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IRRATIONALITY MEASURES FOR DILOGARITHMS OF RATIONAL NUMBERS

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ABSTRACT. Let μ denote the irrationality exponent. For coprime integers a and b with $0 < 2|a| \le b$ we prove an explicit upper bound for $\mu(\mathrm{Li}_2(a/b))$ whenever an explicit margin $\eta(a,b)$ is positive. The margin is the sum of a closed-form arithmetic rate in $\log b$, $\log|a|$ and $\log|b-a|$ and a piecewise affine decay rate in $\log(b/|a|)$. For example $\mu(\mathrm{Li}_2(1/2)) \le 228.26$, $\mu(\mathrm{Li}_2(1/3)) \le 21.52$, $\mu(\mathrm{Li}_2(1/4)) \le 15.16$, $\mu(\mathrm{Li}_2(1/5)) \le 12.90$, $\mu(\mathrm{Li}_2(-1/2)) \le 13.92$ and $\mu(\mathrm{Li}_2(-1/3)) \le 11.89$. As far as we know these are the first irrationality measures for these numbers. For the numerator 2 we obtain finite measures for $\mathrm{Li}_2(2/b)$ for every odd $b \ge 13$ and for $\mathrm{Li}_2(-2/b)$ for every odd $b \ge 11$. As far as we know this proves for the first time that $\mathrm{Li}_2(-2/11)$, $\mathrm{Li}_2(-2/13)$ and $\mathrm{Li}_2(-2/15)$ are irrational. The same holds for $\mathrm{Li}_2(-3/50)$, $\mathrm{Li}_2(4/167)$ and $\mathrm{Li}_2(-4/177)$. For $6 \le b \le 20$ the bounds improve those of Rhin and Viola, for instance from 783.29 to 11.72 at $b = 6$ and from 11.94 to 9.36 at $b = 20$. For $\mathrm{Li}_2(-1/b)$ with $5 \le b \le 20$ they improve the bounds that follow from Rhin and Viola's linear independence measures. The approximations are Tang's normalized Hankel determinants. Liu and Tan and Zhang used them, with an arithmetic rate of the same shape, to prove irrationality of $\mathrm{Li}_2(\pm 1/k)$ and of $\mathrm{Li}_2(p/q)$ in explicit ranges. A measure needs more. We prove that the determinants do not vanish at rational points for a band of primes of positive density, and we prove decay on an exponentially small neighbourhood of $\mathrm{Li}_2(a/b)$ by a dual certificate for a logarithmic energy problem with capped lattice charges. Concavity of the dual and a transport in $\log(b/|a|)$ make the decay bound uniform from finitely many certificates. The proof is formalized in Lean 4.

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1. Introduction

1.1. **The result.** The *irrationality exponent* $\mu(\xi)$ of a real number ξ is the infimum of the real numbers μ such that $|\xi - p/q| > q^{-\mu}$ for all integers p and all sufficiently large integers q. It does not change when ξ is multiplied by a nonzero rational number or translated by a rational number. A finite bound for $\mu(\xi)$ is an irrationality measure for ξ, and it shows that ξ is irrational. The dilogarithm is

$$\mathrm{Li}_2(x) = \sum_{k\geq 1} \frac{x^k}{k^2} \qquad (|x| \leq 1).$$

Let a and b be coprime integers with $b > 0$ and $0 < 2|a| \leq b$. Put

$$\lambda = \frac{a}{b}, \qquad t = \log\frac{b}{|a|}, \qquad s = \operatorname{sign} a,$$

and define the *arithmetic rate*

$$c(a,b) = c_0 + \frac{8}{3}\log b - 6\log|a| - \frac{9}{4}\log|b-a|, \tag{1.1}$$

where

$$c_0 = 28\log 2 - \frac{9}{2}\log 3 - \frac{139}{12} = 2.8810324\ldots \tag{1.2}$$

Let $M = 98/39$. In Section 7 we define, from a finite table of certified data, an explicit function $\underline{D}_s(t,\rho)$ of the sign s, of $t \geq \log 2$ and of $\rho > 0$. It is the largest of finitely many numbers, each of which is affine in t on an interval. The table has 37 nodes and 419 entries. A summary is in Appendix A, and the full table is part of the formal statement.

Theorem 1.1. *Let a, b be coprime integers with $b > 0$ and $0 < 2|a| \leq b$, and let $\rho > 0$. Suppose that $\eta = \underline{D}_s(t,\rho) + c(a,b) > 0$. Then for every*

$$\mu > \rho \cdot \max\Big(\frac{2}{\eta}, \frac{1}{M}\Big)$$

there is an integer q_0 such that

$$\Big|\mathrm{Li}_2\Big(\frac{a}{b}\Big) - \frac{p}{q}\Big| > q^{-\mu}$$

for all integers p and all integers $q \geq q_0$. In particular $\mu(\mathrm{Li}_2(a/b)) \leq \rho\max(2/\eta, 1/M)$.

Theorem 1.1 has been verified in the Lean 4 proof assistant, together with every theorem, proposition and lemma used in its proof. The formal proof has no unproved steps and uses only the standard axioms. Section 10 describes it. The remarks are not part of the formal proof. The use of AI in this work is disclosed at the end of the paper.

The numbers in the following corollaries are evaluations of the formula of Theorem 1.1 at one admissible ρ. They are rounded up in the last digit.

Corollary 1.2. *For the numbers x in Table 1 the bound $\mu(x) \le \mu_0$ holds with the stated μ_0, in the plain form of Theorem 1.1.*

	$x = \mathrm{Li}_2(1/k)$			$x = \mathrm{Li}_2(-1/k)$		
k	μ_0	RV	Hata	μ_0	RV	Hata
2	228.26	–	–	13.92	–	–
3	21.52	–	–	11.89	–	–
4	15.16	–	–	10.98	–	–
5	12.90	–	–	10.37	62.69	228.62
6	11.72	783.29	–	10.01	44.51	55.94
7	11.02	69.69	95.07	9.76	33.76	35.84
8	10.49	47.43	51.08	9.57	26.06	27.39
9	10.14	30.44	33.13	9.42	21.88	22.66
10	9.85	24.04	25.24	9.30	19.24	19.75
11	9.66	19.80	21.19	9.20	17.42	17.77
12	9.54	17.60	18.54	9.11	16.08	16.33
13	9.40	16.29	16.65	9.16	15.05	15.24
14	9.32	15.13	15.27	9.14	14.23	14.42
15	9.32	14.40	14.48	9.16	13.58	13.75
16	9.33	13.71	13.81	9.16	13.06	13.20
17	9.34	13.18	13.24	9.20	12.61	–
18	9.32	12.71	12.76	9.23	12.23	–
19	9.34	12.30	–	9.26	11.90	–
20	9.36	11.94	–	9.29	11.62	–

TABLE 1. Bounds $\mu(x) \le \mu_0$ from Theorem 1.1, with the published bounds. For $\mathrm{Li}_2(1/k)$ the column RV is Rhin and Viola [25, pp. 434–435]. For $\mathrm{Li}_2(-1/k)$ it is $\nu + 1$, where ν is their linear independence measure of 1, $\mathrm{Li}_1(-1/k)$, $\mathrm{Li}_2(-1/k)$ [26, Table 1]. The columns Hata are [11, Tables 1 and 2, p. 386]. Published values are rounded to two decimals.

For the numerator 2 the theorem covers two full ranges of odd denominators.

Corollary 1.3. $\mathrm{Li}_2(2/b)$ *has a finite irrationality measure for every odd* $b \geq 13$*, and* $\mathrm{Li}_2(-2/b)$ *for every odd* $b \geq 11$*. For the numbers* x *in Tables 2 and 3 the bound* $\mu(x) \leq \mu_0$ *holds with the stated* μ_0*. In particular* $\mathrm{Li}_2(-2/11)$*,* $\mathrm{Li}_2(-2/13)$*,* $\mathrm{Li}_2(-2/15)$*,* $\mathrm{Li}_2(-3/50)$*,* $\mathrm{Li}_2(4/167)$ *and* $\mathrm{Li}_2(-4/177)$ *are irrational.*

	$x = \mathrm{Li}_2(2/b)$		$x = \mathrm{Li}_2(-2/b)$	
b	μ_0	entries	μ_0	entries
11	–		110.29	308
13	2408.00	31	65.00	308, 328
15	127.11	13	49.03	328
17	72.41	19	41.07	328, 348
19	53.50	46	36.13	328, 348
21	44.00	43, 62	32.75	328, 348
23	38.02	60	30.28	328, 348
25	34.06	72	28.39	348
27	31.29	69, 91	27.01	348, 369
29	29.14	69, 89	25.87	348, 369
31	27.40	68, 88	24.92	348, 369
33	25.98	87	24.12	348, 369
35	24.85	86, 102	23.42	348, 369
37	23.89	102	22.81	348, 369
39	23.08	117	22.27	348, 369
41	22.41	117, 136	21.80	369

TABLE 2. Bounds $\mu(x) \leq \mu_0$ from Theorem 1.1 for the numerator 2. The columns "entries" give the zero-based indices of the table entries of the candidate. One index is a transport from that entry with $m = 0$. Two indices are an interpolation.

x	μ_0	entries	Tan–Zhang	Liu
$\mathrm{Li}_2(-3/50)$	12476.00	349, 370	$q \geq 60$	$k \geq 62$
$\mathrm{Li}_2(4/167)$	14121.72	137, 147	$q \geq 168$	$k \geq 183$
$\mathrm{Li}_2(-4/177)$	7834.47	137, 147	$q \geq 178$	$k \geq 194$

TABLE 3. Bounds from Theorem 1.1 for larger numerators. The last two columns give the ranges of denominators in which Tan and Zhang [28, Table 1] and Liu [19] prove irrationality for the same numerator.

No irrationality measure for any $\mathrm{Li}_2(\pm 2/b)$ was published before, as far as we know. Liu proves that $\mathrm{Li}_2(2/k)$ is irrational for $k \geq 13$ and $\mathrm{Li}_2(-2/k)$ for $k \geq 16$ [19]. Tan and Zhang prove it for $q \geq 12$ and $q \geq 16$ [28, Table 1]. Their ranges include every denominator, ours only those coprime to the numerator. For $\mathrm{Li}_2(2/b)$ with odd b all three ranges start at $b = 13$. For $\mathrm{Li}_2(-2/b)$ ours starts at $b = 11$, and the first irrationality proofs are those of $\mathrm{Li}_2(-2/11)$, $\mathrm{Li}_2(-2/13)$ and $\mathrm{Li}_2(-2/15)$. The three points of Table 3 lie below the ranges of both Tan and Zhang and Liu, so their irrationality is also new as far as we know. Their margins are below 10^{-2}, and the measures are large. The theorem gives no measure for $\mathrm{Li}_2(2/b)$ with $b \in \{5, 7, 9, 11\}$ or for $\mathrm{Li}_2(-2/b)$ with $b \in \{5, 7, 9\}$.

Remark 1.4 (larger numerators). Evaluating the formula for $3 \leq |a| \leq 7$ and $b < 4000$ gives a positive margin for every b coprime to a from $b = 55, 167, 424, 953, 2046$ on for $a = 3, \ldots, 7$, and from $b = 50, 177, 437, 971, 2069$ on for $a = -3, \ldots, -7$. For $b \geq 4000$ the transport of Remark 1.5 has a positive margin at $b = 4000$, and its margin increases with b. Tan and Zhang prove irrationality from $q = 54, 168, 418, 1038, 2233$ and from $q = 60, 178, 434, 1058, 2256$ [28, Table 1]. So the evaluation gives new irrational values for the numerators -3, ± 4, ± 6 and ± 7 just below their thresholds, and none for 3 and ± 5. Table 3 lists the first of them for -3 and ± 4. This remark rests on a floating-point evaluation of the margin at each b and is not part of the formal proof.

Remark 1.5 (large denominators). For fixed a the margin η tends to infinity with b. Let (ρ_k, D_k) be an entry of the lattice node $\frac{1}{403}$. For $b \geq 403|a|$ the transport with slope $m = 1$ from this entry is a candidate with $D = D_k + t - \log 403$, and $c(a, b) = \frac{5}{12} \log b + O_a(1)$. So $\eta \geq \frac{17}{12} \log b + O_a(1)$, and Theorem 1.1 gives a finite measure for every large b coprime to a. For $a = \pm 1$ a numerical evaluation of the margin for $2 \leq b \leq 403$ finds it positive, with minimum $0.1362\ldots$ at $b = 2$. This remark is not part of the formal proof.

1.2. **Background.** *Classical results.* The values of polylogarithms at algebraic points are values of G-functions. Their arithmetic was studied by Siegel's method and by Padé approximation, and we refer to the introduction of [25] for this history. Apéry proved that $\zeta(2) = \mathrm{Li}_2(1)$ is irrational [1], and Beukers gave his integral proof [3]. Hata constructed rational approximations to $\mathrm{Li}_2(1/s)$ and proved its irrationality for every integer $s \geq 7$ and $s \leq -5$ [11]. He also gave irrationality measures for $7 \leq s \leq 18$ and $-16 \leq s \leq -5$, from $\mu(\mathrm{Li}_2(1/7)) < 95.07$ to $\mu(\mathrm{Li}_2(1/18)) < 12.76$ and from $\mu(\mathrm{Li}_2(-1/5)) < 228.62$ to $\mu(\mathrm{Li}_2(-1/16)) < 13.20$. Rhin and Viola extended their permutation group method from $\zeta(2)$ [24] to double integrals for the dilogarithm [25]. They proved that $\mathrm{Li}_2(1/6)$ is irrational with $\mu(\mathrm{Li}_2(1/6)) < 783.29036$, and they improved Hata's measures for $7 \leq s \leq 18$. Their table gives $\mu(\mathrm{Li}_2(1/s)) < 11.94109$ at $s = 20$, and their general theorem gives $\mu(\mathrm{Li}_2(1/s)) < 16.14$ for all $s \geq 15$. They also proved that $\mathrm{Li}_2(r/s)$ is irrational for every $r \geq 1$ and every s larger than an explicit $s_1(r)$, for

instance for $r = 2$ and $s \geq 51$. Later they proved the linear independence of 1, $\mathrm{Li}_1(r/s)$ and $\mathrm{Li}_2(r/s)$ for positive and negative arguments [26]. For $-20 \leq s \leq -5$ they gave linear independence measures ν of 1, $\mathrm{Li}_1(1/s)$ and $\mathrm{Li}_2(1/s)$. Taking the coefficient of Li_1 to be zero gives $\mu(\mathrm{Li}_2(1/s)) \leq \nu + 1$, for instance 62.69 at $s = -5$ and 11.62 at $s = -20$. These bounds are below Hata's for every $-16 \leq s \leq -5$. For $\zeta(2)$ itself the best measure is now Giraudin's 5.0193784 [10], after Zudilin's 5.095412 [33].

Small arguments. The arguments $\pm 1/2$, $\pm 1/3$, $\pm 1/4$ and $1/5$ are out of reach of the classical forms. Calegari, Dimitrov and Tang recorded the irrationality of $\mathrm{Li}_2(1/2)$ among the open cases [4, Remark 2.8.2 and §15.2]. Prellberg proved that $\mathrm{Li}_2(-1/4)$, $\mathrm{Li}_2(1/5)$ and $\mathrm{Li}_2(-1/3)$ are irrational by refinements of the constructions of Rhin and Viola and of Viola and Zudilin [23]. Qian Tang adapted Fauzan's Hankel determinants for $\zeta(5)$ [7] to the dilogarithm. His Lean formalization proves that $\mathrm{Li}_2(1/2)$ is irrational [29]. Ruiyi Zhang gave an independent proof [32]. His approximations are determinants of moments of $1/(xy - 2)$. He bounds them by a logarithmic energy on an ellipse, certified in interval arithmetic, and proves nonvanishing by a Lucas congruence modulo 83.

The most recent work. Si-Qi Liu extended Tang's method in a Lean formalization [19]. He proves that $\mathrm{Li}_2(1/k)$ is irrational for every $k \geq 2$ and $\mathrm{Li}_2(-1/k)$ for every $k \geq 1$. He also proves irrationality of $\mathrm{Li}_2(\pm p/q)$ for $p \leq 4$ beyond explicit denominators, and for every $p \geq 5$ when $q \geq 0.1851\, p^{84/17}$. Tan and Zhang proved that $\mathrm{Li}_2(p/q)$ is irrational whenever $q > (4e^4/27)p^3$, by the Rhin–Viola integrals [28]. With Liu's construction and new comparison measures they lowered the thresholds for $p \leq 7$, for instance to $q \geq 418$ for $p = 5$. They quote the arithmetic rate of Liu's polynomials as $\frac{8}{3}\log q - \frac{9}{4}\log(q - \epsilon p) - 6\log p + \frac{72}{25}$ for $\mathrm{Li}_2(\epsilon p/q)$ [28, (2.2)]. This is the rate (1.1) with the constant $\frac{72}{25}$ in place of c_0, and $c_0 > \frac{72}{25}$. These results prove irrationality. Neither states an irrationality measure. Theorem 1.1 uses the same polynomials and the same denominators, and it gives a measure. As far as we know, no irrationality measure for $\mathrm{Li}_2(1/k)$ with $2 \leq k \leq 5$, for $\mathrm{Li}_2(-1/k)$ with $2 \leq k \leq 4$, or for any $\mathrm{Li}_2(\pm 2/b)$ was published before. Table 1 compares our bounds with the published ones. At every k where a published bound exists, the best published bound is that of Rhin and Viola, and ours is smaller. For $\mathrm{Li}_2(1/k)$ with $6 \leq k \leq 20$ the improvement runs from 783.29 to 11.72 at $k = 6$ and from 11.94 to 9.36 at $k = 20$. For $\mathrm{Li}_2(-1/k)$ with $5 \leq k \leq 20$ it runs from 62.69 to 10.37 at $k = 5$ and from 11.62 to 9.29 at $k = 20$.

Recent results. Since September 2026 several new results have appeared in this area, some obtained with AI systems. In October 2026 OpenAI proved that Catalan's constant is irrational [21] and that $\mu(\pi) = 2$ [22], both with Lean formalizations. Liu, Jiang and Zhang proved $\mu(\log r) = 2$ for rational $r > 0$, $r \neq 1$ [18]. Their method needs a number that is a period of a one-dimensional group, and we do not know how to apply it to $\mathrm{Li}_2(a/b)$. Fauzan proved that $\zeta(5)$ is irrational [7], and Lai combined Fauzan's determinants

with simultaneous approximations to show that 1, $\zeta(3)$ and $\zeta(4)$ are linearly independent [13].

The method. The measure method we use, a family of integer polynomials whose values near the constant are small and nonzero, is a one-variable form of the criteria of Nesterenko [20] used by Ball and Rivoal [2]. Determinants of growing size go back to Zudilin's determinantal approach [34]. The arithmetic of the coefficients is handled prime by prime, as in the work of Hata [11] and of Rhin and Viola [24, 25]. The size of the determinants is controlled by Andréief's identity [9] and a logarithmic energy problem with an external field [27].

1.3. **The companion papers.** This paper applies one method to a fifth family of constants. The method consists of Hankel or Löwner determinants, denominators bounded prime by prime, decay proved through an energy problem, and a formal proof of every step. The first companion paper [14] refines OpenAI's determinant for Catalan's constant. The second [17] uses Löwner determinants of digamma combinations. The third [15] proves a measure for $\zeta(5)$ with Fauzan's determinants, and the fourth [16] proves measures on the plane spanned by $\zeta(3)$ and π^4 with Lai's determinants. The dual certificate of Section 6 extends the one of [16], and the comparison measures are of the arcsine type used in [15]. We use the definitions of μ, the measure criterion and the potential-theoretic language of these papers. The construction is Tang's, with his notation.

1.4. **What a measure requires.** Tang's polynomials Q_n have degree at most $2n$, and their values at $\xi = \mathrm{Li}_2(\lambda)$ are Hankel determinants of a linear functional. Let p/q be a rational close to ξ. Three estimates are needed at sizes n of order $\log q$.

(i) *Denominators.* A rational factor d_n with $d_n\tilde{Q}_n \in \mathbb{Z}[X]$ primitive and $\log d_n \le -(c+o(1))n^2$, where $\tilde{Q}_n$ is Tang's normalization of Q_n. The factor removes a large common divisor of the coefficients.

(ii) *Decay on a neighbourhood.* $|\tilde{Q}_n(x)| \le e^{-(D-o(1))n^2}$ for every real x with $|x-\xi| \le e^{-(\rho+o(1))n}$, not only at $x=\xi$.

(iii) *Nonvanishing.* $Q_n(p/q) \neq 0$ for some n in a range of length $O(\log q)$ above the size where (i) and (ii) beat q^{2n}.

Then $\mu(\xi) \le \rho \max(2/(D+c), \beta)$, where β measures how sparse the sizes with nonvanishing are. Item (i) is known. It is the rate of Liu's construction [19, 28]. The irrationality proofs of Tang and Liu need decay at the point ξ only, and nonvanishing on the line $n = \ell - 1$ through the primes ℓ. That line has one prime per size, and it would give $\beta = 1$. A measure needs the decay on a neighbourhood, for each a/b, and nonvanishing at sizes of positive density.

1.5. **New ideas.** We keep Tang's determinants and the known arithmetic rate. The proof rests on the following ideas.

(1) *Nonvanishing on a band of primes.* For every prime ℓ with $n/13 < \ell < 4n/3$, outside a finite set and three residues of n, one of Q_n and Q_{n+1} is nonzero at every rational whose denominator is not divisible by ℓ^2 (Theorem 5.1). The proof reduces the Hankel matrix modulo ℓ in a basis adapted to the residue classes of the poles. Its reduction is a product of finitely many Hankel determinants of local weights, the type constants, times a binomial tail in the tied case. The type constants are rational functions of λ. We prove that each is nonzero at every rational λ with $0 < |\lambda| \le \frac{1}{2}$ by its leading terms at 0, 1 and ∞ and one reduction modulo a prime (Theorem 5.3).

(2) *A measure criterion for a band of primes.* The primes of the band have total logarithmic mass $2(\frac{4}{3} - \frac{1}{13})n = Mn$ with multiplicity two. An averaging argument over n removes the exceptional residues, and the measure becomes $\rho \max(2/\eta, 1/M)$ (Theorem 3.2). When $\eta \le 2M$, nonvanishing costs nothing.

(3) *Decay on a neighbourhood by a capped dual.* We write the Hankel determinant at $\xi + \varepsilon$ by Heine's formula as a sum over configurations of $2n$ charges. The charges sit on the positive integers, at most two per site, or on a contour, and the perturbation adds charges at the poles $-j$, at most one per site. A positive measure σ of mass 2 and a level θ give an explicit upper bound for the rate, in which the caps enter as the penalties $2\int (V - 2U^\sigma - \theta)_+$ and $\int (A_\rho - \theta)_+$ (Theorem 6.1). For $\lambda > 0$ we use the lattice alone. For $\lambda < 0$ a bent contour through 0 captures a cancellation that the lattice does not see.

(4) *Uniformity in the argument.* The dual value depends on λ only through $t = \log(1/|\lambda|)$ and the sign, and it is jointly concave in the datum, t and ρ. Lowering the level by $m\Delta$ transports a datum from t to $t + \Delta$ at the cost $(4 + m)\Delta$ in ρ, with gain $(2m - m^2)\Delta$. So 37 certified data give a decay bound for every λ with $|\lambda| \le \frac{1}{2}$, of either sign (Theorem 7.3).

(5) *Rigorous certification of the decay bounds.* Each of the 419 lower bounds for the dual value is proved by finitely many inequalities in rational arithmetic. The potentials and energies of the comparison measures have closed forms, and the penalty integrals are computed from the sign pattern of the field, proved on Taylor cells with explicit remainders (Section 6.6).

The denominator bound of Section 4 has the shape of the rate of Liu's construction. We derive it from one valuation bound valid at every prime (Theorem 4.1) and from Tang's local bounds at the other primes. For the medium primes we evaluate Tang's profile integral in closed form, and this gives the constant $c_0 = 28 \log 2 - \frac{9}{2} \log 3 - \frac{139}{12}$.

1.6. **Organization.** Section 2 recalls Tang's construction. Section 3 proves the measure criterion. Section 4 bounds the denominators, Section 5 proves the nonvanishing, Section 6 proves the decay for one datum, and Section 7 makes it uniform in t. Section 8 proves Theorem 1.1 and Corollary 1.2.

Section 9 proves Tang's local bounds and the closed form of the medium-prime constant. Section 10 describes the formalization. The disclosure of AI use follows it.

1.7. **Notation.** We write v_p for the p-adic valuation and v_p^{G} for the Gauss valuation of a polynomial, the minimum of the valuations of its coefficients. We write $y_+ = \max(y, 0)$. Throughout, $\lambda = a/b$ is as in Theorem 1.1, $\xi = \mathrm{Li}_2(\lambda)$, $t = \log(1/|\lambda|)$ and $s = \operatorname{sign}\lambda$. For a positive measure σ on $\mathbb{C}$ with compact support, $U^\sigma(z) = \int \log(1/|z-y|)\, d\sigma(y)$ is its logarithmic potential and $I(\sigma) = \int U^\sigma d\sigma$ its energy.

2. Tang's construction

We recall the construction of [29], for a rational λ with $0 < |\lambda| < 1$. Put $D_m(x) = \prod_{i=1}^{m}(x+i)$ and

$$\tau_j = \sum_{c=1}^{j} \frac{\lambda^c}{c^2}, \qquad \mathrm{Li}_{-k}(\lambda) = \sum_{x\ge 1} x^k \lambda^x \in \mathbb{Q}.$$

The second sum is a rational function of λ. For a variable X define a $\mathbb{Q}$-linear functional Λ_X on the rational functions whose poles are simple and lie at negative integers by

(2.1)

$$\Lambda_X(x^k) = (k+1)\,\mathrm{Li}_{-k}(\lambda) \ \ (k \ge 0), \qquad \Lambda_X\Big(\frac{1}{x+j}\Big) = j\lambda^{-j}(X - \tau_j) \ \ (j \ge 1),$$

extended by polynomial division and partial fractions. Its values are affine in X. At $X = \xi$ it has the closed form

$$\Lambda_\xi(R) = \sum_{x\ge 1} \lambda^x\, (xR(x))', \tag{2.2}$$

since $(x \cdot x^k)' = (k+1)x^k$ and $\sum_{x\ge1} \lambda^x j(x+j)^{-2} = j\lambda^{-j}(\xi - \tau_j)$. Let

$$W_n(x) = \frac{D_n(x)^3}{D_{4n}(x)}, \qquad Q_n(X) = \det\big[\Lambda_X(W_n(x)\, x^{i+j})\big]_{0\le i,j<2n}.$$

The entries are affine in X, so $\deg Q_n \le 2n$. The function W_n has simple poles at $-j$ for $n < j \le 4n$, with residues

$$r_j = \mathrm{Res}_{x=-j} W_n, \qquad |r_j| = \frac{((j-1)!)^2}{((j-n-1)!)^3 (4n-j)!}. \tag{2.3}$$

With $S_n = (4n)!/(n!)^3$ and $F_n = \prod_{i<2n}(i!)^2$ Tang normalizes

$$\tilde{Q}_n = \frac{S_n^{2n}}{F_n} Q_n \in \mathbb{Q}[X],$$

and he writes $\tilde{Q}_n = \mathrm{cont}(\tilde{Q}_n)\, P_n$ with $P_n \in \mathbb{Z}[X]$ primitive and $\mathrm{cont}(\tilde{Q}_n)$ a positive rational, its content. We put $\tilde{d}_n = 1/\mathrm{cont}(\tilde{Q}_n)$. Then $P_n = \tilde{d}_n \tilde{Q}_n$, and for every rational p/q the number $q^{2n} P_n(p/q)$ is an integer.

Lemma 2.1 (Heine's formula)**.** *For a linear functional Φ on polynomials,*

$$\det[\Phi(x^{i+j})]_{0\le i,j<N} = \frac{1}{N!}\,\Phi^{\otimes N}\Big(\prod_{1\le k<l\le N}(x_k-x_l)^2\Big).$$

Proof. Expand the square of the Vandermonde determinant $\det[x_k^i]$ and apply Φ in each variable. □

By (2.2) the value $Q_n(\xi)$ is the Hankel determinant of the functional $T\mapsto\sum_{x\ge1}\lambda^x(xW_nT)'(x)$. Andréief's identity [9] is the same statement for a functional given by a measure.

3. The measure criterion

The criterion turns decay on a neighbourhood and nonvanishing on a band of primes into a measure. It is the form of the criterion of the companion papers [16] in which the sizes with nonvanishing are selected by primes of positive density.

Definition 3.1. Let $(P_n)_{n\ge1}$ be polynomials with rational coefficients and $0<\alpha<\beta$. A *band nonvanishing datum* for (P_n) on (α,β) consists of a finite set B of primes, sets R_ℓ of at most three residues modulo ℓ, and an integer n_0, with the following property. Let $\ell\notin B$ be a prime and $n\ge n_0$ with $\alpha(n+1)<\ell<\beta n$ and $n \bmod \ell\notin R_\ell$. Then for all integers p and $q>0$ with $\ell^2\nmid q$,

$$P_n(p/q)\neq0 \quad\text{or}\quad P_{n+1}(p/q)\neq0.$$

Theorem 3.2 (measure criterion)**.** *Let ξ be real, $\kappa>0$, $\rho>0$, $\eta>0$ and $0<\alpha<\beta$. Let $P_n\in\mathbb{Z}[X]$ with $\deg P_n\le\kappa n$. Suppose that for every $\varepsilon>0$ and all large n*

$$|P_n(\xi+u)|\le e^{-(\eta-\varepsilon)n^2}\qquad\text{whenever } |u|\le e^{-(\rho+\varepsilon)n},$$

and that (P_n) has a band nonvanishing datum on (α,β). Put $M=2(\beta-\alpha)$. Then for every $\mu>\rho\max(\kappa/\eta,1/M)$ there is q_0 with $|\xi-p/q|>q^{-\mu}$ for all integers p and all $q\ge q_0$.

The proof uses three lemmas. The first is the contrapositive of Definition 3.1.

Lemma 3.3. *Let S be a finite set of primes $\ell\notin B$ with $\alpha(n+1)<\ell<\beta n$, $n\ge n_0$ and $n \bmod \ell\notin R_\ell$. If $P_n(p/q)=P_{n+1}(p/q)=0$ with $q>0$, then $\prod_{\ell\in S}\ell^2$ divides q.*

Lemma 3.4 (averaging)**.** *For integers $N,H\ge1$ there is $n\in[N,N+H)$ with*

$$\sum_{\substack{\ell\le2N+2H\\ n \bmod \ell\in R_\ell}}\log\ell\le3\log(2N+2H)\big(1+\log(2N+2H)\big)+\frac{6(N+H)\log4}{H}.$$

Proof. For a prime ℓ the number of $n \in [N, N+H)$ with $n \bmod \ell \in R_\ell$ is at most $3(H/\ell + 1)$. Summing $\log \ell$ over these pairs gives at most $3H \sum_{\ell \le x} (\log \ell)/\ell + 3 \sum_{\ell \le x} \log \ell$ with $x = 2N + 2H$. The first sum is at most $\log x\,(1 + \log x)$ by the harmonic bound, and the second is at most $x \log 4$ by Chebyshev's bound. Some n is at or below the average. □

Lemma 3.5 (band mass)**.** *For every $\varepsilon > 0$ and all large N there are $n \in [N, (1+\varepsilon)N]$ and a set S as in Lemma 3.3 with $\sum_{\ell \in S} 2 \log \ell \ge (M - \varepsilon)n$.*

Proof. By the prime number theorem, $\sum_{\alpha(n+1) < \ell < \beta n} \log \ell \ge (\beta - \alpha - \varepsilon/4)n$ for large n [5]. Removing the finite set B costs $O(1)$. Lemma 3.4 with $H = \lceil \varepsilon N \rceil$ gives an n for which the primes with $n \bmod \ell \in R_\ell$ have mass $O(\log^2 N) + O(1/\varepsilon) = o(N)$. □

Proof of Theorem 3.2. Put $A = \max(\kappa/\eta, 1/M)$ and fix $\mu > \rho A$. Choose $\delta, \varepsilon > 0$ with $(M - \varepsilon)(1+\delta)A > 1$, $(\eta - 2\varepsilon)(1+\delta) > \eta$ and $(\rho + \varepsilon)(1+\varepsilon)(1+\delta)A + \varepsilon < \mu$. Let q be large, $L = \log q$ and $N = \lceil (1+\delta)AL \rceil$. Take n and S from Lemma 3.5, and suppose $|\xi - p/q| \le q^{-\mu}$.

If $P_n(p/q) = P_{n+1}(p/q) = 0$, then Lemma 3.3 gives $L \ge \sum_S 2 \log \ell \ge (M - \varepsilon)n \ge (M - \varepsilon)(1+\delta)AL > L$. So $P_{n'}(p/q) \ne 0$ for some $n' \in \{n, n+1\}$. Then $q^{\deg P_{n'}} P_{n'}(p/q)$ is a nonzero integer, and $|P_{n'}(p/q)| \ge q^{-\kappa n'}$. On the other hand $\mu L \ge (\rho + \varepsilon) n'$ by the choice of N, so $|p/q - \xi| \le e^{-(\rho+\varepsilon)n'}$, and $|P_{n'}(p/q)| \le e^{-(\eta - \varepsilon)n'^2}$. Hence $(\eta - 2\varepsilon)n' \le \kappa L$ for large q. But $n' \ge (1+\delta)(\kappa/\eta)L$, so $(\eta - 2\varepsilon)(1+\delta) \le \eta$. This contradicts the choice of ε. □

The criterion concludes for all integers p, without coprimality, since Lemma 3.3 and the integrality of $q^{\deg P} P(p/q)$ hold for every fraction.

4. Denominators

In this section we bound the content of $\tilde{Q}_n$. We have $v_p(\mathrm{cont}\, \tilde{Q}_n) = v_p^{\mathrm{G}}(\tilde{Q}_n)$ and

$$\log \tilde{d}_n = -\sum_p v_p(\mathrm{cont}\, \tilde{Q}_n) \log p.$$

So a lower bound for the Gauss valuation at each prime is what is needed.

4.1. **The local bound.** Fix a prime p and put $\alpha_p = v_p(a)$, $e_p = v_p(b)$ and $f_p = v_p(b - a)$. Since a and b are coprime, at most one of them is positive. Put $L_p(n) = \lfloor \log_p(7n - 2) \rfloor$ and

$$A_2(n) = \sum_{k < 2n-1} \Big(n + 1 + \Big\lfloor \frac{k}{3} \Big\rfloor \Big), \qquad S(n) = \sum_{i < 2n} (2i - n + 1)_+.$$

Theorem 4.1 (local bound)**.** *For every prime p and every $n \ge 1$,*

$$v_p^{\mathrm{G}}(\tilde{Q}_n) \ge e_p A_2(n) - \alpha_p(6n^2 + n) - f_p S(n) - 4nL_p(n).$$

At a prime that divides none of a, b and $b-a$ this is Tang's bound $v_p^{\mathrm{G}}(\tilde{Q}_n) \geq -4nL_p(n)$ [29]. We describe the three other cases. Tang writes the Hankel matrix as a Gram matrix in the binomial basis. Its entries are

$$E_{ij} = U_{ij} + \sum_{j'=1}^{4n} \rho_{j'} \binom{-j'}{i} \binom{-j'}{j} j' \lambda^{-j'} (X - \tau_{j'}) \qquad (0 \leq i, j < 2n), \tag{4.1}$$

where U_{ij} is the value of Λ on the polynomial part of $S_n D_n^3 \binom{x}{i}\binom{x}{j}/D_{4n}$, and $\rho_{j'}$ is the residue of $S_n D_n^3/D_{4n}$ at $-j'$. The residues $\rho_{j'}$ are p-integral, and they vanish for $j' \leq n$. The polynomial part enters only through the moments $\mathrm{Li}_{-k}(\lambda)$. In the binomial basis these are $G(k) = r^k/(1-\lambda)$ for $k \geq 1$ and $G(0) = r$, with $r = \lambda/(1-\lambda) = a/(b-a)$. So the three kinds of prime affect three different parts of (4.1).

Lemma 4.2. *Let $f \geq 0$ and $v_p(r) \geq -f$.*

(a) $v_p(G(k)) \geq -(k+1)f$ for every $k \geq 0$.

(b) If g is a polynomial of degree at most d with $v_p(g(i)) \geq \gamma$ for $0 \leq i \leq d$, then the value of Λ on g has valuation at least $\gamma - \lfloor \log_p d \rfloor - (d+1)f$.

(c) For $0 \leq i, j < 2n$, $v_p(U_{ij}) \geq -2L_p(n) - f\,(i+j-n+1)_+$.

Proof. (a) The recurrence $G(k+1) = r(G(k) + \binom{1}{k+1})$ and induction. (b) Expand g in the Newton basis of binomial polynomials. The Newton coefficients have valuation at least γ, the moments lose $(d+1)f$ by (a), and the derivative in (2.1) loses $\lfloor \log_p d \rfloor$. (c) If $i + j < n$, the numerator has degree below $4n$ and the polynomial part is 0. Otherwise the polynomial part has degree at most $i+j-n$ and integral values up to $\lfloor \log_p(3n+i+j) \rfloor \leq L_p(n)$, and (b) applies. □

For $f = 0$ these are Tang's lemmas, and the parameter f is the only change.

Proof of Theorem 4.1. The case $p \mid b-a$. Then λ and λ^{-1} are p-units, and the residue part of E_{ij} has Gauss valuation at least $-2L_p(n)$ by Tang's bound for $\tau_{j'}$. With Lemma 4.2(c), $v_p^{\mathrm{G}}(E_{ij}) \geq -2L_p(n) - f_p\varphi(i+j)$ with $\varphi(m) = (m-n+1)_+$. Since φ is convex, $\varphi(i+j) \leq \frac{1}{2}(\varphi(2i) + \varphi(2j))$. So $v_p^{\mathrm{G}}(E_{ij}) \geq w_i + w_j$ with $w_i = -L_p(n) - \frac{f_p}{2}\varphi(2i)$, and the determinant has Gauss valuation at least $2\sum_i w_i = -4nL_p(n) - f_p S(n)$.

The case $p \mid b$. Then r is a p-unit and $v_p(\lambda^{-1}) = e_p \geq 1$. This is Tang's argument for $p = 2$ at $\lambda = \frac{1}{2}$, with e_p in place of 1 [29]. The Hankel matrix factors as a product of a matrix of values at the poles and a matrix of binomial coefficients. Tang attaches a weight to each of the $2n$ slots of the factorization, ranks the slots by their weights, and applies the Cauchy–Binet expansion. The weights of the pole slots are $e_p \min(n+1+\lfloor (i-1)/3 \rfloor, 4n+1) - 2L_p(n)$, and their sum over the $2n$ ranks is $e_p A_2(n) - 4nL_p(n)$.

The case $p \mid a$. Then r and λ have nonnegative valuation, and $v_p(\lambda^{-j'}) = -\alpha_p j'$. By (4.1) the matrix $[E_{ij}]$ is the product WC of a $2n \times 6n$ matrix W and a $6n \times 2n$ matrix C. The first $2n$ columns of W are the polynomial

parts $U_{i\cdot}$, and the others are $\rho_{j'}\binom{-j'}{i}j'\lambda^{-j'}(X-\tau_{j'})$ for $1\le j'\le 4n$. The rows of C are the unit vectors and the integers $\binom{-j'}{j}$. The slots have weights $-2L_p(n)$ and $-2L_p(n)-\alpha_p j'$. Rank the slot j' as $4n-j'$ and the polynomial slots above them. By the Cauchy–Binet expansion each minor of W uses $2n$ distinct slots, so its weight is at least $\sum_{t<2n}(-2L_p(n)-\alpha_p(4n-t)) = -4nL_p(n)-\alpha_p(6n^2+n)$. □

4.2. **The rate.** The other primes are handled by Tang's estimates [29], which hold for every rational λ with $|\lambda|<1$ at the primes that divide none of a, b and $b-a$. We state them as we use them.

Proposition 4.3 (Tang)**.** *Let $\varepsilon>0$ and $\delta=1/(N+1)$ with $4/(N+1)<\varepsilon/2$. For large n, with the sums over primes p not dividing $ab(b-a)$:*

(a) (small primes) $\sum_{4<p\le\delta n} v_p(\operatorname{cont}\tilde Q_n)\log p \ge (-4\delta-\varepsilon/8)n^2$;
(b) (medium primes) $\sum_{\delta n<p\le n} v_p(\operatorname{cont}\tilde Q_n)\log p \ge (c_{\mathrm{med}}-\varepsilon/8)n^2$, with $c_{\mathrm{med}} = 28\log 2-\frac{9}{2}\log 3-\frac{181}{12}$;
(c) (outer primes) $\sum_{n<p\le 4n} v_p(\operatorname{cont}\tilde Q_n)\log p \ge (\frac{7}{2}-\varepsilon/8)n^2$;
(d) (large primes) $v_p(\operatorname{cont}\tilde Q_n)\ge 0$ for $p>4n$.

Part (a) is the local bound summed with Chebyshev's estimate. In part (b) Tang bounds v_p by a profile $n\,\varpi(p/n)$, where $\varpi(x)$ is an explicit function of $\lfloor 1/x\rfloor$, $\lfloor 2/x\rfloor$ and $\lfloor 4/x\rfloor$. The weighted prime number theorem turns the sum into $n^2\int_\delta^1\varpi$. On each interval where $\lfloor 1/x\rfloor$ is constant the integral is a rational function of the floor value. Partial fractions and harmonic numbers evaluate the sum of these integrals over all intervals as c_{med}, and every partial sum is at least c_{med}. Part (c) counts the multiples of p among the factors of S_n^{2n}/F_n and the poles. Part (d) holds because every factor is then p-integral. Section 9 proves all four parts.

Theorem 4.4 (arithmetic rate)**.** *For every $\varepsilon>0$ there is n_1 such that $\log\tilde d_n\le -(c(a,b)-\varepsilon)n^2$ for $n\ge n_1$ with $\tilde Q_n\ne 0$.*

Proof. Split the primes $p\le 4n$ into the set $\mathcal{S}$ of primes dividing $ab(b-a)$, the primes $p\le 4$, and the three ranges of Proposition 4.3. For n large the last two ranges miss $\mathcal{S}$. By Theorem 4.1, $A_2(n)\ge\frac{8}{3}n^2-O(n)$, $S(n)\le\frac{9}{4}n^2$ and $L_p(n)=O(\log n)$. So

$$\sum_{p\in\mathcal{S}} v_p(\operatorname{cont}\tilde Q_n)\log p \ge \Big(\frac{8}{3}\log b-6\log|a|-\frac{9}{4}\log|b-a|-\frac{\varepsilon}{8}\Big)n^2.$$

The primes $p\le 4$ outside $\mathcal{S}$ cost $O(n\log n)$. Adding Proposition 4.3, with δ so small that $4\delta<\varepsilon/8$, gives the claim, since $\frac{7}{2}+c_{\mathrm{med}}=c_0$. □

Remark 4.5. The rate of Theorem 4.4 is the rate of Liu's construction, quoted by Tan and Zhang [28, (2.2)]. The logarithmic terms are the same. Liu's constant is $\frac{72}{25}=2.88$, and ours is $c_0=\frac{7}{2}+c_{\mathrm{med}}=2.8810324\ldots$, larger by $0.0010324\ldots$.

5. Nonvanishing

5.1. **Statement.** Let $\ell \geq 13$ be a prime. Write $n = q\ell + r$ with $0 \leq r < \ell$, $k = \lfloor 4r/\ell \rfloor$ and $r' = 4r - k\ell$. We call n *tied* at ℓ if $r' < r$. We assume

$$\ell(\ell-1) > 4n, \qquad \text{and} \qquad \ell \leq n \text{ or } 3\ell < 4n < 4\ell. \tag{5.1}$$

For integers $0 \leq \alpha' < \gamma'$ and $m \geq 0$ define the *local weights*

$$w_{\alpha',\gamma'}(u) = \frac{\prod_{m'<\alpha'}(u+m')^2}{\prod_{m'=\alpha'}^{\gamma'-1}(u+m')}, \qquad w^0_{\alpha',\gamma'}(u) = \frac{\prod_{m'=1}^{\alpha'}(u+m')^2}{\prod_{m'=\alpha'+1}^{\gamma'}(u+m')},$$

and the *type constants*

$$K_V(\alpha',\gamma';m) = \det\big[\mathcal{V}(u^{i+i'}w_{\alpha',\gamma'})\big]_{i,i'<m},$$
$$K_U(\alpha',\gamma';m) = \det\big[\mathcal{V}(u^{i+i'+1}w^0_{\alpha',\gamma'})\big]_{i,i'<m},$$

where $\mathcal{V}$ is the functional $f \mapsto \sum_{x\geq 1}\lambda^x f'(x)$ with the coefficient of $\mathrm{Li}_2(\lambda)$ set to 0. Concretely $\mathcal{V}(u^j) = j\,\mathrm{Li}_{1-j}(\lambda)$ and $\mathcal{V}(1/(u+m)) = \lambda^{-m}\tau_m$. These are rational functions of λ. For $q \leq 12$ and $k \leq 3$ let $\mathcal{K}(q,k)$ be the finite list of constants attached to the residue types below, together with $K_V(q+1, 4q+k; 2q+k-1)$ and $K_U(q, 4q+k; 2q+k-1)$ when $k \geq 1$.

Theorem 5.1 (nonvanishing)**.** *Let ℓ and n satisfy (5.1) with $q \leq 12$, and let ℓ divide none of the numerators and denominators of λ and $1-\lambda$. Suppose that every constant of $\mathcal{K}(q,k)$ is an ℓ-adic unit and, if n is tied, that the number $T'(n,\ell)$ defined in (5.2) is not divisible by ℓ. Then $Q_n(x) \neq 0$ for every $x \in \mathbb{Q}$ with $v_\ell(x) \geq -1$.*

The tolerance $v_\ell(x) \geq -1$ allows a rational p/q whose denominator is divisible by ℓ but not by ℓ^2.

5.2. **Residue classes.** For $0 \leq \sigma < \ell$ let a_σ be the number of $i \in [1,n]$ and c_σ the number of $j \in [1,4n]$ with $i \equiv \sigma$, $j \equiv \sigma \pmod{\ell}$. Put $m_\sigma = c_\sigma - 2a_\sigma$, $b_\sigma = c_\sigma - a_\sigma$ and $\nu_\sigma = 3a_\sigma - c_\sigma - 1 + [\sigma = 0]$. Then $\sum_\sigma m_\sigma = 2n$. Under (5.1) we have $c_\sigma \leq \ell - 1$ and $m_\sigma \geq 0$, and the classes fall into five types.

type	classes	(a_σ, c_σ)	m_σ
A	$1 \leq \sigma \leq \min(r,r')$	$(q+1, 4q+k+1)$	$2q+k-1$
B	$r < \sigma \leq r'$	$(q, 4q+k+1)$	$2q+k+1$
C	$r' < \sigma \leq r$	$(q+1, 4q+k)$	$2q+k-2$
D	$\sigma > \max(r,r')$	$(q, 4q+k)$	$2q+k$
zero	$\sigma = 0$	$(q, 4q+k)$	$2q+k$

The types B and C never occur together, and C occurs exactly when n is tied. The type constant of a class of type A, B, C or D is $K_V(a_\sigma, c_\sigma; m_\sigma)$, and that of the zero class is $K_U(a_0, c_0; m_0)$.

5.3. **Proof of Theorem 5.1.** We follow the reduction of the Hankel matrix modulo ℓ in a basis adapted to the classes.

Step 1: the basis. Put $\Pi_\sigma = \prod_{\sigma' \neq \sigma} (x+\sigma')^{m_{\sigma'}}$ and $e_{\sigma,i} = (x+\sigma)^i \Pi_\sigma$ for $i < m_\sigma$. These are $2n$ polynomials of degree below $2n$. Their coefficient matrix T has determinant prime to ℓ, by the Chinese remainder theorem, since the roots $-\sigma$ are distinct modulo ℓ. The Gram matrix $G(x) = [\Lambda_x(W_n e_\alpha e_\beta)]$ satisfies $\det G(x) = (\det T)^2 Q_n(x)$. So it suffices to show $\det G(x) \neq 0$.

Step 2: the ℓ-adic functional. Put $\lambda_\ell = \lambda^\ell$, so that $\lambda_\ell \equiv \lambda \pmod{\ell}$, and note that $\mathrm{Li}_{-k}(\lambda_\ell)$ is ℓ-integral since $1-\lambda_\ell$ is a unit. For a power series $h = \sum H_j y^j$ in $y = \ell u$ with ℓ-integral coefficients put $\mathcal{S}(h) = \sum_j H_j \ell^j \mathrm{Li}_{-j}(\lambda_\ell)$, a convergent ℓ-adic sum. The identity $\lambda \sum_{i \leq j} \binom{j}{i} \mathrm{Li}_{-i}(\lambda) = \mathrm{Li}_{-j}(\lambda) - \lambda$ gives the shift rule $\mathcal{S}(h(\cdot+1)) = \lambda_\ell^{-1}(\mathcal{S}(h) - \lambda_\ell h(1))$. For $\ell \nmid c$ put $E(c) = \mathcal{S}((c+\ell u)^{-2})$. The shift rule gives $E(c+\ell) = \lambda_\ell^{-1} E(c) - (c+\ell)^{-2}$, and from this the sum

$$T(j) = \sum_{\substack{c \leq j \\ \ell \nmid c}} \frac{\lambda^c}{c^2} + \sum_{\substack{0 \leq \sigma < \ell \\ \sigma \not\equiv j}} \lambda^{j-\sigma} E(j-\sigma)$$

does not depend on j. Call its value L. It is ℓ-integral.

Step 3: decomposition by classes. For a rational function F with poles at negative integers and a class σ, expand $(xF)'$ at $x = -\sigma + \ell u$, and let $\mathcal{W}_\sigma(F;x)$ be $\lambda^{-\sigma}$ times the corresponding ℓ-adic sum, with $\ell^2(x-L)$ in place of the value $\mathrm{Li}_2(\lambda_\ell)$. Then $\Lambda_x(F) = \sum_{\sigma < \ell} \mathcal{W}_\sigma(F;x)$. For $F = x^k$ this is the distribution relation $\mathrm{Li}_{-k}(\lambda) = \sum_\sigma \lambda^{-\sigma} \sum_i \binom{k}{i} (-\sigma)^{k-i} \ell^i \mathrm{Li}_{-i}(\lambda_\ell)$, an identity of rational functions. For $F = 1/(x+j)$ it follows from the shift rule and the constancy of T. Since $v_\ell(L) \geq 0$, $x - L$ has valuation at least -1 when x does.

Step 4: local expansion. At $x = -\sigma + \ell u$ a factor $x+i$ with $i \equiv \sigma$ equals $\ell(u+m')$, and the other factors are ℓ-adic units times $1 + O(\ell u)$. So W_n is $\ell^{3a_\sigma - c_\sigma}$ times a unit times $w_\sigma(u)$ times a power series. It follows that $\mathcal{W}_\sigma(W_n e_\alpha e_\beta; x)$ has valuation at least $\nu_\sigma + d_\sigma(\alpha) + d_\sigma(\beta)$, where $d_\sigma(\alpha)$ is the order of vanishing of e_α at $-\sigma$ modulo ℓ. The part proportional to $\ell^2(x-L)$ has one more power of ℓ. This is where the tolerance enters. For two basis vectors of the same class σ the leading term is a unit times $\mathcal{V}(u^{i+i'+[\sigma=0]} w_\sigma)$ reduced modulo ℓ.

Step 5: thresholds. Put $\pi_{\sigma,i} = i + \nu_\sigma/2$ and $K_\sigma = m_\sigma + \nu_\sigma/2$. By the table, all K_σ lie in an interval of length one. A comparison of $\nu_{\sigma''} + d_{\sigma''}(\alpha) + d_{\sigma''}(\beta)$ with $\pi_\alpha + \pi_\beta$ shows that the entries of the scaled matrix $Z_{\alpha\beta} = \ell^{-\pi_\alpha - \pi_\beta} G_{\alpha\beta}$, with half-integers split between rows and columns by the parity of ν_σ, are ℓ-integral. Modulo ℓ the cross-class entries vanish, except in the tied case, where the classes of type C couple to the top row of the zero class.

Step 6: the reduction. If n is not tied, Z mod ℓ is block diagonal over the classes, and its determinant is a product of units times the type constants. If n is tied, the group formed by the classes of type C and the zero class has

a bordered block structure. Its Schur complement gives

$$T'(n,\ell) = c_1 - c_2 B(r,r'), \qquad B(r,r') = \sum_{i=r'+1}^{r} \binom{r}{i} \lambda^{-i}(1-\lambda^{-1})^{r-i}, \tag{5.2}$$

times a unit, where $c_1 = K_U(q, 4q+k; 2q+k)K_V(q+1, 4q+k; 2q+k-2)$ and $c_2 = K_U(q, 4q+k; 2q+k-1)K_V(q+1, 4q+k; 2q+k-1)$. The binomial tail comes from the units of Step 4, which Wilson's theorem evaluates modulo ℓ, and from the polynomial identity

$$\sum_{i=r'+1}^{r} \binom{r}{i} y^i (1-y)^{r-i} = \frac{r!}{r'!\,N!} \sum_{j=0}^{N} (-1)^j \binom{N}{j} \frac{y^{r'+1+j}}{r'+1+j} \qquad (N = r-r'-1).$$

In both cases $\det Z \not\equiv 0 \pmod{\ell}$, so $\det G(x) \neq 0$.

5.4. **The pair rule.** When both n and $n+1$ are tied, a second congruence comes from Pascal's rule. The residue k is the same, and $r'(n+1) = r'(n)+4$. If $T'(n,\ell) \equiv T'(n+1,\ell) \equiv 0$ and $\ell \nmid c_2$, then ℓ divides $R_y(r)$, where $y = \lambda^{-1}$ and

$$\begin{aligned} R_y(r) = {} & 4(1-y)^3 y(4r+1)(4r+2)(4r+3) - 4(1-y)^3 y(r+1)(4r+2)(4r+3) \\ & + 12(1-y)^2 y^2 r(r+1)(4r+3) - 12(1-y) y^3 r(r+1)(3r+1) \\ & + 3y^4 r(3r+1)(3r+2). \end{aligned}$$

As a polynomial in r, $R_y = r(\rho_1(y) + \rho_2(y) r + \rho_3(y) r^2)$ with $\rho_3(y) = -3y(y-4)(27y^2 - 40y + 16)$, and the gcd of ρ_1, ρ_2, ρ_3 in $\mathbb{Z}[y]$ is $3y$. So for every rational $y \neq 0$ the polynomial R_y is nonzero, and it has at most three roots modulo ℓ for all ℓ outside a finite set.

Corollary 5.2. *Suppose that every type constant of $\mathcal{K}(q,k)$, $q \leq 12$, $k \leq 3$, is a nonzero rational at λ. Then the family (P_n) has a band nonvanishing datum on $(\frac{1}{13}, \frac{4}{3})$.*

Proof. Let B be the primes below 13, the primes dividing the numerators and denominators of $6\lambda(1-\lambda)$, of the constants of $\mathcal{K}(q,k)$, of c_2 and of a nonzero coefficient of R_y. Let R_ℓ be the roots of R_y modulo ℓ. If $\frac{n+1}{13} < \ell < \frac{4}{3}n$ and n is large, then n and $n+1$ satisfy (5.1) with $q \leq 12$. If n is not tied, or $T'(n,\ell) \not\equiv 0$, Theorem 5.1 applies at n. Otherwise it applies at $n+1$, or both are exceptional and $n \bmod \ell \in R_\ell$. Since P_n is a nonzero rational multiple of Q_n, the conclusion holds for P_n. □

5.5. **The type constants.** It remains to show that the constants are nonzero. There are 318 of them for $q \leq 12$. Each is a rational function of λ, and $K_V(1,4;2) = -(\lambda+12)(13\lambda-12)/(8\lambda^3(\lambda-1))$ shows that some vanish at rational points. Let $Z_0 = \{-12, -4, -\frac{4}{5}, \frac{12}{13}\}$.

Theorem 5.3. *Every constant of $\mathcal{K}(q,k)$ with $q \leq 12$ and $k \leq 3$ is nonzero at every rational $\lambda \notin Z_0 \cup \{0,1\}$. In particular it is nonzero for $0 < |\lambda| \leq \frac{1}{2}$.*

The proof rests on a principle for rational roots and on the leading terms of the constants at 0, 1 and ∞.

Lemma 5.4. *Let $F \in \mathbb{Z}[z]$ be nonzero and p prime. Let v_0 and v_1 be the orders of F at 0 and 1. Suppose that p does not divide the lowest coefficient of F at 0, the lowest Taylor coefficient of F at 1, or the leading coefficient of F, and that $F \bmod p$ has no root in $\mathbb{F}_p \setminus \{0,1\}$. Then $F(z) \neq 0$ for every $z \in \mathbb{Q} \setminus \{0,1\}$.*

Proof. Let $z = a'/b'$ in lowest terms and $G = b'^{\deg F} F(a'/b')$. If $p \nmid a'b'(b'-a')$, then $G \not\equiv 0$ by the root condition. If $p \mid a'$, then $G = a'^{v_0}(f_{v_0} b'^{\deg F - v_0} + a'(\dots))$ and the bracket is prime to p. The cases $p \mid b'-a'$ and $p \mid b'$ are the same with the Taylor expansion at 1 and the leading coefficient. □

Lemma 5.5. *Let M be an $m \times m$ matrix over the Laurent series in one variable x, and let r_i, c_j be integers with $\operatorname{ord} M_{ij} \geq r_i + c_j$. Let Λ_{ij} be the coefficient of $x^{r_i+c_j}$ in M_{ij}. Then $\det M = x^{\sum r_i + \sum c_j}(\det \Lambda + O(x))$.*

Proof. Multiply by $\operatorname{diag}(x^{-r_i})$ and $\operatorname{diag}(x^{-c_j})$ and reduce to the constant term. □

Proof of Theorem 5.3. Write the weight w of a constant as a polynomial part plus $\sum_m \varrho_m/(u+m)$. Then

$$\mathcal{V}(g\,w) = \sum_m \varrho_m g(-m)\lambda^{-m}\tau_m + \sum_{k\geq 1} k\,p_k\,\mathrm{Li}_{1-k}(\lambda),$$

with p_k the coefficients of the polynomial part of gw. At each of the points $\lambda = 0$, 1, ∞ we choose a basis in which the Hankel matrix has an explicit pattern of orders, and Lemma 5.5 gives the leading coefficient.

- At $\lambda = 0$ we use the Newton basis on the largest poles $m_1 > m_2 > \dots$. Since $\lambda^{-m}\tau_m$ has order $1-m$ with leading coefficient 1, the matrix of leading terms is triangular. The leading coefficient is $\prod_k \varrho_{m_k} \prod_{k<l}(m_k - m_l)^2$.
- At $\lambda = 1$ the pole part is regular with value $\sum_m \varrho_m g(-m) H_m^{(2)}$, where $H_m^{(2)} = \sum_{c\leq m} c^{-2}$, and $\mathrm{Li}_{-j}(\lambda)$ has a pole of order $j+1$ with leading coefficient $j!$. The matrix of leading terms is block triangular. Its determinant is one Hankel determinant of size at most 8 in the numbers $\sum_m \varrho_m (-m)^i H_m^{(2)}$, times products of factorials. This uses $\det[(i+j+\gamma)!]_{i,j<N} = \prod_{k<N} k!(k+\gamma)!$.
- At $\lambda = \infty$ we use $\mathrm{Li}_{-j}(1/w) = (-1)^{j+1}\mathrm{Li}_{-j}(w)$ and the Hermite–Newton basis on the 3-jets of the poles. The matrix of leading terms is block triangular with anti-triangular blocks, of determinant ϱ_m^3 times squares of values of the basis, and one block of size at most 2.

For each constant let $F \in \mathbb{Z}[\lambda]$ be the numerator obtained by clearing the denominators of the moments. The three leading coefficients of F are the closed forms above, up to a power of a fixed integer. For each of 313 of the 318 constants we fix a prime p and check in exact arithmetic that p

divides none of the three leading coefficients and that $F \bmod p$ has no root in $\mathbb{F}_p \setminus \{0,1\}$. Lemma 5.4 then gives the claim. The other five constants have $q \le 1$. For them F has degree at most 10, and an exact factorization shows that its rational roots lie in $Z_0 \cup \{0,1\}$. □

6. Decay

6.1. **The field and the dual value.** Put

$$\Phi_0(z) = 3\int_0^1 \log|z+y|\,dy - \int_0^4 \log|z+y|\,dy.$$

This is the limit of $n^{-1}\log|W_n(nz)| + \log n$. For $1 \le x \le 4$, $\Phi_0(-x) = 2x\log x - 3(x-1)\log(x-1) - (4-x)\log(4-x) + 1$. Let $\kappa_s = 2\pi$ for $s = +1$ and $\kappa_s = \pi$ for $s = -1$. Define on $\{\operatorname{Re} z \ge 0\}$ and on $[1,4]$

$$V(z) = \Phi_0(z) - t\operatorname{Re} z - \kappa_s|\operatorname{Im} z|, \qquad \omega(x) = \Phi_0(-x) + tx.$$

The first is the rate of the weights λ^x and of the kernel of a contour. The second is the rate of the residues $r_j j\lambda^{-j}$ at $j = xn$, by (2.3) and Stirling's formula.

We use two kinds of frame. The *lattice frame* has no contour, and every integer $x \ge 1$ is a site. A *bent frame* is used for $\lambda < 0$. It is a polyline Γ in $\{\operatorname{Re} z \ge 0\}$ from $-i\infty$ to $+i\infty$, symmetric under conjugation, that meets $\mathbb{R}$ only at 0, has no horizontal edge through 0, and ends in vertical rays. We use four bent frames $\Gamma_2, \Gamma_3, \Gamma_4, \Gamma_{15}$. Their upper vertices after 0 are

$$\Gamma_2:\ \tfrac{1}{200} + \tfrac{1}{8}i,\ \tfrac{89}{100} + \tfrac{457}{200}i; \qquad \Gamma_3:\ \tfrac{2}{5} + \tfrac{18}{25}i,\ \tfrac{7}{5} + \tfrac{68}{25}i;$$
$$\Gamma_4:\ \tfrac{6}{25} + \tfrac{3}{10}i,\ \tfrac{167}{100} + \tfrac{263}{100}i; \qquad \Gamma_{15}:\ \tfrac{13}{50} + \tfrac{3}{50}i,\ 2 + 2i,$$

each followed by a vertical ray.

A *dual datum* is a pair (σ, θ) with $\theta \in \mathbb{R}$ and σ a positive measure of mass 2 and compact support whose potential is Hölder continuous. Finite sums of measures on segments with bounded density against the uniform or the arcsine law have this property. Put

$$W_\sigma = V - 2U^\sigma, \qquad A_\rho(x) = \omega(x) - \rho - 2U^\sigma(-x) \quad (1 \le x \le 4).$$

The datum is *admissible* if $W_\sigma \le \theta$ on Γ. For the lattice frame there is no condition. The *dual value* is

$$D_\rho(\sigma,\theta) = -(12\log 2 + 4) - I(\sigma) - 2\theta - 2\int_0^\infty (W_\sigma(y) - \theta)_+\,dy - \int_1^4 (A_\rho(x) - \theta)_+\,dx \tag{6.1}$$

for the lattice frame, and the same without the lattice integral for a bent frame.

Theorem 6.1 (decay)**.** *Let (σ,θ) be admissible for the lattice frame, or for a bent frame when $\lambda < 0$. Then for every $\rho > 0$*

$$\limsup_{n\to\infty} \frac{1}{n^2}\log \sup_{|\varepsilon| \le e^{-\rho n}} |\tilde{Q}_n(\xi + \varepsilon)| \le -D_\rho(\sigma,\theta).$$

The dual value depends on λ only through t and s. For the lattice frame it does not depend on s.

6.2. **Heine's formula with jets.** By (2.1) and (2.2), $\Lambda_{\xi+\varepsilon}(W_nT) = \Lambda_\xi(W_nT) + \varepsilon\sum_{j=n+1}^{4n} r_j j\lambda^{-j}T(-j)$. For the lattice frame $\Lambda_\xi(W_nT) = \sum_{x\ge1}(\alpha_x T(x) + \beta_x T'(x))$ with $\alpha_x = \lambda^x(xW_n)'(x)$ and $\beta_x = \lambda^x x W_n(x)$. So the functional is a sum of jets at the sites and point masses at the poles.

Lemma 6.2. *In Heine's formula for this functional, a configuration contributes only if every site carries at most two variables. A site with one variable contributes $(\alpha_x + \beta_x\partial)$ applied to the square of the Vandermonde product, and a site with two variables u_k, u_l contributes $-2\beta_x^2$ times the product with the factor $(u_k - u_l)^2$ removed.*

Proof. At a site carrying m variables the operator has order at most m. The square of the Vandermonde product vanishes to order $m(m-1)$ on the diagonal, which exceeds m for $m \ge 3$. For $m = 2$ only $\beta_x^2\partial_k\partial_l$ survives, and $\partial_k\partial_l(u_k - u_l)^2 = -2$. □

A derivative brings at most a factor Cn, since $|\alpha_x/\beta_x| \le 5(1 + \log n)$ and the other variables are at distance at least $\frac12$. So every term is bounded by $e^{O(n\log n)}$ times the product of the weights of its charges and the reduced Vandermonde product.

6.3. **The contour.** For a bent frame the sum over the sites is replaced by a contour integral.

Lemma 6.3. *Let $\lambda < 0$, $L = \log|\lambda|$ and T a polynomial. Let Γ_n be $n\Gamma$ with its part in $|\operatorname{Im} z| \le 1$ replaced by the segment $\{\frac12 + iy : |y| \le 1\}$ and two horizontal joins. Then*

$$\sum_{x\ge1} \lambda^x (xW_nT)'(x) = \frac{1}{2\pi i}\int_{\Gamma_n} T(z)\, zW_n(z)|\lambda|^z\, \kappa(z)\, dz,$$

where $\kappa = -k' - Lk$ and $k(z) = \pi/\sin\pi z$, with Γ_n oriented downward, and $|\kappa(z)| \le Ce^{-\pi|\operatorname{Im} z|}$ on Γ_n.

Proof. The kernel k has residue $(-1)^x$ at x. The residue theorem on the region to the right of Γ_n gives the sum with $(H' - LH)k$, $H(z) = zW_n(z)T(z)|\lambda|^z$, since the arcs at infinity contribute 0. An integration by parts moves the derivative to the kernel. On the vertical segment $|\sin\pi z| \ge 1$, on the joins $|\sin\pi z| \ge \sinh\pi$, and for $|\operatorname{Im} z| \ge 1$ the bound follows from $|\sin\pi z| \ge \frac12 e^{\pi|\operatorname{Im} z|}(1 - e^{-2\pi})$. □

6.4. **The weights.**

Lemma 6.4. *(a) For $\operatorname{Re} u \ge 0$ and $|u| \ge \frac12$, $\log|W_n(u)| \le n\Phi_0(u/n) - n\log n + 3\log(|u| + n + 1) + 3\log 2$. (b) For $n < j \le 4n$, $\log|r_j j\lambda^{-j}| = n\,\omega(j/n) - n\log n + O(\log n)$.*

Proof. (a) The function $y \mapsto \log|u+y|$ is nondecreasing on $y \geq 0$. So the sum over $1 \leq i \leq n$ is at most the integral over $[1, n+1]$, and the sum over $1 \leq j \leq 4n$ is at least the integral over $[0, 4n]$. Substitute $y = nx$. (b) Stirling's formula in (2.3). □

6.5. **The dual inequality.** Scale every charge by n. A configuration then consists of $2n$ points s_i: lattice points x/n with at most two per site, contour points, and points $-j/n$ with at most one per pole.

Lemma 6.5. *Let σ be a positive measure of mass 2 and compact support with a Hölder continuous potential. For every configuration of $2n$ points,*

$$\sum_{i \neq j} \log|s_i - s_j| \leq n^2 I(\sigma) - 2n \sum_i U^\sigma(s_i) + o(n^2),$$

uniformly in the configuration, where a confluent pair counts with $\log 2$ in place of $\log 0$.

Proof. Replace each point by the normalized arc measure on a circle of radius n^{-2} around it, and let μ^ϵ be n^{-1} times their sum. Subharmonicity bounds $\log|s_i - s_j|$ by the average over the two circles. The energy of the neutral measure $\mu^\epsilon - \sigma$ is nonnegative [27, Lemma I.1.8], so $-I(\mu^\epsilon) \leq I(\sigma) - 2\int U^\sigma d\mu^\epsilon$. The Hölder continuity of U^σ replaces the averages over the circles by the values at the points with an error $o(n^2)$ in total. □

Proof of Theorem 6.1. By Lemma 2.1 and Lemma 6.2, $|Q_n(\xi+\varepsilon)|$ is at most $e^{O(n \log n)}$ times a sum over configurations of the product of the weights and of the reduced Vandermonde product. By Lemma 6.4 and Lemma 6.3, a lattice or contour charge at s has weight at most $e^{n(V(s) - \log n) + O(\log n)}$, and an atom charge at $-x$ has weight at most $|\varepsilon| e^{n(\omega(x) - \log n) + O(\log n)} \leq e^{n(\omega(x) - \rho - \log n) + O(\log n)}$. Put $f(s) = W_\sigma(s)$ for a lattice or contour charge and $f(-x) = A_\rho(x)$ for an atom charge. The Vandermonde product contributes $n^{2n(2n-1)} = e^{4n^2 \log n + O(n \log n)}$ from the scaling, and the $2n$ weights contribute $e^{-2n^2 \log n}$. By Lemma 6.5 the logarithm of a term is at most

$$n^2 I(\sigma) + n \sum_i f(s_i) + 2n^2 \log n + o(n^2).$$

For any θ,

$$\sum_i f(s_i) \leq 2n\theta + \sum_i (f(s_i) - \theta)_+.$$

Contour charges have $W_\sigma \leq \theta$ by admissibility. Lattice charges occupy each site at most twice, so their excess is at most $2n \int_0^\infty (W_\sigma - \theta)_+ + o(n)$. Atom charges occupy each pole at most once, so theirs is at most $n \int_1^4 (A_\rho - \theta)_+ + o(n)$. The number of configurations is $e^{O(n \log n)}$ after the tails are summed, since $W_\sigma(z) \leq 3\log(|z| + R) - t \operatorname{Re} z - \kappa_s |\operatorname{Im} z| + O(1)$ for large $|z|$. Finally $\log(S_n^{2n}/F_n) = -2n^2 \log n + (12 \log 2 + 4) n^2 + O(n \log n)$, which cancels the power of n and contributes the constant. Adding these bounds gives $\log|\tilde{Q}_n(\xi+\varepsilon)| \leq -D_\rho(\sigma, \theta) n^2 + o(n^2)$. □

6.6. **Certificates.** A dual datum gives a lower bound for the decay rate once three quantities are bounded: the energy $I(\sigma)$, the admissibility on Γ, and the two penalty integrals. Our data are of two shapes.

For the lattice frame, σ is the sum of a measure on $[0, Y]$ and a measure on $[-4, -1]$, the second carrying the share of the poles. Each is the arcsine law of its segment times $|\sum_k h_k e^{ik\phi}|^2$, with $t = \cos\phi$ and rational h_k. Its potential inside and outside the segment and its energy are explicit Chebyshev series, as in [15]. The cross energy of the two segments is a sum of Stieltjes moments. The penalty $\int (W_\sigma - \theta)_+$ is computed from the sign pattern of $W_\sigma - \theta$. Between consecutive rational points where the sign is constant, the integral of $(W_\sigma - \theta)_+$ is either 0 or an explicit integral of $W_\sigma - \theta$. The signs are proved on cells by Taylor polynomials with explicit remainders. Beyond the support, U^σ is monotone, and beyond $y = 60$ the tail bound $W_\sigma \le 3\log(y+1) - 4\log y - ty + 4\log(y+R)$ applies.

For a bent frame, σ is a sum of uniform measures on panels of Γ and on $[-4, -1]$. The admissibility $W_\sigma \le \theta$ is proved on cells along each edge, the energy is a finite sum of explicit panel integrals, and the atom penalty is bounded by a piecewise linear majorant of $\omega - 2U^\sigma(-x)$, which covers every ρ at once.

Every such inequality is a finite computation with rational numbers and rational enclosures of logarithms. These were designed with the interval arithmetic of Arb [12] and are checked in the formal proof by the Lean kernel. The data were found by numerical optimization and play no other role in the proof.

7. Uniformity in the argument

7.1. **Structure of the dual.** For a frame F write $D^F_\rho(\sigma, \theta; t)$ for the dual value (6.1) with t as a parameter, and call the datum admissible at t if $W_\sigma \le \theta$ on the contour.

Lemma 7.1. *Fix a sign and a frame.*

(a) (concavity) The set of $(\sigma, \theta, t, \rho)$ with (σ, θ) admissible at t is convex, and $D^F_\rho(\sigma, \theta; t)$ is jointly concave on it.

(b) (transport) Let $0 \le m \le 1$, with $m = 0$ for a bent frame. If (σ, θ) is admissible at t and $\Delta \ge 0$, then $(\sigma, \theta - m\Delta)$ is admissible at $t + \Delta$, and

$$D^F_{\rho+(4+m)\Delta}(\sigma, \theta - m\Delta; t + \Delta) \ge D^F_\rho(\sigma, \theta; t) + (2m - m^2)\Delta.$$

(c) (monotonicity) $D^F_\rho(\sigma, \theta; t)$ is nondecreasing in ρ.

(d) (sign) For the lattice frame the dual value and admissibility do not depend on the sign.

Proof. (a) The fields $W_\sigma - \theta$ and $A_\rho - \theta$ are affine in $(\sigma, \theta, t, \rho)$, so the positive parts and their integrals are convex, and admissibility is a convex condition.

For $\sigma_\lambda = \lambda\sigma_0 + (1-\lambda)\sigma_1$,

$$I(\sigma_\lambda) = \lambda I(\sigma_0) + (1-\lambda)I(\sigma_1) - \lambda(1-\lambda)I(\sigma_0 - \sigma_1),$$

and $I(\sigma_0 - \sigma_1) \geq 0$ for a neutral signed measure of finite energy [27, Lemma I.1.8]. So $-I$ is concave.

(b) At $t + \Delta$ the fields are $W_\sigma - \Delta \operatorname{Re} z$ and $A_\rho + \Delta x$. On a bent frame $\operatorname{Re} z \geq 0 = m$. On the lattice,

$$\int_0^\infty (W_\sigma - \Delta y - \theta + m\Delta)_+ dy \leq \int_0^\infty (W_\sigma - \theta)_+ dy + \Delta \int_0^\infty (m-y)_+ dy,$$

and the last integral is $m^2/2$. With $\rho + (4+m)\Delta$ in place of ρ, the atom field changes by $\Delta(x-4) \leq 0$ on $[1,4]$. The level term gains $2m\Delta$.

(c) A_ρ decreases in ρ. (d) The lattice frame uses only $|\lambda|$. □

By Lemma 7.1(b), lowering the level by Δ gains two units of mass, and the saturated lattice loses at most $\Delta \int_0^1 2(1-y)\, dy = \Delta$. This is the source of the slope 1 in t.

7.2. **The table.** A *node* is a frame together with a rational number λ_k with $0 < |\lambda_k| \leq \frac{1}{2}$. Its value $t_k = \log(1/|\lambda_k|)$ is the parameter of its entries. An *entry* of a node is a pair of rationals (ρ_k, D_k) for which a dual datum admissible at t_k with $D^F_{\rho_k} \geq D_k$ is certified as in Section 6.6. Our table has 24 lattice nodes and 13 bent nodes, with 151 and 268 entries. Lattice nodes serve both signs, and bent nodes serve $s = -1$. Appendix A lists them.

Definition 7.2. For a sign s, $t > 0$ and $\rho > 0$, a number D is a *candidate* at (s, t, ρ) if it arises in one of two ways from entries of nodes serving the sign s.

(I) (interpolation) Two entries (ρ_i, D_i) and (ρ_j, D_j) of nodes on the same frame with $t_i < t_j$, $t_j - t_i \leq \frac{4}{5}$, $t = \theta t_i + (1-\theta)t_j$ with $\theta \in [0,1]$, and $\theta\rho_i + (1-\theta)\rho_j \leq \rho$. Then $D = \theta D_i + (1-\theta)D_j$.

(II) (transport) One entry (ρ_k, D_k) with $t_k \leq t$ and a slope $m \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ if the node is a lattice node, or $m = 0$, with $\rho_k + (4+m)(t - t_k) \leq \rho$. Then $D = D_k + (2m - m^2)(t - t_k)$.

Let $\underline{D}_s(t, \rho)$ be the largest candidate, or $-\infty$ if there is none.

Theorem 7.3 (uniform decay)**.** *Let λ be rational with $0 < |\lambda| < 1$, $s = \operatorname{sign}\lambda$ and $t = \log(1/|\lambda|)$. If D is a candidate at (s, t, ρ) with $\rho > 0$, then for every $\varepsilon > 0$ and all large n, $|\tilde{Q}_n(\xi + u)| \leq e^{-(D-\varepsilon)n^2}$ whenever $|u| \leq e^{-(\rho+\varepsilon)n}$.*

Proof. For an interpolation candidate, the mixture of the two certified data with weights θ and $1-\theta$ is admissible at t and has dual value at least D at $\theta\rho_i + (1-\theta)\rho_j$, by Lemma 7.1(a). For a transport candidate the transported datum of Lemma 7.1(b) does the same. Lemma 7.1(c) passes to ρ, and Lemma 7.1(d) allows lattice nodes for $s = -1$. Theorem 6.1 with $\rho + \varepsilon$ in place of ρ, and the monotonicity in ρ, give the claim. □

8. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\lambda = a/b$ and $P_n = \tilde{d}_n \tilde{Q}_n \in \mathbb{Z}[X]$, of degree at most $2n$. Since $0 < |\lambda| \le \frac{1}{2}$, Theorem 5.3 shows that all type constants are nonzero at λ, and Corollary 5.2 gives a band nonvanishing datum on $(\frac{1}{13}, \frac{4}{3})$ for (P_n). So $M = 2(\frac{4}{3} - \frac{1}{13}) = \frac{98}{39}$.

Let $D = \underline{D}_s(t, \rho)$, attained by a candidate. By Theorem 7.3, $|\tilde{Q}_n(\xi + u)| \le e^{-(D-\varepsilon)n^2}$ for $|u| \le e^{-(\rho+\varepsilon)n}$ and n large. By Theorem 4.4, $\tilde{d}_n \le e^{-(c(a,b)-\varepsilon)n^2}$ whenever $\tilde{Q}_n \ne 0$. When $\tilde{Q}_n = 0$, $P_n = 0$. Hence

$$|P_n(\xi + u)| \le e^{-(\eta-2\varepsilon)n^2} \qquad (|u| \le e^{-(\rho+\varepsilon)n}),$$

with $\eta = D + c(a, b) > 0$. Theorem 3.2 with $\kappa = 2$ and $(\alpha, \beta) = (\frac{1}{13}, \frac{4}{3})$ gives the measure. □

Proof of Corollary 1.2. Each entry is Theorem 1.1 at one candidate. For example, at $\lambda = \frac{1}{3}$ the lattice node $\frac{1}{3}$ has the entry $\rho = \frac{35}{2}$, $D = -2.624336\ldots$, and $c(1, 3) = 4.251084\ldots$. So $\eta = 1.626747\ldots$ and $\rho \max(2/\eta, 39/98) = 21.5153\ldots$. At $\lambda = -\frac{1}{3}$ the bent node $-\frac{1}{3}$ on Γ_3 has the entry $\rho = 18$, $D = 0.336425\ldots$, and $c(-1, 3) = 2.691502\ldots$, which gives $11.8893\ldots$. At $\lambda = \frac{1}{2}$ the entry $\rho = \frac{31}{2}$, $D = -4.593612\ldots$ and $c(1, 2) = 4.729424\ldots$ give $\eta = 0.135812\ldots$ and $228.2564\ldots$. The other entries of Table 1 are computed in the same way. □

Proof of Corollary 1.3. The ranges. For $\lambda = 2/b$ with odd $b \ge 13$ take the transport with slope $m = 1$ of entry 31, which lies at the lattice node $\frac{2}{13}$. Its margin is $\eta(b) = D_{31} + \log(b/13) + c(2, b)$. At $b = 13$ it is $0.0191\ldots > 0$, and

$$\frac{d\eta}{db} = \frac{1}{b} + \frac{8}{3b} - \frac{9}{4(b-2)} = \frac{17b - 88}{12b(b-2)} > 0 \qquad (b \ge 13).$$

For $\lambda = -2/b$ with odd $b \ge 11$ take the transport with slope $m = 0$ of entry 308, at the bent node $-\frac{2}{11}$. Its margin is $\eta(b) = D_{308} + c(-2, b)$. At $b = 11$ it is $0.4125\ldots > 0$, and

$$\frac{d\eta}{db} = \frac{8}{3b} - \frac{9}{4(b+2)} = \frac{5b + 64}{12b(b+2)} > 0 \qquad (b \ge 11).$$

So the margin stays positive on both ranges, and Theorem 1.1 gives a finite measure.

The tables. Each value is Theorem 1.1 at the indicated candidate. For example entry 308 has $\rho = \frac{91}{4}$ and $D = 5335835236283847/5000000000000000$, and at $\lambda = -\frac{2}{11}$ it gives $\eta = 0.412567\ldots$ and $110.2849\ldots$. At $-\frac{3}{50}$, $\frac{4}{167}$ and $-\frac{4}{177}$ the margins are $0.004394\ldots$, $0.004562\ldots$ and $0.008292\ldots$. A finite measure implies irrationality. □

Remark 8.1 (ranges without a measure). The margin η is at most 0 for every ρ when $|a| \ge 2$ and b is small, for instance for $\mathrm{Li}_2(2/11)$ and $\mathrm{Li}_2(-2/9)$. Near the first denominators with $\eta > 0$ the margin is small and the measure is large. This remark is not part of the formal proof.

9. Proofs of Tang's estimates

Tang's bounds for the denominators exist only as Lean code [29]. In this section we give paper proofs of all of them that Section 4 uses. For the medium primes we also evaluate Tang's profile integral in closed form, which gives the constant c_{med} of Proposition 4.3(b). These are the bound at a prime that divides none of a, b and $b-a$, the bound at a prime that divides b, the bounds at the medium, outer and large primes, and the sums over primes in Proposition 4.3. We also prove Lemma 4.2, which is Tang's lemma with one added parameter. The proofs follow the Lean statements and constants. At two places, the medium primes and the primes dividing b, our argument is shorter than the formal one. The statements are the same, and Remarks 9.13 and 9.10 describe the differences.

Throughout, p is a prime and $v = v_p$, with $v(0) = +\infty$. For $P = A_0 + A_1 X \in \mathbb{Q}[X]$ we write $v_p^{\mathrm{G}}(P) = \min(v(A_0), v(A_1))$. The Gauss valuation satisfies $v_p^{\mathrm{G}}(P+P') \geq \min(v_p^{\mathrm{G}}P, v_p^{\mathrm{G}}P')$ and $v_p^{\mathrm{G}}(PP') \geq v_p^{\mathrm{G}}P + v_p^{\mathrm{G}}P'$. The rational $\lambda = a/b$ is as in Section 4, so $0 < |\lambda| < 1$. We put $r = \lambda/(1-\lambda)$, $L = L_p(n) = \lfloor \log_p(7n-2) \rfloor$ and $q_k = \lfloor kn/p \rfloor$ for $k = 1, 2, 3, 4$.

9.1. **The functional.** For a polynomial Q put

$$\mathcal{G}(Q) = \sum_{x \geq 1} \lambda^x Q(x).$$

The series converges absolutely since $|\lambda| < 1$, and $\mathcal{G}(x^k) = \mathrm{Li}_{-k}(\lambda)$. In this notation the functional (2.1) is

$$\Lambda_X(R) = \mathcal{G}\big((xq)'\big) + \sum_{l=1}^{K} R_l \, l\lambda^{-l}(X - \tau_l) \tag{9.1}$$

for a rational function $R = F/D_K$ with F a polynomial. Here q is the polynomial part of R and R_l is its residue at $x = -l$. Indeed $\Lambda_X(x^k) = (k+1)\,\mathrm{Li}_{-k}(\lambda) = \mathcal{G}((x \cdot x^k)')$. The *binomial moments* are

$$G(k) = \mathcal{G}\Big(\binom{x}{k}\Big) = \sum_{x \geq 1} \lambda^x \binom{x}{k}.$$

So $G(0) = r$ and $G(k) = \lambda^k/(1-\lambda)^{k+1} = r^k/(1-\lambda)$ for $k \geq 1$.

Lemma 9.1. *For every polynomial Q and every integer $M \geq 0$,*

$$\mathcal{G}(Q) = -\sum_{m=0}^{M-1} \lambda^{-m} Q(-m) + \lambda^{-M} \mathcal{G}(Q(x-M)),$$

and $G(k+1) = r\big(G(k) + \binom{1}{k+1}\big)$ for $k \geq 0$.

Proof. Shifting the summation variable gives

$$\mathcal{G}(Q(x-1)) = \lambda \sum_{y \geq 0} \lambda^y Q(y) = \lambda(Q(0) + \mathcal{G}(Q)).$$

So $\mathcal{G}(Q) = \lambda^{-1}\mathcal{G}(Q(x-1)) - Q(0)$, and induction on M gives the first identity. With $Q = \binom{x}{k+1}$ the same shift reads $\mathcal{G}(\binom{x+1}{k+1}) = \lambda^{-1}(G(k+1) - \lambda\binom{1}{k+1})$. Since $\binom{x+1}{k+1} = \binom{x}{k+1} + \binom{x}{k}$, this gives

$$G(k+1)(\lambda^{-1} - 1) = G(k) + \binom{1}{k+1},$$

and $(\lambda^{-1} - 1)^{-1} = r$. □

9.2. **Integer-valued polynomials.** The next lemma collects three classical facts. We use the convention $\lfloor \log_p d \rfloor = 0$ for $d \le 1$.

Lemma 9.2. *Let $Q \in \mathbb{Q}[x]$ have degree at most d, and let $v(Q(i)) \ge \gamma$ for $0 \le i \le d$.*

(a) $v(Q(m)) \ge \gamma$ and $v(\Delta^k Q(m)) \ge \gamma$ for every integer m and every $k \ge 0$, where Δ is the forward difference.

(b) For every integer m and every $s \ge 0$, the coefficient of y^s in $Q(m+y)$ has valuation at least $\gamma - s\lfloor \log_p d \rfloor$. In particular $v(Q'(m)) \ge \gamma - \lfloor \log_p d \rfloor$.

Proof. (a) Newton's formula gives $Q(x) = \sum_{k \le d} \Delta^k Q(0) \binom{x}{k}$ with $\Delta^k Q(0) = \sum_{i \le k} (-1)^{k-i} \binom{k}{i} Q(i)$. The binomial coefficients $\binom{m}{k}$ are integers for every integer m, and differences of values at integers are integral combinations of such values.

(b) By (a), $Q(m+y) = \sum_{k \le d} \Delta^k Q(m) \binom{y}{k}$, so it suffices to show that the coefficient $\beta_{k,s}$ of y^s in $\binom{y}{k}$ satisfies $v(\beta_{k,s}) \ge -s\lfloor \log_p k \rfloor$ for $1 \le k \le d$. For $s = 0$ this holds since $\beta_{k,0} = 0$. From $\binom{y}{k} = \frac{y}{k}\binom{y-1}{k-1}$ and $\binom{y-1}{k-1} = \sum_{i<k} (-1)^{k-1-i}\binom{y}{i}$ we get $\beta_{k,s} = \frac{1}{k}\sum_{i<k} (-1)^{k-1-i}\beta_{i,s-1}$ for $s \ge 1$. Here $\beta_{0,s-1} = [s = 1]$. Since $v(k) \le \lfloor \log_p k \rfloor$, induction on s gives the claim. □

9.3. **Proof of Lemma 4.2.** We prove the three parts of Lemma 4.2, with $f \ge 0$ and $v(r) \ge -f$. For $f = 0$ these are Tang's lemmas.

(a) We have $G(0) = r$. By Lemma 9.1, $G(k+1) = r(G(k) + \binom{1}{k+1})$, and $\binom{1}{k+1} \in \{0, 1\}$. If $v(G(k)) \ge -(k+1)f$, then $v(G(k+1)) \ge -f + \min(-(k+1)f, 0) = -(k+2)f$.

(b) Let g have degree at most d with $v(g(i)) \ge \gamma$ for $0 \le i \le d$. The value of Λ on g is $\mathcal{G}(h)$ with $h = (xg)' = g + xg'$, a polynomial of degree at most d. By Lemma 9.2, $v(h(m)) \ge \gamma - \lfloor \log_p d \rfloor$ for every integer m. Newton's formula and linearity give $\mathcal{G}(h) = \sum_{k \le d} \Delta^k h(0)\, G(k)$, and (a) gives $v(G(k)) \ge -(d+1)f$ for $k \le d$. The same argument shows, for every polynomial h of degree at most d,

$$v(\mathcal{G}(h)) \ge \min_{0 \le i \le d} v(h(i)) - (d+1)f. \tag{9.2}$$

(c) Let q be the polynomial part of $S_n D_n^3 \binom{x}{i}\binom{x}{j}/D_{4n}$. Lemma 9.3(b) below shows that $q = 0$ if $i + j < n$, and otherwise $\deg q \le d = i + j - n$ and

$v(q(m)) \geq -\lfloor \log_p(3n+i+j) \rfloor \geq -L$ for every integer m, since $3n+i+j \leq 7n-2$. Then (b) gives $v(U_{ij}) \geq -L - \lfloor \log_p d \rfloor - (d+1)f \geq -2L - f(i+j-n+1)$. □

9.4. **The entries of the binomial Gram matrix.** Put $F_{ij} = S_n D_n^3 \binom{x}{i}\binom{x}{j}$ for $0 \leq i, j < 2n$, so that the entries (4.1) are $E_{ij} = \Lambda_X(F_{ij}/D_{4n})$.

Lemma 9.3. *Let $0 \leq i, j < 2n$ and $1 \leq l \leq 4n$.*

(a) The residue of F_{ij}/D_{4n} at $-l$ is an integer. It is 0 for $l \leq n$.
(b) The polynomial part q_{ij} of F_{ij}/D_{4n} is 0 if $i+j < n$. Otherwise it has degree at most $i+j-n$, and $v(q_{ij}(m)) \geq -\lfloor \log_p(3n+i+j) \rfloor$ for every prime p and every integer m.
(c) $\tilde{Q}_n = \det[E_{ij}]_{i,j<2n}$. If T is the coefficient matrix of polynomials $e_0, \dots, e_{2n-1}$ of degree below $2n$, then $\det[\Lambda_X(W_n e_i e_j)] = (\det T)^2 Q_n$.
(d) If $4n < p^2$, then $v(S_n^{2n}/F_n) = N_p(n)$ with $N_p(n) = 2n(q_4 - 3q_1) - 2\sum_{i<2n} \lfloor i/p \rfloor$.

Proof. Write $\binom{x+n}{n} = D_n(x)/n!$. Then $F_{ij} = (4n)!\, P_{ij}$ with $P_{ij} = \binom{x+n}{n}^3 \binom{x}{i}\binom{x}{j}$, a polynomial of degree $3n+i+j$ with integer values at the integers.

(a) The residue is $(4n)! P_{ij}(-l) / \prod_{l' \neq l}(l'-l)$, the product over $1 \leq l' \leq 4n$. The product equals $(-1)^{l-1}(l-1)!(4n-l)!$, so the residue is $\pm 4n \binom{4n-1}{l-1} P_{ij}(-l)$, an integer. For $l \leq n$ the factor $D_n(-l)$ vanishes.

(b) Expand $P_{ij} = \sum_{k \leq 3n+i+j} \pi_k \binom{x+4n}{k}$ with $\pi_k \in \mathbb{Z}$. For $k \geq 4n$,

$$(4n)! \binom{x+4n}{k} = D_{4n}(x) \binom{x}{k-4n} \Big/ \binom{k}{4n},$$

and for $k < 4n$ the term has degree below $4n$. So

$$q_{ij} = \sum_{4n \leq k \leq 3n+i+j} \pi_k \binom{x}{k-4n} \Big/ \binom{k}{4n}.$$

By Kummer's theorem $v(\binom{k}{4n}) \leq \lfloor \log_p k \rfloor$, which gives both claims.

(c) The polynomials $\binom{x}{i}$ have leading coefficients $1/i!$. So the coefficient matrix of $(\binom{x}{i})_{i<2n}$ is triangular with determinant $\prod_{i<2n} 1/i!$. The Gram matrix is bilinear in the basis. Hence $\det[E_{ij}] = S_n^{2n} \prod_{i<2n}(i!)^{-2} Q_n = \tilde{Q}_n$, and the second statement holds for the same reason.

(d) For $m < p^2$ we have $v(m!) = \lfloor m/p \rfloor$. Apply this to $(4n)!$, $n!$ and $i!$ with $i < 2n$. □

9.5. **Three determinant lemmas.** The following lemma is used at every prime.

Lemma 9.4. *Let M be an $h \times h$ matrix over $\mathbb{Q}[X]$.*

(a) If $v_p^{\mathrm{G}}(M_{ij}) \geq u_i + u'_j$ for all i, j, then $v_p^{\mathrm{G}}(\det M) \geq \sum_i u_i + \sum_j u'_j$.

(b) *Let $M = WC$ with W of size $h \times I$ and C of size $I \times h$ for a finite index set I. Suppose $v_p^{\mathrm{G}}(W_{is}) \ge \omega_s$ and $v_p^{\mathrm{G}}(C_{sj}) \ge \kappa_s$ for all i, j and $s \in I$. Let $\mathrm{rk} : I \to \mathbb{Z}_{\ge 0}$ be injective and θ a nondecreasing function with $\theta(\mathrm{rk}(s)) \le \omega_s + \kappa_s$ for all s. Then $v_p^{\mathrm{G}}(\det M) \ge \sum_{t<h} \theta(t)$.*

Proof. (a) Each term of the Leibniz expansion is a product of entries $M_{i\pi(i)}$, with Gauss valuation at least $\sum_i u_i + \sum_i u'_{\pi(i)}$.

(b) By the Cauchy–Binet formula $\det M = \sum_S \det W_{\cdot,S} \det C_{S,\cdot}$ over the subsets $S \subset I$ of size h. By (a) with constant row weights, $v_p^{\mathrm{G}}(\det W_{\cdot,S}) \ge \sum_{s\in S} \omega_s$, and similarly for C. So each term has Gauss valuation at least $\sum_{s\in S} \theta(\mathrm{rk}(s))$. The ranks of S are h distinct nonnegative integers. Since θ is nondecreasing, their sum is at least $\sum_{t<h} \theta(t)$. □

The third lemma treats a Hankel matrix plus point masses at nodes that are congruent modulo p.

Lemma 9.5. *Let $H_{ij} = V_{ij} + \sum_\sigma \sum_{t<C_\sigma} \gamma_{\sigma,t}\, x_{\sigma,t}^{i+j}$ for $0 \le i, j < h$, where σ runs over a finite set. Suppose that $v_p^{\mathrm{G}}(V_{ij}) \ge 0$, that the $x_{\sigma,t}$ are distinct integers with $x_{\sigma,t} \equiv x_{\sigma,0} \pmod p$, and that $v_p^{\mathrm{G}}(\gamma_{\sigma,t}) \ge w_{\sigma,t}$ with $w_{\sigma,t}$ nondecreasing in t. Then*

$$v_p^{\mathrm{G}}(\det H) \ge \sum_\sigma \sum_{k<C_\sigma} \min(w_{\sigma,k} + 2k, 0).$$

Proof. Fix σ and write $x_t = x_{\sigma,t}$. Newton interpolation on the nodes $x_0, x_1, \dots$ gives $y^i = \sum_{k\le i} h_{i-k}(x_0, \dots, x_k) \prod_{s<k}(y - x_s)$, where h_m is the complete homogeneous symmetric polynomial. At $y = x_t$ the terms with $k > t$ vanish. Put $A_{i,(\sigma,k)} = p^k h_{i-k}(x_0, \dots, x_k)$ and $T_{k,t} = p^{-k} \prod_{s<k}(x_t - x_s)$. Then $v(A_{i,(\sigma,k)}) \ge k$, the $T_{k,t}$ are p-integral, and $x_t^i = \sum_{k\le t} A_{i,(\sigma,k)} T_{k,t}$. Hence

$$\sum_t \gamma_{\sigma,t} x_t^{i+j} = \sum_{k,k'} A_{i,(\sigma,k)} M^\sigma_{kk'} A_{j,(\sigma,k')}, \qquad M^\sigma_{kk'} = \sum_{t\ge \max(k,k')} T_{k,t} T_{k',t} \gamma_{\sigma,t}.$$

Since w is nondecreasing, $v_p^{\mathrm{G}}(M^\sigma_{kk'}) \ge w_{\sigma,\max(k,k')} \ge \frac12 (w_{\sigma,k} + w_{\sigma,k'})$.

Let I consist of h slots s with $\rho_s = w_s = 0$ and the slots (σ, k) with $\rho = k$ and $w = w_{\sigma,k}$. Then $H = \tilde A \,\mathrm{diag}(V, M)\, \tilde A^T$ with $\tilde A = [\,\mathrm{Id} \mid A\,]$, where M is block diagonal with blocks M^σ. The column s of $\tilde A$ has valuation at least ρ_s, and the entry (s, s') of the middle matrix has Gauss valuation at least $\frac12(w_s + w_{s'})$. By the Cauchy–Binet formula and Lemma 9.4(a), every term of $\det H$ has Gauss valuation at least $\sum_{s\in S}(\rho_s + \frac12 w_s) + \sum_{s\in S'}(\rho_s + \frac12 w_s)$ for two sets $S, S' \subset I$. This is at least $2\sum_{s\in I} \min(\rho_s + \frac12 w_s, 0) = \sum_{\sigma,k} \min(w_{\sigma,k} + 2k, 0)$. □

9.6. **Primes not dividing $ab(b-a)$, and large primes.**

Lemma 9.6. *If $p \nmid ab(b-a)$, then λ, λ^{-1}, $1-\lambda$ and r are p-adic units.*

Proof. We have $\lambda = a/b$, $1 - \lambda = (b-a)/b$ and $r = a/(b-a)$. □

Proposition 9.7. *Let $p \nmid ab(b-a)$ and $n \geq 1$.*

(a) $v_p^{\mathrm{G}}(E_{ij}) \geq -2L$ for all $i, j < 2n$, and $v_p^{\mathrm{G}}(\tilde{Q}_n) \geq -4nL$.

(b) If $p > 4n$, then $v_p^{\mathrm{G}}(E_{ij}) \geq 0$ and $v_p^{\mathrm{G}}(\tilde{Q}_n) \geq 0$.

Proof. (a) By (9.1), $E_{ij} = U_{ij} + \sum_l c_l\, l\lambda^{-l}(X - \tau_l)$ with c_l the residues of Lemma 9.3(a). Lemma 4.2(c) with $f = 0$ gives $v(U_{ij}) \geq -2L$. The c_l are integers and λ^{-1} is a unit. Since λ is a unit and $c \leq 4n \leq 7n - 2$, each term λ^c/c^2 of τ_l has valuation at least $-2L$, so $v_p^{\mathrm{G}}(X - \tau_l) \geq -2L$. Lemma 9.4(a) with $u_i = u'_j = -L$ and Lemma 9.3(c) give the bound for $\tilde{Q}_n$.

(b) Now $(4n)!$, $n!$ and $i!$ for $i < 2n$ are units, and so is every integer in $[1, 4n]$. So $\binom{x}{i} \in \mathbb{Z}_{(p)}[x]$, and division by the monic D_{4n} keeps the polynomial part q_{ij} in $\mathbb{Z}_{(p)}[x]$. Then $(xq_{ij})'$ has p-integral coefficients, so its values at integers are p-integral, and $\mathcal{G}((xq_{ij})')$ is p-integral by (9.2) with $f = 0$. The residues are integers and τ_l is p-integral since $l < p$. □

At a prime $p \nmid ab(b-a)$ with $p > 4n$ we therefore have $v_p(\operatorname{cont} \tilde{Q}_n) \geq 0$. This is Proposition 4.3(d).

9.7. **Primes dividing b.** Let $e = v_p(b) \geq 1$. Then $v(\lambda^{-1}) = e$, $r = a/(b-a)$ is a unit and $v(1/(1-\lambda)) = e$. Put

$$A_2(n) = \sum_{k<2n-1} \Big(n + 1 + \Big\lfloor \frac{k}{3} \Big\rfloor\Big).$$

Proposition 9.8. *If $p \mid b$ and $e = v_p(b)$, then $v_p^{\mathrm{G}}(\tilde{Q}_n) \geq e\, A_2(n) - 4nL$ for every $n \geq 1$.*

The proof writes the Gram matrix as a product WC through slots attached to the poles, as Tang does for $p = 2$ at $\lambda = \frac{1}{2}$. The weight of a pole slot comes from the factor λ^{-m}, which is p-adically small. The factorization rests on a reflection formula for Λ_X.

Lemma 9.9. *Let $K \geq 1$, $R = F/D_K$ with F a polynomial, q the polynomial part of R and c_m the residue of R at $-m$. For $1 \leq m \leq K$ write $R = c_m/y + r_0^{(m)} + r_1^{(m)} y + O(y^2)$ with $y = x + m$. Put $\mathrm{C}_K = \sum_{k=1}^{K} \lambda^{-k}/k^2$. Then*

$$\Lambda_X(R) = -R(0) + \sum_{m=1}^{K} \lambda^{-m}\big(mc_m(X + \mathrm{C}_K) - r_0^{(m)} + mr_1^{(m)}\big) + \mathcal{T}(R),$$

$$\mathcal{T}(R) = \lambda^{-K-1}\mathcal{G}\big(h(x - K - 1)\big) - \sum_{l=1}^{K} c_l\, l \sum_{k=K-l+1}^{K} \frac{\lambda^{-l-k}}{k^2}, \qquad h = (xq)'.$$

Proof. Lemma 9.1 with $M = K + 1$ gives $\mathcal{G}(h) = -\sum_{m=0}^{K} \lambda^{-m} h(-m) + \lambda^{-K-1}\mathcal{G}(h(x - K - 1))$. Since $xq = xR - \sum_l c_l + \sum_l lc_l/(x + l)$, we have $h = (xR)' - \sum_l lc_l/(x + l)^2$. Near $x = -m$, $xR = (y - m)R = -mc_m/y + \text{const} + (r_0^{(m)} - mr_1^{(m)})y + O(y^2)$, so the double pole of $(xR)'$ at $-m$ is mc_m/y^2

and its regular value is $r_0^{(m)} - mr_1^{(m)}$. Hence, with $c_0 = 0$ and $r_0^{(0)} = R(0)$, $r_1^{(0)} = 0$,

$$h(-m) = r_0^{(m)} - mr_1^{(m)} - \sum_{l\neq m} \frac{lc_l}{(l-m)^2} \qquad (0 \le m \le K).$$

On the other side, $l\lambda^{-l}\tau_l = \sum_{m=0}^{l-1} l\lambda^{-m}/(l-m)^2$. Inserting both into (9.1), the double sums combine to

$$\sum_l c_l\, l\Big(\sum_{\substack{0\le m\le K\\ m\neq l}} \frac{\lambda^{-m}}{(l-m)^2} - \sum_{m<l} \frac{\lambda^{-m}}{(l-m)^2}\Big) = \sum_l c_l\, l\lambda^{-l} \sum_{k=1}^{K-l} \frac{\lambda^{-k}}{k^2}.$$

Write the inner sum as $\mathrm{C}_K - \sum_{k=K-l+1}^{K} \lambda^{-k}/k^2$. Collecting terms gives the formula. □

Proof of Proposition 9.8. Apply Lemma 9.9 with $K = 4n$ to $R = F_{ij}/D_{4n}$. We have $R(0) = [i = j = 0]$, since $S_nD_n(0)^3/D_{4n}(0) = 1$ and $\binom{0}{i} = [i = 0]$. For $1 \le m \le n$ the function R has a zero of order at least two at $-m$, so $c_m = r_0^{(m)} = r_1^{(m)} = 0$.

The node terms. Let $n < m \le 4n$ and $y = x + m$. Put $\chi(y) = S_nD_n(x)^3/\prod_{l\neq m}(x+l) = \sum_t \chi_t y^t$, $f_s = [y^s]\binom{x}{i}$ and $g_s = [y^s]\binom{x}{j}$. Then $R = \chi\binom{x}{i}\binom{x}{j}/y$, so c_m, $r_0^{(m)}$ and $r_1^{(m)}$ are the coefficients of y^0, y^1, y^2 in $\chi\binom{x}{i}\binom{x}{j}$. Hence the node term equals $\lambda^{-m}\sum_{s,s'\le 2} f_s \mathcal{K}_{ss'}^{(m)} g_{s'}$ with

$$\mathcal{K}_{ss'}^{(m)} = \sum_{t=0}^{2-s-s'} \chi_t \beta_{t+s+s'}, \qquad \beta_0 = m(X + \mathrm{C}_{4n}),\ \beta_1 = -1,\ \beta_2 = m,$$

and $\mathcal{K}_{ss'}^{(m)} = 0$ for $s + s' > 2$. We bound the factors.

- $v(f_s) \ge -sL$ and $v(g_s) \ge -sL$, by Lemma 9.2(b) with $\gamma = 0$ and $d < 2n$.
- χ_0 is the residue of S_nW_n at $-m$, an integer by Lemma 9.3(a). The quotient $\chi(y)/\chi_0 = \prod_{i\le n}(1 + \frac{y}{i-m})^3 \prod_{l\neq m}(1 + \frac{y}{l-m})^{-1}$ has as coefficient of y^t a polynomial with integer coefficients, homogeneous of degree t, in numbers $1/(i-m)$, $1/(l-m)$ of valuation at least $-\lfloor \log_p 4n \rfloor \ge -L$. So $v(\chi_t) \ge -tL$.
- $v(\lambda^{-k}/k^2) = ek - 2v_p(k) \ge k - 2v_p(k) \ge 0$, since $k \ge p^{v_p(k)} \ge 2^{v_p(k)} \ge 2v_p(k)$. So $v(\mathrm{C}_{4n}) \ge 0$ and $v_p^{\mathrm{G}}(\beta_t) \ge 0$.

Hence $v_p^{\mathrm{G}}(\mathcal{K}_{ss'}^{(m)}) \ge -(2-s-s')L$, and $v_p^{\mathrm{G}}(\lambda^{-m}\sum_{s'} \mathcal{K}_{ss'}^{(m)} g_{s'}) \ge em + (s-2)L$.

The tail. Let $h = (xq_{ij})'$. By Lemma 9.3(b) and Lemma 9.2(b), $v(h(m)) \ge -2L$ for every integer m. Since r is a unit, (9.2) with $f = 0$ shows that $v(\mathcal{G}(h(x-4n-1))) \ge -2L$. In the second part of $\mathcal{T}$, c_l is an integer, $l + k \ge 4n + 1$ and $v(k^2) \le 2L$. So $v(\mathcal{T}(R)) \ge (4n+1)e - 2L$.

The factorization. Let I consist of a constant slot 0, the node slots (m,s) with $n<m\le 4n$ and $0\le s\le 2$, and $2n$ tail slots j'. Put

$$W_{i,0}=[i=0],\qquad C_{0,j}=-[j=0],$$
$$W_{i,(m,s)}=f_s,\qquad C_{(m,s),j}=\lambda^{-m}\sum_{s'}\mathcal{K}^{(m)}_{ss'}g_{s'},$$
$$W_{i,j'}=\mathcal{T}(F_{ij'}/D_{4n}),\qquad C_{j',j}=[j'=j].$$

Then $[E_{ij}]=WC$ by Lemma 9.9. The weights are $\omega_0=\kappa_0=0$, $\omega_{(m,s)}=-sL$, $\kappa_{(m,s)}=em+(s-2)L$, $\omega_{j'}=(4n+1)e-2L$ and $\kappa_{j'}=0$. Rank the slots by $\mathrm{rk}(0)=0$, $\mathrm{rk}(m,s)=3(m-n-1)+s+1$ and $\mathrm{rk}(j')=9n+1+j'$, and put $\theta(0)=-2L$ and $\theta(t)=e\min(n+1+\lfloor(t-1)/3\rfloor,4n+1)-2L$ for $t\ge 1$. Then θ is nondecreasing, and $\theta(\mathrm{rk}(s))=\omega_s+\kappa_s$ for every slot except the constant one, where $\theta(0)\le 0$. By Lemma 9.4(b) and Lemma 9.3(c),

$$v_p^{\mathrm{G}}(\tilde{Q}_n)\ge\sum_{t<2n}\theta(t)=-4nL+e\sum_{k=0}^{2n-2}\Big(n+1+\Big\lfloor\frac{k}{3}\Big\rfloor\Big)=eA_2(n)-4nL,$$

since $n+1+\lfloor(2n-2)/3\rfloor\le 4n+1$. □

Remark 9.10. The formal proof uses the finite identity of Lemma 9.1 and the same slots. It writes the node terms through the derivatives of $1/\prod_{l\ne m}(x+l)$ and keeps the part of C_K beyond K in the tail. The Laurent coefficients used here give the same matrices W and C up to the bookkeeping of these terms. The weights and the bound are Tang's, with e in place of 1.

9.8. **Medium primes.** In this subsection $p\ge 5$ is a prime with $p\nmid ab(b-a)$, $p\le n$ and $4n<p^2$. We prove Tang's bound at such primes.

Proposition 9.11. *Under these assumptions* $v_p^{\mathrm{G}}(\tilde{Q}_n)\ge M_p(n)$, *where*

$$M_p(n)=n(3q_4-6q_1-4q_2-6)+p(q_1(2q_1-q_4+2)+q_2(q_2+1))\\+\min(n-q_1p,\,4n-q_4p)+q_4-2q_1.$$

We use the residue classes of Section 5, now for the prime p. For $0\le\sigma<p$ let a_σ be the number of $i\in[1,n]$ and c_σ the number of $l\in[1,4n]$ with $i\equiv\sigma$, $l\equiv\sigma\pmod p$, and put $\nu_\sigma=3a_\sigma-c_\sigma-1+[\sigma=0]$. For a polynomial $E\in\mathbb{Z}[x]$ and a class σ, the *local order* $o_\sigma(E)$ is the largest $o\ge 0$ with $E(-\sigma+pu)\in p^ou^o\mathbb{Z}_{(p)}[u]$.

Proposition 9.12 (local bound for the entries)**.** *For all* $E,E'\in\mathbb{Z}[x]$,

$$v_p^{\mathrm{G}}(\Lambda_X(W_nEE'))\ge\min_{0\le\sigma<p}\big(\nu_\sigma+o_\sigma(E)+o_\sigma(E')\big).$$

The proof uses the ring $\mathcal{O}$ of rational functions g/h with $g,h\in\mathbb{Z}_{(p)}[u]$, $h(0)$ a unit and all other coefficients of h divisible by p. This ring is closed under sums, products and d/du. Its elements take p-integral values at every integer u, since $h(u)\equiv h(0)\pmod p$. If $g\in\mathcal{O}$ and $k\in\mathbb{Z}$, then $(g(u)-g(-k))/(u+k)\in\mathcal{O}$. Indeed, the numerator of $g(u)-g(-k)$ over the

denominator $h(u)h(-k)$ lies in $\mathbb{Z}_{(p)}[u]$ and vanishes at $-k$, and $h(-k)$ is a unit.

Proof of Proposition 9.12. Let $R = W_n EE'$, let R_l be its residue at $-l$, and let $\mu = \min_\sigma(\nu_\sigma + o_\sigma(E) + o_\sigma(E'))$. By (9.1), $\Lambda_X(R) = \mathcal{G}(\Phi') + \sum_l R_l\, l\lambda^{-l}(X - \tau_l)$ with $\Phi = xq$ and q the polynomial part of R.

The pole terms. Let l be in the class σ. Every factor $i - l$ of $D_n(-l)$ with $i \equiv \sigma$ is divisible by p, and $v(\prod_{l' \neq l}(l' - l)) = c_\sigma - 1$ since $0 < |l' - l| < p^2$. Writing $-l = -\sigma + pu$ with $u \in \mathbb{Z}$ gives $v(E(-l)) \geq o_\sigma(E)$. Hence

$$v(R_l) \geq 3a_\sigma - c_\sigma + 1 + o_\sigma(E) + o_\sigma(E'). \tag{9.3}$$

Moreover $v(l) \geq [\sigma = 0]$, λ is a unit and $v(\tau_l) \geq -2$. So each pole term has Gauss valuation at least $\nu_\sigma + o_\sigma(E) + o_\sigma(E') \geq \mu$.

Splitting by classes. Put $x = -\sigma + pu$. The integers $x \geq 1$ are exactly the numbers $pu - \sigma$ with $0 \leq \sigma < p$ and $u \geq 1$. So $\mathcal{G}(Q) = \sum_\sigma \lambda^{-\sigma} \mathcal{G}_p(Q(-\sigma + pu))$ for every polynomial Q, where $\mathcal{G}_p$ is $\mathcal{G}$ with λ^p in place of λ. Since λ is p-integral, $\lambda^p \equiv \lambda \pmod p$, so $1 - \lambda^p$ is a unit and $\lambda^p/(1 - \lambda^p)$ is p-integral. So (9.2) for λ^p with $f = 0$ gives $v(\mathcal{G}_p(g)) \geq \min_{0 \leq i \leq \deg g} v(g(i))$ for every polynomial g. With $\phi_\sigma(u) = \Phi(-\sigma + pu)$ we have $\Phi'(-\sigma + pu) = p^{-1}\phi'_\sigma(u)$. So it suffices to show $v(\phi'_\sigma(i)) \geq \mu + 1$ for all σ and all integers $i \geq 0$.

The local form. Fix σ. Let K_σ be the set of $k \geq 0$ with $1 \leq \sigma + pk \leq 4n$ and $K'_\sigma \subset K_\sigma$ those with $\sigma + pk \leq n$. At $x = -\sigma + pu$ the factors $x + l$ with $l = \sigma + pk$ equal $p(u + k)$, and the others are $(l - \sigma) + pu$ with $l - \sigma$ a unit. Hence

$$D_n(x) = p^{a_\sigma} \prod_{k \in K'_\sigma} (u + k)\, N(u), \qquad D_{4n}(x) = p^{c_\sigma} \prod_{k \in K_\sigma} (u + k)\, M(u),$$

with $N, M \in \mathbb{Z}[u]$ of the form required for denominators in $\mathcal{O}$. Write $E(x) = p^o u^o H(u)$ and $E'(x) = p^{o'} u^{o'} H'(u)$ with $o = o_\sigma(E)$, $o' = o_\sigma(E')$ and $H, H' \in \mathbb{Z}_{(p)}[u]$. Then

$$\Psi(u) := xR(x) = p^\epsilon \frac{g(u)}{\prod_{k \in S}(u + k)}, \qquad S = K_\sigma \setminus K'_\sigma,$$

with $\epsilon = 3a_\sigma - c_\sigma + o + o' + [\sigma = 0]$ and $g = p^{-[\sigma=0]}(-\sigma + pu)u^{o+o'} \prod_{k \in K'_\sigma}(u + k)^2 N^3 HH'/M \in \mathcal{O}$. The elements of S are distinct integers in $[0, p - 1]$, since $\sigma + pk \leq 4n < p^2$. So $1/\prod_{k \in S}(u + k) = \sum_{k \in S} \eta_k/(u + k)$ with $\eta_k = 1/\prod_{k' \in S, k' \neq k}(k' - k)$ a unit. Hence

$$\Psi = p^\epsilon A + p^\epsilon \sum_{k \in S} \frac{\eta_k g(-k)}{u + k}, \qquad A = \sum_{k \in S} \eta_k \frac{g(u) - g(-k)}{u + k} \in \mathcal{O}.$$

Cancellation of the poles. From $\Phi = xR - \sum_l R_l + \sum_l lR_l/(x + l)$ we get

$$\phi_\sigma = \Psi - \sum_l R_l + \sum_{l \equiv \sigma} \frac{lR_l}{p(u + k_l)} + \sum_{l \not\equiv \sigma} \frac{lR_l}{(l - \sigma) + pu},$$

with $l = \sigma + pk_l$. The residues R_l with $l \leq n$ vanish, since $D_n(-l) = 0$, so the third sum has poles only at the points $u = -k$ with $k \in S$. The function ϕ_σ

is a polynomial, and the last sum has no poles at integers. So the principal parts at the points $u = -k$, $k \in S$, cancel between Ψ and the third sum. Therefore

$$\phi_\sigma = p^\epsilon A - \sum_l R_l + \sum_{l \not\equiv \sigma} \frac{l R_l}{(l-\sigma) + pu},$$

$$\phi'_\sigma(u) = p^\epsilon A'(u) - \sum_{l \not\equiv \sigma} \frac{p\, l R_l}{((l-\sigma) + pu)^2}.$$

At an integer u the first term has valuation at least $\epsilon = \nu_\sigma + o + o' + 1 \geq \mu + 1$, since $A' \in \mathcal{O}$. By (9.3) for the class of l, each term of the sum has valuation at least $1 + v(R_l) \geq \mu + 3$. □

Remark 9.13. The formal proof reaches the same entry bound through an explicit decomposition of the polynomial and pole parts over $\mathbb{Z}_p$, by classes of poles, with jets of the basis at each class. The argument above uses only values at integers and the ring $\mathcal{O}$. The statement and the weights are Tang's.

The basis. Put $h_\sigma = c_\sigma - 2a_\sigma$ for $\sigma \neq 0$, $h_0 = q_3 - q_1$ and $\delta = q_4 - q_1 - q_3$. Then $\delta \in \{0, 1\}$. Since $p \leq n$ we have $a_\sigma \leq q_1 + 1 \leq 2q_1 \leq c_\sigma/2$, so $h_\sigma \geq 0$. Also $\sum_\sigma h_\sigma + \delta = 4n - 2n = 2n$, since $\sum_\sigma c_\sigma = 4n$, $\sum_\sigma a_\sigma = n$, $a_0 = q_1$ and $c_0 = q_4$. Put $\Pi = \prod_\sigma (x + \sigma)^{h_\sigma}$ and

$$e_{\sigma,t} = (x + \sigma)^t \prod_{\sigma' \neq \sigma} (x + \sigma')^{h_{\sigma'}} \quad (0 \leq t < h_\sigma),$$

together with $e_* = \Pi$ when $\delta = 1$. These are $2n$ monic integer polynomials of degree below $2n$.

Lemma 9.14. *The coefficient matrix T of this basis satisfies $v(\det T) = 0$. The local orders satisfy $o_\sigma(e_{\sigma,t}) \geq t$, $o_{\sigma'}(e_{\sigma,t}) \geq h_{\sigma'}$ for $\sigma' \neq \sigma$, and $o_{\sigma'}(e_*) \geq h_{\sigma'}$.*

Proof. Suppose an $\mathbb{F}_p$-combination $\sum \xi_{\sigma,t} e_{\sigma,t} + \xi_* e_*$ vanishes modulo p. The polynomial e_* is monic of degree $2n - 1$, and the others have smaller degree, so $\xi_* = 0$. Reduce modulo Π and use the Chinese remainder theorem for the coprime factors $(x + \sigma)^{h_\sigma}$ over $\mathbb{F}_p$. In the component of σ only the $e_{\sigma,t}$ survive, and they are $(x + \sigma)^t$ times a unit. They are independent, so all ξ vanish. The local orders follow from $x + \sigma' = (\sigma' - \sigma) + pu$ at $x = -\sigma + pu$, which equals pu for $\sigma' = \sigma$. □

Proof of Proposition 9.11. Put $T_\sigma = \nu_\sigma + 2h_\sigma$. For $\sigma \neq 0$, $T_\sigma = c_\sigma - a_\sigma - 1$, and $c_\sigma - a_\sigma$ counts the $l \equiv \sigma$ in an interval of length $3n$, so $q_3 - 1 \leq T_\sigma \leq q_3$. For $\sigma = 0$, $T_0 = 3q_1 - q_4 + 2(q_3 - q_1) = q_3 - \delta$. Give $e_{\sigma,t}$ the weight $w = \nu_\sigma + 2t$ and e_* the weight $q_3 - 1$. We claim $w(e) \leq \nu_{\sigma'} + 2o_{\sigma'}(e)$ for every basis vector e and every class σ'. For $e_{\sigma,t}$ and $\sigma' = \sigma$ this follows from $o_\sigma(e_{\sigma,t}) \geq t$. In the other cases $o_{\sigma'}(e) \geq h_{\sigma'}$, so the right side is at least $T_{\sigma'} \geq q_3 - 1$. The left side is $q_3 - 1$ for e_*, and at most $\nu_\sigma + 2(h_\sigma - 1) = T_\sigma - 2 \leq q_3 - 2$ for $e_{\sigma,t}$. By Proposition 9.12 the Gram matrix of the basis has entries with

$v_p^{\mathrm{G}} \ge \frac{1}{2}(w(e)+w(e'))$. Lemma 9.4(a), Lemma 9.3(c),(d) and Lemma 9.14 give $v_p^{\mathrm{G}}(\tilde{Q}_n) \ge \sum w + N_p(n)$.

We compute $\sum w$. For $\sigma \neq 0$ the class gives $\sum_{t<h_\sigma}(\nu_\sigma + 2t) = h_\sigma(\nu_\sigma + h_\sigma - 1) = (c_\sigma - 2a_\sigma)(a_\sigma - 2)$. For $\sigma = 0$, with $q_4 = q_1 + q_3 + \delta$, the class and e_* give $h_0(3q_1 - q_4 + h_0 - 1) + \delta(q_3 - 1) = (q_4 - 2q_1)(q_1 - 1)$. So

$$\sum w = \sum_\sigma (c_\sigma - 2a_\sigma)(a_\sigma - 2) + q_4 - 2q_1.$$

Write $n = q_1 p + \rho_1$, $2n = q_2 p + \rho_2$ and $4n = q_4 p + \rho_4$ with $0 \le \rho_k < p$. Index the classes by $j \in [1,p]$. Then $a = q_1 + [j \le \rho_1]$ and $c = q_4 + [j \le \rho_4]$, and summing over j gives

$$\sum_\sigma (c_\sigma - 2a_\sigma)(a_\sigma - 2) = p(q_4 - 2q_1)(q_1 - 2) + (q_4 - 4q_1 + 2)\rho_1 + (q_1 - 2)\rho_4 + \min(\rho_1, \rho_4).$$

Also $\sum_{i<2n}\lfloor i/p \rfloor = pq_2(q_2-1)/2 + q_2\rho_2$. Inserting $\rho_k = kn - q_k p$ and collecting terms gives $\sum w + N_p(n) = M_p(n)$. □

9.9. **Outer primes.** Let $p \ge 5$, $p \nmid ab(b-a)$ and $n < p \le 4n$. Then $4n < p^2$. Put

$$\Omega_p(n) = \begin{cases} 3n - p + 2, & 3p \le 4n, \\ 5p - 7n + 1, & 4n < 3p,\ 2p \le 3n, \\ p - n + 1, & 3n < 2p,\ p \le 2n, \\ 3p - 7n - 1, & 2n < p \le 3n, \\ 2n - 1, & 3n < p \le 4n. \end{cases}$$

Proposition 9.15. *Under these assumptions* $v_p^{\mathrm{G}}(\tilde{Q}_n) \ge \Omega_p(n)$.

Proof. The raw bound. The Hankel entry is $\Lambda_X(W_n x^{i+j}) = V_{ij} + \sum_l \gamma_l (-l)^{i+j}$ with $\gamma_l = \varrho_l\, l\lambda^{-l}(X - \tau_l)$, where ϱ_l is the residue of W_n at $-l$ and V_{ij} is Λ_X of the polynomial part of $D_n^3 x^{i+j}/D_{4n}$. That polynomial part has integer coefficients, so $v_p^{\mathrm{G}}(V_{ij}) \ge 0$ as in Proposition 9.7(b). For $n < l \le 4n$, $\varrho_l = \pm(l-1)!^2/((l-n-1)!^3(4n-l)!)$, so

$$v(\varrho_l) = 2\Big\lfloor \frac{l-1}{p} \Big\rfloor - 3\Big\lfloor \frac{l-1-n}{p} \Big\rfloor - \Big\lfloor \frac{4n-l}{p} \Big\rfloor,$$

and $v_p^{\mathrm{G}}(X - \tau_l) \ge -2[p \le l]$. Put $w(l) = v(\varrho_l) + [p \mid l] - 2[p \le l]$ for $l > n$ and $w(l) = 16n + 4$ for $l \le n$, where $\gamma_l = 0$. In each class order the poles by decreasing l. Then w is nondecreasing along each class: if $n < l$ and $l + p \le 4n$, the three floors change by $+2$, -3 and $+1$, so $w(l+p) - w(l) = -2[l < p] \le 0$, and $w(l) \le 16n + 4$ always. Lemma 9.5 with the nodes $-l$ gives $v_p^{\mathrm{G}}(Q_n) \ge \sum_\sigma \sum_{k<C_\sigma} \min(w(l_{\sigma,k}) + 2k, 0)$, where $l_{\sigma,k}$ is the $(k+1)$-st largest pole of the class and C_σ the size of the class.

The slot weights. Let N_σ be the number of poles $l \le n$ in the class. For $k < C_\sigma - N_\sigma$ and $l = l_{\sigma,k}$ the class has k poles above l, so $\lfloor (4n-l)/p \rfloor = k$, $\lfloor (l-1)/p \rfloor = C_\sigma - 1 - k$ for $\sigma \neq 0$, and $\lfloor (l-1-n)/p \rfloor = C_\sigma - 1 - k - N_\sigma$.

This gives $w(l) = 3N_\sigma - C_\sigma + 1 + [\sigma = 0] - 2[p \le l]$ for $\sigma \neq 0$, and the same formula holds for $\sigma = 0$. The slots with $k \ge C_\sigma - N_\sigma$ contribute 0.

The classes. Let $L_4 = q_4 \in \{1, 2, 3\}$ and $\rho = 4n - L_4 p$. Index the classes by $j \in [1, p]$. For $j < p$ we have $N = [j \le n]$, since $p > n$, and $C = L_4 + [j \le \rho]$. The smallest pole of the class is $j < p$, and all others exceed p. For $j = p$ we have $N = 0$ and $C = L_4$. Summing $\min(w + 2k, 0)$ over the slots gives, per class,

class	$j \le n$	C	contribution
(i)	yes	$L_4 + 1$	$-(L_4 - 1)$
(ii)	yes	L_4	$-\max(L_4 - 2, 0)$
(iii)	no	$L_4 + 1$	$-3L_4$
(iv)	no	L_4	$-3(L_4 - 1)$
(v) $j = p$	no	L_4	$-1, -2, -4$ for $L_4 = 1, 2, 3$

For instance in type (iii) the weights are $-L_4 - 2 + 2k$ for $k < L_4$ and L_4 for $k = L_4$, with sum of negative parts $-3L_4$. Also $N_p(n) = 2nL_4 - 2\max(2n - p, 0)$, since $\lfloor i/p \rfloor \le 1$ for $i < 2n < 2p$.

The five ranges. In each range of Ω we count the classes and add $N_p(n)$.

- $3p \le 4n$: $L_4 = 3$, $\rho = 4n - 3p < n$. Types (i), (ii), (iv) occur ρ, $n - \rho$, $p - 1 - n$ times. The sum is $-2\rho - (n - \rho) - 6(p - 1 - n) - 4 = n - 3p + 2$, and $N_p(n) = 2n + 2p$.
- $4n < 3p$, $2p \le 3n$: $L_4 = 2$, $\rho = 4n - 2p \ge n$. Types (i), (iii), (iv) occur n, $\rho - n$, $p - 1 - \rho$ times. The sum is $-n - 6(\rho - n) - 3(p - 1 - \rho) - 2 = 3p - 7n + 1$, and $N_p(n) = 2p$.
- $3n < 2p \le 4n$: $L_4 = 2$, $\rho < n$. Types (i), (ii), (iv) occur ρ, $n - \rho$, $p - 1 - n$ times. The sum is $-\rho - 3(p - 1 - n) - 2 = -p - n + 1$, and $N_p(n) = 2p$.
- $2n < p \le 3n$: $L_4 = 1$, $\rho = 4n - p \ge n$. Types (i), (iii), (iv) occur n, $3n - p$, $2p - 4n - 1$ times. The sum is $-3(3n - p) - 1$, and $N_p(n) = 2n$.
- $3n < p \le 4n$: $L_4 = 1$, $\rho < n$. The sum is -1, and $N_p(n) = 2n$.

In each case the sum plus $N_p(n)$ is $\Omega_p(n)$. Lemma 9.3(d) finishes the proof. □

9.10. **The profile.** For $0 < x \le 1$ define

$$\varpi(x) = 3\Big\lfloor\frac{4}{x}\Big\rfloor - 6\Big\lfloor\frac{1}{x}\Big\rfloor - 4\Big\lfloor\frac{2}{x}\Big\rfloor - 6 + x\Big(\Big\lfloor\frac{1}{x}\Big\rfloor\Big(2\Big\lfloor\frac{1}{x}\Big\rfloor - \Big\lfloor\frac{4}{x}\Big\rfloor + 2\Big) + \Big\lfloor\frac{2}{x}\Big\rfloor\Big(\Big\lfloor\frac{2}{x}\Big\rfloor + 1\Big)\Big) + \min\Big(1 - x\Big\lfloor\frac{1}{x}\Big\rfloor,\, 4 - x\Big\lfloor\frac{4}{x}\Big\rfloor\Big).$$

At $x = p/n$ we have $\lfloor k/x \rfloor = q_k$. Hence

$$M_p(n) = n\varpi(p/n) + q_4 - 2q_1 \ge n\varpi(p/n), \tag{9.4}$$

since $q_4 \ge 4q_1$. The function ϖ is piecewise affine. Let $A \ge 1$ be an integer and $A < 1/x < A + 1$. Put $\upsilon = 1/x - A$. Then $\lfloor 1/x \rfloor = A$, $\lfloor 2/x \rfloor = 2A + \lfloor 2\upsilon \rfloor$,

$\lfloor 4/x\rfloor = 4A + \lfloor 4\upsilon\rfloor$, $1 - xA = x\upsilon$ and $4 - x\lfloor 4/x\rfloor = x\{4\upsilon\}$. Substituting gives the following table.

υ	$\varpi(x)$
$(0, \frac{1}{4})$	$(2A^2 + 3A)x - 2A - 5$
$(\frac{1}{4}, \frac{1}{3})$	$(2A^2 - A - 1)x - 2A + 1$
$(\frac{1}{3}, \frac{1}{2})$	$(2A^2 + 2A)x - 2A - 2$
$(\frac{1}{2}, \frac{2}{3})$	$(2A^2 + 2A)x - 2A$
$(\frac{2}{3}, \frac{3}{4})$	$(2A^2 + 5A + 2)x - 2A - 3$
$(\frac{3}{4}, 1)$	$(2A^2 + A - 1)x - 2A + 3$

The minimum is $x\upsilon$ when $\lfloor 4\upsilon\rfloor \le 3\upsilon$, that is on the first, third and fifth intervals, and $x\{4\upsilon\}$ otherwise.

The integral of the profile over each interval $[1/(A+1), 1/A]$ is a rational function of A. Its sum over A is the medium-prime constant. Partial fractions reduce the sum to harmonic numbers $H_m = \sum_{k\le m} 1/k$.

Lemma 9.16. *For every integer $A \ge 1$,*

$$\int_{1/(A+1)}^{1/A} \varpi(x)\,dx = \mathcal{I}(A) := -\frac{384A^4 + 768A^3 + 563A^2 + 179A + 21}{\mathcal{D}(A)} < 0,$$

where $\mathcal{D}(A) = A(A+1)(2A+1)(3A+1)(3A+2)(4A+1)(4A+3)$. For every integer $N \ge 1$,

$$\int_{1/(N+1)}^{1} \varpi(x)\,dx = \sum_{A=1}^{N} \mathcal{I}(A) \ge c_{\mathrm{med}} = 28\log 2 - \frac{9}{2}\log 3 - \frac{181}{12}.$$

Proof. The interval $\upsilon \in (\alpha, \beta)$ corresponds to $x \in (1/(A+\beta), 1/(A+\alpha))$, and $\int_u^v (cx + d)\,dx = c(v^2 - u^2)/2 + d(v - u)$. The endpoints are $\frac{1}{A}$, $\frac{4}{4A+1}$, $\frac{3}{3A+1}$, $\frac{2}{2A+1}$, $\frac{3}{3A+2}$, $\frac{4}{4A+3}$ and $\frac{1}{A+1}$. Adding the six integrals over the common denominator $\mathcal{D}(A)$ gives $\mathcal{I}(A)$. The numerator and $\mathcal{D}(A)$ are positive for $A \ge 1$, so $\mathcal{I}(A) < 0$.

The partial fraction decomposition of $\mathcal{I}(A)$ is

$$\mathcal{I}(A) = -\frac{7}{2}\Big(\frac{1}{A} + \frac{1}{A+1}\Big) + \frac{16}{4A+1} + \frac{16}{4A+3} + \frac{4}{2A+1} - \frac{9/2}{3A+1} - \frac{9/2}{3A+2},$$

as one checks by clearing denominators. For $M \ge 1$ put

$$C(M) = 16H_{4M} - 4H_{2M} - \frac{9}{2}H_{3M} - \frac{15}{2}H_M + \frac{7}{2M} - \frac{181}{12}.$$

With $H_4 = \frac{25}{12}$, $H_3 = \frac{11}{6}$, $H_2 = \frac{3}{2}$ and $H_1 = 1$ we get $C(1) = 0$. We claim $C(A+1) - C(A) = \mathcal{I}(A)$ for $A \ge 1$. Indeed $H_{4A+4} - H_{4A} = \sum_{i=1}^{4} \frac{1}{4A+i}$, and similarly for $3A$, $2A$ and A. In the difference, the terms with denominators $4A+2$ and $2A+1$ give $\frac{16}{4A+2} - \frac{4}{2A+1} = \frac{4}{2A+1}$. The terms with denominators $4A+4$, $2A+2$, $3A+3$ and $A+1$ give

$$\Big(4 - 2 - \frac{3}{2} - \frac{15}{2} + \frac{7}{2}\Big)\frac{1}{A+1} = -\frac{7}{2(A+1)},$$

and with $-\frac{7}{2A}$ this is $-\frac{7}{2}(\frac{1}{A}+\frac{1}{A+1})$. The remaining terms are $\frac{16}{4A+1}+\frac{16}{4A+3}-\frac{9/2}{3A+1}-\frac{9/2}{3A+2}$. So the difference is the partial fraction form of $\mathcal{I}(A)$. By induction, $\sum_{A=1}^{N}\mathcal{I}(A)=C(N+1)$.

As $M\to\infty$, $H_m=\log m+\gamma+o(1)$ with Euler's constant γ. The coefficients of H in $C(M)$ sum to $16-4-\frac{9}{2}-\frac{15}{2}=0$, so γ and $\log M$ cancel, and

$$C(M)\to 16\log 4-4\log 2-\frac{9}{2}\log 3-\frac{181}{12}=c_{\mathrm{med}}.$$

Since every $\mathcal{I}(A)$ is negative, the partial sums $C(N+1)$ decrease to c_{med}. So each of them is at least c_{med}. □

Remark 9.17. The integrals $\mathcal{I}(A)$ are Tang's, and he bounds their sum below by $-\frac{523}{840}$. The closed form of the sum is our derivation. Numerically $c_{\mathrm{med}}=-0.6189675\ldots$, and $\frac{7}{2}+c_{\mathrm{med}}=c_0$.

9.11. **Sums over primes.** We use the prime number theorem in the form $\vartheta(x)=\sum_{p\le x}\log p=x+o(x)$ [5]. Abel summation gives $\sum_{p\le x}p\log p=x\vartheta(x)-\int_2^x\vartheta(t)\,dt=\frac{1}{2}x^2+o(x^2)$. The prime powers $p^k\le x$ with $k\ge 2$ satisfy $\sum\log p=\sum_{k\ge 2}\vartheta(x^{1/k})=O(\sqrt{x}\log^2 x)$.

Lemma 9.18. *Let $0<u<v$ and $\alpha,\beta\in\mathbb{R}$. Then $\sum_{un<p\le vn}(\alpha p+\beta n)\log p=(\frac{\alpha}{2}(v^2-u^2)+\beta(v-u))n^2+o(n^2)$, and the same holds with strict inequalities at either end.*

Proof. This follows from the two asymptotic formulas. Changing an endpoint adds or removes at most one prime, at a cost of $O(n\log n)$. □

Proof of Proposition 4.3. Let S be the set of primes dividing $ab(b-a)$. For large n we have $\delta n>\max S$, $\delta n\ge 5$ and $n>4(N+1)^2$. Every prime $p>\delta n$ then satisfies $p\notin S$, $p\ge 5$ and $p^2>n^2/(N+1)^2>4n$. Recall that $v_p(\operatorname{cont}\tilde{Q}_n)=v_p^{\mathrm{G}}(\tilde{Q}_n)$.

(a) By Proposition 9.7(a),

$$\sum_{4<p\le\delta n,\,p\notin S}v_p(\operatorname{cont}\tilde{Q}_n)\log p\ge -4n\sum_{p\le\delta n}L_p(n)\log p.$$

Since $\lfloor\log_p(7n-2)\rfloor-1\le\#\{k\ge 2:p^k\le 7n\}$, the last sum is at most $\vartheta(\delta n)+\sum_{p^k\le 7n,\,k\ge 2}\log p=\delta n+o(n)$.

(b) By Proposition 9.11 and (9.4), the sum is at least

$$\sum_{\delta n<p\le n}n\varpi(p/n)\log p.$$

The interval $[\delta,1]$ is the union of $6N$ intervals on whose interiors ϖ is affine. The function ϖ is bounded on $[\delta,1]$. We apply Lemma 9.18 on each interval. The primes at the endpoints cost $O(n\log n)$. This gives

$$\sum_{\delta n<p\le n}n\varpi(p/n)\log p=n^2\int_\delta^1\varpi+o(n^2).$$

Lemma 9.16 with $\delta = 1/(N+1)$ gives (b).

(c) By Proposition 9.15 the sum is at least $\sum_{n<p\le 4n} \Omega_p(n) \log p$. On the five ranges Ω is $\alpha p + \beta n$ plus a constant, and the constants contribute $O(n)$. By Lemma 9.18 the sum is n^2 times

$$\int_1^{4/3} (3-x)\,dx + \int_{4/3}^{3/2} (5x-7)\,dx + \int_{3/2}^{2} (x-1)\,dx + \int_2^3 (3x-7)\,dx + \int_3^4 2\,dx \\ = \frac{11}{18} + \frac{1}{72} + \frac{3}{8} + \frac{1}{2} + 2 = \frac{7}{2},$$

up to $o(n^2)$.

(d) This is Proposition 9.7(b). □

The proof of Theorem 4.4 uses one more step. Let $p > 4n$. For n large, $p \notin S$. So $v_p(\tilde{d}_n) = -v_p(\operatorname{cont} \tilde{Q}_n) \le 0$, and

$$\log \tilde{d}_n = \sum_p v_p(\tilde{d}_n) \log p \le - \sum_{p \le 4n} v_p(\operatorname{cont} \tilde{Q}_n) \log p.$$

The primes $p \le 4n$ split into S, the primes $p \le 4$ outside S and the three ranges of Proposition 4.3, as in the proof of Theorem 4.4.

10. Formalization

Theorem 1.1 is formalized in Lean 4 [6] with Mathlib [30], together with every theorem, proposition and lemma its proof uses. The remarks are not part of the formalization. The public statement is the theorem `Li2IrrationalityMeasure.li2_measure_general` in `SolutionLi2.lean`, against the statement file `ChallengeLi2.lean`, of the repository

`https://github.com/slitsey/catalan-irrationality-measure`

Its statement uses only Mathlib. The number $\mathrm{Li}_2(a/b)$ is written as the series $\sum_{k\ge 0} (a/b)^{k+1}/(k+1)^2$, the rate $c(a,b)$ is written out, and the table of Definition 7.2 is given as rational literals. The conclusion is $q^{-\mu} < |\mathrm{Li}_2(a/b) - p/q|$ for all integers p and all sufficiently large q. The modules contain no `sorry` and no `native_decide`, and `#printaxioms` reports only `propext`, `Classical.choice` and `Quot.sound`.

Tang's construction and his local bounds at the primes prime to $ab(b-a)$ are ported from his formalization [29] under the Apache License 2.0. The closed form of the medium-prime constant in Proposition 4.3(b) is formalized on top of them. The ported files are listed in the repository's `NOTICE`. Tang's own files adapt parts of Firsching's formalization of Fauzan's proof [8]. The prime number theorem is taken from the PNT+ project [31]. The finite checks of Theorem 5.3 and of the certificates of Section 6.6 are carried out by the Lean kernel in rational arithmetic. Their data were computed with floating-point optimization and the interval arithmetic of Arb [12], and they play no role in the proof beyond satisfying the checks.

Disclosure of AI use

The mathematics, the Lean formalization, the certificate generators and the text of this paper were produced by AI systems at the request of the author. The author posed the problems, directed the work and is responsible for the paper. The systems were Claude Opus 5.5, Claude Sonnet 5.5 and Claude Haiku 5.5 (Anthropic), running in Claude Code, and GPT-6.1 Sol (OpenAI), running in Codex. The construction and the irrationality proof of $\mathrm{Li}_2(1/2)$ are due to Tang [29].

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Appendix A. The table of nodes

Table 4 summarizes the 37 nodes of Definition 7.2: the node λ_k ($\pm$ for a lattice node, which serves both signs), its frame, t_k, the number of entries, the range of ρ_k and the largest D_k. The exact rational entries are part of the formal statement.

Table 4. The nodes.

node	frame	t_k	entries	ρ_k	$\max D_k$
$\pm 1/2$	lattice	0.6931	6	14.5–16	−4.5932
$\pm 1/3$	lattice	1.0986	1	17.5	−2.6243
$\pm 1/4$	lattice	1.3863	1	18.5	−1.6650
$\pm 1/5$	lattice	1.6094	1	20	−0.9524
$\pm 1/6$	lattice	1.7918	1	20.5	−0.5379
$\pm 2/13$	lattice	1.8718	14	22.3–23.55	−0.1476
$\pm 2/15$	lattice	2.0149	7	22.5–24.5	0.2020
$\pm 1/8$	lattice	2.0794	1	22	0.1472
$\pm 2/17$	lattice	2.1401	6	23.5–25	0.4905
$\pm 2/19$	lattice	2.2513	14	23.95–25.2	0.7348
$\pm 1/10$	lattice	2.3026	1	22.9	0.5739
$\pm 2/23$	lattice	2.4423	14	24.8–26.05	1.1294
$\pm 2/25$	lattice	2.5257	15	25.2–26.45	1.2933
$\pm 1/14$	lattice	2.6391	1	23.4	0.8978
$\pm 2/33$	lattice	2.8034	14	26.45–27.7	1.8041
$\pm 2/37$	lattice	2.9178	16	26.95–28.2	2.0011
$\pm 2/39$	lattice	2.9704	18	27.2–28.45	2.0904
$\pm 1/20$	lattice	2.9957	1	23.5	0.8591
$\pm 1/26$	lattice	3.2581	3	24–30	2.5675

TABLE 5. The nodes (continued).

node	frame	t_k	entries	ρ_k	$\max D_k$
$\pm 1/33$	lattice	3.4965	4	24.5–30.5	2.8818
$\pm 1/55$	lattice	4.0073	3	28.5–33.5	3.6539
$\pm 1/90$	lattice	4.4998	3	31–36	4.3096
$\pm 1/148$	lattice	4.9972	3	33.5–38.5	4.9298
$\pm 1/403$	lattice	5.9989	3	38.5–43.5	6.0988
$-1/2$	Γ_2	0.6931	21	13.5–18.5	0.1226
$-1/3$	Γ_3	1.0986	21	15–20	0.3465
$-1/4$	Γ_4	1.3863	21	16–21	0.5109
$-1/5$	Γ_4	1.6094	20	17.25–22	0.7657
$-2/11$	Γ_{15}	1.7047	20	19.75–24.5	1.0672
$-1/6$	Γ_{15}	1.7918	20	18.25–23	0.9669
$-2/15$	Γ_{15}	2.0149	20	21–25.75	1.4102
$-2/25$	Γ_{15}	2.5257	21	23–28	1.9449
$-1/13$	Γ_{15}	2.5649	20	20–24.75	1.3212
$-1/20$	Γ_{15}	2.9957	21	20–25	1.0338
$-2/41$	Γ_{15}	3.0204	21	25–30	2.4113
$-2/65$	Γ_{15}	3.4812	21	27.5–32.5	2.8658
$-1/33$	Γ_{15}	3.4965	21	21.5–26.5	1.2118

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