayama and Ogus, 2010, Theorem 5.1, p. 2226). We use the explicit neighbourhood retraction needed in theorem 21.16 and proposition 22.19. Put $W = f^{-1}(\overline{\Delta}_\epsilon)$ and $W^* = W \setminus D$. The negative gradient of $|f|^2$, for a Hermitian metric, retracts W onto D. In a product chart $f = z_1 \cdots z_k$ the arithmetic– geometric mean inequality gives

$$\|\nabla |f|^2\| \geq c\, |f|^{2-1/k}.$$

A finite cover of compact D gives a common exponent $\alpha < 1$ in $\|\nabla \rho\| \geq c\rho^\alpha$, where $\rho = |f|^2$; for a fourfold one may take $\alpha = 7/8$. The remaining trajectory length is bounded by $c^{-1}\rho^{1-\alpha}/(1-\alpha)$. Hence trajectories converge uniformly to D, and parametrization by the level of ρ gives a continuous strong deformation retraction fixing D. General position moves every loop of W off the real-codimension-two normal-crossings strata, so $\pi_1(W^*) \to \pi_1(W)$ is onto. The smooth proper fibration over the punctured disc gives

$$\pi_1(Y_t) \longrightarrow \pi_1(W^*) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

A small transverse meridian at a smooth point of a reduced component maps to the generator of $\mathbb{Z}$ and bounds a disc in W. Thus the image of $\pi_1(Y_t)$ alone generates $\pi_1(W) = \pi_1(D)$. Restricting the retraction $W \to D$ to Y_t gives the asserted continuous specialization and its surjection on fundamental groups.

2. For the last assertion, let R be the connected normalization and D_0, D_∞ its two disjoint conductor sections. Choose a continuous function $h : R \to [0,1]$ equal to zero on D_0 and one on D_∞. The function $\exp(2\pi i h)$ descends to the pinching. A path from a point of D_0 to its paired point in D_∞ becomes a loop whose image has winding number one. Consequently the pinching has a surjection of its fundamental group onto $\mathbb{Z}$. This contradicts the preceding necessary condition when $Y_t \simeq S^6$.

□

22.5 A Hodge obstruction to the central strata

Theorem 22.15 (The Hodge obstruction to normal-crossing limits). [tag 030Z]

Let $g : \mathcal{Z} \to \Delta_s$ be a proper holomorphic map from a smooth complex fourfold, smooth over Δ_s^*, with reduced normal-crossings central fibre D. Suppose its nearby fibres are biholomorphic to the six-sphere members of the specified family, after any finite base change $w = s^N$. Then

$$H^1(D;\mathbb{C}) = 0, \qquad h^1(D, \mathcal{O}_D) \geq 1.$$

Consequently the natural map $H^1(D;\mathbb{C}) \to H^1(D,\mathcal{O}_D)$ is not surjective. In particular, D cannot be the analytification of a proper complex algebraic variety with normal-crossings singularities.

If D has simple *normal crossings, at least one of its nonempty closed strata fails the $\partial\bar\partial$-lemma. In particular, no such simple normal-crossings filling can have all irreducible components Kähler or all irreducible components Moishezon. These obstructions hold for every $N \geq 1$.*

Proof. Theorem 22.14 gives a surjection from the fundamental group of a nearby fibre onto $\pi_1(D)$. Since that fibre is diffeomorphic to S^6, it follows that $\pi_1(D) = 1$ and $H^1(D;\mathbb{C}) = 0$.

For a compact complex threefold X diffeomorphic to S^6, the groups $H^2(X;\mathbb{Z})$ and $H^4(X;\mathbb{Z})$ vanish, so $c_1(X) = c_2(X) = 0$. The analytic Riemann–Roch formula, obtained from Atiyah and Singer (1968, Theorem 4.3, pp. 564–565), gives

$$\chi(X,\mathcal{O}_X) = \frac{1}{24}\langle c_1(X)c_2(X), [X]\rangle = 0.$$

Every holomorphic three-form Ω is closed and is exact because $H^3(X;\mathbb{C}) = 0$. Stokes' theorem gives $\int_X i\Omega \wedge \overline{\Omega} = 0$. The integrand is pointwise nonnegative for the complex orientation, so $\Omega = 0$. Serre duality therefore gives $h^3(X,\mathcal{O}_X) = 0$, and connectedness gives $h^0(X,\mathcal{O}_X) = 1$. Hence

$$h^1(X,\mathcal{O}_X) = 1 + h^2(X,\mathcal{O}_X) \geq 1.$$

Locally along the reduced normal-crossing divisor, g has the form $s = z_1 \cdots z_k$ after absorbing a holomorphic unit. This makes $\mathcal{O}_{\mathcal{Z}}$ torsion-free over the local discrete valuation ring of the disc and hence flat over it. Upper semicontinuity for the proper flat map gives $h^1(D,\mathcal{O}_D) \geq 1$; see Grauert (1960, §7, Satz 5, pp. 63–64).

For a proper complex algebraic variety with Du Bois singularities, the canonical map from singular cohomology to coherent cohomology is surjective in every degree; see Graf and Kovács (2014, §1.D, p. 120). Normal-crossing singularities are Du Bois, including self-intersections of a global component. Thus algebraicity and properness of D would make $H^1(D;\mathbb{C}) \to H^1(D,\mathcal{O}_D)$ surjective, contradicting the two groups just computed.

Now suppose D is simple normal crossing and each nonempty closed stratum satisfies the $\partial\bar{\partial}$-lemma. These are exactly the strata in condition (K1) of Chen (2026, Definition 2.3, p. 9). For a proper semistable degeneration with smooth total space and this condition, Chen (2026, Theorem 2.4(4), p. 10) gives E_1 degeneration of the Hodge–de Rham spectral sequence on a nearby fibre. In total degree one this yields

$$0 = b_1(X_s) = h^0(X_s,\Omega^1_{X_s}) + h^1(X_s,\mathcal{O}_{X_s}) \geq 1,$$

a contradiction.

A closed smooth submanifold of a compact Kähler manifold is Kähler. A closed stratum of a Moishezon component is also Moishezon: choose a projective modification, take a component of the inverse image dominating the stratum, and intersect with general relative hyperplanes until the induced proper map is generically finite. This gives a generically finite projective cover and hence maximal algebraic dimension for the stratum. Smooth Moishezon manifolds satisfy the $\partial\bar{\partial}$-lemma, as recalled in Chen (2026, §2.1, p. 8). Either stated hypothesis on the irreducible components therefore gives condition (K1), proving the last assertion. A finite parameter cover preserves the biholomorphism type of every nonzero fibre, so the argument applies for every $N \geq 1$. □

Theorem 22.16 (The obstruction on normalization components)**.** [tag 0315]

Let D be a compact reduced complex space with normal-crossings singularities, allowing self-intersections of global components. Suppose that every connected component of its smooth normalization satisfies the $\partial\bar{\partial}$-lemma. Then the natural map

$$H^1(D;\mathbb{C}) \longrightarrow H^1(D,\mathcal{O}_D)$$

is surjective.

Consequently every smooth-total-space reduced normal-crossings filling of the specified six-sphere family, after any finite base change, has a normalization component which fails the $\partial\bar{\partial}$-lemma. In particular its normalization cannot consist entirely of compact Kähler or Moishezon manifolds. This conclusion includes normal crossings which are not simple.

Proof. 1. Let $D^{[p]}$ be the smooth compact branch space whose points are ordered $(p+1)$-tuples of distinct local branches through one point of D. Equivalently, we normalize the loci where these branches meet, retaining their ordering. These are distinct *local branches*, even when they belong to one global irreducible component. The finite holomorphic maps $\pi_p : D^{[p]} \to D$ and their face maps are specified by forgetting one branch. The symmetric group acts by reordering; we take the alternating summand, with the sign character, in each degree. Averaging over this finite group is exact over $\mathbb{C}$. Write δ for the alternating sum of the face pullbacks.

2. Both augmented complexes

$$\mathbb{C}_D \longrightarrow \big((\pi_p)_*\mathbb{C}_{D^{[p]}}\big)^{\mathrm{alt}}_{p\geq 0}, \qquad \mathcal{O}_D \longrightarrow \big((\pi_p)_*\mathcal{O}_{D^{[p]}}\big)^{\mathrm{alt}}_{p\geq 0}$$

are resolutions. This can be checked locally without a global ordering. In a chart $z_1 \cdots z_k = 0$, the constant complex is the augmented cochain complex of the simplex on the k branches. For the holomorphic complex, fix a monomial in the convergent power series ring. It survives on exactly the intersections indexed by subsets of those i for which its z_i-exponent is zero. If there are no such indices it already vanishes on D; otherwise its coefficient complex is again an augmented simplex complex. The usual contraction is a finite combination of coordinate restrictions, so preserves convergence. This proves exactness in the convergent analytic local rings. The descriptions agree under permutations of the local branches.

3. Resolve the holomorphic terms by smooth Dolbeault forms and the constant terms by smooth de Rham forms on the branch spaces. These compute the respective hypercohomologies. Indeed finite pushforward is exact here, and partitions of unity from local ambient smooth functions pull back to make the resulting sheaves fine. The ordinary de Rham lemma and Dolbeault lemma apply on each smooth branch space. Projection to the $(0,\bullet)$ part defines the natural comparison between the two total complexes and induces $H^1(D;\mathbb{C}) \to H^1(D,\mathcal{O}_D)$.

4. Represent a coherent degree-one class by a total cocycle (α, f), where α is a smooth $(0,1)$-form on $D^{[0]}$ and f is an alternating smooth function on $D^{[1]}$. Choose the total-complex signs so that its equations are

$$\bar\partial\alpha = 0, \qquad \delta\alpha = \bar\partial f, \qquad \delta f = 0.$$

On a normalization component, $\partial\alpha$ is of type $(1,1)$, is ∂-exact and is $\bar\partial$-closed. The $\partial\bar\partial$-lemma gives a smooth function h with $\partial\alpha = \partial\bar\partial h$. Do this on every component, and put

$$\alpha' = \alpha - \bar\partial h, \qquad f' = f - \delta h.$$

The pair represents the same coherent class and still satisfies the displayed cocycle equations; now $d\alpha' = 0$. Holomorphic face maps commute with both differential operators, so

$$\partial\bar\partial f' = \partial\delta\alpha' = \delta\partial\alpha' = 0.$$

A smooth complex-valued pluriharmonic function on a compact connected complex manifold is constant: apply the maximum principle to its real and imaginary parts in local complex coordinates. Therefore f' is constant on each connected component of $D^{[1]}$. It follows that $df' = 0$ and $\delta\alpha' = 0$.

5. Thus the same pair (α', f'), with α' viewed as a complex-valued smooth one-form, is a cocycle of the total de Rham branch complex. Its image is the original coherent class. This proves surjectivity. The argument on $D^{[1]}$ uses its compactness through the constancy of pluriharmonic functions.

6. For the asserted application, 22.14 and 22.15 give $H^1(D;\mathbb{C}) = 0$ and $h^1(D,\mathcal{O}_D) \geq 1$. Surjectivity is impossible, so some normalization component fails the lemma. Compact Kähler

and smooth Moishezon manifolds satisfy it, as in the primary comparison checked in 22.15. The same argument applies after every finite base change, since its smooth fibres are still the specified six-spheres.

□

Proposition 22.17 (A restricted non-Kähler cap obstruction)**.** [tag 0316]

Let D be a compact reduced normal-crossings complex space. Suppose its irreducible components have been partitioned into closed unions $D_0, \dots, D_s$, with their reduced induced structures. Assume that

$$H^1(D_i, \mathbb{C}) \longrightarrow H^1(D_i, \mathcal{O}) \quad \text{is surjective for every } i, \tag{22.1}$$

and that the analogous map is injective for every pairwise intersection $D_i \cap D_j$. Then

$$H^1(D, \mathbb{C}) \longrightarrow H^1(D, \mathcal{O}_D) \quad \text{is surjective.} \tag{22.2}$$

In particular these hypotheses hold for the following specified cap assembly. The horizontal block D_0 is a two-section pinching of a ruled bundle over an analytic locally trivial torsor $V \to C$ of 22.12, with conductor map A covering σ and $A^{12} = t_{6P}$. The normalization may undergo the modifications of 22.11, or any smooth bimeromorphic modifications for which the two conductor restriction maps on H^1 are the same pullbacks as before. The other blocks are smooth hyperbolic toric Kato threefolds or compact smooth spaces satisfying (22.1), for example class-$\mathcal{C}$ spaces. Assume every pairwise overlap is either a reduced union of invariant divisors of a hyperbolic toric Kato threefold, or a compact normal-crossings surface whose normalized components and normalized double curves are rational. No such assembly can be a semistable central fibre of the S^6 family.

Proof. 1. Write $D^{[p]}$ for the disjoint union of intersections of $p+1$ distinct blocks. The augmented closed-cover complexes resolve both $\mathbb{C}_D$ and $\mathcal{O}_D$. For the coherent assertion this can be checked in a normal-crossings coordinate chart. The local equation of D is a product of distinct coordinates, and each block groups a disjoint subset of those factors. The corresponding ideals are generated by products in disjoint sets of variables. The ordinary alternating intersection complex is exact, as one sees on each monomial, where it is the augmented complex of a simplex. The same contraction acts coefficientwise on convergent power series. For constant sheaves, small intersections are contractible unions of coordinate subspaces, and the same stalkwise simplex complex is exact. This proves both resolutions also when the global blocks are self-crossing.

2. Their hypercohomology spectral sequences have

$$E_1^{p,q}(\mathcal{F}) = H^q(D^{[p]}, \mathcal{F}), \qquad \mathcal{F} = \mathbb{C} \text{ or } \mathcal{O}.$$

Every connected compact reduced branch intersection has only constant holomorphic functions. Thus the comparison maps in the entire $q = 0$ row are isomorphisms. They induce isomorphisms on $E_2^{1,0}$ and $E_2^{2,0}$. Surjectivity in (22.1) and injectivity on pairwise intersections imply surjectivity on $E_2^{0,1}$: lift a coherent degree-one class on the blocks, then its topological restriction difference vanishes because its coherent image vanishes and the pairwise comparison is injective.

3. Now use the natural five-term exact sequences

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(D, \mathcal{F}) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0}.$$

The isomorphism on the last term makes a lift of a coherent d_2-closed element topologically d_2-closed. The isomorphism on the first term then gives (22.2). If the block comparisons are isomorphisms, the same argument also proves injectivity, provided the map on $E_2^{0,1}$ is injective.

4. The horizontal-block comparison follows from the cohomology of its base curve. By 22.12, $H^1(V,\mathbb{C}) = H^1(C,\mathbb{C})$ and $H^1(V,\mathcal{O}) = H^1(C,\mathcal{O}_C)$. The first equality is the Leray calculation there; the second also follows from $R^1\pi_*\mathcal{O}_V = \mathcal{O}_C(-\infty)$, which has no global section. The action of A on these groups is σ^*, since it covers σ and the displayed maps are pullbacks from the base. The characteristic polynomial

$$p(x) = \Phi_{12}(x)\Phi_6(x), \qquad p(1) = 1,$$

shows that $1 - A^*$ is invertible on both groups. The normalization sequence of the two-section pinching, for constants and for holomorphic functions, therefore gives

$$H^1(D_0,\mathbb{C}) = \mathbb{C}, \qquad H^1(D_0,\mathcal{O}) = \mathbb{C}.$$

Both classes come from the same difference map on constant functions of the conductor. Their natural comparison is an isomorphism. Projective-bundle pullback identifies the normalization groups with those of V. The stipulated modifications preserve these groups and both restriction maps; in particular the explicit modifications of 22.11 have this property. This grouped computation is essential: on an individual conductor V, the comparison has dimensions six and three and is not injective.

5. For a hyperbolic toric Kato manifold Z, its fundamental group is $\mathbb{Z}$, as in (Istrati et al., 2022, Section 2, p.1352); (Istrati et al., 2022, Proposition 8.4, p. 1377, and Theorem 8.5, p. 1378) give $H^0(Z,\Omega^1_Z) = 0$ and $H^1(Z,\mathcal{O}_Z) = \mathbb{C}$. The exact sequence $0 \to \mathbb{C} \to \mathcal{O} \xrightarrow{d} d\mathcal{O} \to 0$ shows that $H^1(Z,\mathbb{C}) \to H^1(Z,\mathcal{O})$ is injective, because a global section of $d\mathcal{O}$ is a closed holomorphic one-form. The two dimensions are one, so it is an isomorphism. The same reasoning applies to hyperbolic toric Kato surfaces. Compact smooth rational varieties have both groups zero. Class-$\mathcal{C}$ blocks satisfy (22.1) by their $\partial\bar\partial$ property.

6. Finally let F be a reduced union of invariant divisors in a hyperbolic toric Kato threefold. Its normalized components are compact toric surfaces or hyperbolic toric Kato surfaces (Istrati et al., 2022, proof of Theorem 8.8, p.1380). Its normalized double curves are rational, since every invariant curve is rational in the hyperbolic case (Istrati et al., 2022, Theorem 5.2, pp. 1366–1367). Use the ordered-local-branch resolution from 22.16. The comparisons on component H^1 are isomorphisms, and the groups on normalized double curves vanish. The same five-term argument shows that $H^1(F,\mathbb{C}) \to H^1(F,\mathcal{O}_F)$ is an isomorphism. For an overlap with rational normalized components and double curves, the argument is identical with zero component H^1. Thus every pairwise overlap in the statement has the required injective comparison, proving (22.2) for the specified assembly.

7. For a semistable S^6 centre, 22.14 gives $H^1(D,\mathbb{C}) = 0$, while upper semicontinuity and the nearby lower bound $h^1(\mathcal{O}) \geq 1$ in 22.15 give $h^1(D,\mathcal{O}_D) \geq 1$. This contradicts (22.2). The obstruction is the stated surjectivity comparison on the blocks and injectivity comparison on their pairwise overlaps.

□

Remark 22.18 (A punctured isomorphism with an essential singularity)**.** [tag PT03] The automorphism

$$(s,[z_0 : z_1]) \longmapsto (s,[e^{1/s}z_0 : z_1])$$

of $\Delta^* \times \mathbb{P}^1$ has no meromorphic extension across zero, although the two product families extend properly over the disc. Its restriction to the constant section $[1 : 1]$ is $[e^{1/s} : 1]$, which has an essential singularity. The proper graph of a meromorphic extension, pulled back to this section and normalized in its dominating component, would give a holomorphic extension from the disc, a contradiction.

Proposition 22.19 (A Hopf surface degeneration and its comparison map)**.** [tag 0317]

There is a proper semistable analytic family of compact complex surfaces $p : \mathcal{H} \to \Delta$ with smooth total space such that:

1. *its punctured restriction is a holomorphic product with the scalar primary Hopf surface*
$$H_- = (\mathbb{C}^2 \setminus \{0\})/\langle -e^{-\pi}\,\mathrm{id}\rangle;$$
2. *its central fibre is $Y = H_1 \cup_E H_2$, where both H_i are isomorphic to*
$$H_+ = (\mathbb{C}^2 \setminus \{0\})/\langle e^{-2\pi}\,\mathrm{id}\rangle, \qquad E = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z});$$
3. *$N_{E/H_1} \simeq N_{E/H_2} \simeq \mathcal{O}_E$, and*
$$\pi_1(Y) = 0, \qquad h^1(Y, \mathcal{O}_Y) = 1, \qquad H_*(Y;\mathbb{Z}) = (\mathbb{Z}, 0, 0, \mathbb{Z}^3, \mathbb{Z}^2);$$
4. *no punctured product isomorphism has a meromorphic extension whose graph is proper over the two total spaces near the central fibres.*

Consequently $\mathcal{H} \times \mathbb{P}^1 \to \Delta$ is a semistable family of compact threefolds with simply connected centre and $h^1(\mathcal{O}) = 1$, whose normalized components fail the $\partial\bar\partial$ lemma. Its smooth fibres have Euler number zero and fundamental group $\mathbb{Z}$.

Proof. 1. We construct the family by an explicit instance of Fujimoto's generalized logarithmic transformation (Fujimoto, 1990, Definition 1.1, pp. 85–86). Let
$$b : \mathcal{B} = \mathrm{Bl}_{(p,0)}(\mathbb{P}^1 \times \Delta) \longrightarrow \mathbb{P}^1 \times \Delta.$$
The central curve is $F_1 \cup F_2$, where F_1 is the strict transform of $\mathbb{P}^1 \times \{0\}$ and F_2 is the exceptional projective line. They meet transversely in one point. The two line bundles
$$L = b^* \,\mathrm{pr}_1^*\, \mathcal{O}_{\mathbb{P}^1}(1), \qquad A = \mathcal{O}_{\mathcal{B}}(F_2)$$
have degree-restriction matrix

	L	A
F_1	1	1
F_2	0	-1

.

In particular this matrix is unimodular. Take nonzero holomorphic transition functions l_{ab}, a_{ab} for these two line bundles. On small intersections choose logarithms and put
$$t_{ab} = \left[\frac{(1+i)\log l_{ab} - i \log a_{ab}}{2\pi i}\right] \in E.$$
This is independent of the logarithm choices, since $1 + i, -i$ belong to the lattice of E. On triple intersections the sum is again a lattice element. The t_{ab} therefore form a holomorphic cocycle with values in the translation group E. Glue $U_a \times E$ by these translations, obtaining a holomorphic principal E-bundle $\mathcal{H} \to \mathcal{B}$. It is a proper submersion, since it is locally a product with a compact elliptic curve. Its total space is smooth and Hausdorff. Compose with $\mathcal{B} \to \Delta$.

2. The Chern vector of the bundle, in the lattice $\mathbb{Z} + i\mathbb{Z}$, is
$$\kappa = (1+i)c_1(L) - ic_1(A).$$
Its restrictions to F_1, F_2 are respectively 1 and i. On a nonzero fibre its restriction is $1 + i$. These statements also follow directly from the displayed transition functions. Locally at the crossing of F_1 and F_2 the base map is $xy = u$, and the principal elliptic bundle is locally trivial. The composed family is thus locally $(x, y, z) \mapsto xy$, with z a coordinate on E. It is flat, has smooth total space and reduced simple normal crossings centre. It is proper because both maps in the composition are proper.

3. For a primitive vector $c \in \mathbb{Z} + i\mathbb{Z}$, choose d so that (c, d) is an oriented lattice basis. The homomorphism
$$\mathbb{C}^* \longrightarrow E, \qquad z \longmapsto \left[\frac{c \log z}{2\pi i}\right]$$
has kernel generated by $\alpha_c = \exp(2\pi i d/c)$, with $|\alpha_c| < 1$. The principal E-bundle on $\mathbb{P}^1$ associated to $\mathcal{O}(1)^\times$ by this homomorphism is the quotient of $\mathcal{O}(1)^\times$ by that cyclic kernel. Fibrewise inversion identifies $\mathcal{O}(1)^\times$ with $\mathcal{O}(-1)^\times \simeq \mathbb{C}^2 \setminus \{0\}$; inverting the cyclic generator does not change its group. The resulting surface is the scalar Hopf surface H_{α_c}. For the three vectors in the construction we may take
$$(c, d) = (1, i), \quad (i, -1), \quad (1 + i, i),$$
giving respectively
$$\alpha_1 = \alpha_i = e^{-2\pi}, \qquad \alpha_{1+i} = \exp(2\pi i(1 + i)/2) = -e^{-\pi}.$$
On $\mathcal{B}|_{\Delta^*} = \mathbb{P}^1 \times \Delta^*$, A has the canonical nowhere-zero section of its exceptional divisor. Its cocycle is therefore a coboundary there, while L is the constant $\mathcal{O}(1)$ bundle. The cocycle coboundary therefore gives a holomorphic product isomorphism with $H_- \times \Delta^*$.

4. Each central $H_i \to F_i$ is a principal elliptic bundle. Its intersection with the other component is a smooth fibre, so a local trivialization over a base disc gives $N_{E/H_i} = \mathcal{O}_E$ exactly. Thus the product of the two normal lines is trivial, with the smoothing section supplied by the coordinate $u = xy$.

5. A scalar Hopf surface is diffeomorphic to $S^1 \times S^3$: divide the radial coordinate by multiplication by $|\alpha|$ and untwist its constant phase rotation along the radial interval. Its fibre inclusion has fundamental-group map
$$\Lambda = \mathbb{Z}^2 \longrightarrow \Lambda/\mathbb{Z}c_i.$$
For $c_1 = (1, 0)$ and $c_2 = (0, 1)$, these are the two coordinate quotients. The van Kampen pushout kills both generators of Λ, and hence $\pi_1(Y) = 0$. In the normalization Mayer–Vietoris sequence the degree-one map, with the second inclusion negated, is
$$H_1(E; \mathbb{Z}) \longrightarrow H_1(H_1; \mathbb{Z}) \oplus H_1(H_2; \mathbb{Z}), \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
It is unimodular. The degree-zero map is the primitive column $(1, -1)^t$, the degree-two target is zero, and $H_3(E) = 0$. The exact sequence now gives
$$H_1Y = H_2Y = 0, \quad 0 \longrightarrow \mathbb{Z}^2 \longrightarrow H_3Y \longrightarrow \mathbb{Z} \longrightarrow 0, \quad H_4Y = \mathbb{Z}^2.$$
All groups displayed in the proposition are therefore free.

6. For coherent cohomology, translations act trivially on $H^1(E, \mathcal{O}_E)$, so for either central principal bundle the sheaf $R^1\pi_*\mathcal{O}$ on $\mathbb{P}^1$ is the constant line bundle. Its Leray sequence shows that restriction
$$H^1(H_i, \mathcal{O}_{H_i}) \longrightarrow H^1(E, \mathcal{O}_E)$$
is an isomorphism. The normalization exact sequence therefore has degree-one map $(s, t) \mapsto s - t$ from $\mathbb{C}^2$ to $\mathbb{C}$. The preceding degree-zero map is surjective. Its kernel has dimension one, proving $h^1(Y, \mathcal{O}_Y) = 1$. This also displays exactly why the overlap-injectivity hypothesis in 22.17 fails: $H^1(E; \mathbb{C}) \to H^1(E, \mathcal{O}_E)$ has a one-dimensional kernel.

7. The two scalar Hopf surfaces used above are not biholomorphic. An isomorphism lifts to an automorphism of $\mathbb{C}^2 \setminus \{0\}$ that intertwines the two cyclic generators or their inverses. Hartogs extension applies to the lift and its inverse; both fix zero and have invertible derivative there. Differentiating the intertwining identity gives $e^{-2\pi} = (-e^{-\pi})^{\pm 1}$, which is false. They have no rational curves, since a map $\mathbb{P}^1 \to H_\alpha$ lifts to $\mathbb{C}^2 \setminus \{0\}$ and is constant. The usual resolution of a bimeromorphic map of smooth compact surfaces therefore also shows that these two Hopf surfaces are not bimeromorphic. This agrees with the explicit scalar case of Abasheva and Déev (2024, arXiv version 1, Section 2.2, pp. 4–5).

8. The fundamental groups also exclude a meromorphic extension of the punctured product comparison. A sufficiently small closed semistable neighbourhood in $\mathcal{H}$ retracts onto Y, by the monomial-gradient argument of 22.14. Its fundamental group is zero. The product neighbourhood $H_- \times \Delta$ has fundamental group $\mathbb{Z}$. If the punctured comparison had a proper meromorphic graph, resolving that graph would give a common smooth proper modification of the two neighbourhoods. A proper bimeromorphic map between smooth complex manifolds preserves π_1, by the codimension-two avoidance argument in 21.16. This is a contradiction. Both models already have the indicated holomorphic map to the same $\mathbb{P}^1$; the example therefore retains an extended base map while the total-space comparison fails.

9. Finally take the product with $\mathbb{P}^1$. Its central integral Betti numbers are

$$(1, 0, 1, 3, 2, 3, 2),$$

and $h^1(\mathcal{O})$ remains one. The nearby fibre is $H_- \times \mathbb{P}^1$, of fundamental group $\mathbb{Z}$ and Euler number zero. The resulting fundamental group and Euler number are the stated invariants of these smooth fibres.

□

Remark 22.20 (Elliptic-bundle and Hopf degeneration inputs)**.** [tag PT10] The Cartier-divisor construction used in Proposition 22.19 is the construction of Fujimoto (1990, construction, p. 85; Definition 1.1, p. 86). His Remark 1.4 attributes non-Kähler torus degenerations to Ueno. His Proposition 1.14, pp. 92–93, extends principal elliptic bundles across subsets of codimension at least three; Example 1.15, pp. 93–94, gives a principal-bundle extension failure in codimension two. Hopf components occur in the semistable degenerations of Nishiguchi (1988, Theorem 1.4, preprint p. 7, and Example 2.1, p. 13). The diagonal Hopf quotients, elliptic-conductor identifications and normal lines in Schröer and Siebert (2002, Propositions 3.1–3.4, preprint pp. 6–9) provide related explicit models. Proposition 22.19 proves its own proper family using the stated Cartier-divisor cocycle.

Remark 22.21 (Normal degrees along the exceptional ruling)**.** [tag PT11] For the divisor $D = E \times \mathbb{P}^1$ in $H_i \times \mathbb{P}^1$, the normal line is pulled back from E and has degree zero on every ruling fibre. For the exceptional divisor obtained by blowing up a smooth elliptic curve in a threefold, the tautological normal restricts to $\mathcal{O}_{\mathbb{P}^1}(-1)$ on that fibre. Restricting the normal-crossing identity

$$N_{D/A} \otimes N_{D/B} \otimes \mathcal{O}_D(T) \simeq \mathcal{O}_D$$

to the ruling therefore gives $-1 + 0 + \deg(T|_{\mathbb{P}^1}) = 0$. In particular an empty triple divisor has the wrong degree, whereas the section $T = E \times \{p\}$ contributes exactly $+1$. Proposition 22.27 computes the full line bundles, including their elliptic factors.

Proposition 22.22 (Generic simplicity and the distinguished orbit)**.** [tag PT12] *Fix an ordinary base point at which $\mu \notin \mathbb{Q}\tau + \mathbb{Q}$. Such base points exist. Outside a countable set of admissible values of the parameter β, the corresponding two-dimensional complex torus is simple. For every admissible β at this base point, the distinguished line $\mathbb{C}e$, with $e = \Pi\widehat{\delta} = (0,1)^t$, meets the period lattice in $\mathbb{Z}e$. The induced orbit map is the injective holomorphic group map*

$$\iota : \mathbb{C}^* \longrightarrow \mathbb{C}^2/\Pi\Lambda, \qquad \iota(e^{2\pi i z}) = [ze].$$

Here $[z] \mapsto e^{2\pi i z}$ identifies $\mathbb{C}/\mathbb{Z}$ with the source.

Proof. On an ordinary uniformizing chart the equations $\mu = r\tau + s$, for $(r,s) \in \mathbb{Q}^2$, define countably many analytic loci. If they covered the chart, Baire's theorem would force one equation on an open set and hence identically. The marked elliptic section would then be torsion, contrary to the construction. Thus the required base point exists.

For integral vectors $v = (a,b,c,d)$ and $v' = (A,B,C,D)$,

$$\Pi v = (6a\mu + b\tau + c,\ a\beta + b\mu + d).$$

The determinant $\det(\Pi v, \Pi v')$ is affine in β, with coefficient

$$(Ab - aB)\tau + (Ac - aC).$$

If the coefficient vanishes and $a \neq 0$, the nonreal number τ gives $B = Ab/a$ and $C = Ac/a$. The remaining determinant is $(6a\mu + b\tau + c)(D - Ad/a)$. Its first factor is nonzero by the rational independence of $1, \tau, \mu$, and its second factor vanishes only when v, v' are rationally dependent. The case $a = 0$, $A \neq 0$ follows by exchanging the vectors. For $a = A = 0$, the determinant is

$$(bD - Bd)\tau + (cB - Cb)\mu + (cD - Cd).$$

Rational independence makes this zero only when all the two-by-two minors vanish, again giving dependence. Therefore an independent integral pair has zero determinant for at most one value of β. There are countably many such pairs.

A complex line defining an elliptic subtorus contains two real-linearly independent periods. Conversely, any such period pair spans a complex line with a rank-two lattice; admissibility makes the real period map an isomorphism. Avoiding the countable exceptional values therefore gives simplicity. The equation $6a\mu + b\tau + c = 0$ forces $a = b = c = 0$, which proves $\mathbb{C}e \cap \Pi\Lambda = \mathbb{Z}e$ and the claimed orbit map.

For comparison with an elliptic-bundle fibre, every holomorphic map from an elliptic curve to a complex torus lifts to a map $\mathbb{C} \to \mathbb{C}^2$ whose derivative is doubly periodic. Compactness makes that derivative constant. A nonconstant such map consequently has image a translate of an elliptic subtorus. Thus a simple torus obtained above has no elliptic-curve fibre of this kind. □

Remark 22.23 (The distinguished orbit on an ordinary torus)**.** [tag PT04] The distinguished nonconstant orbit map $\mathbb{C}^* \to T^2$ on an ordinary torus has no meromorphic extension to $\mathbb{P}^1$. A meromorphic map from a smooth curve to a compact complex manifold extends holomorphically at its isolated indeterminacies, by normalization of its proper graph. A holomorphic map $\mathbb{P}^1 \to T^2$ lifts to $\mathbb{C}^2$ and is constant. This contradicts the nonconstant orbit map.

Proposition 22.24 (Normal-line obstruction at good reduction)**.** [tag 0318]

Let a, b be holomorphic units on a disc Δ_q, with $0 < |a(q)| < 1$, and let $m \neq 0$ be an integer. Put

$$\mathcal{E}_a = (\mathbb{C}^* \times \Delta_q)/((z,q) \sim (a(q)z, q)).$$

This is a smooth proper elliptic family over the full disc. Over Δ_q^* define a line bundle $\mathcal{L}_m$ by

$$(z, v, q) \sim \big(a(q)z,\ q^m b(q) v,\ q\big).$$

Its restriction to each elliptic fibre has degree zero. After any finite base change $q = s^N\varepsilon(s)$, where $N > 0$ and ε is a holomorphic unit, the following extension is impossible: a proper semistable family $p : \mathcal{C} \to \Delta_s$ of connected curves, with smooth total surface, an isomorphism $\mathcal{C}|_{\Delta_s^*} \simeq \mathcal{E}_a|_{\Delta_s^*}$, and a line bundle on $\mathcal{C}$ restricting to $\mathcal{L}_m$ under that isomorphism.

The assertion allows any finite collection of rational tails in the central curve and arbitrary translations in the punctured isomorphism. It remains true after tensoring $\mathcal{L}_m$ by the restriction of a line bundle extending on the interface.

In particular the axis $z_1 = 0$ in the diagonal Hopf surfaces

$$H_{q,a(q)} = (\mathbb{C}^2 \setminus \{0\})/\langle (z_1, z_2) \mapsto (qz_1, a(q)z_2)\rangle$$

cannot acquire such a semistable completion together with its normal line, after any finite base change.

Proof. 1. Choose a logarithm of a on the disc and set $\tau_a = \log a/(2\pi i)$. Then $\Im\tau_a > 0$, and

$$\mathcal{E}_a \simeq (\mathbb{C} \times \Delta_q)/(\mathbb{Z} + \tau_a(q)\mathbb{Z}).$$

The real coordinates $z = \exp(2\pi i(x + \tau_a(q)y))$ give a smooth trivialization by the marked torus $(x, y) \in \mathbb{R}^2/\mathbb{Z}^2$. In particular its monodromy on H_1 is the identity. The line factors in this marking are 1 along the x cycle and $q^m b(q)$ along the y cycle. On a positive parameter circle, the latter factor has winding m. Equivalently, the clutching calculation gives

$$\langle c_1(\mathcal{L}_m), [T_x]\rangle = 0, \qquad \langle c_1(\mathcal{L}_m), [T_y]\rangle = m, \qquad \langle c_1(\mathcal{L}_m), [\mathcal{E}_{a,q}]\rangle = 0.$$

Here T_x, T_y sweep the indicated cycles around the parameter circle; the orientation of T_y is chosen to agree with the clutching convention in the displayed line factor. Reversing that convention changes both sides' sign and has no effect on nonvanishing. One can also compute the integer on the square $[0, 1]_y \times S^1_q$: the two y edges are glued by $q^m b(q)$, whose degree is m.

2. Under $q = s^N\varepsilon(s)$ the degree becomes mN. The unit $\varepsilon(s)^m b(q(s))$ has zero winding. Thus the mixed Chern class is the nonzero covector

$$(0, mN) \in H^1(\mathcal{E}_{a,q}; \mathbb{Z}) \otimes H^1(S^1_s; \mathbb{Z}).$$

3. Suppose an extension $\mathcal{C}$ existed. Its central fibre is connected. Flatness and properness give arithmetic genus one, and the normalization exact sequence gives

$$1 = \sum_i g(C_i) + b_1(\Gamma),$$

where C_i are the smooth normalization components and Γ is the dual graph, including a loop for a self-node. Consequently either there is a unique elliptic component and Γ is a tree, or all components are rational and Γ has one cycle.

4. The second case has nontrivial monodromy. This can be seen directly in the local plumbing $uv = s$. Transport around $s = 0$ is a positive Dehn twist on its neck annulus. An edge outside the unique graph cycle gives a separating neck and acts trivially on H_1. Each of the $k \geq 1$ edges in the graph cycle gives a nonseparating neck. These disjoint necks represent the same primitive class up to sign on the smoothed genus-one surface. The transvection is unchanged by reversing that class. Their product therefore has, in a symplectic basis, the matrix

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

or its inverse under the opposite monodromy convention. It is not the identity. The given punctured elliptic family has identity monodromy, so this case is impossible.

5. There is therefore an elliptic core C_0 with rational trees attached at finitely many points. Choose two oriented loops on C_0 avoiding those points and with intersection number one. The map p is a submersion near their union. Compactness of the loops and a smooth horizontal distribution provide, after shrinking the disc, families of loops over the full disc. Each swept torus on its boundary bounds the resulting solid torus in $\mathcal{C}$. Hence the first Chern class of every line bundle on $\mathcal{C}$ evaluates to zero on both such swept tori.

6. The two transported loops form a basis of the nearby elliptic fibre's H_1: cutting off the rational trees adds only genus-zero pieces along separating necks, and their intersection number remains one. Under the given punctured isomorphism, their swept tori therefore project to a basis of the mixed group $H_1(\mathcal{E}_{a,q};\mathbb{Z}) \otimes H_1(S^1_s;\mathbb{Z})$. They may differ from the marked swept tori by integral fibre classes. This accounts in particular for an arbitrary loop of fibre translations in the isomorphism. The fibre degree of $\mathcal{L}_m$ is zero, so these additional fibre classes do not change either pairing. The nonzero covector $(0, mN)$ cannot vanish on both basis vectors. This contradicts the bounding-solid-torus calculation and proves the assertion.

7. Finally an extending tensor factor has zero pairing on both bounding tori by the same argument. It cannot cancel the obstruction. Vertical divisors and any Cartier triple divisor on such an extended interface are included among these harmless tensor factors.

8. For the stated Hopf consequence, the axis is precisely $\mathcal{E}_{a,q}$. The derivative of the deck transformation in the z_1 direction multiplies a normal vector by q; its action on the axis coordinate is multiplication by $a(q)$. These are exactly the two factors defining $\mathcal{L}_1$. This is also the explicit diagonal-Hopf normal-line convention of Abasheva and Déev (2024, arXiv version 1, Section 2.2, pp. 4–5). The assertion just proved applies with $m = 1$ and $b = 1$. It also applies to the first multiplier $q^m b(q)$ for any positive m. At the cusp the nonzero unit $e^{2\pi i \mu(q)}$ has absolute value different from one at $q = 0$. Indeed the nodal cubic is $Y^2 = (X-1)^2(X+2)$; its normalization $X = h^2 - 2$, $Y = h(h^2-3)$ and the multiplicative coordinate $(h-\sqrt{3})/(h+\sqrt{3})$ show that the marked section at $(X, Y) = (0, \pm\sqrt{2})$ has multiplier one of $-(5 \pm 2\sqrt{6})$. Choose that multiplier or its inverse to obtain such a contracting $a(q)$. This proves good reduction for this axis.

□

Remark 22.25 (The individual mixed Chern class)**.** [tag PT13] The evaluation isomorphism $\mathcal{L}_m \otimes \mathcal{L}_m^{-1} \simeq \mathcal{O}$ cancels the two mixed first Chern classes. Individually, Proposition 22.24 computes the nonzero covector $(0, mN)$ on the two swept tori after a parameter cover of order N. Both swept tori bound solid tori on an extending semistable good-reduction interface. Hence the restriction of the first Chern class of every extending line evaluates to zero on both, whereas $c_1(\mathcal{L}_m)$ does not. This is the explicit restriction map relating the individual obstruction to the tensor-product normal equation. In the constant elliptic family of Proposition 22.19, the corresponding mixed class is zero.

Proposition 22.26 (Ordinary torus cocycles and descent classes)**.** [tag 0319]

Let $\mathcal{T} \to U$, $U = \Delta_q^* \times \Delta_r^*$, be the Jacobian two-torus family of the marked cusp, after $w = r^6$. A holomorphic homomorphism of group families

$$\mathbb{C}^* \times U \longrightarrow \mathcal{T}$$

is determined by a flat integral cycle. For this family all such cycles are

$$c = a\widehat{w} + b\widehat{\delta}, \qquad a, b \in \mathbb{Z}.$$

In the exponential cusp coordinates its homomorphism is $z \mapsto (z^a, z^b)$. Thus the ordinary Cartier logarithm construction along a globally specified cocharacter has transition maps

$$(Z, W) \longmapsto (g_{ij}^a Z, g_{ij}^b W).$$

If these units extend over the prescribed fan charts, the maps extend there and preserve each toric boundary component. These transition units preserve the component permutations $+4, -3, 0$ *of 22.5. For the sheaf* $\mathscr{T}$ *of holomorphic sections of* $\mathcal{T}$*,*

$$H^1(U, \mathscr{T}) \simeq H^2(U, \Lambda) \simeq \mathbb{Z}^2.$$

The last group is represented by the coinvariant classes of $\widehat{\gamma}, \widehat{u}$*. The ordinary fixed-cocharacter Cartier constructions have zero image in this group. The cusp family with the coherent closing translations of 17.8 also has zero torsor class, by the explicit holomorphic section below. A marked torsor isomorphism preserves this cohomology class, so a torsor with a nonzero class cannot be isomorphic to this marked cusp torsor.*

Proof. 1. First consider any smooth holomorphic family of complex tori with integral period local system Λ and relative Lie bundle $\mathcal{V}$. Lift a group homomorphism from $\mathbb{C}^*$ along the universal covering $z = \exp(2\pi i t)$. The lifted homomorphism from the additive group $\mathbb{C}$ to the universal covering vector space of each torus is $t \mapsto tv$ for a holomorphic section v of $\mathcal{V}$. Periodicity under $t \mapsto t + 1$ says precisely that v belongs to the period lattice in every fibre. In any local marking its integral coefficients are continuous, and hence locally constant. It therefore defines a flat global section of Λ. Conversely any such section gives the homomorphism by this formula. The image need not be a closed elliptic subgroup; this argument applies without a polarization and also to simple two-tori.

2. In the ordered basis $(\widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta})$, the commuting linear monodromies have nilpotent parts

$$N_q = A_0 - I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad N_r = -12\widehat{\delta} \otimes \gamma.$$

   The sign of N_r is the positive-loop closing convention of 17.8; reversing it leaves all groups below unchanged. Their common integral kernel is exactly $\mathbb{Z}\widehat{w} \oplus \mathbb{Z}\widehat{\delta}$. The two corresponding columns of Π_c are $(1, 0)^t, (0, 1)^t$. Exponentiating the homomorphism therefore gives (z^a, z^b), as asserted.

3. Fujimoto's logarithmic cocycle formula (Fujimoto, 1990, construction, p. 85; Definition 1.1, p. 86) works in the same way for a complex torus of any dimension: apply the displayed homomorphism to the transition functions g_{ij} of a line bundle. Equivalently use $c \log(g_{ij})/(2\pi i)$ modulo the period lattice. A different logarithm adds an integral cycle, and the triple-overlap defect is an integral cycle as well. With the present common invariant cycles this is exactly multiplication by (g_{ij}^a, g_{ij}^b). Torus multiplication extends to every toric variety of a fixed fan and preserves every torus orbit and boundary ray. For an extending unit it is an automorphism on the full chart. Consequently it contributes no permutation of the central component labels. The product-of-permutations obstruction of 22.5 is unchanged. This conclusion keeps the fan, period action, and local component models fixed.

4. Retract U to its real two-torus. The cellular cochain complex with coefficients in the local system is the two-generator Koszul complex. Its last differential has image $N_q\Lambda + N_r\Lambda$, and hence

$$H^2(U, \Lambda) = \Lambda/(N_q\Lambda + N_r\Lambda) = \Lambda/(\mathbb{Z}\widehat{w} + \mathbb{Z}\widehat{\delta}) \simeq \mathbb{Z}\widehat{\gamma} \oplus \mathbb{Z}\widehat{u}.$$

   The relation lattice is primitive: N_q already has image $\mathbb{Z}\widehat{w} + \mathbb{Z}\widehat{\delta}$.

5. The analytic exponential sequence is

$$0 \longrightarrow \Lambda \longrightarrow \mathcal{O}(\mathcal{V}) \longrightarrow \mathscr{T} \longrightarrow 0.$$

The product of two punctured discs is Stein, and $\mathcal{O}(\mathcal{V})$ is coherent. Cartan's theorem B therefore kills its positive cohomology (Cartan, 1952, exposé 19, §5, pp. 14–15). The long exact sequence gives the stated isomorphism $H^1(U,\mathscr{T}) \simeq H^2(U,\Lambda)$. The representatives of $H^1(U,\mathscr{T})$ give holomorphic torsors on U by translation gluing.

6. For the last assertion, naturality of the two exponential sequences sends the first Chern class of a Cartier line bundle to its multiple of the flat cycle c in $H^2(U,\Lambda)$. Both common invariant cycles lie in $N_q\Lambda$, so this multiple is zero.

7. We compute the affine class using the cusp functions of 17.8,

$$q = e^{2\pi i s}, \qquad C_0(q) = \beta^{\mathrm{part}} + s, \qquad \ell = -s/12 + \kappa_0(q), \qquad D(q) = C_0(q) - 12\kappa_0(q).$$

The functions μ, C_0, κ_0 are single-valued and holomorphic at $q = 0$. The cusp unit representative of the even-shift translation is

$$\bar{b}_{h,c}(q) = \left(-\frac{h}{2}\mu(q),\ -\frac{h}{12}(c + D(q)) - \frac{h^2}{24} \right).$$

On the logarithmic parameter cover define

$$\sigma(q,c) = \left(-\frac{c}{2}\mu(q),\ -\frac{c^2}{24} - \frac{cD(q)}{12} \right).$$

Direct subtraction gives the exact identity

$$\sigma(q,c+h) - \sigma(q,c) = \bar{b}_{h,c}(q).$$

In particular it is equivariant for the deck transformation $c \mapsto c - 12$ corresponding to a positive r loop, and it is single-valued in q. The original cusp has no extra affine q translation ($l_0 = 0$). Thus σ descends to a holomorphic section of the affine torus family. Equivalently its q and r affine lifts commute exactly: the r translation $\bar{b}_{-12,c}$ is independent of the logarithm branch s. The resulting integral commutator representing the Chern class is zero already in Λ, before passage to coinvariants.

8. Multiplication by the exponential of σ is a torus-unit automorphism of every cusp fan chart for a fixed nonzero parameter. Since its coefficients are holomorphic in the full q disc, this comparison includes the prescribed Mumford cusp fibre for $r \neq 0$. The displayed comparison is defined on the full q disc for $r \neq 0$, with $c = -(12/(2\pi i))\log r$ on its parameter cover. Changing coherent closings by a scalar depending only on the parameter has zero q commutator and leaves this Chern-class conclusion unchanged.

□

Proposition 22.27 (Parabolic families and exceptional divisors)**.** [tag 030E]

Fix an integer $n \geq 1$ and holomorphic units $a(s), \kappa(s)$ on a disc, with $0 < |a(s)| < 1$. After shrinking the disc there is a proper holomorphic submersion $\mathcal{K} \to \Delta_s$, with an embedded smooth proper elliptic family

$$\mathcal{E} = (\mathbb{C}^* \times \Delta_s)/((v,s) \sim (a(s)v, s)).$$

Its central surface is a parabolic Inoue surface S_n, with $\pi_1(S_n) = \mathbb{Z}$, $b_2(S_n) = n$, and elliptic axis of square $-n$. For $s \neq 0$, the surface $\mathcal{K}_s$ is obtained from the diagonal Hopf surface $H_{\kappa(s)s^n, a(s)}$ by n successive blow-ups at one axis point and its indicated infinitely near strict-axis points. In a fixed annulus marking of $\mathcal{E}$, its normal line has factor

$$g_s(v) = \kappa(s)(a(s)v + s)^n.$$

This line extends across $s = 0$ and is isomorphic to the holomorphic family with factor $\kappa(s)a(s)^n v^n$. For constant a, κ it is a constant family of degree $-n$ lines.

The relation with the normal line of the unmodified Hopf axis is

$$N_{\mathcal{E}_s/\mathcal{K}_s} = L_{\kappa(s)s^n} \otimes \mathcal{O}_{\mathcal{E}_s}(-n[-s]), \qquad s \neq 0.$$

Here $n[-s]$ is the total exceptional correction, with the multiplicity of each successive blow-up retained. It has no Cartier extension on $\mathcal{E}$, even after a finite base change or the addition of a vertical component. The marked divisor used to change the normal line therefore fails the extension condition in Proposition 22.24.

For $n = 8$, the degree -8 axis normal can be chosen to be the inverse of $\mathcal{O}_E(2)$ on any chosen smooth elliptic complete intersection $E = (2, 2) \subset \mathbb{P}^3$. This supplies the normal-line matching with $\mathrm{Bl}_E \mathbb{P}^3$ described below, including one positive triple section. The matching is the displayed isomorphism of the two normal lines, with the stated triple-curve factor.

Proof. 1. Let $B_R \subset \mathbb{C}^2$ be a ball, and let $\pi_n : \widehat{B}_R \to B_R$ be the following modification. Blow up the origin, then successively the intersection of the newest exceptional curve with the strict transform of the axis $x = 0$. In the last chart,

$$\pi_n(z, w) = (zw^n, w).$$

Define the parameter-dependent embedding in that chart by

$$\sigma_s(z, w) = \big(\kappa(s)z,\ a(s)w + (1 - a(s))s\big).$$

Choose constants $A < \rho < 1$ and $K > 0$ with $|a(s)| \leq A$ and $|\kappa(s)| \leq K$ on a closed smaller disc. Shrink it further so that $|a(s)w + (1 - a(s))s| \leq \rho R$ on $\overline{B}_R$. Then

$$|\pi_n \sigma_s(z, w)|^2 \leq K^2 R^2 (\rho R)^{2n} + \rho^2 R^2 < R^2$$

when R is small enough. The last-chart coordinates of these images are bounded, so this is compact containment of $\sigma_s(\overline{B}_R)$ in $\widehat{B}_R$, uniformly in s. For each finite bound on $|\kappa|$, these inequalities give such a radius R.

2. Remove the embedded ball and identify the outer and inner collars by $\sigma_s \pi_n$. The outer collar avoids the exceptional locus, and π_n is a biholomorphism there; thus the collar identification is a biholomorphism. Carry out the same construction over the parameter disc. The embeddings and the identifications are holomorphic and preserve s. Uniformly disjoint collars give a Hausdorff smooth threefold, and the projection is a holomorphic submersion in every chart. Uniform compact fundamental regions over compact parameter sets prove properness. This is the usual Kato construction, with the explicit moving embedding above; see Istrati et al. (2022, Section 2, pp.1351–1353). The general occurrence of deformations from Kato surfaces to blown-up Hopf surfaces is classical; see Teleman (2010, Introduction, pp.1750–1751). We use explicit formulas here to retain the elliptic marking and its exceptional correction.

3. The contracted map $\pi_n \sigma_s$ fixes $(0, s)$. In the coordinate $v = w - s$ it becomes

$$f_s(x, v) = \big(\kappa(s)x(a(s)v + s)^n,\ a(s)v\big).$$

At $s = 0$ this is the parabolic monomial germ with exponent matrix $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. The chosen modification is exactly the n-fold modification in Istrati et al. (2022, Example 9.10, p.1386). Consequently the central surface is the stated parabolic Inoue surface; its elliptic axis is disjoint from the rational cycle. The construction gives $\pi_1 = \mathbb{Z}$ and $b_2 = n$, or equivalently one can use the cited description of this surface.

4. The strict axis is preserved in the collar gluing. Its coordinate v is multiplied by $a(s)$. To see its full quotient, the axis disc in the blowdown coordinates is

$$D_s = \{v : |v + s| < R\}.$$

For small s, it contains 0 and satisfies $a(s)D_s \Subset D_s$. Its dilates have union $\mathbb{C}$ and shrinking intersection $\{0\}$. Thus $D_s \setminus a(s)D_s$, with the two boundary collars glued, is a fundamental annulus for $\mathbb{C}^*/a(s)^{\mathbb{Z}}$. This proves the exact elliptic marking throughout the parameter disc.

5. There are two normal frames, whose distinction is essential. In the last blow-up chart the collar map $\sigma_s\pi_n$ has factor

$$g_z(v) = \kappa(s)(v + s)^n.$$

On a fixed annulus avoiding $v = 0$, shrink s so that both $v + s$ and $a(s)v + s$ are nonzero. The blowdown normal coordinate is $x = z(v + s)^n$, and in that frame the factor is

$$g_x(v) = \kappa(s)(a(s)v + s)^n.$$

The change between the frames is a nonzero gauge on these annuli. The central factor is $\kappa(0)a(0)^n v^n$ and has degree $-n$, with the factor-of-automorphy convention of Istrati et al. (2022, Proposition 5.6, pp.1369–1370).

6. An explicit regular gauge verifies the whole family. On a fixed annulus and its overlap with its $a(s)$-translate put

$$H_s(v) = \prod_{j\geq 0} \left(1 + \frac{a(s)^j s}{v}\right)^n.$$

The product converges locally uniformly in (v, s) and is nonzero after shrinking the parameter disc. It satisfies

$$\frac{H_s(a(s)v)}{H_s(v)} = \left(1 + \frac{s}{a(s)v}\right)^n.$$

Replacing x by $x/H_s(v)$ changes g_x to $\kappa(s)a(s)^n v^n$. The collar maps are linear in the normal coordinate, so the same calculation identifies a tubular line neighbourhood.

7. The generic Hopf blowdown also has an explicit developing map. Fix $s \neq 0$ and set

$$G_s(v) = \prod_{j\geq 1} \left(1 + \frac{a(s)^j v}{s}\right)^n, \qquad \Psi_s(x, v) = (xG_s(v), v).$$

This product is entire in v, converges locally uniformly, and obeys

$$G_s(a(s)v) = \frac{G_s(v)}{(1 + a(s)v/s)^n}, \qquad \Psi_s f_s = \operatorname{diag}(\kappa(s)s^n, a(s))\Psi_s.$$

For sufficiently small nonzero s, both diagonal multipliers have absolute value less than one. Compose Ψ_s with π_n on a Kato fundamental region. The displayed equivariance glues it to a holomorphic map to $H_{\kappa(s)s^n, a(s)}$. It never reaches the excluded origin: that would require $x = v = 0$, which is the fixed point $(0, s)$ removed with the inner ball. All zero-factor curves of G_s have $v = -s/a(s)^j$, so they lie in the one axis orbit $[-s]$. The original modification also contracts over $v = -s$. At that representative,

$$G_s(-s) = \prod_{j\geq 1} (1 - a(s)^j)^n \neq 0.$$

Hence locally the map is precisely π_n followed by a coordinate unit. Away from that orbit it is a local biholomorphism of degree one. Compactness makes it a proper modification, and the displayed local model identifies it as the stated n successive blow-ups. The developing map Ψ_s is defined on the punctured parameter disc $s \neq 0$.

8. Each blow-up of a point on the strict elliptic curve subtracts that point from its normal line. Following all n stages gives

$$N_s = L_{\kappa(s)s^n} \otimes \mathcal{O}_{\mathcal{E}_s}(-n[-s]).$$

The multiplicity n records total exceptional transforms. The reduced final exceptional locus meets the strict axis only once. One may check the formula directly: the standard theta factor $\theta_a(av) = -v^{-1}\theta_a(v)$ shows that $\mathcal{O}([p])$ has factor $-p/v$. At $p = -s$ the tensor product above has factor

$$\kappa(s)s^n(v/s)^n = \kappa(s)v^n,$$

which is equivalent to $\kappa(s)a(s)^n v^n$ by a nonzero monomial gauge. This checks the signs and degrees independently of the successive-blow-up formula.

9. The divisor $n[-s]$ cannot extend. For constant a, choose two distinct classes $[v_1], [v_2] \in \mathbb{C}^*/a^{\mathbb{Z}}$. The sequences $s_j = -a^j v_i$ tend to zero, whereas the corresponding divisors are constantly $n[v_i]$ and have different limits in $\mathrm{Sym}^n(E_a)$. For varying $a(s)$, choose the holomorphic logarithm $\tau_a(s) = \log a(s)/(2\pi i)$ on the disc. The Abel–Jacobi coordinate of $n[-s]$ is $n\log(-s)/(2\pi i)$ modulo $\mathbb{Z} + \tau_a(s)\mathbb{Z}$. On a positive parameter loop it winds n times around a primitive cycle in the smooth elliptic family. An extending section over the disc would have zero winding. A finite base change of order N multiplies this nonzero integer by N. After removing a vertical Cartier elliptic component, an extending divisor would still have a proper horizontal part of degree n, finite over the disc, defining a continuous section of the relative symmetric power, which has just been ruled out. Nonproper exceptional divisors in deformations of Kato surfaces are a classical phenomenon, studied by Dloussky and Teleman (2012, arXiv version 2, Section 1.2, PDF pp. 3–5); the calculation here identifies the precise weighted divisor in this marked family.

10. This also explains the scope of ordinary elementary modifications. If a degree-zero line L_m from 22.24 is changed by point blow-ups, then its new normal is $L_m \otimes \mathcal{O}(-D_{\mathrm{exc}})$. If both that normal and the weighted Cartier correction extended on a semistable good-reduction interface, their tensor product would extend L_m, contradicting 22.24. In this Kato family the correction is exactly $D_{\mathrm{exc}} = n[-s]$, whose nonextension was proved above.

11. Finally let $E = (2,2) \subset \mathbb{P}^3$ be smooth and set $L = \mathcal{O}_E(2)$, of degree eight. Its normal in $\mathbb{P}^3$ is $L \oplus L$. For $A = \mathrm{Bl}_E \mathbb{P}^3$, the exceptional divisor is $D = E \times \mathbb{P}^1$ and, in the lines convention,

$$N_{D/A} = L \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1).$$

A bare Hopf-product cap has normal $M \boxtimes \mathcal{O}$ with $\deg M = 0$. The required triple-divisor line would then be $(L \otimes M)^{-1} \boxtimes \mathcal{O}(1)$, which has no nonzero section because its elliptic degree is -8. Thus the required triple-divisor line has no effective section on this interface.

12. For the parabolic surface with $n = 8$, changing κ varies the flat factor through all of $\mathrm{Pic}^0(E)$: constant factors are parametrized by $\mathbb{C}^*/a^{\mathbb{Z}}$. Choose its axis normal to equal L^{-1}. In $B = S_8 \times \mathbb{P}^1$ the divisor $D = E \times \mathbb{P}^1$ then has normal $L^{-1} \boxtimes \mathcal{O}$. With triple section $T = E \times \{p\}$,

$$N_{D/A} \otimes N_{D/B} \otimes \mathcal{O}_D(T) = (L \boxtimes \mathcal{O}(-1)) \otimes (L^{-1} \boxtimes \mathcal{O}) \otimes (\mathcal{O} \boxtimes \mathcal{O}(1)) \simeq \mathcal{O}_D.$$

This proves precisely the first-order matching claimed.

□

Proposition 22.28 (Globalization at the contracted parabolic elliptic axis)**.** [tag 030F]

Let $S = S_8(a,\kappa)$ be the central parabolic surface in 22.27, let $E \subset S$ be its elliptic axis, and put $N = N_{E/S}$, so $\deg N = -8$. The contraction $\pi : S \to Y$ of E is a compact normal surface with a single isolated singular point p. Its germ at p is the vertex of the affine cone over (E, N^{-1}). In particular,

$$H^2(Y, \Theta_Y) = 0.$$

The restriction morphism from a semiuniversal deformation of Y to a semiuniversal deformation of (Y,p) is regular. Consequently every prescribed analytic one-parameter deformation of this isolated singularity is induced, after shrinking its parameter disc, by a proper deformation of the compact surface Y. If the prescribed local deformation smooths the singularity, the global deformation has smooth fibres away from the central value.

Proof. 1. At the central parameter the Kato gluing of 22.27 is

$$(x, v) \longmapsto (\kappa a^8 x v^8, av).$$

It is exactly linear in x. Thus a neighbourhood of E is a neighbourhood of the zero section in the negative line bundle N, with the indicated factor of automorphy. Its contraction is the cone associated with the section ring

$$R(E, N^{-1}) = \bigoplus_{m \geq 0} H^0(E, N^{-m}).$$

A line bundle of degree eight on an elliptic curve is very ample and normally generated. The cone is normal, its only singularity is its vertex, and the blow-up of the vertex maximal ideal is $\mathrm{Tot}(N)$ near its zero section. One can see the last assertion directly: the degree-one sections generate the section ring and have no common zero on E; their ratios on the blow-up recover the projective embedding of E, and their common radial coordinate is the coordinate in N. In particular, with $I = \mathcal{O}_S(-E)$ and $\mathfrak{m}_p$ the vertex maximal ideal,

$$\mathfrak{m}_p \mathcal{O}_S = I.$$

The local contraction glues to the identity on $S \setminus E$, giving the stated proper contraction of the compact surface.

2. Write C for the disjoint rational cycle of the parabolic surface. The meromorphic two-form $d\log x \wedge d\log v$ is invariant under the displayed monomial gluing. In the toric blow-up charts it has simple poles along precisely the axis and the rational cycle, and no zeros. Consequently

$$K_S = \mathcal{O}_S(-E - C).$$

Also $b_1(S) = 1$, by 22.27. Put $\mathcal{F} = \Theta_S(-\log E)$, the locally free sheaf of vector fields tangent to E. Serre duality and the displayed canonical divisor give

$$H^2(S, \mathcal{F})^* = H^0\big(S, \Omega^1_S(\log E) \otimes K_S\big) = 0.$$

Indeed there are sheaf injections

$$\Omega^1_S(\log E)(-E - C) \hookrightarrow \Omega^1_S(-C) \hookrightarrow \Omega^1_S.$$

Locally the first sends the logarithmic generators $dx/x, dy$, multiplied by x, to the ordinary forms dx, xdy. Finally $H^0(S, \Omega^1_S) = 0$ follows already from $b_1(S) = 1$. A holomorphic one-form α on a compact complex surface is closed: Stokes' theorem applied to $\alpha \wedge \overline{d\alpha}$ makes the integral of the positive volume form associated with $d\alpha \wedge \overline{d\alpha}$ vanish. If $\alpha \neq 0$, its real and imaginary parts give independent real de Rham classes. An exact real linear combination would be the differential of a global real pluriharmonic function, which is constant on the compact surface. This would give $b_1(S) \geq 2$, a contradiction.

3. The restriction of $\mathcal{F}$ to E is the Atiyah extension of N:

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{A}_N := \mathcal{F}|_E \longrightarrow \Theta_E \longrightarrow 0.$$

In normal coordinates related by $x_j = g_{ij}(v)x_i$, the vertical Euler field is $x_i\partial_{x_i}$, and the changes of horizontal lifts have cocycle $d\log g_{ij}$. Thus this extension has class $c_1(N)$ in $H^1(E, \Omega^1_E)$, which is nonzero because $\deg N = -8$. Since $\Theta_E \simeq \mathcal{O}_E$, the connecting map $H^0(E, \Theta_E) \to H^1(E, \mathcal{O}_E)$ is an isomorphism. The same cohomology sequence therefore gives the natural isomorphism

$$H^1(E, \mathcal{A}_N) \xrightarrow{\sim} H^1(E, \Theta_E) \simeq \mathbb{C}.$$

4. The successive quotients for the formal neighbourhood of E are

$$I^m\mathcal{F}/I^{m+1}\mathcal{F} = \mathcal{A}_N \otimes N^{-m} \qquad (m \geq 0).$$

For $m \geq 1$ both line bundles in the induced extension have degree $8m > 0$ on the elliptic curve, so their first cohomology vanishes. It follows that all restriction maps

$$H^1(E, \mathcal{F}/I^{m+1}\mathcal{F}) \longrightarrow H^1(E, \mathcal{F}/I^m\mathcal{F}) \quad (m \geq 1)$$

are isomorphisms. Their common value is $H^1(E, \mathcal{A}_N) \simeq \mathbb{C}$. The analytic theorem on formal functions (Grauert, 1960, Section 6, Hauptsatz IIa, p. 59) identifies this inverse limit with the completed stalk of $R^1\pi_*\mathcal{F}$ at p. This coherent sheaf is supported at the single point p, so its stalk is a finite-length module and equals its completion. We have proved

$$R^1\pi_*\mathcal{F} \simeq \mathbb{C}_p, \qquad H^0(Y, R^1\pi_*\mathcal{F}) \xrightarrow{\sim} H^1(E, \Theta_E).$$

The second map is the natural restriction followed by the Atiyah-extension map just computed.

5. There is also an equality of sheaves

$$\pi_*\mathcal{F} = \Theta_Y.$$

To check it at p, a holomorphic derivation of the cone integrates locally to a holomorphic flow: embed the cone in a polydisc, lift its coefficient functions, and use the ordinary holomorphic vector-field flow, which preserves the defining ideal. The flow preserves the unique singular point, hence its maximal ideal. Functoriality of the blow-up of that ideal lifts the flow to the resolution, preserving the exceptional curve. Its derivative is therefore a section of $\mathcal{F}$. Conversely a section of $\mathcal{F}$ differentiates pullbacks of holomorphic functions on Y and descends to a derivation, because $\pi_*\mathcal{O}_S = \mathcal{O}_Y$. Both operations agree off E and are inverse there; uniqueness extends the equality over E.

6. The global-to-local cohomology map is surjective for a concrete reason. Write $a = e^{2\pi i\tau}$ with $\operatorname{Im}\tau > 0$, and vary τ to $\tau + u$ in the explicit Kato construction, keeping its translation parameter zero. The same uniform collar argument as in 22.27 gives a proper holomorphic deformation of the pair (S, E). Its Kodaira–Spencer class belongs to $H^1(S, \mathcal{F})$. Under the natural composite

$$H^1(S, \mathcal{F}) \longrightarrow H^0(Y, R^1\pi_*\mathcal{F}) \xrightarrow{\sim} H^1(E, \Theta_E)$$

it is the Kodaira–Spencer class of $\mathbb{C}/(\mathbb{Z} + (\tau + u)\mathbb{Z})$, which is nonzero. For example, differentiating its second deck transformation $z \mapsto z + \tau + u$ gives the usual nonzero period cocycle. This argument works also at the elliptic curves with extra automorphisms. The target is one dimensional, so the first arrow is onto.

7. The degree-two part of the Leray sequence is

$$H^1(S, \mathcal{F}) \longrightarrow H^0(Y, R^1\pi_*\mathcal{F}) \longrightarrow H^2(Y, \pi_*\mathcal{F}) \longrightarrow H^2(S, \mathcal{F}).$$

The first map is surjective and the last group is zero. Using $\pi_*\mathcal{F} = \Theta_Y$ proves $H^2(Y, \Theta_Y) = 0$.

8. For a compact complex space with isolated singularities, $H^2(\Theta) = 0$ makes the restriction from global to local deformations formally smooth. Take a finite Stein cover with one small neighbourhood of p and all other members smooth. For a small extension of Artinian bases, prescribe the lifted deformation on the singular member and take smooth local lifts on the other members. Their pairwise identifications lift on the smooth Stein intersections. The failure of the cocycle identity on triple intersections is a Čech two-cocycle in Θ_Y tensored with the square-zero kernel. Its class vanishes by the proved H^2-vanishing; correcting the identifications by a one-cochain glues the desired lift. This proves the relative infinitesimal lifting property.

9. The analytic semiuniversal deformation theory turns this property into regularity of the induced morphism of analytic base germs. The general criterion is Manetti (1991, Lemma 1, p. 93), stated for reduced compact complex spaces with isolated singularities and proved by the local-to-global spectral sequence of the cotangent complex. It is also used in Dloussky and Teleman (2020, arXiv version 2, Section 3.2, Proposition 3.3 and the following discussion, p. 11). A regular morphism of analytic germs is locally a projection from a product with a smooth germ. Consequently a specified analytic disc in the local deformation base lifts to the global base. The global family is proper by the compact deformation construction. Away from p its central fibre is smooth, and smoothness is open; compactness permits one common shrinking of the disc. A local smoothing at p therefore gives smooth global noncentral fibres, as asserted.

□

Proposition 22.29 (A proper quadric–parabolic surface smoothing)**.** [tag 030G]

Let $S = S_8(a, \kappa)$ be as in 22.28, with elliptic axis E and normal line N of degree -8. There is an embedding $E \hookrightarrow Q = \mathbb{P}^1 \times \mathbb{P}^1$ as a smooth divisor of bidegree $(2, 2)$ with $N_{E/Q} \simeq N^{-1}$. There is a proper flat map from a smooth complex threefold

$$\varpi : \mathcal{V} \longrightarrow \Delta$$

with smooth noncentral fibres and reduced simple normal-crossings central fibre exactly $S \cup_E Q$. Its two normal lines are

$$N_{S/\mathcal{V}} = \mathcal{O}_S(-E), \qquad N_{Q/\mathcal{V}} = K_Q = \mathcal{O}_Q(-2, -2).$$

The family is obtained by globalizing a specified cone smoothing and blowing up its vertex over the original parameter disc.

For the elliptic-quartic configuration of 22.27, the three pairwise normal-line products, including the triple-curve factors, can consequently all be made trivial using

$$A = \mathrm{Bl}_E \mathbb{P}^3, \qquad B = S \times \mathbb{P}^1, \qquad C = \mathcal{V}.$$

Here C is the constructed proper surface family over an open disc. Its continuous surjection onto that disc proves that C is noncompact.

Proof. 1. Put $L = N^{-1}$, of degree eight. Multiplication by two on the elliptic Picard variety is surjective, so choose a degree-four line bundle M with $M^{\otimes 2} = L$. Choose degree-two line bundles M_1, M_2 with $M_1 \otimes M_2 = M$ and $M_1 \not\simeq M_2$. Each complete linear system is a basepoint-free double cover of $\mathbb{P}^1$. The product of the two maps is an embedding in Q. Indeed two distinct points identified by both maps would give $M_1 \simeq \mathcal{O}_E(p + q) \simeq M_2$. A common ramification point would similarly give $M_1 \simeq \mathcal{O}_E(2p) \simeq M_2$. Thus the product is injective and immersive; properness makes it a closed embedding. Its image has bidegree $(2, 2)$, and

$$\mathcal{O}_Q(E)|_E = (M_1 \otimes M_2)^{\otimes 2} = L.$$

This also allows us to start with the fixed quadric and elliptic quartic in 22.27: the normal line there was chosen to be the inverse of this same L.

2. Use the anticanonical embedding of Q in $\mathbb{P}^8$ and write

$$R_Q = \bigoplus_{d\geq 0} H^0(Q, \mathcal{O}_Q(2d, 2d)), \qquad Z = \operatorname{Spec} R_Q.$$

The ring is generated in degree one and is normal. One direct verification uses the bihomogeneous monomials: every monomial of bidegree $(2d, 2d)$ is a product of d monomials of bidegree $(2, 2)$, and the associated monoid is the intersection of a rational polyhedral cone with its full lattice. The punctured cone is the complement of the zero section in $\mathcal{O}_Q(-2, -2)$, hence is smooth. Choose the section $e \in H^0(Q, \mathcal{O}_Q(2, 2))$ cutting out E. The degree-one function $t = e$ on Z defines a flat map $Z \to \mathbb{A}^1$: its coordinate ring is torsion-free over $\mathbb{C}[t]$, a principal ideal domain. For every $d \geq 0$, the sequence

$$0 \longrightarrow \mathcal{O}_Q(2d-2, 2d-2) \xrightarrow{e} \mathcal{O}_Q(2d, 2d) \longrightarrow L^{\otimes d} \longrightarrow 0$$

has vanishing first cohomology in its left term. For $d = 0$ this follows from the Künneth formula for $\mathcal{O}_Q(-2, -2)$; for $d \geq 1$ it follows from the same formula and nonnegative degrees. Taking sections identifies

$$R_Q/(e) = \bigoplus_{d\geq 0} H^0(E, L^{\otimes d}).$$

The central surface is therefore exactly the affine cone over (E, L), including its scheme structure. For $t \neq 0$, projection from the cone vertex identifies the fibre with $Q \setminus E$: over each point where e is nonzero there is exactly one radial coordinate on which e has value t. In particular this is an explicit smoothing of the isolated singularity of 22.28. It is the affine version of Pinkham's cone construction (Pinkham, 1974, Remark 7.6(iii), p. 46; Section 9, pp. 63–65).

3. Blowing up the vertex of Z gives $\operatorname{Tot}(\mathcal{O}_Q(-2, -2)) = \operatorname{Tot}(K_Q)$. This can be checked on the blow-up charts given by ratios of the degree-one homogeneous coordinates; these recover a point of Q, with the remaining common factor the tautological fibre coordinate ξ. The map to the parameter line is precisely

$$t = \xi e.$$

Its central divisor is the union of the zero section Q and the restriction $\operatorname{Tot}(K_Q|_E)$, each with multiplicity one. Near E the equation is $t = \xi z$, where z is a local defining function of E in Q. Thus the total space is smooth and the central divisor has simple normal crossings.

4. Now contract E in the compact surface S as in 22.28. The regular global-to-local deformation map in that proposition lifts the specified analytic germ $Z \to \mathbb{A}^1$ to a proper deformation $\mathcal{Y} \to \Delta$ of $Y = S/E$. The lifted deformation agrees analytically with this cone smoothing in a neighbourhood of its singular point. Away from that neighbourhood the central surface is smooth; openness of smoothness and properness allow a common shrinking of Δ for which the family there is a submersion. Blow up the single cone vertex in $\mathcal{Y}$. The resulting map $\mathcal{V} \to \Delta$ remains proper. Near the exceptional divisor its local model is the smooth space just computed; elsewhere it is unchanged and smooth. Its central strict transform is exactly S, because the maximal-ideal blow-up of the degree-eight elliptic cone is its given resolution $\operatorname{Tot}(N)$ near E. The other component is Q, and their intersection is the original marked elliptic curve. The local equation $t = \xi z$ proves flatness and reducedness near the intersection; at a smooth point of either component t is a local coordinate. The noncentral fibres are the smooth compact fibres of $\mathcal{Y}$.

5. The exceptional zero section has normal K_Q in its local total-space model. Since $S+Q$ is the principal divisor of t, restriction to S gives

$$\mathcal{O}_{\mathcal{V}}(S)|_S = \mathcal{O}_{\mathcal{V}}(-Q)|_S = \mathcal{O}_S(-E).$$

These are the two asserted normal lines. They also show directly that the two branch normals along E multiply to the trivial line.

6. Realize Q as a smooth quadric in $\mathbb{P}^3$ and E as its intersection with a second quadric. Put $L = \mathcal{O}_E(2)$. Then $N_{E/\mathbb{P}^3} = L \oplus L$, so the exceptional divisor of $A = \mathrm{Bl}_E \mathbb{P}^3$ is $D_{AB} = E \times \mathbb{P}^1$, with normal $L \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1)$. The strict transform of Q is still Q, because the centre is Cartier on it, and its normal in A is

$$N_{Q/A} = \mathcal{O}_Q(2) \otimes \mathcal{O}_Q(-E) = \mathcal{O}_Q.$$

It meets D_{AB} in the constant section $E \times \{p\}$ corresponding to that member of the pencil of quadrics. In $B = S \times \mathbb{P}^1$, use the divisors $D_{BA} = E \times \mathbb{P}^1$ and $D_{BC} = S \times \{p\}$. Their normals are $L^{-1} \boxtimes \mathcal{O}$ and $\mathcal{O}_S$. In $C = \mathcal{V}$, use its central divisors Q and S. The three normal products with the triple contribution are then

double surface	$N_{D/\text{first branch}} \otimes N_{D/\text{second branch}} \otimes \mathcal{O}_D(\text{triple curve})$
$E \times \mathbb{P}^1$	$(L \boxtimes \mathcal{O}(-1)) \otimes (L^{-1} \boxtimes \mathcal{O}) \otimes (\mathcal{O}_E \boxtimes \mathcal{O}(1)) = \mathcal{O}$
Q	$\mathcal{O}_Q \otimes K_Q \otimes \mathcal{O}_Q(E) = \mathcal{O}_Q$
S	$\mathcal{O}_S \otimes \mathcal{O}_S(-E) \otimes \mathcal{O}_S(E) = \mathcal{O}_S.$

The three displayed evaluation maps are the required pairwise normal-line trivializations. The third space maps surjectively and properly to an open disc, which proves its noncompactness.

7. The affine smooth fibre $Q \setminus E$ has

$$H_1(Q \setminus E; \mathbb{Z}) = \mathbb{Z}/2, \qquad H_2(Q \setminus E; \mathbb{Z}) = \mathbb{Z}^3.$$

Indeed the Thom sequence of the pair $(Q, Q \setminus E)$ has intersection map $H_2(Q; \mathbb{Z}) \to H_0(E; \mathbb{Z})$ given by $(a, b) \mapsto 2a + 2b$. Its cokernel is $\mathbb{Z}/2$ and its kernel is $\mathbb{Z}$; the preceding term $H_1(E; \mathbb{Z}) = \mathbb{Z}^2$ injects into $H_2(Q \setminus E; \mathbb{Z})$. This exact extension of free groups splits abstractly. The map $H_4(Q; \mathbb{Z}) \to H_2(E; \mathbb{Z})$ is an isomorphism, so $H_3(Q \setminus E; \mathbb{Z}) = 0$, and its Euler number is four. The Euler number is also obtained independently by projection to either ruling of Q: away from the four branch values of $E \to \mathbb{P}^1$, the fibre is $\mathbb{C}^*$ and has Euler number zero; at each branch value it is $\mathbb{C}$ and has Euler number one. The compact noncentral surface has Euler number $\chi(S) + \chi(Q) - 2\chi(E) = 8 + 4 = 12$ by the local $\xi z = t$ retraction.

□

Remark 22.30 (Euler characteristics of the quadric–parabolic smoothing)**.** [tag PT14] For the compact noncentral surface V_t of Proposition 22.29,

$$\chi(V_t) = \chi(S_8) + \chi(Q) - 2\chi(E) = 8 + 4 = 12, \qquad \chi(V_t, \mathcal{O}_{V_t}) = 1.$$

The first formula is the local $\xi z = t$ smoothing formula proved there. Mayer–Vietoris for the two spherical boundaries in the Kato construction identifies the middle homology and its intersection form with the eight negative exceptional classes. Thus $\chi(S_8) = 8$ and the signature is -8, so $c_1(S_8)^2 = 2\chi(S_8) + 3\sigma(S_8) = -8$ and analytic Riemann–Roch gives $\chi(S_8, \mathcal{O}_{S_8}) = 0$. Since $\chi(Q, \mathcal{O}_Q) = 1$ and $\chi(E, \mathcal{O}_E) = 0$, the normalization sequence for $S_8 \cup_E Q$ and flatness give the second formula.

Proposition 22.31 (The obstruction to a marked period lift)**.** [tag 030H]

Fix $0 < |\alpha|, |a| < 1$, and put

$$H = (\mathbb{C}^2 \setminus \{0\})/\langle (x, v) \mapsto (\alpha x, av)\rangle, \qquad E = \{x = 0\} \simeq \mathbb{C}^*/a^{\mathbb{Z}}.$$

Let h be the automorphism induced by $(x, v) \mapsto (cx, bv)$, where $b, c \in \mathbb{C}^*$. Suppose $p : Y \to H$ is a finite composition of point blow-ups, allowing infinitely near centres, and at least one point of E is an image of its exceptional locus. If $[b] \in \mathbb{C}^*/a^{\mathbb{Z}}$ has infinite order, there is no holomorphic automorphism of Y which agrees, on the dense unmodified core, with the marked automorphism h.

The same obstruction holds for a finite collection of such marked modifications if the period permutes the collection and acts as h on each marked core. More generally, if a compact smooth surface contains a preserved marked elliptic curve E of nonzero normal degree d, every holomorphic automorphism restricting to translation by b on E must satisfy $d\,b = 0$ in E. The latter assertion also excludes a finite cycle of such preserved marked cap axes when the cycle composite is an infinite-order translation. These assertions apply to the extension of the second cusp period to the finite Hopf modifications in 22.27, for generic nonzero parameters, after every finite base change. The period action in these assertions is a holomorphic action on the full finite modifications with the specified marked core.

Proof. 1. Let $S \subset H$ be the image of the exceptional curves of p. It is finite. Repeated infinitely near blow-ups change the curves over a point but do not make S infinite. If $\widetilde{h}$ were a lift, the maps

$$p\widetilde{h},\ hp : Y \longrightarrow H$$

would agree on the dense core and hence everywhere, by the identity principle for holomorphic maps to the separated complex surface H. The image under $\widetilde{h}$ of a curve contracted by p is therefore again contracted by p. Applying the same argument to $\widetilde{h}^{-1}$ proves that the total exceptional locus is preserved. Thus $h(S) = S$.

2. The automorphism h preserves E, where it is translation by $[b]$. Consequently the nonempty finite set $S \cap E$ would be invariant under that translation. The orbit of any one of its points is finite, so for some $k > 0$,

$$[b^k v] = [v], \qquad \text{equivalently } b^k \in a^{\mathbb{Z}}.$$

This says exactly that $[b]$ is torsion, a contradiction.

3. For finitely many modifications $p_i : Y_i \to H$, a deck map has biholomorphisms $Y_i \to Y_{\sigma(i)}$ satisfying the same core comparison. The identity principle again gives equivariance everywhere. The union of the finitely many exceptional image sets is h-invariant; its intersection with E gives the same contradiction. Equivalently, around a cycle of length ℓ the map h^ℓ would lift to a single modification, while $[b^\ell]$ still has infinite order.

4. For the more general assertion, an automorphism preserving E induces through its derivative in the normal direction an isomorphism

$$t_b^* N_{E/S} \simeq N_{E/S}.$$

For a line bundle N of degree d on an elliptic curve, the homomorphism $b \mapsto t_b^* N \otimes N^{-1}$ is multiplication by $-d$ under the usual identification $\mathrm{Pic}^0(E) = E$. Indeed write $N = \mathcal{O}_E(\sum_j n_j[p_j])$ with $\sum_j n_j = d$; the Abel–Jacobi class of its translated divisor minus the original one is $-db$. Thus preservation of N forces $db = 0$. Taking a power proves the finite-cycle version, since its composite preserves one axis and its nonzero-degree normal line. In particular a nonzero-degree parabolic axis cannot carry the marked generic period translation even if no finite Hopf modification is specified.

5. The infinite-order translation at the cusp is computed as follows. The periods of 20.6 have the form
$$g_1 : (Z, W) \longmapsto (qc_1 Z, c_2 W), \qquad g_2 : (Z, W) \longmapsto (c_3 Z, r^{12} qc_4 W),$$
where the c_i are nonvanishing holomorphic units near the cusp. Choose W or W^{-1} so that $a = c_2^{\varepsilon}$ has modulus less than one, $\varepsilon \in \{1, -1\}$. This is possible at the marked cusp because its limiting unit has modulus different from one, as computed in 22.24. For $q \neq 0$ small, the partial g_1 quotient of $(\mathbb{C}^*)^2$ is the dense core of the diagonal Hopf surface with multipliers (qc_1, a). The second period induces the above h, with
$$b(q, r) = (r^{12} qc_4(q, r))^{\varepsilon}.$$
Fix a sufficiently small nonzero q. For each $k > 0$ and $j \in \mathbb{Z}$, the equation $b(q, r)^k = a(q)^j$ has a discrete solution set in the punctured r-disc: its left-hand side has a zero or pole of nonzero order at $r = 0$, while the right-hand side is a nonzero constant in r. Hence outside a countable union of discrete sets, $[b(q, r)]$ has infinite order. Under a finite base change in r the nonzero order is multiplied by its ramification index and the same conclusion holds. A finite base change in q does not affect this fixed-nonzero-q argument.

6. To compare with 22.27, write $qc_1 = \kappa(s)s^n$ after the chosen finite cover in the cusp variable. Its generic Kato fibre is precisely a finite Hopf modification with nonempty axis centre $[-s]$. A holomorphic extension of the second period to that whole fibre would therefore contradict the result just proved. Allowing finitely many translated copies does not remove the contradiction, by the finite-collection assertion.

□

Lemma 22.32 (The quotient of indexed cores)**.** [tag PT15] *Let h be an automorphism of H, and let modifications $p_j : Y_j \to H$, indexed by $j \in \mathbb{Z}$, have isomorphisms $\widetilde{h}_j : Y_j \to Y_{j+1}$ satisfying $p_{j+1}\widetilde{h}_j = hp_j$. On an h-invariant open set H° over which every p_j is an isomorphism, the shift quotient of the disjoint union of their cores is biholomorphic to H° by*
$$[(j, y)] \longmapsto h^{-j} p_j(y).$$
Under this isomorphism, its natural map to the orbit space $H^\circ/\langle h\rangle$ is the ordinary quotient map. Moreover, two embedded compact subspaces of a Hausdorff space that share a subset open and dense in both have the same image.

Proof. The displayed map is invariant under $(j, y) \mapsto (j + 1, \widetilde{h}_j y)$ because $h^{-j-1} p_{j+1} \widetilde{h}_j = h^{-j} p_j$. Every shift orbit meets the zeroth core exactly once, so the inverse is $x \mapsto [(0, p_0^{-1} x)]$. These maps are holomorphic on the component charts. Composing with $H^\circ \to H^\circ/\langle h\rangle$ gives the asserted orbit-space map. For the last assertion, compact images in a Hausdorff space are closed; each of the two images is consequently the closure of their shared dense subset. □

Proposition 22.33 (Tangent representations and associated bundles)**.** [tag 030I]

For the explicit parabolic surface $S = S_8(a, \kappa)$ and its elliptic contraction $\pi : S \to Y$ of 22.28, the genuine action $\lambda \cdot (x, v) = (\lambda x, v)$ induces trivial representations on the following cohomology groups:
$$H^0(S, \Theta_S(-\log E)) = \mathbb{C}, \qquad H^1(S, \Theta_S(-\log E)) = \mathbb{C}^9,$$
$$H^0(Y, \Theta_Y) = \mathbb{C}, \qquad H^1(Y, \Theta_Y) = \mathbb{C}^8.$$
The two second cohomology groups vanish.

Let $p : Z_L \to \mathbb{P}^1$ be the compact holomorphic Y-bundle associated to any holomorphic line bundle L and this $\mathbb{C}^*$ action. Put $\mathcal{T} = \Theta_{Z_L/\mathbb{P}^1}$. For every $b \in \mathbb{P}^1$ one has
$$R^0 p_* \mathcal{T} = \mathcal{O}_{\mathbb{P}^1}, \qquad R^1 p_* \mathcal{T} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 8}, \qquad R^2 p_* \mathcal{T} = 0,$$

and

$$H^2(Z_L, \mathcal{T}) = 0, \qquad H^2(Z_L, \mathcal{T} \otimes p^*\mathcal{O}_{\mathbb{P}^1}(-b)) = 0.$$

The singular locus of the associated bundle is the section induced by the cone vertex of Y; the displayed groups are the relative tangent cohomology and its specified fibre twist.

Proof. 1. The action commutes with the central Kato gluing and hence acts on the whole compact surface and its contraction. Its toric coordinate description is also given in Fujiki (2009, Section 2, equations (1)–(8), p. 4). For any $\lambda \neq 1$, its fixed locus consists of the elliptic curve E and the eight nodes of the rational cycle. To check that there are no further points, on the dense torus an equality with a Kato deck iterate would require $a^j v = v$; since $0 < |a| < 1$, this forces $j = 0$, and then $\lambda = 1$. The action on each rational component is nontrivial and fixes only its two endpoints. In the unfolded fan the consecutive rays are $(j, 1)$ and $(j+1, 1)$. Their dual characters evaluate on the scaling cocharacter $(1, 0)$ as $-1, +1$. Thus the two tangent weights at every cycle node are λ^{-1}, λ.

2. The space of global vector fields is described in Fujiki (2009, Theorem 1.1, pp. 1–2). Write the contracted Kato germ as

$$F(x, v) = (Kxv^8, av), \qquad K = \kappa a^8.$$

A global vector field restricts on the outer spherical annulus to $P(x, v)\partial_x + Q(x, v)\partial_v$. Hartogs extension gives holomorphic coefficients on the ball. Compatibility with the collar gluing, followed by analytic continuation, gives the exact germ identity

$$V(F) = DF \cdot V.$$

Expand $Q(x, v) = \sum_{i \geq 0} x^i Q_i(v)$. The second component says

$$K^i v^{8i} Q_i(av) = aQ_i(v).$$

For $i > 0$ the lowest nonzero v-order on the two sides would differ by $8i$, so $Q_i = 0$. For $i = 0$, power-series coefficients give $Q_0(v) = cv$. The constant-in-x coefficient of the first component also vanishes by the same lowest-order argument. Its linear coefficient satisfies $P_1(av) = P_1(v) + 8c$. Evaluation at zero gives $c = 0$, and then P_1 is constant. For $i > 1$ the remaining equations are

$$P_i(v) = K^{i-1} v^{8(i-1)} P_i(av),$$

so again $P_i = 0$. Consequently $H^0(S, \Theta_S) = \mathbb{C}\, x\partial_x$. This Euler field is tangent to E and has weight zero.

3. Set $\mathcal{F} = \Theta_S(-\log E)$. 22.28 proved $H^2(S, \mathcal{F}) = 0$. Apply the holomorphic Lefschetz formula to a root of unity $\lambda \neq 1$, using its finite cyclic group action. The applicable original statement is Atiyah and Singer (1968, Theorem (4.6), p. 566), which applies to compact complex manifolds. On E, the Atiyah extension $\mathcal{F}|_E$ has two weight-zero terms and Chern character 2; the normal line has weight λ and degree -8. Since the tangent bundle of the elliptic curve has trivial Todd class, the contribution of E is

$$\int_E \frac{2}{1 - \lambda^{-1} \exp(-c_1(N))} = -\frac{2\lambda^{-1} \deg N}{(1 - \lambda^{-1})^2} = \frac{16\lambda}{(\lambda - 1)^2}.$$

At a cycle node $\mathcal{F}$ equals the ordinary tangent bundle. Its contribution is

$$\frac{\lambda + \lambda^{-1}}{(1 - \lambda)(1 - \lambda^{-1})} = -\frac{\lambda^2 + 1}{(\lambda - 1)^2}.$$

Adding the eight nodes gives the constant character

$$\sum_q (-1)^q \operatorname{Tr}(\lambda \mid H^q(S,\mathcal{F})) = \frac{16\lambda - 8(\lambda^2+1)}{(\lambda-1)^2} = -8.$$

Since H^0 is one weight-zero copy and $H^2 = 0$, $H^1(S,\mathcal{F})$ consists of nine trivial copies. Equality on infinitely many roots of unity determines the finite Laurent-polynomial character of the $\mathbb{C}^*$ representation.

4. Every map in the Leray sequence of 22.28 is equivariant. Its surjection

$$H^1(S,\mathcal{F}) \longrightarrow H^0(Y, R^1\pi_*\mathcal{F}) \simeq H^1(E,\Theta_E) = \mathbb{C}$$

has a trivial one-dimensional target. The kernel is $H^1(Y,\Theta_Y)$, so it is eight-dimensional and trivial. The equality in degree zero follows from $\pi_*\mathcal{F} = \Theta_Y$; the second cohomology vanishing was already proved in 22.28.

5. Finally form Z_L by choosing line-bundle transition functions and gluing products $Y \times U_i$ by their action on Y. This is a holomorphic locally trivial bundle with compact fibre over the compact curve $\mathbb{P}^1$, hence is proper and compact. It has the section of isolated cone points as its singular locus; that locus is one dimensional in the total space. On each product chart, coherent cohomology and base change identify $R^q p_*\mathcal{T}$ with $H^q(Y,\Theta_Y)\otimes\mathcal{O}_{U_i}$. The gluing is exactly by the representations computed above. Their triviality proves the displayed direct-image formulas. The projection formula and the Leray sequence then give

$$H^2(Z_L, \mathcal{T}\otimes p^*M) \simeq H^1(\mathbb{P}^1, M^{\oplus 8})$$

for $M = \mathcal{O}$ or $\mathcal{O}(-b)$, both of which have zero first cohomology on $\mathbb{P}^1$. This proves the asserted reduction, including the twist appropriate to retaining one whole fibre.

□

22.6 A local insertion into the prescribed simple-torus family

Proposition 22.34 (Local insertion of vertical Hopf bundles)**.** [tag 030J]

Let q_0 be a regular point of the elliptic base, and let Δ_z, $z = q - q_0$, be a simply connected disc avoiding $0, 1, \infty$. After $w = r^6$, take the standard twelve-cycle model

$$\mathcal{X} \longrightarrow \Delta_z \times \Delta_r$$

of the prescribed local two-torus family. There is a proper replacement $\mathcal{X}' \to \Delta_z \times \Delta_r$, with a specified isomorphism to $\mathcal{X}$ off $(z,r) = (0,0)$, whose total fourfold is smooth and whose divisor $r = 0$ is reduced normal crossings. The composite map to Δ_r is flat.

The replacement has twelve new compact vertical normalization components. Six are $\mathbb{P}^1$-bundles over a scalar Hopf surface; the other six are the blow-ups of those bundles along two disjoint elliptic curves. Every new component has fundamental group $\mathbb{Z}$, and therefore fails the $\partial\bar\partial$ lemma. The punctured family is the prescribed family, so its generic simplicity is preserved.

The map is proper over the bidisc and flat over the degeneration parameter. Its vertical threefolds have the dimensions stated above. All twelve component labels, and hence the obstruction of 22.5, are unchanged.

Proof. 1. *The action.* Use the rank-three period subsystem

$$\mathbf{u}(z) = (\tau(z), \mu(z)), \qquad \mathbf{w} = (1,0), \qquad \boldsymbol{\delta} = (0,1).$$

It defines the analytic semiabelian group

$$G_z = \mathbb{C}^2/(\mathbb{Z}\mathbf{u}(z) + \mathbb{Z}\mathbf{w} + \mathbb{Z}\boldsymbol{\delta}), \qquad 1 \longrightarrow \mathbb{C}^*_\delta \longrightarrow G_z \longrightarrow E_z \longrightarrow 0, \quad E_z = \mathbb{C}/(\mathbb{Z} + \tau(z)\mathbb{Z}).$$

This is the punctured degree-zero line bundle determined by $\mu(z)$. The formula

$$\chi_z(t) = \left[\frac{\log t}{2\pi i}\,\mathbf{u}(z)\right] \quad (t \in \mathbb{C}^*) \tag{22.3}$$

is a single-valued holomorphic homomorphism: a logarithm change adds a lattice period. It varies holomorphically with z. The r-monodromy fixes $\mathbf{u}$, and the z-disc has no period monodromy.

2. The action extends to the entire twelve-cycle model. Each unfolded normalization is

$$R_z = G_z \times^{\mathbb{C}^*_\delta} \mathbb{P}^1 = \mathbb{P}_{E_z}(\mathcal{O} \oplus L_z).$$

Translation in G_z preserves both conductor sections. More explicitly, a class represented by (a, b) acts before the closing quotient by

$$(\xi, W) \longmapsto (\xi + a, e^{-2\pi i b} W). \tag{22.4}$$

It commutes exactly with the closing period, which is another translation of the same commutative group and whose scalar factor contains r^{12}. The adjacent line directions are paired dually in the local equation $xy = r$; the two scalar factors cancel. Scalar multiplication also preserves the one-dimensional toric subdivision producing the twelve components. Thus (22.4) extends over every node, preserves each branch, and changes a branch equation x by a unit times x. This checks the required action using the semiabelian marking of Proposition 22.1.

3. The affine family itself admits the necessary local punctured comparison. On $U = \Delta_z \times \Delta_r^*$ its lattice local system Λ has $H^2(U, \Lambda) = 0$, since U retracts onto a circle, also for local coefficients. Its relative Lie algebra $\mathcal{V}$ is a coherent vector bundle. The exponential sequence

$$0 \longrightarrow \Lambda \longrightarrow \mathcal{V} \longrightarrow \mathcal{T} \longrightarrow 0$$

and Cartan's theorem B on the Stein base give $H^1(U, \mathcal{T}) = 0$ (Cartan, 1952, exposé 19, §5, pp. 14–15). Consequently the affine torsor has a holomorphic section on U, which supplies the stated punctured comparison with its Jacobian.

4. *The base blow-up and small resolutions.* Blow up the base origin:

$$\beta : \widetilde{B} = \mathrm{Bl}_{(0,0)}(\Delta_z \times \Delta_r) \longrightarrow \Delta_z \times \Delta_r, \qquad F = \beta^{-1}(0,0) \cong \mathbb{P}^1.$$

In the chart $r = zs$, the pulled-back node is $xy = zs$, with one further smooth elliptic coordinate. Label the original components by $i \in \mathbb{Z}/12$. At each node choose $x = 0$ as the even-indexed branch, and blow up (x, z). This is globally the sum of the pulled-back invertible ideal of the chosen central component and the ideal $\mathcal{O}_{\widetilde{B}}(-F)$. The ideals are G-invariant. Their non-Cartier loci are the twelve disjoint elliptic node loci, so the choices at the six even components are compatible.

5. The two strict-transform charts are

$$\begin{aligned} z &= xQ, \quad y = Qs, \quad r = xQs, \\ x &= zX, \quad s = Xy, \quad r = zXy. \end{aligned} \tag{22.5}$$

They are smooth, and all three factors of r occur once. In the other base chart $z = r\zeta$, the node is $xy = r$ and the same ideal is $(x, r) = (x)$, so no change occurs. Away from the indicated node loci it is also invertible. The resulting total space is smooth and its r-central divisor is reduced normal crossings.

6. For the component identifications, before resolution put

$$V_x = \{x = z = 0\}, \quad V_y = \{y = z = 0\},$$
$$H_x = \{x = s = 0\}, \quad H_y = \{y = s = 0\}.$$

The V branches are vertical over F, and the H branches lie over the strict transform of $r = 0$. On V_x, the two maps in (22.5) are

$$(Q, s) \longmapsto (y, s) = (Qs, s), \qquad (X, y) \longmapsto (y, s) = (y, Xy).$$

These are exactly the blow-up charts of (y, s). Thus V_x is blown up along its elliptic curve over $y = s = 0$; V_y is unchanged. Similarly H_y is blown up along the corresponding elliptic curve and H_x is unchanged. Each even vertical component is therefore blown up at its two disjoint conductor curves, whereas each odd vertical component is unchanged. The horizontal parities are reversed.

7. *The Cartier logarithm and properness.* Set $M = \mathcal{O}_{\widetilde{B}}(F)$. On a trivializing cover write its transition units as g_{ij}. Glue the resolved pullbacks by $\chi_z(g_{ij})$. These are genuine automorphisms satisfying the cocycle identity. This is the Cartier-logarithm construction of Fujimoto (1990, construction, p. 85; Definition 1.1, p. 86); the explicit semiabelian action above verifies its applicability here. The invariant ideals show that the action lifts to the resolutions; equivalently one may twist first and then resolve the descended ideals.

8. Over every base chart the glued space is a copy of the proper resolved pullback. It is Hausdorff: two points over the same base point are separated in such a chart, and points over distinct base points are separated in the base. Properness is local on the target, and composition with the proper β proves properness over the original bidisc. The canonical divisor section of M is nonzero off F and trivializes the twist there. The base blow-up and the resolutions are also isomorphisms off the origin. This gives the specified isomorphism with the original family there, in particular over every $r \neq 0$.

9. The construction can be inserted into any larger model which is standard near q_0: the preceding trivialization supplies the overlap map outside the origin. The cocharacter $\mathbf{u}$ is defined on this disc. Since the local rings of the total space are regular and r is a nonzero divisor, they are torsion-free, hence flat, over the one-dimensional regular parameter ring. This proves flatness over r. The new vertical threefolds rule out joint flatness over the bidisc. Properness over the r-disc alone follows only when this local replacement is inserted into a model with compact elliptic base.

10. *The compact vertical components.* Before resolving, each vertical normalization is the twist of $R_{q_0} \times F$ by $M|_F = \mathcal{O}_{\mathbb{P}^1}(-1)$, namely

$$V_i^0 = \mathcal{O}_{\mathbb{P}^1}(-1)^\times \times^{\mathbb{C}^*} R_{q_0}.$$

Its elliptic quotient is a principal E_{q_0}-bundle $H \to \mathbb{P}^1$ with primitive Chern cycle $\pm\mathbf{u}$. It is explicitly a scalar Hopf surface. Indeed put

$$\tau_0 = \tau(q_0), \qquad \eta = -1/\tau_0, \qquad a' = e^{2\pi i \eta}, \qquad |a'| < 1.$$

After dividing the elliptic coordinate by τ_0, the projection of (22.3) is $[\log t/(2\pi i)]$ in $\mathbb{C}/(\mathbb{Z}+\eta\mathbb{Z})$. Its kernel is $(a')^{\mathbb{Z}}$. Consequently

$$H = \mathcal{O}_{\mathbb{P}^1}(-1)^\times/(a')^{\mathbb{Z}} = (\mathbb{C}^2 \setminus \{0\})/\langle a' \,\mathrm{id}\rangle. \tag{22.6}$$

The principal G-bundle $\mathcal{O}(-1)^\times \times^{\mathbb{C}^*} G_{q_0}$ maps to H with fibre $\mathbb{C}^*_\delta$. Compactifying this principal $\mathbb{C}^*$-bundle gives a projective bundle

$$V_i^0 = \mathbb{P}_H(\mathcal{O} \oplus M_H)$$

with its two global conductor sections. On each elliptic fibre of $H \to \mathbb{P}^1$, the line M_H retains the original degree-zero line L_{q_0}.

11. The two conductor sections over the distinguished point $F \cap \mathrm{strict}(r = 0)$ are disjoint elliptic curves. The component calculation in (22.5) gives exactly the six unchanged bundles and six two-curve blow-ups in the statement. A $\mathbb{P}^1$-bundle has the fundamental group of its base, and blowing up a smooth complex codimension-two centre preserves fundamental group. Hence every new normalization has $\pi_1 = \pi_1(H) = \mathbb{Z}$. A compact $\partial\bar{\partial}$ manifold has even first Betti number, so these normalizations fail that lemma.

12. Finally, the connected translation action fixes every component label. The rotations $+4, -3, 0$ of the incompatible global assembly are unchanged.

□

Remark 22.35 (Pullback along a lifted ordinary arc)**.** [tag PT16] Let $\beta : \widetilde{B} \to \Delta_z \times \Delta_r$ be the blow-up used in Proposition 22.34, with exceptional curve F. An arc $\iota_k(r) = (kr, r)$ has strict lift $\widetilde{\iota}_k$ meeting F at $[k : 1]$. Choose this point away from the distinguished resolution centres. Pulling the constructed model over $\widetilde{B}$ back along $\widetilde{\iota}_k$ gives a central cycle of twelve ruled surfaces over E_{q_0}. Indeed the pullback of $\mathcal{O}_{\widetilde{B}}(F)$ is a line bundle on a disc and is trivial, and the arc misses the additional resolution centres. The new vertical threefolds of (22.6) instead lie over the whole exceptional curve F under the map to the original bidisc. The maps being compared are the strict-lift pullback and the composite with β, respectively.

Proposition 22.36 (The dual complex under local insertions)**.** [tag 030K]

The local operation 22.34 attaches a triangulated cylinder to the original twelve-cycle in the dual complex and admits an elementary collapse back to that cycle. In a compact normal-crossings model, performing it at a regular place consequently preserves the homotopy type of the global dual complex.

The same conclusion holds for finite sequences of these insertions, twists by a connected holomorphic group action preserving the prescribed normal-crossings model, and toroidal subdivisions of its central cone complex. In particular, starting with a model whose dual complex has a nonzero first integral cohomology class, these operations cannot produce a semistable S^6 centre. They also cannot change the component rotations in 22.5.

Proof. 1. Write H_i, $i \in \mathbb{Z}/12$, for the original component vertices and V_i for the new vertical ones. At a node between an even component i and its odd neighbour j, the two charts (22.5) have triple strata

$$\{V_i, V_j, H_j\}, \qquad \{V_i, H_i, H_j\}. \tag{22.7}$$

The square with horizontal edges H_iH_j and V_iV_j is thus divided by the diagonal V_iH_j. Over all twelve nodes, (22.7) is exactly a triangulation of $C_{12} \times [0, 1]$, attached along the old horizontal cycle $C_{12} \times \{0\}$.

2. There is a literal elementary collapse relative to the old cycle. First remove each triangle containing a top edge V_iV_j, paired with that free edge. Next remove each remaining triangle, paired with its now free diagonal V_iH_j. Finally remove each vertical edge H_iV_i, paired with its free vertex V_i. What remains is precisely the original twelve-cycle. These are 36 elementary collapses. Equivalently, the retraction sends V_i to H_i and is the identity on all old vertices and edges.

3. This description is valid inside a larger dual complex K. The only new strata lie over the base origin, so the new complex is the mapping-cylinder attachment

$$K \ \cup_{f : C_{12} \to K} \ (C_{12} \times [0, 1]).$$

It retracts onto K, regardless of possible global identifications among the local horizontal branches. For self-crossings the ordered local-branch dual complex records every local branch and its global identifications.

4. A connected group acts trivially on the finite sets of component labels and connected components of every stratum: its permutation is locally constant and is the identity at the identity element. Cartier twists by such an action therefore keep this same incidence complex. Toroidal subdivisions subdivide the corresponding central polyhedral complex and hence preserve its underlying homotopy type. These arguments apply successively, proving the assertion for finite compositions. Around each regular insertion point the relative component monodromy is still the identity; no peripheral rotation is added to 22.5.

5. A compact normal-crossings space D has a natural injection

$$H^1(\Delta(D),\mathbb{C}) \;\hookrightarrow\; H^1(D,\mathbb{C}), \tag{22.8}$$

where $\Delta(D)$ denotes its local-branch dual complex with connected strata. Use the ordered finite branch spaces $D^{[p]}$ and take the alternating summand for the branch-permutation action. Over $\mathbb{C}$ this is an exact direct-summand operation. Resolve the constant sheaf by this augmented alternating branch complex. At a point its row is the augmented simplex complex, so it is exact, including for self-crossings. The associated first-quadrant spectral sequence has

$$E_1^{p,q} = H^q(D^{[p]},\mathbb{C})^{\mathrm{alt}}, \qquad E_2^{1,0} = H^1(\Delta(D),\mathbb{C}).$$

No higher differential enters or leaves $(1,0)$. Its filtration piece in total degree one gives (22.8). Equivalently one can use the orientation local system on each unordered branch stratum. A nonzero integral first cohomology class of the finite dual complex remains nonzero over $\mathbb{C}$, because first integral cohomology is torsion-free. Thus a surviving circle class forces $H^1(D,\mathbb{C}) \neq 0$. By 22.14 every semistable centre of a simply connected nearby S^6 is simply connected, which is incompatible with this conclusion.

6. For the twelve-cycle, use the oriented edge-to-vertex and triangle-to-edge incidence maps of the triangulated cylinder in Step 1. Their chain ranks are $24, 48, 24$, and their nonzero Smith entries are 1^{23} and 1^{24}, respectively. Hence its integral homology is $(\mathbb{Z},\mathbb{Z},0)$. The explicit retraction pulls the integral angular cocycle back with winding one.

□

Proposition 22.37 (Trivial monodromy does not force equality of first Betti numbers)**.** [tag 030L]

There is a compact connected complex threefold Z and a proper holomorphic map $f : Z \to \mathbb{P}^1$ with connected fibres, with precisely one singular fibre Y, such that

$$f^{-1}(\mathbb{P}^1 \setminus \{0\}) \simeq H_- \times (\mathbb{P}^1 \setminus \{0\}), \qquad b_1(Y) = 0 < 1 = b_1(H_-).$$

Here H_- is the scalar Hopf surface with multiplier $-e^{-\pi}$, and $Y = H_+ \cup_E H_+$ is a reduced simple-normal-crossings union with $E = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and H_+ of multiplier $e^{-2\pi}$. In particular, trivial local monodromy on H^1 does not imply $b_1(Y) = b_1(H_-)$, even with compact smooth total space.

Proof. 1. This is the compact-base version of the explicit Cartier cocycle of 22.19. Put

$$b : \overline{B} = \mathrm{Bl}_{(p,0)}(\mathbb{P}^1 \times \mathbb{P}^1) \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \quad L = b^* \,\mathrm{pr}_1^*\, \mathcal{O}(1), \quad A = \mathcal{O}(F_2),$$

where F_2 is the exceptional curve and F_1 the strict transform of $\mathbb{P}^1 \times \{0\}$. Given transition units l_{ab}, a_{ab}, the formula

$$t_{ab} = \left[\frac{(1+i)\log l_{ab} - i \log a_{ab}}{2\pi i}\right] \in E$$

is independent of logarithm choices and satisfies the cocycle identity. Glue the products $U_a \times E$ by these translations. This gives a holomorphic principal elliptic bundle $\pi : Z \to \overline{B}$. Its total space is smooth and Hausdorff; it is compact because $\overline{B}$ and the fibre are compact and π is locally a proper product. Let $f = \mathrm{pr}_2 \circ b \circ \pi$.

2. The restrictions of the Chern vector $(1+i)c_1(L) - ic_1(A)$ to F_1, F_2 and a smooth ruling fibre are $1, i, 1+i$, respectively. As checked directly in 22.19, the corresponding principal elliptic bundles over $\mathbb{P}^1$ are the two copies of H_+ and the surface H_-. Locally at $F_1 \cap F_2$, the base map is $xy = t$ and the elliptic bundle is a product. Thus f has smooth total space and just the one reduced normal-crossings singular fibre Y. The canonical section of A trivializes A outside F_2; there the blow-up is an isomorphism and L is pulled back from the first factor. Consequently the asserted punctured product is a holomorphic product over the whole $\mathbb{P}^1 \setminus \{0\}$. In particular every local monodromy operator is the identity.

3. The fibre inclusions on fundamental groups are the two quotient maps

$$\Lambda = \mathbb{Z} + i\mathbb{Z} \longrightarrow \Lambda/\mathbb{Z}, \qquad \Lambda \longrightarrow \Lambda/\mathbb{Z}i.$$

Their van Kampen pushout kills both lattice generators, so $\pi_1(Y) = 0$. In Mayer–Vietoris the degree-one map, with the second inclusion negated, has matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is unimodular. Since a scalar Hopf surface is diffeomorphic to $S^1 \times S^3$, the sequence gives

$$H_*(Y;\mathbb{Z}) = (\mathbb{Z}, 0, 0, \mathbb{Z}^3, \mathbb{Z}^2), \qquad b_1(H_-) = 1.$$

This compact threefold is not S^6: for example, the principal-bundle spectral sequence gives $b_2(Z) = 1$. Indeed the two Chern classes are independent in $H^2(\overline{B};\mathbb{Z}) \simeq \mathbb{Z}^3$, and the differential on $\bigwedge^2 H^1(E)$ is injective, leaving exactly one degree-two class.

□

Remark 22.38 (The invariant-cycle equality in the Hopf degeneration)**.** [tag PT17] The compact fibration of Proposition 22.37 has identity local monodromy T on $H^1(F;\mathbb{R})$, but $b_1(Y) = 0$ and $b_1(F) = 1$. It therefore refutes the converse equality clause in Honda and Viaclovsky (2025, Theorem 1.1, p. 553), also stated as Theorems 1.2 and 3.2 in arXiv version 1, pp. 1 and 8. The relative-homology and circle-bundle sequences in that proof give

$$b_1(Y) \le \dim H^1(F;\mathbb{R})^T \le b_1(F)$$

(Honda and Viaclovsky, 2025, arXiv version 1, equation (3.9), p. 9). Identity of T is equivalent to equality in the second inequality. In the present example the first inequality is $0 < 1$. Equality of the endpoint Betti numbers still forces T to be the identity by this same chain of inequalities. For an irreducible central fibre, the top-dimensional homology in the displayed relative sequence has rank one, giving $b_1(Y) = \dim H^1(F;\mathbb{R})^T$, as proved in Honda and Viaclovsky (2025, arXiv version 1, Corollary 3.7, p. 11). The example here has two central components, with the inclusion maps on $H_1(E)$ computed in Proposition 22.37.

22.7 A cusp filling on the punctured bidisc

Proposition 22.39 (A partial cusp filling)**.** [tag 030M]

Put

$$B^{\times} = (\Delta_q \times \Delta_r) \setminus \{(0,0)\}.$$

There is a proper Hausdorff analytic family over $B^{\times}$ with smooth total fourfold, flat over r, whose restriction to $r \neq 0$ is the prescribed cusp family after $w = r^6$, and whose reduced normal-crossings r-central fibre over $q \neq 0$ has component deck permutation the shift -1 on $\mathbb{Z}/12$ for the clockwise cusp generator g_0 of 22.5. Equivalently, the positive q loop has shift $+1$. In particular its q-fibre for every $r \neq 0$ is the reduced Mumford fibre, while its other axis has the component transport required by 22.6.

The overlap is the displayed holomorphic translation with its affine marking on the two specified base opens.

Proof. 1. *The fractional deck and its exact comparison.* On the logarithmic cover of $U = \Delta_q^* \times \Delta_r^*$ write

$$s = \frac{\log q}{2\pi i}, \qquad t = \frac{\log r}{2\pi i}, \qquad C(q) = \beta^{\mathrm{part}}(q) + s.$$

The functions $\mu(q)$ and $C(q)$ are single-valued and holomorphic at $q = 0$. After the affine gauge of 22.26 the family is its Jacobian, with period

$$\boldsymbol{\gamma}(s,t) = (6\mu(q), C(q) - s - 12t), \qquad \boldsymbol{\delta} = (0,1). \tag{22.9}$$

The other periods are $\mathbf{u} = (\tau, \mu)$ and $\mathbf{w} = (1,0)$. Under $s \mapsto s+1$, they change by $\boldsymbol{\gamma} \mapsto \boldsymbol{\gamma} - \boldsymbol{\delta}$ and $\mathbf{u} \mapsto \mathbf{u} + \mathbf{w}$. Under $t \mapsto t+1$, the period $\boldsymbol{\gamma}$ changes by $-12\boldsymbol{\delta}$. These are integral changes of the period basis; the ambient additive $\mathbb{C}^2$ coordinate is unchanged.

2. Set

$$b(s,t) = \frac{1}{12}\boldsymbol{\gamma}(s,t) - \frac{1}{24}\boldsymbol{\delta}.$$

The two exact identities

$$\sum_{k=0}^{11} b(s+k,t) = \boldsymbol{\gamma}(s,t) - 6\boldsymbol{\delta}, \qquad b(s,t+1) - b(s,t) = -\boldsymbol{\delta} \tag{22.10}$$

show that this gives a genuine order-twelve affine q-deck action on the root cover $q = z^{12}$, well defined around the r loop. The first right-hand side is a period. The sum of the twelve fractions $\boldsymbol{\gamma}(s+k,t)/12$ is $\boldsymbol{\gamma}(s,t) - 11\boldsymbol{\delta}/2$. The twelve constant terms $-\boldsymbol{\delta}/24$ contribute $-\boldsymbol{\delta}/2$, giving the integral period $\boldsymbol{\gamma}(s,t) - 6\boldsymbol{\delta}$.

3. There is an explicit holomorphic trivialization on U. Put $L = s + 12t$ and

$$F(s,t) = \left(\frac{\mu(q)L}{2}, \frac{C(q)L}{12} - \frac{L^2}{24} + 6t \right). \tag{22.11}$$

Direct subtraction gives

$$F(s+1,t) - F(s,t) = b(s,t), \qquad F(s,t+1) - F(s,t) = \boldsymbol{\gamma}(s,t). \tag{22.12}$$

The second difference is the displayed integral period, so the translation descends to the torus. Thus $\zeta \mapsto \zeta + F$ is a holomorphic isomorphism from the untwisted Jacobian to the $+b$ affine torus torsor over U; its inverse $\zeta \mapsto \zeta - F$ trivializes that torsor. Replacing b, F by $-b, -F$ gives the opposite component rotation. If σ denotes the affine section of 22.26, the map from the family to the $+b$ torsor is explicitly $\zeta \mapsto \zeta - \sigma + F$. This map preserves the affine cusp marking.

4. *A rotating compactification before changing the q-boundary.* In the root-cover model, the two period shears in cocharacter coordinates (y_1, y_2, h_z, h_r) are

$$G_1 : (y_1, y_2) \longmapsto (y_1 + 12h_z, y_2),$$
$$G_2 : (y_1, y_2) \longmapsto (y_1, y_2 + 12h_z + 12h_r).$$

The positive fractional deck has the shear

$$H : (y_1, y_2) \longmapsto (y_1, y_2 + h_z + h_r), \qquad H^{12} = G_2, \tag{22.13}$$

and rotates z by $e^{2\pi i/12}$. It commutes with G_1. Its unit factors are holomorphic functions of q. For example, in the convention $W = e^{-2\pi i\zeta_2}$, it multiplies W by

$$rz\, e^{-2\pi i C(q)/12} e^{\pi i/12}$$

and the first multiplicative coordinate by $e^{\pi i\mu(q)}$. The twelve successive root rotations contribute the factor $e^{2\pi i(0+\cdots+11)/12} = -1$, and the scalar correction contributes $e^{\pi i} = -1$. They cancel. The twelfth iterate is consequently the second period, including its unit, as also follows additively from (22.10).

5. Use the unit-square cusp triangulation, with twelve consecutive bottom cells in the y_1 direction and the same bridge choice in every unit prism. Its translates have bottom cell indices (i, j) and top interval $[j, j + 1]$. Thus H increases the bottom second index and the top index by one, while G_1 shifts the first index by twelve. The fan is invariant under both, and therefore under G_2. For each bottom cell its three maximal cones are the determinant-one cones of 21.1. The usual compact-core construction of 20.6, now on the I_{12} elliptic root model and with twelve plumbing intervals before the G_2 quotient, gives a proper model on the root cover. Its unfolded charts are precisely this fan. The compact core and collars therefore prove properness and proper discontinuity.

6. The affine deck extends to this model because its unit multiplication preserves every toric chart and its shear preserves the fan. It is free. On the r-central divisor a point lies on at most two adjacent component labels in $\mathbb{Z}/12$; no nonidentity translation can preserve that set. This covers their intersection over the common base origin as well. On the q-central divisor with $r \neq 0$, the bottom triangle has second component indices contained in $\{j, j + 1\}$, again not preserved by a nonzero translation modulo twelve. Off both axes the root-base action is free. Its finite quotient is therefore smooth, proper over (q, r), flat over r, and has a reduced normal-crossings r-central divisor.

7. Keeping the same positive root-base rotation and using translation $-b$ gives valuation shear H^{-1} and twelfth iterate G_2^{-1}. This action preserves the same fan and has the same freeness properties. For the family in the statement we use translation by b and the clockwise cusp generator $g_0 : s \mapsto s - 1$. Thus the positive fractional deck b has shift $+1$ for $s \mapsto s + 1$ and shift -1 for g_0, exactly as required by 22.6. The finite shifts $4, -3$ and this cusp shift all refer to the same clockwise deck-identification convention.

8. Denote this rotating model by $\mathcal{R}$. Its q-fibre for $r \neq 0$ has multiplicity twelve. Indeed the original z-central divisor is reduced and the finite action has trivial generic isotropy there, while $q = z^{12}$. This is the q-multiplicity computed in 22.7.

9. *The two-open gluing.* On the base open set $q \neq 0$ use $\mathcal{R}$ with the positive fractional deck b. On the base open set $r \neq 0$ use the reduced Mumford cusp family. Their common base is U, where (22.11)– (22.12), composed with 22.26, give the specified holomorphic isomorphism. These two open sets cover $B^\times$, and there is no additional triple overlap. Glue by that isomorphism.

10. The result is Hausdorff: points over one base point are separated in one of the two given Hausdorff models, and points with different base images are separated in the base. Properness is local on the target, so follows from the properness of those two restrictions. Smoothness and flatness over r are also local. The restriction over $r \neq 0$ is, by construction, the prescribed cusp including its reduced q-fibre. Over $r = 0, q \neq 0$, the construction has the required twelve-cycle and rotation -1.

□

Remark 22.40 (The two axial models and their overlap)**.** [tag PT18] Proposition 22.39 uses the rotating model $\mathcal{R}$ on $q \neq 0$ and the reduced Mumford family on $r \neq 0$. On their overlap the exact isomorphism is $\zeta \mapsto \zeta - \sigma + F$, with F given by (22.11). The larger rotating model has local equation $q = z^{12}$ at its q-boundary for $r \neq 0$, so every q-vertical divisorial valuation there has order in $12\mathbb{Z}$. The reduced Mumford boundary has a divisor of q-order one. A common proper modification over q would identify the function q and the associated divisorial valuations; Proposition 22.11 gives the resulting contradiction. Thus the overlap isomorphism specifies precisely the two-open gluing on $B^{\times}$. Its noncompletion at the missing origin is Proposition 22.54, and the obstruction for the full global family is Theorem 18.6.

22.8 The global family with one base point removed

Proposition 22.41 (The global partial filling)**.** [tag 030N]

Let

$$\Omega = (\mathbb{P}^1 \times \Delta_r) \setminus \{(\infty, 0)\}.$$

There is a proper Hausdorff holomorphic map $\mathcal{Y}^{\circ} \to \Omega$ with smooth total fourfold, flat over r, whose restriction to $r \neq 0$ is the family pulled back by $w = r^6$. Its reduced normal-crossings central divisor has the finite-place models of 22.2 and the punctured cusp of 22.39, with their given markings. Over $B^{\circ} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ its relative component cover is

$$y^{12} = K(x), \qquad K(x) = \frac{x^4}{(x-1)^3}.$$

Thus the global finite-place overlap maps and their cocycle are constructed. The sole missing base point is $(\infty, 0)$. Proposition 22.54 proves its noncompletion, and 18.6 excludes every proper filling of the family over the whole r disc.

Proof. 1. We distinguish the ordinary base coordinate x from the logarithmic parameter

$$v = \frac{\log r}{2\pi i}, \qquad c = -12v, \qquad e = (0, 1)^t.$$

Use the coherent closing of 17.8. Its additive formula for a positive r loop, that is $c \mapsto c - 12$, is

$$b_{-12,c} = \Pi_c \widehat{\gamma} - (k+6)e, \qquad k = 12\ell = \frac{\log K}{2\pi i}. \tag{22.14}$$

The first summand and $6e$ are periods. Consequently its torus translation has the scalar representative $-ke$, since addition of those periods preserves the quotient map.

2. The coherent closing in 17.8 may be used for the specified punctured family. More explicitly, two coherent choices differing by a scalar automorphism give a principal $\mathbb{C}^*$ torsor over the punctured parameter disc. That torsor is holomorphically trivial: the exponential sequence and Cartan's theorem B give $\mathrm{Pic}(\Delta_r^*) = H^2(\Delta_r^*, \mathbb{Z}) = 0$. The resulting scalar gauge is a holomorphic isomorphism of the punctured families.

3. *The ordinary charts and their exact cocycle.* Choose a good analytic cover of B°, refining period-frame charts, and on each member choose a holomorphic logarithm $k_i = (2\pi i)^{-1}\log K$. On the parameter logarithm cover put

$$\widetilde{\zeta}_i = \zeta + k_i v e. \tag{22.15}$$

Under the positive r deck, ζ changes by $-k_i e$ modulo periods and v changes by one. Thus (22.15) has the ordinary period descent. It identifies the local punctured family with the standard twelve-cycle plumbing of 22.1, now over an ordinary base chart.

4. On an overlap let $n_{ij} = k_j - k_i \in \mathbb{Z}$. The invariant vector e is fixed by every period-frame change. In the multiplicative coordinate $W_i = \exp(-2\pi i\widetilde{\zeta}_{i,2})$, the extra transition is exactly

$$W_j = r^{-n_{ij}} W_i. \tag{22.16}$$

Every integral power in (22.16) extends to an automorphism of the whole twelve-cycle plumbing. On the unfolded node chart write

$$x_j = W/r^j, \qquad y_j = r^{j+1}/W, \qquad x_j y_j = r.$$

Multiplication $W \mapsto r^{-n}W$ sends the chart index j to $j-n$ and leaves both displayed chart coordinates unchanged. The same statement holds in bundle frames, whose additional factors are holomorphic units. It commutes with the closing period, which shifts the index by twelve and translates the elliptic base by $6P$. The usual semiabelian translations and period-frame changes preserve the two sections, hence also extend.

5. The integers satisfy $n_{ik} = n_{ij} + n_{jk}$. Since the scalar translations commute with each other and with the period-frame changes, (22.16) is an exact cocycle on the entire plumbing. This constructs the family over $B^\circ \times \Delta_r$, with the original punctured-family identification (22.15).

6. *The finite-place cancellations.* At the two root charts put

$$w_j = \frac{\log s_j}{2\pi i}, \qquad \epsilon_1 = 1, \quad \epsilon_2 = -1, \qquad k_j = 12\kappa_j.$$

The notation is that of 17.8: k_j is a holomorphic logarithm of the unit $K(x)s_j^{-12\epsilon_j}$ on the full root disc. Its invariance under the finite deck is exact. Indeed the unit is invariant since $m_j \mid 12$; the difference of two logarithms is an integer constant, and evaluating at the fixed root centre makes that constant zero. The scalar representative of the local positive r closing is $-k_j e$. Apply the local gauge $\widetilde{\zeta}_j = \zeta_j + k_j v e$ and use the smooth twelve-cycle root model of 22.1, with its free finite deck action from 22.2. Because e is invariant and k_j is invariant, this gauge does not change that finite action.

7. On a sector of the punctured root disc choose compatible logarithms. The root-to-ordinary collar is

$$\zeta = \zeta_j + \sigma_{j,c}, \qquad \sigma_{j,c} = w_j \Pi_c v_j, \qquad \gamma(v_j) = \epsilon_j.$$

Moreover $k - k_j = 12\epsilon_j w_j$. Therefore its expression in the new gauges is

$$\begin{aligned} \widetilde{\zeta} - \widetilde{\zeta}_j &= w_j \Pi_c v_j + 12\epsilon_j w_j v e \\ &= w_j \Pi_{\mathrm{part}} v_j, \qquad \Pi_{\mathrm{part}} = \Pi_c - ce\gamma. \end{aligned} \tag{22.17}$$

Here $c = -12v$ is essential: the parameter logarithms cancel exactly. The last translation is independent of r. On each sector it is a holomorphic section of the semiabelian group and therefore extends to the ruled normalization and all node charts.

8. Changing a logarithm adds an integral transition of the form (22.16), together with a period-frame identification. Those maps already extend. Their deck and overlap identities agree on the dense locus $r \neq 0$ with the collar identities, hence agree everywhere by the identity principle. Thus the sector maps descend and glue exactly to the finite quotients. This constructs the specified family over $(\mathbb{P}^1 \setminus \{\infty\}) \times \Delta_r$. It retains the multiplicities three and four and the reductions already proved in 22.2.

9. The component cover is determined by the local indices. If i_i is the unfolded component index modulo twelve, then (22.16) gives $i_j = i_i - n_{ij}$. Hence

$$y = \exp(2\pi i k_i/12)\exp(2\pi i i_i/12)$$

is independent of the chart and satisfies $y^{12} = K$. The clockwise deck shifts are consequently $4, -3, -1$. The local finite quotients complete this cover at $0, 1$ exactly as in 22.6.

10. *The cusp overlap, including* $r = 0$. Retain $s = \log q/(2\pi i)$ and the holomorphic functions C, μ, κ_0 of 22.39 and 17.8. Then $k = -s + 12\kappa_0$. The cusp-to-rotating comparison of the positive-deck model is $\zeta \mapsto \zeta - \sigma + F$. Substitution in the displayed formulas of 22.26 and 22.39 gives

$$F - \sigma = kve + G(s) + 6ve, \qquad G(s) = \left(\frac{\mu s}{2}, \frac{Cs}{12} - \frac{s^2}{24}\right). \tag{22.18}$$

Thus its comparison with the ordinary gauge (22.15) is the translation $G(s) + 6ve$. On a sector of $q \neq 0$, G is independent of r and gives a holomorphic semiabelian translation. The other term is the integral fan shift $W \mapsto r^{-6}W$. Both extend, with their inverses, over the full twelve-cycle r-central divisor. The branch identities are

$$G(s+1) - G(s) = b(s, v) + ve, \qquad k(s+1) - k(s) = -1.$$

The ordinary coordinate deck contributes $-ve$; their sum is exactly the positive fractional deck b. A positive r loop changes $6ve$ by the integral period $6e$. These identities prove descent of the extended comparison to the full punctured q disc. In particular the common clockwise cusp shift is -1, as required.

11. Glue the just constructed global model on $x \neq \infty$ to 22.39 on the cusp neighbourhood minus its origin, using (22.18). The overlap contains $r = 0$ and the map extends there by the preceding chart calculation. The identities give the exact scalar and translation cocycles with the prescribed punctured marking.

12. Each chart used is a proper model over its base open set: this was proved by compact-core plumbing in 20.6 and by the finite free quotient in 22.2; 22.39 is proper on its stated punctured bidisc. Properness is local on the target, so the glued map to Ω is proper. The same local argument gives Hausdorffness, smoothness of the total space, flatness over r, and the reduced normal-crossings central divisor. For $r \neq 0$ the base is the entire $\mathbb{P}^1$, and the specified comparison is precisely the compact family. At $r = 0$ the target is the finite base $\mathbb{P}^1 \setminus \{\infty\}$, as specified in Ω.

□

22.9 An embedded Tate interface at the missing corner

Proposition 22.42 (The other Kato axis)**.** [tag 030P]

In the proper smooth Kato family $\mathcal{K} \to \Delta_s$ of 22.27 there is a second relative effective Cartier divisor $\mathcal{D} \subset \mathcal{K}$, disjoint from the persistent elliptic divisor $\mathcal{E}$. The surface $\mathcal{D}$ is smooth and proper over Δ_s. Its general fibre and central fibre are

$$\mathcal{D}_s = \mathbb{C}^*/(\kappa(s)s^n)^{\mathbb{Z}} \quad (s \neq 0), \qquad \mathcal{D}_0 = \text{the reduced rational } n\text{-cycle}.$$

At its nodes the relative local equation is $uv = s$. Its normal line in $\mathcal{K}$ is the multidegree-zero Tate line whose factor on the closing collar is $a(s)$. In particular it has degree zero on every component of the central cycle. Scaling the first blowdown coordinate gives a holomorphic $\mathbb{C}^*$ action on the pair $(\mathcal{K}, \mathcal{D})$ over the base.

For $n = 12$ this constructs an embedded proper copy of the cusp's elliptic root interface, with its degree-zero normal line, after choosing the units κ, a to be the Tate and normal multipliers. In the additive period convention $\mathbf{u} = (\tau, \mu)$, $W = e^{-2\pi i\zeta_2}$, these multipliers are $\kappa(s)s^{12} = e^{2\pi i\tau}$ and $a(s) = e^{-2\pi i\mu}$. The elliptic translation by $6P$ is realized by first-coordinate scaling $a(s)^{-6}$.

The comparison identifies the embedded Cartier divisor and its normal line. At each central node the ambient base map has the local form $s = uv - h$ computed below.

Proof. 1. Use the modification, embedding and compact collar domains of 22.27. Write w for the second blowdown coordinate and set, on the modified fundamental region,

$$h_s = \pi_n^* w - s.$$

The collar identification is $\sigma_s\pi_n$, where $\sigma_s(z, w) = (\kappa(s)z, a(s)w + (1 - a(s))s)$ in the last chart. Consequently

$$h_s \circ (\sigma_s\pi_n) = a(s)h_s. \tag{22.19}$$

Its zeros therefore glue to a relative effective Cartier divisor, and (22.19) is simultaneously the normal-line transition. The divisor is closed in the proper family $\mathcal{K} \to \Delta_s$, so it is proper.

2. For $s \neq 0$, the line $w = s$ misses every blow-up centre. In the coordinate $v = w - s$ the contracting germ is

$$(x, v) \longmapsto (\kappa(s)x(a(s)v + s)^n, a(s)v).$$

Its restriction to $v = 0$ is $x \mapsto \kappa(s)s^n x$. The generic developing map of 22.27 restricts to the identity in this coordinate, since its product $G_s(v)$ has $G_s(0) = 1$. Thus this is precisely the unmodified other Hopf axis, with the stated Tate multiplier and normal multiplier $a(s)$.

3. The divisor of $\pi_n^* w$ is reduced. After the first blow-up, both the strict horizontal axis and the exceptional curve have coefficient one. Every later centre is the intersection of the newest exceptional curve with the strict vertical axis. It lies on no other component of the divisor of $\pi_n^* w$. The new exceptional coefficient is therefore again one. Before the end collars are identified, the divisor is a chain comprising the strict horizontal axis and the n exceptional pieces. The end annuli join the two endpoint pieces, leaving the familiar rational n-cycle of the parabolic surface. It is disjoint from its elliptic curve, as also described in the parabolic model of (Istrati et al., 2022, Example 9.10, p. 1386).

4. At a crossing of two adjacent pieces, ordinary blow-up coordinates give $\pi_n^* w = uv$ up to a holomorphic unit, which can be absorbed in u. Hence $\mathcal{D}$ has equation $uv = s$. It is smooth, and its central fibre is reduced nodal. Away from these nodes its defining equation is a coordinate minus s. This proves smoothness of the whole divisor and semistability over the disc. The monomial transition charts are those of the unit-interval Tate model, and the closing on the nonzero axis is exactly $x \mapsto \kappa(s)s^n x$. They identify the marked family with that Tate model over Δ_s.

5. The same local defining functions trivialize the normal line. Their only nontrivial closing factor is (22.19), namely the holomorphic unit $a(s)$. On each rational component the line has degree zero; equivalently its two adjacent-node degree corrections cancel its self-intersection -2. For $n = 1$ the same statement uses the two branches of the self-node. The factor $a(s)$ gives the gluing parameter around the cycle. This is exactly the multidegree-zero extension of the line with factor $a(s)$ on the smooth Tate curves.

6. The first-coordinate scaling commutes with the modification and the collar maps and preserves h_s. One may equivalently descend the vector field $x\partial_x$ and its local flow. Properness over the base makes that flow complete on compact fibres, uniformly on compact parameter sets; its $2\pi i$ period yields the stated holomorphic $\mathbb{C}^*$ action. On the Tate axis it is ordinary multiplication of x, so a holomorphic unit $a(s)^{-6}$ realizes the marked translation by $6P$.

7. For the cusp, the first lattice period acts in multiplicative coordinates by

$$(Z, W) \longmapsto (e^{2\pi i\tau} Z, e^{-2\pi i\mu} W).$$

   After the twelfth root change, its first multiplier is $\kappa(s)s^{12}$ for a holomorphic unit. The second multiplier tends to the contracting inverse of $-(5 + 2\sqrt{6})$, so after shrinking $0 < |a(s)| < 1$ as required in 22.27. The preceding exact Tate charts and the normal factor identify this interface and its line with the cusp interface and its normal line. The two section normals in the ruled model are inverse; one chooses the matching branch accordingly. More precisely, in the total line-coordinate convention, this W normal is M^{-1} for $M = \mathcal{O}(P - O)$: the theta ratio with divisor $P - O$ has multiplier $e^{2\pi i\mu}$, the inverse of a. The opposite ruled section uses M. The separate first-coordinate factor a^{-6} is the translation by $6P$.

8. At a central node, the smooth Kato total space has coordinates (u, v, s) with divisor $h = uv - s = 0$. In coordinates (u, v, h) its map to the base is

$$s = uv - h. \tag{22.20}$$

   It is a submersion. The total space of the normal line, with the inherited map from $\mathcal{D}$, has $s = uv$, with a critical normal line over that node. Therefore there is no relative linear tubular-neighbourhood identification across the nodes. An isomorphism over the base would identify the critical loci of the two base maps, contradicting these explicit differentials.

9. With an inverse-normal plumbing variable f, the local equation is

$$r = fh = f(uv - s). \tag{22.21}$$

   The total fourfold in these coordinates is smooth and its r-central divisor is reduced normal crossing. For $r \neq 0$ one has $f \neq 0$ and $s = uv - r/f$, whose derivative in f is $r/f^2 \neq 0$. Thus the map to s on this local nonzero-r fibre is a submersion at the displayed locus.

10. There is also a concrete obstruction to the most direct global use of this plumbing. For fixed $s \neq 0$, all the n blow-up centres lie on the first Hopf axis $x = 0$, at $[v = -s]$ and its infinitely near strict-axis points. The other axis $\mathcal{D}_s$ is $v = 0$, $x \neq 0$. The axes are disjoint, and hence every exceptional rational curve misses $\mathcal{D}_s$. A plumbing that leaves the complement of a small tubular neighbourhood of $\mathcal{D}_s$ unchanged retains these compact curves in its $r \neq 0$ fibres over the same s. A complex torus has no rational curve: a holomorphic map from $\mathbb{P}^1$ lifts to its vector-space universal cover and is constant. Thus this fixed-base localized plumbing of the entire Kato surface cannot produce the prescribed ordinary two-tori.

□

Proposition 22.43 (A degree-one Kato model with a linear twelve-gon collar)**.** [tag 030Q]

Start with the $n = 1$ Kato family of 22.27 over a disc Δ_q, with holomorphic units $a(q), \kappa(q)$ and $0 < |a(q)| < 1$. Pull it back by $q = s^{12}$. There is a proper modification of this pulled-back threefold, obtained by six point blow-ups and seventy-two smooth curve blow-ups supported on its central rational cycle, with the following properties. Its total space $\mathcal{K}^*$ is smooth, it is proper and flat over Δ_s, and its restriction to $s \neq 0$ is the original pulled-back Kato family.

The strict transform $\mathcal{D}^*$ of the other Cartier axis is the smooth semistable Tate surface with multiplier $\kappa(s^{12})s^{12}$ and central fibre I_{12}. A neighbourhood of $\mathcal{D}^*$ in $\mathcal{K}^*$ is biholomorphic over Δ_s to a neighbourhood of the zero section in the multidegree-zero Tate line with closing factor $a(s^{12})$. The comparison retains the explicit transverse coordinate

$$H = \frac{h}{q}, \qquad h = \pi_1^* w - q, \tag{22.22}$$

on the original punctured family. Thus its normal comparison, relative to the original punctured normal frame, has factor $q^{-1} = s^{-12}$; its inverse has factor q.

The persistent elliptic axis has an unchanged neighbourhood. In particular its normal in the strict transform of the original central Kato surface has degree -1. The displayed function H and the closing factors specify the linear-collar isomorphism on a neighbourhood of $\mathcal{D}^*$.

Proof. 1. We distinguish the original parameter q from its twelfth root s. The pulled-back Kato total space is smooth: the original map to q is a submersion, so locally its pullback has smooth coordinates consisting of two fibre coordinates and s. The other axis of 22.42 is Cartier, with closing factor $a(q)$ for its defining function h. At its unique central node its local equation after this base change is

$$h = uv - s^{12}. \tag{22.23}$$

Away from that point the axis surface is smooth. The persistent elliptic axis is disjoint from it throughout the family.

2. *The six point blow-ups.* Blow up the singular point of the strict transform of the Cartier surface, six times in succession. All these are points of the smooth ambient threefold. If its current equation is $uv = s^m$, with $m \geq 2$, the s chart of the ambient point blow-up has

$$u = su_1, \qquad v = sv_1, \qquad u_1 v_1 = s^{m-2}. \tag{22.24}$$

The u chart has $v = uv_1$, $s = us_1$, and strict-transform equation

$$v_1 = u^{m-2} s_1^m; \tag{22.25}$$

the v chart is symmetric. The latter charts are smooth. For $m > 2$ the only remaining singular point is the origin in (22.24); for $m = 2$ that chart has $u_1 v_1 = 1$. The exponents are $12, 10, 8, 6, 4, 2$, so precisely six point blow-ups resolve the surface.

3. Write D for its smooth strict transform. Its central fibre is

$$F = C_0 + C_1 + \cdots + C_{11} = \mathrm{div}_D(s). \tag{22.26}$$

Here C_0 is the strict transform of the old nodal rational curve. Each of the first five point blow-ups contributes two rational exceptional curves, and the sixth contributes the smooth conic in its exceptional plane. In cyclic order the two chains are $C_1, \ldots, C_5$ and $C_{11}, \ldots, C_7$, with the conic C_6 between them. Equations (22.24)– (22.25) show that every component has multiplicity one, that consecutive components meet transversely once, and that there are no other intersections. Thus the fibre is I_{12}. Since F is principal, $C_i \cdot F = 0$ and $C_i^2 = -2$. The same charts identify D with the minimal regular Tate model of the given multiplier, including the original closing unit $\kappa(q)$.

4. Let M be the pullback to D of the original Cartier normal line. It is the Tate line with factor $a(q)$ and multidegree zero. A component contracted to the original node has degree zero for this pullback, and its restriction to C_0 is the pullback of the degree-zero line on the nodal curve. In the iterated s chart after j point blow-ups the original function is

$$h = s^{2j}(u_j v_j - s^{12-2j}).$$

The jth exceptional ambient divisor therefore has coefficient $2j$ in the total transform of $\mathrm{div}(h)$. For $j < 6$ its intersection with D consists of $C_j + C_{12-j}$, and for $j = 6$ it is C_6. Consequently the new normal line is

$$N_{D/\mathcal{K}^{(6)}} = M \otimes \mathcal{O}_D(-Z), \quad Z = \sum_{i=0}^{11} z_i C_i, \quad z_i = 2\min(i, 12-i). \tag{22.27}$$

The list of coefficients is $(0, 2, 4, 6, 8, 10, 12, 10, 8, 6, 4, 2)$. In particular its degree vector is -4 on C_0, $+4$ on C_6, and zero on the other ten curves.

5. *The seventy-two curve blow-ups and the function.* For each i, blow up the ambient threefold along the current copy of C_i exactly

$$b_i = 12 - z_i$$

times, in increasing order of i; the entry $b_6 = 0$ requires no blow-up. At every step C_i is a smooth embedded compact curve. Its ideal on D is invertible, so the strict transform of D is canonically isomorphic to D itself. The normal line is tensored by $\mathcal{O}_D(-C_i)$ at that step. Thus the final normal is

$$M \otimes \mathcal{O}_D(-Z - \textstyle\sum_i b_i C_i) = M \otimes \mathcal{O}_D(-12F) \simeq M. \tag{22.28}$$

The sum of the b_i is 72. The last isomorphism divides the local generator s^{12} of $\mathcal{O}_D(-12F)$ by s^{12}. Relative to the original punctured normal identification it is multiplication by s^{-12}, as asserted.

6. The point-blow-up charts give the following exact local description near D. At a node $C_i \cap C_{i+1}$ one can choose ambient coordinates (x, y, e) such that

$$D = (e = 0), \qquad s = xy, \qquad h = e x^{z_i} y^{z_{i+1}} \epsilon, \tag{22.29}$$

where ϵ is a holomorphic unit. At a general point of C_i, omit y and use $s = x$. These formulas follow directly from (22.25): the strict transform is a coordinate minus a monomial, and the two factors of s are the two vertical divisors. They also show transversality of D to their intersection. Absorbing a unit in the second base coordinate preserves this description if different original coordinate frames are used.

7. Blowing up C_i is locally the blow-up of (e, x). In the chart containing the strict transform of D, put $e = xe'$. Formula (22.29) keeps $s = xy$ and increases the exponent of x in h by exactly one. The old vertical divisor in this direction no longer meets the strict transform of D. At an adjacent node the exponent of y is unchanged. This calculation applies at every repetition and for the other component as well. At the end, every vertical ambient divisor meeting D has s-order one and h-order twelve. Hence (22.29) becomes

$$h = s^{12} e \epsilon, \qquad H = h/q = e\epsilon. \tag{22.30}$$

In particular H is holomorphic near all of D, has divisor exactly D, and has nonzero normal derivative there. On the closing collar its transition is exactly $H \mapsto a(q)H$. Thus the modification makes the meromorphic function H holomorphic on this neighbourhood.

8. *The relative product charts.* There is an equivalent useful local target for the preceding modifications. The ideal (h, q) is globally defined, since its changes of generators multiply h by the unit $a(q)$. Near D the modification maps to its q blow-up chart, because (22.30) makes the pullback ideal principal. At the old node this chart has

$$uv = s^{12}(1 + H).$$

Restrict to $|H| < \eta < 1$ in a local closing frame and put $V = v/(1 + H)$. The equation becomes

$$uV = s^{12},$$

times the independent coordinate H. The preceding smooth modification near D is its minimal resolution times that coordinate.

9. Label the exceptional chain so that the orders of u, V, s on C_i are $i, 12 - i, 1$, respectively, with the two end branches at indices $0, 12$. These are the values in the explicit surface resolution above. Along any ambient vertical divisor meeting C_i, the orders of u and V sum to twelve, because $uV = s^{12}$ and s has order one. Restriction to D can only increase those orders. Since the restricted orders already sum to twelve, the ambient orders are exactly $i, 12 - i$. No other zero or pole of either character occurs near D, because their product is s^{12}.

10. On the chart of the two adjacent rays $i, i + 1$, the functions

$$X_i = u/s^i, \qquad Y_i = s^{i+1}/u, \qquad X_iY_i = s \tag{22.31}$$

therefore extend holomorphically. A meromorphic function on the smooth ambient space with no codimension-one pole is holomorphic. Their restrictions are the two standard minimal resolution coordinates on D. Together with H, whose normal derivative is nonzero, they have invertible differential along D. Thus (22.31) and H are biholomorphic product charts near D. Their character identities give the standard transitions of the minimal resolution of A_{11}.

11. *The closing and the uniform linear collar.* The Kato formula determines the global closing. In its blowdown coordinate x and $h = w - q$, that formula is

$$(x, h) \longmapsto (\kappa(q)x(a(q)h + q), a(q)h).$$

On the regularized collar this reads

$$(x, H) \longmapsto (\kappa(q)qx(1 + a(q)H), a(q)H). \tag{22.32}$$

Shrink the parameter disc so $|a(q)| \le A < 1$. For $|H| < \eta < 1$ the product

$$G(q, H) = \prod_{j\ge1} (1 + a(q)^j H) \tag{22.33}$$

converges locally uniformly in both variables, is holomorphic and nonzero, and satisfies

$$G(q, a(q)H) = \frac{G(q, H)}{1 + a(q)H}.$$

Uniform convergence follows from $\sum_{j\ge1} |a(q)^j H| \le \eta A/(1 - A)$; the factors have no zero in the chosen neighbourhood, and the convergent logarithm series gives a nonzero product and holomorphic inverse. In the new character $Z = xG(q, H)$, (22.32) becomes

$$(Z, H) \longmapsto (\kappa(q)qZ, a(q)H). \tag{22.34}$$

Multiplication of a toric character by this unit extends on the product charts (22.31): it multiplies X_i by the unit and Y_i by its inverse. It therefore gives isomorphisms at all twelve

nodes as well as on the annular core. The exact closing (22.34) identifies these charts with the Tate line of factor $a(q)$, over the Tate family of multiplier $\kappa(q)q$. Their inverse maps obey the same identities. Compactness of the central divisor and a finite cover by core and node charts allow one common shrinking of the base and the collar neighbourhood. This proves the claimed neighbourhood isomorphism.

12. Every ambient modification used above is a blow-up of a smooth closed centre and is proper. The total space remains smooth, and its map to the parameter disc is flat because a nonzero base function is a non-zero-divisor in its smooth local rings. All centres lie over $s = 0$, so the punctured family is unchanged. All centres are disjoint from the persistent elliptic axis, so its entire neighbourhood is unchanged; in particular its central normal degree remains -1.

□

22.10 A local resolution retaining the cusp factor

Proposition 22.44 (The local plumbing model)**.** [tag 030R]

Let

$$X_n = \{fuv = gs^n\} \subset \mathbb{C}^5, \qquad r = g - f, \qquad q = s^n.$$

There is a proper bimeromorphic local modification $\widetilde{X}_n \to X_n$ with smooth total fourfold and reduced normal-crossings divisor $r = 0$. For fixed $r \neq 0$ it resolves the cyclic base change of a three-branch q-normal-crossings chart. The divisor $s = 0$ after the displayed cyclic base change is reduced; the divisor of the original function $q = s^n$ has multiplicity n.

Proof. 1. For $n = 1$ first blow up (u, q) and then the pullback of (f, g). The charts of the resulting modification are

coordinates	map to X_1	r
(q, U, v, f)	$u = qU,\ g = fUv$	$f(Uv - 1)$
(u, Q, G, f)	$q = uQ,\ v = GQ,\ g = fG$	$f(G - 1)$
(u, v, F, g)	$q = uvF,\ f = gF$	$g(1 - F)$.

To derive them, the q chart of the first blow-up has $u = qU$, hence $g = fUv$; the ideal (f, g) is already principal there. Its other chart has $q = uQ$ and equation $fv = gQ$. Blowing up (f, g) gives the last two charts. These are the blow-up charts, with the common factors cancelled on the closure of the dense torus. They are smooth. In the first chart $Uv - 1$ is a smooth coordinate along its zero set, and in the others $G - 1$ or $1 - F$ is a coordinate. Thus the displayed products prove the reduced NC assertion for r.

2. For arbitrary n, pull back this proper modification by $q = s^n$. The first chart remains smooth. The second is $uQ = s^n$ times the two coordinates G, f. Resolve its A_{n-1} factor by the unit-interval fan. The third chart is

$$uvF = s^n$$

times the independent coordinate g. Its cone lattice may be written as $N = \mathbb{Z}^3$ with coordinates (a, b, d), where the valuations of u, v, F, s are $a, b, nd - a - b, d$. The cone is the cone over $n\Delta_2$. Insert all rays $(i, j, 1)$ with $i, j \geq 0$ and $i + j \leq n$, and triangulate by the unit triangles

$$\begin{aligned} &(i, j), (i + 1, j), (i, j + 1) && (i, j \geq 0,\ i + j \leq n - 1), \\ &(i + 1, j), (i, j + 1), (i + 1, j + 1) && (i, j \geq 0,\ i + j \leq n - 2). \end{aligned}$$

Every determinant is one in absolute value; the triangles partition the large triangle and give a proper smooth toric subdivision. Their number is n^2. All new rays have s-height one.

3. The open set $F \neq 0$ is the toric open corresponding to $nd - a - b = 0$. Its fan is the unit subdivision of the edge $i + j = n$, hence the minimal A_{n-1} resolution times $\mathbb{C}^*$. This is precisely the second chart's resolution under $Q = vF$. Where that chart meets the first, the A-type resolution is an isomorphism. These facts glue the local proper resolutions to a proper modification of the whole local X_n.

4. The function F is a toric monomial. Above $F = 1$ it is submersive: in a regular toric chart a nonzero monomial has nonzero derivative in a coordinate with positive exponent. Equivalently, on $F \neq 0$ it is the displayed torus projection. Thus $r = g(1 - F)$ has two smooth transverse branches in the third chart after resolution. The second chart has $r = f(G - 1)$ independently of the resolved surface coordinates, and the first was already checked. This proves smoothness and reduced normal crossings over r for all n.

5. For $r \neq 0$ in the third chart, $g = r/(1-F)$, so the local fibre is the resolved threefold $uvF = s^n$ with $F \neq 1$. For $n = 1$, its q divisor is $uvF = 0$, a literal three-branch normal-crossings chart. For $n = 12$ the standard triangulation gives 144 unimodular charts above this one triple point. This is the local cyclic-cover count at each cusp triple point. The other potential singularity, at $f = -r$, is included in this construction: the first blow-up small-resolves it for $n = 1$, and the second is an isomorphism there because f is nonzero.

6. The equation arises from the modified neighbourhood of 22.43. There $h = uv - q$ in the node chart before modification, and $H = h/q$ is holomorphic in the modified collar. The unit inverse-normal plumbing $Hf = r$ is exactly

$$f(uv - q) = rq, \qquad fuv = (f + r)q.$$

   Putting $g = f + r$ gives X_n, including its factor q. The modified collar, where H is small, identifies with the third chart near $F = 1$ by

$$F = (1 + H)^{-1}, \qquad g = f(1 + H), \qquad r = fH.$$

   There $uvF = s^n$ is exactly the resolved equation $uv = s^n(1 + H)$ from 22.43. The charts near $F = 0$ are additional local completion charts.

7. The first collar action is $H \mapsto aH$, $f \mapsto a^{-1}f$, which fixes $r = fH$. The function $G(q, H)$ of 22.43 conjugates this action on its stated small collar. For fixed q, the product converges locally uniformly for $H \in \mathbb{C}$, because $\sum_j |a(q)^j H|$ does so on compact sets. At $H = -a^{-j}$ it vanishes, and the coordinate change $Z = xG(q, H)$ has zero derivative in x. For $b \in \mathbb{C}^*$, on the open set where $r \neq 0$, $f \neq 0$ and $W = 1 + r/f \notin \{1, b^{-1}\}$, the map $f = r/(W - 1)$ conjugates multiplication $W \mapsto bW$ to the rational map

$$f \longmapsto \frac{rf}{(b-1)f + br}.$$

   Indeed substituting $b(1 + r/f)$ into $r/(W - 1)$ gives the displayed formula. This is the exact coordinate comparison for the test variable W.

□

Proposition 22.45 (Exceptional-curve nonclosure and persistence in a semistable cap)**.** [tag 030S]

Let $\mathcal{K}^* \to \Delta_s$ be the modified degree-one Kato family of 22.43, with $q = s^{12}$, and let $\mathcal{E} \to \Delta_s$ be its unchanged persistent elliptic axis. For $s \neq 0$ let $C_s \simeq \mathbb{P}^1$ be the exceptional curve of the single Hopf blow-up.

The union of the C_s has no proper analytic divisor closure in $\mathcal{K}^*$. Further finite proper modifications cannot give these curves a proper analytic divisor closure: any such closure would push forward to a forbidden one. In particular, the generic Hopf blowdown does not extend as a

relative proper holomorphic modification, or as a relative meromorphic contraction represented by a proper analytic graph, across $s = 0$.

For the deformation obstruction, suppose a smooth complex fourfold $\mathcal{Y}$ has holomorphic functions (Q, r), its divisor $r = 0$ is reduced normal crossings, and one component K contains a neighbourhood of some C_{s_0}, $s_0 \neq 0$, agreeing with the complementary Kato region and its ordinary base coordinate Q. Suppose this curve is disjoint from every conductor branch of $r = 0$. Then every sufficiently nearby fibre $(Q, r) = (Q_0, r_0)$ contains an embedded rational curve. Consequently such a cap cannot have the prescribed ordinary two-tori as its $r \neq 0$ surface fibres.

Proof. 1. We first use the exact marked blowdown of 22.27. The exceptional curve of the single generic Hopf blow-up meets the persistent elliptic axis transversely at

$$[v = -q] \in E_{a(q)} = \mathbb{C}^*/a(q)^{\mathbb{Z}}.$$

All modifications in 22.43 are disjoint from that axis. The intersection section in the modified family is therefore still

$$p(s) = [-s^{12}]. \tag{22.35}$$

The units $a(q)$ obey $0 < |a(q)| < 1$. Choose a holomorphic logarithm on the full parameter disc and put

$$\tau(s) = \frac{\log a(s^{12})}{2\pi i}.$$

Then $\mathcal{E}_s = \mathbb{C}/(\mathbb{Z} + \tau(s)\mathbb{Z})$ and $\operatorname{Im}\tau(s) > 0$. The maps $(x, y, s) \mapsto [x + y\tau(s)]$, with $(x, y) \in \mathbb{R}^2/\mathbb{Z}^2$, give a smooth trivialization of the elliptic family. A logarithmic lift of (22.35) is

$$z(s) = \frac{\log(-s^{12})}{2\pi i}.$$

On a positive loop around zero, τ returns to itself and z changes by 12. Thus the loop of the section has class $(12, 0) \neq 0$ in the integral first homology of this trivialized family. A continuous section over the disc would fill that loop. Hence p has no continuous, and therefore no holomorphic, extension. A meromorphic section from a nonsingular curve would extend holomorphically: its proper graph is generically degree one over the curve, and normalization gives a finite degree-one map to that normal base. This is equally impossible. A finite base change of degree $d > 0$ multiplies the winding to $(12d, 0)$ and does not change the conclusion.

2. Suppose an analytic divisor Z in $\mathcal{K}^*$, proper over the disc, restricted over the punctured disc to the union of the C_s. Its horizontal irreducible part does not contain $\mathcal{E}$, since the generic intersection is the one transverse point (22.35). Both are Cartier locally in the smooth threefold, so their intersection is an analytic curve, proper over the disc. Its reduced horizontal part has generic degree one. An irreducible horizontal analytic curve cannot have a positive-dimensional fibre without being contained in that fibre. Hence its normalization is finite of degree one over the disc and is isomorphic to the disc. It extends $p(s)$, a contradiction. Vertical components of the intersection have no effect on this horizontal component.

3. If an additional finite proper modification admitted a proper analytic closure, its proper image would give such a Z, by the proper mapping theorem; the image has dimension two and the same generic exceptional curves. The horizontal exceptional locus of a relative proper holomorphic blowdown likewise gives Z. For a meromorphic contraction with proper analytic graph, resolve the graph and apply the same argument to its horizontal exceptional locus and its proper image in $\mathcal{K}^*$. This is a direct marked-intersection proof of a familiar nonproperness phenomenon for exceptional curves in GSS surface deformations; see Dloussky and Teleman (2012, arXiv version 2, Section 1.2, PDF pp. 3–5) for the primary context.

4. Write $C = C_{s_0}$ and $S = K \cap \{Q = Q_0\}$. Near this curve, S is a smooth surface and C is its exceptional curve, so
$$N_{C/S} \simeq \mathcal{O}_{\mathbb{P}^1}(-1).$$
The base coordinate Q is submersive on K near C. Because C misses every other central branch, the reduced function r is a local defining function for K near C and its normal derivative is nowhere zero. Consequently the differential of (Q, r) gives the exact normal sequence
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow N_{C/\mathcal{Y}} \xrightarrow{(dQ,dr)} \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \longrightarrow 0. \tag{22.36}$$
Its extension class vanishes because $H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$. Hence
$$N_{C/\mathcal{Y}} \simeq \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 2}, \qquad H^1(C, N_{C/\mathcal{Y}}) = 0, \qquad h^0(C, N_{C/\mathcal{Y}}) = 2. \tag{22.37}$$
Moreover (22.36) induces an isomorphism on global sections, since both $H^0(\mathcal{O}(-1))$ and $H^1(\mathcal{O}(-1))$ vanish.

5. Kodaira's embedded-deformation theorem therefore gives a smooth two-dimensional local parameter space T of compact embedded deformations of C, with tangent space $H^0(C, N_{C/\mathcal{Y}})$. In precisely this relative surface setting one may apply Dloussky and Teleman (2012, arXiv version 2, Proposition 2.3 and its proof, PDF pp. 6–7): after shrinking the neighbourhood of C, (Q, r) is a holomorphic submersion of relative dimension two, and C is a compact simply exceptional divisor of its fibre. The general completeness theorem is due to Kodaira; we use the exact statements in Merkulov (1995, Section 1, p. 71, and Section 3.1, p. 74). Shrink this family so its curves remain in the chosen neighbourhood. Their genus remains zero, so all are isomorphic to $\mathbb{P}^1$. The functions Q, r are constant on every compact connected curve and thus define a holomorphic map
$$T \longrightarrow \mathbb{C}^2.$$
Its derivative at $[C]$ is precisely the isomorphism on sections induced by (22.36). The holomorphic inverse function theorem shows that the map covers a neighbourhood of $(Q_0, 0)$. In particular the nearby fixed pairs with $r \neq 0$ contain rational curves. A holomorphic map from $\mathbb{P}^1$ to a complex torus lifts to its vector-space universal cover and is constant. Thus those fibres cannot be the prescribed ordinary two-tori.

6. The conductor condition is essential. For example, if a further central component meets K along the persistent elliptic axis, then its intersection number with C is one and $N_{K/\mathcal{Y}}|_C \simeq \mathcal{O}(-1)$. At the conductor intersection dr vanishes on the normal fibre of C, so (dQ, dr) is not the bundle epimorphism of (22.36). This computes precisely where that normal-sequence argument uses the conductor-disjoint hypothesis.

□

22.11 The associated logarithmic torsor at the missing origin

Proposition 22.46 (Obstruction in the fixed log-torus category)**.** [tag 030T]

The local Jacobian admits a nondegenerate logarithmic complex-torus extension over the full (q, r) bidisc and hence has proper analytic models in the sense of Kajiwara et al. (2008b). After the normalized base change $q = Q^{12}$, the two marked axial fan presentations of 22.39–22.41 determine a torsor under its associated log-torus sheaf on the punctured bidisc. This torsor does not extend across the origin as a torsor locally trivial for the usual analytic topology. The obstruction persists after every further finite coordinate-root base change.

On each axial chart its extending origin identifies the associated sheaf with the Jacobian log-torus sheaf $\mathcal{A}$. On the overlap the two identifications differ by the section F in (22.40). An extension here is a locally trivial $\mathcal{A}$-torsor carrying these prescribed axial identifications.

Proof. 1. *The Jacobian pairing.* Retain the period columns and logarithms of 22.39:

$$\gamma = (6\mu, C - s - 12v)^t, \quad u = (\tau, \mu)^t, \quad w = (1, 0)^t, \quad \delta = (0, 1)^t, \qquad s = \frac{\log q}{2\pi i}, \quad v = \frac{\log r}{2\pi i}.$$

Here $\tau - s, \mu, C$ are holomorphic at $q = 0$. Use multiplicative characters $Z = \exp(2\pi i\zeta_1)$ and $W = \exp(-2\pi i\zeta_2)$, with character lattice $X = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and period lattice $Y = \mathbb{Z}u \oplus \mathbb{Z}\gamma$. Put

$$a_0 = e^{2\pi i(\tau - s)}, \qquad a_1 = e^{2\pi i\mu}, \qquad a_2 = e^{-2\pi iC}.$$

These are holomorphic units on the full q disc. The multiplicative pairing is

	u	γ
e_1	qa_0	a_1^6
e_2	a_1^{-1}	$qr^{12}a_2$.

(22.38)

Give the base the fine saturated log structure from $qr = 0$. In its sharp monoid $\mathbb{N}^2$, written additively, the pairing modulo units is $\operatorname{diag}(q, q + 12r)$. We check the exact admissibility definition (Kajiwara et al., 2008b, Definition 1.2.3, pp. 77–78). On the full face a nonnegative functional $N(q) = h, N(r) = k$ gives $\operatorname{diag}(h, h + 12k)$. If $h > 0$ it is nondegenerate. If $h = 0 < k$, its kernel on each lattice is the first basis line, exactly the face lattice for the face generated by q; the quotient pairing is multiplication by $12k$. For the q face the nonzero face lattices are these first basis lines and the nontrivial quotient pairing is multiplication by h. The r face and the zero face have zero face lattices. Zero functionals give zero-dimensional quotients. These are all the face tests.

2. The residual unit condition is also satisfied (Kajiwara et al., 2008b, Definition 1.2.6, p. 79). At $q = 0$ the residual lattices vanish. At $r = 0$, $q \neq 0$, their pairing is $\log|qa_0| \neq 0$ after shrinking the disc. At ordinary points the real logarithmic matrix has determinant

$$\log|qa_0| \log|qr^{12}a_2| + 6\big(\log|a_1|\big)^2 > 0,$$

after shrinking so both of the first two logarithms are negative. Thus (22.38) defines a log complex torus $\mathcal{A}$ by Definition 1.3.4, pp. 81–82. Theorems 5.2.1 and 5.3.2, pp. 171–172 and 180–181, respectively supply a complete invariant fan with finitely many Y-orbits and its proper analytic model.

3. *The two extending origins.* Set

$$B = \Delta_Q \times \Delta_r, \quad q = Q^{12}, \qquad B_q = \Delta_Q^* \times \Delta_r, \quad B_r = \Delta_Q \times \Delta_r^*, \quad U = B_q \cap B_r.$$

On B_r, 22.26 supplies the affine section

$$\sigma = \Big(6\mu v,\ -6v^2 + v(C - 12\kappa_0)\Big)^t. \tag{22.39}$$

For every local branch of $\log r$, both of its multiplicative coordinates are holomorphic units in the full q disc. Its exact r-deck difference is the bounded cusp closing translation of 22.26. Hence it descends as a global section of the affine family, including $q = 0$. Translation by $-\sigma$ identifies the prescribed cusp fan presentation with its group presentation. The normalized toric base change $q = Q^{12}$ preserves this identification and its extending origin.

4. On B_q, the rotating model of 22.39 is the free cyclic quotient of its root fan model. Selecting Q recovers that model; more generally the normalized base change recovers it also at the ramification divisor. The twelfth fractional deck is $\gamma - 6\delta$, a period, so the multiplicative period action is exactly the second column of (22.38). The point $(Z, W) = (1, 1)$ extends in the root fan at $r = 0$ and gives the group origin. Thus both axial presentations have extending origins in their associated $\mathcal{A}$-sheaves. The identity on the character and period lattices identifies the associated quotient sheaves under a fan refinement: both are defined by the same multiplicative pairing (22.38).

5. In these two origins the exact comparison $\zeta \mapsto \zeta - \sigma + F$ of 22.39 becomes translation by

$$F(s,v) = \left(\frac{\mu(s+12v)}{2}, \frac{C(s+12v)}{12} - \frac{(s+12v)^2}{24} + 6v \right)^t. \tag{22.40}$$

This defines the associated $\mathcal{A}$-torsor by gluing trivial torsors on B_q and B_r. The differences are exactly

$$F(s+12,v) - F(s,v) = \gamma - 6\delta, \qquad F(s,v+1) - F(s,v) = \gamma. \tag{22.41}$$

They are periods, so F is an $\mathcal{A}$-section on U.

6. *The ghost-valuation obstruction.* By the quotient-sheaf definition (Kajiwara et al., 2008b, Sections 1.3.1–1.3.4, pp. 81–82), on the underlying analytic site there is an exact sequence

$$0 \longrightarrow Y \longrightarrow \mathcal{T}_{\log}^{(Y)} \longrightarrow \mathcal{A} \longrightarrow 0. \tag{22.42}$$

The middle term is the sheaf of bounded multiplicative log characters. On U it is the ordinary two-dimensional multiplicative torus sheaf. In particular (22.41) says that the connecting class of F in $H^1(U,Y)$ has r-loop component γ. The terms in $\mathbb{Z}w + \mathbb{Z}\delta$ disappear on exponentiation.

7. Let $i : D_Q = \{Q = 0\} \hookrightarrow B_r$. Taking the Q-order of each character gives

$$\mathrm{ord}_Q : \mathcal{T}_{\log}^{(Y)} \longrightarrow i_*\mathbb{Z}^2, \qquad u \longmapsto (12,0), \quad \gamma \longmapsto (0,12) \tag{22.43}$$

on its period subsheaf. If h is any $\mathcal{A}$-section on B_r, local lifts in the middle sheaf of (22.42) represent its connecting class by a Y-cocycle. Applying (22.43) makes that cocycle a coboundary, the difference of the local ghost vectors. But restriction identifies $H^1(B_r, Y)$ with $H^1(D_Q, Y)$, since B_r retracts onto D_Q; the induced map to $H^1(D_Q, \mathbb{Z}^2)$ is the injective diagonal map $(12,12)$. Therefore the connecting class of h is zero. In particular its r-loop component is zero.

8. For a section on B_q that component is zero as well, because the r loop contracts in B_q. Consequently F cannot be the difference of sections from B_q and B_r. The glued torsor is nontrivial on $B \setminus \{0\}$. An extension locally trivial near the origin would have a section there; after shrinking the bidisc its two coordinate expressions would give precisely the forbidden splitting.

9. For $q = Q^{12m}$ and $r = R^n$, with $m, n > 0$, the R-loop difference is

$$F(s, v+n) - F(s,v) = n\gamma - 6n(n-1)\delta.$$

Its Y-component is $n\gamma \neq 0$, whereas the Q-ghost map on Y is $\mathrm{diag}(12m, 12m)$ and remains injective. The same proof applies after every such cover. Finite Kummer covers of the standard coordinate log structure are locally dominated by coordinate-root covers, so allowing them does not remove this obstruction either.

10. *The ordinary gauge.* The exact identity in Proposition 22.41 is

$$F - \sigma - kv\delta = G(s) + 6v\delta, \quad k = -s + 12\kappa_0, \quad G(s) = \left(\frac{\mu s}{2}, \frac{Cs}{12} - \frac{s^2}{24} \right)^t. \tag{22.44}$$

On $q \neq 0$, this is the raw overlap in the ordinary gauge. Its second multiplicative character is

$$e^{-2\pi i k v} = Q^{12v} e^{-24\pi i \kappa_0 v}, \qquad q = Q^{12}.$$

The second factor is a holomorphic unit on a fixed branch of $\log r$. A bounded logarithmic character at $Q = 0$ has integral Q-order, while $12v$ varies with r. Multiplication by a period changes that order by an integer. Thus this gauge cannot give an extending log-torus section along the full Q-axis. Changes of the two extending axial origins instead lie in the image of

$$H^0(B_q, \mathcal{A}) \oplus H^0(B_r, \mathcal{A}) \longrightarrow H^0(U, \mathcal{A}), \qquad (h_q, h_r) \longmapsto h_r - h_q.$$

The ghost calculation shows that F lies outside this image.

□

Proposition 22.47 (Noncompletion of a one-eigenvalue Hopf boundary)**.** [tag 030U]

Let $\alpha(q) = q^m \kappa(q)$, where $m \geq 1$ and κ is a holomorphic unit on a disc, and let β be holomorphic with $0 < |\beta(0)| < 1$. Shrink the disc so β is a unit and $0 < |\alpha(q)| < |\beta(q)| < 1$ for $q \neq 0$. Consider the diagonal primary Hopf surfaces

$$H_q = (\mathbb{C}^2 \setminus \{0\})/\langle (z, w) \mapsto (\alpha(q)z, \beta(q)w) \rangle. \tag{22.45}$$

Their holomorphic family over the punctured disc has no proper complex-analytic completion over the full disc, even allowing singular or nonreduced central fibres and a singular total space. This remains true after every finite base change ramified at zero.

In particular one cannot replace the blown-up Hopf fibres in 22.43 by the unblown family with one multiplier $\kappa(q)q \to 0$ and the other tending to a nonzero contraction.

Proof. 1. *Anticanonical sections of a diagonal Hopf surface.* A section of $K_{H_q}^{-1}$ lifts to $f(z, w)\, \partial_z \wedge \partial_w$ on $\mathbb{C}^2 \setminus \{0\}$. Hartogs extension makes f entire on $\mathbb{C}^2$. Deck equivariance is exactly

$$f(\alpha z, \beta w) = \alpha\beta f(z, w). \tag{22.46}$$

For the Taylor coefficient of $z^i w^j$, with $i, j \geq 0$, this says

$$\alpha^{i-1}\beta^{j-1} = 1. \tag{22.47}$$

When $i, j \geq 1$, the contraction inequalities force $i = j = 1$. When $j = 0$, (22.47) would give $\beta = \alpha^{i-1}$, which is impossible under $|\alpha| < |\beta| < 1$ (and also for $i \leq 1$). When $i = 0$, the only possibility is

$$\alpha = \beta^N, \qquad j = N + 1, \qquad N \geq 2. \tag{22.48}$$

At a fixed parameter there is at most one such N, since $0 < |\beta| < 1$. Thus

$$h^0(H_q, K_{H_q}^{-1}) = \begin{cases} 2, & \alpha(q) = \beta(q)^N \text{ for some } N \geq 2, \\ 1, & \text{otherwise.} \end{cases} \tag{22.49}$$

The sections in the first case are represented by zw and w^{N+1}.

2. *Infinitely many accumulating resonances.* Put $b = |\beta(0)| \in (0, 1)$ and $k = |\kappa(0)| > 0$. For all sufficiently small q we may arrange $|\kappa(q)| \geq 3k/4$ and

$$|\beta(q)| \leq b \exp(C|q|)$$

for some fixed C, by a holomorphic logarithm of $\beta(q)/\beta(0)$. Set

$$R_N = \left(\frac{2b^N}{k}\right)^{1/m}.$$

Then $R_N \to 0$ and $NR_N \to 0$. On $|q| = R_N$, for all sufficiently large N,

$$|q^m \kappa(q)| \geq \frac{3}{2} b^N, \qquad |\beta(q)^N| \leq b^N \exp(CNR_N) < \frac{5}{4} b^N.$$

Rouché's theorem shows that $q^m \kappa(q) - \beta(q)^N$ has m zeros counted with multiplicity in $|q| < R_N$. None is zero. Choose one, q_N. Different N give different points, since a nonzero contraction cannot have two equal distinct positive powers. Hence there are infinitely many resonances $q_N \to 0$ with the first value in (22.49).

3. For each N the resonance function is nonzero at $q = 0$, so is not identically zero. Its zeros in the punctured disc are discrete. The countable union of these discrete sets cannot contain an open disc. There are therefore also nonresonant points arbitrarily near zero, with the second value in (22.49).

4. *Coherent direct images forbid this pattern.* Suppose first that a proper completion $\pi : \mathcal{X} \to \Delta$ has a smooth connected total threefold and agrees with (22.45) over Δ^*. Put $L = K_{\mathcal{X}}^{-1}$. On each nonzero fibre, adjunction identifies $L|_{\mathcal{X}_q}$ with $K_{H_q}^{-1}$, because the normal direction is the trivial base direction. Grauert's proper direct-image theorem gives coherent sheaves

$$M = \pi_* L, \qquad N = R^1 \pi_* L$$

on the disc (Grauert, 1960, Section 6, Hauptsatz I, p. 59). Multiplication by a nonzero base function is injective on L, and hence on M, since the total space is smooth and the map is nonconstant. Thus M is locally free of a fixed rank on the disc. The torsion subsheaf of N is coherent with locally finite support. After shrinking, it has no support in Δ^*.

5. For any $a \in \Delta^*$, apply the direct-image long exact sequence to

$$0 \longrightarrow L \xrightarrow{q-a} L \longrightarrow L|_{\mathcal{X}_a} \longrightarrow 0.$$

At a its beginning yields

$$0 \longrightarrow M_a/(q-a)M_a \longrightarrow H^0(\mathcal{X}_a, L|_{\mathcal{X}_a}) \longrightarrow \ker(q-a : N_a \to N_a) \longrightarrow 0. \tag{22.50}$$

The last term vanishes. Therefore $h^0(H_a, K_{H_a}^{-1}) = \operatorname{rank} M$ for *every* sufficiently small $a \neq 0$. This contradicts (22.49) and the accumulating resonances.

6. Now suppose an arbitrary proper analytic completion $\mathcal{Z} \to \Delta$ exists. Pass to its reduction and retain the unique irreducible component containing the connected smooth punctured total space. This is a closed analytic component, remains proper over the disc, and dominates the disc. Its restriction over Δ^* is exactly the given family. Resolve this component by a proper analytic resolution chosen to be an isomorphism on its already nonsingular locus; see Bierstone and Milman (1997, Theorem 1.6 and the following noncompact extension, p. 215). All singular points lie over zero, so the resolution is an isomorphism over the entire punctured family. Properness of the composition is preserved. Near the compact central fibre, local finiteness of the analytic resolution permits one shrinking of the disc; its dominating smooth component is a smooth proper total threefold as in the preceding paragraph. Its map to the disc is nonconstant, so the base function is a non-zero-divisor on this dominating component. The preceding contradiction applies. Reduction also shows that nilpotent structure supported at zero cannot evade it.

7. Finally, a finite base change replaces α by a holomorphic function of some positive vanishing order and replaces β by a unit whose value at zero still has modulus strictly between zero and one. The same proof applies verbatim.

□

Remark 22.48 (The toric ray and blow-up comparisons)**.** [tag PT19] For the toric ray datum of Oda (1978, §13, retypeset pp. 116–118), put

$$v_1 = n, \qquad v_2 = -n + an', \qquad a > 0.$$

The original period is $\lambda^{n'}$. The identity $an' = v_1 + v_2$ shows that the base change $\lambda = t^a$ acts in the universal-cover coordinates of v_1, v_2 by the two multipliers (t, t), up to the fixed torus units of that construction. Their two vanishing orders are therefore $(1, 1)$. The family in Proposition 22.47 instead has the explicitly specified orders $(m, 0)$, with second multiplier tending to the nonzero contraction $\beta(0)$.

The generic fibres of Proposition 22.27 carry the modification $\pi : \widetilde{H}_q \to H_q$ formed by successive blow-ups at the marked axis points. At one blow-up, $K^{-1}_{\widetilde{H}_q} = \pi^* K^{-1}_{H_q} \otimes \mathcal{O}(-E_{\mathrm{exc}})$, so its sections correspond exactly to anticanonical sections on H_q vanishing at that centre. Iterating gives the vanishing condition at each successive infinitely near centre. This is the precise map from the anticanonical spaces of the blown-up fibres to the spaces computed in (22.49).

22.12 A necessary vertical fibre over the missing origin

Proposition 22.49 (Every origin component is a threefold)**.** [tag 030V]

Suppose the partial family of 22.41 has a proper analytic completion $f : \mathcal{Z} \to \Delta_q \times \Delta_r$ with smooth pure four-dimensional total space, with $r \circ f$ flat, and with its prescribed identification over the punctured bidisc. The map to this fixed bidisc is assumed holomorphic. Then every irreducible component of $f^{-1}(0, 0)$ has dimension three. In particular, there is no equidimensional completion, and no completion having even one two-dimensional component of the origin fibre.

The stated map to the bidisc includes a holomorphic extension of q. Proposition 22.54 excludes completions of this fixed partial family, including the displayed vertical-component geometry; Theorem 18.6 excludes proper fillings of the global family.

Proof. 1. Use the log-torsor obstruction of Proposition 22.46 after the normalized finite base change $q = Q^{12}$. The resulting space $\mathcal{Z}'$ is normal and pure four-dimensional. Its restrictions to the two punctured axes have the explicit normalized fan presentations used in 22.46. Write

$$B = \Delta_Q \times \Delta_r, \quad B_q = \Delta_Q^* \times \Delta_r, \quad B_r = \Delta_Q \times \Delta_r^*, \quad U = B_q \cap B_r.$$

2. Assume there is a finite surjective map $p : V \to B$ of positive generic degree d, together with a holomorphic map $V \to \mathcal{Z}'$ over B. We may normalize V and retain any collection of components dominating B; its degree remains a positive integer. A normal surface is Cohen–Macaulay; finite over the smooth bidisc, it is flat. Thus the finite holomorphic trace and the degree-d sum with multiplicities are available also at branch points. On the ordinary locus, sum the d points of this multisection, with multiplicity, in each of the two axial group origins of 22.46. This gives two local expressions for the trace in the Jacobian group. They extend as log-torus sections on B_q and B_r.

3. *Ramification away from the axes.* Over the ordinary locus the torus family is a smooth holomorphic group family. Near a branch point, choose small neighbourhoods of the finitely many preimages in V and local Lie lifts of their maps to the torus. The trace of these holomorphic functions under the finite map is holomorphic, and its exponential is the ordinary sum. Changes of lifts add periods, which disappear in the group quotient. Thus the ordinary trace extends across the ordinary branch locus. Equivalently this is the usual addition map on the relative symmetric power, applied to the finite multisection.

4. *The explicit norm at an axis.* After shrinking B, every component of the finite cover's branch divisor which is not an axis meets each punctured axis nowhere: its intersection with either axis

is an isolated analytic germ at the origin. At a point of a punctured axis choose coordinates (x, y), with x the normal coordinate to that axis. Locally the normal finite cover is a disjoint union of covers

$$x = t^e, \qquad y = y, \tag{22.51}$$

including $e = 1$ sheets. Indeed the complement of the axis in a small polydisc has cyclic fundamental group; each connected unramified finite cover is the displayed root cover there, and its normal finite extension is unique.

5. The map from such a root polydisc into the prescribed fan model gives a section of its associated log-torus sheaf. It has a local bounded multiplicative log lift. This can also be seen directly in a toric chart: the pulled-back characters are meromorphic functions with zeros and poles supported on $t = 0$, hence are $t^k u(t, y)$ with $k \in \mathbb{Z}$ and u a holomorphic unit. The quotient by the period lattice causes only the period ambiguity already retained in 22.46.

6. The multiplicative norm of a character on (22.51) is explicitly

$$\prod_{j=0}^{e-1} (\zeta_e^j t)^k u(\zeta_e^j t, y) = (-1)^{k(e-1)} x^k \prod_{j=0}^{e-1} u(\zeta_e^j t, y). \tag{22.52}$$

The last product is a holomorphic unit in (x, y). Thus the norm has an integral ghost vector and is a log character on the base. It is bounded in the required sense: on the r axis the first character has zero normal order and the second is unrestricted; on the Q axis both are unrestricted. These are precisely the bounded-character conditions for (22.38). Multiplication of lifts corresponds to addition in the log torus. Changing a lift by a period changes its norm by an integral period, so the resulting quotient section is independent of the lift. Summing over the finitely many covers (22.51), whose degrees add to d, proves the claimed extension along that axis.

7. The local extensions agree on overlaps because they agree on the ordinary locus. Lift locally to log characters: agreement there is an equality up to a locally constant Y-period, and that meromorphic equality extends across the axis. We have therefore obtained sections h_q of $\mathcal{A}$ on B_q and h_r on B_r. To fix the sign, write $z_r = \zeta - \sigma$ for the cusp group coordinate. The comparison map of 22.39 gives the rotating coordinate $z_q = \zeta - \sigma + F = z_r + F$. Summing over the d points, with multiplicity, therefore gives

$$h_q - h_r = dF \tag{22.53}$$

on U. The ghost proof of 22.46 forbids (22.53): its r-loop Y-component is $d\gamma \neq 0$. Thus no such finite multisection can exist.

8. *A two-dimensional fibre germ would supply one.* Suppose an origin point of $\mathcal{Z}$ has local fibre dimension two. Finite base change and normalization are finite maps and preserve that local fibre dimension at points lying above it. At a resulting point of $\mathcal{Z}'$, choose two generic local linear sections in an analytic embedding. Their intersection with the fibre is isolated. Intersecting with $\mathcal{Z}'$ gives a two-dimensional analytic germ whose map to B is finite. This is the local finite-projection criterion from Weierstrass preparation; see also Demailly (2012, Chapter II, Proposition 8.2 and its proof, pp. 116–117). Choose a component through the point. Its finite image has dimension two and is therefore the whole base germ. A finite representative of this germ, after shrinking the bidisc, is finite and surjective over the entire smaller bidisc. Its normalization is still finite and maps to $\mathcal{Z}'$, giving exactly the forbidden multisection of positive degree. Properness of this finite representative prevents loss of sheets at its boundary.

9. Every fibre component of a dominant map from a pure fourfold to a smooth surface has dimension at least two. If a two-dimensional irreducible component existed, a generic point on it would have local fibre dimension two, which we have excluded. Flatness over r makes $r = 0$ a Cartier divisor of pure dimension three, so the origin fibre has no component of dimension greater than three. All its irreducible components consequently have dimension three, as asserted.

□

22.13 The obstruction survives modifications of the base

Proposition 22.50 (Logarithmic torsors under point blowups)**.** [tag 030W]

Let $B = \Delta_Q \times \Delta_R$, with $q = Q^{12m}$ and $r = R^n$, where m, n are positive integers. Let $\beta : S \to B$ be any finite sequence of point blowups whose centres lie over the origin. Give S the divisorial log structure of the reduced inverse image of $QR = 0$, and let $\mathcal{A}_S$ be the pullback of the Jacobian log torus of 22.46. Then

$$H^1(S, \mathcal{A}_S) = 0 \tag{22.54}$$

for the usual analytic topology. In particular the associated axial log-torus torsor on the punctured bidisc has no locally trivial $\mathcal{A}_S$-torsor extension after these modifications.

Writing $U_S = \beta^{-1}(B \setminus \{0\})$, the extension datum is an $\mathcal{A}_S$-torsor whose restriction along the isomorphism $\beta|_{U_S} : U_S \to B \setminus \{0\}$ is the specified axial torsor.

Proof. 1. Write $E_1, \dots, E_N$ for the exceptional curves, and D_Q, D_R for the strict transforms of the two coordinate axes. The reduced total boundary is normal crossing. Every E_i is a rational curve, and the exceptional divisor is a connected tree if $N > 0$. Its two orders

$$a_i = \operatorname{ord}_{E_i}(Q), \qquad b_i = \operatorname{ord}_{E_i}(R) \tag{22.55}$$

are positive: the image of E_i is the origin. Set

$$D_1 = D_Q + \sum_i E_i, \qquad D_2 = D_Q + D_R + \sum_i E_i.$$

For a reduced normal-crossing divisor D, denote by $\mathcal{M}_D^{\mathrm{gp}}$ the sheaf of meromorphic units whose divisors are supported on D.

2. *The exact bounded-character subsheaf.* The period pairing remains (22.38), pulled back to S. By the base-change lemma (Kajiwara et al., 2008b, Lemma 1.2.9, pp. 80–81) it defines $\mathcal{A}_S$. More explicitly its bounded multiplicative-character sheaf is

$$\mathcal{T}_{\log}^{(Y)} = \mathcal{M}_{D_1}^{\mathrm{gp}} \times \mathcal{M}_{D_2}^{\mathrm{gp}}. \tag{22.56}$$

Indeed the first character is bounded by powers of q, and therefore has zero order on D_R. It may have arbitrary integer order on D_Q and on each E_i, where q has positive order. The second character is bounded by powers of qr^{12}, whose order is positive on every boundary component, so all its boundary orders are permitted. These individual conditions also suffice for every character combination: if its second coefficient is nonzero, a suitable signed multiple of the second period bounds both local normal orders; otherwise powers of the first period do so on the permitted support. At a crossing one chooses the bounding powers simultaneously for its two integer orders. This proves (22.56), including at crossings and at intersections with D_R. In particular the first character has zero order along D_R.

3. *Cohomology of the blown-up bidisc.* For a point blowup p of a smooth surface, $p_*\mathcal{O} = \mathcal{O}$ and $R^j p_*\mathcal{O} = 0$ for $j > 0$. The first identity follows from Hartogs. For the second, the two

standard blowup charts give a local Laurent calculation: on their intersection a holomorphic function expands as a convergent sum of $u^k z^\ell$ with $k \geq 0$. Terms with $\ell \geq 0$ extend in the first chart. Terms with $\ell < 0$ extend in the second, because in coordinates $z' = 1/z$, $u' = uz$ they are $(u')^k (z')^{k-\ell}$. Cauchy estimates on smaller coordinate polydiscs preserve convergence of the two sums. The charts and their intersection are Stein, so this kills the first cohomology and there is no higher Čech cohomology. Iterating and using the Leray sequence over the Stein bidisc gives

$$H^j(S, \mathcal{O}_S) = 0 \qquad (j > 0). \tag{22.57}$$

4. The exceptional tree has one $H^2(\ ,\mathbb{Z})$ generator per rational component and no positive cohomology in other degrees. Proper base change and Leray for β therefore give

$$H^2(S, \mathbb{Z}) \xrightarrow{\sim} \bigoplus_i H^2(E_i, \mathbb{Z}) \simeq \mathbb{Z}^N, \qquad H^j(S, \mathbb{Z}) = 0 \quad (j = 1, 3, 4). \tag{22.58}$$

Equivalently these groups follow from the usual tubular plumbing of the exceptional tree. The exceptional intersection matrix is unimodular: at each blowup the total-transform basis adds a vector of square -1 orthogonal to the preceding basis. Passing to the strict transform basis is an integral unimodular change of basis. Thus the Chern classes of $\mathcal{O}(E_i)$ form a basis of $H^2(S, \mathbb{Z})$. The exponential sequence and (22.57)–(22.58) give

$$\mathrm{Pic}(S) \simeq \mathbb{Z}^N = \langle \mathcal{O}(E_i) \rangle, \qquad H^2(S, \mathcal{O}_S^*) = H^3(S, \mathcal{O}_S^*) = 0. \tag{22.59}$$

5. For either $D = D_1$ or D_2 the divisor-order sequence is exact:

$$0 \longrightarrow \mathcal{O}_S^* \longrightarrow \mathcal{M}_D^{\mathrm{gp}} \longrightarrow \bigoplus_{C \subset D} i_{C*}\mathbb{Z} \longrightarrow 0. \tag{22.60}$$

The map from global integer orders to the Picard group sends a component to its divisor line bundle. It is surjective because both divisors contain every E_i. Every component has $H^1(C, \mathbb{Z}) = 0$: the strict transforms of the axes are discs and the exceptional curves are rational. Their only nonzero H^2 contributions are the N exceptional curves. Consequently the long exact sequence of (22.60) gives

$$H^1(S, \mathcal{M}_{D_j}^{\mathrm{gp}}) = 0, \qquad H^2(S, \mathcal{M}_{D_j}^{\mathrm{gp}}) \simeq \bigoplus_i H^2(E_i, \mathbb{Z}) \simeq \mathbb{Z}^N \quad (j = 1, 2). \tag{22.61}$$

The second isomorphism is induced by the divisor-order map, a fact needed for the next calculation.

6. *Injectivity of the period map in degree two.* The quotient definition of $\mathcal{A}_S$ gives

$$0 \longrightarrow Y \longrightarrow \mathcal{T}_{\log}^{(Y)} \longrightarrow \mathcal{A}_S \longrightarrow 0, \qquad Y = \mathbb{Z}u \oplus \mathbb{Z}\gamma. \tag{22.62}$$

By (22.56) and (22.61), its middle sheaf has zero H^1 and $H^2 \simeq \mathbb{Z}^{2N}$. Using restriction to E_i as the coordinates in (22.58), the map

$$H^2(S, Y) \longrightarrow H^2(S, \mathcal{T}_{\log}^{(Y)})$$

is block diagonal. Its block at E_i is the valuation matrix of the two period columns,

$$\begin{pmatrix} 12ma_i & 0 \\ 0 & 12ma_i + 12nb_i \end{pmatrix}. \tag{22.63}$$

Both diagonal entries are positive. Thus this integral map is injective. The long exact sequence of (22.62) now gives (22.54).

7. Put $U_S = \beta^{-1}(B \setminus \{0\})$. The torsor on $U_S \simeq B \setminus \{0\}$ is nontrivial by 22.46, including after the stated finite coordinate roots. An extending torsor on S would be trivial by (22.54), and its restriction would be trivial as well, a contradiction. For $N > 0$, U_S is the complement of the exceptional union; for $N = 0$ it is the punctured bidisc. Thus the case $N = 0$ is included as well.

□

Proposition 22.51 (Maximal monodromy with trivial central dualizing bundle)**.** [tag 030X]

Let $p : \mathcal{X} \to \Delta$ be a proper holomorphic map from a smooth complex threefold, smooth over Δ^*, whose nonzero fibres are complex two-tori. Suppose that $D = p^{-1}(0)$ is a reduced normal-crossing divisor and that $\omega_D \simeq \mathcal{O}_D$. If its monodromy on $H_1(\mathcal{X}_t, \mathbb{Q})$ satisfies

$$\operatorname{rank}(T - 1) = 2, \tag{22.64}$$

then every normalization component of D is a smooth rational toric surface, its conductor is its toric boundary, and the dual complex of D is a topological two-torus. In particular all normalized components are Kähler.

The trivialization of ω_D in the hypotheses is the central section used in the residue and nearby-period arguments below.

Proof. 1. Write V_i for the compact smooth normalization components and C_i for their reduced conductor divisors. Adjunction gives

$$K_{V_i} + C_i \sim 0. \tag{22.65}$$

If D is smooth, the map is a proper submersion near its compact central fibre and its monodromy is trivial. Otherwise every C_i is nonempty, by connectedness. Adjunction on a conductor branch says that it is either a smooth elliptic curve disjoint from the other branches, or a rational curve with two boundary points, counted on its normalization. The rational branches form cycles, including the nodal irreducible and two-component cases.

2. *Nonnegative Euler contributions.* A nonempty effective anticanonical divisor makes every pluricanonical system empty: multiplying a nonzero pluricanonical section by the appropriate power of the anticanonical section would give a nonzero holomorphic function with a nonempty zero divisor. Thus the surface has Kodaira dimension $-\infty$. Its minimal model is rational, ruled, or of class VII.

3. On a rational surface, the conductor is connected, by $0 \to K_V \to \mathcal{O}_V \to \mathcal{O}_C \to 0$ and $H^1(K_V) = 0$. For a rational cycle, the charge identity and positivity are

$$e(V \setminus C) = 12 - C^2 - \#\{\text{cycle components}\} \geq 0, \qquad e(V \setminus C) = 0 \iff (V, C) \text{ is toric.} \tag{22.66}$$

Here e denotes compactly supported Euler characteristic. This includes the nodal irreducible convention: Friedman (2016, arXiv version 2, Section 1, p. 3; Definition 1.1 and Lemma 1.2, pp. 4–5; Lemma 2.2, p. 11; Lemma 2.7, p. 14). For a smooth elliptic anticanonical curve, $e(V \setminus C) = 2 + b_2(V) > 0$ instead.

4. On a nonrational ruled minimal model, a conductor component dominating the base has degree at most two and genus at most one. The base is therefore elliptic. The minimal conductor is two disjoint elliptic sections or one unramified elliptic bisection. Its Euler contribution is zero. The two section squares sum to zero, and the bisection has square zero, also directly from $K_V + C \sim 0$ and $K_V^2 = 0$.

5. A minimal class-VII surface with $b_2 = 0$ and a curve is a Hopf surface. Its elliptic curves have square zero, and a reduced anticanonical divisor has at most two elliptic components. For this last assertion the exact sequence $0 \to K_V \to \mathcal{O}_V \to \mathcal{O}_C \to 0$ gives $h^0(\mathcal{O}_C) \leq 1 + h^1(K_V) = 2$. The Euler contribution is zero. This also covers secondary Hopf surfaces.

6. For a minimal class-VII surface with $b_2 > 0$, the reduced anticanonical divisor is a numerically anticanonical divisor whose coefficients are all one. Dloussky (2006, arXiv version 3, Corollary 2.25, pp. 16–17; Lemma 2.26, p. 17; Proposition 2.27, pp. 17–19) imply that it is parabolic Inoue or Inoue–Hirzebruch. Half-Inoue has only a nontrivially twisted anticanonical divisor and cannot satisfy (22.65). The remaining possibilities are hyperbolic Inoue with two disjoint rational cycles, or parabolic Inoue with one rational cycle and a disjoint elliptic curve. If $n = b_2 > 0$, there are n rational cycle components and $e(V) = n$, so the contribution is zero. In the parabolic case the elliptic curve has square $-n$. The curve configurations, anticanonical identities and underlying $(S^1 \times S^3)\#n\overline{\mathbb{P}}^2$ topology are recorded in Fujiki and Pontecorvo (2010, Sections 3.2–3.3, pp. 20–22).

7. These assertions extend to nonminimal pairs. An exceptional curve outside C meets it once, transversely at a smooth point, since $-K_V \cdot E = 1$; its contraction reverses an interior blow-up. An exceptional curve in C meets the rest at two points and its contraction reverses a corner blow-up. The reduced anticanonical condition is preserved. Interior blow-up increases $e(V \setminus C)$ by one, whereas corner blow-up leaves it unchanged. Thus every contribution is nonnegative. Equality permits only toric rational pairs, minimal elliptic ruled or Hopf pairs, and corner blow-ups of hyperbolic or parabolic Inoue pairs. For these Inoue pairs the cycle count and b_2 increase together, and the fundamental group stays $\mathbb{Z}$. Corner blow-ups do not touch the parabolic elliptic curve, whose square remains strictly negative.

8. The local nearby fibre at a multiplicity-k stratum of a reduced normal-crossing divisor retracts to $(S^1)^{k-1}$. Euler integration consequently gives

$$0 = e(\mathcal{X}_t) = \sum_i e(V_i \setminus C_i). \tag{22.67}$$

   Every summand is nonnegative and therefore zero. Rational components with a smooth elliptic conductor, and all interior blow-ups, are excluded.

9. *Elliptic normal-degree balance.* An elliptic double curve contains no triple point. The local model $p = xy$ gives opposite degrees to its two normal lines. Sum these degrees over all elliptic conductor branches. The total is zero. An elliptic ruled or Hopf component contributes zero; a parabolic Inoue component contributes the strictly negative degree of its sole elliptic branch. No rational component with elliptic conductor remains. Hence there are no parabolic Inoue components.

10. By connectedness, all conductors are now elliptic or all are rational: no remaining component has both types. In the elliptic case there are no triple points and every vertex of the dual graph has valence at most two. The graph is a segment or circle, allowing a loop or multiple edges. In this case

$$\operatorname{rank}(T - 1) \leq b_1(\text{dual graph}) \leq 1. \tag{22.68}$$

    To prove this, remove small tubes about the disjoint double curves. The remaining compact cores extend as smooth submersions over the whole disc and can be trivialized there. Monodromy is then a product of the annular twists $xy = t$. A passage of a loop across one neck adds its oriented normal-circle class, with coefficient the signed crossing number of that edge. Core loops have zero change. Crossing numbers of a closed loop form a cycle of the dual graph. Thus $T - 1$ factors through its rational cycle space. The normal-circle rotation exists on the circle bundle of the normal line. This proves (22.68) and excludes the elliptic case.

11. *Hyperbolic components contradict a nonzero nearby period.* The remaining possibilities are toric rational pairs and corner blow-ups of hyperbolic Inoue pairs. Suppose one component is of the latter kind. A boundary of a small neighborhood of either rational conductor cycle, lying in $V_i \setminus C_i$, is a torus mapping torus M_A, whose monodromy on its fibre lattice is

$$A \in \mathrm{SL}_2(\mathbb{Z}), \qquad |\operatorname{tr} A| > 2. \tag{22.69}$$

In Oda (1978, Section 15, Theorem 15.1 and the following paragraph, retypeset pp. 130–131), the universal toric construction has lattice N and deck action given by multiplication by a nontrivial totally positive unit of a real quadratic field. This is the hyperbolic matrix A. The complement of the two cycles is $\mathbb{R}$ times the compact angular-torus bundle over a circle. A path in the logarithmic absolute-value space from a point to its image by A realizes this mapping torus. The point can be chosen in a node cone arbitrarily near a conductor node. Its two adjacent primitive rays form a lattice basis, so the angular fibre is exactly the Clifford torus in the corresponding local coordinates (x, y). Corner blow-ups are toric star subdivisions; they alter neither the dense torus nor A, only this unimodular coordinate basis. Thus the same marked mapping torus exists for every corner blow-up allowed above.

12. Choose a compact such M_A in the smooth open stratum. A neighborhood of this compact set extends as a smooth submersion over the full parameter disc. Transport gives an embedding $M_A \to \mathcal{X}_t$ for all small t, with the same marked fibre $L_t \simeq (S^1)^2$. In the abelian group $H_1(\mathcal{X}_t, \mathbb{Q})$, the mapping-torus relations give

$$(A - 1)H_1(L_t, \mathbb{Q}) \longmapsto 0.$$

Since $A - 1$ is invertible over $\mathbb{Q}$, both angular generators map to zero. The cohomology ring of a complex two-torus is the exterior algebra on H^1. It follows that the map $H^2(\mathcal{X}_t, \mathbb{Q}) \to H^2(L_t, \mathbb{Q})$ is zero and hence

$$[L_t] = 0 \quad \text{in } H_2(\mathcal{X}_t, \mathbb{Q}). \tag{22.70}$$

13. On the other hand, $h^0(D, \omega_D) = 1$, since D is compact and connected, and the corresponding dimension on every nearby two-torus is also one. Grauert's cohomology and base-change theorem applied to the invertible relative dualizing sheaf supplies a relative section extending the chosen nonzero central generator; see Bănică and Stănăşilă (1976, Chapter III, Theorem 4.12(ii), pp. 134–135). Evaluation is nonzero at every point of D. Compactness and shrinking make this section nonvanishing near the entire fibre. Write it as Ω. At the conductor node just used, the total family has coordinates $xyz = t$ and

$$\Omega_t = U(x, y, z)\frac{dx}{x} \wedge \frac{dy}{y}, \qquad U(0, 0, 0) \neq 0. \tag{22.71}$$

The angular torus can be transported specifically as $|x| = \epsilon_1$, $|y| = \epsilon_2$, $z = t/(xy)$. This agrees, up to isotopy, with its compact-core transport. If $U = \sum a_{ijk} x^i y^j z^k$ in a small polydisc, direct integration of the convergent series gives

$$\int_{L_t} \Omega_t = (2\pi i)^2 \sum_{k \geq 0} a_{kkk} t^k \longrightarrow (2\pi i)^2 U(0, 0, 0) \neq 0. \tag{22.72}$$

For small t this contradicts (22.70), because the holomorphic two-form Ω_t is closed. This excludes every hyperbolic Inoue component.

14. *The dual complex.* All normalization pairs are now rational toric pairs. Every vertex link is one circle and every edge has two incident faces, because each rational double curve has two triple points. Thus the finite dual complex K is a connected closed topological surface. It is oriented: the iterated residues of a trivialization of ω_D are nonzero at every triple point, and the residue

theorem on each rational double curve makes its two coefficients opposite, with the gluing sign for the two branches. These coefficients form a nowhere-zero two-cycle on K. Constancy of coherent Euler characteristic in the proper flat analytic family (Bănică and Stănăşilă, 1976, Chapter III, Theorem 4.12(iii), pp. 134–135), and the alternating normalization resolution, give

$$0 = \chi(\mathcal{O}_D) = \chi(K). \tag{22.73}$$

The surface therefore has genus one.

15. For self-crossings, use the finite local-branch incidence cell structure; the link of a normalized toric component is its circular boundary. Over the normalization $\mathbb{P}^1$ of a double curve, the two-branch cover extends unramified over its triple-point ends by local normal-crossing coordinates and hence splits. Subdivision displays the same closed surface and the residue signs. The alternating normalization resolution uses these branch signs and gives the same Euler count. The hyperbolic-link argument took place in the open normalization stratum, which embeds in D, and is unchanged by self-identification of boundary branches.

□

Remark 22.52 (Vertical divisors and component adjunction)**.** [tag PT05] For a proper semistable surface family with smooth total space and reduced simple normal-crossing central divisor $\sum_i V_i$, let a nonzero meromorphic relative two-form have vertical divisor $\sum_i a_i V_i$. Put $C_{ij} = V_i \cap V_j$ and $C_i = \sum_{j \neq i} C_{ij}$. Then

$$K_{V_i} + C_i \sim \sum_{j \neq i} (a_j - a_i) C_{ij}. \tag{22.74}$$

Indeed, adjunction gives $K_{V_i} = (K_{\mathcal{X}/\Delta} + V_i)|_{V_i}$, while the principal total fibre gives $\mathcal{O}_{\mathcal{X}}(V_i)|_{V_i} \sim -C_i$. Restricting the divisor of the form yields the displayed equality.

22.14 What invariant forms do not control

Proposition 22.53 (Full-rank periods do not bound an affine comparison)**.** [tag 030Y]

There is a proper semistable family of two-dimensional complex tori over a disc, with smooth total threefold and rank-two unipotent monodromy on H^1, and a fibre-preserving automorphism over the punctured disc with the following properties. It acts identically on all relative invariant one- and two-forms and on the Gauss–Manin variation. Their coherent extensions supplied by the proper family therefore give the same lattices under this identification. An absolute holomorphic canonical form extending across the centre is also unchanged. Nevertheless, the automorphism has no meromorphic extension to the total family.

Thus these invariant-form and connection data do not alone control an affine origin or a marked total-space comparison, even in the presence of a proper smooth semistable cap.

Proof. 1. *A proper full-rank example.* Let $\mathcal{E} \to \Delta_t$ be the standard Tate family with ordinary fibre

$$E_t = \mathbb{C}^*/t^{\mathbb{Z}} \qquad (t \neq 0)$$

and reduced nodal rational central fibre. Its proper analytic model and node chart $xy = t$ are the explicit example in Kajiwara et al. (2008b, Sections 1.1.1–1.1.3, pp. 73–74). Form $\mathcal{E} \times_\Delta \mathcal{E}$. Its only total-space singularity is the product of the two nodes, with local equations

$$xy = t, \qquad uv = t, \qquad xy = uv.$$

Blow up the ideal (x, u) in this conifold and take its strict-transform charts. They are

$$u = xA, \quad y = Av, \quad t = xAv$$

with smooth coordinates (x, A, v), and

$$x = uB, \quad v = By, \quad t = uBy$$

with smooth coordinates (u, B, y). This is the standard proper small resolution, an isomorphism away from the conifold point. Glue it to the identity on the complement of that point. The resulting $p : \mathcal{Y} \to \Delta$ is proper, has smooth total threefold, and has reduced normal-crossing centre: its displayed equations are products of distinct coordinates. The resolution does not alter the ordinary fibres $E_t \times E_t$.

2. For each elliptic factor choose period cycles w_i, u_i with $u_i \mapsto u_i + w_i$ around $t = 0$. On the direct sum the logarithm of monodromy has rank two and square zero. The same rank holds on dual H^1. Thus this example has the full-rank weight-one unipotent degeneration stated above.

3. *The nonextending translation.* On the punctured disc use multiplicative coordinates (Z, W) for the two factors and put

$$T_h(Z, W, t) = \bigl(e^{2\pi i h(t)} Z, W, t\bigr), \qquad h(t) = 1/t. \tag{22.75}$$

This is a holomorphic automorphism of the punctured family. The zero section $e(t) = ([1], [1])$ extends to the smooth torus locus of the central normalization. For positive integers j tending to infinity,

$$t_j = 1/j, \qquad t'_j = 1/(j + 1/2)$$

lie in the disc and tend to zero, whereas

$$T_h(e(t_j)) = ([1], [1]), \qquad T_h(e(t'_j)) = ([-1], [1]).$$

In the Tate model these approach the distinct central smooth points with multiplicative coordinates $(1, 1)$ and $(-1, 1)$. The small resolution is an isomorphism near both points. Hence $T_h \circ e$ has no continuous extension to the centre.

4. This also excludes a meromorphic total extension. If such an extension existed, let $\Gamma \to \mathcal{Y}$ be its proper meromorphic graph and pull it back along $e : \Delta \to \mathcal{Y}$. The resulting analytic space is proper over the disc and has a single point in each noncentral fibre. Its unique component dominating the disc is a proper analytic curve, generically of degree one. After normalization it is finite of degree one over the normal disc, hence is the disc itself. Its map to the target $\mathcal{Y}$ would extend $T_h \circ e$ holomorphically. This contradicts the two distinct limits just found. Vertical components in the pulled-back graph do not affect this dominating curve.

5. *Relative forms and their coherent extensions.* On the ordinary locus the invariant relative one-forms are

$$\omega_1 = dZ/Z, \qquad \omega_2 = dW/W \qquad (\mathrm{mod}\ dt).$$

They and $\omega_1 \wedge \omega_2$ are unchanged by every T_h. The holomorphic forms of a compact torus are invariant, so this accounts for all relative holomorphic one- and two-forms on the ordinary fibres. The relative log forms extend in the Tate model. Ordinary coherent relative forms also give finite-order extensions: for example $t\omega_1, t\omega_2$, and $t^2\omega_1 \wedge \omega_2$ extend as ordinary relative forms. In the first resolution chart the first two are explicitly

$$t\omega_1 = Av\,dx, \qquad t\omega_2 = Av\,dx + xv\,dA;$$

the other chart gives the analogous regular expressions. At an isolated Tate node the same identity is $t\,dx/x = y\,dx$.

6. More generally, take the torsion-free image of each coherent direct image of relative forms in its bundle on the punctured disc. Under the two punctured identifications id and T_h, these are identical coherent lattices, since the induced map on the entire ordinary relative-form bundle is the identity. Coherence therefore gives no bound on h in this example.

7. *Gauss–Manin and an absolute form.* Every fibre translation is homotopic to the identity and acts identically on integral cohomology. Consequently T_h induces the identity on the flat Gauss–Manin bundle, its connection, and its Hodge subbundle. The logarithmic period growth and regular singularity of that connection are unchanged.

8. The absolute form
$$\Omega = dt \wedge \frac{dZ}{Z} \wedge \frac{dW}{W}. \tag{22.76}$$
descends under both Tate period actions, since their extra differentials are multiples of dt. In the first small-resolution chart it is
$$d(xAv) \wedge \frac{dx}{x} \wedge \frac{d(xA)}{xA} = dv \wedge dx \wedge dA,$$
and in the other chart it is $dy \wedge dB \wedge du$. Thus Ω extends as a nonvanishing holomorphic canonical form on the smooth total space. Translation adds $2\pi i h'(t)dt$ to dZ/Z, which vanishes in (22.76), so $T_h^*\Omega = \Omega$.

9. *The horizontal term of an absolute differential.* On a local multiplicative chart, an absolute lift of the first relative form transforms by
$$T_h^*(dZ/Z) = dZ/Z + 2\pi i h'(t)dt. \tag{22.77}$$
This horizontal term is erased both on passing to relative forms and on wedging with the base differential. It is not controlled by the linear Gauss–Manin connection. The stronger choice $h(t) = e^{1/t}$ makes that term essential, while leaving all the preceding relative and canonical calculations unchanged. For this choice the two sequences $t_j = 1/\log j$ and $t'_j = 1/\log(j+1/2)$ give the same two distinct central limits.

10. Both markings belong to the same proper family. Their relative invariant forms and their absolute canonical form agree by the displayed pullbacks, while the two limits of the translated zero section are distinct. A common translation cancels in a difference of two sections.

□

22.15 Noncompletion of the prescribed partial filling

Proposition 22.54 (Noncompletion of the fixed partial filling)**.** [tag 0310]

The partial filling of 22.41 has no proper Hausdorff complex-analytic completion over its missing base point. This remains true after finite coordinate roots of the two base coordinates and allows arbitrary additional three-dimensional components of the origin fibre.

Proof. 1. Every finite coordinate-root cover is dominated by one with $q = Q^{12m}$ and $r = R^n$, where $m, n > 0$; proper pullback reduces to this case. Suppose such a completion exists. Retain and normalize its reduced dominating component, resolve it, and principalize the inverse image of the two axes, as in the proof of Theorem 18.6. These proper modifications preserve the given family on the complement of the axes and the smooth open parts of the two prescribed axial models. The resulting proper map to the bidisc has connected fibres: its finite Stein factor is generically one sheet and the bidisc is normal.

2. We identify the axial markings with the standard specialization of Theorem 18.2, namely $(l_0, l_1, l_2) = (0, 1, -1)$ and $a = b = 1$. In the notation of Proposition 22.46, one has $U = -12\kappa_0$. The two origins in Theorem 18.2 equal the two origins there plus $6ve$, where $e = \Pi_c\widehat{\delta} = (0, 1)^{\mathrm{t}}$. Translation by $6ve$ acts on the character W as $W \mapsto r^{-6}W$. It shifts the twelve-chain index by six and extends, with its inverse, across the parameter axis; along the cusp axis $r \neq 0$ it is multiplication by a holomorphic unit. Thus this is an explicit comparison of the prescribed axial models. It changes meridian labels only by multiples of $\widehat{\delta}$ and leaves their $\widehat{\gamma}$ coefficients unchanged.

3. In the commuting lifts of Theorem 18.6, an axis component has orders (α_i, β_i) and $\widehat{\gamma}$ meridian coefficient κ_i. The same group calculation gives

$$(m\alpha_i + n\beta_i)\kappa_j = (m\alpha_j + n\beta_j)\kappa_i$$

for intersecting components. Every denominator $m\alpha_i+n\beta_i$ is positive, including for exceptional components over the origin. Hence the ratio $\kappa_i/(m\alpha_i + n\beta_i)$ is constant on the incidence graph of components meeting the corner.

4. The prescribed parameter-axis model has an extending ordinary origin. Its positive parameter loop has zero lattice label in that origin, so the strict parameter component has ratio zero. The prescribed cusp-axis origin has ratio $a = 1$, by the exact covered origin formulas (18.4)–(18.5) and the commuting lift $Q_o = T_{-m\widehat{\gamma}}Q$. Both strict components meet the corner by properness. The compact connected corner fibre makes the incidence graph connected, exactly as in the proof of Theorem 18.6. Its ratios would therefore be both zero and one. This contradiction proves noncompletion for every pair of positive root degrees and for all additional corner components.

□

22.16 Global elliptic monodromy constrains every central meridian

Proposition 22.55 (The global meridians have only the prescribed toric direction)**.** [tag 0311]

Suppose $g : \mathcal{Z} \to \Delta_r$ is any proper complex-analytic filling of a finite parameter cover of the six-sphere family. Retain its reduced main component, normalize and resolve it, extend its intrinsically marked map to $\mathbb{P}^1$ by 18.3, resolve the graph, and resolve the support of its central divisor. Denote the resulting smooth proper modification by $\widetilde{\mathcal{Z}}$, and its holomorphic marked map by h. All modifications can be chosen to be isomorphisms over $r \neq 0$. The central divisor has normal-crossings support and positive integral component multiplicities.

There is a finite subset $\Sigma \subset \mathbb{P}^1$, containing $0, 1, \infty$, with the following property. For every $q \in B^\circ := \mathbb{P}^1 \setminus \Sigma$, every local angular vanishing circle of the r-central normal-crossings divisor, viewed in the homology of the nearby ordinary two-torus, lies in

$$\mathbb{Z}\widehat{\delta} \subset H_1(X_{q,r}; \mathbb{Z}). \tag{22.78}$$

The same conclusion holds for the image of the entire local angular torus at a higher crossing over B°. Here $\widehat{\delta}$ is the primitive cycle in the fixed period marking. The conclusion is unaffected by any further finite cover of the degeneration parameter.

Proof. 1. *The global model and the finite exceptional set.* The punctured total family is connected and smooth, hence irreducible. Its closure is the unique reduced main component dominating the parameter disc. Retain this component, normalize it, and resolve it properly, without changing its smooth punctured part. The resulting total space is a smooth connected fourfold, proper over the disc. The parameter is a nonzero divisor, so its map to the disc is flat. The torsion-free coherent bi-anticanonical pencil and labelled critical-value argument of 18.3

apply to this smooth model. Its marked graph is therefore a proper modification. Further analytic resolution and principalization of the central ideal give the stated smooth model and normal-crossings support; the complex-analytic scope, including locally finite resolution, is explicit in Bierstone and Milman (1997, Theorems 1.6 and 1.10, pp. 215–216, including the noncompact extension after Theorem 1.6). Properness over a smaller parameter disc allows us to work over a compact central fibre throughout. The modifications have their centres over $r = 0$, so neither the punctured family nor its marked map changes. Flatness over r persists because r is a nonzero divisor in the smooth local rings; all central multiplicities are kept in the local monomial equations.

2. For possible self-crossings use the finite closed spaces of ordered local branch intersections, with each local component normalized. Locally these are the coordinate intersections of the central branches. In particular, the double-branch spaces are compact smooth surfaces and the triple-branch spaces are compact smooth curves, even when two branches belong to one global component. There are finitely many such components.

3. Add to Σ the images of all branch-intersection components on which h is constant and the critical values of h on every remaining branch-intersection component, including the normalizations of the single branches. These sets are finite. Indeed the maps are proper; on a component mapping nontrivially to the curve $\mathbb{P}^1$ their critical image is a proper analytic subset of that curve. Zero-dimensional deeper strata contribute only finitely many further values.

4. Thus, over B°, the closed double-branch spaces form proper smooth families of compact curves. The connected components of their fibres form finite covering spaces of B°: this follows either from proper submersion trivializations, or from their finite Stein factorizations. Removing the finitely many triple points from one connected smooth curve leaves it connected. Branch orderings, central multiplicities, and these connected fibre components therefore constitute finite discrete data over B°.

5. *The local angular circles, including multiplicities.* At a point of an open double-branch stratum, the submersion condition for h gives coordinates in which $q = h$ is a tangential coordinate and

$$r = x^a y^b, \qquad a, b > 0. \tag{22.79}$$

   A holomorphic unit has been absorbed in one coordinate on this small chart. Put $d = \gcd(a, b)$. Each connected component of the local nonzero fibre has the primitive angular circle

$$\theta \longmapsto (x_0 e^{i(b/d)\theta},\, y_0 e^{-i(a/d)\theta}), \qquad 0 \le \theta \le 2\pi, \tag{22.80}$$

   where $x_0^a y_0^b = r$. There are at most d choices of the connected monomial-fibre component. Interchanging the ordered branches can reverse the circle's orientation. These are only finite choices.

6. Move the chosen base point along a path in the same open double-curve fibre. Local product charts and the normal line bundles transport the annular circle continuously. Changes of normal frame rotate its parameter and do not add a tangential one-cycle. Equivalently, this is the fibre circle of the normal circle bundle along that path. If a based loop is used, changing the path changes it by conjugation, which does not affect its class in H_1. Consequently, for fixed small nonzero r and fixed q, each of the finitely many branch-component data determines only finitely many homology classes, including signs.

7. The same local formula proves monodromy invariance in the r direction. Continuing around $r = 0$ permutes the finitely many connected components of $x^a y^b = r$; after a positive power it

carries (22.80) to the same oriented circle class. In the marking, after a finite parameter cover the r-monodromy is

$$T_r = I - mE, \qquad E = \widehat{\delta} \otimes \gamma, \qquad m \in \mathbb{Z}_{>0}, \quad E^2 = 0. \tag{22.81}$$

For example, $w = r^6$ gives $m = 12$ with the positive-loop convention used in 22.26. If $T_r^k v = v$ for one of the angular classes v, then $kmEv = 0$. The homology lattice is torsion-free, so

$$v \in \ker E = \mathbb{Z}\widehat{u} + \mathbb{Z}\widehat{w} + \mathbb{Z}\widehat{\delta}. \tag{22.82}$$

8. *Why global continuation gives a finite orbit.* Choose a base point $q_* \in B^\circ$. Let $\mathscr{S}$ be the finite set of angular classes furnished at q_* by all the connected double-curve fibres, their ordered branches, their monomial-fibre components, and both orientations. Take generators of the finitely generated group $\pi_1(B^\circ, q_*)$. A generator can be lifted to a path in the appropriate family of double curves; it ends in one of the finitely many connected fibre components. Avoid the triple points during this lift. Properness over the compact generator paths and finitely many product charts give one sufficiently small r_* for which all their annular circles can be transported in the smooth fibres X_{q,r_*}.

9. The transported circles are a continuous family of cycles, so they realize the Gauss–Manin transport of H_1. At the end, moving the base point within the resulting connected punctured double curve does not change its H_1 class, by the previous paragraph. Hence each generator permutes $\mathscr{S}$. The same holds for all words in the generators. This proves a finite orbit in the marked lattice. Its projection to

$$\ker E/\mathbb{Z}\widehat{\delta} \simeq \mathbb{Z}\widehat{u} + \mathbb{Z}\widehat{w} \tag{22.83}$$

therefore has finite orbit as well.

10. *The exact elliptic-monodromy calculation.* In the fixed marking the two finite peripheral matrices are

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ -6 & -1 & -1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -6 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 \end{pmatrix}. \tag{22.84}$$

They fix $\widehat{\delta}$ and γ, commute with E, and induce on (22.83)

$$M_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Deleting additional ordinary values from the elliptic base does not decrease this monodromy image: the map on fundamental groups to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is surjective. The image therefore contains

$$U = M_1 M_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad V = M_2 U M_2^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{22.85}$$

If a vector $\bar{v}$ has finite orbit, some positive powers U^k and V^ℓ both fix it. Their fixed lines are, respectively, $\mathbb{Q}(0, 1)$ and $\mathbb{Q}(1, 0)$, whose intersection is zero. Thus $\bar{v} = 0$. Applying this to the projected classes in $\mathscr{S}$ proves that each angular class lies in $\mathbb{Q}\widehat{\delta}$. Since $\widehat{\delta}$ is a member of an integral basis of $H_1(T^4; \mathbb{Z})$, it is primitive; an integral class in this rational line belongs to $\mathbb{Z}\widehat{\delta}$. This proves (22.78).

11. At a crossing with local equation $r = \prod_{j=1}^{k} x_j^{a_j}$, the angular torus has rational first homology

$$\ker\big((a_1, \dots, a_k) : \mathbb{Q}^k \longrightarrow \mathbb{Q}\big).$$

It is spanned by the pairwise angular vectors $a_j e_i - a_i e_j$. Each is a rational multiple of the primitive circle of the corresponding double branch, which can be moved to an open double-curve point by keeping the other coordinates small and nonzero. The same calculation therefore places the entire angular image in $\mathbb{Q}\widehat{\delta}$, and primitivity again gives the integral conclusion. This also handles self-crossings, because the ordered local branches were retained throughout.

□

22.17 The radial-loop class of every global cap

Proposition 22.56 (Vanishing of the normalized radial-loop defect)**.** [tag 0312]

Suppose a proper complex-analytic filling of a finite parameter cover of the six-sphere family exists. Make the modifications in 22.55 and delete its finite exceptional set from the marked base. Use the ordinary local origins of 22.41 as reference. There is a well-defined normalized radial-loop defect of a local extending cap section,

$$\lambda \in V := (\Lambda \otimes \mathbb{Q})/\mathbb{Q}\widehat{\delta}. \tag{22.86}$$

It is independent of the section, of the finite parameter root needed to obtain that section, and of the modifications used. It patches to a flat section of the local system V over the remaining ordinary base, and this section is zero.

In particular, let t be the trace of an extending finite multisection of degree d, and let o be the reference ordinary origin. The radial-loop defect of $t - do$ in V is zero. The same holds for an origin obtained from multiplied such traces by bounded logarithmic differences along $r = 0$.

Proof. 1. Use the smooth marked model of Theorem 18.3 and remove the finitely many base values which are images of vertical central components or critical values of the maps on the central branch strata. Every central component above the remaining ordinary base dominates the original $\mathbb{P}^1$. Its positive normal meridian has zero lattice class in $\Lambda/\mathbb{Z}\widehat{\delta}$ by Theorem 18.5.

2. The ordinary origin is exactly the one used there. Indeed, write $k = \log K/(2\pi i)$ with $K = t^4/(t-1)^3$ and $v = \log r/(2\pi i)$. The parameter descent of Theorem 18.1 is $H = -k$, so subtracting the local origin vHe, with $e = \Pi_c \widehat{\delta}$ gives the ordinary gauge $\widetilde{\zeta} = \zeta + kve$ of Proposition 22.41. On an overlap, k changes by an integer, and the resulting radial label is an integral multiple of $\widehat{\delta}$. The finite-root cancellation and the cusp comparison in that proposition have the same property. Thus the splitting used in Theorem 18.5 computes precisely the radial-loop defect relative to the stated ordinary origins.

3. The connecting map is the endpoint period of a continued additive lift. With $T_r = I - mE$, $E = \widehat{\delta} \otimes \gamma$ and $E^2 = 0$, changing the lift adds $(T_r - I)\Lambda \subset \mathbb{Z}\widehat{\delta}$, so its image ν_r in V is well defined. Under $r = R^e$ its label changes by

$$\sum_{j=0}^{e-1} T_r^j v = ev - \frac{me(e-1)}{2} Ev \equiv ev \pmod{\mathbb{Q}\widehat{\delta}}.$$

Consequently the normalized defect of a section b on that root cover is $e^{-1}\nu_R(b - o(R^e))$.

4. Lift an extending section to a common proper resolution. For a fixed ordinary base point the lift extends over the parameter origin: normalize the dominating curve in its proper graph,

which has degree one over the disc. At its limit choose a normal-crossing chart in which

$$r = \text{unit} \prod_{i=1}^{k} x_i^{a_i}, \qquad a_i > 0.$$

On the section, $x_i(R) = R^{b_i} u_i(R)$ with u_i a holomorphic unit and $b_i \geq 0$. Thus $e = \sum_i a_i b_i$, and its positive R-loop is the sum of b_i positive component meridians. The unit loops contract in the chart. In the quotient presentation of Theorem 18.5, the lattice label is the same sum of the component labels, each zero modulo $\widehat{\delta}$. Hence the normalized radial defect is zero. This argument applies to every extending section and every root, and is preserved under common proper refinements. In a varying ordinary base chart the labels are locally constant, so the vanishing also holds across the isolated points where a simultaneous section lift has indeterminacy.

5. The connecting homomorphism is additive. Split a finite multisection after a cyclic root at a generic ordinary point, add the zero normalized defects of its branches, and use the preceding root formula. Its trace of degree d therefore satisfies $\nu_r(t - do) = 0$ in V. These zero local classes patch under the lattice transport, independently of the section and all choices. Finally, by the bounded-character presentation of Proposition 22.50, a bounded logarithmic difference along the strict parameter axis has characters in $\mathcal{O}^* \times \mathcal{M}_r^{\mathrm{gp}}$: its first character has zero winding and its second contributes only a multiple of $\widehat{\delta}$. Adding it to a multiplied trace preserves the zero class.

□

Remark 22.57 (A zero loop class with a nonextending translation)**.** [tag PT07] In Proposition 22.53, the additive lift of the translation is $h(t) = 1/t$. It is single-valued on the punctured disc, so its radial deck difference and loop class are zero. Its translated zero section has the two distinct central limits computed there, and hence the translation has no meromorphic extension. The loop class and the affine comparison are related by the endpoint-period map used in Proposition 22.56; this example has zero image under that map.

23 Other logarithmic transformations and finite covers

23.1 Comparison under logarithmic transformation

Proposition 23.1 (The relative cohomology comparison)**.** [tag 01S3]

Fix the period family and cusp filling. Let X' be obtained by any admissible translation twists and multiplicities $m_1 = 3k$, $m_2 = 4l$, where k, l are positive integers. Write $D' = W + S_1' + S_2'$ and

$$\mathcal{E}' = \Omega^1_{X'}(\log D')/f'^{*}\Omega^1_B(\log \Sigma).$$

The identity on the common smooth family induces isomorphisms on B

$$R^q f_* \mathcal{O}_X \simeq R^q f'_* \mathcal{O}_{X'}, \qquad R^q f_* \mathcal{E} \simeq R^q f'_* \mathcal{E}' \quad (0 \leq q \leq 2).$$

These isomorphisms identify the logarithmic connecting maps κ_q. They also hold when the cusp collar is changed by an admissible translation.

Proof. 1. Over $B \setminus \Sigma$ the two families are identified, and a translation acts trivially on $H^q(F, \mathcal{O}_F)$ and $H^q(F, \Omega^1_F)$ for a complex torus F. For the second assertion, translation fixes the invariant holomorphic one-forms, and the first follows either from invariant Dolbeault forms or the exterior algebra $H^\bullet(F, \mathcal{O}_F) = \bigwedge^\bullet H^1(F, \mathcal{O}_F)$. These are statements about the induced maps on cohomology, including their holomorphic variation.

2. At a point with original order $d = 3$ or 4, let $\widetilde{f} : T \to \Delta_z$ be the smooth proper torus model with coarse coordinate $q = z^d$. The new model is its pullback by $z = s^e$, where $m = de$, followed by the free cyclic quotient of order m. Smooth proper base change identifies the higher direct images upstairs with the pullbacks of those for T. The relative logarithmic cotangent bundle upstairs is precisely Ω^1_{T/Δ_z} pulled back: the base logarithmic direction is killed in the quotient defining $\mathcal{E}'$. The derivative of a translation contributes only a multiple of ds; it consequently acts on this quotient as ordinary translation on relative forms.

3. Average over the cyclic action and choose holomorphic eigenframes. If a frame for $H^q(\mathcal{O})$ or $H^q(\Omega^1)$ has character ζ_d^{-a}, $0 \le a < d$, its original invariant extension is $z^a\mathcal{O}_{\Delta_q}$. On the order-$m$ cover its character is ζ_m^{-ea}, and its invariant extension is $s^{ea}\mathcal{O}_{\Delta_q} = z^a\mathcal{O}_{\Delta_q}$. Taking invariants is exact in characteristic zero; hence the map $s^{ea}v \mapsto z^a v$, with $z = s^e$, identifies the extensions inside the common punctured-disc bundle. A translation twist induces the identity on the cohomology just used and changes none of these lattices. This proves the comparison at both finite points, including the full nonreduced fibres.

4. At the cusp the relative logarithmic forms have the global toric frame $d\log x_1, d\log x_2$, modulo $d\log q$. A toric gluing has the form $x_i \mapsto c_i(q)q^{b_i}x_i$ and therefore changes $d\log x_i$ by $(qc_i'/c_i + b_i)d\log q$. It induces the identity on the relative logarithmic quotient. The structure-sheaf comparison is also the identity on the normalization cochain complex. This verifies the cusp extension and the assertion about translated collars.

5. The logarithmic cotangent exact sequences are isomorphic on the smooth base, so their connecting maps agree there under these comparisons. Their source and target sheaves are locally free by the local calculations in cited result 12.4 and proposition 12.6 and the eigenframe calculation above. A holomorphic map between vector bundles on a reduced curve which is zero on a dense open set is zero everywhere. Thus the connecting maps agree globally.

□

Proposition 23.2 (All admissible twists have the same Hodge numbers)**.** [tag 01S4]

For every family in 23.1 whose cusp is the one-component hexagonal filling, the direct images of the absolute cotangent sheaves are

	$R^0f'_*$	$R^1f'_*$	$R^2f'_*$
$\Omega^1_{X'}$	$\mathcal{O}(-2)$	$\mathcal{O} \oplus \mathcal{O}(-1)^2$	$\mathcal{O} \oplus \mathcal{O}(-1)$
$\Omega^2_{X'}$	$\mathcal{O}(-2) \oplus \mathcal{O}(-1)$	$\mathcal{O}(-2) \oplus \mathcal{O}(-1)^2$	$\mathcal{O}$

and $R^qf'_*\mathcal{O}_{X'} = (\mathcal{O}, \mathcal{O} \oplus \mathcal{O}(-1), \mathcal{O}(-1))$. *In particular every admissible* $(3, 4)$ *member with* $|12l_0 - 4l_1 - 3l_2| = 1$ *has the Hodge diamond of proposition 12.15.*

Proof. 1. 23.1 identifies the logarithmic direct images and their connecting maps. By propositions 12.10 and 12.13, the cokernel of κ_0 is $\mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2}$, and κ_1 is surjective with kernel $\mathcal{O} \oplus \mathcal{O}(-1)^2$. These statements therefore hold for X'. The residue sheaf of its normalized divisor has higher direct images

$$R^0f'_*\mathcal{O}_{\widetilde{D}'} = \mathbb{C}_{p_0} \oplus \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2}, \quad R^1f'_*\mathcal{O}_{\widetilde{D}'} = \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2}, \quad R^2 = 0.$$

Indeed the finite reductions are bielliptic, and the cusp normalization is rational. The base logarithmic residue is evaluation with weights $(1, m_1, m_2)$; its kernel is $\mathcal{O}(1)(-\Sigma) = \mathcal{O}(-2)$. On H^1 the residue of the invariant class $\gamma^{0,1}$ is $m_j\gamma^{0,1}|_{S'_j} \neq 0$. The other eigenline has zero residue. The local rank-one image of κ_0 is its kernel, as follows either from the residue diagram of proposition 12.14 or from the eigenframe calculation in 23.1. Thus the two finite skyscrapers cancel exactly. This proves the first row, with no surviving torsion.

2. The proper flat map is Gorenstein of relative dimension two. Relative duality for $F = \Omega^1_{X'}$, using $F^* \otimes \omega_{X'/B} = \Omega^2_{X'} \otimes f'^*\mathcal{O}(2)$, gives the second row because all sheaves in the first row are locally free. Coherent Leray on $\mathbb{P}^1$ gives every Hodge number. In particular $h^{p,0} = 0$ for $p > 0$ follows from these direct images and duality, as the degree-zero entries and their duals show.

□

23.2 Higher multiplicities and the anticanonical ring

Proposition 23.3 (Canonical bundles for all coprime multiplicities)**.** [tag 01S5]

Let $X = X_{3k,4l}(u)$ with admissible translation data as in Viaclovsky (2026a, Theorem 1.1, p. 1, and Proposition 2.3, pp. 5–6), $\gcd(3k, 4l) = 1$, and any nonzero u in its construction domain. Then

$$K_X = f^*\mathcal{O}_B(-1) \otimes \mathcal{O}_X\big((k-1)S_1 + (3l-1)S_2\big).$$

Moreover $(h^{0,q})_{q=0}^3 = (1, 1, 0, 0)$, $h^{p,0} = 0$ for $p > 0$, and $(h^{1,1}, h^{1,2}) = (2, 1)$ for every such pair. The relative comparison maps in Proposition 23.1 identify the cohomological calculation for all these multiplicities.

Proof. 1. The cited proposition supplies exactly the free cyclic quotients used in 23.1; its Theorem 1.1 supplies the stated sphere topology for its admissible choices. Proposition 23.2 computes the cohomology by the relative cotangent comparison. For the canonical bundle we check the natural evaluation $f^*f_*\omega_{X/B} \to \omega_{X/B}$. The invariant relative determinant has weights $a_1 = 2k$ and $a_2 = l$ under the clockwise generators of orders $3k$ and $4l$. The pullback of a canonical form is $F(s)\,ds \wedge e$, where the character of $ds \wedge e$ is ζ_m^{1+a}. Therefore its invariant coefficients are s^{m-1-a} times an invariant function and a nowhere vanishing unit. Since the quotient of total spaces is étale, the evaluation vanishes to order $m-1-a$ on the reduced fibre downstairs. These orders are $k-1$ and $3l-1$. At the cusp the form $d\log x_1 \wedge d\log x_2 \wedge dq$ is a nowhere vanishing canonical frame in every smooth toric chart, so no cusp component occurs.

2. By 23.1 and relative duality, $f_*\omega_{X/B} = (R^2f_*\mathcal{O}_X)^* = \mathcal{O}(1)$. The evaluation divisor just computed therefore gives $\omega_{X/B} = f^*\mathcal{O}(1) \otimes \mathcal{O}((k-1)S_1 + (3l-1)S_2)$. Tensoring by $f^*K_B = f^*\mathcal{O}(-2)$ proves the formula.

□

Proposition 23.4 (The complete graded anticanonical ring)**.** [tag 01S6]

Put $m_1 = 3k$, $m_2 = 4l$, and $d = 4l + 3k - kl$. Choose integers r, s with $m_2r + m_1s = 1$ and set $\mathcal{L} = \mathcal{O}_X(rS_1 + sS_2)$. There is a graded algebra isomorphism

$$\bigoplus_{n\geq 0} H^0(X, -nK_X) \simeq \bigoplus_{n\geq 0} \mathbb{C}[x, y]_{dn}, \qquad \deg x = m_2, \quad \deg y = m_1,$$

where the right-hand summand is assigned degree n. For negative weights the polynomial space is zero. Explicitly

$$h^0(X, -nK_X) = \max\left\{1 + n - \left\lceil \frac{n(k-1)}{3k} \right\rceil - \left\lceil \frac{n(3l-1)}{4l} \right\rceil, 0\right\}.$$

The integer d is nonzero for admissible pairs. If $d < 0$ the anticanonical ring is $\mathbb{C}$; if $d > 0$ it is the indicated weighted Veronese ring. For $(k, l) = (1, 1)$ it is $\mathbb{C}[U, V]$, $\deg U = 1$, $\deg V = 2$.

Proof. 1. The divisor of $f^*(t/(t-1))$ is $m_1S_1 - m_2S_2$. It follows, retaining this rational function as the comparison, that $\mathcal{L}^{m_2} \simeq \mathcal{O}(S_1)$, $\mathcal{L}^{m_1} \simeq \mathcal{O}(S_2)$ and $\mathcal{L}^{m_1m_2} \simeq f^*\mathcal{O}(1)$. 23.3 then gives $-K_X = \mathcal{L}^d$.

2. For every integer power of $\mathcal{L}$, a meromorphic function representing a section of $\mathcal{O}(aS_1+bS_2)$ is holomorphic on every regular compact fibre and is constant there. It descends to the base on the smooth locus. On a cyclic cover $q=s^m$, the finite pole bound gives a Laurent expansion bounded below, and invariance leaves only powers divisible by m. Thus descent extends meromorphically across the finite point, with the exact valuation inequality $m\,\mathrm{ord}_q(h)+a\geq 0$. The identical argument at the reduced cusp has $m=1$. Conversely each base function satisfying these bounds pulls back to an allowed section. Consequently, as subsheaves of meromorphic functions,

$$f_*\mathcal{O}(aS_1+bS_2)=\mathcal{O}_B\left(\lfloor a/m_1\rfloor\, p_1+\lfloor b/m_2\rfloor\, p_2\right).$$

This proves the displayed dimension formula using 23.3 and $h^0(\mathbb{P}^1,\mathcal{O}(e))=\max\{e+1,0\}$.

3. For the algebra, let x,y be the canonical sections of $\mathcal{O}(S_1),\mathcal{O}(S_2)$ under the chosen identifications with powers of $\mathcal{L}$. Their powers x^{m_1},y^{m_2} pull back two independent sections of $\mathcal{O}_B(1)$, with zeros at the two distinct points. For any integer j, choose the unique residues $0\leq a_0<m_1$, $0\leq b_0<m_2$ for which $j-m_2a_0-m_1b_0=em_1m_2$ is integral. The valuation formula identifies $H^0(\mathcal{L}^j)$ with $x^{a_0}y^{b_0}H^0(\mathcal{O}_B(e))$. When $e\geq 0$ a basis is

$$x^{a_0+m_1i}y^{b_0+m_2(e-i)},\qquad 0\leq i\leq e;$$

when $e<0$ there are no sections. These are exactly the monomials of weighted degree j. Their multiplication is ordinary polynomial multiplication, proving the algebra assertion in every degree. When $k=l=1$, the weight-$6n$ monomials are generated by y^2 and x^3, of regraded degrees $1,2$. Finally, $d=0$ would imply $k\mid 4l$ and $l\mid 3k$; coprimality forces $k=l=1$, which instead gives $d=6$.

□

Remark 23.5 (Six multiplicity pairs)**.** [tag PI18]

The formulas of propositions 23.3 and 23.4 give the following values, where r_1,r_2 are the canonical-divisor coefficients and $P_n^-=h^0(-nK_X)$:

(k,l)	(m_1,m_2)	(r_1,r_2)	$(\deg x,\deg y)$	d	P_1^-	P_2^-	P_3^-
$(1,1)$	$(3,4)$	$(0,2)$	$(4,3)$	6	1	2	2
$(1,2)$	$(3,8)$	$(0,5)$	$(8,3)$	9	1	1	2
$(3,1)$	$(9,4)$	$(2,2)$	$(4,9)$	10	0	1	1
$(1,4)$	$(3,16)$	$(0,11)$	$(16,3)$	15	1	1	1
$(5,1)$	$(15,4)$	$(4,2)$	$(4,15)$	14	0	1	1
$(1,5)$	$(3,20)$	$(0,14)$	$(20,3)$	18	1	1	1

Each entry follows by substitution in the displayed floor formula, or by counting monomials of weights (m_2,m_1).

23.3 Finite fundamental groups and universal covers

Lemma 23.6 (Contraction of the cusp tube)**.** [tag PI19] *The reduced one-component hexagonal cusp fibre W in the proper local filling has a closed tube $N=\{|t_c|\leq\epsilon\}$ that strongly deformation retracts onto W. The retraction fixes W pointwise.*

Proof. Choose $\epsilon>0$ small enough that the compact tube is covered by finitely many normal-crossing charts $t_c=z_1\cdots z_r$, $1\leq r\leq 3$. Off the divisor, take the real vector field whose z_i component is

$$-\frac{z_i|z_i|^{-2}}{\sum_{j=1}^{r}|z_j|^{-2}},$$

with its other coordinate components zero. It satisfies $\dot{t}_c = -t_c$. Each component has absolute value at most $(\sum_j |z_j|^{-2})^{-1/2} \le \min_j |z_j| \le |t_c|^{1/r}$, so the norm is at most $C|t_c|^{1/r}$. A finite partition of unity preserves $\dot{t}_c = -t_c$ and gives a global field of norm at most $C'|t_c|^{1/3}$ after shrinking the tube so that $|t_c| \le 1$. Extend the field by zero on W.

Its flow Φ_T on $N \setminus W$ satisfies $t_c(\Phi_T(x)) = e^{-T} t_c(x)$. It therefore remains in the compact tube and exists for every $T \ge 0$. Its length after time T is at most

$$3C'|t_c(x)|^{1/3} e^{-T/3} \le 3C'\epsilon^{1/3} e^{-T/3}.$$

The flow converges uniformly to a continuous map $r : N \to W$, with $r|_W = \mathrm{id}_W$. The same estimate gives continuity at W after extending every Φ_T by the identity there. The map

$$H(x,u) = \begin{cases} \Phi_{-\log(1-u)}(x), & 0 \le u < 1, \\ r(x), & u = 1 \end{cases}$$

is continuous at $u = 1$ by the uniform tail estimate. It has $H(x,0) = x$, $H(x,1) \in W$, and $H(w,u) = w$ for $w \in W$, which proves the strong deformation retraction. □

Proposition 23.7 (Connected cyclic covers and the cusp comparison)**.** [tag 01S7]

For an admissible $(3,4)$ *member with* $n = |12l_0 - 4l_1 - 3l_2| > 1$*, let* $\pi : Y \to X$ *be the universal cover and* $g = f\pi$*. It is a finite étale holomorphic map of degree* n*. Every fibre of* g *is connected; its finite reductions are bielliptic, its multiplicities are* $3, 4$*, and its cusp* W_n *normalizes to a disjoint union of* n *copies of* dP_6*. For* $\mathcal{E}_X$ *of Definition 12.3 and the identical quotient* $\mathcal{E}_Y$ *on* Y*, pullback induces isomorphisms*

$$R^q f_* \mathcal{O}_X \simeq R^q g_* \mathcal{O}_Y, \qquad R^q f_* \mathcal{E}_X \simeq R^q g_* \mathcal{E}_Y.$$

Proof. 1. The smooth family over the punctured base has zero section, so its fundamental group is $\Lambda \rtimes \langle x, y \rangle$, where x, y are the analytic peripheral generators fixed by the root-deck convention and act by A_1, A_2. On a finite filling the lifted affine generator has m_jth power translation by v_j: its m_j translations add to v_j because $A_j v_j = v_j$. Thus filling the normal disc imposes $x^3 = t_{v_1}$ and $y^4 = t_{v_2}$, with the signs fixed by the clockwise convention of Definition 2.1.

2. The toric covering space of the cusp tube is simply connected. Indeed its dense torus has cocharacter lattice $\mathbb{Z}^3$; a loop may be perturbed into that torus, and the normal discs of the three rays $(0,1), (e_1,1), (e_2,1)$ kill a basis of this lattice. This argument applies also to the infinite fan, since a compact loop and a homotopy meet only finitely many of its locally finite charts. The same conclusion holds for the small tube: if a loop in $|t_c| < \epsilon$ bounds a disc in the toric space, the torus action $(x_1, x_2, t_c) \mapsto (x_1, x_2, a t_c)$ for a sufficiently small positive a moves that compact disc into the tube, and the scaling homotopy of its boundary stays in the tube. The cusp quotient therefore has fundamental group $\Lambda/\Lambda_{\mathrm{tor}}$, and its toric meridian bounds the normal disc at fixed x_1, x_2. A translated collar makes this meridian $t_\mu y^{-1} x^{-1}$, where $\gamma(\mu) = l_0$. Van Kampen consequently kills Λ_{tor} and imposes $xy = t_\mu$.

3. The normal closure of Λ_{tor} in the fibre lattice is $\ker \gamma$: it contains $\hat{w}, \hat{\delta}$ and $A_1\hat{w} + \hat{w} = \hat{u}$, and all these relations lie in $\ker \gamma$. The remaining generator c, the image of $\hat{\gamma}$, is central because $\gamma A_j = \gamma$. The presentation is thus $\langle c, x, y \mid c \text{ central},\ xy = c^{l_0},\ x^3 = c^{l_1},\ y^4 = c^{l_2} \rangle$. Eliminating y makes it abelian, with relation matrix

$$\begin{pmatrix} 3 & -l_1 \\ 4 & l_2 - 4l_0 \end{pmatrix}, \qquad \det = -p.$$

Its first Smith divisor is one because its entries include three and four; hence the group is $\mathbb{Z}/|p|$. After imposing $c = 1$ the quotient has $x^3 = x^4 = 1$, so it is trivial. Thus the image of

the regular-fibre lattice, containing c, generates the entire group $\mathbb{Z}/n$. The same is true of each reduced special fibre. At a finite point the cyclic model is a smooth torus family over a disc, with a free affine finite-group action. An equivariant smooth trivialization, obtained by averaging a horizontal connection, contracts the disc to its centre and descends to a deformation retraction of the local total space onto S_j. Thus the image of the regular-fibre lattice in the local fundamental group is contained in the image of $\pi_1(S_j)$; the same holds after mapping to $\pi_1(X)$. At the cusp $\hat{\gamma}$ survives in $\Lambda/\Lambda_{\mathrm{tor}}$. The explicit normal-crossing contraction of the tube onto W in Lemma 23.6 identifies their fundamental groups, so the same generator is in the image of $\pi_1(W)$. The covering-space criterion therefore makes the inverse image of every fibre connected. Admissibility gives $3 \nmid l_1$ and odd l_2, so $\gcd(n, 12) = 1$. The étale cover preserves fibre multiplicities. On a finite reduction it cannot kill the nontrivial order-3 or order-4 linear holonomy, since the covering degree is prime to it; the resulting finite torus quotient is again bielliptic.

4. Normalization commutes with the finite étale base change $Y \to X$. The pullback of the simply connected normalization dP_6 is its trivial degree-n cover. The double curves and triple points also pull back étale, and their incidence maps are the lifts of the original incidence maps. The lifted incidence maps therefore make the dual complex a degree-n covering of the original real two-torus. It is connected because W_n is connected. The normalization resolution for $\mathcal{O}_{W_n}$ has acyclic components, double curves and point strata. Its global cochain complex is consequently the cellular cochain complex of this dual torus, with $n, 3n, 2n$ cells. Thus its coherent cohomology is $(1, 2, 1)$, and pullback from W is an isomorphism over $\mathbb{C}$: on a finite cover of real tori it is induced by a full-rank map of their integral first cohomology lattices.

5. Modulo $d \log q$, the two toric forms $d \log x_i$ are invariant under all cusp gluings as computed in 23.1. Hence $\mathcal{E}_X|_W = \mathcal{O}_W^2$ and $\mathcal{E}_Y|_{W_n} = \mathcal{O}_{W_n}^2$, and the same pullback argument applies to their cohomology. Over a regular fibre, an isogeny of complex tori induces isomorphisms on $H^q(\mathcal{O})$ and $H^q(\Omega^1)$ over $\mathbb{C}$. At a finite fibre, make the cyclic base change. The map is an isogeny of the smooth torus families and induces isomorphisms of their cohomology bundles. Taking the same cyclic invariant eigenspaces descends these isomorphisms, including all nilpotent layers of the scheme fibre. Grauert base change at the cusp, and this explicit calculation at the finite points, prove the asserted bundle maps are isomorphisms on all of B.

□

Proposition 23.8 (Cotangent direct images and Hodge numbers of the covers)**.** [tag 01S8]
Write $T_n = \mathbb{C}_{p_0}^{n-1}$. Then

	$R^0 g_*$	$R^1 g_*$	$R^2 g_*$
Ω^1_Y	$\mathcal{O}(-2)$	$\mathcal{O} \oplus \mathcal{O}(-1)^2 \oplus T_n$	$\mathcal{O} \oplus \mathcal{O}(-1)$
Ω^2_Y	$\mathcal{O}(-2) \oplus \mathcal{O}(-1)$	$\mathcal{O}(-2) \oplus \mathcal{O}(-1)^2$	$\mathcal{O} \oplus T_n$

Writing V for the locally free quotient, the splittings form an affine space under $\mathrm{Hom}(V, T_n)$*, by the exact sequence in the proof. The only nonzero Hodge numbers of Y are*

$$h^{0,0} = h^{0,1} = h^{3,2} = h^{3,3} = 1, \quad h^{1,2} = h^{2,1} = 1, \quad h^{1,1} = h^{2,2} = n + 1.$$

The original finite-fundamental-group threefold X itself has the same Hodge diamond as the simply connected $(3, 4)$ member.

Proof. 1. 23.7 identifies the logarithmic cotangent sequence on cohomology, and therefore its connecting maps and direct images, with those of the original family. The normalized residue sheaf differs only at the cusp: its degree-zero cohomology there is now $\mathbb{C}^n$. The residue of a base logarithmic form is the diagonal vector $(1, \ldots, 1)$, since each cusp component has

multiplicity one. Its cokernel is T_n. The degree-one residues at the finite fibres cancel the same two skyscrapers as in 23.2. The long exact sequence therefore gives

$$0 \longrightarrow T_n \longrightarrow R^1 g_* \Omega^1_Y \longrightarrow \mathcal{O} \oplus \mathcal{O}(-1)^2 \longrightarrow 0$$

and the remaining entries of the first row. The extension splits because $\mathrm{Ext}^1(V, T_n) = H^1(V^* \otimes T_n) = 0$ for a vector bundle V on B.

2. For a sheaf F locally free on the total space, relative duality in its derived form gives

$$Rg_*(F^* \otimes \omega_{Y/B}) \simeq R\mathcal{H}om_B(Rg_*F, \mathcal{O}_B)[-2].$$

For $F = \Omega^1_Y$, the torsion in $R^1 g_* F$ contributes $\mathcal{E}xt^1_B(T_n, \mathcal{O}_B)$ in degree two of the right side. This is a length-$(n-1)$ skyscraper at p_0. After tensoring by $\mathcal{O}(-2)$ it still has that length. The locally free parts dualize in complementary degrees, giving exactly the second row; the extension by the locally free quotient splits for the same reason as above. Coherent Leray computes the displayed Hodge numbers. For X with one-component cusp, 23.2 already applies independently of its fundamental group.

□

Proposition 23.9 (Canonical rings and rational topology of the covers)**.** [tag 01S9]
For X and Y as above,

$$K_Y = \pi^* K_X = \mathcal{O}_Y(-2S'_2), \quad \bigoplus_{m \geq 0} H^0(Y, -mK_Y) = \mathbb{C}[U, V], \quad \deg(U, V) = (1, 2).$$

The same ring holds for X. Over $\mathbb{C}$, the Betti numbers of Y are $(1, 0, n-1, 0, n-1, 0, 1)$; its Frölicher sequence degenerates at E_2. For every nontrivial character of the deck group, the character part of Dolbeault cohomology consists of one class in $(1, 1)$ and one in $(2, 2)$ and no others.

Proof. 1. The canonical formula pulls back through an étale map, and $4S'_2 = g^* p_2$. 23.7 gives $g_* \mathcal{O}_Y = \mathcal{O}_B$. The valuation argument in 23.4 therefore applies unchanged and proves the ring statement with all multiplication maps.

2. The comparisons of 23.7 are pullback maps and identify the logarithmic cohomology bundles with the invariant part; the deck group acts trivially on those bundles. At the cusp its action on the n normalization components is the regular permutation representation. The quotient by the diagonal has each nontrivial character once. Thus the first-row additional skyscraper in 23.8 has precisely these characters, and duality gives the conjugate characters in its second-row additional skyscraper. All other cohomology is invariant and equal to that of X. Coherent Leray now proves the assertion about the Dolbeault character spaces.

3. The de Rham complex and its Frölicher sequence split under the idempotents of the finite deck group. A nontrivial character has just the two stated Dolbeault entries. No differential can connect them or connect either to another nonzero entry, so both survive. The invariant summand is the de Rham complex of X by finite étale descent and averaging. Its rational topology is computed by the following attachment maps. In the remainder of this proof all homology and cohomology coefficients are $\mathbb{Q}$.

4. Let N be a small closed cusp tube, $M = \partial N$, and $U = X \setminus \operatorname{int} N$. The space U is covered by two pieces retracting onto S_1, S_2, whose intersection retracts onto a regular fibre F. The retractions are obtained from the equivariant smooth torus trivializations used in 23.7. The maps $F \to S_j$ are the central finite covering maps; translation of their origins induces the

identity on their rational cohomology. Finite-cover transfer identifies their pullbacks with the invariant subspaces for A_j. Put

$$\psi_1 = 2u + w + 3\delta, \qquad \psi_2 = u + w + 2\delta, \qquad \eta = u \wedge w + 6\gamma \wedge \delta.$$

Direct substitution in A_1, A_2 shows that the invariant degree-one spaces are $\langle \gamma, \psi_j \rangle$ and the invariant degree-two spaces are $\langle \gamma \wedge \psi_j, \eta \rangle$. Their dimensions are two: the nontrivial eigenvalues of A_j on the complementary plane are the inverse primitive third or fourth roots, so the exterior-square fixed space has exactly the two displayed generators. In the transfer-normalized dual homology bases, the two Mayer–Vietoris maps are therefore the following matrices:

$$\alpha_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 \\ -1 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & -6 & -1 & 0 & 0 \end{pmatrix}.$$

The second matrix uses the ordered wedge basis $(\hat{\gamma}\hat{u}, \hat{\gamma}\hat{w}, \hat{\gamma}\hat{\delta}, \hat{u}\hat{w}, \hat{u}\hat{\delta}, \hat{w}\hat{\delta})$. Both matrices have rank three, and $\ker \alpha_1 = \mathbb{Q}\nu_0$, with $\nu_0 = -\hat{u} - \hat{w} + \hat{\delta}$. Consequently

$$H_1(U) = \mathbb{Q}, \qquad 0 \longrightarrow \operatorname{coker} \alpha_2 = \mathbb{Q} \longrightarrow H_2(U) \xrightarrow{\partial} \mathbb{Q}\nu_0 \longrightarrow 0. \tag{23.1}$$

The generator $\Theta \in H^1(U)$ is normalized by $\Theta|_F = \gamma$. The finite affine generators have $3\bar{g}_1 = l_1 c$, $4\bar{g}_2 = l_2 c$ in rational homology, where c is the image of $\hat{\gamma}$ and $\Theta(c) = 1$. For the toric meridian $a_0 = t_\mu y^{-1} x^{-1}$ this gives the explicit collar value

$$\Theta(j_* a_0) = l_0 - l_1/3 - l_2/4 = p/12. \tag{23.2}$$

5. Lemma 23.6 gives the strong deformation retraction $N \to W$, fixing W pointwise.

6. The constant-sheaf normalization resolution of W has strata dP_6, three copies of $\mathbb{P}^1$, and two points. Its row of degree-zero cohomology is

$$\mathbb{Q} \xrightarrow{0} \mathbb{Q}^3 \xrightarrow{(a,b,c)\mapsto(a-b+c,\,a-b+c)} \mathbb{Q}^2.$$

The strata have no degree-one cohomology. This resolution therefore gives $H_1(N) = H_1(W) = \mathbb{Q}^2$, $H_4(N) = \mathbb{Q}[W]$ and $H_5(N) = 0$; it also gives $e(W) = 6 - 3 \cdot 2 + 2 = 2$. Poincaré–Lefschetz duality for the compact smooth manifold with boundary (N, M) now shows that $H_1(M) \to H_1(N)$ and $H_2(M) \to H_2(N)$ are surjective. For degree one the next relative group is $H_1(N, M) = H^5(N) = 0$. For degree two the map to $H_2(N, M) = H^4(N)$ is the intersection pairing with $[W]$, which is zero because W is the divisor of the global function t_c on N. The Wang sequence for the boundary mapping torus, with $\operatorname{rank}(M_0 - I) = 2$, gives $\dim H_1(M) = 3$. Its kernel to $H_1(N)$ is exactly $\mathbb{Q}a_0$: this meridian is nonzero under projection to the base circle and bounds its normal disc in N.

7. We exhibit two classes in $K_2 = \ker(H_2(M) \to H_2(N))$. For $\nu = \hat{w}, \hat{\delta}$, let ξ_ν be the two-torus obtained by translating the toric meridian around the angular circle S^1_ν. In the chart $t_c = z_0 z_1 z_2$ corresponding to the rays $(0,1), (e_1,1), (e_2,1)$, these are the boundaries of $D^2_{z_0} \times S^1_{z_i}$ with the other coordinate fixed and the radii chosen so that $|t_c| = \epsilon$ on the boundary. They therefore bound in N with the specified toric meridian. Cutting the meridian into the two arcs of the cover defining (23.1), the two cylinders have end circles differing by $(I - A_1^{-1})\nu$. With this orientation of the Mayer–Vietoris boundary, matrix multiplication gives

$$\partial j_* \xi_{\hat{w}} = -\nu_0, \qquad \partial j_* \xi_{\hat{\delta}} = 0. \tag{23.3}$$

The affine translation of either endpoint has no effect on its homology class, which explains why this boundary formula is the same for every translation twist.

8. Choose $\mathcal{A} \in H^2(U)$ restricting to η on F; it exists by Mayer–Vietoris because the two transfer-normalized classes η have equal restrictions. Fibrewise translation by $S^1_{\hat{\delta}}$ defines a global smooth action on U: $A_j\hat{\delta} = \hat{\delta}$, and all affine translations commute with this circle. Slant product of its action map with the circle defines $\iota : H^2(U) \to H^1(U)$. Naturality on F gives $\iota\mathcal{A}|_F = \iota_{\hat{\delta}}\eta = -6\gamma$. The restriction $H^1(U) \to H^1(F)$ is injective by (23.1), so $\iota\mathcal{A} = -6\Theta$. By the definition of slant product,

$$\langle \mathcal{A}, j_*\xi_{\hat{\delta}} \rangle = \langle \iota\mathcal{A}, j_*a_0 \rangle = -p/2. \tag{23.4}$$

We may add the class pulled back from $(\mathbb{Q}\nu_0)^*$ under the boundary map in (23.1) so that $\langle \mathcal{A}, j_*\xi_{\hat{w}} \rangle = 0$; this does not change its restriction to F or (23.4). In the coordinates on $H_2(U)$ given by ∂/ν_0 and this normalized pairing, the two attachment cycles have matrix

$$\begin{bmatrix} j_*\xi_{\hat{w}} & j_*\xi_{\hat{\delta}} \end{bmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -p/2 \end{pmatrix}. \tag{23.5}$$

It is invertible over $\mathbb{Q}$ because admissibility implies $p \neq 0$. Thus the image of K_2 already fills $H_2(U)$. Likewise (23.2) says the image of $\ker(H_1(M) \to H_1(N))$ is nonzero in $H_1(U)$. Using the surjectivity to N just proved, Mayer–Vietoris for $X = U \cup N$ now gives $H_1(X) = H_2(X) = 0$. Every other fibre has Euler number zero, so additivity over the three special fibres and the smooth torus bundle gives $e(X) = 2$. Poincaré duality then gives $b_4 = b_5 = 0$ and $b_3 = 0$. This proves the asserted rational homology of X for all admissible twists; the only changing determinant in (23.2)–(23.5) is a nonzero constant multiple of p.

9. Proposition 13.2, applied to this rational topology and the same Hodge table, gives degeneration of the invariant summand at E_2. Summing it with the nontrivial character spaces proves the Betti numbers and the degeneration statement for Y. In particular $e(Y) = 2n$, as also follows from the covering degree.

□

23.4 CM and torsion

Cited result 23.10 (The CM torsion fibres)**.** [tag 000U]

There is no $t \in \mathbb{C} \setminus \{0, 1\}$ for which E_t has complex multiplication and P_t is torsion. On the smooth models at $t = 0, 1$ after base change of orders $3, 4$, respectively, the limiting points have orders $3, 2$. Thus these are the only CM torsion points on the completed family.

Proof. Theorem 4.15(a) of (Bourdon et al., 2017, arXiv v3, p. 23) applies to an elliptic curve with CM by any imaginary quadratic order over $F = \mathbb{Q}(j(E))$. Its torsion group is one of $0, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/6, (\mathbb{Z}/2)^2$. Here $F = \mathbb{Q}(t)$ because $j = 1728t$, and the equation and section of definition 2.6 are defined over F. On the smooth base, $\psi_2(P)$ and $\psi_3(P)$ are nonzero. Thus only orders four and six remain.

The order-four value has $j = 55296/5$. The order-six values have irreducible monic polynomial

$$X^2 - \frac{10395648}{11}X + \frac{24461180928}{11}.$$

None is an algebraic integer. The integrality of singular moduli for every imaginary quadratic order (Sutherland, 2023, Theorem 20.12 and Corollary 20.13, Lecture 20, p. 6) therefore excludes CM in all these cases.

At zero, the changes $t = s^3$, $x = s^4X$, $y = s^6Y$ give

$$Y^2 = X^3 - \tfrac{3}{4}s(s^3 - 1)X + \tfrac{1}{4}(s^3 - 1)^2, \qquad P_s = (0, (s^3 - 1)/2).$$

The central equation is $Y^2 = X^3 + 1/4$, and the limiting point $(0, -1/2)$ has order three by its horizontal tangent. At one, $t = 1 + s^4$, $x = s^2 X$, $y = s^3 Y$ give

$$Y^2 = X^3 - \tfrac{3}{4}(1+s^4)^3 X + \tfrac{1}{4}(1+s^4)^4 s^2, \qquad P_s = (0, (1+s^4)^2 s/2).$$

The central equation is $Y^2 = X^3 - 3X/4$, and $(0,0)$ has exact order two. Both central discriminants are nonzero. Their CM orders are $\mathbb{Z}[\rho]$ and $\mathbb{Z}[i]$, and their norm values are those of proposition 4.3. □

24 Global heights and the eleven rational CM fibres

Remark 24.1 (Arithmetic normalization)**.** [tag IA25] The arithmetic equation, invariant differential, place weights, and height normalization are those of definition 2.6.

Proposition 24.2 (The global height formula)**.** [tag 000V]

For every number field K, every $t \in K \setminus \{0,1\}$, and the section just specified,

$$\sum_{\sigma : K \hookrightarrow \mathbb{C}} \log N(P_{\sigma(t)}) = \sum_{v \nmid \infty} n_v \lambda_v(P_t) - [K:\mathbb{Q}]\widehat{h}_O(P_t). \tag{24.1}$$

Here the local heights are invariant under change of Weierstrass model and satisfy, when $nP \neq O$,

$$\lambda_v(nP) = n^2 \lambda_v(P) - \log|\psi_n(P)|_v + \frac{n^2-1}{12}\log|\Delta|_v. \tag{24.2}$$

At a complex embedding $\lambda_v(P) = -\log N(P)$; complex embeddings are counted separately in (24.1). For rational t, put $A_t = \exp(\sum_p \lambda_p(P_t))$. Then A_t is a positive real algebraic number and

$$N(P_t) = A_t e^{-\widehat{h}_O(P_t)}, \qquad D_{\mathrm{part}}(t) = \frac{6}{\pi}\left(\log\frac{A_t}{2\sqrt{3}} - \widehat{h}_O(P_t)\right).$$

Thus $N(P_t)$ is algebraic if and only if $e^{\widehat{h}_O(P_t)}$ is algebraic.

Proof. 1. The sigma expression for the archimedean local height is

$$\lambda_v(P) = -\log|\sigma(u)| + \frac{1}{2}\Re(u\eta(u)) - \frac{1}{12}\log|\Delta|.$$

The sigma–theta relation and the Legendre relation identify this with $-\log N$ exactly, including its constant; this is also the normalization in cited result 4.6. The division identity $\sigma(nu) = (-1)^{n-1}\psi_n(P)\sigma(u)^{n^2}$ gives (24.2). At finite places the same identity is the defining multiplication law of the canonical Néron function. Its normalization is uniquely determined by continuity away from O, its logarithmic pole at O, and the multiplication law. These are the three properties printed in DeMarco and Mavraki (2020, Section 2.1, arXiv v2, pp. 5–6); their duplication law includes the discriminant term $\frac{1}{4}\log|\Delta|_v$. For the homothety $(x, y, \Delta, \psi_n) \mapsto (c^{-2}x, c^{-3}y, c^{-12}\Delta, c^{1-n^2}\psi_n)$, the change in the two logarithmic terms of (24.2) is $-(1-n^2)\log|c|_v - (n^2-1)\log|c|_v = 0$. The pole normalization is transported by the same formal-parameter map.

2. Sum the duplication law over places. The product formula cancels the division polynomial and discriminant terms, so that the sum $H(P) = [K:\mathbb{Q}]^{-1}\sum_v n_v\lambda_v(P)$ satisfies $H(2P) = 4H(P)$. The pole normalization implies $H(P) = \frac{1}{2}h(x(P)) + O_E(1)$. Iterating multiplication by two therefore identifies H with $\widehat{h}_O$. Moving the archimedean terms to the other side gives (24.1). For a finite place, a multiple of P lies in the identity component of a minimal model. There the local height is a rational multiple of $\log p$; the multiplication law shows the same for P. Only finitely many places contribute. This proves the assertion about A_t and both of its consequences.

□

Proposition 24.3 (Finite local heights)**.** [tag 000W]

Let $E/\mathbb{Q}$ be a global minimal Weierstrass model. If mP is in the identity component at p, and $mP \neq O$, then

$$\frac{\lambda_p(P)}{\log p} = \frac{v_p(\Delta)}{12} + \frac{\frac{1}{2}\max(0, -v_p(x(mP))) - v_p(\psi_m(P))}{m^2}. \tag{24.3}$$

For $m = 1$ we read $\psi_1 = 1$. This formula, with exact rational coordinates and valuations, gives all the finite contributions in the tables below.

Proof. 1. On the identity component of a minimal model the canonical Néron function is $\frac{1}{2}\log\max(1, |x|_p) - \frac{1}{12}\log|\Delta|_p$. This follows either from the formal parameter at O followed by the group law on the smooth locus, or from the uniqueness properties used above. Substitute it for $\lambda_p(mP)$ in (24.2) and divide by $m^2 \log p$. The contribution $v_p(\Delta)/(12m^2)$ combines with the discriminant term to give $v_p(\Delta)/12$. Membership in the identity component is checked by reduction to the nonsingular locus of the minimal cubic. Thus the test is exact also at 2 and 3.

□

Proposition 24.4 (The CM models and their height identities)**.** [tag 000X]

The eleven rational CM fibres have the global minimal models, section coordinates, component classes, and finite local heights in the following tables. On each model $a_1 = a_2 = 0$, and we write $[a_3, a_4, a_6]$ for $y^2 + a_3 y = x^3 + a_4 x + a_6$. The displayed point is $(0, y_P)$. For row i, put $t_i = j_i/1728$ and

$$u_i = -\frac{3t_i(2y_{P,i} + a_{3,i})}{4a_{4,i}}.$$

The exact pointed isomorphism to the arithmetic equation is

$$X = u_i^2 x, \qquad Y = u_i^3(y + a_{3,i}/2).$$

It sends the displayed point to P_{t_i} and pulls back $dX/(2Y)$ to $u_i^{-1}dx/(2y + a_{3,i})$. The constants A_t in increasing row order are

$$\sqrt{6}, \quad 2^{2/3}3^{1/4}, \quad 2^{3/4}, \quad 2, \quad 2, \quad \sqrt{3}, \quad 1, \quad 3^{5/12}, \quad 1, \quad 1, \quad 1.$$

Row	j	a_3	a_4	a_6	y_P
1	8000	0	-5880	153664	392
2	54000	0	-7260	234256	484
3	287496	0	-2156	38416	196
4	16581375	0	-7732620	8275904784	90972
5	-3375	0	-140	784	-28
6	-32768	1	-3234	72630	-270
7	-884736	1	-38	90	-10
8	-12288000	1	-1920270	1024288020	-32005
9	-884736000	1	-379260	89898842	-9482
10	-147197952000	1	-347045930	2488447882842	-1577482
11	-262537412640768000	1	-75065352465446420	791602953315560584426684 2	-2813543945482

Row	p	Type	Component	Order	$\lambda_p/\log p$
1	2	III^*	c_P	2	$\frac{1}{2}$
1	3	I_0^*	0	1	$\frac{1}{2}$
1	7	I_0^*	c_P	2	0
2	2	I_4^*	c_P	2	$\frac{2}{3}$

Row	p	Type	Component	Order	$\lambda_p/\log p$
2	3	III	0	1	$\frac{1}{4}$
2	11	I_0^*	c_P	2	0
3	2	I_5^*	c_P	2	$\frac{3}{4}$
3	7	I_0^*	c_P	2	0
4	2	I_8^*	c_P	2	1
4	3	I_0^*	c_P	2	0
4	7	III	c_P	2	0
4	19	I_0^*	c_P	2	0
5	2	I_8^*	c_P	2	1
5	7	III	c_P	2	0
6	3	I_0^*	0	1	$\frac{1}{2}$
6	7	I_0^*	c_P	2	0
6	11	III	c_P	2	0
7	19	III	c_P	2	0
8	3	IV	0	1	$\frac{5}{12}$
8	11	I_0^*	c_P	2	0
8	23	I_0^*	c_P	2	0
9	3	I_0^*	c_P	2	0
9	7	I_0^*	c_P	2	0
9	43	III	c_P	2	0
10	7	I_0^*	c_P	2	0
10	31	I_0^*	c_P	2	0
10	67	III	c_P	2	0
11	7	I_0^*	c_P	2	0
11	11	I_0^*	c_P	2	0
11	19	I_0^*	c_P	2	0
11	127	I_0^*	c_P	2	0
11	163	III	c_P	2	0

Proof. 1. Substitution verifies the displayed pointed isomorphisms and $\Delta(E_t) = u_i^{12}\Delta(E_i)$. Tate's reduction algorithm on each integral equation gives the minimality, fibre types and component groups in the table. For each bad prime, rational tangent addition computes mP, the two partial derivatives of the minimal cubic test membership in its nonsingular locus, and the division recurrence computes ψ_m. Formula (24.3) then proves each entry. Outside the listed primes the discriminant is a unit and P is integral, so the local height is zero. Multiplying $p^{\lambda_p/\log p}$ gives the listed constants A_t.

2. The component labels are defined as follows. The label 0 means the identity component, and c_P is the class represented by the explicitly displayed section P. At an odd prime of type I_0^* this nonidentity class is the outer component indexed by the root zero of the residual Tate cubic; the scaling $x = pX$ identifies that root. When there are two rational components, c_P is their unique nonzero class. The only remaining ambiguity is row 5 at 2, where the component group is $(\mathbb{Z}/2)^2$. Here let $T = (-14, 0)$; we label it with the basis (c_P, c_T). Indeed $P + T = (18, 64)$ after the corresponding choice of sign, and $P, T, P + T$ all reduce to the singular point, whereas $2P = (25/4, 99/8)$ reduces to the identity component. Thus the three nonzero classes are explicitly separated.

□

Remark 24.5 (Numerical sizes of the CM heights)**.** [tag IA02]

The following approximations have absolute error less than 10^{-8}, by the enclosures in

proposition 24.7.

i	$\widehat{h}_O(P_i)$	$\log N(P_i)$
1	0.68444759	0.21143215
2	0.64179980	0.09495139
3	0.53353021	−0.01366983
4	0.99406687	−0.30091969
5	0.46736568	0.22578150
6	0.42860917	0.12069697
7	0.09064584	−0.09064584
8	0.73662083	−0.27886571
9	0.60182962	−0.60182962
10	1.00264929	−1.00264929
11	2.16548925	−2.16548925

Proposition 24.6 (The maximum as a height inequality)**.** [tag 000Y]

For $t \in K \setminus \{0,1\}$,

$$\widehat{h}_O(P_t) \geq \frac{1}{[K:\mathbb{Q}]} \sum_{v \nmid \infty} n_v \lambda_v(P_t) - \frac{1}{2}\log 2. \tag{24.4}$$

The inequality is strict on this smooth base. Over $\mathbb{Q}$ it is $\widehat{h}_O(P_t) \geq \log A_t - \frac{1}{2}\log 2$. The generic height is $\widehat{h}_O(P) = 1/12$, and the variation divisor is $D_P = (\infty)/12$. Consequently

$$\widehat{h}_O(P_t) = \frac{1}{12}h(t) + O(1) \qquad (t \in \overline{\mathbb{Q}} \setminus \{0,1\}),$$

where the constant is uniform over algebraic t and is for the fixed family and section.

Proof. 1. Apply the analytic maximum theorem 6.6 at every embedding and substitute into (24.1). Equality would require $j(\sigma(t)) = 1728$ for every embedding, hence $t = 1$, which has been excluded.

2. For the generic local heights at 0 and 1, the minimal discriminant orders are 8 and 3, and the section meets the nonidentity component of types IV^* and III. Their local corrections are $2/3$ and $1/4$, respectively, so their normalized local heights are zero. At infinity, put $s = 1/t$, $X = s^{-2}x$, $Y = s^{-3}y$. The new equation is

$$y^2 = x^3 - \tfrac{3}{4}(1-s)x + \tfrac{1}{4}(1-s)^2, \qquad P = (0,(1-s)/2), \qquad \Delta = 27s(1-s)^3.$$

At $s = 0$ the node is $(1/2, 0)$, whereas $P = (0, 1/2)$ is nonsingular. Its height contribution is $1/12$. At all other function-field places the point is integral and the discriminant is a unit. This proves $D_P = (\infty)/12$ and the generic height.

3. Theorem 1.1 of DeMarco and Mavraki (2020, arXiv v2, p. 2) applies with $K = \mathbb{Q}$, $B = \mathbb{P}^1$, the displayed elliptic surface and rational section, since the generic height is positive. It equips this divisor with continuous adelic semipositive metrics whose associated height is exactly $\widehat{h}_O(P_t)$. Comparing with the usual height for $(\infty)/12$ gives the bounded difference. This is the specialization here of the variation theorem of Tate, with the local refinements of Silverman reproduced in DeMarco and Mavraki (2020, Theorems 2.2–2.3, arXiv v2, pp. 7–8): outside finitely many places the local correction is zero, and at each remaining place it extends continuously after subtracting its prescribed logarithmic singularity. Our inequality is an explicit archimedean estimate with constant $\frac{1}{2}\log 2$.

4. The same generic coefficient appears in (Call and Silverman, 1993, Theorem 4.1 and the following remark, pp. 183–184): for $T = \mathbb{P}^1$ and $[2]^*(O) \sim 4(O)$, the specialization ratio tends to $1/12$. The bounded remainder used here is the adelic-height conclusion applied in the preceding step.

□

Proposition 24.7 (Explicit height enclosures)**.** [tag 000Z]

For each of the eleven rational CM parameters in proposition 24.4, the Fourier and product algorithms below give rational interval enclosures of $\log N(P)$ *and* $\widehat{h}_O(P)$ *of radius less than* 10^{-236}.

Proof. 1. The CM discriminants, in the order of 24.4, are

$$-8, -12, -16, -28, -7, -11, -19, -27, -43, -67, -163.$$

Enumeration of primitive reduced forms (a, b, c), $|b| \le a \le c$, $b^2 - 4ac = D$, with the usual nonnegative boundary convention, gives exactly one form of each discriminant, always with $a = 1$. The range is finite and proved by $a \le \sqrt{|D|/3}$. Thus the CM theorem used in 23.10 makes the corresponding singular modulus an integer. The Fourier enclosures below place it within $1/4$ of the specified j, proving its identity.

2. Write $\tau = iT$ for the first four entries and $\tau = 1/2 + iT$ for the other seven, with

$$(T_1, T_2, T_3, T_4) = (\sqrt{2}, \sqrt{3}, 2, \sqrt{7}), \qquad T_k = \sqrt{|D_k|}/2 \quad (k \ge 5).$$

Put $Q = e^{-2\pi T}$, $q = Q$ or $-Q$ accordingly, $M = 350$, and $Y = 2/5$. On $0 \le y \le Y$,

$$\frac{\wp(1/2 - iy)}{\pi^2} = \frac{1}{\cosh^2 \pi y} - \frac{1}{3} + 8 \sum_{n \ge 1} \frac{nq^n}{1 - q^n} \{1 - (-1)^n \cosh(2\pi n y)\}. \tag{24.5}$$

For $a = Qe^{2\pi Y}$, the absolute tail after M in $\wp$ is at most

$$\frac{16\pi^2}{1 - Q} \frac{a^{M+1}\{M + 1 - Ma\}}{(1 - a)^2} < 10^{-875}.$$

The proof is $1 + \cosh(2\pi n y) \le 2e^{2\pi n Y}$ and $\sum_{n > M} n a^n = a^{M+1}(M + 1 - Ma)/(1 - a)^2$. A numerical zero approximation selects a rational centre y_0. At the two rational endpoints $y_0 \pm 10^{-240}$ the truncated expression plus this infinite tail is respectively strictly positive and strictly negative. These strict endpoint signs are evaluated with outward error bounds. Continuity produces a zero within each bracket. Every zero of $\wp$ is one of the pair $\pm P$, and N agrees on that pair, so either zero gives the same value of N.

3. The product formula on this bracket has positive real factors:

$$\begin{aligned} \log N = {} & \log(2 \cosh \pi y) - \frac{\pi T}{6} - \frac{\pi y^2}{T} \\ & + \sum_{n \ge 1} \{\log(1 + q^n e^{2\pi y}) + \log(1 + q^n e^{-2\pi y})\}. \end{aligned}$$

Since $|\log(1 + x)| \le |x|/(1 - |x|)$ for real $|x| < 1$, the omitted tail is bounded by

$$\frac{2e^{2\pi Y} Q^{M+1}}{(1 - Q)(1 - e^{2\pi Y} Q^{M+1})} < 10^{-1260}.$$

The finite sums, elementary functions and tails are evaluated by outward interval arithmetic. The radius and sign comparisons are strict. The resulting radius is less than 10^{-236}. Adding the exact finite local logarithms of 24.4 and subtracting this enclosure gives the claimed height bound.

4. The singular-modulus certificate uses $E_4 = 1 + 240\sum n^3q^n/(1-q^n)$ and $\Delta_0 = q\prod(1-q^n)^{24}$. Their respective error bounds, after truncation at M, are

$$\frac{240(M+1)^3Q^{M+1}(1+4Q+Q^2)}{(1-Q)^5}, \qquad \frac{24Q^{M+1}}{(1-Q)(1-Q^{M+1})}$$

for E_4 and for the logarithm of the product. The first uses $(M+1+k)^3 \le (M+1)^3(k+1)^3$ and the differentiated geometric series; the second uses the same logarithm bound. Thus E_4^3/Δ_0 is enclosed by the stated finite sums and tails.

□

24.1 Ranks, saturation, and central derivatives

Remark 24.8 (The real period and diagonal height pairing)**.** [tag IA26] Write E_i for the eleven minimal models in proposition 24.4. The full real period and the height relative to (O) are those of definition 2.6. Pullback of the normalized Poincaré bundle along the diagonal gives $[2]^*\mathcal{O}(O) \otimes \mathcal{O}(O)^{-2} \simeq \mathcal{O}(2O)$. Consequently the diagonal Néron–Tate pairing used in the leading-term formula is $2\widehat{h}_O$.

Proposition 24.9 (Mordell–Weil ranks and primitive sections)**.** [tag 0010]

The algebraic and analytic ranks agree for the eleven curves. They are two for $j = 8000$ and $j = -12288000$, and one for all nine other values. For every rank-one curve the section P is a primitive generator of $E(\mathbb{Q})/E(\mathbb{Q})_{\mathrm{tors}}$. Its index is therefore one. For the two rank-two curves the index of $\mathbb{Z}P$ in this free quotient is infinite.

i	j	$\operatorname{rank} E(\mathbb{Q})$	$[E(\mathbb{Q})/E(\mathbb{Q})_{\mathrm{tors}} : \mathbb{Z}P]$
1	8000	2	∞
2	54000	1	1
3	287496	1	1
4	16581375	1	1
5	−3375	1	1
6	−32768	1	1
7	−884736	1	1
8	−12288000	2	∞
9	−884736000	1	1
10	−147197952000	1	1
11	−262537412640768000	1	1

Arithmetic verification. 1. Exact 2-descent gives matching lower and upper rank bounds for all eleven equations. For each of the nine rank-one sections, height bounds restrict its possible index to a finite interval; reduction maps at good primes test divisibility by every prime in that interval and every prime dividing a component-group order. These saturation tests give index one in all nine cases.

2. The integer parts of the bounds for the indices in the subgroup of points having good reduction everywhere are respectively

$$1, 1, 1, 2, 2, 2, 3, 5, 5$$

in the rank-one order $i = 2, 3, 4, 5, 6, 7, 9, 10, 11$; the algorithm also checks the primes dividing component-group orders. The underlying height lower bound is the one in Cremona and Siksek (2006, Theorem 1, PDF p. 8), applied to a minimal Weierstrass equation and the subgroup $E(\mathbb{Q}) \cap E^0(\mathbb{R}) \cap \bigcap_p E^0(\mathbb{Q}_p)$. For the largest curve the lower bound on the nonzero height in this subgroup is $0.651940632595967949\ldots$, its regulator is $17.323913974368488616\ldots$, and the resulting index bound is $5.154885040175890631\ldots$. The finite-field saturation maps for $2, 3, 5$

already have full rank at residue primes $17, 17, 59$. The index bounds and full-rank reduction maps certify primitivity.

3. The analytic ranks follow from the explicit derivative sums and bounds proved below. The two rank-two bases furnished by descent are $(-7, 441), (105, 833)$ for $i = 1$, and $(0, 32004), (22218, 3305456)$ for $i = 8$.

□

Lemma 24.10 (The CM coefficient series)**.** [tag IA08]

For $s = 11, 43, 67, 163$, let E_s be the respective rational elliptic curve 121b1*,* 1849a1*,* 4489a1*, or* 26569a1*. The coefficients of its normalized newform are*

$$b_n = \sum_{a^2=n,\ a>0} \left(\frac{a}{s}\right) a + \sum_{\substack{u,v>0,\ u\equiv v\ (2)\\ u^2+sv^2=4n}} \left(\frac{(s+1)u/2}{s}\right) u.$$

Quadratic twisting multiplies b_n by the corresponding quadratic character.

Proof. For $s = 11, 43, 67, 163$, put $F_s = \mathbb{Q}(\sqrt{-s})$ and $\mathfrak{p} = (\sqrt{-s})$. These fields have class number one and units $\{\pm 1\}$. The character

$$\psi((\alpha)) = \left(\frac{\alpha \bmod \mathfrak{p}}{s}\right) \alpha$$

is well defined because $(-1/s) = -1$. It has infinity type $(1, 0)$ and finite conductor $\mathfrak{p}$. Hecke's theta construction places $\sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N\mathfrak{a}}$ in $S_2(\Gamma_0(s^2))$: the nebentypus is $\chi_{-s}(n)(n/s) = 1$. This exact special case is also written in Darmon (2009, §3, pp. 4–5). Grouping conjugate generators $\alpha = (u + v\sqrt{-s})/2$ gives the integral formula for the coefficients:

$$b_n = \sum_{a^2=n,\ a>0} \left(\frac{a}{s}\right) a + \sum_{\substack{u,v>0,\ u\equiv v\ (2)\\ u^2+sv^2=4n}} \left(\frac{(s+1)u/2}{s}\right) u.$$

For the four curves in the statement, exact point counts at good primes, the bad-prime Euler factors, and multiplicativity give elliptic-curve coefficients equal to b_n for $1 \le n \le 10000$.

The difference of the two weight-two forms has integral Fourier coefficients. Apply Sturm's congruence theorem (Stein, 2007, Theorem 9.18, p. 171) to this difference modulo each rational prime. Agreement through $\lfloor kI/12 \rfloor$, where

$$I = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p \mid N} (1 + 1/p),$$

forces every coefficient to vanish modulo every prime; the index is also stated in (Stein, 2007, Corollary 9.20, p. 174). An integer divisible by every prime is zero, so the forms are equal over $\mathbb{C}$. For $k = 2, N = s^2$ the four bounds are

$$\left\lfloor \frac{s^2 + s}{6} \right\rfloor = 22, 315, 759, 4455,$$

all below 10000. Applying the exact quadratic twist character then identifies every coefficient in both the target and auxiliary streams, including coefficients past the tested range.

□

Proposition 24.11 (Central derivatives with explicit tails)**.** [tag 0011]

The first derivative ratios are enclosed by the following values with absolute error less than 10^{-8}*. The zeros in rows one and eight are exact.*

i	$L'(E_i,1)/\Omega_i$
1	0
2	5.13439843
3	2.13412085
4	31.81013979
5	1.86946271
6	3.42887338
7	0.36258335
8	0
9	9.62927392
10	16.04238858
11	1247.32180615

For the same full real periods,

$$L''(E_1,1)/\Omega_1 = 28.71408723\ldots, \qquad L''(E_8,1)/\Omega_8 = 78.03222042\ldots,$$

again with absolute error less than 10^{-8}*. The leading Taylor coefficients in these two rows are half the second derivatives.*

Derivative formulas and error control. 1. Let N be the conductor, $c = 2\pi/\sqrt{N}$, and $L(E,s) = \sum_{n\ge1} a_n n^{-s}$. Put

$$E_1(x) = \int_x^\infty \frac{e^{-u}}{u}\,du, \qquad G(x) = \int_x^\infty e^{-u}\frac{\log(u/x)}{u}\,du.$$

Splitting the Mellin integral at $1/\sqrt{N}$ and using the functional equation gives, for root number -1,

$$L'(E,1) = 2\sum_{n\ge1}\frac{a_n}{n}E_1(cn). \tag{24.6}$$

For root number $+1$ and $L(E,1)=0$, it gives

$$L''(E,1) = 4\sum_{n\ge1}\frac{a_n}{n}G(cn). \tag{24.7}$$

One can verify the factors directly by writing the completed Mellin integral as an integral over $[1,\infty)$ of $(u^{s-1}+wu^{1-s})f(iu/\sqrt{N})$: its first derivative has the factor $2\log u$ when $w=-1$, and its second derivative the factor $2\log^2 u$ when $w=1$. Integration by parts gives the two formulas above. The vanishing of the lower derivatives removes all derivatives of the gamma and conductor factors.

2. Hasse's bound and multiplicativity imply $|a_n| \le d(n)\sqrt{n} \le 2n$. Also $E_1(x) \le e^{-x}/x$ and $G(x) \le e^{-x}/x^2$. The latter follows on writing $u = x+v$ and using $\log(1+v/x) \le v/x$. Consequently the absolute tails after $n = M$ are at most

$$\frac{4e^{-c(M+1)}}{c(M+1)(1-e^{-c})}, \qquad \frac{8e^{-c(M+1)}}{c^2(M+1)^2(1-e^{-c})}, \tag{24.8}$$

respectively. For the first ten curves take $M = \lceil 180/c \rceil$ and compute the coefficients exactly. For the second-derivative kernel use the convergent series

$$G(x) = \frac{1}{2}(\log x+\gamma)^2 + \frac{\pi^2}{12} + \sum_{k\ge1}\frac{(-x)^k}{k^2 k!}.$$

After $k > x$ its terms decrease in absolute value. The alternating remainder was cut below 10^{-140}. At most 740 terms were needed; the largest intermediate term was less than $1.4 \cdot 10^{72}$. Evaluating these finite sums with the rounding allowance below controls this cancellation. The largest infinite-tail bound in (24.8) is $3.46 \cdot 10^{-77}$.

3. For $i = 11$, generate the coefficients by the CM theta series. The curve is the quadratic twist, by $d = 185801 = 7 \cdot 11 \cdot 19 \cdot 127$, of the conductor-$163^2$ curve

$$y^2 + y = x^3 - 2174420x + 1234136692.$$

Write $D = 163$, $\chi_D(a) = (a/D)$, and $\chi_d(n) = (d/n)$. The exact coefficient a_n is the sum of

$$\chi_d(n)\chi_D(a)a \quad (n = a^2,\ a > 0)$$

and

$$\chi_d(n)\chi_D((D+1)u/2)u \quad \left(n = \frac{u^2 + Dv^2}{4},\ u, v > 0,\ u \equiv v \pmod 2\right).$$

This is the Hecke-character theta series for $\mathcal{O}_{\mathbb{Q}(\sqrt{-D})}$, grouping conjugate algebraic integers and then applying the quadratic twist; see Darmon (2009, §3, pp. 4–5). The finite modular-form comparison in lemma 24.10 proves the global identification of this coefficient formula.

4. To accelerate the sum, in a block $n = n_0 + r$, $|r| \le R$, form the exact integer moments $M_k = \sum a_n r^k$. For $x = cn_0$ write

$$e_k = e^{-x}(-x)^k/k!, \quad g_0 = e^{-x}, \quad g_k = e_k - g_{k-1}, \quad \phi_0 = E_1(x), \quad \phi_k = -\phi_{k-1} - g_{k-1}/k.$$

Then the block contribution to $L'/2$ is $\sum_k \phi_k M_k / n_0^{k+1}$. This follows by Taylor expansion of $E_1(x(1+s))/(1+s)$. On $|s| = 1/2$ its modulus is at most $2E_1(x/2)$, which is less than 14 throughout the block range. We take enough terms that $(2R/n_0)^{k+1} < 10^{-85}$. The Cauchy remainder, summed using the bound $\sum |a_n| < 2 \cdot 10^{12}$ and $n_0 > 7700$, is less than 10^{-75}. The values of E_1 at consecutive block centres are continued by the analogous Taylor expansion; after at most 1024 such continuations, enclose E_1 directly again. The continuation bound is 10^{-110} per normalized expansion. All moments are exact integers.

5. Taking $R = 96$ and cutoff $cM \ge 190$ uses 915818441 coefficients and gives an infinite tail less than $3.10 \cdot 10^{-78}$. In the recurrences, $|e_k| \le 1$, $|g_k| \le 1$, and $|\phi_k| < 20$. There are fewer than $5 \cdot 10^6$ blocks and at most 1024 consecutive continuations. Bounding each normalized moment by $\sum |a_n|(R/n_0)^k$ controls accumulated rounding. The elementary evaluations are carried out to make that contribution less than 10^{-90}; adding the Taylor remainder gives the derivative allowance used below.

6. Enclose the real periods from the exact minimal invariants.

For rows $1, 2, 3, 4$ put respectively $T = \sqrt{2}, \sqrt{3}, 2, \sqrt{7}$ and $\tau = iT$.

For rows $5, \dots, 11$ put $T = \sqrt{7}/2, \sqrt{11}/2, \sqrt{19}/2, \sqrt{27}/2, \sqrt{43}/2, \sqrt{67}/2, \sqrt{163}/2$ and $\tau = 1/2 + iT$. Let $r = e^{-2\pi T}$ and $q = e^{2\pi i\tau}$. The identity $E_4 = 1 + 240\sum_{n\ge1} \sigma_3(n)q^n$ and $\sigma_3(n) < 2n^3$ give, after $M = 100$ terms, the tail bound

$$480r^A \left\{ \frac{A^3}{1-r} + \frac{3A^2 r}{(1-r)^2} + \frac{3Ar(1+r)}{(1-r)^3} + \frac{r(1+4r+r^2)}{(1-r)^4} \right\}, \qquad A = 101.$$

This follows by expanding $(A+j)^3$ and summing each geometric moment. All eleven parameters satisfy $1 - 1008r/(1-32r) > 0$; the bound $\sigma_5(n) < 2n^5$ and $n^5 \le 32^{n-1}$ therefore proves $E_6(\tau) > 0$. The exact Weierstrass invariant relations yield $\kappa = 2\pi(E_4/c_4)^{1/4}$. For $c_6 > 0$ the real lattice generator has length κ. For $c_6 < 0$ the lattice is rotated by i: its real generator

has length $T\kappa$ in the rectangular case and $2T\kappa$ in the other case. Indeed the real vectors in $i(\mathbb{Z}+iT\mathbb{Z})$ are $T\mathbb{Z}$, while those in $i(\mathbb{Z}+(1/2+iT)\mathbb{Z})$ are $2T\mathbb{Z}$. Multiplication by the number $n_{\mathbb{R}}=2$ for $\Delta>0$ and $n_{\mathbb{R}}=1$ for $\Delta<0$ gives the full real period used in the statement.

Evaluate these operations with outward intervals. For the first ten derivatives enlarge the explicit tail by one percent and add 10^{-100} for arithmetic and kernel-series errors. This last bound is conservative: fewer than $5\cdot 10^5$ coefficients and 740 series terms occur, the intermediate terms are below $1.4\cdot 10^{72}$, and elementary precision chosen to bound accumulated rounding, together with the 10^{-140} kernel remainders, stays below this allowance. For the last derivative add $1.1\cdot 10^{-75}$, covering the block Taylor and rounding bounds above. Division by the strictly positive period intervals gives the stated enclosures.

7. The exact root numbers are $+1$ for $i=1,8$ and -1 otherwise. All points are nontorsion by cited result 23.10. For a CM curve, the Coates–Wiles theorem, in the form Rubin (1999, Theorem 10.1, p. 47), says that a nonzero central Hecke L-value forces the Mordell–Weil group over the CM field to be finite. Rubin (1999, Proposition 5.3, p. 15) supplies, for a nonmaximal CM order, an isogenous curve with maximal order: the quotient is by the Galois-stable subgroup attached to the conductor ideal. The isogeny induces an isomorphism on Mordell–Weil groups tensored with $\mathbb{Q}$ and on their rational Tate modules, preserving ranks and local L-factors. Since $L(E/K,s)=L(E/\mathbb{Q},s)^2$ for the CM field K, nontorsion therefore forces $L(E,1)=0$. This proves the needed central vanishing also in the two sign-$+1$ cases. The functional equation then forces $L'(E,1)=0$ for those two curves. The positive first or second derivatives just evaluated prove exactly the asserted analytic ranks.

□

Proposition 24.12 (Rational leading factors and finite Tate–Shafarevich groups)**.** [tag 0012]
For each of the nine rank-one curves there is a positive rational number q_i such that

$$\frac{L'(E_i,1)}{\Omega_i}=q_i\widehat{h}(P_i). \tag{24.9}$$

Their Tate–Shafarevich groups over $\mathbb{Q}$ are finite. Consequently the corresponding D_{part} is an explicit logarithm minus $(6/(\pi q_i))L'(E_i,1)/\Omega_i$. For the two rank-two curves, $L'(E_i,1)=0$ while $\widehat{h}(P_i)>0$.

Proof. 1. Gross–Zagier's theorem applies because each curve is modular and $L(E_i,1)=0$. CM modularity follows from its Hecke-character theta series. Their Theorem I.7.3 (Gross and Zagier, 1986, Chapter I, Theorem 7.3, preprint p. 12) gives a rational point R and a nonzero rational factor relating $L'(E,1)$ to its canonical height times the real period. The differentials in the two conventions are rational multiples: write $\omega'=a\omega_E$ with $a\in\mathbb{Q}^*$. A primitive real period for ω' is $|a|\Omega_i/n_{\mathbb{R}}$, where $n_{\mathbb{R}}\in\{1,2\}$ is the number of real components. Thus the change to the full real period multiplies the rational factor by $|a|/n_{\mathbb{R}}$. As $L'(E_i,1)>0$, R has infinite order. Primitivity gives $R=mP_i+T$, $m\neq 0$, T torsion; its height is $m^2\widehat{h}(P_i)$. This proves (24.9), including positivity.

2. The rank-one Gross–Zagier–Kolyvagin implication asserts rank one and finite Tate–Shafarevich group when the central zero is simple. One can use Kolyvagin's Heegner-point theorem as stated in Gross (1991, Theorem 1.3, pp. 235–236), together with the nonzero Gross–Zagier Heegner height; a modern primary research statement of the rank conclusion is Jetchev et al. (2009, Theorem 1.1, p. 2 of arXiv v1). For each of the nine rows, the associated weight-two newform has sign -1 by proposition 24.11. Apply the nonvanishing theorem (Bump et al., 1990, Introduction, Theorem (ii), pp. 543–544) with $S=\{2\}\cup\{p : p\mid N_i\}$, where N_i is the conductor. It gives a negative fundamental discriminant D such that every prime in S splits in $K=\mathbb{Q}(\sqrt{D})$ and $L(E_i^D,1)\neq 0$. Since 2 splits, $D\equiv 1 \pmod 8$, so $D\neq -3,-4$ and

$\mathcal{O}_K^\times = \{\pm 1\}$, as required in Gross's setup. The Gross–Zagier formula (Gross and Zagier, 1986, Chapter V, Theorem 2.1, preprint p. 127; proof of I.7.3, p. 129) then gives a nonzero height for the Heegner trace, because $L'(E_i/K,1) = L'(E_i,1)L(E_i^D,1) \neq 0$. Thus Gross's Theorem 1.3 gives finite $\mathrm{Sha}(E_i/K)$. The restriction map

$$\mathrm{Sha}(E_i/\mathbb{Q}) \longrightarrow \mathrm{Sha}(E_i/K)$$

has kernel contained in $H^1(\mathrm{Gal}(K/\mathbb{Q}), E_i(K))$ by restriction–inflation. This last group is finite: the Mordell–Weil group is finitely generated, and its degree-one cohomology for this order-two group is finitely generated and killed by two. The finite kernel and finite image prove the claimed finiteness over $\mathbb{Q}$. For the two other rows the established equations are $L'(E_i,1) = 0$ and $\widehat{h}_O(P_i) > 0$; therefore an equation $L'(E_i,1)/\Omega_i = q\widehat{h}_O(P_i)$ with $q > 0$ would contradict these exact values. Their rank-two regulator, for any free basis P, Q, is the determinant $\widehat{h}_O(P)\widehat{h}_O(Q) - \langle P, Q\rangle^2$.

□

Remark 24.13 (The analytic leading quotients)**.** [tag IA04]

For the nine rank-one rows, define

$$Q_i = \frac{L'(E_i,1)/\Omega_i}{2\widehat{h}_O(P_i)\prod_p c_p/\#E_i(\mathbb{Q})_{\mathrm{tors}}^2}.$$

The exact factors in cited result 24.14 and proposition 24.16 give $Q_i = 1$ for $i = 2,3,4,5,6,7,9,10$ and $Q_{11} = 9$. In rows $3,5,7$, cited result 24.14 also computes $\#\,\mathrm{Sha}(E_i/\mathbb{Q}) = 1$. The remaining Tate–Shafarevich orders are discussed in section D.

Cited result 24.14 (Three exact leading factors)**.** [tag 0013]

For $i = 3,5,7$, namely $j = 287496, -3375, -884736$, one has

$$\mathrm{Sha}(E_i/\mathbb{Q}) = 0, \qquad \frac{L'(E_i,1)}{\Omega_i} = 4\widehat{h}_O(P_i).$$

Thus the factors $q_3 = q_5 = q_7 = 4$ are determined exactly by the integer bounds in the cited application. For the smallest of these examples, $E : y^2 + y = x^3 - 38x + 90$ and $P = (0,9)$, this gives the exact reduction

$$N(P) = \exp\left(-\frac{L'(E,1)}{4\Omega_E}\right).$$

Proof. Miller's theorem (Miller, 2011, Definition 1.1 and Theorem 1.2, arXiv v3, p. 2; Table 9, p. 23) applies to analytic rank at most one and conductor less than 5000, outside the listed exceptional curve–prime pairs. The present conductors are $3136, 3136, 361$; none occurs in that exception table, and their analytic ranks are one by proposition 24.11. The theorem applies to every curve in each isogeny class.

For these three rows, $2\prod_p c_p/\#E(\mathbb{Q})_{\mathrm{tors}}^2 = 4$, and proposition 24.9 proves primitivity. Therefore $L'(E_i,1)/(4\Omega_i\widehat{h}_O(P_i))$ is the positive integer $\#\,\mathrm{Sha}(E_i/\mathbb{Q})$. To isolate it, complete the square and write the duplication map as relatively prime integral quartics $[F(X,Z) : G(X,Z)]$. The extended Euclidean algorithm gives integer homogeneous cubics A_0, B_0, A_1, B_1 and $D \neq 0$ with

$$A_0F + B_0G = DZ^7, \qquad A_1F + B_1G = DX^7.$$

Let C dominate the coefficient one-norms of F, G and the sums $\|A_j\|_1 + \|B_j\|_1$. For coprime a, b, put $H = \max(|a|,|b|)$ and $d = \gcd(F(a,b), G(a,b))$. The two identities give $d \mid D$. Using the identity whose right side contains the coordinate of size H also gives $\max(|F(a,b)|, |G(a,b)|) \geq |D|H^4/C$. After division by d, the projective size is at least H^4/C and at most CH^4. Therefore

$$|h(x(2Q)) - 4h(x(Q))| \leq \log C.$$

Telescoping yields

$$\left|\widehat{h}_O(P) - \frac{h(x(2^6P))}{2\cdot 4^6}\right| \le \frac{\log C}{6\cdot 4^6}.$$

The exact constants for rows $3, 5, 7$ are 26610617537591808, 188716953600, and 285931133. Six rational doublings and outward logarithm enclosures, together with the derivative tails of proposition 24.11, give respective quotient intervals 1 ± 0.003, 1 ± 0.003, and 1 ± 0.009. Each contains only the integer one. Finally $A_t = 1$ in row seven by proposition 24.4, giving the displayed formula for $N(P)$. □

Remark 24.15 (Ordinary CM leading-term theorems)**.** [tag IA05]

The applicable ordinary CM results concern a prime p at which E has good ordinary reduction, equivalently p splits in the CM field. One formulation of Rubin's rank-one formula is Bertolini et al. (2012, Theorem 1, p. 262): for a curve over $\mathbb{Q}$ with CM by the maximal order, an ordinary prime prime to the conductor, and a simple central zero, an appropriate Katz p-adic L-value is, up to a nonzero algebraic factor, the square of a formal logarithm of a nontorsion rational point divided by a p-adic period. The analytic-rank-one input holds by proposition 24.11. The formal logarithm, Katz value and period in this formula are the p-adic quantities specified in that theorem.

For the p-part of the real leading-term formula, a modern precise version is Castella (2025, Theorem A, p. 2 of the author version). In addition to analytic rank one it requires full CM, abelian torsion field over the CM field, $p \nmid 6h_K$, splitting of p, conductor prime to p, and that the different of K divide the Hecke conductor exactly. For the maximal odd-discriminant cases here, the latter follows from the exponent two of the ramified prime in the rational conductor. The nonmaximal cases of discriminant -12 and -28 can be passed through a 2-isogeny at $p > 3$. The discriminant -16 case requires care: the direct application of this version has a different-conductor hypothesis at 2 that fails, so this formulation does not apply directly to that case.

For the last curve, 3 is inert in $\mathbb{Q}(\sqrt{-163})$ and is excluded by the stated $p > 3$ hypotheses. The exact analytic quotient $Q_{11} = 9$ is proved in proposition 24.16; the remaining group-order question is recorded in section D.

24.2 Exact leading factors from a genus-character denominator bound

Proposition 24.16 (The remaining rational leading factors)**.** [tag 0014]

Use the full real period and the absolute height of definition 2.6.

For the six rank-one rows $i = 2, 4, 6, 9, 10, 11$, one has

$$\frac{L'(E_i, 1)}{\Omega_i} = q_i \widehat{h}_O(P_i), \qquad (q_2, q_4, q_6, q_9, q_{10}, q_{11}) = (8, 32, 8, 16, 16, 576).$$

Together with 24.14, this determines every rank-one factor:

i	2	3	4	5	6	7	9	10	11
q_i	8	4	32	4	8	4	16	16	576.

The certified auxiliary ratios and denominator bounds used to identify the six additional factors are as follows. Here D is the auxiliary negative fundamental discriminant, $r_i = L(E_i^D, 1)/\Omega_{i,D}$,

and the denominator of q_i is at most B_i.

i	D	r_i	B_i	q_i	$\lvert a_i\rvert$
2	-23	64	256	8	32
4	-31	64	256	32	64
6	-19	8	32	8	8
9	-19	16	64	16	16
10	-3	400	1600	16	80
11	-3	64	256	576	192

The last column is the absolute genus-trace integer in the proof. These identities determine the analytic BSD quotients exactly: they are one for $i = 2, 4, 6, 9, 10$ and nine for $i = 11$.

Proof. 1. **A parametrization under quadratic twisting.** Let $E_0/\mathbb{Q}$ have conductor N_0, and suppose a parametrization $\phi : X_0(N_0) \to E_0$ sends ∞ to O and satisfies

$$\phi^*\omega_0 = c_0 f_0(q)\, dq/q,$$

where ω_0 is a minimal differential and f_0 is the normalized newform. Let b be an odd fundamental discriminant coprime to N_0, and suppose that the minimal twist has conductor $N_0 b^2$ and exact invariants

$$c_4(E_0^b) = b^2 c_4(E_0), \qquad c_6(E_0^b) = b^3 c_6(E_0).$$

Then E_0^b has a parametrization from $X_0(N_0 b^2)$ with the same Manin factor c_0.

2. Indeed, put $m = |b|$ and, using addition on E_0, define

$$F(z) = \sum_{a \bmod m} \chi_b(a)\phi(z + a/m).$$

For $\gamma = \left(\begin{smallmatrix} A & B \\ C & D_0 \end{smallmatrix}\right)$ in $\Gamma_0(N_0 m^2)$ choose $a' \equiv aD_0^2 \pmod m$. Direct multiplication gives $T_{a/m}\gamma T_{a'/m}^{-1} \in \Gamma_0(N_0)$, where T_t denotes translation by t. Because $\chi_b(D_0^2) = 1$, the sum is invariant. Each summand is an algebraic morphism on a common congruence cover; invariance therefore descends the sum to $X_0(N_0 m^2)$ over $\mathbb{Q}(\zeta_m)$. Its expansion at ∞ gives $F^{\sigma_u} = \chi_b(u)F$ for $\sigma_u(\zeta_m) = \zeta_m^u$. Quadratic descent gives the required map to E_0^b over $\mathbb{Q}$, with ∞ mapping to O. On differentials, the Gauss-sum identity gives

$$F^*\omega_0 = c_0 \tau(\chi_b) f_b(q)\, dq/q, \qquad \tau(\chi_b)^2 = b.$$

For the twist isomorphism $\iota : E_0^b \to E_0$ the exact invariant identities imply $\iota^*\omega_0 = \sqrt{b}\,\omega_b$ up to sign. Cancellation proves the assertion.

3. If $c_0 = 1$, let $t_b = \#E_0^b(\mathbb{Q})_{\mathrm{tors}}$ and $n_b = \#\pi_0(E_0^b(\mathbb{R}))$. Manin–Drinfeld (Drinfeld, 1973, Section 1, unnumbered theorem, p. 83 of the Russian original) says that the image of the rational cuspidal divisor $[0] - [\infty]$ is torsion. Its elliptic logarithm, along the imaginary axis, is $\pm L(E_0^b, 1)$. Consequently $t_b L(E_0^b, 1)$ is an integral multiple of the positive real lattice period Ω_b^+, and

$$t_b n_b \frac{L(E_0^b, 1)}{\Omega_b} \in \mathbb{Z}, \qquad \Omega_b = n_b \Omega_b^+. \tag{24.10}$$

4. **The six small base curves and Heegner hypotheses.** Use the following data; $E_i = E_0^d$ and $K = \mathbb{Q}(\sqrt{D})$.

i	E_0	N_0	d	D
2	576e2	576	−11	−23
4	3136r2	3136	57	−31
6	121b1	121	21	−19
9	1849a1	1849	21	−19
10	4489a1	4489	217	−3
11	26569a1	26569	185801	−3

All six base Manin factors are one. For the first five bases this follows from the exact modular-symbol algorithm. For 26569a1, the value one and modular degree 59368 are explicitly tabulated by Belabas et al. (2019, Appendix B, p. 102, unnumbered 26569a1/26569a2 block); their Algorithm 1 describes the computation. Exact rational transformations check both c_4, c_6 scaling identities for $b = d, dD$. Tate's algorithm and rational torsion computations give both conductor identities and torsion orders; quadratic residue symbols check the splitting conditions.

5. In every row d, D, dD are fundamental discriminants, $\gcd(d, D) = \gcd(dD, N_0) = 1$, and every prime dividing N_0 splits in K. The genus character for $K(\sqrt{d})/K$ factors through $\mathrm{Pic}(\mathcal{O}_{|d|})$ and has conductor exactly $|d|$. On ideals prime to d it is

$$\chi(\mathfrak{a}) = \chi_d(N\mathfrak{a}).$$

For a unit of the order locally at a prime $p \mid d$, its norm is a square modulo p, so this character is trivial on the order's units. It is nontrivial on the full local unit group: at a split prime the norm is multiplication on two copies of $\mathbb{Q}_p^\times$, and at an inert prime the residue-field norm onto $\mathbb{F}_p^\times$ is surjective. Since p is odd and unramified in K, this also proves that the conductor contains every $p \mid d$ exactly once. Thus the primitive ring-class-character hypotheses of Cai et al. (2014, Theorem 1.1, pp. 2524–2525) hold. Their $\mu(N_0, D) = 0$ and $u = [\mathcal{O}_{|d|}^\times : \mathbb{Z}^\times] = 1$; the latter remains true for $D = -3$ because $|d| > 1$.

6. Let $P_\mathfrak{a}$ be the corresponding Heegner points and put

$$Y = \sum_{[\mathfrak{a}]} \chi(\mathfrak{a})\phi(P_\mathfrak{a}).$$

The sum is taken in $E_0(\overline{\mathbb{Q}})$ with coefficients $\chi(\mathfrak{a}) \in \{\pm 1\}$, so $Y \in E_0(\overline{\mathbb{Q}})$. It is fixed by $\ker \chi$ and anti-invariant under the other automorphism of $K(\sqrt{d})/K$. Twisting therefore gives $Y_d \in E_i(K)$. Artin factorization gives

$$L(s, E_0, \chi) = L(E_i, s)L(E_i^D, s).$$

The intervals below prove $L(E_i^D, 1) > 0$, hence the auxiliary curve has rank zero. The rank-zero implication used here is the same unconditional modular/CM rank implication used in proposition 24.11, proposition 24.12. Thus $E_i(K)$ has rank one. 24.9 proves that P_i is primitive over $\mathbb{Q}$. Since $Y_d - \overline{Y_d}$ is torsion and $Y_d + \overline{Y_d}$ is rational, there is an integer a_i with

$$2Y_d = a_i P_i \pmod{E_i(K)_{\mathrm{tors}}}, \qquad \widehat{h}_O(Y_d) = \frac{a_i^2}{4}\widehat{h}_O(P_i).$$

7. **The exact height and period normalization.** Write $V_0 = \mathrm{area}(\mathbb{C}/\Lambda_0)$ for the period area of the minimal differential. Integrating $\phi^*\omega_0$ gives

$$4\pi^2(f_0, f_0)_{\Gamma_0(N_0)} = \frac{\deg \phi}{c_0^2} V_0.$$

The degree in the denominator of CST's Theorem 1.1 cancels, giving

$$L'(E_i,1)L(E_i^D,1)=\frac{2V_0}{c_0^2|d|\sqrt{|D|}}\,\widehat{h}_K(Y). \tag{24.11}$$

Here $\widehat{h}_K$ is the diagonal Néron–Tate pairing relative to K. This convention is explicit in Yuan et al. (2013, §7.1.1, pp. 218–219, Theorem 7.1.2): their underlying height is normalized by $[L:K]^{-1}$ over a field of definition L, and their pairing is $\widehat{h}_{\mathcal{L}}(x+y)-\widehat{h}_{\mathcal{L}}(x)-\widehat{h}_{\mathcal{L}}(y)$.

8. For $\mathcal{L}=\mathcal{O}(O)$, pull the normalized Poincaré bundle back under $1\times\lambda_{\mathcal{L}}$. The result is $m^*\mathcal{L}\otimes p_1^*\mathcal{L}^{-1}\otimes p_2^*\mathcal{L}^{-1}$. On the diagonal its canonical height is that of $[2]^*\mathcal{L}\otimes\mathcal{L}^{-2}\simeq\mathcal{L}^2$, namely $2\widehat{h}_O$ in the absolute convention. Changing from the absolute convention to relative heights over K multiplies by $[K:\mathbb{Q}]=2$. Therefore

$$\widehat{h}_K(Y)=4\widehat{h}_O(Y_d)=a_i^2\widehat{h}_O(P_i).$$

9. A real-adapted basis of Λ_0 gives

$$\Omega_i\Omega_{i,D}=\begin{cases}2V_0/(|d|\sqrt{|D|}), & \Delta(E_0)<0,\\ 4V_0/(|d|\sqrt{|D|}), & \Delta(E_0)>0.\end{cases}$$

Indeed, a triangular real lattice has one real component and its purely imaginary primitive period has twice the height of its nonreal basis vector. A rectangular lattice has two real components. Twisting by the two opposite signs interchanges the real and imaginary directions, and the exact minimal differential scalings in step 2 supply $1/(|d|\sqrt{|D|})$. Substituting into (24.11) gives

$$q_ir_i=\begin{cases}a_i^2/c_0^2, & \Delta(E_0)<0,\\ a_i^2/(2c_0^2), & \Delta(E_0)>0.\end{cases} \tag{24.12}$$

In particular, if $r_i=A_i/B_i'$ is reduced and $c_0=1$, the denominator of q_i divides $2|A_i|$. We use the weaker upper bound $B_i=4|A_i|$ in the interval certificate.

10. **Global identification of the CM coefficient series.** Apply lemma 24.10 with $s=11,43,67,163$, and then multiply the nth coefficient by the exact quadratic twist character. This identifies both target and auxiliary series.

11. **Directed enclosures of both L-sums.** For $c=2\pi/\sqrt{N}$, the root-number $+1$ formula is

$$L(E,1)=2\sum_{n\ge1}a_n\frac{e^{-cn}}{n},$$

while for root number -1 the first derivative is (24.6). Generate the coefficients in the smaller cases by point counts and the Euler-product recurrence. The largest auxiliary stream uses $s=163$, signed twist -557403, conductor 8254937936042721, and $M=867617471$; the target stream uses $s=163, d=185801, M=289205824$. Both use the cutoff $cM\ge60$ up to rounding of the integer cutoff, with the displayed integer M used in every tail bound.

12. Evaluate the finite sums with outward interval arithmetic. In a block centred at b with radius $R=128$, set $S_k=\sum a_n(n-b)^k\in\mathbb{Z}$. For $g(z)=e^{-cz}/z$, put

$$C_0=1,\quad C_k=C_{k-1}c/k,\qquad T_0=1/b,\quad T_k=(T_{k-1}+C_k)/b.$$

The coefficient of $(z-b)^k$ is $(-1)^ke^{-cb}T_k$. For $h(z)=E_1(cz)/z$, put

$$e_0=E_1(cb),\quad e_k=(-1)^ke^{-cb}T_{k-1}/k\ (k\ge1),\qquad h_0=e_0/b,\quad h_k=(e_k-h_{k-1})/b.$$

The block sums are the finite sums of these interval coefficients times the exact integers S_k.

13. On $|z-b| = b/2$, the two kernels have bounds

$$|g(z)| \le \frac{2e^{-cb/2}}{b} =: U_g, \qquad |h(z)| \le \frac{4e^{-cb/2}}{cb^2} =: U_h.$$

For the second bound, use $E_1(cz) = \int_1^\infty e^{-czt}\,dt/t$ when $\Re z > 0$. Thus truncation at degree k contributes at most

$$U_g \frac{(2R/b)^{k+1}}{1-2R/b} \sum |a_n| \quad \text{or} \quad U_h \frac{(2R/b)^{k+1}}{1-2R/b} \sum |a_n|.$$

Select the degree to make the geometric factor small, then evaluate the displayed remainder bound with upward rounding. For the derivative stream, the initial E_1 in each segment is evaluated using correctly rounded $-\operatorname{Ei}(-x)$ and monotonicity. Subsequent centres, spaced by $W = 257$, use

$$E_1(c(b+W)) = E_1(cb) - \int_0^W g(b+x)\,dx.$$

Integrating the same Taylor expansion gives the interval for this advance. Its omitted part is bounded by $WU_g(2W/b)^{k+1}/(1-2W/b)$ and added to both endpoints. The first 10280 terms are summed directly. The remaining coefficients are partitioned into segments whose boundaries are multiples of 257.

14. Hasse's bound and multiplicativity give $|a_n| \le 2n$. The infinite tails, including the factor two, are consequently at most

$$\frac{4e^{-c(M+1)}}{1-e^{-c}}, \qquad \frac{4e^{-c(M+1)}}{c(M+1)(1-e^{-c})},$$

respectively. Evaluate both tails with a lower bound for c and upper bounds for the remaining positive expressions, then add the block remainders and elementary-evaluation errors. For row eleven the central-value tail is below $5.07 \cdot 10^{-19}$, and the derivative tail is below $2.82 \cdot 10^{-21}$. These modest precisions already suffice for rational separation.

15. **Periods, heights, and exact rational separation.** Apply the real-period algorithm in proposition 24.11 separately to each target and auxiliary minimal equation, using $\tau = i\sqrt{s}$ for $i = 2, 4$ and $\tau = (1+i\sqrt{s})/2$ for $i = 6, 9, 10, 11$, where $s = 3, 7, 11, 43, 67, 163$ respectively. These parameters and their class-number-one identification are proved in proposition 24.7. In both equations the exact c_4, c_6 determine the same homothety rule $\kappa = 2\pi(E_4/c_4)^{1/4}$; the signs of c_6 and Δ select the real generator and number of components. Thus the E_4 tail, positive E_6 bound and real-lattice transport of proposition 24.11 give both full-period intervals. Use the height intervals of proposition 24.7.

16. First, the intervals for r_i isolate a unique integer after multiplication by the exact $t_b n_b$ in (24.10). This proves the six auxiliary rational values from the displayed integrality and interval bounds. Next form the interval $L'(E_i, 1)/(\Omega_i \widehat{h}_O(P_i))$. Outward enlargement of the intervals gives the following rational error bounds:

i	$\lvert r_i - r_i^{\text{listed}}\rvert$ is at most	$\lvert q_i - q_i^{\text{listed}}\rvert$ is at most
2	$4 \cdot 10^{-22}$	$2 \cdot 10^{-53}$
4	$4 \cdot 10^{-20}$	$3 \cdot 10^{-45}$
6	$4 \cdot 10^{-22}$	$2 \cdot 10^{-53}$
9	$5 \cdot 10^{-21}$	$4 \cdot 10^{-53}$
10	$3 \cdot 10^{-20}$	10^{-23}
11	$7 \cdot 10^{-15}$	$8 \cdot 10^{-19}$

Each last-column bound is strictly less than $1/B_i$. A rational number with denominator at most B_i differing from the listed integer would differ by at least $1/B_i$. Equation (24.12) therefore makes every listed q_i exact. The same equation gives the listed $|a_i|$ by integer square-root checks.

□

24.3 The three smooth torsion fibres

Proposition 24.17 (The rational order-four fibre)**.** [tag 0015]

At $t = 32/5$ a global minimal model and the image of the section are

$$E : y^2 = x^3 - 2x + 1, \qquad P = (0, 1), \qquad \Delta = 80.$$

The conductor is 40, $E(\mathbb{Q}) = \mathbb{Z}/4\mathbb{Z}$ is generated by P, and $\widehat{h}(P) = 0$. The index of $\mathbb{Z}P$ in the quotient modulo torsion is one. The complete bad-prime data are

p	$\operatorname{ord}_p \Delta$	*type*	*component of P*	$\lambda_p(P)$
2	4	III	*unique nonzero class, order 2*	$\log 2/12$
5	1	I_1	*identity*	$\log 5/12$.

All other finite local heights vanish. Consequently

$$\log N = \frac{\log 10}{12}, \qquad D_{\mathrm{part}} = \frac{6}{\pi} \log \frac{10^{1/12}}{2\sqrt{3}}.$$

The central values satisfy

$$\frac{L(E, 1)}{\Omega} = \frac{1}{8}, \qquad \frac{L'(E, 1)}{\Omega} = \frac{\gamma + \log(2\pi/\sqrt{40})}{8},$$

where γ is Euler's constant. In particular the analytic rank is zero.

Proof. 1. From the displayed minimal coordinates to the family coordinates of definition 2.6, the map is $X = (24/5)^2 x$, $Y = (24/5)^3 y$. It sends $(0, 1)$ to $P_{32/5}$ and pulls the family differential back to $(5/24)\, dx/(2y)$. Tate's algorithm gives the listed minimal model, conductor and local data. Since P has order four, $3P = -P$. The local division identity consequently gives

$$\lambda_p(P) = \left(\frac{v_p(\Delta)}{12} - \frac{v_p(\psi_3(P))}{8} \right) \log p.$$

Here $\psi_3(P) = -4$, proving both local coefficients exactly. Exact 2-descent gives rank zero, and the torsion calculation gives $\mathbb{Z}/4\mathbb{Z}$ generated by P. The plus modular symbol from 0 to ∞ is $-1/4$ in the primitive real-period normalization. There are two real components, so $L(E, 1)/\Omega = 1/8$ for the full real period. Differentiating the completed functional equation at its central point gives the displayed formula for $L'(E, 1)/\Omega$.

□

Proposition 24.18 (The conjugate order-six fibres)**.** [tag 0016]

Put $s = \sqrt{3}$ and $t_\pm = (3008 \pm 1728s)/11$. Over $K = \mathbb{Q}(s)$, a global minimal model for the fibre at t_- is

$$E_- : y^2 = x^3 - (18 + 10s)x + 21 + 12s, \qquad P_- = (0, 3 + 2s).$$

The model and point for t_+ are its conjugates. The discriminant of this model is $1032912 + 596352s$. Define

$$\mathfrak{p}_2 = (2, 1 + s), \qquad \mathfrak{p}_3 = (s), \qquad \mathfrak{p}_{11,-} = (11, s + 5), \quad \mathfrak{p}_{11,+} = (11, s - 5).$$

The bad-prime data of E_- are

v	e_v	$v(\Delta)$	*type*	*component order of* P_-	$\lambda_v(P_-)$
$\mathfrak{p}_2$	2	8	IV^*	3	0
$\mathfrak{p}_3$	2	3	III	2	0
$\mathfrak{p}_{11,-}$	1	1	I_1	1	$\log 11/12$.

For E_+ replace $\mathfrak{p}_{11,-}$ by $\mathfrak{p}_{11,+}$. The nonzero component at $\mathfrak{p}_3$ is unique; the two nonzero classes at $\mathfrak{p}_2$ are interchanged by $P \mapsto -P$ and the class of P is a generator. All omitted local heights vanish. Both points have order six, and

$$\log N(P_-) = \frac{1}{12}\log(17+10\sqrt{3}),$$
$$\log N(P_+) = \frac{1}{12}\log(10\sqrt{3}-17),$$
$$\sum_{\sigma:K\hookrightarrow\mathbb{R}} \log N(P_\sigma) = \frac{\log 11}{12} = \sum_{v<\infty} n_v\lambda_v(P) - 2\widehat{h}(P).$$

In particular $A_\mp = (17+10\sqrt{3})^{1/12}$ and $(10\sqrt{3}-17)^{1/12}$ respectively give $D_{\mathrm{part}}(t_\mp) = (6/\pi)\log(A_\mp/(2\sqrt{3}))$.

The two j-invariants are $1728t_\pm \in K\setminus\mathbb{Q}$. An elliptic curve over $\mathbb{Q}$ has rational j, so these fibres admit no model over $\mathbb{Q}$. Over K, both curves have rank one and torsion group $\mathbb{Z}/6\mathbb{Z}$. Thus the index of the torsion section in the free quotient is infinite. For example $(1,-1-s)$ has infinite order on E_-.

Proof. 1. For t_-, put $u = 48(-64+37s)/11$. The map from the displayed minimal model to the family equation is $X = u^2x$, $Y = u^3y$, with differential $dX/(2Y) = u^{-1}dx/(2y)$. It is obtained by multiplying the first global minimalization scale by the unit $(2+s)^{-3}$. Hence it preserves all finite minimal valuations. The conjugate scale gives E_+. Tate's algorithm yields the displayed table. Its discriminant ideal is $\mathfrak{p}_2^8\mathfrak{p}_3^3\mathfrak{p}_{11,-}$ and its conductor ideal is $\mathfrak{p}_2^2\mathfrak{p}_3^2\mathfrak{p}_{11,-}$.

2. As $5P = -P$, the normalized local heights are computed exactly by

$$\frac{\lambda_v(P)}{\log p} = \frac{1}{e_v}\left(\frac{v(\Delta)}{12} - \frac{v(\psi_5(P))}{24}\right).$$

The valuations of $\psi_5(P)$ at the three listed primes are $16, 6, 0$, respectively. This proves the table, including zero contribution at the two additive primes. Here $f_v = 1$ at all three primes, so weighting by $n_v = e_vf_v$ gives precisely $\log 11/12$. The exact torsion values proved in Proposition 4.9 have product $(17+10\sqrt{3})(10\sqrt{3}-17) = 11$, proving the archimedean side.

3. For the rank, translate the rational two-torsion point to zero. The resulting equation is

$$y^2 = x^3 - (9+6s)x^2 + (45+26s)x.$$

The 2-isogeny descent gives $[E(K) : \widehat{\phi}E'(K)] = 2$ and $[E'(K) : \phi E(K)] = 4$. Every surviving covering has a point; every excluded covering has a local obstruction at $\mathfrak{p}_2$ or $\mathfrak{p}_3$. The quotient formula $2^{r+2} = 2\cdot 4$ proves $r = 1$. The torsion calculation gives $\mathbb{Z}/6\mathbb{Z}$, generated by P, independently of the rank calculation.

□

Remark 24.19 (The number-field period convention)**.** [tag IA06]

For $K=\mathbb{Q}(\sqrt{3})$, put

$$\Omega_{\mathrm{prod}}=\prod_{\sigma:K\hookrightarrow\mathbb{R}}\int_{E_\sigma(\mathbb{R})}|\omega_\sigma|,\qquad \Omega_K=\frac{\Omega_{\mathrm{prod}}}{\sqrt{12}}.$$

The two conjugate equations give the same Hasse–Weil L-function over K. Its analytic conductor is $12^2\cdot 396=57024$, its gamma parameters are $(0,0,1,1)$, and its sign is -1. The exact conversion of derivative ratios is $L'(E/K,1)/\Omega_K=\sqrt{12}\,L'(E/K,1)/\Omega_{\mathrm{prod}}$.

24.4 Integral and Gaussian descriptions

Proposition 24.20 (The Mahler-measure identities)**.** [tag 001T]

Let $\Lambda\subset\mathbb{C}$ be a lattice of area A, let v be a zero of $\wp_\Lambda$, and let $d\mu=A^{-1}dA_z$ be Haar probability measure on $\mathbb{C}/\Lambda$. Put $U(z)=\log N(z)$ and $L=U(v)$. Then

$$6L=\int_{\mathbb{C}/\Lambda}\log\left|\frac{\wp'(z)^2}{\wp(z)^3}\right|d\mu(z),\tag{24.13}$$

$$N(v)^2=|\Delta(\Lambda)|^{1/6}\exp\left(-\int_{\mathbb{C}/\Lambda}\log|\wp(z)|d\mu(z)\right).\tag{24.14}$$

If $C(u)=1-(g_2/4)u^2-(g_3/4)u^3$, the pushforward of $d\mu$ under $u=1/\wp(z)$ is

$$d\nu(u)=\frac{1}{2A}\frac{dA_u}{|u|\,|C(u)|}.\tag{24.15}$$

In particular its unnormalized integral is exactly $2A$, and

$$N(v)^6=4\exp\left(\int_{\mathbb{C}}\log|C(u)|d\nu(u)\right).$$

Writing e_1,e_2,e_3 for the three branch values and $p(x)=\int\log|x-\wp(z)|d\mu(z)$, the maximum theorem is equivalently

$$\int\log|C|d\nu\le\log 2,\qquad p(0)\ge\frac{p(e_1)+p(e_2)+p(e_3)}{3}-\frac{\log 2}{3}.$$

Equality in either inequality occurs exactly when $g_3=0$.

Proof. 1. The mean of U is zero by the theta product calculation in proposition 4.2, and near zero $U(z)=\log|z|+\frac{1}{12}\log|\Delta|+o(1)$. The divisor of $\wp$ is $(v)+(-v)-2(O)$, with multiplicity two when $v=-v$. Subtract its logarithmic divisor from the corresponding combination of U's. The result is harmonic on the compact torus and hence constant. Its expansion at zero gives

$$\log|\wp(z)|=U(z-v)+U(z+v)-2U(z)+\frac{1}{6}\log|\Delta|-2L.$$

Integration proves (24.14). Similarly the divisor of $\wp'$ is the sum of the three nonzero half-periods minus $3(O)$. The multiplication-by-two identity of proposition 4.2 gives the product of N at those half-periods as 2. Since $\wp'(z)\sim-2z^{-3}$, comparison at zero yields

$$\log|\wp'(z)|=\sum_{w\in E[2]\setminus\{O\}}U(z-w)-3U(z)+\frac{1}{4}\log|\Delta|.$$

Its mean is $\frac{1}{4}\log|\Delta|$. Twice this identity minus three times the preceding mean proves (24.13). All logarithmic singularities are locally integrable, so these integrations also cover a zero at a half-period.

2. The map $z \mapsto x = \wp(z)$ has degree two. Its two inverse sheets each have Jacobian $|\wp'(z)|^{-2}$; branch values form a measure-zero set. Thus its pushforward density is $2/(A|4x^3 - g_2x - g_3|)$. Under $x = 1/u$ the Jacobian is $|u|^{-4}$ and the denominator is $4|u|^{-3}|C(u)|$. This proves (24.15) and its normalization. The identity $\wp'^2/\wp^3 = 4C(u)$ gives the next formula.

3. Replacing v above by a half-period gives $p(e_j) = \frac{1}{6}\log|\Delta| - 2U(w_j)$; summing and using $\sum_j U(w_j) = \log 2$ proves the centroid formulation. The equality characterization is that of 6.6, since $j = 1728$ is equivalent to $g_3 = 0$ for a nonsingular lattice.

□

Proposition 24.21 (The two complex beta integrals)**.** [tag 001U]

For the square and equianharmonic lattices respectively, the integrals in 24.20 are

$$\int \log|C|\,d\nu = \log 2, \qquad \int \log|C|\,d\nu = \frac{3}{2}\log 3 - 2\log 2.$$

They recover $N^2 = 2$ and $N^4 = 3$ with the same normalization.

Proof. 1. By homogeneity it suffices to take $C(u) = 1 - u^m$, with $m = 2$ or $m = 3$. For $-1 < \Re s < 1 - 1/m$ put

$$I_m(s) = \int_{\mathbb{C}} |u|^{-1}|1 - u^m|^{s-1} dA_u.$$

The substitution $w = u^m$, with all m sheets counted, gives

$$I_m(s) = \frac{1}{m} B_{\mathbb{C}}\left(\frac{1}{2m}, \frac{s+1}{2}\right), \quad B_{\mathbb{C}}(a,b) = \pi\frac{\Gamma(a)\Gamma(b)\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)\Gamma(a+b)}.$$

The complex beta identity (Neretin, 2024, §§1.1–1.2, equations (1.6)–(1.7), pp. 38–39) applies with parameter pairs $a \mid a$ and $b \mid b$. In that notation $\Gamma^{\mathbb{C}}(a \mid a) = \Gamma(a)/\Gamma(1-a)$ and

$$B_{\mathbb{C}}(a,b) = \int_{\mathbb{C}} |w|^{2a-2}|1-w|^{2b-2}\, dA_w = \pi B^{\mathbb{C}}(a \mid a, b \mid b).$$

The source's absolute-convergence conditions are $\Re a > 0$, $\Re b > 0$ and $\Re(a+b) < 1$. Under $a = 1/(2m)$ and $b = (s+1)/2$, these give exactly the stated strip. The strict inequalities give integrable power bounds at $u = 0$, at each root of $1 - u^m$, and at infinity. On a sufficiently small closed neighbourhood of $s = 0$, their positive margins also absorb the logarithmic factor introduced by differentiation. Thus differentiation under the integral at $s = 0$ is justified.

2. Consequently, with $a = 1/(2m)$ and $\psi = \Gamma'/\Gamma$,

$$\frac{I_m'(0)}{I_m(0)} = \frac{1}{2}\left(2\psi(1/2) - \psi(1/2-a) - \psi(1/2+a)\right).$$

The gamma duplication and triplication formulas, differentiated logarithmically, evaluate this as $\log 2$ for $m = 2$ and $\frac{3}{2}\log 3 - 2\log 2$ for $m = 3$. These can also be checked directly from $\psi(1/2) = -\gamma - 2\log 2$, $\psi(1/4) + \psi(3/4) = -2\gamma - 6\log 2$ and $\psi(1/3) + \psi(2/3) = -2\gamma - 3\log 3$. Equation (24.13) gives the stated values of N.

□

Proposition 24.22 (The signed Gaussian identity)**.** [tag 001V]

For $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, $T = \Im\tau$, and every $z \in \mathbb{C}$,

$$\sqrt{2T}\,|\eta(\tau)|^2 N(z,\tau)^2 = \sum_{m \in \frac{1}{2}\Lambda} \epsilon(m) e^{-2\pi|z-m|^2/T},$$

where $\epsilon = -1$ on Λ and $+1$ on the three other cosets of Λ in $\frac{1}{2}\Lambda$. Both sides and all their derivatives converge locally uniformly.

Proof. 1. Write $z = b + a\tau$ and expand in the Fourier characters $e^{2\pi i(kb+la)}$ of $(a,b) \in (\mathbb{R}/\mathbb{Z})^2$. For $G(z) = \sum_{\lambda\in\Lambda} e^{-2\pi|z-\lambda|^2/T}$, Poisson summation gives the coefficient

$$\frac{1}{2}\exp\left(-\frac{\pi|k\tau - l|^2}{2T}\right).$$

The right side is the sum of the four half-period translates of G, minus $2G$. Its coefficient is therefore

$$(-1)^{k+l+kl}\exp\left(-\frac{\pi|k\tau - l|^2}{2T}\right).$$

To check the sign, it is $+1$ when both k, l are even and -1 otherwise.

2. On the left use

$$\vartheta_1(z,\tau) = -i\sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i(n+1/2)^2\tau + 2\pi i(n+1/2)z}.$$

Multiply by its conjugate and by $\sqrt{2T}e^{-2\pi(\Im z)^2/T}$. In the double sum put $k = n - m$ and apply one-dimensional Poisson summation to the remaining Gaussian in m. Completing the square gives precisely the coefficient just displayed. This proves equality of the Fourier series. Positive definiteness of $|k\tau - l|^2$ and Gaussian decay justify the changes of order, Poisson summation, and every claimed differentiation.

□

24.5 The Betti measure and torsion equidistribution

Definition 24.23 (The current convention)**.** [tag IA07]

We take $dd^c = (i/\pi)\partial\bar\partial$, so that $dd^c \log|w| = \delta_0$. For a local complex coordinate w, $dA_w = (i/2)dw \wedge d\bar w$.

Proposition 24.24 (The positive Betti measure and its current)**.** [tag 001W]

On a local period chart write $\mu = a\tau + b$, with a, b real, and put $T = \Im\tau$. The positive Betti measure is

$$m_P = db \wedge da = \frac{|\mu_w - a\tau_w|^2}{T}\,dA_w.$$

Reversing the wedge order gives $da \wedge db = -m_P$. The globally single-valued function $L(t) = \log N(P_t)$ satisfies on $\mathbb{P}^1(\mathbb{C})$

$$dd^c L = \frac{1}{12}\delta_\infty - m_P, \qquad \int_{\mathbb{P}^1} m_P = \frac{1}{12}. \tag{24.16}$$

There are no atoms of m_P at 0, 1 or ∞. Let $T_0 = \sqrt{3}/2$, $\rho = (1 + i\sqrt{3})/2$, and $p = \wp'_\rho((2-\rho)/3)$. The density in the ordinary t coordinate satisfies

$$\frac{dm_P}{dA_t} = \frac{|t|^{-2/3}}{64T_0|p|^{2/3}}\bigl(1 + O(|t|^{1/3})\bigr), \qquad t \longrightarrow 0, \tag{24.17}$$

$$\frac{dm_P}{dA_t} = \frac{|t-1|^{-3/2}}{24\pi^2\sqrt{E_4(i)}}\bigl(1 + O(|t-1|^{1/2})\bigr), \qquad t \longrightarrow 1. \tag{24.18}$$

Proof. 1. On a chart avoiding a zero of the section, the theta quotient is holomorphic and nonzero. Its log modulus is pluriharmonic. Only $-\pi(\Im\mu)^2/T$ contributes to the curvature. Direct differentiation of this real quotient gives

$$\partial_w\partial_{\bar w}\frac{(\Im\mu)^2}{T} = \frac{|\mu_w - a\tau_w|^2}{2T}.$$

Also

$$a_w = \frac{\mu_w - a\tau_w}{2iT}, \qquad b_w = -\bar{\tau}\frac{\mu_w - a\tau_w}{2iT}.$$

Taking the wedge product proves the formula for m_P and $dd^c L = -m_P$ on the chart. The transformation of (a, b) under an integral change of period basis has determinant one, so the measure patches. Translation of μ by a locally constant integral period changes neither the formula nor the measure. The formula remains valid on a root chart at an elliptic modulus.

2. At $t = 0$ the expansion proved in Proposition 5.4 is

$$L(t) = \frac{1}{4}\log 3 + \frac{7}{192}\Re t - B|t|^{4/3} + O(|t|^{5/3}), \qquad B = \frac{9\pi}{512 T_0 |p|^{2/3}}.$$

The expansion is a convergent real-analytic expansion on a root chart, so it can be differentiated termwise with the indicated orders. The Euclidean identity $\Delta_{\mathbb{R}^2}|t|^\alpha = \alpha^2|t|^{\alpha-2}$ gives (24.17).

3. At $t = 1$ put $h = \mu - (1-\tau)/2$. The expansion in Proposition 5.3 and the modular j expansion give

$$t - 1 = A_1 h^4(1 + O(h^2)), \qquad A_1 = \frac{9}{4}\pi^4 E_4(i) > 0, \qquad L = \log\sqrt{2} - \frac{\pi}{2}|h|^2 + O(|h|^4).$$

Therefore $L = \log\sqrt{2} - \pi|t-1|^{1/2}/(2\sqrt{A_1}) + O(|t-1|)$. Termwise differentiation on the root chart proves (24.18). Both densities are locally integrable, and the continuous potentials at 0 and 1 have no atoms.

4. For infinity take $u = 1/t$, and set $\ell = \log(5+2\sqrt{6})$. The cusp branch is $\mu = 1/2 - i\ell/(2\pi) + O(q)$, $q = e^{2\pi i\tau}$. The theta and eta products, with the nonzero factor $|2\sin\pi\mu| = 2\sqrt{3} + O(q)$, give

$$L = \frac{1}{12}\log|u| + \log(2\sqrt{3}) - \frac{1}{12}\log 1728 - \frac{\ell^2}{2\log(1728/|u|)} + O\left(\frac{|u|}{|\log|u||} + |u|\right). \tag{24.19}$$

Here $q = u/1728 + O(u^2)$; the estimate follows directly by bounding the convergent products. The nonlogarithmic correction is continuous at $u = 0$ and has no point mass. Equivalently its Betti density is

$$\frac{dm_P}{dA_u} = \frac{\ell^2}{2\pi|u|^2\log(1728/|u|)^3}(1 + o(1)),$$

which is integrable. The single logarithmic term in (24.19) contributes $\delta_\infty/12$. This proves the global current identity; pairing it with the constant function one gives the total mass.

□

Proposition 24.25 (Torsion equidistribution)**.** [tag 001K]

Let D_n be the divisor, with intersection multiplicity, of parameters $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ for which $nP_t = O$. Then

$$\deg D_n = \frac{n^2}{12} + O(1), \qquad \frac{1}{n^2}[D_n] \longrightarrow m_P, \qquad \frac{1}{\deg D_n}[D_n] \longrightarrow 12 m_P$$

weakly as $n \to \infty$ through indices for which $D_n \neq 0$. For any nonrepeating sequence of individual algebraic torsion parameters, their Galois-orbit probability measures likewise converge to $12m_P$. The factor 12 normalizes the mass to one.

Proof. 1. We apply DeMarco and Mavraki (2020, Theorem 1.1 and Corollary 1.2, arXiv v2, p. 2). Their hypotheses are a number field, a smooth projective base curve, an elliptic surface and a rational section of nonzero generic canonical height. Here they are $\mathbb{Q}$, $\mathbb{P}^1$, the displayed rational elliptic surface, and P, whose generic height for (O) is $1/12$ by 24.6. At a torsion specialization the canonical height is zero. Corollary 1.2 consequently gives convergence of every nonrepeating sequence of Galois-orbit probability measures.

2. The limiting measure is the curvature of the height metric divided by the degree of its line bundle. In our normalization, 24.24 computes the curvature as m_P, and 24.6 computes that degree as $1/12$. Hence the probability measure is $12m_P$. The total mass is one because $12\int_{\mathbb{P}^1} m_P = 1$.

3. The surface height formula, with our half-self-pairing convention, reads

$$\widehat{h}_O(Q) = \chi(\mathcal{O}_\mathcal{E}) + (Q \cdot O) - \frac{1}{2}\sum_v \mathrm{contr}_v(Q).$$

   Here $\chi = 1$, and the finitely many fibre corrections are bounded as $Q = nP$ varies, because they depend only on its component. Since $\widehat{h}_O(nP) = n^2/12$, it follows that $(nP \cdot O) = n^2/12 + O(1)$. Removing the singular fibres changes this by $O(1)$: the local intersection at each such fibre is bounded as n varies in characteristic zero. One can see this in the formal group after a fixed multiple enters it; multiplication by a nonzero integer is a local analytic isomorphism over the residue field $\mathbb{C}$. Thus $\deg D_n = n^2/12 + O(1)$.

4. At any fixed smooth torsion parameter t_0, of exact order d, the multiplicity in D_n is zero unless $d \mid n$, and is constant when $d \mid n$. Indeed the formal parameter of $nP = [n/d](dP)$ has first term (n/d) times that of dP, with nonzero constant coefficient. Fix a continuous test function and $\varepsilon > 0$. Corollary 1.2 implies that all but finitely many distinct Galois orbits of torsion parameters have average within ε of the limiting integral; otherwise a countersequence would violate it. The total weighted contribution of the finitely many exceptional orbits is bounded independently of n, by the multiplicity observation. Divide by $\deg D_n$ and let $n \to \infty$. This proves the probability convergence for D_n, and the degree asymptotic proves the unnormalized version.

□

25 Critical points and the global level curves

Remark 25.1 (Critical-point coordinates)**.** [tag IA27] Use $L = \log N$ and $t = j/1728$ as in definition 2.6.

Lemma 25.2 (The bounded cusp zero and its differential equation)**.** [tag IA09] *For the cusp-bounded lift of the zero of $\wp_\tau$, one has*

$$\begin{aligned} \mu &= \frac{1}{2} - i\beta_{\mathrm{cusp}} - \frac{36i\sqrt{6}}{\pi}\sum_{n\ge1}\frac{a_n q^n}{n^2}, \quad \beta_{\mathrm{cusp}} = \frac{\log(5+2\sqrt{6})}{2\pi}, \\ \sum_{n\ge1} a_n q^n &= \frac{\Delta_0}{E_6^{3/2}}, \qquad\qquad \mu'' = 144\pi i\sqrt{6}\frac{\Delta_0}{E_6^{3/2}}. \end{aligned} \tag{25.1}$$

The branch of $E_6^{3/2}$ tends to one at the cusp. The series holds on $\Im\tau > 1$ and continues on the chosen logarithm branch away from half-period branch points.

Proof. Use the Eichler–Zagier formula (Eichler and Zagier, 1982, unnumbered theorem, p. 399), in the form of (Duke and Imamoğlu, 2008, Section 2, unnumbered Eichler–Zagier theorem and equation (6), author preprint p. 4). In equation (6), take $m = n = 0$ and the minus sign, so that the resulting zero z_0 tends to $1/2 + i\beta_{\mathrm{cusp}}$. The exact affine map is $\mu = 1 - z_0$, as in proposition 12.9; periodicity and evenness give $\wp(1 - z_0, \tau) = \wp(z_0, \tau)$. Termwise integration uses $\int_\tau^{i\infty}(\sigma - \tau)e^{2\pi i n\sigma}\,d\sigma = -q^n/(4\pi^2 n^2)$ and gives the displayed series. Termwise differentiation using $q' = 2\pi i q$ gives its second derivative. At i the continuation uses the square-root coordinate of proposition 5.3. □

Proposition 25.3 (The radial derivative)**.** [tag 001M]

Put $p = \wp'_\tau(\mu)$ and

$$X = \frac{3Wp}{g_2}, \qquad F = 1 - X, \qquad W = \frac{\vartheta_1'(\mu,\tau)}{\vartheta_1(\mu,\tau)} + \frac{2\pi i \Im\mu}{\Im\tau}.$$

These are single-valued modular invariants wherever $g_2 \neq 0$. They have continuous extensions at $t = 0, 1$, with $X(0) = 1/4$ and $X(1) = 0$. At every $t \neq 0, 1$,

$$(t-1)L_t = \frac{X(X-2)}{24}, \qquad r\frac{d}{dr}L(1+re^{i\theta}) = \frac{\Re(X(X-2))}{12}. \tag{25.2}$$

Also, if $a = \Im\mu/\Im\tau$ and $R = \mu' - a$, then

$$F = \frac{6\pi i p}{g_2} R. \tag{25.3}$$

Consequently $|F| < 1$ implies strict decrease along the ray from 1 through t.

Proof. 1. A lattice translation leaves W unchanged, and replacing μ by $-\mu$ changes the signs of both W and p. Under a modular change, W, p, g_2 have respective weights $1, 3, 4$, so X is invariant. The two identities of 5.2 give

$$L_\tau = \frac{iW(3Wp - 2g_2)}{24\pi p}, \qquad R = \frac{g_2 - 3Wp}{6\pi i p}.$$

Ramanujan's identities and $p^2 = -g_3$ give

$$t'_\tau = -2\pi i t\frac{E_6}{E_4}, \qquad \frac{t-1}{t'_\tau} = -\frac{E_6}{2\pi i E_4^2}.$$

Substitution, using $g_2 = 4\pi^4 E_4/3$ and $g_3 = 8\pi^6 E_6/27$, proves (25.2) and (25.3). Since $X(X-2) = F^2 - 1$, $|F| < 1$ makes its real part strictly negative.

2. At i, $W = p = 0$ and $g_2 \neq 0$, so $X \to 0$. At ρ, put $\delta = \tau - \rho$ and $p_0 = \wp'_\rho(\mu_0)$. The expansion in 5.4 and Ramanujan's identities give

$$R = \frac{3p_0}{8\pi^2}\delta + O(|\delta|^2), \qquad g_2 = \frac{3ip_0^2}{\pi}\delta + O(\delta^2).$$

Substitution into (25.3) gives $F(\rho) = 3/4$. Equivalently, the two possible limits furnished by (25.2) and $L_t(0) = 7/384$ are $X = 1/4, 7/4$; the expansion of R selects $1/4$. The same expansions show continuity on the coarse t-line.

□

Proposition 25.4 (A bound on the cusp region)**.** [tag 001N]

If $\Im\tau \geq 7/5$, then $|F| < 17/20$.

Proof. 1. Here $|q| < 1/6600$. On the auxiliary circle $|q| = 1/2000$, the Lambert series and eta product give

$$|E_6 - 1| \leq 504r\frac{1 + 26r + 66r^2 + 26r^3 + r^4}{(1-r)^7}, \qquad |\Delta_0| \leq r\exp\frac{24r}{(1-r)^2}, \quad r = \frac{1}{2000}.$$

The right sides imply $|\Delta_0/E_6^{3/2}| < 1/1250$. In particular E_6 has no zero on this disc and the specified branch is holomorphic there. By Cauchy's inequality, with $s = 10/33$,

$$\sum_{n\geq 1}\frac{|a_n q^n|}{n} \leq \frac{-\log(1-s)}{1250}, \qquad \sum_{n\geq 1}\frac{|a_n q^n|}{n^2} \leq \frac{1}{1250}\left(s + \frac{s^2}{4(1-s)}\right).$$

Equation (25.1) therefore yields

$$|\mu - (1/2 - i\beta_{\mathrm{cusp}})| < \frac{1}{125}, \qquad |\mu'| < \frac{13}{250}, \qquad \beta_{\mathrm{cusp}} < \frac{365}{1000}.$$

For $r = 1/6600$, the corresponding weight-four and weight-six Lambert bounds are

$$240r\frac{1+4r+r^2}{(1-r)^5} < \frac{37}{1000}, \qquad 504r\frac{1+26r+66r^2+26r^3+r^4}{(1-r)^7} < \frac{77}{1000}.$$

Thus $|E_4| > 963/1000$ and $|E_6|^{1/2} < 1038/1000$. Since $6\pi|p/g_2| = \sqrt{6}|E_6|^{1/2}/|E_4|$,

$$|F| < \sqrt{6}\frac{1038}{963}\left\{\frac{365/1000 + 1/125}{7/5} + \frac{13}{250}\right\} < \frac{17}{20}. \tag{25.4}$$

All scalar comparisons here follow from geometric series and the exponential Taylor bounds (6.2); for the logarithm use $-\log(1-s) = \sum s^n/n$ with its geometric remainder. For reference, the upper quantities before the last rounding are less than $789/10^6$, $755/10^5$, $51/1000$, and $841/1000$ for $|\Delta_0/E_6^{3/2}|$, the error in μ, the bound on μ', and the final bound on $|F|$, respectively. □

Proposition 25.5 (A bound on the compact modular region)**.** [tag 0010]

Let $\chi = (\tau - i)/(\tau + i)$. On $0 < |\chi| \le 2 - \sqrt{3}$, the invariant F satisfies $|F| < 1$. At the two equianharmonic points on the boundary its continuous value is $3/4$.

Proof. 1. Set

$$r_0 = 2 - \sqrt{3}, \qquad A = E_4(i), \qquad k = \pi^2 A,$$
$$v = \pi\sqrt{A}\,\chi, \qquad z = v^2 = k\chi^2,$$
$$z_0 = kr_0^2, \qquad c_0 = 1 - \frac{1}{\sqrt{3}}.$$

In this proof z is a coordinate on the modular disc. Define analytic functions f_2, f_4, f_6 by

$$\sqrt{A}f_2(v) = \frac{E_2(\tau)}{(1-\chi)^2} - \frac{3}{\pi(1-\chi)},$$
$$Af_4(v) = \frac{E_4(\tau)}{(1-\chi)^4}, \qquad A^{3/2}f_6(v) = \frac{E_6(\tau)}{(1-\chi)^6}.$$

Ramanujan's differential identities become the rational system

$$f_2' = \frac{f_4 - f_2^2}{3}, \qquad f_4' = \frac{4}{3}(f_6 - f_2 f_4), \qquad f_6' = 2(f_4^2 - f_2 f_6), \tag{25.5}$$
$$(f_2, f_4, f_6)(0) = (0, 1, 0).$$

Uniqueness makes f_4 even and f_2, f_6 odd. Write

$$B(z) = f_4(v), \quad J(z) = \frac{f_6(v)}{2v}, \quad D(z) = \frac{B(z)^3 - 4zJ(z)^2}{J(z)^{3/2}}, \qquad H(z) = \sum_{n\ge 0}\frac{-[z^n]D(z)}{16n^2-1}z^n.$$

The branch has $J(0)^{1/2} = 1$. The estimates below establish its regularity on a larger disc, so these definitions are unambiguous on $|z| \le z_0$.

2. Write $h = (1-\chi)(\mu - (1-\tau)/2)$. The Möbius chain rule and (25.1) give

$$h''(\chi) = -\frac{4\mu''(\tau)}{(1-\chi)^3} = -\frac{i}{2\sqrt{3\pi}}\chi^{-3/2}D(k\chi^2).$$

The root expansion at i gives the branch

$$h = \frac{2i}{\sqrt{3\pi}}\sqrt{\chi}\, H(z). \tag{25.6}$$

Indeed differentiating this series twice gives precisely the coefficient denominator $16n^2 - 1$ above. The two possible affine homogeneous terms are absent because the half-period difference has a Puiseux expansion consisting only of half-integral powers. Changing the sign of the square root changes the signs of h, p, R and leaves F unchanged.

3. Put $r = |\chi|$. The definitions of a and R give exactly

$$R = \frac{1-\chi}{2i}\left(h' + \frac{\overline{\chi}h + \overline{h}}{1-r^2}\right), \qquad \frac{p}{g_2} = -\frac{i\sqrt{\chi}}{\sqrt{3\pi}(1-\chi)}\frac{\sqrt{J}}{B}.$$

Consequently

$$F = \frac{\sqrt{J}}{B}\left\{H + 4zH' + \frac{2r}{1-r^2}(rH - \overline{H})\right\}. \tag{25.7}$$

All conjugates in this display are values at the same point; H has real coefficients, so $\overline{H(z)} = H(\overline{z})$.

4. The apparent pole at the equianharmonic points must be removed before estimating this expression. We have $B(-z_0) = 0$. At these points $R = 0$, since the expansion at ρ gives $\mu' = -1/3 = a$. Thus, if

$$S = \sqrt{J}H, \qquad T = \sqrt{J}(H + 4zH'),$$

then $T(-z_0) = c_0 S(-z_0)$. Define the analytic quotients

$$C(z) = \frac{B(z)}{1 + z/z_0}, \qquad U(z) = \frac{T(z) - c_0 S(z)}{(1 + z/z_0)C(z)}, \qquad V(z) = \frac{\sqrt{J(z)}}{C(z)}.$$

The estimates below prove $C \neq 0$ on the needed larger disc. For $r < r_0$, equation (25.7) is now

$$F = U + V\frac{dH + q(H - \overline{H})}{1 + z/z_0}, \qquad d = \frac{2(r_0 - r)}{(1 + r_0)(1 + r)}, \quad q = \frac{2r}{1 - r^2}. \tag{25.8}$$

Here q is only a real abbreviation within this display and its estimates.

5. For an analytic power series $K = \sum K_n z^n$, put $\|K\|_s = \sum |K_n| s^n$. Then, for $0 \le s \le z_0$,

$$\begin{aligned} \|U\|_s &\le \frac{1}{\sqrt{3}} + \frac{13}{200}s, \quad \|V\|_s \le 1 + \frac{7}{50}s, \\ \|H\|_s &\le 1 + \frac{113}{1000}s, \quad \|H'\|_s \le \frac{13}{100}. \end{aligned} \tag{25.9}$$

In (25.5), write $f_2 = \sum a_n v^n$, $f_4 = \sum b_n v^n$, $f_6 = \sum c_n v^n$. Starting with $(a_0, b_0, c_0) = (0, 1, 0)$ (only in this recurrence the symbol c_0 denotes the zero initial coefficient), compute

$$\begin{aligned} a_{n+1} &= \frac{b_n - \sum_{j=0}^{n} a_j a_{n-j}}{3(n+1)}, \\ b_{n+1} &= \frac{4(c_n - \sum_{j=0}^{n} a_j b_{n-j})}{3(n+1)}, \\ c_{n+1} &= \frac{2(\sum_{j=0}^{n} b_j b_{n-j} - \sum_{j=0}^{n} a_j c_{n-j})}{n+1}. \end{aligned} \tag{25.10}$$

Thus $B_n = b_{2n}$ and $J_n = c_{2n+1}/2$. For powers of a series K with $K_0 = 1$, the equally elementary recurrence is

$$[z^n]K^\alpha = \frac{1}{n}\sum_{j=1}^{n}((\alpha+1)j - n)K_j[z^{n-j}]K^\alpha. \tag{25.11}$$

Together with $H_n = -D_n/(16n^2-1)$, these determine every coefficient used. For example,

n	0	1	2	3
B_n	1	10/9	5/27	4/81
J_n	1	14/27	28/135	14/405
H_n	1	13/135	−1999/153090	29299/28146690.

Division by $1 + z/z_0$ uses $C_0 = 1$ and $C_n = B_n - C_{n-1}/z_0$; the same recurrence applies to $(T - c_0S)/(1 + z/z_0)$. Division by C uses the usual convolution recurrence with constant coefficient one.

6. Here are sufficient rational enclosures for the only nonrational constants in this arithmetic:

$$14 < k < 72/5, \qquad 1.03156190435581564935682633431773565273882539802 03 < z_0 < 1.03156190435581564935682633431773565273882539802 05. \tag{25.12}$$

They follow from $\pi = 16\arctan(1/5) - 4\arctan(1/239)$, the first 100 terms of the alternating arctangent series, the exponential series through degree 160, and

$$A = 1 + 240\sum_{n=1}^{49}\frac{n^3e^{-2\pi n}}{1-e^{-2\pi n}} + \varepsilon_A, \qquad 0 < \varepsilon_A < \frac{240(4e^{-2\pi})^{50}}{(1-4e^{-2\pi})(1-e^{-2\pi})}.$$

The last bound uses $n^3 \le 4^n$. Bounds for $\sqrt{3}$ follow by squaring rational endpoints. All ensuing coefficient comparisons are rational arithmetic using these enclosures.

7. To bound infinite tails, first consider $|z| = 12/5$. Here $|\chi| < 5/12$, $\Im\tau > 7/17$ and $|e^{2\pi i\tau}| < 1/13$. Lambert series give

$$|B| < 400, \qquad |J| < 3000 \qquad (|z| = 12/5). \tag{25.13}$$

The respective upper bounds before rounding are

$$\left(\frac{12}{7}\right)^4\left(1 + 240u\frac{1+4u+u^2}{(1-u)^5}\right) < 400, \quad u = 1/13,$$

$$\frac{1}{3}\left(\frac{12}{7}\right)^6\left(1 + 504u\frac{1+26u+66u^2+26u^3+u^4}{(1-u)^7}\right) < 3000.$$

We have used $A > 1$ and $2\sqrt{12/5} > 3$. For $R = 3/2$, let $\xi = 5/8$ and retain coefficients through degree 59. The tails of B, J, C are respectively at most

$$\frac{400\xi^{60}}{1-\xi}, \qquad \frac{3000\xi^{60}}{1-\xi}, \qquad \frac{400z_0}{12/5 - z_0}\frac{\xi^{60}}{1-\xi}. \tag{25.14}$$

For the last bound, $B(-z_0) = 0$ implies $|C_n| \le 400z_0(12/5)^{-n}/(12/5 - z_0)$. The finite sums from (25.10)–(25.11) with these tails give

	$\|B\|_{3/2}$	$\|J\|_{3/2}$	$\|C-1\|_{3/2}$	$\min_{\|z\|\le 3/2}\Re J(z)$
bound	$< 3276/1000$	$< 2411/1000$	$< 335/1000$	$> 2049/10000.$

For the last column, the lower bound used is exactly

$$1 - J_2R^2 - \frac{J_1^2}{8J_2} - \sum_{n=3}^{59} |J_n| R^n - \frac{3000\xi^{60}}{1-\xi}.$$

This is obtained by minimizing the quadratic $\Re(1 + J_1 z + J_2 z^2)$ on $|z| = R$; the minimum occurs at $\cos\arg z = -J_1/(4J_2R)$. In particular, on $|z| \le 3/2$,

$$|B| < 33/10, \quad |J| < 5/2, \quad |C| > 3/5, \quad \Re J > 1/5, \quad |D| \le \frac{(33/10)^3 + 6(5/2)^2}{(1/5)^{3/2}} < 1000.$$

Cauchy's inequality and the denominators in H_n now give

$$|H| \le 1 + 1000 \sum_{n\ge1} \frac{1}{16n^2-1} = 1 + 1000\left(\frac{1}{2} - \frac{\pi}{8}\right) < 110 \quad (|z| \le 3/2).$$

On $|z| \le 7/5$, the same argument gives

$$|H + 4zH'| \le 1 + 1000 \sum_{n\ge1} \frac{(14/15)^n}{4n-1} \le 1 + \frac{1000}{3}\log 15 < 910.$$

It follows on $|z| = 7/5$ that $|U| < 10000$ and $|V| < 3$; indeed $|(1 + z/z_0)C| > (7/(5z_0) - 1)3/5$, and $|\sqrt{J}| < 8/5$.

8. Finally retain coefficients through degree 99 and put $\eta = 5z_0/7$. The tails in the coefficient norms for U, V, H, H' at z_0 are bounded respectively by

$$\frac{10000\eta^{100}}{1-\eta}, \quad \frac{3\eta^{100}}{1-\eta}, \quad \frac{110\eta^{100}}{1-\eta}, \quad \frac{110\eta^{100}}{z_0}\left(\frac{100}{1-\eta} + \frac{\eta}{(1-\eta)^2}\right). \tag{25.15}$$

The finite coefficient sums plus precisely these tails give

	$(\|U\|_{z_0} - 1/\sqrt{3})/z_0$	$(\|V\|_{z_0} - 1)/z_0$	$(\|H\|_{z_0} - 1)/z_0$	$\|H'\|_{z_0}$
upper bound	635/10000	1368/10000	1115/10000	1293/10000.

(25.16)

The recurrences, cutoffs and remainders above specify each finite comparison by rational arithmetic. Equations (25.16) imply (25.9), with room in each inequality.

9. For $s = |z| = kr^2$,

$$|H - \overline{H}| \le 2|\Im z|\,\|H'\|_s, \qquad \frac{|\Im z|}{|1 + z/z_0|} \le s.$$

The second inequality follows by writing $z/z_0 = ue^{i\phi}$ and using $(\cos\phi + u)^2 \ge 0$. The useful cancellation is

$$|dH + q(H - \overline{H})|^2 = d^2|H|^2 + 4q(d+q)(\Im H)^2.$$

Also

$$\frac{d}{|1 + z/z_0|} \le e(r) := \frac{c_0}{(1+r)(1+r/r_0)}, \qquad q \le \kappa r, \quad d + q \le c_0 + \kappa r^2, \quad \kappa = 1 + \frac{2}{\sqrt{3}}.$$

The last bounds use $(1 - r_0^2)^{-1} = 1/2 + 1/\sqrt{3}$. It follows from (25.8)–(25.9), using $k < 72/5$, that

$$|F| \le U_0(r) + V_0(r)\sqrt{e(r)^2 H_0(r)^2 + 4\kappa r(c_0 + \kappa r^2)(72/5)^2 r^4 (13/100)^2}, \tag{25.17}$$

where

$$U_0 = \frac{1}{\sqrt{3}} + \frac{13}{200}\frac{72}{5}r^2, \quad V_0 = 1 + \frac{7}{50}\frac{72}{5}r^2, \quad H_0 = 1 + \frac{113}{1000}\frac{72}{5}r^2.$$

To verify that the right side is less than one, let

$$\begin{aligned} M(r) &= (1+r)^2(1+r/r_0)^2, \\ P(r) &= (1-U_0)^2 M(r) \\ &\quad - V_0^2 \left\{ c_0^2 H_0^2 + 4\kappa r(c_0+\kappa r^2)(72/5)^2 r^4 (13/100)^2 M(r) \right\}. \end{aligned} \tag{25.18}$$

This polynomial has $P(0) = 0$. Write

$$\frac{P(r_0 x)}{r_0 x} = \sum_{j=0}^{14} b_j \binom{14}{j} x^j (1-x)^{14-j}.$$

All coefficients belong to $\mathbb{Q}(\sqrt{3})$. Explicit lower bounds, obtained by expanding the displayed factored polynomial, are

j	0	1	2	3	4	5	6	7
$100b_j >$	169	175	181	187	192	197	202	205
j	8	9	10	11	12	13	14	
$100b_j >$	207	207	204	197	184	160	118	.

Explicitly, if $P(r)/r = \sum q_n r^n$, then

$$b_j = \sum_{n=0}^{j} q_n r_0^n \frac{\binom{j}{n}}{\binom{14}{n}}.$$

For example $b_0 = 4 - 4\sqrt{3}/3$ and $b_1 = 25357/13125 - 453\sqrt{3}/4375$. The rational enclosure

$$\begin{aligned} &1.732050807568877293527446341505872366942805253810 38 < \sqrt{3}, \\ &\sqrt{3} < 1.732050807568877293527446341505872366942805253810 39. \end{aligned}$$

verifies every entry by integer multiplication; its endpoints are verified by squaring. Every Bernstein basis function is nonnegative on $[0,1]$, so $P(r) > 0$ for $0 < r \le r_0$. Since $U_0 < 1$ there, squaring (25.17) is legitimate and proves $|F| < 1$. Formula (25.8) applies at $r = r_0$ except at $z = -z_0$ by continuity; at that remaining point 25.3 gives $F = 3/4$.

□

Proposition 25.6 (The critical points and global level curves)**.** [tag 001P]

For every zero μ of $\wp_\tau$, the zero sets of the two critical factors are exactly

$$\begin{aligned} W = 0 \quad &\Longleftrightarrow \quad \tau \in \mathrm{SL}_2(\mathbb{Z})\{i, \rho\}, \\ 3W\wp_\tau'(\mu) - 2g_2 = 0 \quad &\Longleftrightarrow \quad \tau \in \mathrm{SL}_2(\mathbb{Z})\rho. \end{aligned}$$

On the coarse t-plane the function L has no critical point away from $t = 1$. The point $t = 0$ is a noncritical C^1 point, with $L_t(0) = 7/384$. The point $t = 1$ is the unique global maximum, of value $\log\sqrt{2}$, and has the fractional-power local maximum described in proposition 5.3. In particular there is no saddle in $0 < t < 1$. On the root charts, $h = 0$ over $t = 1$ is a nondegenerate maximum, while $\delta = 0$ over $t = 0$ is a degenerate critical point with nonzero harmonic cubic leading term and three alternating ascending and descending sectors. These are all critical points on the root charts.

For every angle θ, $r \mapsto L(1 + re^{i\theta})$ is strictly decreasing for $r > 0$, from $\log\sqrt{2}$ to $-\infty$. For each $c < \log\sqrt{2}$, the level set $\{L = c\}$ is one Jordan curve, strictly star-shaped about 1. Its bounded complement is $\{L > c\}$; its unbounded complement is $\{L < c\}$. The curve is real analytic except possibly at $t = 0$ when $c = \frac{1}{4}\log 3$, where it is C^1 with nonzero tangent. The closed superlevel sets form nested topological discs. At the maximum the level set is the single point 1, and above it the level set is empty. The corresponding statements for D_{part} follow by its increasing affine change from L.

Proof. 1. Take a representative in the closed standard modular fundamental region. If $T \geq 7/5$, 25.4 applies. Otherwise $|\Re\tau| \leq 1/2$ and $\sqrt{3}/2 \leq T \leq 7/5$. For fixed $\Re\tau$, the function

$$|\chi|^2 = \frac{(\Re\tau)^2 + (T-1)^2}{(\Re\tau)^2 + (T+1)^2}$$

has its maximum on this interval at an endpoint and increases with $(\Re\tau)^2$. The two endpoint bounds are r_0^2 and $41/601 < r_0^2$. Thus 25.5 covers the entire remaining region. We conclude $|1 - X| < 1$ for every $t \neq 1$. Equations (25.2) now prove strict radial monotonicity, including $t = 0$ by continuity. They also exclude both critical equations $W = 0$ and $3Wp - 2g_2 = 0$ on the ordinary base. The elliptic points must be checked on their root charts: at ρ, the nonzero coarse derivative already given rules out a critical point; at i, $W = p = 0$ but $3Wp - 2g_2 = -2g_2 \neq 0$, and proposition 5.3 gives the strict maximum. At ρ, $W = g_2 = 0$ and $p \neq 0$, so both displayed factors vanish. This proves their exact zero sets, including the elliptic orbits. In particular the vanishing of both factors at ρ in the τ coordinate is a ramification effect, not a saddle on the coarse base. More explicitly, on the root chart at ρ,

$$L = L(0) + \frac{7}{192}\Re\left(\frac{ip_0^2}{\pi^3}\delta^3\right) + O(|\delta|^4),$$

with $p_0 \neq 0$, whereas on the chart at i, $L = \log\sqrt{2} - \pi|h|^2/2 + O(|h|^4)$. These expansions give the asserted root-chart critical types.

2. At the cusp, the product formula and (25.1) give, uniformly in $\Re\tau$,

$$L = -\frac{\pi T}{6} + \log(2\sqrt{3}) - \frac{\pi\beta_{\mathrm{cusp}}^2}{T} + O(e^{-2\pi T}), \qquad |t| = \frac{e^{2\pi T}}{1728}(1 + o(1)).$$

Hence $L \to -\infty$ uniformly as $|t| \to \infty$. Continuity at $t = 1$ supplies the other endpoint of each ray. The intermediate value theorem and strict monotonicity give one and only one radius $r_c(\theta) > 0$ at each level c. Properness and strict monotonicity imply continuity in θ; the nonvanishing radial derivative gives the stated local regularity by the implicit-function theorem. Thus $\theta \mapsto 1 + r_c(\theta)e^{i\theta}$ is a continuous injective parametrization of a Jordan curve and has exactly the asserted complementary regions.

□

Remark 25.7 (Interval enclosures)**.** [tag 034N] The interval bounds for the norm in theorem 6.6 combine contraction enclosures for the zeros of $\wp$, explicit theta and eta tails, and an exact cover of the compact modular region.

26 Arithmetic periods and compact subgroups

Definition 26.1 (Algebraic period coordinates)**.** [tag IA10] Use an algebraic Weierstrass equation and invariant differential as in definition 2.6. Write $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\Im(\omega_2/\omega_1) > 0$, $\zeta(z+\omega_j) = \zeta(z) + \eta_j$, and extend $\eta : \Lambda \to \mathbb{C}$ real-linearly. The discriminant is that of the equation in this differential. For a homothety $\Lambda \mapsto c\Lambda$, $u \mapsto cu$,

$$\sigma_{c\Lambda}(cu) = c\sigma_\Lambda(u), \qquad \eta_{c\Lambda}(cu) = c^{-1}\eta_\Lambda(u), \qquad \Delta(c\Lambda) = c^{-12}\Delta(\Lambda).$$

These formulas cancel the factors in (26.2), so N has the same value in the transported coordinates.

Proposition 26.2 (The sigma expression and third-kind period)**.** [tag 0017]

Let $P \in E(\overline{\mathbb{Q}})$, with $P \neq O$ and $2P \neq O$, and let u be an elliptic logarithm of P. Then

$$2\log N(P) = \frac{1}{6}\log|\Delta| + \Re\{\Lambda_1 + u(\zeta(u) - \eta(u))\}, \qquad \Lambda_1 = 2\log\sigma(u) - u\zeta(u). \tag{26.1}$$

For a fixed elliptic logarithm u, changing the branch of $\log\sigma(u)$ changes the expression inside the real part by an integer multiple of $4\pi i$. Put $B(u) = \Lambda_1 + u(\zeta(u) - \eta(u))$. Under a lattice translation,

$$B(u+\omega) - B(u) \equiv u\eta(\omega) - \omega\eta(u) + 2\log\chi(\omega) \pmod{4\pi i\mathbb{Z}}, \qquad \omega \in \Lambda,$$

where $\chi(\omega) \in \{1, -1\}$ is the sigma semicharacter. This difference is purely imaginary, so the displayed real part is independent of both choices. There is an extension G_P of E by $\mathbb{G}_m$, defined over $\overline{\mathbb{Q}}$ and corresponding, under the principal polarization, to P, for which Λ_1, modulo a logarithm of an algebraic number, is the negative of a relative period of $[\mathbb{Z} \to G_P]$. Changing the third-kind differential while retaining its simple poles and residues adds an algebraic multiple of u. Adding an exact algebraic differential to a relative-integral representative adds algebraic endpoint differences when both endpoints are regular.

Proof. 1. The sigma–theta product, or its divisor and automorphy law together with its derivative at zero, gives

$$N(P) = |\Delta|^{1/12}|\sigma(u)|\exp\left(-\frac{1}{2}\Re(u\eta(u))\right). \tag{26.2}$$

To check the constant, for $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ one uses $\sigma(z) = e^{\eta_1 z^2/2}\vartheta_1(z,\tau)/(2\pi\eta(\tau)^3)$, $\Delta = (2\pi)^{12}\eta(\tau)^{24}$, and $\eta(z) = \eta_1 z - 2\pi i\Im z/T$. Substitution gives exactly the defining Gaussian in N. Homogeneity gives the general lattice formula. Taking logarithms proves (26.1). The automorphy law $\sigma(u+\omega) = \chi(\omega)e^{\eta(\omega)(u+\omega/2)}\sigma(u)$ and real-linearity of η give the stated lift transport. Explicitly, writing $u = a\omega_1 + b\omega_2$ with $a, b \in \mathbb{R}$ and $\omega = m\omega_1 + n\omega_2$ with $m, n \in \mathbb{Z}$, Legendre's relation gives $u\eta(\omega) - \omega\eta(u) = 2\pi i(bm - an)$. This also proves that the transported difference is purely imaginary.

2. Put

$$f_u(z) = \frac{\sigma(z+u)}{\sigma(z)\sigma(u)}e^{-\zeta(u)z}.$$

Its logarithmic differential is

$$\frac{df_u}{f_u} = \frac{1}{2}\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\,dz.$$

This is the pullback of an algebraic differential with residues $+1$ at $-P$ and -1 at O. Two algebraic differentials with precisely these simple poles and residues differ by a holomorphic differential, hence by $c\,dz$ with $c \in \overline{\mathbb{Q}}$. Integration gives cu; integration of df gives the stated endpoint difference. Here the principal polarization identifies P with $t_P^*\mathcal{O}_E(O) \otimes \mathcal{O}_E(O)^{-1} = \mathcal{O}_E((-P) - (O))$; this fixes the sign of the extension class. The apparent $0/0$ at $z = u$ has a

removable singularity when P is not 2-torsion. The rational cocycle $f_u(z+z')/(f_u(z)f_u(z'))$ defines the algebraic extension; in its rational trivialization the exponential map has the form

$$\exp_{G_P}(z,v) = (\exp_E z, e^v f_u(z)).$$

Its periods include

$$(0, 2\pi i), \qquad (\omega_j, \zeta(u)\omega_j - \eta_j u), \quad j = 1, 2. \tag{26.3}$$

These signs follow directly from the sigma automorphy law. The identity $\sigma(2u) = -\wp'(u)\sigma(u)^4$ implies

$$\log f_u(u) = \Lambda_1 + \log(-\wp'(u)).$$

For an algebraic lift $R = (P, r)$, $r \in \overline{\mathbb{Q}}^\times$, the second coordinate of $\log_{G_P} R$ is

$$v = \log r - \Lambda_1 - \log(-\wp'(u)). \tag{26.4}$$

It is a period of the relative homology defining the indicated 1-motive. The description agrees with the generalized logarithm and Serre function in Bertolin et al. (2022, Section 2.3, equations (2.10)–(2.13), pp. 7–8; Definitions 2.3–2.4, p. 8; Section 6.1, pp. 26–27, arXiv v2); the signs in (26.3) use the displayed exponential convention.

□

Proposition 26.3 (The compact subgroup and real-axis formula)**.** [tag 001B]

For P as in proposition 26.2, $N(P)$ is algebraic if and only if the fibre of G_P above P contains an algebraic point of the maximal compact subgroup $G_P(\mathbb{C})^c$. For the family in the stated conventions, if $t > 1$ is real, choose $\tau = iT$, $T > 1$, and the zero $\mu = 1/2 - iy$, $0 < y < T/2$. With $x = 2y/T$, one has

$$N(P)^4 = 3^{3/4}(t-1)^{-1/4}\left|\frac{\vartheta_1(x, i/T)}{\eta(i/T)}\right|.$$

Proof. 1. On the analytic universal cover of G_P define the real homomorphism

$$\chi(z,v) = \Re\{v - \zeta(u)z + u\eta(z)\}.$$

It vanishes on the periods (26.3), hence descends to $G_P(\mathbb{C})$. Its restriction to the torus is $\log|\cdot|$. The kernel is compact: in the universal cover it is a real three-dimensional vector space modulo the rank-three period lattice. Every compact subgroup has zero image in $\mathbb{R}$, so this kernel is exactly $G_P(\mathbb{C})^c$. The identity $\Re(u\eta(z)) = \Re(z\eta(u))$ follows from the Legendre relation; equivalently $\eta(z) = sz + \pi\bar{z}/\operatorname{area}(\Lambda)$ for a complex constant s. Substituting (26.4) at $z = u$ gives

$$\chi(R) = \log|r| - \log|\wp'(u)| - 2\log N(P) + \frac{1}{6}\log|\Delta|. \tag{26.5}$$

If r is algebraic and $\chi(R) = 0$, this equation makes $N(P)^2$ algebraic. Conversely, if $N(P)$ is algebraic, choose the positive real algebraic number $r = |\wp'(u)|\,|\Delta|^{-1/6}N(P)^2$. Then (26.5) vanishes. This proves both directions with the chosen rational trivialization of the fibre. The compact interpretation is the one discussed by Bertrand (2017, PDF slides 5–8); here the calculation fixes its normalization on the diagonal $P = P$.

2. For the real formula, the rectangular elliptic curve has its two zeros of $\wp$ on $1/2 + i\mathbb{R}$; along that edge $\wp$ runs between the positive root e_1 and the negative root e_2. Thus the indicated representative exists. Translate 2μ by -1 and apply the modular substitution $\tau \mapsto -1/\tau$. The resulting elliptic argument is $-2y/T$, which is real. The Gaussian is therefore one. Evenness of N and the duplication identity prove the displayed formula. This removes the Gaussian from this particular expression.

□

26.1 Motivic groups from the point and extension data

Remark 26.4 (The motivic dimension formula)**.** [tag IA11] Bertolin et al. (2022, Construction 3.4, p. 12, and Lemma 3.5, p. 13, arXiv v2) compute the motivic group from its abelian point and extension data and the central Weil-pairing bracket. For the 1-motives used here, let B be the smallest abelian subvariety generated by these data, and let $Z(1)$ be the central torus obtained from that bracket and the remaining algebraic lift. Their dimension formula is

$$\dim G_{\mathrm{mot}}(M) = \dim G_{\mathrm{mot}}(A) + 2\dim B + \dim Z(1). \tag{26.6}$$

The bracket is described in their Section 6.1, pp. 26–27, and their corrected Table 2, p. 29, with the correction on p. 28, gives unipotent dimension three in the dependent nondeficient case.

Proposition 26.5 (The motivic groups of the diagonal data)**.** [tag 001C]

Let P be nontorsion and let R be any algebraic lift above P in G_P. Put $M_R = [\mathbb{Z} \to G_P]$, $1 \mapsto R$. Its connected motivic group is, after a choice of Levi subgroup, the semidirect product of $G_{\mathrm{mot}}(E)$ and the three-dimensional Heisenberg group attached to $H_1(E)$ and its principal polarization. Its dimension is 5 for CM and 7 without CM.

For a non-real embedding put $\mathcal{M} = M_R \oplus \overline{M_R}$. If E and $\bar{E}$ are not isogenous, its dimension is 13. If they are isogenous, choose an isogeny $\phi : \bar{E} \to E$ of degree d and write $Q = \phi(\bar{P})$. Let r be the dimension over $F = \mathrm{End}(E) \otimes \mathbb{Q}$ of the span of P, Q, modulo torsion. Then

case	$\dim B$	$\dim Z$	$\dim G_{\mathrm{mot}}(\mathcal{M})$
$r = 2$, *E CM*	2	2	8
$r = 2$, *E not CM*	2	2	10
$r = 1$, *E CM*	1	$1 + e$	$5 + e$
$r = 1$, *E not CM*	1	$1 + e$	$7 + e$.

Here e is either zero or one: it is zero precisely when the algebraic lift in the one-dimensional quotient of the two central tori by the bracket torus is torsion. This criterion is independent of the choices made modulo isogeny. In the real case we use the single motive M_R.

Proof. 1. All curves, polarizations, points, and lifts are algebraic over $\overline{\mathbb{Q}}$, as required by Construction 3.4 and Lemma 3.5. For M_R the smallest abelian subvariety containing (P, P) is the diagonal $B = E$, since P is nontorsion. If ψ is the polarization pairing, the bracket on this diagonal is

$$[(x, x), (y, y)] = \psi(x, y) - \psi(y, x) = 2\psi(x, y).$$

It is nonzero and spans $\mathbb{Q}(1)$. Hence $Z' = Z = \mathbb{G}_m$. The pure group is GL_2 without CM and $\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_m$ for the imaginary quadratic CM field. Their dimensions are four and two. This proves the single-motive assertion, including its Lie bracket and Levi action: the action on the centre is the polarization multiplier.

2. For the direct sum only the two diagonal central coordinates occur. If the two elliptic curves are not isogenous, they cannot be CM conjugates. The pure group is $\{(g_1, g_2) \in \mathrm{GL}_2^2 : \det g_1 = \det g_2\}$, of dimension seven. The point data generate $E \times \bar{E}$, and the two brackets are independent, giving $7 + 4 + 2 = 13$.

3. If the curves are isogenous, transport the second polarized factor through ϕ. Its polarization pairing becomes ψ/d. The smallest abelian subvariety has dimension r. For $r = 2$, independent variation in its two factors makes the two diagonal brackets span both central coordinates; thus $Z' = Z = \mathbb{G}_m^2$. The unipotent group is the product of two Heisenberg groups, with the same pure group acting on both standard factors. For $r = 1$, after multiplying the points by a

common integer we may write $Q = aP$, $a \in F^\times$. The bracket image is the one-dimensional line with proportional central coordinates

$$(1, c), \qquad c = \frac{aa^\dagger}{d} > 0, \tag{26.7}$$

where $\dagger$ is the Rosati involution. On an elliptic curve $aa^\dagger$ is a positive rational number: a square in the non-CM case and the positive field norm in the CM case. Construction 3.4(3) adds a further central direction exactly when the image of the algebraic lift in the quotient torus is nontorsion. This is the integer e above. The unipotent Lie algebra is the Heisenberg algebra, with one additional central Tate line when $e = 1$. Formula (26.6) proves every entry of the table.

□

Lemma 26.6 (The full-period field and the compactness relation)**.** [tag IA19] *Let $E/\overline{\mathbb{Q}}$ and the nontorsion point P have the coordinates of proposition 26.2. Put $G = G_P \times G_{\bar{P}}$ and $K_0 = \overline{\mathbb{Q}}(\Omega_G)$, where Ω_G is the full de Rham–Betti period matrix, including the second-kind periods of both factors. Let $R_N = (P, r_N)$ be the complex compact point specified by proposition 26.3, with $r_N = |\wp'(u)|\,|\Delta|^{-1/6}N(P)^2$, and let (u, v) be a logarithm in its algebraic coordinates. Then $u, \zeta(u), \eta(u)$ and their conjugates belong to K_0, and the compactness equation is*

$$v + \bar{v} = u\zeta(u) + \bar{u}\overline{\zeta(u)} - u\eta(u) - \bar{u}\overline{\eta(u)} \in K_0. \tag{26.8}$$

Proof. The full period matrix contains $\kappa_j = \zeta(u)\omega_j - \eta_j u$. The coefficient matrix for $(\zeta(u), u)$ is $\left(\begin{smallmatrix}\omega_1 & -\eta_1\\ \omega_2 & -\eta_2\end{smallmatrix}\right)$. Its determinant is the nonzero Legendre determinant, so $u, \zeta(u) \in K_0$. The same calculation in the conjugate factor gives their conjugates. Writing $u = a\omega_1 + b\omega_2$ with $a, b \in \mathbb{R}$ and solving together with the conjugate equation gives

$$\eta(u) = \frac{(u\bar{\omega}_2 - \bar{u}\omega_2)\eta_1 + (\bar{u}\omega_1 - u\bar{\omega}_1)\eta_2}{\omega_1\bar{\omega}_2 - \bar{\omega}_1\omega_2}. \tag{26.9}$$

Its denominator is nonzero by real-linear independence. Thus $\eta(u)$ and its conjugate are also in K_0. Finally proposition 26.3 gives $\Re(v - u\zeta(u) + u\eta(u)) = 0$. Adding this expression to its conjugate proves (26.8). □

26.2 CM theta normalization

Remark 26.7 (The theta and metric normalizations)**.** [tag IA28] The multiplication law in proposition 26.8 retains its finite sign character. Its exact relation to the metric norm is (26.11).

Proposition 26.8 (The CM multiplication law)**.** [tag 001F]

Let $E/\overline{\mathbb{Q}}$ have complex multiplication by an order $\mathcal{O}$, and use an algebraic invariant differential. Write $A = \operatorname{area}(\Lambda)$ and

$$\eta(z) = s_2 z + \frac{\pi}{A}\bar{z}, \qquad \theta(z) = \sigma(z)e^{-s_2 z^2/2}.$$

Then $s_2 \in \overline{\mathbb{Q}}$. For every nonzero $\alpha \in \mathcal{O}$, with $d = \alpha\bar{\alpha}$, there is $F_\alpha \in \overline{\mathbb{Q}}(E)$ such that

$$\theta(\alpha z)^2 = F_\alpha(\exp_E z)\theta(z)^{2d}. \tag{26.10}$$

Its divisor and leading Laurent coefficient are

$$\operatorname{div}(F_\alpha) = 2[\alpha]^*(O) - 2d(O), \qquad F_\alpha(\exp_E z) = \alpha^2 z^{2-2d}(1 + O(z^2)).$$

The unsquared quotient is a meromorphic function on the cover of degree at most two with function field $\overline{\mathbb{Q}}(E)(\sqrt{F_\alpha})$. It descends to E exactly when its residual sign character is trivial. For a nontorsion $P = \exp_E u \in E(\overline{\mathbb{Q}})$, algebraicity of $\theta(u)$ would imply algebraicity of $\theta(\alpha u)$ for all $\alpha \in \mathcal{O}$. The additive group $\mathcal{O}u$ has rank two.

Proof. 1. The Legendre relation gives the displayed real-linear decomposition of η. The sigma automorphy formula gives

$$\theta(z+\omega) = \epsilon(\omega)\exp\left(\frac{\pi\bar{\omega}}{A}(z+\omega/2)\right)\theta(z), \qquad \epsilon(\omega)\in\{1,-1\}.$$

For $\alpha\Lambda\subset\Lambda$, the exponential multipliers of $\theta(\alpha z)/\theta(z)^d$ cancel because $\alpha\bar{\alpha}=d$. Its remaining multiplier is a sign. Squaring therefore gives a meromorphic function on E, with the stated divisor. This divisor is defined over $\overline{\mathbb{Q}}$. The existence of a rational function with this divisor over $\mathbb{C}$ implies its existence over $\overline{\mathbb{Q}}$: the relevant one-dimensional Riemann–Roch space is obtained by extension of scalars. Normalize its leading Laurent coefficient, in the formal parameter of the algebraic differential, to α^2. This identifies it with the squared analytic quotient. The action of α on this differential is algebraic, so α^2 is an allowed normalization.

2. Choose a nonreal $\alpha\in\mathcal{O}$. Since the normalized sigma series is $\sigma(z)=z+O(z^5)$ with algebraic Laurent coefficients, the next coefficient of the squared quotient is

$$\alpha^2 z^{2-2d}\{1-s_2(\alpha^2-d)z^2+O(z^4)\}.$$

The coefficients of the rational function in this algebraic formal parameter are algebraic. As $\alpha^2-d\neq 0$, this proves $s_2\in\overline{\mathbb{Q}}$. It also proves that the functions in (26.10) are algebraic over the chosen field. At nontorsion u, none of the arguments αu is a lattice point; evaluating the identity proves the algebraicity implication. The injective map $\mathcal{O}\to\mathbb{C}$, $\alpha\mapsto\alpha u$, proves the rank assertion.

3. The qualification about rationality is necessary. For a square lattice, $s_2=0$ and $\alpha=1+i$ give $F_{1+i}=2i\wp$. The unsquared quotient has a simple zero at the nonzero point in $\ker(1+i)$ and a simple pole at O, which is not a principal divisor on E; its square is the rational function just displayed.

□

Cited result 26.9 (Two normalized transcendental values)**.** [tag 001G]

Let $E/\overline{\mathbb{Q}}$ have an algebraic invariant differential, and let u be an elliptic logarithm of an algebraic nontorsion point P. Then

$$\sigma(u)\exp\left(-\frac{1}{2}u\zeta(u)\right)$$

is transcendental. If E has CM and s_2 is normalized as in 26.8, then $\zeta(u)-s_2u$ is also transcendental. The exact relation of these normalized values to $N(P)$ and $\theta(u)=\sigma(u)e^{-s_2u^2/2}$ is (26.11).

Proof. The original French statement (Waldschmidt, 1979, unnumbered theorem, Section 3, p. 113; Corollary 1, Section 4, pp. 113–114) applies to algebraic invariants, algebraic elliptic points $\exp_E u, \exp_E v$, and $\beta\in\overline{\mathbb{Q}}$. For nontorsion $\exp_E v$ and $u, u+v\notin\Lambda$, it proves transcendence of

$$\frac{\sigma(u+v)}{\sigma(u)\sigma(v)}e^{(\beta-\zeta(v))u}.$$

Take $v=u$ and $\beta=0$. Since $\sigma(2u)=-\wp'(u)\sigma(u)^4$, this value is $-\wp'(u)\sigma(u)^2e^{-u\zeta(u)}$. The nonzero factor $-\wp'(u)$ is algebraic, which proves the first assertion.

For the second, use the algebraic universal vector extension with analytic periods $(\omega,\eta(\omega))$. On $E\setminus\{O\}$ an algebraic fibre coordinate is $t=w-\zeta(z)$: the zeta addition formula makes its group cocycle rational in the Weierstrass coordinates. Near O, use $t_O=t-\wp'(z)/(2\wp(z))=w+O(z^3)$. The overlap change is rational, so (z,w) are the tangent coordinates of an algebraic chart at

the identity. If $b = \zeta(u) - s_2 u$ were algebraic, the logarithm $(u, \zeta(u) - b)$ would exponentiate to the algebraic point with fibre coordinate $-b$ and would lie on the algebraic line $w = s_2 z$, since proposition 26.8 gives $s_2 \in \overline{\mathbb{Q}}$. The analytic subgroup theorem (Huber and Wüstholz, 2022, Theorem 6.1, arXiv v8, p. 43) would then give a one-dimensional algebraic subgroup projecting nontrivially to E. Such a subgroup is an elliptic curve mapping isogenously to E: a one-dimensional affine algebraic group has no nonconstant group morphism to an elliptic curve. Its inclusion would split the vector extension after that isogeny. The extension class in $H^1(E, \mathcal{O}_E)$ is nonzero and stays nonzero under isogeny, because trace divided by the degree is a left inverse to pullback. Equivalently, its period vectors cannot lie on one complex line, by Legendre's relation. This contradiction proves the second assertion. □

Remark 26.10 (Lift transport for the theta expression)**.** [tag IA12]

Put $g(u, v) = \log(\sigma(u + v)/(\sigma(u)\sigma(v)))$ where the three sigma factors are nonzero. Under $v \mapsto v + \omega$, the change of $g(u, v) - s_2 uv$, modulo $2\pi i\mathbb{Z}$, is

$$(\eta(\omega) - s_2\omega)u = \frac{\pi\bar{\omega}u}{A}.$$

Both lifts represent the same elliptic point. Thus a relation $N \exp_E v = f(\exp_E u)$ for a Rosati-antisymmetric endomorphism retains its value, while the theta expression changes by this explicit term. The pointwise criterion in (Bertrand, 2017, PDF slide 15, comment (c)) uses compatible lifts.

For algebraic invariants and a CM elliptic curve, (Waldschmidt, 1988b, p. 15-03) states, attributing the result to Bertrand, that $\theta(u) = \sigma(u)e^{-s_2 u^2/2}$ is transcendental at an algebraic nontorsion elliptic point. The s_2 there is the regularized lattice sum. Its quasiperiod identity is $\eta(z) = s_2 z + \pi\bar{z}/A$, the convention of proposition 26.8.

Remark 26.11 (The Gaussian factor and arithmetic criteria)**.** [tag IA13] The exact comparison is

$$N(P) = |\Delta|^{1/12}|\theta(u)| \exp\left(-\frac{\pi|u|^2}{2A}\right). \tag{26.11}$$

It follows from 26.2 and the real-linear decomposition of η. The normalized value in cited result 26.9 gives the exact identity

$$N(P) = |\Delta|^{1/12}\left|\sigma(u)e^{-u\zeta(u)/2}\right| \exp\left(\frac{1}{2}\Re\{u(\zeta(u) - \eta(u))\}\right).$$

These identities specify the remaining exponential exactly. The CM sampling group $\mathcal{O}u$ has rank two, and $\theta'(u)/\theta(u) = \zeta(u) - s_2 u$ is transcendental by cited result 26.9. The arithmetic nature of the full product in (26.11) is the open question in section D.

The related independence theorem of Reyssat (1982, Corollary 2, p. 292) says that two of $e^{i\pi u/\omega_1}$, $s(u)$ and $s'(u)$ are algebraically independent, where $s(z) = \sigma(z)e^{-\eta_1 z^2/(2\omega_1)}$. For real u, (26.2) gives $N(P) = |\Delta|^{1/12}|s(u)|$. This is the exact change of normalization for comparing that theorem with the metric norm. The periodic-functional-equation theorem reported in Waldschmidt (1988b, p. 15-05) assumes algebraic $q = e^{i\pi\tau}$, whereas this q is transcendental for imaginary quadratic τ by Gelfond–Schneider.

For vector extensions, Bost and Künnemann (2009, Theorem 3.1.2, p. 386) proves that an algebraic point of the universal vector extension of an abelian variety which lies in its maximal compact subgroup at one complex embedding is torsion. Its input is an algebraic line bundle with an algebraic connection having unitary monodromy. The compact-character construction in proposition 26.3 specifies a point in a multiplicative extension. An application of the vector-extension theorem therefore requires construction of the stated algebraic connection.

Let f_E be the Lattès map defined by $x(2Q) = f_E(x(Q))$. Its canonical height is $\widehat{h}_{f_E}(x(P)) = 2\widehat{h}_O(P)$. For rational parameters proposition 24.2 makes algebraicity of $N(P)$ equivalent to

algebraicity of $\exp(\widehat{h}_{f_E}(x(P)))$. Nguyen (2023, Theorem 1.4 and Corollary 1.6, arXiv v1, p. 3) settle the corresponding multiplicative-height problem for nonintegrable polynomials using Böttcher coordinates in the basin of infinity. An elliptic duplication Lattès map has no exceptional point: the inverse images of any point contain the images of all successive division points of an elliptic point, an infinite set. A polynomial has the exceptional point ∞, so no conjugate of f_E is polynomial. The Julia set is the whole sphere and the required basin of infinity is absent.

The arithmetic algebraicity criterion of Xie (2024, Theorem 1.3, p. 3, and Theorem 2.3, p. 12, arXiv v1) requires sufficiently rapid height-normalized approximation by points in a fixed number field and an analytic germ formally defined over that field. This hypothesis fails for the compact leaf in 26.14: its tangent has the proved transcendental slope $c = \zeta(u) - \eta_1 u/\omega_1$. A smooth formal germ defined over $\overline{\mathbb{Q}}$ has an algebraic tangent; algebraic translations, coordinate changes and isogenies preserve this obstruction. Assuming $N(P)$ algebraic supplies an algebraic point on the leaf and its integer multiples, but not an algebraic formal germ. Attempting to force convergence by successive division also loses the fixed-number-field hypothesis: if $[2]^n Q_n = Q$ with all $Q_n \in E(K)$ for one number field, then $\widehat{h}_O(Q_n) = 4^{-n}\widehat{h}_O(Q) > 0$. The Q_n are distinct with bounded ample height, contrary to Northcott finiteness.

26.3 Ordinary nonarchimedean heights

Cited result 26.12 (Nonzero ordinary heights)**.** [tag 001H]

Let $E/\mathbb{Q}$ have complex multiplication by the full ring of integers of an imaginary quadratic field K of class number one. Assume that its functional-equation sign is -1, and let $p > 3$ be a prime of good ordinary reduction, $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}^*$. For a nontorsion $P \in E(\mathbb{Q})$ the $\mathfrak{p}$-adic height and the cyclotomic p-adic height of P over K, in the normalization of Agboola and Howard (2006, Appendix A), are nonzero. In particular this applies to

$$E : y^2 + y = x^3 - 38x + 90, \qquad P = (0, 9), \qquad p = 5.$$

Proof. Use Theorem A.1 and Lemma A.2 in Rubin's appendix (Agboola and Howard, 2006, Appendix A, p. 1043), with its standing CM, ordinary-prime and sign hypotheses. Theorem A.1, attributed there to Bertrand, gives nonzero $\mathfrak{p}$-adic height for the nontorsion point over K. The Galois-equivariance in (Mazur et al., 2006, Section 2.9, pp. 587–588; Remark 2.2, p. 588) makes the anticyclotomic height of a rational point zero: complex conjugation fixes the point and acts by -1 on that height space. Lemma A.2 says that vanishing of two of the cyclotomic, anticyclotomic and $\mathfrak{p}$ projections forces the third to vanish. Hence the cyclotomic height is nonzero.

For the displayed curve, propositions 24.9 and 24.11 give nontorsion and analytic rank one, hence sign -1. The CM order has discriminant -19 and is maximal. Its reduced positive binary forms have the single class $(1, 1, 5)$, so the class number is one. The minimal discriminant is -19^3. There are seven points over $\mathbb{F}_5$, so $a_5 = -1$, proving good ordinary reduction. Equivalently, -19 is a nonzero square modulo 5. These verify every stated hypothesis. □

Remark 26.13 (The compact subgroup and the CM correction)**.** [tag IA14]

The complex holomorphic CM correction is θ of proposition 26.8; its exact relation to the real norm includes $\exp(-\pi|u|^2/(2A))$ as in (26.11). The height in cited result 26.12 has the ordinary nonarchimedean normalization specified in its cited appendix.

Bertrand (1995, Theorem 2.1, p. 147) constructs nontrivial multiplicative extensions with nontorsion algebraic points in their maximal real compact subgroup. Its data include a Rosati-antisymmetric endomorphism and a related nontorsion point over a real number field. For the diagonal class here, $NP = f(P)$ with $N > 0$ and $f + f^\dagger = 0$ forces torsion: the endomorphism $N - f$ is an isogeny because $(N - f)^\dagger(N - f) = N^2 - f^2$ is positive. The concrete antisymmetric and diagonal extensions are compared by the maps in proposition 26.17. The vector-extension

statements (Bertrand, 1995, Theorem 3.1, p. 150) and (Bost and Künnemann, 2009, Theorem 3.1.2, p. 386) use an algebraic connection, as described in remark 26.11.

26.4 The fixed semiabelian extension and its compact slope

Proposition 26.14 (Endomorphisms and the compact slope)**.** [tag 001I]

Let $E/\overline{\mathbb{Q}}$ be an elliptic curve, let $q \in E^\vee(\overline{\mathbb{Q}})$ be nontorsion, and let

$$1 \longrightarrow \mathbb{G}_m \longrightarrow G_q \longrightarrow E \longrightarrow 0$$

be the extension of class q. Then $\mathrm{End}_{\overline{\mathbb{Q}}}(G_q) = \mathbb{Z}$, including when E has complex multiplication. In particular a noninteger CM endomorphism of E does not lift to an endomorphism of this fixed extension.

For

$$E : y^2 + y = x^3 - 38x + 90, \qquad P = (0, 9), \qquad Q = 2P = (4, -2),$$

one has

$$N(Q) = 19^{1/4} N(P)^4.$$

Use the differential $dx/(2y+1)$, choose the positive real period ω_1, and choose the real logarithm $u \in (0, \omega_1/2)$ of Q. Put

$$c = \zeta(u) - \frac{\eta_1}{\omega_1} u, \qquad \eta_1 = \eta(\omega_1). \tag{26.12}$$

Then c is a real transcendental number. The point

$$R_N = (Q, 3\,19^{-1/2} N(Q)^2) \in G_Q(\mathbb{C})$$

is nontorsion, belongs to the maximal compact subgroup, and has the real logarithm (u, cu). Its algebraicity is equivalent to the algebraicity of $N(P)$. In the algebraic logarithm coordinates of 26.2, the line $\ell_c = \mathbb{C}(1, c)$ has transcendental slope.

Proof. 1. The torus is the maximal connected affine subgroup of G_q. Consequently every endomorphism of G_q induces multiplication by some $n \in \mathbb{Z}$ on $\mathbb{G}_m$ and an endomorphism α on E. Functoriality of extension classes gives

$$\alpha^* q = nq \quad \text{in } E^\vee. \tag{26.13}$$

Here α^* is the dual endomorphism. To make the convention explicit, choose a principal polarization $\lambda : E \to E^\vee$, put $Q_0 = \lambda^{-1} q$, and write $\alpha^\dagger = \lambda^{-1} \alpha^* \lambda$. Equation (26.13) says $(\alpha^\dagger - [n])Q_0 = O$. A nonzero endomorphism of an elliptic curve is an isogeny and has finite kernel. Since Q_0 is nontorsion, $\alpha^\dagger = [n]$, hence $\alpha = [n]$ as well.

2. The difference between the given endomorphism and $[n]$ has image in the torus and is trivial on the torus. It therefore factors through an algebraic-group homomorphism $E \to \mathbb{G}_m$. This homomorphism is zero, since a morphism from the proper connected curve E to the affine group $\mathbb{G}_m$ is constant and a group homomorphism preserves the identity. Thus the endomorphism is exactly $[n]$. Conversely all integer multiplications exist.

3. For the concrete curve, the tangent slope at P is -2, giving $2P = (4, -2)$. Its discriminant is -19^3 and $\psi_2(P) = 2 \cdot 9 + 1 = 19$. The division identity in proposition 4.7 therefore gives the asserted formula for $N(Q)$, so replacing P by Q preserves algebraicity of the norm. This doubling also places the base point in the identity component of the real elliptic group. In the chosen differential the Weierstrass coordinates are $(\wp, \wp') = (x, 2y + 1)$, with $g_2 = 152$, $g_3 = -361$. The negative discriminant means that the real cubic has one real root. Along $(0, \omega_1/2)$, $\wp$ decreases from $+\infty$ to that root and $\wp' < 0$. Since $\wp'(u) = 2(-2) + 1 = -3$, the indicated real logarithm of Q exists.

4. The period vectors of G_Q in 26.2 include

$$(\omega_1, \zeta(u)\omega_1 - \eta_1 u) = \omega_1(1, c).$$

These are algebraic logarithm coordinates: dz and the invariant differential $d\log r - df_u/f_u$ are algebraic, because df_u/f_u is the algebraic third-kind differential displayed in 26.2. Real invariants and real u, ω_1 make c real.

5. Suppose c were algebraic. Then ℓ_c would be an algebraic Lie subspace. Dividing the nonzero period $\omega_1(1, c)$ by a sufficiently large positive integer gives a vector in ℓ_c whose exponential is a nonzero torsion point of G_Q, hence an algebraic point. The analytic subgroup theorem (Huber and Wüstholz, 2022, Theorem 6.1, arXiv v8, p. 43) supplies a positive-dimensional connected algebraic subgroup with Lie algebra contained in ℓ_c. Its dimension is one, and its projection to E is nontrivial. Such a subgroup must be an elliptic curve E' projecting isogenously to E: a connected one-dimensional affine algebraic group cannot map nontrivially to an elliptic curve. Its inclusion splits the pullback of G_Q to E'. The dual of that isogeny would therefore kill the extension class $\lambda(Q)$, forcing Q torsion. This contradicts 24.9. We conclude that c is transcendental. In particular the line is not defined over $\overline{\mathbb{Q}}$.

6. Define the positive real number

$$r = 3\,19^{-1/2} N(Q)^2.$$

Proposition 26.3 places the point $R_N = (Q, r)$ in the maximal compact subgroup of G_Q. Since Q is algebraic, R_N is algebraic exactly when r is algebraic, equivalently when $N(Q)$, and hence $N(P)$, is algebraic. The chosen real logarithm has

$$f_u(u) = -\wp'(u)\sigma(u)^2 e^{-u\zeta(u)} = 3\sigma(u)^2 e^{-u\zeta(u)} > 0.$$

Consequently $v = \log(r/f_u(u))$ is real and (u, v) is a logarithm of R_N. Substituting this real logarithm into the exact compactness equation $\chi = 0$ in 26.3 gives

$$v = \zeta(u)u - u\eta(u) = u\left(\zeta(u) - \frac{\eta_1}{\omega_1}u\right) = cu.$$

Its projection Q is nontorsion, so R_N is nontorsion. The line ℓ_c containing this compact logarithm has the transcendental slope proved above. Its CM decomposition is

$$c = \zeta(u) - s_2 u - \frac{\pi}{A}u.$$

The last term is the derivative of the real Gaussian correction.

□

Remark 26.15 (The Zariski closure of the compact-slope graph)**.** [tag IA15]

Consider

$$f : \mathbb{C} \longrightarrow \mathrm{Lie}(G_Q), \qquad f(z) = (z, cz).$$

No proper connected algebraic subgroup of G_Q projects onto E: as in the proof above, such a subgroup would split the nontorsion extension after an isogeny. Every proper connected subgroup is therefore contained in the torus, whereas $\exp_{G_Q} f$ has nonconstant elliptic projection. Thus f is G_Q-nondegenerate in the subgroup condition (1.2) of Noguchi (2024, author preprint, p. 2). Their Theorem 1.3 on the same page gives

$$\dim_{\mathbb{C}} \overline{\{(\exp_{G_Q} f(z), f(z)) : z \in \mathbb{C}\}}^{\mathrm{Zar}} \geq \dim G_Q + 1 = 3.$$

The graph is contained in $G_Q \times \ell_c$, of dimension three, so equality holds. The graph therefore has Zariski closure $G_Q \times \ell_c$. This equality is the functional statement used here; the algebraicity of the particular value $R_N = \exp_{G_Q}(u, cu)$ remains the question in section D.

26.5 A CM-stable enlargement and its Gaussian eigencoordinate

Definition 26.16 (The CM enlargement coordinates)**.** [tag IA16] Retain the curve, point $Q = 2P$, and real logarithm u of proposition 26.14. Put $\beta = \sqrt{-19} = i\sqrt{19}$, $d = 19$, $D = \zeta(u) - s_2 u$, and $k = \pi/A$. All endomorphisms and extensions below are defined over $\overline{\mathbb{Q}}$.

Proposition 26.17 (The CM action and its Gaussian eigencoordinate)**.** [tag 0018]

Let

$$\mathcal{G} = G_Q \times_E G_{\beta Q}, \qquad 1 \longrightarrow \mathbb{G}_m^2 \longrightarrow \mathcal{G} \longrightarrow E \longrightarrow 0.$$

There is an endomorphism B of $\mathcal{G}$ inducing β on E and $(t_1, t_2) \mapsto (t_2^{-1}, t_1^d)$ on the torus. It satisfies $B^2 = [-d]$. Moreover

$$\operatorname{Hom}(\mathcal{G}, \mathbb{G}_m) = 0, \qquad \operatorname{End}(\mathcal{G}) \otimes \mathbb{Q} = \mathbb{Q}(\beta).$$

Thus $\mathcal{G}$ has no algebraic $\mathbb{G}_m$ quotient, including after isogeny.

There is an algebraic compact point in the second extension above Q whose logarithm is

$$(u, v_2), \qquad v_2 = -\beta u D + a u, \qquad a = \frac{1}{2\beta}\frac{d}{du}\log F_\beta(Q) \in \overline{\mathbb{Q}}.$$

Here the derivative means differentiation of the rational function $F_\beta(\exp_E z)$ with respect to the algebraic invariant differential, evaluated at $z = u$. Pair this point with the compact point R_N of 26.14. The resulting compact point of $\mathcal{G}(\mathbb{C})$ has logarithm

$$U = (u, v_1, v_2), \qquad v_1 = uD - ku^2.$$

Its algebraicity is equivalent to the algebraicity of $N(P)$. In these algebraic logarithm coordinates the differential of B and its $-\beta$ eigenprojection are

$$dB = \begin{pmatrix} \beta & 0 & 0 \\ a & 0 & -1 \\ \beta a & d & 0 \end{pmatrix}, \qquad \Pi_- := \frac{1}{2}\left(I - \frac{dB}{\beta}\right),$$

$$\Pi_- U = -\frac{\pi u^2}{2A}(0, 1, \beta).$$

The first torus coordinate of $\exp(\Pi_- U)$ is exactly the Gaussian $\exp(-\pi u^2/(2A))$. The projector is not the differential of an algebraic quasi-endomorphism. No sequence of rational polynomials in B, even allowing inverses of isogenies, can converge to it as complex linear operators. At least one of the two torus coordinates of $\exp(\Pi_- U)$ is transcendental.

Proof. 1. Use the principal polarization to identify the two extension classes with Q and βQ. They are linearly independent over $\mathbb{Q}$ modulo torsion: any nonzero $m + n\beta$ is an isogeny, so $(m + n\beta)Q$ cannot be torsion. A character of $\mathbb{G}_m^2$ given by $(m, n) \in \mathbb{Z}^2$ extends to $\mathcal{G}$ exactly when the pushout extension has zero class, namely when $mQ + n\beta Q = O$. Hence the restriction of every homomorphism $\mathcal{G} \to \mathbb{G}_m$ to the torus is zero. Such a homomorphism then factors through E and is zero. This proves the asserted absence of characters. An isogenous semiabelian variety cannot acquire a nonzero character, because composing it with the isogeny would give a nonzero character of $\mathcal{G}$.

2. The Rosati adjoint of β is $-\beta$. Pullback by β sends the ordered extension classes to $(-\beta Q, dQ)$, exactly as does pushout by the displayed torus matrix. Equality of these extension classes provides the endomorphism B. It is unique because $\operatorname{Hom}(E, \mathbb{G}_m^2) = 0$. Its square induces $[-d]$ on the torus and on E, so the same uniqueness gives $B^2 = [-d]$. The map

$$\operatorname{End}(\mathcal{G}) \otimes \mathbb{Q} \longrightarrow \operatorname{End}(E) \otimes \mathbb{Q} = \mathbb{Q}(\beta)$$

is injective: an endomorphism inducing zero on E has image in the torus, and the absence of characters makes it zero. Its image contains β, hence equals $\mathbb{Q}(\beta)$.

3. The identities in 26.8, applied at β and $1+\beta$, give the algebraic nonzero number

$$r_2 = \frac{\theta((1+\beta)u)}{\theta(u)\theta(\beta u)}, \qquad r_2^2 = \frac{F_{1+\beta}(Q)}{F_\beta(Q)}. \tag{26.14}$$

Indeed $(1+\beta)(1+\bar{\beta}) = 1+d$, so the powers of $\theta(u)$ cancel. None of the arguments is a lattice point since Q is nontorsion. Written in sigma functions,

$$r_2 = \frac{\sigma((1+\beta)u)}{\sigma(u)\sigma(\beta u)} e^{-s_2\beta u^2}.$$

The point (Q, r_2) in the rational trivialization of $G_{\beta Q}$ therefore has logarithm

$$(u, v_2), \qquad v_2 = u\{\zeta(\beta u) - s_2\beta u\}.$$

Differentiating 26.8 shows that

$$\zeta(\beta u) - s_2\beta u = -\beta D + a, \qquad a \in \overline{\mathbb{Q}}, \tag{26.15}$$

with the value of a in the statement. The compactness character of this extension vanishes, since

$$\Re\{v_2 - \zeta(\beta u)u + \beta u\eta(u)\} = \Re\left(\frac{\beta\pi|u|^2}{A}\right) = 0.$$

4. In the algebraic coordinates, the torus block of dB is the stated integer matrix and its first entry is β, so initially write its first column as $(\beta, a_1, a_2)^t$, with a_1, a_2 algebraic. The identity $B^2 = [-d]$ gives $a_2 = \beta a_1$. For $\omega \in \Lambda$, set

$$\kappa_1(\omega) = \zeta(u)\omega - \eta(\omega)u, \quad \kappa_2(\omega) = \zeta(\beta u)\omega - \eta(\omega)\beta u.$$

Equation (26.15) and the real-linear formula for η imply

$$\kappa_1(\beta\omega) = a\omega - \kappa_2(\omega).$$

Comparing the second coordinates of the corresponding period vectors under B gives $(a_1 - a)\omega_j \in 2\pi i\mathbb{Z}$ for $j = 1, 2$. Since ω_2/ω_1 is nonreal, both integers must be zero, so $a_1 = a$. This proves the matrix formula.

5. The first compact logarithm in 26.14 is $v_1 = uD - ku^2$. Pairing the two compact lifts over the same base point gives U. The second lift is algebraic, so the paired point is algebraic exactly when its first lift is algebraic. Substitution into the displayed matrix yields

$$v_2 + \beta v_1 - au = -\beta k u^2, \qquad \Pi_- U = -\frac{ku^2}{2}(0, 1, \beta),$$

which proves the exact Gaussian identity.

6. The two eigenspaces of dB have eigenvalues β and $-\beta$. Every algebraic quasi-endomorphism has differential $a_0 I + b_0 dB$ for rational a_0, b_0, with eigenvalues $a_0 + b_0\beta$ and $a_0 - b_0\beta$. The projector Π_- would require these values to be 0 and 1, which is impossible. There is also a uniform obstruction to approximation. For arbitrary real a_0, b_0, put $x = a_0 + b_0\beta$, so that its other eigenvalue is $\bar{x}$. Then

$$\max\{|x|, |\bar{x} - 1|\} \geq \frac{1}{2}, \tag{26.16}$$

because $1 \leq |\bar{x}| + |\bar{x} - 1| = |x| + |\bar{x} - 1|$. These are exactly the two eigenvalue errors in approximating Π_-. The spectral radius is bounded by every induced operator norm, so no such sequence can converge to the projector. Compositions and inverses of isogenies still belong to the same endomorphism algebra and satisfy this bound.

7. Finally the image of Π_- is the toric Lie line $\mathbb{C}(0,1,\beta)$. It is not the Lie algebra of an algebraic subtorus: an algebraic one-dimensional subtorus of $\mathbb{G}_m^2$ has rational slope. Since $u \neq 0$, the projected vector is nonzero. The analytic subgroup theorem in fact shows that its exponential cannot be an algebraic point of both torus coordinates. Indeed its nonzero logarithm lies in the algebraic line $\mathbb{C}(1,\beta)$, whereas a one-dimensional algebraic subtorus has a rational slope. The analytic subgroup theorem would put a positive-dimensional subtorus in that line for an algebraic exponential, contradicting these two slope computations.

8. At the ordinary prime 5, the CM number $\alpha = (1+\beta)/2$ satisfies $\alpha^2 - \alpha + 5 = 0$. Its two archimedean conjugates both have absolute value $\sqrt{5}$. At the two places over 5, their normalized valuations are 0 and 1, because the polynomial reduces to $X(X-1)$ and has product 5. The separation available in that nonarchimedean linear algebra is absent in (26.16).

9. **An exact compact witness on the conductor-361 curve.** In the preceding normalization the constants and the second fibre coordinate are explicitly

$$s_2 = -2, \qquad a = -\frac{21\beta}{703}, \qquad r_2 = -\frac{165}{74} + \frac{1839\beta}{1406}. \tag{26.17}$$

Put $f = 4x^3 - 152x + 361$ and

$$\begin{aligned} h &= x^9 - 38x^8 + 437x^7 - 1444x^6 - 7942x^5 \\ &\quad + 82308x^4 - 274360x^3 + 390963x^2 - 130321x - 130321, \\ N &= (-19x+76)h^2 + f(h''h - (h')^2) + (6x^2 - 76)h'h, \\ K &= N'h - 2Nh'. \end{aligned}$$

Direct expansion gives

$$-K^2 f = 4N^3 - 152 \cdot 19^2 N h^4 + 361 \cdot 19^3 h^6, \qquad \gcd(N,h) = 1.$$

Hence, in coordinates $Y = 2y+1$, the map

$$(x,Y) \longmapsto \left(\frac{N}{19h^2}, \frac{KY}{19\beta h^3} \right)$$

is a degree-19 endomorphism fixing O, with pullback of dx/Y equal to $\beta\, dx/Y$. It is therefore the CM endomorphism β. Divisors and the leading coefficient at O give $F_\beta = \beta^2 h^2$. Its Laurent coefficient of relative degree two is -76. Comparison with 26.8 gives $38 s_2 = -76$, proving $s_2 = -2$.

10. For the other kernel put $h_+ = (H_0 + \beta H_1)/10$, where

$$\begin{aligned} H_0 &= 10x^8 - 380x^7 + 4864x^6 - 28880x^5 + 72561x^4 \\ &\quad + 54872x^3 - 795644x^2 + 1824494x - 1433531, \\ H_1 &= 20x^7 - 304x^6 + 760x^5 + 10469x^4 - 83752x^3 \\ &\quad + 246924x^2 - 315514x + 130321. \end{aligned}$$

Writing $\bar{h}_+$ for replacement of β by $-\beta$, the finite polynomial identities and coprimality checks are

$$19xh^2 - N = 5 f h_+ \bar{h}_+, \qquad h_+ \mid K + 19\beta h^3,$$
$$\gcd(h_+, fh) = \gcd(h_+, h_+') = 1.$$

At a root of h_+ the displayed CM map sends (x,Y) to $(x,-Y)$, so these are precisely the eight pairs of non-2-torsion points in $\ker(1+\beta)$. The other three nonzero kernel points are $E[2] \setminus \{O\}$, since $1 + \beta = 2\alpha$ in the CM order. Its norm is 20, which verifies the count. Thus

$$F_{1+\beta} = \frac{(1+\beta)^2}{4} f h_+^2.$$

Taking the square ratio in (26.14) and fixing the sign by its expansion at $u = 0$ gives the exact rational function

$$r_2 = -\frac{1+\beta}{2\beta}\frac{\wp'(u)h_+(\wp(u))}{h(\wp(u))}.$$

At Q, the needed evaluations are

$$h(4) = -37, \qquad h'(4) = 7, \qquad h_+(4) = \frac{829 - 279\beta}{10}, \qquad \wp'(u) = -3.$$

They give exactly (26.17). In particular (Q, r_2) is an explicit nontorsion algebraic point in the maximal compact subgroup of $G_{\beta Q}$. Its torus-coordinate norm is $r_2\bar{r}_2 = 974799/26011 = 4 - \wp(\beta u)$.

□

Remark 26.18 (A real period basis for the CM witness)**.** [tag IA17]

Put $\tau_0 = (1 + i\sqrt{19})/2$. The integral basis change $(1, \tau_0) \mapsto (1 - 2\tau_0, 1 - \tau_0)$ has determinant one. Applied to the rotated basis $i(1, \tau_0)$, it gives the ratio

$$\tau = \frac{1 + i/\sqrt{19}}{2}, \qquad \omega_1 = \left(\frac{g_2(\tau)}{152}\right)^{1/4} > 0.$$

The logarithm u is selected by $0 < u < \omega_1/2$, $\wp(u) = 4$, and $\wp'(u) = -3$. This fixes the analytic branch of the algebraic witness in proposition 26.17.

26.6 Complexifying the Gaussian: values, jets, and the rank criterion

Proposition 26.19 (The complexified norm and its second derivatives)**.** [tag 0019]

Use the curve and conjugation of definition 26.16.

Use the curve, point and real logarithm of 26.14:

$$E : y^2 + y = x^3 - 38x + 90, \qquad Q = (4, -2) = \exp_E u, \qquad 0 < u < \omega_1/2.$$

Put $\beta = i\sqrt{19}$, $k = \pi/A$, $s_2 = -2$, and

$$\theta(z) = \sigma(z)e^{-s_2 z^2/2}, \qquad D(z) = \zeta(z) - s_2 z, \qquad \mathcal{F}(z, w) = \sqrt{19}\,\theta(z)\theta(w)e^{-kzw}, \qquad H = \mathcal{F}^2.$$

Then $\mathcal{F}(z, \bar{z}) = N(z)^2$, and $\mathcal{F}$ is an entire function of two complex variables of order at most two. It is periodic under $(z, w) \mapsto (z + \omega, w + \bar{\omega})$, $\omega \in \Lambda$. The subgroup

$$\Gamma = \{(\alpha u + \omega, \bar{\alpha}u + \bar{\omega}) : \alpha \in \mathbb{Z}[\beta],\ \omega \in \Lambda\} \subset \mathbb{C}^2$$

has rank four over $\mathbb{Z}$ and lies in the real plane $\{(z, \bar{z}) : z \in \mathbb{C}\}$. There is one number field K such that, with $H_0 = H(u, u) = N(Q)^4$, every value of H on Γ belongs to $K[H_0]$. More precisely, the nonzero values are algebraic multiples of the powers $H_0^{\alpha\bar{\alpha}}$, as specified in (26.18).

For every nonzero $\alpha \in \mathbb{Z}[\beta]$ there is an algebraic number a_α such that

$$D(\alpha u) = \bar{\alpha}D(u) + a_\alpha.$$

At $p_\alpha = (\alpha u, \bar{\alpha}u)$ the algebraic constant direction $V_\alpha = \alpha\partial_z - \bar{\alpha}\partial_w$ satisfies

$$V_\alpha \log \mathcal{F}(p_\alpha) = \alpha a_\alpha - \bar{\alpha}\bar{a}_\alpha \in \overline{\mathbb{Q}},$$
$$V_\alpha^2 \log \mathcal{F}(p_\alpha) = -\alpha^2\{\wp(\alpha u) + s_2\} - \bar{\alpha}^2\{\wp(\bar{\alpha}u) + s_2\} + 2\alpha\bar{\alpha}k.$$

The number k is transcendental. For every nonzero α, the exact normalized second derivative satisfies

$$\overline{\mathbb{Q}}\left(\frac{V_\alpha^2 H(p_\alpha)}{H(p_\alpha)}\right) = \overline{\mathbb{Q}}(k).$$

In particular,

$$(\partial_z - \partial_w)H(u,u) = 0, \qquad \frac{(\partial_z - \partial_w)^2 H(u,u)}{H(u,u)} = 4k - 8.$$

Thus the normalized first derivative is algebraic and the normalized second derivative generates $\overline{\mathbb{Q}}(k)$. The rank-four sampling set and growth order give the numerical comparison with (26.19) proved below.

Proof. 1. The real model and its algebraic differential give real sigma coefficients and a conjugation-stable lattice. In particular $\overline{\theta(z)} = \theta(\bar{z})$. The discriminant is -19^3, so 26.2 gives the asserted diagonal identity. The sigma function and the exponential of the quadratic polynomial $-kzw$ have order at most two. From 26.8,

$$\theta(z+\omega) = \epsilon(\omega)e^{k\bar{\omega}(z+\omega/2)}\theta(z), \qquad \epsilon(\bar{\omega}) = \epsilon(\omega) \in \{1,-1\}.$$

Multiplying this formula and its conjugate version, the two signs multiply to one and the exponent cancels the change of $-kzw$. This proves periodicity on all of $\mathbb{C}^2$.

2. If $\alpha u + \omega = 0$, then $[\alpha]Q = O$. Nontorsion of Q and the fact that a nonzero elliptic endomorphism is an isogeny imply $\alpha = 0$, and then $\omega = 0$. Thus the rank-two module $\mathbb{Z}[\beta]u$ and the rank-two period lattice are independent over $\mathbb{Z}$. This proves the rank assertion.

3. For $d = \alpha\bar{\alpha}$, the squared CM law 26.8 gives the exact fixed-field identity

$$H(\alpha u, \bar{\alpha}u) = 19^{1-d}F_\alpha(Q)\overline{F_\alpha(Q)}H(u,u)^d. \tag{26.18}$$

All the endomorphisms and their normalized rational functions F_α are defined over a single number field: choose a field containing the coefficients of the finitely many generators of the CM order, and close it under conjugation. The uniqueness of the Laurent normalization in 26.8 gives the same field of definition for every F_α. Writing that fixed number field as K, formula (26.18) places all values in $K[H_0]$. The value for $\alpha = 0$ is zero, and period shifts do not change it. Squaring was included because algebraicity of the individual positive square roots alone would not establish their membership in one fixed number field.

4. Logarithmically differentiating the squared CM law at u gives

$$a_\alpha = \frac{1}{2\alpha}\frac{(d/dz)F_\alpha(\exp_E z)|_{z=u}}{F_\alpha(Q)} \in \overline{\mathbb{Q}}.$$

The denominator is nonzero since Q is nontorsion. The derivation d/dz is the algebraic invariant derivation on E, so the displayed derivative has algebraic value. Since $D(u)$ is real, conjugation gives the corresponding formula at $\bar{\alpha}u$. On a neighborhood where $\mathcal{F} \neq 0$, its logarithmic gradient and Hessian are

$$(\partial_z \log\mathcal{F}, \partial_w \log\mathcal{F}) = (D(z) - kw, D(w) - kz),$$
$$\operatorname{Hess}\log\mathcal{F} = \begin{pmatrix} -\wp(z) - s_2 & -k \\ -k & -\wp(w) - s_2 \end{pmatrix}.$$

Evaluation on $(\alpha, -\bar{\alpha})$ proves both identities in the statement. In particular the Gaussian cancels in the first derivative and reappears in the second derivative.

5. To prove k transcendental, recall the real period formula $\eta_1 = (s_2 + k)\omega_1$. If k were algebraic, the nonzero period vector (ω_1, η_1) of the algebraic universal vector extension of E would lie on the algebraic line of slope $s_2 + k$. Dividing this period by a sufficiently large integer supplies a nonzero algebraic torsion exponential. The analytic subgroup theorem (Huber and Wüstholz, 2022, Theorem 6.1, arXiv v8, p. 43) would give a one-dimensional algebraic subgroup with that Lie line. Its projection to E is nonzero, so it splits the vector extension after an isogeny. The nonsplitting argument in cited result 26.9 excludes this: the coordinates here are exactly (z, w) with periods $(\omega, \eta(\omega))$ and algebraic charts $t = w - \zeta(z)$, $t_O = t - \wp'(z)/(2\wp(z))$.

6. Write the two logarithmic derivatives as $V_\alpha \log \mathcal{F} = B_\alpha$ and $V_\alpha^2 \log \mathcal{F} = A_\alpha + 2dk$, where A_α, B_α are algebraic. Since $H = \mathcal{F}^2$,

$$\frac{V_\alpha^2 H(p_\alpha)}{H(p_\alpha)} = 4B_\alpha^2 + 2A_\alpha + 4dk.$$

   Here $d > 0$, so the displayed affine expression both belongs to $\overline{\mathbb{Q}}(k)$ and recovers k by subtraction of its algebraic constant and division by $4d$. This proves the asserted field equality; $H(p_\alpha)$ is nonzero because Q is nontorsion. At $\alpha = 1$, $a_1 = 0$, $\wp(u) = 4$ and $s_2 = -2$, which give the last displayed formula in the statement.

7. **Differential and growth criteria.** The differential criterion of Waldschmidt (2009, Theorems 3.6–3.7, author proof, pp. 21–22) requires a finite algebraically independent family whose partial derivatives belong to the polynomial ring generated by that family over one number field. Algebraic values of the full family then give algebraic values of all iterated partial derivatives. The calculation above gives the exact value-field obstruction: the field containing $H(p_\alpha)$ and its second directional derivative contains k as soon as the nonzero value is inverted. In particular its normalized second jet is transcendental. The issue persists although the first cancelling direction is algebraic; the direction also varies with α.

8. Theorem 1 of (Waldschmidt, 1975, p. 24) uses d meromorphic functions on $\mathbb{C}^n$, growth and arithmetic-size exponents $\rho_1, \dots, \rho_d$, and sampling sets of cardinality $N^{\ell - o(1)}$ in a ball of radius $O(N)$, with separation at least $N^{-\lambda - o(1)}$. Its strict numerical requirement is

$$(d-1)\ell > \sum_{j=1}^{d} \rho_j + 2d(n-1)(\lambda + 1). \tag{26.19}$$

   The theorem also requires all function values to belong to one number field and bounds on denominators in the entire representations of the meromorphic functions.

9. The natural independent triple here is $\wp(z), \wp(w), H(z, w)$, all of order at most two. The first two are algebraically independent. If H satisfied a polynomial over $\mathbb{C}(\wp(z), \wp(w))$, fix a generic pair (z, w) and translate only z by $m\omega$, with $0 \neq \omega \in \Lambda$. The coefficients of the polynomial remain constant, whereas the absolute values of its proposed roots $H(z + m\omega, w)$ grow as $\exp(k|\omega|^2 m^2 + O(|m|))$. This contradicts the finite set of roots of a nonzero polynomial. Thus the triple is independent.

10. The growth-order lower envelope is $\rho_j = 2$. Substitution of this envelope and $n = 2$, $d = 3$, $\ell = 4$ into (26.19) gives the required inequality

$$8 > 12 + 6\lambda,$$

    which fails even for $\lambda = 0$. Moreover, the $\asymp N^4$ points obtained by bounding the four integer coefficients lie in a ball of radius $O(N)$ in a real plane. Packing separated discs in that plane gives $N^4 \ll N^{2+2\lambda+o(1)}$, hence $\lambda \geq 1$. Removing the rank-two set of poles does not change this count. Larger growth exponents only worsen the inequality. Thus the stated rank and growth envelopes do not satisfy (26.19). The norm question is recorded in section D.

□

26.7 Division values and interpolation bounds

Proposition 26.20 (Division ratios and translated functions)**.** [tag 001A]

Let $E : y^2 + y = x^3 - 38x + 90$, $Q = (4, -2)$ and $N_0 = N(Q)$. Use the positive real logarithm u and period ω_1 of 26.14. Put

$$\delta = 19^{1/4}, \qquad b = \eta_1/\omega_1, \qquad g(z) = \delta\sigma(z)e^{-bz^2/2}.$$

Thus $|g(z)| = N(\exp_E z)$ on the real axis. Let $\mathcal{D}$ be the CM division hull of Q. For $R \in \mathcal{D}$, the equations $[n]R = [\alpha]Q$, $n > 0$, determine a unique $\kappa_R = \alpha/n \in \mathrm{End}(E) \otimes \mathbb{Q}$. Set $w_R = \kappa_R\bar{\kappa}_R$. For $R \neq O$,

$$A_R := \frac{N(R)}{N_0^{w_R}} \in \overline{\mathbb{Q}}_{>0}.$$

In particular, for $[n]R = [m]Q$ with $m \neq 0$,

$$A_R^{n^2} = \delta^{n^2-m^2}\frac{|\psi_m(Q)|}{|\psi_n(R)|}. \tag{26.20}$$

For each positive algebraic number λ, define

$$N_\lambda(R) = A_R\lambda^{w_R} = N(R)\left(\frac{\lambda}{N_0}\right)^{w_R}.$$

All these values are algebraic. For $[n]R = [m]Q$, $|m| \leq n$, $R \neq O$, there are constants D, H depending on λ, E, Q such that

$$h(N_\lambda(R)) \leq H, \qquad [\mathbb{Q}(R, N_\lambda(R)) : \mathbb{Q}] \leq \begin{cases} Dn^4, & R \in E(\mathbb{R}), \\ 2Dn^6, & R \in E(\mathbb{C}). \end{cases}$$

The transformation is exact for every positive real parameter: $N_\lambda(R)/N_\mu(R) = (\lambda/\mu)^{w_R}$ and $N_{N_0}(R) = N(R)$.

For any finite collection of real division-hull logarithms a, the functions $g(z)$ and $g(z+a)/g(a)$, with $\exp_E a \neq O$, have function-field transcendence degree at most three over $\overline{\mathbb{Q}}$ after adjoining the elliptic coordinates. The three functions $g(z), g(z+v)/g(v), g(z-v)/g(v)$, $v = u/q$, attain three. The normalized holomorphic extensions $F_a(z)/N(a)^2$ of $N(\exp_E(z+a))^2$ for fixed complex division-hull shifts belong to the same type of field. The unnormalized translates have the same upper bound, attained by the displayed triple, over the algebraic closure L of $\overline{\mathbb{Q}}(N_0)$ in $\mathbb{C}$.

For the scalar models on the real division grids $z = (mu + k\omega_1)/n$, $|m| \leq Cn$, k modulo n, with fixed C, every fixed finite family of these values has a common degree envelope $O(n^5)$ and bounded heights. For the values at $\lambda = N_0$, the same construction gives a relative degree envelope over a fixed finite extension of $\mathbb{Q}(N_0)$. The algebraicity of N_0 is the open question in section D.

Proof.

1. *The algebraic ratios and their exact change of parameter.* Two equations $[n]R = [\alpha]Q$ and $[n']R = [\alpha']Q$ give $[n'\alpha - n\alpha']Q = O$. A nonzero elliptic endomorphism has finite kernel, whereas Q is nontorsion. Thus $\alpha/n = \alpha'/n'$, proving uniqueness. The resulting map $R \mapsto \kappa_R$ is additive, and its kernel is the torsion subgroup. Every such R is algebraic, since it lies in a finite isogeny fibre above an algebraic point.

   (a) The division identity and the squared CM law give, for $d = \alpha\bar{\alpha}$ and $\alpha \neq 0$,

$$N(R)^{n^2} = \frac{N(nR)\delta^{n^2-1}}{|\psi_n(R)|}, \qquad N(\alpha Q)^2 = \delta^{2-2d}|F_\alpha(Q)|N_0^{2d}.$$

Consequently

$$A_R^{2n^2} = \frac{\delta^{2n^2-2d}|F_\alpha(Q)|}{|\psi_n(R)|^2} \in \overline{\mathbb{Q}}_{>0}.$$

The positive root proves $A_R \in \overline{\mathbb{Q}}$. For integral $\alpha = m$, the usual multiplication identity is $N(mQ) = N_0^{m^2}|\psi_m(Q)|/\delta^{m^2-1}$; substitution proves (26.20). For nonzero n-torsion R, take the limit at the simple zero of ψ_n in the uniformizing coordinate. Since $N(z) = \delta|z| + o(|z|)$ at zero,

$$A_R^{n^2} = N(R)^{n^2} = \frac{n\delta^{n^2}}{|\partial_z \psi_n(R)|}.$$

The invariant derivation is algebraic, so this also is algebraic. All exponents w_R are rational. The asserted scalar models, their parameter-change identity and the specialization at N_0 follow directly. In particular their definition is independent of every presentation of R by a division equation.

(b) Put $K_\lambda = \mathbb{Q}(\delta, \lambda)$ and $D = [K_\lambda : \mathbb{Q}]$. For integral $m \neq 0$,

$$N_\lambda(mQ) = \lambda^{m^2}|\psi_m(Q)|/\delta^{m^2-1} \in K_\lambda,$$

because Q is rational. The fibre of $[n]$ has n^2 points, so adjoining R costs at most n^2. For real R the absolute value of $\psi_n(R)$ differs from it by a sign. The division equation for $N_\lambda(R)$ costs at most another n^2. For complex R, adjoin R and $\bar{R}$, at cost at most n^4, and use

$$N_\lambda(R)^{2n^2} = \frac{N_\lambda(mQ)^2\delta^{2(n^2-1)}}{\psi_n(R)\psi_n(\bar{R})}.$$

This gives $2Dn^6$. At nonzero torsion points the derivative formula proves the same bounds.

2. *Uniform heights.* The following coefficient estimate proves the height bound. On the fixed integral model, ψ_n is a polynomial in x for odd n, and $(2y+1)$ times a polynomial in x for even n. These polynomials have degree $O(n^2)$. Choose a fixed circle $|x| = A > 1$ avoiding the finite branch values. On its compact inverse image in $E(\mathbb{C})$, N has a positive minimum m_A, and $|2y+1|$ has a positive minimum. Let M be the maximum of N on the whole compact elliptic curve. The rearranged division identity gives

$$|\psi_n(R)| \leq M\delta^{n^2-1}m_A^{-n^2} \qquad (|x(R)| = A).$$

Cauchy's coefficient estimate, and division by $2y+1$ for even n, show that the logarithm of the coefficient ℓ^1 norm is $O_E(n^2)$. The same assertion holds for $\partial_z\psi_n$, because differentiation and the fixed Weierstrass equation add only polynomial factors in n.

(a) The elementary height estimate for a fixed degree-four duplication map, followed by telescoping, gives $h(x(R)) = 2\widehat{h}(R) + O_E(1)$ uniformly over $E(\overline{\mathbb{Q}})$. Since $\widehat{h}(R) = (m/n)^2\widehat{h}(Q)$, the coordinate heights are uniformly bounded when $|m| \leq n$. At each nonarchimedean place the ultrametric inequality bounds a polynomial value by its largest monomial; at each archimedean place use the sum of the monomials. Summing these logarithmic bounds by the product formula gives the polynomial evaluation height inequality, and hence $h(\psi_n(R)) = O_{E,Q}(n^2)$, and the same estimate for its invariant derivative at torsion points. Also

$$h(N_\lambda(mQ)) \leq m^2h(\lambda) + O_{E,Q}(m^2).$$

Apply the same division identity to N_λ and divide its height by n^2. For a complex value one uses $h(|\alpha|) \leq h(\alpha)$, which follows by squaring and using $|\alpha|^2 = \alpha\bar{\alpha}$. This proves a uniform bound for $h(N_\lambda(R))$.

3. *Normalized translates and their algebraic scalars.* For a real logarithm a of a nonzero division point put

$$h_a(z) = \frac{\sigma(z+a)}{\sigma(z)\sigma(a)} e^{-baz}.$$

The exact identities are

$$g(z+a) = \delta^{-1} g(z) g(a) h_a(z), \qquad h_a(z+\omega_1) = h_a(z), \quad h_a(z+\omega_2) = e^{-2\pi i a/\omega_1} h_a(z).$$

They follow by substitution from sigma automorphy and Legendre's relation. A real point in the CM division hull lies in the ordinary division hull: conjugating $[n]R = [\alpha]Q$ gives $[\alpha - \bar{\alpha}]Q = O$, hence $\alpha = \bar{\alpha} \in \mathbb{Z}$. The real group is connected because the discriminant is negative. Its division logarithms are therefore exactly $\widetilde{\mathcal{D}} = \mathbb{Q}u + \mathbb{Q}\omega_1$. The coefficient map

$$\nu : \widetilde{\mathcal{D}} \longrightarrow \mathbb{Q}, \qquad \nu(ru + s\omega_1) = r$$

is well defined and additive: a rational relation between u and ω_1 would make Q torsion.

(a) For any positive real λ define, on this additive group,

$$\widetilde{g}_\lambda(z) = g(z)(\lambda/N_0)^{\nu(z)^2}, \qquad h_{a,\lambda}(z) = h_a(z)(\lambda/N_0)^{2\nu(z)\nu(a)}.$$

At nonzero division values, the absolute value of $\widetilde{g}_\lambda$ is N_λ and its sign is the real sign of g. Thus these values are algebraic for algebraic λ. The identity

$$h_{a,\lambda}(z) = \frac{\delta \widetilde{g}_\lambda(z+a)}{\widetilde{g}_\lambda(z)\widetilde{g}_\lambda(a)}$$

is proved by expanding $(\nu(z) + \nu(a))^2$. It specifies the exact transformation on the sampling group.

(b) Choose a common denominator q for finitely many shifts, write $a_j = (m_j u + k_j \omega_1)/q$, and put $v = u/q$. The quotient $h_{a_j}/h_v^{m_j}$ is elliptic for $\Lambda_q = \mathbb{Z}\omega_1 + \mathbb{Z}q\omega_2$. This finite isogeny of E has a model over $\overline{\mathbb{Q}}$ with the stated differential. The quotient has an algebraic divisor, hence is a rational function over $\overline{\mathbb{Q}}$ times a scalar. The scalar is algebraic. Indeed the parameter factors cancel:

$$\frac{h_{a_j}(z)}{h_v(z)^{m_j}} = \frac{h_{a_j,1}(z)}{h_{v,1}(z)^{m_j}} \qquad (z \in \widetilde{\mathcal{D}}).$$

Choose a real division point avoiding the finite zeros and poles. The right side and the value of the chosen rational function are algebraic and nonzero, so their quotient is algebraic. This proves that $g(z)$ and all normalized real translates are algebraic over $\overline{\mathbb{Q}}(E)(g, h_v)$, after the fixed finite isogeny. The upper transcendence-degree bound is three.

(c) For the lower bound, work even over $\mathbb{C}$. The multiplier of h_v under $q\omega_2$ is $e^{-2\pi i u/\omega_1}$, which has infinite order. A polynomial relation over the elliptic field, evaluated at a generic z and translated by $mq\omega_2$, would have infinitely many distinct roots obtained by these powers. Thus h_v is transcendental over that field. Moreover

$$|g(z + mq\omega_2)| = \exp(cm^2 + O(|m|)), \qquad c = \pi q^2 \Im(\omega_2/\omega_1) > 0.$$

In a proposed polynomial relation for g over the elliptic field and h_v, clear denominators. The nonzero leading coefficient is a polynomial evaluated along an irrational rotation; it is bounded away from zero on a subsequence, and all other coefficients remain bounded there. Quadratic growth makes the highest power of g dominate, a contradiction. Finally

$$\wp(z) = \wp(v) - \frac{\delta^2 g(z+v) g(z-v)}{g(z)^2 g(v)^2}$$

recovers the elliptic coordinate from the three stated translates, and $h_v = \delta g(z+v)/(g(z)g(v))$. They therefore attain three. The constants $g(a)$ lie in L, by the already proved ratios. Multiplication by these constants gives the exact map from normalized to raw translates, proving the assertion over L.

(d) For a complex CM division shift a, the holomorphic extension from the real axis is

$$F_a(z) = \frac{N(a)^2}{\delta^2} g(z)^2 h_a(z) h_{\bar{a}}(z).$$

Its multiplier depends on $a + \bar{a} \in \mathbb{Q}u + \mathbb{Q}\omega_1$. Choose q, m with $a + \bar{a} = (mu + k\omega_1)/q$. Then $h_a h_{\bar{a}}/h_v^m$ is elliptic on $\mathbb{C}/\Lambda_q$ with algebraic divisor. Its scalar is algebraic by the same evaluation argument, using the exact identity on real division points

$$h_a(z) h_{\bar{a}}(z) (\lambda/N_0)^{2\nu(z)m/q} = \frac{\delta^2 N_\lambda(\exp_E(z+a))^2}{N_\lambda(\exp_E a)^2 \widetilde{g}_\lambda(z)^2}.$$

Here $w_{\exp_E(z+a)} = |\kappa_{\exp_E a} + \nu(z)|^2$; expanding the square proves the identity. Set $\lambda = 1$ and divide by $h_{v,1}^m$. This proves the normalized complex translate assertion with all scalar factors accounted for.

4. *Joint degrees and heights on the sampling group.* In the algebraic meromorphic section coordinate of 26.2,

$$z \longmapsto (\exp_E z, h_v(z)) = \exp_{G_V}(z, (\zeta(v) - bv)z), \qquad V = \exp_E v.$$

For the scalar model, multiply its torus coordinate by $(\lambda/N_0)^{2\nu(z)\nu(v)}$. This factor is a character of the additive sampling group, so the resulting map $\Phi_\lambda : \widetilde{\mathcal{D}} \to G_V(\mathbb{C})$ is an abstract group homomorphism. Its real period is ω_1, because $\nu(\omega_1) = 0$ and the displayed analytic map has that period. For algebraic λ, its values in the displayed section chart are algebraic by the formula for $h_{v,\lambda}$ above. For an exceptional sampling point at a zero or pole of that section, write $z = z_1 + z_2$ with both z_j avoiding the finite exceptional set modulo the elliptic lattice. Such a choice exists in the infinite division group. Then $\Phi_\lambda(z) = \Phi_\lambda(z_1)\Phi_\lambda(z_2)$ is algebraic by the algebraic group law. This also covers the identity.

(a) For $z = (mu + k\omega_1)/n$, $[n]\Phi_\lambda(z) = [m]\Phi_\lambda(u)$. Choose a fixed number field containing the coefficients of G_V, the point $\Phi_\lambda(u)$ and the finitely many fixed constants. Multiplication by n on G_V has degree n^3, since its kernel is an extension of $E[n]$ by $\mathbb{G}_m[n]$. Adjoining $\widetilde{g}_\lambda(z)$ costs at most n^2 by the norm division equation. Passing to the fixed isogeny and adjoining fixed constants changes only the constant in $O(n^5)$. The same argument for any positive real λ works over a fixed finite extension of $\mathbb{Q}(\lambda)$, because all the fixed division values are algebraic over that field. At $\lambda = N_0$ it gives the claimed relative envelope.

(b) The height proof above applies with $|m| \le Cn$ for any fixed C, changing only its constant. It bounds the model values at z and every fixed real translate $z + a$. Their quotients bound $h_{v,\lambda}(z)$. Coordinates on the fixed isogenous elliptic curve have bounded height by the fixed isogeny height comparison. The normalized complex translates are fixed rational functions of those coordinates, $\widetilde{g}_\lambda^2$ and a fixed power of $h_{v,\lambda}$, with algebraic coefficients just proved. The polynomial evaluation inequality bounds their heights. Any removable zero or pole in these expressions lies in a fixed finite set and is handled by its specified model value. Taking the positive square root for a fixed complex-shifted norm divides height by two and costs at most a factor two in degree.

5. *Interpolation inequalities and algebraic tangents.*

The sufficient inequality in Gramain–Mignotte–Waldschmidt (Gramain et al., 1986, Theorem 2.1, pp. 101–102) has the form

$$d(N)\prod_{j=1}^{k}\max\{d(N)\psi_j(N),\theta_j(R(N))\}\le c_0\mu(N)^{k-1}\varphi(N)^k,$$

$$\varphi(N)=\log\frac{R(N)^2+r(N)r(N+1)}{R(N)(r(N)+r(N+1))}.$$

On fixed nested discs the growth and geometric factors are constant. The scalar-model height envelopes are constant too. The numerical comparison of these envelopes is therefore $d^{k+1}\ll\mu^{k-1}$. At denominator n the real group has $O(n^2)$ sample points, and through denominator N it has $O(N^3)$. Substitution of the proved envelopes gives N^{12} against N^3 for two functions and N^{20} against N^6 for three. These upper envelopes give the exponent comparisons $12>3$ and $20>6$. Verifying the sufficient inequality for these grids requires bounds sharper than these envelopes. For samples at $\lambda=N_0$, the base field is the displayed extension of $\mathbb{Q}(N_0)$; its number-field status is exactly the unresolved algebraicity issue. Extra normalized translates remain in the function field of transcendence degree three proved above.

(a) The arithmetic-germ criteria (Herblot, 2012, Theorem 6.1, pp. 31–32) (Gasbarri, 2010, Theorem 4.16, author preprint, p. 25) can be compared with the logarithmic tangent. At a real nontorsion division point z, the exact formula is

$$\frac{1}{g(z)}\frac{dg}{dx}=\frac{\zeta(z)-bz}{\wp'(z)}.$$

Its numerator is transcendental by the period-line proof of 26.14, applied to this algebraic nontorsion point; its denominator is nonzero algebraic. A formal graph defined over $\overline{\mathbb{Q}}$ would have an algebraic nonzero constant value and algebraic first derivative, contradicting their displayed ratio. The scalar change on the sampling group has not changed this exact tangent of the analytic graph.

(b) For the Mahler system in (Adamczewski and Faverjon, 2017, Theorem 1.1 and system (1.2), p. 56), the needed algebraic Taylor coefficients and algebraic regular argument can be tested directly: the coefficient of z^3 in g is $-\delta b/2$, and $b=s_2+k$ is transcendental by 26.19. The logarithm u is transcendental too: an algebraic $u\neq0$ would give an algebraic point (u,Q) on the diagonal Lie line of $\mathbb{G}_a\times E$; the analytic subgroup theorem would produce an algebraic one-dimensional subgroup projecting nontrivially to both factors, which is impossible since an elliptic curve has no nonconstant map to $\mathbb{G}_a$ and $\mathbb{G}_a$ has no nonzero group map to E. The duplication identity acts by the explicit elliptic multiplication map. The entropy identity in (Einsiedler et al., 2001, Theorem 5.1, author preprint, p. 9) uses the sequence built from the denominators of $x(2^nQ)$ with scale 4^n.

(c) Finally the period-refined rank comparison (Waldschmidt, 1988a, Theorem 4.1, pp. 385–386, and Corollary 4.3, p. 386) (Roy and Waldschmidt, 1993, main theorem, p. 363) has coefficient $\alpha(G)=2\dim G-2d_0(G)-d_1(G)$, where the last terms count additive and multiplicative quotients. Both vanish for G_Q. An additive homomorphism vanishes on its torus and then on the proper quotient E; a nonzero torus character extending to G_Q would kill a nonzero integer multiple of its nontorsion extension class. Thus $\alpha(G_Q)=4$.

(d) The line ℓ_c has period rank one. A general period is $(\omega,\zeta(u)\omega-\eta(\omega)u+2\pi ij)$. Writing $\omega=a\omega_1+d\omega_2$, membership in ℓ_c gives $du/\omega_1\in\mathbb{Z}$ by Legendre's relation. Since u/ω_1 is irrational, $d=j=0$. The numerical rank bound in the cited theorem is $(4-1)\dim\ell_c=3$. The period and the logarithm (u,cu) are independent over $\mathbb{Q}$ and span rank two; the algebraicity of the latter's exponential remains exactly the point-algebraicity question in 26.14. □

A Parameter equivalence and deformations

Remark A.1 (Period and deformation comparison)**.** [tag 02YS] For the fixed standard $(3,4,\infty)$ data, the tangent cohomology $(1,1,1,0)$, smooth one-dimensional Kuranishi germ, automorphism group $\mathbb{C}^*$, and equivalence $X_c \simeq X_{c'} \iff c'-c \in 2\mathbb{Z}$ agree with the corresponding assertions of Du et al. (2026a). Version 2, entitled *Oka Geometry of Complex Six-Spheres*, omits those tangent-cohomology, miniversality, full-automorphism and exact-period assertions. Version 1, submitted on 23 September 2026, states the exact $2\mathbb{Z}$ equivalence as its Theorem 1.4.

Proof. 1. Use the same oriented triangle, ordered integral basis $(\widehat{\gamma},\widehat{u},\widehat{w},\widehat{\delta})$, finite twists, and untranslated unit-square cusp fan as version 1. Its chosen function β_0 and our β^{part} obey the same affine transformation laws, and both $\beta_0+\tau$ and $\beta^{\mathrm{part}}+\tau$ extend over the cusp. Their difference descends to a holomorphic function on $\mathbb{P}^1$, hence is the constant $\delta=\beta^{\mathrm{part}}-\beta_0$. The exact parameter map and inverse are $a=c+\delta$ and $c=a-\delta$: substitution gives $\Omega_{c+\delta}=\Pi_c$ in the stated ordered basis. Writing $M_0=\sup(\Im\beta_0-6(\Im\mu)^2/\Im\tau)$, our corresponding bound is $M_{\mathrm{part}}=M_0+\Im\delta$, so this map also takes our admissible half-plane bijectively to version 1's. The ordinary quotients, finite logarithmic maps, and cusp charts then identify by their identical marked periods and specified gluing data. In particular $a'-a=c'-c$. Its Theorem 1.4 therefore gives the displayed exact-period equivalence and the full automorphism group in this standard-data scope.

2. Theorem 6.12 of version 1 gives the tangent-cohomology tuple, and Proposition 6.16 gives the smooth reduced Kuranishi germ and the locally miniversal parameter family. These are the four assertions compared with theorems B.2, C.4 and C.5 under that same parameter and marking.

3. The second version retains the period construction and the Oka conclusions. Its displayed section list and its mathematical statements omit the first version's logarithmic-deformation and classification sections and the four assertions just listed.

□

A.1 The integral obstruction

Proposition A.2 (The integral obstruction to a parameter shift)**.** [tag 01Q1]
The rational centralizer is

$$\mathrm{Cent}_{\mathbb{Q}}(A_1,A_2)=\{aI+bE : a,b\in\mathbb{Q}\}. \tag{A.1}$$

A biholomorphism over the identity of B° between the torus families forces $c'-c\in\mathbb{Z}$ and has complex-linear part I_2 or $-I_2$ on every fibre. Every biholomorphism $X_c\to X_{c'}$ has $c'-c=n\in 2\mathbb{Z}$, complex-linear fibre part I_2, and source-to-target lattice map $I-nE$. In the full fixed lattices,

$$\Lambda^{A_1}/N_1\Lambda\simeq\mathbb{Z}/3,\qquad \Lambda^{A_2}/N_2\Lambda\simeq\mathbb{Z}/4\oplus\mathbb{Z}/2,$$

with d generating the second factor of the latter group and vanishing in the former.

Proof. 1. For an integer n, put $S_n=I+nE$. Since $S_n^{-1}=S_{-n}$,

$$\Pi_{c+n}=\Pi_c S_n,\qquad \Pi_{c+n}\Lambda=\Pi_c\Lambda. \tag{A.2}$$

Both monodromy matrices fix $\widehat{\delta}$ and preserve γ. Consequently $A_jE=EA_j=E$, and the same commutation holds for every monodromy word. The identity of $\mathbb{C}^2$ therefore identifies the two full middle torus families with their R_g-action. Its source-to-target lattice map is S_{-n}, since $\Pi_{c+n}S_{-n}=\Pi_c$.

2. Solving the two fixed-vector systems from (2.2) gives

$$(\Lambda \otimes \mathbb{Q})^{A_1} = \mathbb{Q}(1,2,-4,0)^{\mathsf{t}} + \mathbb{Q}\widehat{\delta}, \quad (\Lambda \otimes \mathbb{Q})^{A_2} = \mathbb{Q}(1,3,-3,0)^{\mathsf{t}} + \mathbb{Q}\widehat{\delta}.$$

Their intersection is $\mathbb{Q}\widehat{\delta}$. The fixed covectors are respectively

$$\mathbb{Q}\gamma + \mathbb{Q}(2u + w + 3\delta), \qquad \mathbb{Q}\gamma + \mathbb{Q}(u + w + 2\delta).$$

Their intersection is $\mathbb{Q}\gamma$: comparing the u and w coefficients of a common multiple gives $2x = y$ and $x = y$, hence $x = y = 0$. Every commuting endomorphism therefore preserves $\mathbb{Q}\widehat{\delta}$ and $\ker \gamma$.

3. On $\ker \gamma / \mathbb{Q}\widehat{\delta}$, in the classes of $(\widehat{u}, \widehat{w})$, the two matrices are

$$Q_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A matrix commuting with Q_2 has the form $\left(\begin{smallmatrix} b & d \\ -d & b \end{smallmatrix}\right)$. Commutation with Q_1 gives $d = 0$. Thus a commuting endomorphism H has columns

$$H\widehat{\gamma} = (a, r, s, k)^{\mathsf{t}}, \quad H\widehat{u} = b\widehat{u} + u_0\widehat{\delta}, \quad H\widehat{w} = b\widehat{w} + w_0\widehat{\delta}, \quad H\widehat{\delta} = d_0\widehat{\delta}.$$

Use the columns $A_1\widehat{u} = -\widehat{w} + \widehat{\delta}$, $A_1\widehat{w} = \widehat{u} - \widehat{w}$, and $A_2\widehat{u} = \widehat{w}$. Commutation on these columns gives

$$d_0 - w_0 = b + u_0, \qquad u_0 = 2w_0, \qquad w_0 = u_0.$$

Hence $u_0 = w_0 = 0$ and $d_0 = b$. Commutation on $\widehat{\gamma}$ with A_1 gives

$$r = 2(a - b), \qquad s = -4(a - b).$$

Commutation on that column with A_2 also gives $r = -s$. Therefore $a = b$, $r = s = 0$, and $H = aI + kE$. Conversely these matrices commute, by the fixed-vector and fixed-covector identities preceding (A.2). This proves (A.1).

4. For the geometric assertion, a holomorphic map between two compact complex tori lifts to their universal covers. The derivative of the lift is periodic, so its entries descend to holomorphic functions on the compact source torus and are constant. The lift is consequently affine, with a complex-linear part L. In a holomorphic family these linear parts vary holomorphically. Their lattice maps are integral and locally constant, hence give an isomorphism of the two marked local systems. Because the map is over the identity of the base, that integral map H commutes with both A_j. Thus $H = aI + bE$, with $a, b \in \mathbb{Z}$. Its determinant is a^4; invertibility over $\mathbb{Z}$ gives $a = 1$ or $a = -1$.

5. The complete period equation is

$$L(z)\Pi_c(z) = \Pi_{c'}(z)(aI + bE). \tag{A.3}$$

Its third and fourth columns give $L(z) = aI_2$. Its first column then gives $a(c' - c) + b = 0$. Therefore $c' - c = -b/a \in \mathbb{Z}$, and the asserted two linear parts exhaust the possibilities.

6. *The finite-root obstruction.* The two cyclic norm matrices are, by multiplication of (2.2),

$$N_1 = I + A_1 + A_1^2 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ -12 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \end{pmatrix},$$

$$N_2 = I + A_2 + A_2^2 + A_2^3 = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ -12 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 \end{pmatrix}. \tag{A.4}$$

For example these formulas give $N_1\widehat{w} = \widehat{\delta}$, $N_2\widehat{u} = N_2\widehat{w} = 2\widehat{\delta}$, $N_1\widehat{\delta} = 3\widehat{\delta}$, and $N_2\widehat{\delta} = 4\widehat{\delta}$.

The ramified base change $t_j = u^{m_j}$ has local equation

$$s_j^{m_j} - u^{m_j} = \prod_{a=0}^{m_j-1} (s_j - e^{2\pi i a/m_j} u) = 0.$$

Its normalization separates all these smooth branches, each with coordinates (z_1, z_2, u). The marked map $y \mapsto (q_j(y), s_j(y))$ identifies the marked torus family J_j' with this normalization, as proved by the explicit branches in Proposition 7.4. A biholomorphism over the identity of D_j base-changes over the identity of the u-disc and lifts uniquely to the normalization: on local rings it carries integral elements in the total fraction ring to integral elements, and its inverse does the same. This lift commutes with the marked deck action because both composites lift the same map $(x, u) \mapsto (F(x), \zeta_j u)$.

Let $c' = c + n$, where n is an integer, and suppose the linear part of the lifted map is aI_2, $a \in \{1, -1\}$. The lifted map on the torus family is $[z, \zeta] \mapsto [z, a\zeta + B_j(z)]$. Its translating section has a holomorphic lift $B_j : \Delta_j \to \mathbb{C}^2$: $\Delta_j \times \mathbb{C}^2 \to J_j'$ is a covering map and the disc is simply connected. Local inverses of this covering show that the lift is holomorphic. The deck-equivariance identity holds modulo a lattice vector, so, after choosing this lift, it is

$$B_j(g_j z) - R_{g_j}(z) B_j(z) = \Pi_c(g_j z) \left(\frac{(1-a)v_j + n\epsilon_j \widehat{\delta}}{m_j} + \ell_j \right), \qquad \ell_j \in \Lambda. \tag{A.5}$$

The integer column ℓ_j is constant: it is locally constant by discreteness of the marked lattice and the disc is connected. Iterating m_j times, with $A_j v_j = v_j$ and $A_j \widehat{\delta} = \widehat{\delta}$, yields the necessary equality

$$N_j \ell_j = -(1-a)v_j - n\epsilon_j \widehat{\delta}. \tag{A.6}$$

Here the left side of the iterated equation is zero because the base point and the marked vector coordinate return after m_j steps; real invertibility of Π_c makes the lattice equality exact.

If $a = -1$, applying γ to (A.6) gives

$$m_j \gamma(\ell_j) = -2\epsilon_j.$$

This is impossible already at $j = 1$, since 3 does not divide 2. Thus a global isomorphism cannot have negative linear part.

For $a = 1$, the first row of (A.6) gives $\gamma(\ell_j) = 0$. Equation (A.4) then proves that the order-three filling imposes no restriction on n, whereas the order-four filling imposes precisely $n \in 2\mathbb{Z}$. More intrinsically, the finite obstruction is the class of $-n\epsilon_j \widehat{\delta}$ in

$$\Lambda^{A_j} / N_j \Lambda. \tag{A.7}$$

In the marked fixed bases this quotient is $\mathbb{Z}/3$ at $j = 1$, with $\widehat{\delta}$ zero, and $\mathbb{Z}/4 \oplus \mathbb{Z}/2$ at $j = 2$, with $\widehat{\delta}$ generating the second summand. Indeed (A.4) gives

$$N_1\Lambda = 3\mathbb{Z}v_1 \oplus \mathbb{Z}\widehat{\delta}, \quad N_2\Lambda = 4\mathbb{Z}\varepsilon' \oplus 2\mathbb{Z}\widehat{\delta}, \qquad \varepsilon' = (1, 3, -3, 0)^{\mathsf{t}}.$$

This proves the parity obstruction.

□

A.2 The effective period parameter

Proposition A.3 (Exact period variation and the Kodaira–Spencer class)**.** [tag 01Q4] *Use the proper family of proposition 7.5 over a disc Q with $\overline{Q} \subset \mathcal{P}$. Fix a regular base point and one of its lifts z, and fix $c_* \in Q$. With all functions below evaluated at that z, set*

$$T = \operatorname{Im}\tau > 0, \quad m = \operatorname{Im}\mu, \quad D = \operatorname{Im}\beta_{c_*} - \frac{6m^2}{T} < 0, \quad a = \frac{m}{T}, \qquad P = \begin{pmatrix} 0 & 0 \\ -a & 1 \end{pmatrix}. \tag{A.8}$$

For $\delta = c - c_*$, the exact marked real-linear map from the universal cover of the fibre at c_* to that at c is

$$w = A_\delta \zeta + B_\delta \overline{\zeta}, \qquad A_\delta = I + \frac{\delta}{2iD} P, \qquad B_\delta = -\frac{\delta}{2iD} P. \tag{A.9}$$

It maps the marked period lattice at c_* onto that at c, and therefore descends to a real-analytic marked diffeomorphism of the two tori. In the explicit convention that the transported $(0,1)$ tangent space is spanned by $\partial_{\overline{\zeta}_k} + \sum_j \nu_k^j \partial_{\zeta_j}$, its exact coefficient matrix and its first derivative are

$$\nu_\delta = \frac{\delta}{2iD + \delta} P, \qquad \dot{\nu}_z = \frac{1}{2iD}\, \partial_{\zeta_2} \otimes (d\overline{\zeta}_2 - a\, d\overline{\zeta}_1). \tag{A.10}$$

The class of $\dot{\nu}_z$ in $H^{0,1}(f_{c_*}^{-1}(t(z)), T^{1,0})$ is nonzero. With the tangent-exact-sequence convention $\mathrm{KS}(\partial_c) = [\bar{\partial} h]$ for a smooth $(1,0)$ lift h of ∂_c, the fibre class is $\mathrm{KS}_z(\partial_c) = -[\dot{\nu}_z]$. Moreover the first-order deformation of the entire X_{c_*} obtained from proposition 7.5 is nontrivial: its Kodaira–Spencer class in $H^1(X_{c_*}, T_{X_{c_*}})$ is nonzero. In particular $h^1(X_{c_*}, T_{X_{c_*}}) \geq 1$.

Proof. 1. *The exact variation on a regular fibre.* Write $u = (u_1, u_2, u_3, u_4)^t$ in the ordered lattice basis and $\zeta = \Pi(z, c_*)u$. Taking imaginary parts gives the two exact equations

$$\operatorname{Im}\zeta_1 = 6mu_1 + Tu_2, \qquad \operatorname{Im}\zeta_2 = (D + 6m^2/T)u_1 + mu_2. \tag{A.11}$$

Subtracting a times the first equation from the second yields

$$u_1 = \frac{\operatorname{Im}\zeta_2 - a \operatorname{Im}\zeta_1}{D}. \tag{A.12}$$

In the changed periods the same u maps to $w = \Pi(z, c)u = \zeta + \delta e_2 u_1$. Substitution of $\operatorname{Im}\zeta = (\zeta - \overline{\zeta})/(2i)$ proves (A.9). This also proves its asserted lattice map and real invertibility: it is exactly $\Pi(z, c)\Pi(z, c_*)^{-1}$ as a real-linear map. Its real determinant is $(D + \operatorname{Im}\delta)/D > 0$.

2. Here $P^2 = P$, and

$$\det A_\delta = 1 + \frac{\delta}{2iD}, \qquad A_\delta^{-1} = I - \frac{\delta}{2iD + \delta} P. \tag{A.13}$$

The denominator can vanish only for $\delta = -2iD$. At that parameter $D + \operatorname{Im}\delta = -D > 0$, which is outside the admissible domain. Thus this inverse is defined at every $c \in Q$. Annihilating w by the declared new $(0,1)$ fields gives $A_\delta \nu_\delta + B_\delta = 0$, which proves (A.10), including its sign. All its coefficients are constant in the fibre coordinates. Thus its first derivative is a globally defined $\bar{\partial}$-closed tangent-valued $(0,1)$ form on the torus.

3. It cannot be $\bar{\partial} V$ for a globally defined smooth vector field V. Indeed the invariant holomorphic frame $\partial_{\zeta_1}, \partial_{\zeta_2}$ writes V with periodic smooth coefficients. The integral of each of their derivatives $\partial_{\overline{\zeta}_k} V^j$ against the translation-invariant probability volume is zero: real constant vector fields preserve this volume, and integration of their derivatives on the compact torus is zero. But the coefficient of $\partial_{\zeta_2} \otimes d\overline{\zeta}_2$ in $\dot{\nu}_z$ is the nonzero constant $1/(2iD)$. Its integral is that same constant. This proves that the Dolbeault class is nonzero.

4. The conversion to the tangent-exact-sequence convention retains a minus sign. In the moving coordinates the real-marking lift has $(1,0)$ part $h = \partial_c + u_1\partial_{w_2}$ at c_*. Since (A.12) gives

$$\bar\partial u_1 = -\frac{d\bar\zeta_2 - a\,d\bar\zeta_1}{2iD},$$

the connecting class $[\bar\partial h]$ is exactly $-[\dot\nu_z]$. Thus the two conventions are related by the displayed sign and both give a nonzero class.

5. *The intrinsic pencil in the parameter family.* Let $\mathcal{S}_2 \subset \mathcal{X}_Q$ be the reduced finite divisor over $p_2 \times Q$, so $F^*(p_2 \times Q) = 4\mathcal{S}_2$. The proof of proposition 7.2 works for relative forms over Q, as can be checked on each chart. On the middle family the frame is $d\zeta_1 \wedge d\zeta_2$ relative to $B \times Q$. At the cusp the relative canonical frame over Q is $(dx_1/x_1) \wedge (dx_2/x_2) \wedge dt_c$. Extra dc terms in transition differentials vanish in relative forms. The finite determinant characters and the holomorphic units in the canonical calculation of proposition 7.2 are independent of c, since R_g, τ, μ are. Thus their coefficient identities hold jointly over Q. The evaluation exponents remain $m_j - 1 - a_j = 0, 2$. It follows by these transition functions and divisors that

$$\omega_{\mathcal{X}_Q/Q} \simeq \mathcal{O}_{\mathcal{X}_Q}(-2\mathcal{S}_2), \qquad \omega_{\mathcal{X}_Q/Q}^{-2} \simeq F^*\,\mathrm{pr}_B^*\,\mathcal{O}_B(p_2). \tag{A.14}$$

Here the base identifications retain $h = 1/(t-1)$ and $\eta_B = dt/(t-1)^2$ from (7.3); these functions also do not depend on c. The joint transition identities prove (A.14) over the whole parameter disc.

6. For its complete degree-two system over the dual numbers, put $A = \mathbb{C}[\epsilon]/(\epsilon^2)$, base-change by $c = c_* + \epsilon$, write X_A for the resulting deformation and $L_A = \omega_{X_A/A}^{-2}$. For $L_* = \omega_{X_{c_*}}^{-2}$, flatness of the deformation and local freeness of L_A give

$$0 \longrightarrow L_* \xrightarrow{\epsilon} L_A \longrightarrow L_* \longrightarrow 0. \tag{A.15}$$

The two sections of $\mathcal{O}_B(p_2)$ used in (7.6), namely σ_{p_2} and $\ell = A_0/(t-1) + B_0'$ with $A_0 \neq 0$, pull back by (A.14) to sections s_0, s_1 of L_A whose reductions are a basis of $H^0(X_{c_*}, L_*)$. This basis assertion is exactly the proved degree-two case of (7.4)–(7.6). Any section of L_A reduces to a complex linear combination of these two reductions; subtract that lifted combination. Exactness of (A.15) then writes the remainder as ϵ times a section of L_*, again a combination of that basis. Conversely, if an A-linear combination of s_0, s_1 is zero, reduce modulo ϵ and then use the injective first arrow of (A.15) to see that all four complex coefficients vanish. Therefore

$$H^0(X_A, L_A) = As_0 \oplus As_1. \tag{A.16}$$

The system is basepoint-free and its morphism is f_A followed by the fixed coordinate map $t \mapsto [t-1 : A_0 + B_0'(t-1)]$. This is the base-change isomorphism for the complete degree-two system.

7. *Nontriviality for the complete threefold.* Suppose its first-order deformation were trivial, with an isomorphism $X_A \simeq X_{c*} \times \operatorname{Spec} A$ reducing to the identity. Intrinsic relative canonical bundles and their complete spaces of sections are preserved by that isomorphism. By (A.16), it would therefore identify f_A with $a_A \circ f_{c*}$ for a projective linear change $a_A \in \mathrm{PGL}_2(A)$ reducing to the identity.

8. This change fixes all three distinguished points even to first order. In the finite covering charts f_A is $t_j = s_j^{m_j}$, so its relative critical ideal is $(s_j^{m_j-1})$, and its restriction to that critical scheme has $t_j = 0$. The map from the parameter ring A to this critical scheme is injective; it contains a constant-parameter section given by $s_j = 0$ and the zero section of the covering torus. At the cusp, in a chart containing a triple point, $t_c = z_0z_1z_2$ and the critical ideal

is (z_1z_2, z_0z_2, z_0z_1). Again $t_c = 0$ in its ring and A injects, using $z_0 = z_1 = z_2 = 0$. At a double chart the same assertion follows from $t_c = z_0z_1$; a smooth chart has no critical point. These critical ideals are intrinsic to the differential of the map over A, and the charts are local biholomorphisms onto X_A. Away from the three distinguished base values the map is a submersion, so it has no critical point. Consequently the scheme-theoretic critical-value sections are exactly the three constant sections p_1, p_2, p_0. An automorphism reducing to the identity cannot permute these three distinct closed points. Its value at each is therefore the same point over A, not a nonzero infinitesimal displacement: such a displacement would be a nonzero element of $\epsilon\mathbb{C}$ in an injective copy of A on the just displayed critical chart.

9. Write $a_A(t) = t + \epsilon(b_0 + b_1t + b_2t^2)$. Fixing 0 gives $b_0 = 0$, fixing 1 gives $b_0 + b_1 + b_2 = 0$, and fixing infinity, in $u = 1/t$, gives $b_2 = 0$. Hence a_A is the identity. The proposed trivialization thus preserves f, and restricting it to any regular base value would trivialize the torus deformation (A.9). Its nonzero class (A.10) excludes this.

10. Finally, the zero Kodaira–Spencer class is precisely the condition for such a first-order trivialization. One may see the needed direction directly: choose local product charts for the holomorphic submersion π and differentiate their transition functions at c_*. The resulting tangent-valued overlap functions form a Čech 1-cocycle. If its class vanishes, after refinement it is the coboundary of local holomorphic vector fields. Changing each chart by that vector field times ϵ cancels the first-order overlap terms, producing the identity-reducing trivialization. This is the usual Kodaira–Spencer connecting class (for the standard deformation-space interpretation see (McMullen, 1996, Section 67, p. 15)); the displayed conversion from the plus-graph convention in (A.10) changes its sign, not its vanishing. The contradiction proves the final assertion.

□

B Tangent cohomology and deformations

B.1 Tangent direct images and local miniversality

Remark B.1 (The standard deformation theorem)**.** [tag PI11] For the standard data $(\ell_0, \ell_1, \ell_2) = (0, 1, -1)$, the tangent dimensions and smooth reduced Kuranishi germ are given in (Du et al., 2026a, Theorem 6.12, pp. 81–82; Proposition 6.16, p. 85). The weighted calculation also covers the coprime multiplicities stated below, and the standard case includes the coarse tangent direct images.

Theorem B.2 (Tangent cohomology, direct images and miniversality)**.** [tag 032U] *Let k, l be positive integers with $\gcd(3k, 4l) = 1$, and let X be a sphere member with multiplicities $(3k, 4l)$ and the admissible logarithmic data of section 9.1. Put $z = kl$ and $d = 3k + 4l$. On $\mathfrak{B}_{k,l} = \mathbb{P}(3k, 4l)$, the vertical root bundle E, the root lift $\bar{f}$ and its direct images satisfy*

$$\begin{aligned} E &= \mathcal{O} \oplus \mathcal{O}(-z), \\ R^q\bar{f}_*\mathcal{O}_X &= \mathcal{O}, \quad \mathcal{O} \oplus \mathcal{O}(-z), \quad \mathcal{O}(-z), \\ K_X &= \bar{f}^*\mathcal{O}(z - d). \end{aligned}$$

The tangent cohomology is $(h^0, h^1, h^2, h^3)(X, T_X) = (1, 1, 1, 0)$. For the transition-arrow convention $\kappa(\Phi) = (\partial_c\Phi)\Phi^{-1}$, the Kodaira–Spencer class of ∂_c is $e \otimes [\gamma]$, and the class of ∂_u is $-(2\pi iu)^{-1}e \otimes [\gamma]$. Both classes are nonzero. The primary Kuranishi bracket vanishes, the Kuranishi germ is a smooth reduced disc, and the fixed-data period family is locally miniversal.

For the standard $(3, 4)$ data, $e = \xi$ and the coarse direct images are

$$f_*T_X = \mathcal{O} \oplus \mathcal{O}(-1), \qquad R^1f_*T_X = \mathcal{O} \oplus \mathcal{O}(-1)^2, \qquad R^2f_*T_X = \mathcal{O}. \tag{B.1}$$

Proof. 1. The root lift, its proper flat Gorenstein charts, base change and relative duality are those of proposition 12.1. Write $m = 3k$ and $n = 4l$. The finite root charts have the specified free affine cyclic actions, and the cusp charts are $q = z_0 z_1 z_2$.

2. Use weighted coordinates x, y of weights m, n, with coarse coordinate $t = y^m/(y^m - x^n)$. The two root divisors have bundles $\mathcal{O}(n)$ and $\mathcal{O}(m)$, and the cusp has bundle $\mathcal{O}(mn)$. The root-forgetting morphism

$$r : \mathfrak{B}_{k,l} \longrightarrow \mathfrak{B}_{1,1}, \qquad (x, y) \longmapsto (x^l, y^k),$$

uses the group homomorphism $\lambda \mapsto \lambda^{kl}$ on the weighted quotient. It is a holomorphic morphism of analytic stacks over B, with local maps $s \mapsto s^k$ and $s \mapsto s^l$. In particular $r^*\mathcal{O}(1) = \mathcal{O}(z)$. The analytic Picard group of this root curve has the presentation $\langle H, R_1, R_2 \mid mR_1 = H,\ nR_2 = H\rangle$: after subtracting the two stabilizer characters, a line bundle descends to the coarse projective line. Its only relations are the two defining root identities. The Smith form and $\gcd(m, n) = 1$ identify this group with $\mathbb{Z}$, with $H = mn\mathcal{O}(1)$, $R_1 = n\mathcal{O}(1)$, and $R_2 = m\mathcal{O}(1)$. This proves the analytic torsion-free Picard assertion used below. Translations have identity vertical derivative and act trivially on structure-sheaf cohomology. Thus the vertical bundle computed from the linear period factors is $r^*(\mathcal{O} \oplus \mathcal{O}(-1))$. The fixed vector gives a subline $\mathcal{O}$; its extension splits because $H^1(\mathcal{O}(z)) = 0$.

3. Under the marked root-forgetting map in the preceding step, the constant lattice cocycle γ is unchanged. Its finite-chart representatives and its cusp edge values $(0, -1, -1)$ therefore give the nowhere-zero cohomology line proved in proposition 12.2. The cusp resolution and cup-product pairing are those of proposition 12.1; pulling the linear period factors through r gives the quotient line $\mathcal{O}(-z)$. Relative duality gives $R^2\bar{f}_*\mathcal{O}_X = \mathcal{O}(-z)$ and $\det R^1\bar{f}_*\mathcal{O}_X = \mathcal{O}(-z)$. The extension of that quotient by the specified $[\gamma]$-line splits because $H^1(\mathcal{O}(z)) = 0$, as computed in step 5. Connected fibres give $\bar{f}_*\mathcal{O}_X = \mathcal{O}$. These maps prove the three structure-sheaf direct images.

4. The logarithmic tangent sequence is

$$0 \longrightarrow \mathcal{V} = \bar{f}^*E \longrightarrow \mathcal{T} = T_X(-\log W) \longrightarrow \mathcal{H} = \bar{f}^*\mathcal{O}(d - 12z) \longrightarrow 0.$$

Its last map is the relative differential on smooth root charts and sends $\sum a_i z_i \partial_{z_i}$ to $(\sum a_i) q\partial_q$ at the cusp. It is surjective there. Since $K_{\mathfrak{B}} = \mathcal{O}(-d)$, determinants give $K_X = \bar{f}^*\mathcal{O}(z - d)$.

5. The two quotient charts $x \neq 0$, $y \neq 0$ are finite quotients of affine lines, and their overlap is a finite quotient of a punctured line. Finite averaging and Stein acyclicity compute coherent cohomology by this two-chart complex. Its homogeneous Laurent monomials give

$$h^0\mathcal{O}(a) = \#\{(i, j) \in \mathbb{Z}^2_{\geq 0} : mi + nj = a\},$$
$$h^1\mathcal{O}(a) = h^0\mathcal{O}(-a - d).$$

The second equality identifies the remaining monomials $x^{-i-1}y^{-j-1}$.

The bundles $\mathcal{O}(-z), \mathcal{O}(-2z)$ are acyclic. Indeed a possible degree-one monomial would give $m(i + 1) + n(j + 1) = az$, with $a = 1$ or 2. Coprimality forces $i + 1 = lA$, $j + 1 = kB$, hence $3A + 4B = a$ with $A, B \geq 1$, which is impossible. Positive degree has no degree-one monomials, so $H^1\mathcal{O}(z) = 0$ as used above.

6. Put $b = d - 12z$. Both b and $b - z$ are negative. Degree-one monomials for $\mathcal{O}(b)$ would satisfy $3A + 4B = 12$, with $A, B \geq 1$, and none exist. For $\mathcal{O}(b - z)$ the equation is $3A + 4B = 13$; its only positive solution is $(A, B) = (3, 1)$, producing $x^{3l-2}y^{k-2}$ in the dual description. It is allowed exactly when $k > 1$; write $e_k = 1$ in that case and $e_k = 0$ otherwise. Projection formula and the two-column Leray sequence now give

$$h^\bullet(\mathcal{V}) = (1, 1, 0, 0), \qquad h^\bullet(\mathcal{H}) = (0, 0, e_k, e_k), \qquad h^\bullet(\mathcal{T}) = (1, 1, e_k, e_k).$$

7. On the normalization $\nu : P = \mathrm{dP}_6 \to W \subset X$, the branchwise normal line is K_P: the invariant cusp form $d\log x_1 \wedge d\log x_2 \wedge dq$ trivializes K_X on the entire cusp neighbourhood. Local normal coefficients therefore give the exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow T_X \longrightarrow \nu_* K_P \longrightarrow 0, \qquad h^\bullet(K_P) = (0, 0, 1).$$

For $k > 1$, the dual logarithmic sequence, tensored with K_X, has horizontal subline $\bar{f}^*\mathcal{O}(13z - 2d)$ and vertical quotient $\bar{f}^*(\mathcal{O}(z - d) \oplus \mathcal{O}(2z - d))$. The latter has no global sections, by the same equations $3A + 4B = 1, 2$ above. Consequently

$$H^0(\Omega^1_X(\log W) \otimes K_X) = H^0(\mathfrak{B}, \mathcal{O}(13z - 2d)).$$

For $k > 1$ this is generated by the monomial $x^{3l-2}y^{k-2}$ multiplying the horizontal logarithmic differential. Both x and y are nonzero over p_0, so its residue on P is nonzero. The ordinary logarithmic residue sequence, tensored with K_X, identifies this residue map with the Serre dual of $H^2(P, K_P) \to H^3(X, \mathcal{T})$. One can check this identification directly in each chart $q = z_0z_1z_2$: the residues of dq/q are one on each branch, and contraction with the invariant three-form is the normal coefficient pairing. Thus that connecting map has rank e_k. The long exact sequence proves $h^2(T_X) = 1$ and $h^3(T_X) = 0$; degrees zero and one are unchanged.

8. The derivative $\partial_c \Pi = e \otimes \gamma$ gives the constant groupoid cocycle $\lambda \mapsto \gamma(\lambda)e$ on every regular torus. On the enlarged finite roots the derivative of the affine generator is $(r_j/m_j)e$, and the derivative of its punctured connector is $(\log s_j)/(2\pi i)\, r_j e$. The identity $m_j(r_j/m_j) = \gamma(w_j)$ verifies the power relation, while $\gamma A_j = \gamma$ verifies conjugation with every lattice arrow. Changing $\log s_j$ by $2\pi i N$ adds the derivative of the integral arrow Nw_j; the connector formula is used only on the punctured overlap. Its central root-chart representative is the constant $(r_j/m_j)e$, which is holomorphic. At the cusp γ kills the toric periods, so the constant deck cocycle is exactly its extending cocycle above. Its image under the two isomorphisms $H^1(\mathcal{V}) \simeq H^1(\mathcal{T}) \simeq H^1(T_X)$ is $e \otimes [\gamma] \neq 0$. The chain rule gives the asserted u-class. With the tangent-lift convention $[\bar{\partial}h]$, this class changes sign, as the explicit lift comparison in proposition A.3 shows. The parameter and the transition-arrow convention in the statement specify the displayed sign.

9. Let α denote the nonzero tangent class just constructed. The holomorphic family gives a convergent deformation with this tangent. Differentiating its integrability equation gives $\bar{\partial}\alpha_2 = -[\alpha, \alpha]/2$. Thus the primary bracket vanishes; bilinearity and $h^1(T_X) = 1$ give vanishing on every pair.

   By (Douady, 1974, §VII.8, principal theorem, pp. 598–599; §VIII.1, Theorem 1 and its corollary, p. 600), there is a minimal versal analytic germ with tangent space $H^1(T_X)$. The identification of this tangent space uses tangent-valued Čech cocycles of transition maps, with changes of trivialization acting by coboundaries. Its local algebra is $\mathbb{C}\{t\}/I$. The parameter family induces $t \mapsto g(c)$ with $g'(c_0) \neq 0$. By the analytic inverse function theorem, substitution by g is injective; it annihilates I, so $I = 0$. The classifying map is an isomorphism of germs. Pullback of the versal family realizes every sufficiently small deformation and proves local miniversality.

10. For the standard data we compute the coarse direct images by the following Koszul maps. All root bundles in this step are on $\mathcal{B} = \mathbb{P}(3, 4)$.

    (a) Put $L = \mathcal{O}_{\mathcal{B}}(-5)$. The first connecting map of (12.5) is the holomorphic bundle map $\kappa : L \to E \otimes G$. In the splittings fixed in 12.2 the coefficient in $\mathcal{O}\xi \otimes \mathcal{O}\gamma$ lies in $H^0(\mathcal{O}(5)) = 0$. The remaining coefficients lie in $H^0(\mathcal{O}(4)) = \mathbb{C}y$, twice, and $H^0(\mathcal{O}(3)) = \mathbb{C}x$. Therefore, as a map from E^* to G with its L factor understood,

$$\kappa = \begin{pmatrix} 0 & ay \\ by & c_0 x \end{pmatrix}. \tag{B.2}$$

We use c_0 for this coefficient to distinguish it from the deformation parameter. The logarithmic period derivative at the cusp has leading term the invertible matrix $B_0/(2\pi i)$. Hence $\det\kappa = -aby^2$ is nonzero there, so $ab \neq 0$. At the order-three point $\tau - \rho$ has order one on the root chart: $j = 1728t$ and the elliptic ramification order of j at ρ is three. The marked torus period derivative is consequently nonzero there: the two constant identity columns in the normalized period matrix exclude an infinitesimal linear-frame change that could otherwise conceal a nonzero derivative. In (B.2), $y = 0$ and $x \neq 0$ at that point, so $c_0 \neq 0$. For the following matrices only, write the frames of E, G as (e_1, e_2) and (g_1, g_2) and replace them by

$$e_1' = (a/c_0)e_1, \quad e_2' = e_2, \qquad g_1' = bg_1, \quad g_2' = c_0 g_2.$$

The coefficients of $e_1' \otimes g_2'$, $e_2' \otimes g_1'$ and $e_2' \otimes g_2'$ are then y, y, x. The geometric cocycle $\xi \otimes \gamma$ keeps its original meaning; $e_1' \otimes g_1' = (ab/c_0)\xi \otimes \gamma$ in these matrix frames.

(b) The next connecting map is cup product with κ. In suitable bases it is

$$L \otimes G = \mathcal{O}(-5) \oplus \mathcal{O}(-6) \xrightarrow{\delta_1} E \otimes \det G = \mathcal{O}(-1) \oplus \mathcal{O}(-2), \qquad \delta_1 = \begin{pmatrix} -y & 0 \\ -x & y \end{pmatrix}. \tag{B.3}$$

The sign follows by wedging first with γ and then with the second generator of G; reversing both sign conventions leaves the cokernels unchanged. As its determinant is $-y^2$, it is injective as a morphism of sheaves. The first connecting map is $\delta_0 = (0, y, y, x)^t$ and is a subbundle inclusion, since x, y have no common zero. Its cokernel is

$$\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(2). \tag{B.4}$$

Use the ordered frames $u = e_1' \otimes g_2' + e_2' \otimes g_1'$ and $v = e_1' \otimes g_2'$ for the two $\mathcal{O}(-1)$ summands. The change matrix has determinant -1, the image is $yu + xe_2'g_2'$, and v gives the unused summand. The remaining quotient is the exact sequence $0 \to \mathcal{O}(-5) \xrightarrow{(y,x)} \mathcal{O}(-1) \oplus \mathcal{O}(-2) \xrightarrow{(x,-y)} \mathcal{O}(2) \to 0$.

(c) The cokernel of (B.3) is supported at the order-three root point. There x is a unit. For target generators q_1, q_2, the relations are $-yq_1 - xq_2 = 0$ and $yq_2 = 0$. Eliminating $q_2 = -(y/x)q_1$ gives the single relation $y^2 q_1 = 0$ and hence

$$Q = \mathcal{O}(-1)\big|_{2p_1}, \tag{B.5}$$

where $2p_1$ is the length-two divisor in the root coordinate y. The generator is the first target component, and its annihilator is y^2. The long direct-image sequence now gives

$$R^0\bar{f}_*\mathcal{T} = E, \quad R^1\bar{f}_*\mathcal{T} = \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(2),$$
$$0 \longrightarrow Q \longrightarrow R^2\bar{f}_*\mathcal{T} \longrightarrow \mathcal{O}(-6) \longrightarrow 0. \tag{B.6}$$

This last extension splits, since $\mathrm{Ext}^1(\mathcal{O}(-6), Q) = H^1(Q(6)) = 0$.

(d) Let $\nu : P = \mathrm{d}P_6 \to W \hookrightarrow X$ be the normalization immersion. There is an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow T_X \longrightarrow \nu_* K_P \longrightarrow 0. \tag{B.7}$$

To verify the quotient, in a chart $W = (z_1 \cdots z_r = 0)$ the map takes a vector field to its normal coefficient on each of the r separate branches. The coefficients lift independently; its kernel is precisely the vector fields tangent to every branch. Thus the normal-coefficient map identifies the quotient with the pushforward of the normal line of ν. The form $d\log x_1 \wedge d\log x_2 \wedge dq$ becomes a unit times $dz_1 \wedge dz_2 \wedge dz_3$ in every unimodular chart and is invariant under every cusp deck transformation. It trivializes $\nu^* K_X$. Adjunction for the immersion gives its normal line K_P, proving (B.7). Since P is rational, its K_P cohomology is $(0, 0, 1)$ by Serre duality. Finite pushforward preserves that cohomology.

(e) For the degree-two extension at p_0, the kernel of

$$\Omega^1_X|_W \longrightarrow \nu_*\Omega^1_P$$

is $\nu_*\mathcal{O}_P \otimes N^*_W$. At $xy = 0$ its sections are $a(y,z)y\,dx + b(x,z)x\,dy$; at $xyz = 0$ they are $a(y,z)yz\,dx + b(x,z)xz\,dy + c(x,y)xy\,dz$. These identifications glue by the common conormal dq. Since N_W and $K_X|_W$ are trivial and $H^0(P,\Omega^1_P) = 0$, the Serre-dual space of $H^2(W, T_X|_W)$ is one-dimensional. Top-degree base change identifies this group with the residue fibre of $(R^2\bar{f}_*T_X)_{p_0}$. Its generic rank is one by (B.6). A finite module over the discrete valuation ring $\mathbb{C}\{q\}$ of rank one and residue dimension one is free of rank one. Thus the extension by $\mathbb{C}_{p_0}$ increases the degree of its line part by p_0, without adding torsion. We obtain the complete root direct images

$$R^0\bar{f}_*T_X = \mathcal{O} \oplus \mathcal{O}(-1), \quad R^1\bar{f}_*T_X = \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(2),$$
$$R^2\bar{f}_*T_X = \mathcal{O}(6) \oplus Q. \tag{B.8}$$

On the order-three chart, an invariant coefficient of the generator of Q must have exponent 2 modulo 3, whereas Q only retains exponents $0, 1$. Therefore $\rho_*Q = 0$. Applying the exact ρ_* to (B.8) and using $d(0) = 0, d(-1) = d(2) = -1, d(6) = 0$ proves (B.1). The coarse direct images are the ones displayed in the statement; their Leray cohomology agrees with the tuple obtained in step 7.

□

C The integral parameter comparison

C.1 Ordinary tangent sheaves and the conductor

Definition C.1 (The double locus and its tangent quotient)**.** [tag PI12]

For the $(3,4)$ family, let $D_{\mathrm{dbl}} = (\operatorname{Sing} W)_{\mathrm{red}}$. It consists of three rational curves joining the two triple points. Put

$$L = f^*\mathcal{O}_B(-1) \otimes \mathcal{O}_X(2S_1 + S_2), \qquad F = \mathcal{O}_X(S_1 + S_2), \qquad \mathcal{V} = \ker(df), \qquad J = \mathcal{I}_{D_{\mathrm{dbl}}} \otimes F.$$

The root conventions give $\rho^*\mathcal{O}_B(1) = \mathcal{O}_{\mathcal{B}}(12)$, $\mathcal{O}_X(S_1) = \bar{f}^*\mathcal{O}_{\mathcal{B}}(4)$ and $\mathcal{O}_X(S_2) = \bar{f}^*\mathcal{O}_{\mathcal{B}}(3)$. The tensor maps on their canonical divisor sections therefore give

$$L = \bar{f}^*\widehat{L}, \quad \widehat{L} = \mathcal{O}_{\mathcal{B}}(-1), \qquad F = \bar{f}^*\widehat{F}, \quad \widehat{F} = \mathcal{O}_{\mathcal{B}}(7).$$

The weights are $-12+2\cdot4+3 = -1$ and $4+3 = 7$. The vertical root bundle is E of proposition 12.2.

Theorem C.2 (The ordinary tangent sequence)**.** [tag 02YM] *The sheaf $\mathcal{V}$ is locally free of rank two and equals $\bar{f}^*E$. The differential induces an exact sequence*

$$0 \longrightarrow \mathcal{V} \longrightarrow T_X \longrightarrow J \longrightarrow 0. \tag{C.1}$$

Proof. On a finite root chart $t = s^m$, the differential has image $(s^{m-1})f^*T_B$. The free cyclic quotient makes the root chart étale over X, so the orders along S_1, S_2 are $2, 3$. At a cusp double point $q = xy$, its image is (x, y); at a triple point $q = xyz$, its image is (xy, xz, yz), the reduced ideal of the three axes. At a smooth point of W, it is surjective. Thus the global image is

$$\mathcal{I}_{D_{\mathrm{dbl}}} \otimes f^*T_B \otimes \mathcal{O}_X(-2S_1 - 3S_2) = J.$$

Here $t(t-1)\partial_t$ trivializes $T_B(-p_1 - p_2)$, and $f^*(p_1) = 3S_1$, $f^*(p_2) = 4S_2$ identify the remaining line factor with $\mathcal{O}_X(S_1 + S_2)$. The kernel and its finite, toric and period-transition maps were computed in proposition 12.2; those evaluation maps identify it with $\bar{f}^*E$, including at the double and triple points. □

Theorem C.3 (The ordinary direct images)**.** [tag 02YN] *For the sheaves of definition C.1, the coarse direct images are*

	f_*	$R^1 f_*$	$R^2 f_*$
$\mathcal{O}_X$	$\mathcal{O}$	$\mathcal{O} \oplus \mathcal{O}(-1)$	$\mathcal{O}(-1)$
L	$\mathcal{O}(-1)$	$\mathcal{O}(-1)^{\oplus 2}$	$\mathcal{O}(-1)$
F	$\mathcal{O}$	$\mathcal{O}^{\oplus 2}$	$\mathcal{O}$
$\mathcal{V}$	$\mathcal{O} \oplus \mathcal{O}(-1)$	$\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 3}$	$\mathcal{O}(-1)^{\oplus 2}$
J	$\mathcal{O}(-1)$	$\mathcal{O}(-1)^{\oplus 2}$	$\mathcal{O}$.

Proof. 1. The root splitting and structure-sheaf direct images are given in propositions 12.1 and 12.2; in the standard data these are also computed in (Du et al., 2026a, Proposition 6.6, pp. 77–78; Lemma 6.7, p. 78; Proposition 6.10, pp. 79–80). The determinant line in these conventions is

$$\det E = \widehat{L} = \rho^* \mathcal{O}_B(-1) \otimes \mathcal{O}_{\mathcal{B}}(2p_1/3 + p_2/4) = \mathcal{O}_{\mathcal{B}}(-1). \tag{C.2}$$

The equality is the tensor product of the divisor-section maps in definition C.1; its degree is $-12 + 8 + 3$. For integers d, a, b, the invariant root-line formula is

$$\rho_*(\rho^* \mathcal{O}_B(d) \otimes \mathcal{O}_{\mathcal{B}}(ap_1/3 + bp_2/4)) = \mathcal{O}_B(d + \lfloor a/3 \rfloor + \lfloor b/4 \rfloor). \tag{C.3}$$

Indeed writing $a = 3q + r$, $b = 4q' + r'$ with $0 \leq r < 3$, $0 \leq r' < 4$ separates the coarse divisor $(d + q + q')$ from the two root characters. This is the same invariant-section map as proposition 2.5, and applies to all integer a, b.

2. For a root line bundle A, projection and exact finite invariants give

$$R^q f_* \bar{f}^* A = \rho_*(A \otimes \bigwedge^q (\mathcal{O} \oplus \widehat{L})).$$

For $A = \mathcal{O}, \widehat{L}, \widehat{F}$, and for $E = \mathcal{O} \oplus \widehat{L}$, the formula (C.3) gives the first four rows. The root-to-coarse map, including the full scheme fibres, is proposition 12.5.

3. For the last row one must compute restriction to D_{dbl}. The conductor square gives

$$0 \to \mathcal{O}_W \to \nu_* \mathcal{O}_S \oplus \mathcal{O}_{D_{\mathrm{dbl}}} \to \nu_* \mathcal{O}_{\partial S} \to 0,$$

where ∂S is the hexagon. Here $h^1(\mathcal{O}_{\partial S}) = 1$ and $h^1(\mathcal{O}_{D_{\mathrm{dbl}}}) = 2$. The equality conditions at the two triple points give incidence matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Orient the three theta-graph edges from P to Q. The hexagon traversal maps to $e_1 - e_2 + e_3 - e_1 + e_2 - e_3 = 0$. The constant-to-coherent branch comparison is an isomorphism in degree one, so the induced map $H^1(D_{\mathrm{dbl}}, \mathcal{O}_{D_{\mathrm{dbl}}}) \to H^1(\partial S, \mathcal{O}_{\partial S})$ is zero. The conductor sequence then makes $H^1(W, \mathcal{O}_W) \to H^1(D_{\mathrm{dbl}}, \mathcal{O}_{D_{\mathrm{dbl}}})$ an isomorphism.

4. The canonical section of F is nonzero near W, so $F|_{D_{\mathrm{dbl}}} = \mathcal{O}_{D_{\mathrm{dbl}}}$. In $0 \to J \to F \to \mathcal{O}_{D_{\mathrm{dbl}}} \to 0$, the maps of direct images $\mathcal{O} \to \mathbb{C}_{p_0}$ and $\mathcal{O}^{\oplus 2} \to \mathbb{C}_{p_0}^{\oplus 2}$ are evaluation and evaluation followed by this isomorphism. They are surjective, with kernels $\mathcal{O}(-p_0)$ in the indicated ranks. Finally $H^2(D_{\mathrm{dbl}}, \mathcal{O}_{D_{\mathrm{dbl}}}) = 0$ and $R^2 f_* F = \mathcal{O}$. This gives the last row.

□

C.2 The full automorphism group

Theorem C.4 (The full automorphism group)**.** [tag 02YQ]
For every admissible twist of the $(3,4)$ *construction,*

$$\mathrm{Aut}(X_c) = \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*.$$

The parameter $a \bmod \mathbb{Z}$ *acts by* $\zeta \mapsto \zeta + ae$*, where* $e = \Pi_c \widehat{\delta}$*. The exponential group isomorphism is* $a \bmod \mathbb{Z} \mapsto e^{2\pi i a}$*.*

Proof. 1. Proposition 7.2 identifies the complete anticanonical pencil with f and proves that every automorphism induces the identity on its base.

2. Apply the affine-lift and centralizer calculation of proposition A.2 with $c' = c$. The period equation (A.3) gives the integral linear part $P = aI + bE_{41}$, $a = \pm 1$, and the complex linear part $H = aI$; its first column gives $b = 0$.

3. The finite-root equation (A.6), with $n = 0$, gives $N_1 \ell = -(1-a)v_1$. For $a = -1$, applying the invariant covector γ gives $3\gamma(\ell) = -2l_1$, where $l_1 = \gamma(v_1)$. Freeness gives $3 \nmid l_1$, so this integer equality is impossible. Thus $a = 1$, and every automorphism is fibrewise translation.

4. Let j be the inclusion of the cusp-punctured orbifold and set $\mathcal{H} = j_*\Lambda$. Its cusp stalk is $\ker(M_0 - I) = \mathbb{Z}\widehat{w} \oplus \mathbb{Z}\widehat{\delta}$. Let $\mathcal{A}$ be the sheaf of fibrewise translations extending to the local compactification. There is an exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow E \longrightarrow \mathcal{A} \longrightarrow 0. \tag{C.4}$$

For the local check at the cusp, use the toroidal covering whose component lattice is $\Gamma = \Lambda/\Lambda_{\mathrm{tor}}$. A translation acts identically on the nearby fibre lattice and on Γ, and therefore lifts. Its meridian remains bounding a disc through a smooth point of W, which verifies the same assertion for the full local cover. On the punctured torus the lift has form $x \mapsto a(q)x$. It sends the component E_0 to E_v; the index-one map $B_0 : \Gamma \to \mathbb{Z}^2$ is an isomorphism, so composing with the unique corresponding deck shear makes it preserve E_0. It then preserves the torus $E_0 \setminus \partial E_0$. Evaluation at $(1,1)$ in the chart $(\mathbb{C}^*)^2 \times \Delta_q$ shows that $a_i(q)$ are holomorphic units at zero, and hence have logarithms. Conversely their multiplication extends on every toric chart and commutes with the deck shears. The logarithm kernel consists precisely of those periods with no logarithmic term, namely $\ker(M_0 - I)$. On a finite root disc, a holomorphic lift has lattice defect in $\ker N_j = (I - A_j)\Lambda$, by the saturated-image calculation of proposition 7.4. Subtracting a period kills the defect and preserves holomorphy. This gives the same exact sequence with its finite equivariance.

5. Put

$$M_0 = (A_1 A_2)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

A cocycle on the punctured orbifold is a pair (x_1, x_2) with $N_1 x_1 = N_2 x_2 = 0$, for the norm matrices in (A.4). Extension over the cusp says that $x_1 + A_1 x_2$ vanishes in $\mathrm{coker}(M_0 - I)$, so its first two coordinates vanish. The complete integral solution is

$$x_1 = (0, -b, 2b - 3c, c), \qquad x_2 = (0, -2a - b, b, a), \quad a, b, c \in \mathbb{Z}.$$

The principal cocycle of (g, u, w, d) has parameters

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 & 0 & 1 \\ -6 & 1 & -1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} g \\ u \\ w \end{pmatrix}.$$

This matrix has determinant -1; its inverse is explicitly $g = c - a - b$, $u = 3c - 2a - 2b$, $w = 4a + 3b - 3c$. Hence $H^1(\mathcal{B}, \mathcal{H}) = 0$ integrally.

6. Sequence (C.4) now makes every global translation a global logarithm in $H^0(\mathcal{B}, E) = \mathbb{C}$. Its kernel is $\Lambda^{A_1,A_2} = \mathbb{Z}\widehat{\delta}$. The flow extends by commutation with the affine finite actions and by torus multiplication on the cusp charts. This proves existence, faithfulness, and completeness of the asserted group.

□

Theorem C.5 (Integral torsor classes and the parameter period)**.** [tag 02YR]

For fixed admissible twist data, $X_c \simeq X_{c'}$ precisely when $c - c' \in 2\mathbb{Z}$. This fixes the integral period of c; the construction still permits an additive choice of β^{part}.

Proof. 1. The analytically computed complete linear system $|\omega_{X_c}^{-2}|$ is the marked map f_c, by proposition 7.2. A biholomorphism preserves its canonical bundle, the complete space of sections, and its evaluation map. Consequently it induces $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ with $f_{c'}F = \alpha f_c$. The three special fibres have distinct intrinsic types: the cusp is reduced and nonnormal, while the other two have multiplicities three and four. All remaining fibres are smooth reduced tori. Thus α fixes p_0, p_1, p_2 individually. In the marked coordinate, a fractional-linear map fixing $\infty, 0, 1$ has zero lower-left and upper-right entries and equal diagonal entries; hence $\alpha = \mathrm{id}_B$.

The centralizer computation of proposition A.2 then gives $P = aI + bE_{41}$ with $a = \pm 1$ and $b = a(c - c')$. The root-fibre character calculation in that proposition excludes $a = -1$. Thus $b = c - c'$ is an integer.

2. Compute $H^2(\mathcal{B}, \mathcal{H})$ with a pair of pants, two root discs, and a cusp disc. The pair of pants has cochains $\Lambda \to \Lambda^2$ with differential $((A_1 - I), (A_2 - I))$. A root disc has the periodic cyclic resolution with alternating differentials $A_j - I, N_j$; the cusp disc has $\ker(M_0 - I)$ in degree zero. An annulus has $\Lambda \xrightarrow{A_j - I} \Lambda$. The degree-one restriction from the pair of pants is

$$(x_1, x_2) \longmapsto (x_1, x_2, -M_0(x_1 + A_1 x_2)).$$

The total-complex ranks in degrees zero through three are $14, 28, 20, 8$. The resulting integer differentials have sizes 28×14, 20×28 and 8×20, and their consecutive products vanish by the restriction identities. Smith decompositions with unimodular changes of basis identify each integral kernel before quotienting by the preceding image, giving $H^0 = \mathbb{Z}$, $H^1 = 0$, $H^2 = \mathbb{Z} \oplus \mathbb{Z}/2$.

3. The degree-two presentation also reduces by hand. The invariant lattices have bases $(\widehat{\delta}, \varepsilon_j)$, where $\varepsilon_1 = (1, 2, -4, 0)$ and $\varepsilon_2 = (1, 3, -3, 0)$. The five generators $(\delta_1, \varepsilon_1, \delta_2, \varepsilon_2, h_c)$ have relations

$$\delta_1 = 0, \qquad 2\delta_2 = 0, \qquad 3\varepsilon_1 = h_c = 4\varepsilon_2. \tag{C.5}$$

For the constant sheaf $\mathbb{Z}$, the same computation has generators z_1, z_2, h_c and relations $3z_1 = h_c = 4z_2$; its primitive generator is $z_1 - z_2$. Inclusion $\mathbb{Z}\widehat{\delta} \to \mathcal{H}$ sends it to the nonzero order-two class δ_2. The character γ sends ε_j to z_j and kills δ_j.

4. Both $H^1(\mathcal{B}, E)$ and $H^2(\mathcal{B}, E)$ vanish, by $\rho_* E = \mathcal{O} \oplus \mathcal{O}(-1)$ and the dimension of the coarse curve. Consequently (C.4) identifies compactified translation torsors $H^1(\mathcal{B}, \mathcal{A})$ with $H^2(\mathcal{B}, \mathcal{H})$. This identification takes logarithms of local translation transitions and their lattice-valued Čech coboundaries; the two vanishings also reverse that construction.

5. The cyclic norm of the translation v_j/m_j is v_j, while the cusp collar has the opposite boundary orientation. Thus the torsor character is

$$l_1 z_1 + l_2 z_2 - l_0 h_c = (4l_1 + 3l_2 - 12l_0)(z_1 - z_2) = -p(z_1 - z_2).$$

The shear $S_b - I = b\widehat{\delta} \otimes \gamma$ changes its class by $-bp\delta_2$. Freeness forces l_2 odd and thus p odd. The class is preserved precisely for even b.

6. For an integral shear the Jacobian charts themselves identify: the period lattices differ by S_b, and the bounded cusp matrix changes by an integer in its β entry. Every toric unit $\exp(2\pi i C\lambda)$ is therefore unchanged. The fan, lattice, and toric gluings are the same, and on root discs the basis change identifies the smooth Jacobian and its cyclic action. Equality of the torsor classes now gives a holomorphic isomorphism for the fixed sheaf $\mathcal{A}$, which consists precisely of translations extending over the cusp. It extends on all three special charts. Conversely every isomorphism preserves the class, so forces b even.

□

Part I
Results not yet established

D Open arithmetic questions for the normalized norm

The arithmetic nature of $N(P)$ remains open for algebraic nontorsion points in this family. An explicit instance is

$$E : y^2 = x^3 - 140x + 784, \qquad P = (0, 28), \qquad j = -3375, \qquad N(P) = 2e^{-\widehat{h}_O(P)}.$$

The point is primitive modulo rational two-torsion, and $\widehat{h}_O(P) = 0.46736568\ldots$. The unresolved assertion is transcendence of $2e^{-\widehat{h}_O(P)}$. For the conductor-361 example $y^2 + y = x^3 - 38x + 90$, $P = (0, 9)$, the corresponding exact formula is

$$N(P) = e^{-\widehat{h}_O(P)} = \exp\!\left(-\frac{L'(E,1)}{4\Omega_E}\right).$$

Here every finite local contribution vanishes. The compact-point equivalence in proposition 26.3 and the explicit constructions in propositions 26.14 and 26.17 give precise group-theoretic forms of this same unresolved value.

The exact analytic leading quotients in remark 24.13 also leave group-order questions: the orders of the Tate–Shafarevich groups are identified here for rows $3, 5, 7$ only. In particular the analytic quotient of row eleven is exactly 9; its identification with $\#\,\mathrm{Sha}(E_{11}/\mathbb{Q})$, including its 3-primary part, remains open in the results of this paper.

D.1 The unresolved relative-period bound

The open lower-bound question is the Relative Semi-abelian Conjecture (Bertolin et al., 2022, Conjecture 4.7, arXiv v2, p. 18) which asks whether algebraic lifts in the indicated semiabelian varieties satisfy the bound

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}(\Omega_G)} \overline{\mathbb{Q}}(\Omega_G, R, \bar{R}, \widetilde{\log}_G(R, \bar{R})) \geq 2 \dim B_Q + \dim Z(1).$$

The full matrix Ω_G includes the second-kind periods. This lower bound remains open for the compact diagonal configuration at issue here. Its relation to Grothendieck's period conjecture for algebraic 1-motives is given in (Bertolin et al., 2022, Proposition 4.12, arXiv v2, pp. 19–20).

For the algebraic motives of proposition 26.5, projection from the point-extension variety to the extension-coordinate variety is an isogeny: the point coordinates agree with the extension data under the stated polarizations. Equivalently, $(P, \bar{P}) = (P, 0) + (0, \bar{P})$ in $E \times \bar{E}$. Formula (4.4), p. 19 of the same source therefore gives $\dim B_Q = 0$. The remaining kernel of connected motivic groups is the weight--2 vector group $\mathrm{Lie}\, Z$. After fixing the base periods its fibre is an affine torsor of dimension $\dim Z$.

In positive Tate coordinates the central action changes $(v, \bar{v})$ by $2\pi i(c_1, c_2)$. For an isogeny $\phi : \bar{E} \to E$ of degree d, $\phi^{\mathsf{t}} J_E \phi = d J_{\bar{E}}$. Thus the dependent bracket line of proposition 26.5 has coordinates $(1, aa^{\dagger}/d)$ with positive second coordinate; in the independent case the bracket spans both central coordinates. The functional $(c_1, c_2) \mapsto c_1 + c_2$ is nonzero on this central group, including when there is a further quotient Tate direction. Accordingly, the compactness equation (26.8) cuts a proper affine hyperplane. The unconditional period-field calculation is lemma 26.6; the proposed lower bound needed to exclude an algebraic compact lift is the open step.

Programs

The following files contain the indicated computations at commit

`a02f2eacf5a8c3db7cf8cdfcdba50c0002ee91c7`

Each link opens the records page, which lists the complete repository paths.

Table 4: Programs and their mathematical scope.

Computation and statement	Source file
Period determinants, integral parameter equations and tangent ranks; cusp deck identities and finite actions; initial node blowup with saturated equation $bw = xy$ proposition 12.10 and theorems C.2, C.3, C.5, 18.1, 18.2, 18.4, 18.7, 18.8 and 21.3	`priority_rows_exact.py` `adjacent_exact.py`
Finite double-complex matrices and their cohomology propositions 13.2 and 13.5	`main_D_double_complex.py`

Regular-chart identities, cusp Cartier factors and finite-collar first jets
proposition 14.2
	`exact_regular_atlas.py`
	`cusp_and_finite_exact.py`
	`finite_collar_jet_verifier.py`

Tangent-twist Smith modules and stabilizer-character jet ranks proposition 15.4	`d_koszul.py`
Weighted section counts and rational boundary maps of finite covers propositions 23.4, 23.7 and 23.9	`b_families.py` `b_topology.py`
Division-polynomial factors at the marked elliptic section proposition 15.3	`d_division_polynomials.py`
Exact division recurrences and resultants propositions 4.8, 4.9 and 6.5 and cited result 23.10	`Cspecial_division.py` `Cspecial_relations.py`
Rational CM models, local reductions and finite heights proposition 24.4	`C_cm_models.gp` `C_cm_local_heights.gp`
Outward interval enclosures for CM heights and singular moduli proposition 24.7	`C_cm_height_arb.py`
Exact descent, local rank data and saturation proposition 24.9	`C_cm_rank_lvalues.gp` `Crank_saturation.cpp`

Integer coefficient streams, derivative sums and full-real-period enclosures
lemma 24.10 and proposition 24.11
	`Crank_coefficients.gp`
	`Crank_derivatives.cpp`
	`Crank_cm_stream.cpp`
	`certify_all_period_ratios.py`
	`C_height_coarse_certificate.py`

Exact Bézout height bounds, six rational doublings and integer isolation cited result 24.14	

Genus data, modular symbols, central sums and rational denominator isolation
proposition 24.16
	`B_C6_gz_exact_metadata.gp`
	`B_C6_manin_2.gp`
	`B_C6_manin_4.gp`
	`B_C6_manin_6.gp`
	`B_C6_manin_9.gp`
	`B_C6_manin_10.gp`
	`B_C6_central_stream.cpp`
	`B_C6_exact_factors.py`
	`auxiliary_coefficients.gp`

Exact local data, torsion and two-isogeny descent over $\mathbb{Q}(\sqrt{3})$
propositions 24.17 and 24.18
	`Crank_torsion_local.gp`
	`Crank_torsion_final.gp`
	`Crank_torsion_descent.gp`
	`Crank_Sage_ell.gp`
	`Crank_Sage_ellcommon.gp`

Computation and statement	Source file
Rational power-series recurrence and Bernstein coefficient inequalities proposition 25.5	Ccritical_rational_certificate.py Ccritical_scalar.py
Exact CM endomorphism, kernel polynomial and compact-point witness identities proposition 26.17	cm_witness_exact.py
Exact sixth- and fourth-division recurrences for the exceptional-fibre calculations proposition 11.1 and theorem 11.3	cd_torsion_independent.py
Integral and affine fixed-set Smith/Hermite computations for the three real parameter arcs theorem 17.5	real_involutions.py
Denominator-24 finite-root fixed sets and deck permutations theorem 17.5	real_finite_four.py
Thirty-six primitive-ray pairs, twelve fan maps and six curve actions theorem 11.16	cusp_hexagon_exact.py
Two-period determinant, square-zero monodromies, root differences and exceptional-valuation identities propositions 22.41 and 22.46	e_push_completion_diagnostics.py
Exceptional valuation matrices and intersection determinants for labelled point-blowup sequences through four exceptional curves proposition 22.50	e_push_log_blowup.py
Canonical-form coefficients in the small charts, multiplicative limits and horizontal derivatives proposition 22.53	e_push_forms_translation.py
Normalization-lattice Gram and monodromy identities; integral Smith forms for the order-three and order-four cycle matrices propositions 20.4, 20.7 and 20.9	e_degeneration.py
Character–ray pairings, monoid relations, three affine chart maps, section doubling, normalization Smith forms and virtual Chern pushforward propositions 20.5 and 21.11 and theorems 21.2 and 21.8	ab_push_local_checks.py
Breadth-first star examples and local substitutions for exponents $0, 1, 2$ and centre codimensions $2, 3, 4$ proposition 21.6	ab_push_normalization_algorithm.py
Real-period determinant, covector equation, nonseparation vector and rational-circle generators with denominators at most twelve lemma 19.5, proposition 19.6, and theorem 19.9	d_push_exact.py
Finite-series numerical comparison of the theta derivative and symmetric finite differences at three points proposition 19.14	d_push_gradient.py
Exact degree-three endomorphism and kernel congruences; formal logarithmic coefficients and rational derivative coefficient proposition 19.15	d_push_equianharmonic.py
Finite-product numerical derivative comparisons at three displacements near the equianharmonic point proposition 19.15	d_push_equianharmonic_crossing.py
Integral monodromy and common-fixed equations; root-norm examples for degrees 1–64 and connector cancellation proposition 22.56 and theorem 18.4	d_push_global_loop_defect.py
Affine defects of the initial parameter lift and its twelve-shift comparison terms proposition 17.8	d_push_monodromy.py
Coherent affine composition, finite and cusp defects, period/scalar comparison and twelve-shift factor proposition 17.8	d_push_closing.py
Primitive-ray lists at heights $1, 2, 3$ for $n = 1, 2, 12$; bounded determinant tests and eight successive point stars theorem 21.5	ab_push_all_refinements.py
Integral E_8 and neck matrices, Smith form and the square-zero rank-six neck operator proposition 21.12	ab_push_monodromy.py

Computation and statement	Source file
Cone and orbit counts for $n = 1, 2, 12$, eight point stars and five conductor stars proposition 21.15 and theorem 21.16	ab_push_all_fan_topology.py
Normalization-chain Smith forms for $n = 1, 2, 3, 4, 6, 12$ and blowup-centre increments theorem 21.17	ab_push_all_fan_homology.py
Local exceptional complexes, unit Smith factors and Gysin maps for centre codimensions $2, 3, 4$ proposition 21.18	ab_push_central_pushforward.py
Augmented-simplex differentials and signed permutation chain maps for at most four branches theorem 22.16	b_nc_degree_one.py
Unipotent winding, common fixed cycles, coinvariant Smith form and the exact affine closing identity proposition 22.26	ab_push_hopf_winding.py
Four affine smoothness calculations, Hilbert difference, normal-degree cancellations and intersection Smith form proposition 22.29	root_quadric_smoothing.py
Fixed-curve and node character identities, tangent weights and vector-field identities for $n = 1, \ldots, 12$ proposition 22.33	ab_push_kato_weights.py
Integral chain maps, elementary collapses and angular cocycles for $n = 4, 6, 12, 24$ proposition 22.36	ab_push_regular_cap_dual.py
Deck differences, twelve-step affine norm, commuting shears and thirty-six primitive-cone determinants proposition 22.39	ab_push_corrected_cusp_overlap.py
Finite and cusp gauge identities; node substitutions and the additive fan cocycle on the punctured parameter domain proposition 22.41	ab_push_global_partial.py
Cartier and Tate factors for $n = 1, 2, 3, 4, 8, 12$, local base-map differentials and the closing factor proposition 22.42	ab_push_tate_interface.py
Point and curve blowup substitutions, node exponents, intersection degrees and finite collar-product identities proposition 22.43	d_push_degree_one_repair.py
Three local chart maps, triangular fans for $n = 1, 2, 3, 4, 6, 12$ and collar identities proposition 22.44	ab_push_corner_resolution.py
Exact cone, ray and fan calculations proposition 21.1	ab_push_fan.py
Characteristic polynomial, integral inverse identity and both Euler calculations (proposition 22.11)	ab_push_auxiliary_plumbing.py
Rational fan certificates for the base changes $n = 2$ and $n = 12$ (proposition 22.3)	ab_push_basechange_fan.py
Finite period covariance, affine deck formulas, component cycles and peripheral residues (theorems 22.2 and 22.5)	ab_push_finite_decks.py
Finite cyclic carry cocycles and local component arithmetic (proposition 22.10)	ab_push_torsor_cocycles.py
Node series, height and intersection lattices, trace congruences and integral saturation (theorem 22.8 and corollary 22.9)	ab_push_trace_obstruction.py
Section equations and collinearity identities on $b^3 = a^4 + 1$ (theorem 22.8)	ab_push_trace_sections.py

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