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IRRATIONALITY MEASURES FOR $a\zeta(3)+b\pi^4$ AND $\zeta(3)/\pi^4$

SEAN LITSEY

Abstract. Let $\mu(\xi)$ denote the irrationality exponent of a real number ξ. We prove that $\mu(a\zeta(3)+b\pi^4)\le 27.49$ for all integers a and b with $b\neq 0$, and that $\mu(\zeta(3)/\pi^4)\le 35.35$. The second bound extends to every quotient $(a\zeta(3)+b\pi^4)/(c\zeta(3)+d\pi^4)$ with rational a,b,c,d and $ad-bc\neq 0$. Lai proved that 1, $\zeta(3)$ and π^4 are linearly independent over $\mathbb{Q}$ with Hankel determinants $J_n(a,b,X)$. We keep his determinants and prove the three estimates that a measure needs. First, the determinants are small on a neighbourhood of the constant, uniformly in the direction (a,b). The proof uses a positive measure on a complex contour whose potential has a closed form. Second, the denominators are bounded prime by prime by an exact law for the p-adic valuation. The integral of this law over the primes has a closed form. Third, J_n does not vanish at sizes n tied to several families of primes. The main family consists of the primes ℓ with $8n/3<\ell<3n-1$. For these primes a reduced Gram matrix is block triangular modulo ℓ. Its half-integral block is a Hankel determinant of Bernoulli-type moments, which we evaluate as a product of norms of Wilson polynomials. A counting argument with the prime number theorem in progressions turns this nonvanishing into a measure. The best published bound on the plane was 66.3, and no measure was known for $\zeta(3)/\pi^4$. The proofs are formalized in Lean 4.

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1. Introduction

1.1. **The result.** The *irrationality exponent* $\mu(\xi)$ of a real number ξ is the infimum of the real numbers μ such that $|\xi - p/q| > q^{-\mu}$ for all integers p and all sufficiently large integers q. It does not change when ξ is multiplied by a nonzero rational number or translated by a rational number. A finite bound for $\mu(\xi)$ is an irrationality measure for ξ, and it shows that ξ is irrational. We write

$$\zeta(3) = \sum_{k\geq 1} \frac{1}{k^3} = 1.20205\,69031\ldots, \qquad \zeta(4) = \sum_{k\geq 1} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

Theorem 1.1. *(a) Let a and b be integers with $b \neq 0$. There is an integer q_0 such that*

$$\left|a\zeta(3) + b\pi^4 - \frac{p}{q}\right| > q^{-27.49}$$

for all integers p and all integers $q \geq q_0$. In particular $\mu(a\zeta(3) + b\pi^4) \leq 27.49$.

(b) There is an integer q_0 such that

$$\left|\frac{\zeta(3)}{\pi^4} - \frac{p}{q}\right| > q^{-35.35}$$

for all integers p and all integers $q \geq q_0$. In particular $\mu(\zeta(3)/\pi^4) \leq 35.35$.

Since μ is invariant under multiplication by nonzero rationals, part (a) bounds the exponent of every number $r\zeta(3) + r'\pi^4$ with $r, r' \in \mathbb{Q}$ and $r' \neq 0$. The integer q_0 of part (a) depends on a and b. The exponent is also invariant under rational Möbius maps, so part (b) extends to every quotient $(a\zeta(3) + b\pi^4)/(c\zeta(3) + d\pi^4)$ with $a, b, c, d \in \mathbb{Q}$ and $ad - bc \neq 0$.

The proof gives the exponents 27.4849 and 35.3492, with the constants of Table 1. Theorem 1.1 has been verified in the Lean 4 proof assistant, together with every theorem, proposition and lemma used in its proof. The formal proof has no unproved steps and uses only the standard axioms. Section 11 describes it. The remarks are not part of the formal proof. The formalization is available at `https://github.com/slitsey/catalan-irrationality-measure`. The use of AI in this work is disclosed at the end of the paper.

1.2. **Background.** *The numbers $\zeta(3)$ and π^4.* Euler showed that $\zeta(4) = \pi^4/90$, and π^4 is transcendental. Apéry proved that $\zeta(3)$ is irrational [2], and Beukers gave his integral proof [4]. Rhin and Viola introduced a method of permutation groups for $\zeta(2)$ [30], and with it they proved $\mu(\zeta(3)) \le 5.513891$ [31]. For π^4 the best bound is due to Marcovecchio and Zudilin, $\mu(\pi^4) = \mu(\zeta(4)) \le 12.51085940\ldots$ [26]. For the other odd values the classical results concern families. Rivoal [32] and Ball and Rivoal [3] proved that infinitely many of the numbers $\zeta(2k+1)$ are irrational. Zudilin proved that one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational [38, 39]. Fischler, Sprang and Zudilin raised the number of irrational values among $\zeta(3),\ldots,\zeta(s)$ [10]. Fischler proved that many of the odd zeta values are linearly independent over $\mathbb{Q}$ [8].

The plane spanned by $\zeta(3)$ and π^4. Whether 1, $\zeta(3)$ and π^4 are linearly independent over $\mathbb{Q}$ was a classical open question. Dauguet and Zudilin studied simultaneous rational approximations to $\zeta(2)$ and $\zeta(3)$ of the hypergeometric type, and they noted that the independence of 1, $\zeta(2)$ and $\zeta(3)$ remained open [5]. Fauzan proved that $\zeta(5)$ is irrational with rationally normalized Hankel determinants [7]. Lai combined Fauzan's determinant method with the simultaneous approximations of Zudilin and of Dauguet and Zudilin, and proved that 1, $\zeta(3)$ and π^4 are linearly independent [19]. As a corollary $\zeta(3)/\pi^4$ is irrational. Leibtag, Barkan, Kalisch, Weinbaum, Shalyt and Kaminer connected the classical constructions through conservative matrix fields [21]. They gave another proof of this independence and proved $\mu(A\zeta(3)+B\pi^4) < 66.3$ for every $(A,B) \in \mathbb{Q}^2 \setminus \{(0,0)\}$. They also proved $\mu(\zeta(2)) < 4.62675$ and $\mu(\zeta(3)) < 5.4197$. Fischler, Lai, Sprang and Zudilin proved that many more odd zeta values are irrational [9]. As far as we know, 66.3 is the best published measure on the plane, and no irrationality measure for $\zeta(3)/\pi^4$ was known. Part (a) improves the bound to 27.49 for $b \neq 0$. Part (b) is, as far as we know, the first measure for $\zeta(3)/\pi^4$. On the line $b = 0$ the number is a rational multiple of $\zeta(3)$, whose measure is much smaller. On the line $a = 0$ the bound of Marcovecchio and Zudilin is smaller than ours.

Recent results. Since September 2026 several new results have appeared in this area, some obtained with AI systems. In October 2026 OpenAI proved that Catalan's constant is irrational [28] and that $\mu(\pi) = 2$ [29], both with Lean formalizations. Liu, Jiang and Zhang proved $\mu(\log r) = 2$ for rational $r > 0$, $r \neq 1$ [25]. Giraudin improved the bounds for $\mu(\zeta(2))$ and $\mu(\zeta(3))$, and he gave a bound for $\mu(\zeta(5))$ [12, 13, 14]. Q. Tang proved the linear independence of 1, $\zeta(2)$ and $\zeta(3)$ in Lean [34].

The method. The measure method we use, a family of integer polynomials whose values near the constant are small and nonzero, is a one-variable form of the criteria of Nesterenko [27] used by Ball and Rivoal [3]. The arithmetic of the coefficients is handled prime by prime, as in the work of Hata [16], Rhin and Viola [30, 31] and Zudilin [39]. Determinants of growing size go back to Zudilin's determinantal approach [40]. Their size is controlled by Andréief's identity [11] and a logarithmic energy problem with an external field [33].

1.3. **The companion papers.** This paper applies one method to a fourth set of constants. The method consists of Hankel or Löwner determinants, denominators bounded prime by prime, decay proved through an energy problem, and a formal proof of every step. The first companion paper [22] refines OpenAI's determinant for Catalan's constant. The second [24] uses Löwner determinants of digamma combinations and proves measures on three planes of numbers. Its third plane is the plane of this paper, since $\pi^4 = 90\zeta(4)$. The third [23] proves a measure for $\zeta(5)$ with Fauzan's determinants. Its decay argument, a comparison measure with closed-form potentials checked on cells, is the closest relative of Section 4. We use the definitions of μ, the measure criterion and the potential-theoretic language of these papers. The construction here is Lai's, with his notation. From Section 1.4 on we write the points of the plane as $a\zeta(3) + b\zeta(4)$, as Lai does. Since $\pi^4 = 90\zeta(4)$, the number $a\zeta(3) + b\pi^4$ of Theorem 1.1(a) is $a\zeta(3) + 90b\zeta(4)$.

1.4. **What a measure requires.** Lai's proof uses three facts about his determinants $J_n = J_n(a, b, a\zeta(3) + b\zeta(4))$. A rational normalizing factor m_n with $\log m_n \le (30.83 + o(1))n^2$ makes $m_n J_n(a, b, X)$ a polynomial with integer coefficients. The decay $|J_n| \le e^{-31.61n^2}$ holds at the constant. And $J_n \neq 0$ for $n = 2(\ell - 1)/9$ at a prime $\ell \equiv 1 \pmod 9$ with $\ell > |b|$. If $a\zeta(3) + b\zeta(4)$ were rational, a nonzero integer would be smaller than 1.

A measure needs more. Let p/q be a rational close to the constant. Then $q^{3n} m_n J_n(a, b, p/q)$ is an integer, and three estimates are needed at sizes n of order $\log q$.

(i) *Denominators.* The bound $\log m_n \le (\Lambda + o(1))n^2$.
(ii) *Decay on a neighbourhood.* $|J_n(a, b, X)| \le e^{-(D-o(1))n^2}$ for every real X within $e^{-(\rho+o(1))n}$ of the constant, not only at the constant.
(iii) *Nonvanishing near the point.* $J_n(a, b, p/q) \neq 0$ for one size n in a range of length $O(\log q)$ above the size where (i) and (ii) beat q^{3n}.

Lai's nonvanishing holds only at sizes $n \gg |b|$. For the ratio $\zeta(3)/\pi^4$ the rational p/q enters through the direction: $q^{3n} J_n(1, -p/q, 0) = J_n(q, -p, 0)$, so the coefficient $b = -p$ grows with q. So the decay must be uniform in the direction (a, b), and nonvanishing must hold at sizes of order $\log q$ for coefficients of size q. The exponent that results is $\rho(3/(D - \Lambda) + \beta)$, where β measures the sparsity of the sizes at which nonvanishing is proved.

1.5. **New ideas.** We keep Lai's determinants unchanged. The proof rests on the following ideas.

(1) *Decay on a complex contour by weak duality.* Lai writes the functional as an integral over a vertical line and bounds J_n by Heine's formula with the modulus of a complex weight. We prove a general upper bound of energy type for this situation. A positive measure σ of mass 3 in the complex plane and an upper bound W^* for $V - 2U^\sigma$ on an admissible contour give $|J_n| \le e^{-(D-o(1))n^2}$ with $D = -\kappa - I(\sigma) - 3W^*$. The perturbation of X adds atoms at the negative integers. They are controlled by the

potential of σ on the negative axis, and this gives the neighbourhood exponent ρ. The bound is uniform in the direction (a,b) (Theorem 4.1). We apply it to a contour with two straight arms and a measure on two segments whose density against the arcsine law is an explicit square. Its potentials have closed forms in real arithmetic, and the inequality is checked on 40 cells (Theorem 4.3).

(2) *An exact law for the denominators.* For every prime $\sqrt{8n} < p < 3n$ we improve Lai's bound for $v_p(J_n)$ by two explicit integers x_1 and x_2, determined by the ties among Lai's increments (Theorems 5.9 and 6.1). The first gain is a rank bound for a reduced Gram form modulo p. Its weights are values of one polynomial on residue classes, so a power sum over $\mathbb{F}_p$ forces a kernel. The second gain comes from a basis whose roots are shifted by multiples of p, so that the leading local forms are orthogonal. The remaining terms cancel modulo p^2, because a Faulhaber sum and the harmonic weights of the unit parts have opposite totals. For $3n < p \le 6n$ similar bases give Lai's case bounds for the polynomial in three variables and an extra $7n-2p$ in the range $3n < p \le 7n/2$ (Theorems 5.5 and 5.6).

(3) *The integral of the law in closed form.* On $p \le 3n$ the law is piecewise affine in n/p on 22 pieces. Its integral against the prime distribution telescopes to a sum over 23 nodes, and Gauss's digamma theorem gives a closed form in logarithms (Theorem 7.6).

(4) *Nonvanishing in a band of primes.* For every prime ℓ with $8n/3 < \ell < 3n-1$, outside a finite set, we prove $J_n(a,b,X) \neq 0$ at every point with $a \in \mathbb{Z}_{(\ell)}$, b an ℓ-adic unit and $\ell^3 X \in \mathbb{Z}_{(\ell)}$ (Theorem 8.5). So a prime is good for every size n in an interval, and not for one size only. The proof shows that a reduced Gram matrix is nonsingular modulo ℓ. This matrix is block triangular. Its integral blocks are small matrices of rational constants with explicit determinants (Lemmas 8.9, 8.10 and 8.11). Its half-integral block is, modulo ℓ, the Hankel matrix of the functional $x^e \mapsto eB_{e-1}$ against the cube of a falling factorial (Lemma 8.12). We obtain this by Faulhaber's formula and four sums of harmonic numbers, and not by analysis.

(5) *The Hankel determinant in closed form.* The determinant of the half-integral block is an explicit product of factorial ratios, the norms of two families of Wilson polynomials, and all its prime factors are smaller than ℓ (Lemma 8.14). We prove it by finite algebra: a Pearson equation for the functional, a Beta-type evaluation, Gram–Schmidt for the orthogonal polynomials and the Pfaff–Saalschütz summation. No integral is used.

(6) *Nonvanishing at prime-tied sizes.* We extend Lai's auxiliary-prime argument to every rational X whose denominator is not divisible by ℓ^4, and to the ratio direction (Theorem 8.1). We find a second family of primes with exact nonvanishing, $3\ell = 7n+1$ and $3\ell = 7n+2$ (Theorem 8.3). There the reduction modulo ℓ leads to a Hankel determinant of a weight on an interval of residues. By duality for complementary subsets its

value is a determinant of size 1 or 2 of power sums of a polynomial over $\mathbb{F}_\ell$, equal to $-1/49$ or -1.

(7) *A criterion with bands.* Suppose that nonvanishing is proved for families of primes in residue classes, and that each prime is good for the sizes n in an interval. The prime number theorem in progressions and an inclusion–exclusion count over the classes turn this into a measure (Theorem 3.1). A prime in several families is counted once. The band of item 4 covers every prime between $8n_0/3$ and $3N$ for a window of sizes $n_0 \le n \le N$.

	Λ	D	ρ	nonvanishing
Lai [19]	30.83	31.61	–	$n = 2(\ell-1)/9$, $\ell > \lvert b\rvert$
This paper	28.68723663	33.4401	42.94856306	band $\frac{8}{3}n < \ell < 3n-1$, $\ell \equiv 1$ (18), $\ell \equiv 3, 5$ (7)

TABLE 1. The constants per n^2 (and per n for ρ). Lai's decay holds at the constant only. The exponent is $\rho(3/(D-\Lambda)+\beta)$ with $\beta = 7/800$ for the plane and $\beta = 9593/50000$ for the ratio.

1.6. **Organization.** Section 2 recalls Lai's construction. Section 3 proves the measure criterion. Section 4 proves the decay. Sections 5, 6 and 7 bound the normalizing factor: the local bounds, the second level of the law, and the prime sum. Section 8 proves the nonvanishing, in Lai's sizes, on two lines, and in the band of primes with its Hankel determinant. Section 9 completes the proof, and Section 10 compares the constants with numerical values. Section 11 describes the formalization. The disclosure of AI use follows it.

1.7. **Notation.** For a prime p we write v_p for the Gauss valuation on $\mathbb{Q}(a, b, X)$: the p-adic valuation extended to polynomials by the minimum over the coefficients, with $v_p(a) = v_p(b) = v_p(X) = 0$. We write $\mathbb{Z}_{(p)}$ for the rationals with $v_p \ge 0$, $H_m = \sum_{i \le m} 1/i$ and $H_m(j) = \sum_{i \le m} i^{-j}$. We write $\{y\} = y - \lfloor y \rfloor$ and $y_+ = \max(y, 0)$. Chebyshev's function in a progression is $\theta(x; M, r) = \sum_{\ell \le x,\ \ell \equiv r\ (M)} \log \ell$, summed over primes ℓ. The logarithmic potential and energy of a positive measure σ on $\mathbb{C}$ are

$$U^\sigma(s) = \int \log \frac{1}{|s-u|}\, d\sigma(u), \qquad I(\sigma) = \iint \log \frac{1}{|s-u|}\, d\sigma(s)\, d\sigma(u).$$

2. Lai's construction

We recall the construction of [19, §2], with the variable X in place of the constant. Throughout $n \ge 1$. Write $(x)_m = x(x+1)\cdots(x+m-1)$ and put

$$R_n(t) = \frac{(6n)!^2}{(2n)!^3} \cdot \frac{(t-2n)_{2n}^3}{(t)_{6n+1}^2}.$$

It has zeros of order 3 at $t=1,\dots,2n$, double poles at $t=0,-1,\dots,-6n$, and $R_n(t)=O(t^{-6n-2})$. For polynomials P of degree less than $6n$ Lai defines

$$L_n(P) = -3a\sum_{m\ge1}(R_nP)'(m) + b\sum_{m\ge1}(R_nP)''(m), \tag{2.1}$$

and

$$J_n = \frac{\det\bigl[L_n(t^{i+j})\bigr]_{0\le i,j<3n}}{\prod_{i<3n} i!^2}.$$

Put

$$c_k = \binom{6n}{k}^2\binom{2n+k}{k}^3, \qquad d_k = 5H_k - 3H_{2n+k} - 2H_{6n-k} \qquad (0\le k\le 6n).$$

Proposition 2.1 (Lai)**.** *For P of degree less than $6n$,*

$$L_n(P) = \sum_{k=0}^{6n} c_k\bigl(U_kP'(-k) + V_kP(-k)\bigr), \tag{2.2}$$

where $U_k = -3aH_k(2) - 2bH_k(3)$ and $V_k = U_kd_k + 6\bigl(X - aH_k(3) - bH_k(4)\bigr)$ at $X = a\zeta(3)+b\zeta(4)$.

Proof. The partial fraction expansion of R_nP has the form $\sum_k\bigl(A_{2,k}(t+k)^{-2} + A_{1,k}(t+k)^{-1}\bigr)$, since $\deg(R_nP)\le -2$. Here $A_{2,k} = c_kP(-k)$, because $(t+k)^2R_n$ takes the value c_k at $t=-k$. The logarithmic derivative of $(t+k)^2R_n$ at $-k$ is $-3(H_{2n+k}-H_k)-2(H_{6n-k}-H_k) = d_k$, so $A_{1,k} = c_k(P'(-k)+d_kP(-k))$. Also $\sum_k A_{1,k}=0$. Now $\sum_{m\ge1}(m+k)^{-s} = \zeta(s) - H_k(s)$. Differentiating the expansion once and twice and summing over m, the terms in $\zeta(2)$ cancel because $\sum_k A_{1,k}=0$. The term $2b\zeta(3)\sum_k A_{1,k}$ cancels for the same reason. What remains is $6(a\zeta(3)+b\zeta(4))\sum_k A_{2,k}$ plus the stated rational part. □

We take (2.2) with X an indeterminate as the definition of a $\mathbb{Q}[a,b,X]$-valued functional, and we write $J_n(a,b,X)$ for the determinant above with this functional. The entries are linear forms in (a,b,X), so $J_n(a,b,X)\in\mathbb{Q}[a,b,X]$ is homogeneous of degree $3n$:

$$J_n(\lambda a,\lambda b,\lambda X) = \lambda^{3n}J_n(a,b,X). \tag{2.3}$$

At $X = a\zeta(3)+b\zeta(4)$ it is Lai's J_n. The two families of this paper are

$$Q_n(X) = J_n(a,b,X) \quad \text{for fixed } a,b\in\mathbb{Z}, \qquad f_n(y) = J_n(1,-y,0).$$

Both have degree at most $3n$. By (2.3),

$$q^{3n}Q_n(p/q) = J_n(qa,qb,p), \qquad q^{3n}f_n(p/q) = J_n(q,-p,0). \tag{2.4}$$

The constant of Q_n is $a\zeta(3)+b\zeta(4)$. The constant of f_n is $y^* = \zeta(3)/\zeta(4) = 1.11062\ldots$, and $\zeta(3)/\pi^4 = y^*/90$. Indeed $f_n(y) = J_n(1,-y,X_y+t)$ with $X_y = \zeta(3)-y\zeta(4)$ and $t = -X_y = \zeta(4)(y-y^*)$. So f_n is small near y^* when J_n is small near the constant, uniformly in the direction.

3. The measure criterion

The criterion of the companion papers [22, 23] assumes nonvanishing at every large size. Here nonvanishing is known only at the sizes n tied to primes ℓ in some residue classes. For some families each prime is tied to one size, and these sizes have density zero. For others each prime is tied to an interval of sizes of length proportional to ℓ. In both cases the primes carry enough logarithmic mass. If $P_n(p/q) = 0$ at every tied size in a window, then a power of every tied prime divides q, and the product of these powers exceeds q. The criterion makes this precise.

Fix a modulus M. A *tied family* consists of a set S of unit classes modulo M, slopes $0 < c \le c'$, an integer $C \ge 0$, an exponent $e \ge 1$ and a finite set E of primes. For every prime $\ell \notin E$ with $\ell \bmod M \in S$ it has a set $R(\ell)$ of sizes $n \ge 1$, with the following property. For every real $y \ge 1$ with $cy \le \ell \le c'y$ the set $R(\ell)$ contains an integer n with $|n - y| \le C$. A *line* has $c = c'$ and $R(\ell) = \{n(\ell)\}$ with $\ell = c\,n(\ell) + d$. A *band* has $c < c'$, and $R(\ell)$ is the set of all n with $cn < \ell < c'n - d$. For a list of tied families $i = 1, \dots, r$ and a nonempty subset T of indices put

$$\delta_T = \frac{\#\{s \in (\mathbb{Z}/M)^\times : s \in S_i \text{ for all } i \in T\}}{\varphi(M)},$$

$$\lambda_T(u, w) = \Big(w \min_{i \in T} c_i' - u \max_{i \in T} c_i\Big)_+,$$

and

$$\mathcal{M}(u, w) = \sum_{\emptyset \ne T \subseteq \{1, \dots, r\}} (-1)^{|T|+1} \Big(\min_{i \in T} e_i\Big) \delta_T \, \lambda_T(u, w).$$

Theorem 3.1 (measure criterion)**.** *Let ξ be real, $\kappa \ge 1$ an integer, and $\Lambda < D$, $\rho > 0$, $\beta > 0$, $K > 0$ real numbers. Let $(P_n)_{n \ge 1}$ be polynomials with rational coefficients and $\deg P_n \le \kappa n$. Suppose that for every $\varepsilon > 0$ and all large n the following hold.*

- *(a) There is a positive rational number d_n with $\log d_n \le (\Lambda + \varepsilon)n^2$ and $q^{\kappa n} d_n P_n(p/q) \in \mathbb{Z}$ for all integers p and $q > 0$.*
- *(b) $|P_n(\xi + t)| \le e^{-(D-\varepsilon)n^2}$ for every real t with $|t| \le e^{-(\rho+\varepsilon)n}$.*

Let $W(p, q)$ be positive integers with $W(p, q) \le Kq$ whenever $|\xi - p/q| < 1$. Let tied families $i = 1, \dots, r$ be given, each with a relation $\mathrm{Bad}_i(\ell, p, q)$ such that:

- *(c) for every prime $\ell \notin E_i$ with $\ell \bmod M \in S_i$, every $n \in R_i(\ell)$, and all p and $q > 0$, either $P_n(p/q) \ne 0$ or $\mathrm{Bad}_i(\ell, p, q)$ holds, and $\mathrm{Bad}_i(\ell, p, q)$ implies $\ell^{e_i} \mid W(p, q)$;*
- *(d) $\mathcal{M}(u_0, u_0 + \beta) > 1$, where $u_0 = \kappa/(D - \Lambda)$.*

Then for every $\mu > \rho(\kappa/(D - \Lambda) + \beta)$ there is q_0 such that $|\xi - p/q| > q^{-\mu}$ for all integers p and all integers $q \ge q_0$.

The fraction p/q need not be in lowest terms, and no irrationality of ξ is assumed. Condition (a) holds for every representation of the fraction, and so does (c).

Proof. Put $\eta = D-\Lambda$ and fix $\mu > \rho(\kappa/\eta+\beta)$. Choose $\varepsilon>0$ so small that $\mathcal{M}(u_\varepsilon, w_\varepsilon) > 1+\varepsilon$ and $\rho_\varepsilon(u_\varepsilon+\beta+\varepsilon) < \mu$, where $u_\varepsilon = (\kappa+\varepsilon)/(\eta-2\varepsilon)+\varepsilon$, $w_\varepsilon = u_0+\beta-\varepsilon$ and $\rho_\varepsilon = \rho+\varepsilon$. This is possible because $\mathcal{M}$ is continuous: it is a finite sum of functions $(w\gamma-u\gamma')_+$.

Let q be large and suppose $|\xi-p/q| \le q^{-\mu}$. Put $L=\log q$,

$$n_0 = \frac{(\kappa+\varepsilon)L+C}{\eta-2\varepsilon}, \qquad N = n_0+(\beta+\varepsilon)L,$$

with a constant C chosen below. Then $\rho_\varepsilon N \le \mu L$ for large q, so $|\xi-p/q| \le e^{-\rho_\varepsilon n}$ for every $n \le N$.

Step 1: no tied size in the window is nonzero. Let $n \in [n_0, N]$ be large and suppose $P_n(p/q) \neq 0$. By (a) the number $z = q^{\kappa n} d_n P_n(p/q)$ is a nonzero integer, so $|z| \ge 1$. By (b) with $t = p/q-\xi$, $|z| \le q^{\kappa n} e^{(\Lambda+\varepsilon)n^2-(D-\varepsilon)n^2}$. Hence $(\eta-2\varepsilon)n \le \kappa L$. This contradicts $n \ge n_0$ when $C>0$.

Step 2: one divisibility. Let U_i be the set of primes $\ell \notin E_i$ with $\ell \bmod M \in S_i$ for which $R_i(\ell)$ meets $[n_0, N]$, and $U = \bigcup_i U_i$. For $\ell \in U$ let $e(\ell)$ be the largest e_i with $\ell \in U_i$. By Step 1 and (c), $\ell^{e_i} \mid W(p,q)$ for every i with $\ell \in U_i$, so $\ell^{e(\ell)} \mid W(p,q)$. The primes are distinct, so

$$\sum_{\ell \in U} e(\ell)\log\ell \le \log W(p,q) \le L + \log K. \tag{3.1}$$

Step 3: the prime count. Let $\ell \notin E_i$ be a prime with $\ell \bmod M \in S_i$ and $c_i(n_0+C_i) < \ell \le c_i'(N-C_i)$. Then the interval $[\ell/c_i', \ell/c_i] \cap [n_0+C_i, N-C_i]$ is not empty. Take y in it. By the property of the family, $R_i(\ell)$ contains an integer n with $|n-y| \le C_i$, and this n lies in $[n_0, N]$. So $\ell \in U_i$. For large q the set U_i therefore contains the primes of the class with ℓ/L in the interval $J_i = (c_i u_\varepsilon, c_i' w_\varepsilon]$. For a fixed class s the finitely many endpoints of the J_i with $s \in S_i$ cut $(0,\infty)$ into intervals. On each the set of i with J_i containing it is constant, and so is the largest e_i. The prime number theorem in arithmetic progressions, $\theta(x; M, s) = x/\varphi(M) + o(x)$ for every unit class s, counts the primes of each class in each interval. Removing the finite sets E_i costs $O(1)$. Summing,

$$\sum_{\ell\in U} e(\ell)\log\ell \ge \Big(\frac{1}{\varphi(M)} \sum_s \int_0^\infty \max_{i:\, s\in S_i} e_i \mathbf{1}_{J_i}(x)\,dx - \frac{\varepsilon}{2}\Big)L$$

for large q. For nonnegative numbers x_i, $\max_i x_i = \sum_{T\neq\emptyset} (-1)^{|T|+1} \min_{i\in T} x_i$. With $x_i = e_i \mathbf{1}_{J_i}(x)$, the minimum over T is $(\min_T e_i)\mathbf{1}_{\bigcap_T J_i}(x)$, and $\bigcap_T J_i$ has length $\lambda_T(u_\varepsilon, w_\varepsilon)$. Averaging over s gives the integral $\mathcal{M}(u_\varepsilon, w_\varepsilon) > 1+\varepsilon$. So the left side of (3.1) is at least $(1+\varepsilon/2)L$ for large q, which contradicts (3.1).

Hence $|\xi-p/q| > q^{-\mu}$ for all large q. The constant C absorbs $\log K$ and the finitely many small n. □

When the windows of different families do not overlap, $\mathcal{M}(u, u+\beta) = \sum_i e_i\delta_i((c_i'-c_i)u + c_i'\beta)$. The inequality (d) then says that the tied primes in a window of length βL carry more than L of logarithmic mass.

The second part of Theorem 1.1 is stated for $\zeta(3)/\pi^4$. We apply the criterion to y^* and use the following step.

Lemma 3.2. *Let x be real, $r \neq 0$ rational and $\mu' < \mu$. Suppose that $|x - p/q| > q^{-\mu'}$ for all integers p and all large q. Then $|rx - p/q| > q^{-\mu}$ for all integers p and all large q.*

Proof. Write $r = u/v$ with integers $u \neq 0$ and $v > 0$. Then $|rx - p/q| = (|u|/v)\,|x - vp/(uq)|$, and $vp/(uq)$ is a fraction with denominator $|u|q$. For large q this is at least $(|u|/v)(|u|q)^{-\mu'}$, which exceeds $q^{-\mu}$ for large q. □

4. Decay on a neighbourhood

This section proves condition (b) of Theorem 3.1 for both families, uniformly in the direction. The method is the one of the $\zeta(5)$ paper [23]. The determinant is an average over configurations of points. The average is bounded by an energy problem with an external field. A single positive measure σ then gives an upper bound by weak duality. Two features are new. The functional is an integral over a complex contour rather than a real interval, and the perturbation of X adds point masses at the negative integers. We first prove a general bound (Theorem 4.1) and then check its hypotheses for an explicit contour and measure (Theorem 4.3).

4.1. **The field.** Put $(K, B, e, r) = (6, 2, 3, 2)$. These are the scaled numbers of poles and zeros of R_n and their orders. For $s \in \mathbb{C}$ put

$$\Phi(s) = r(K\log K - K) - e(B\log B - B) + e\int_0^B \log|s-x|\,dx - r\int_0^K \log|s+x|\,dx,$$

$$V(s) = \Phi(s) - 2\pi|\operatorname{Im} s|, \qquad W_\sigma(s) = V(s) - 2U^\sigma(s).$$

By Stirling's formula $\log|R_n(ns)| = n\Phi(s) + O(\log n)$ away from the poles and zeros. The term $-2\pi|\operatorname{Im} s|$ is the decay of $\pi\cot\pi s$ and of its derivatives off the real axis. For $x \in [0, K]$ put

$$\omega(x) = r\big(K\log K - x\log x - (K-x)\log(K-x)\big) + e\big((B+x)\log(B+x) - x\log x - B\log B\big).$$

Then $\log c_k \le n\,\omega(k/n)$ for $0 \le k \le 6n$, by the entropy bound for binomial coefficients. Finally put $\kappa = 27/2 - 9\log 3$ and $c(a,b) = 3|a| + |b|$.

A *contour* is a polyline Γ in the half-plane $\operatorname{Re} s > 0$, symmetric under conjugation. It runs from $-i\infty$ to $+i\infty$, its two unbounded edges are vertical rays, and it meets $\mathbb{R}$ at one point $x_0 \in (0, B)$.

Theorem 4.1. *Let Γ be a contour. Let σ be a positive measure on $\mathbb{C}$ of mass 3 with compact support, invariant under conjugation, with $\sigma(D(s,r)) \le A\sqrt{r}$ for every disc of radius $r \le r_0$, for some $A, r_0 > 0$. Let W^*, a^* and ρ be real numbers with*

$$W_\sigma(s) \le W^* \quad (s \in \Gamma), \qquad a^* \le W^* + \rho,$$
$$\omega(x) - 2U^\sigma(-x) \le a^* \quad (0 \le x \le 6).$$

Put $D = -\kappa - I(\sigma) - 3W^*$. Then for every $\varepsilon > 0$ there is n_0 such that

$$|J_n(a,b,a\zeta(3)+b\zeta(4)+t)| \le c(a,b)^{3n} e^{-(D-\varepsilon)n^2}$$

for all $n \ge n_0$, all real a, b with $c(a,b) > 0$, and all real t with $|t| \le c(a,b)e^{-\rho n}$.

The bound is uniform in (a,b) and t. This is what the ratio family needs.

Proof. Fix n and write $c = c(a,b)$. We take $X = a\zeta(3) + b\zeta(4) + t$, and we write L_X for the functional at X.

Step 1: the contour representation. Put $\cot_\pi(z) = \pi\cot\pi z$ and $K_{a,b}(z) = 3a\cot'_\pi(z) + b\cot''_\pi(z)$. Let $F = R_n P$ with $\deg P < 6n$. Since $F = O(z^{-2})$, the sums in (2.1) converge. The residue of $F^{(j)}\cot_\pi$ at an integer m is $F^{(j)}(m)$, and on a closed contour $\oint F^{(j)}\cot_\pi = (-1)^j \oint F \cot_\pi^{(j)}$. Let Γ_n be the contour $n\Gamma$, moved by at most 1 near the crossing so that it meets $\mathbb{R}$ at a point $v \notin \mathbb{Z}$ with $0 < v < 2n+1$. The integers m with $1 \le m < v$ are zeros of R_n of order 3, so their residues vanish. Integrating over a large rectangle and letting it grow,

$$L_{X_0}(P) = \frac{1}{2\pi i}\int_{\Gamma_n} R_n(z)P(z)K_{a,b}(z)\,dz, \qquad X_0 = a\zeta(3)+b\zeta(4),$$

with Γ_n oriented downward. By (2.2) the dependence on X is $L_X(P) = L_{X_0}(P) + 6t\sum_k c_k P(-k)$. So L_X is integration against the complex measure $\nu = \nu_\Gamma + \nu_{\mathrm{at}}$, where ν_Γ is the weight on Γ_n and $\nu_{\mathrm{at}} = 6t\sum_k c_k\delta_{-k}$.

Step 2: Heine's formula. For a complex measure ν with finite moments, Andréief's identity [11] gives

$$\det\Big[\int z^{i+j}d\nu\Big]_{i,j<M} = \frac{1}{M!}\int\cdots\int\prod_{i<j}(z_i-z_j)^2\,d\nu(z_1)\cdots d\nu(z_M).$$

With $M = 3n$ we obtain $|J_n| \le (M!\prod_{i<M} i!^2)^{-1}\int\prod_{i<j}|z_i-z_j|^2\,d|\nu|^M$.

Step 3: the weights. On Γ_n we have $|K_{a,b}(z)| \le c\,C(1+|z|)^C e^{-2\pi|\operatorname{Im} z|}$, uniformly in (a,b). Together with Stirling's formula this gives, for $z = ns$ with $s \in \Gamma$,

$$d|\nu_\Gamma|(z) \le c\,Cn^C(1+|s|)^C e^{nV(s)}\,|dz|.$$

The atoms satisfy $6|t|c_k \le 6c\,e^{-\rho n + n\omega(k/n)}$. In the scaled variable $s = z/n$, a point at an atom $-k$ has weight at most $c\,e^{n(\omega(k/n)-\rho)}$. Two points at the same atom contribute nothing, because the Vandermonde factor vanishes.

Step 4: the energy bound. Let υ be the measure $n^{-1}\sum_i \delta_{z_i/n}$ of mass 3. Every configuration with k points on the contour and $3n-k$ points at the atoms has integrand at most

$$c^{3n} n^{9n^2} \exp\Big(n^2\Big[\int \widetilde{W}\, d\upsilon + \iint_{s\neq u} \log|s-u|\, d\upsilon(s)\, d\upsilon(u)\Big] + O(n\log n)\Big),$$

where $\widetilde{W} = V$ on Γ and $\widetilde{W}(-x) = \omega(x) - \rho$ at the atoms. Now $\int \widetilde{W}\, d\upsilon - I(\upsilon) = \int(\widetilde{W} - 2U^\sigma)\, d\upsilon - I(\upsilon-\sigma) + I(\sigma)$. By the two hypotheses and $a^* - \rho \le W^*$, we have $\widetilde{W} - 2U^\sigma \le W^*$ on the support of υ. The measure $\upsilon - \sigma$ has total mass 0. After replacing each point mass by the uniform measure on a circle of radius n^{-2}, its energy is nonnegative [33, Lemma I.1.8]. Let $\tilde\upsilon$ be the smoothed measure and $\delta = n^{-2}$. The mean of $\log|z-u|$ over the circle of radius δ about u is $\log\max(|z-u|,\delta)$. Hence the energy of $\tilde\upsilon$ is at most the energy of υ off the diagonal plus the self-energy of the circles, which is $3n\cdot n^{-2}\log(1/\delta) = O(n^{-1}\log n)$. For the potential, $0 \le U^\sigma(s) - \int U^\sigma d(\text{circle}) = \int_{|u-s|<\delta} \log(\delta/|u-s|)\, d\sigma(u)$. By $\sigma(D(s,r)) \le A\sqrt{r}$ this is at most $\int_0^\infty \sigma(D(s,\delta e^{-\tau}))\, d\tau \le 2A\sqrt{\delta} = 2A/n$. So $\int U^\sigma d\tilde\upsilon \ge \int U^\sigma d\upsilon - 6A/n$, and the bracket is at most $3W^* + I(\sigma) + o(1)$.

Step 5: the normalization and the tail. The integral over the unbounded edges converges, because W_σ tends to $-\infty$ linearly in $|\operatorname{Im} s|$ there and the factors $(1+|s|)^C$ are polynomial. The number of choices of atoms is at most $(6n+1)^{3n} = e^{O(n\log n)}$. Finally $2\sum_{i<3n}\log i! = 9n^2\log(3n) - \frac{27}{2}n^2 + O(n\log n)$. So

$$\begin{aligned}\log|J_n| &\le 3n\log c + n^2\big(3W^* + I(\sigma) + 9\log n - 9\log(3n) + \tfrac{27}{2}\big) + o(n^2)\\ &= 3n\log c - (D - o(1))n^2.\end{aligned}$$

All the error terms are uniform in (a,b) and t. □

Remark 4.2. In the formal proof the contour hypothesis is stated for each large n separately: the contour Γ_n may depend on n, with a bounded number of vertices, and $W_\sigma \le W^* + \eta$ is required on its pieces for every $\eta > 0$. On the two vertical rays it also requires a linear decrease of W_σ in $|\operatorname{Im} s|$, which for a measure of compact support is the fact used in Step 5. The statement above, with one fixed contour, is the case we use.

4.2. **The contour and the measure.** We now choose Γ and σ. The line $\operatorname{Re} s = x_0$ used by Lai gives a value of D below the value we need. A contour that bends to the right in the upper half-plane lowers the field V along it, and a measure on a segment of this contour can follow the bend.

The contour. The upper half of Γ is the segment from $x_0 = 8/5$ to $P = 21/10 + \frac{11}{5}i$, followed by the vertical ray $x_1 + iy$, $y \ge 11/5$, with $x_1 = 21/10$. The lower half is its conjugate.

The measure. Let $S_+ = [A_0, A_1]$ with

$$A_0 = \tfrac{161}{100} + \tfrac{11}{250}i, \qquad A_1 = \tfrac{83}{40} + \tfrac{209}{100}i,$$

a segment of the first edge of Γ. Write $M=(A_0+A_1)/2$ and $h=(A_1-A_0)/2$, and let ω_0 be the arcsine probability measure on $[-1,1]$. Put

$$H(z)=2^{-24}\big(107224+76785z+240184z^2+539459z^3-565327z^4 \\ +3717555z^5-9212693z^6-6236781z^7+16850383z^8\big),$$

and $f(\cos\varphi)=s_0|H(e^{i\varphi})|^2$, where $s_0>0$ is chosen so that the constant term is $c_0=3/2$. Thus $f=\sum_{k\le 8}c_kT_k$ with rational coefficients, and $f\ge 0$ on $[-1,1]$. Let s_+ be the image of $f\,d\omega_0$ under $x\mapsto M+hx$, let s_- be its conjugate, and put $\sigma=s_++s_-$. Then σ has mass $2c_0=3$.

The constants. $W^*=-130471/10000$, $\rho=42.94856306$.

Theorem 4.3. *For Γ, σ, W^* and ρ as above, the hypotheses of Theorem 4.1 hold, and $I(\sigma)\le 208871/10^5$. Hence $D\ge 33.4401$, and for every $\varepsilon>0$ and all large n,*

$$|J_n(a,b,a\zeta(3)+b\zeta(4)+t)|\le c(a,b)^{3n}e^{-(33.4401-\varepsilon)n^2}$$

for all real a,b with $c(a,b)>0$ and all $|t|\le c(a,b)e^{-42.94856306\,n}$.

The constant follows from $D=-\kappa-I(\sigma)-3W^*\ge -\frac{27}{2}+9\log 3-2.08871+39.1413>33.4401$. The proof occupies the rest of this subsection. It combines closed forms for the potentials (Lemmas 4.5 and 4.6), a cover of the compact part of Γ by 40 cells (Proposition 4.7), the energy (Lemma 4.9), the atoms (Lemma 4.10) and the tail (Lemma 4.8). By symmetry under conjugation, $W_\sigma(\bar s)=W_\sigma(s)$, so it suffices to treat the upper half of Γ. The mass bound $\sigma(D(s,r))\le A\sqrt{r}$ holds because a disc of radius r meets each segment in an interval of length at most $2r$, whose arcsine mass is $O(\sqrt{r})$, and f is bounded.

Remark 4.4. The cell computation gives $I(\sigma)\le 2.0886595$, and hence $D\ge 33.44015$. The formal proof records the rounded values $I(\sigma)\le 2.08871$ and $D\ge 33.4401$, which are all the proof of Theorem 1.1 uses. The paper uses the same values.

Lemma 4.5 (the field)**.** *Put $\kappa_V=12\log 3+6\log 2-6$ and, with principal logarithms,*

$$\widetilde\Psi(s)=e\big[s\log s-s-(s-B)(\log(B-s)+i\pi)+(s-B)\big] \\ -r\big[(s+K)\log(s+K)-(s+K)-s\log s+s\big]+2\pi is.$$

(i) $V(s)=\kappa_V+\operatorname{Re}\widetilde\Psi(s)$ *for* $\operatorname{Im}s\ge 0$.

(ii) $\widetilde\Psi$ *is analytic off* $(-\infty,0]\cup[B,\infty)$, *and* $\widetilde\Psi''(s)=5/s+3/(B-s)-2/(s+K)$.

(iii) For $m\ge 2$, $|\widetilde\Psi^{(m)}(s)|\le(m-2)!\big(5|s|^{1-m}+3|s-B|^{1-m}+2|s+K|^{1-m}\big)$, *and* $|\widetilde\Psi'(s)|\le 5|\log|s||+3|\log|B-s||+2|\log|s+K||+11\pi$.

Proof. For $\operatorname{Im}s>0$, $\int_0^B\log(s-x)\,dx=s\log s-s-(s-B)\log(s-B)+(s-B)$ with $\log(s-B)=\log(B-s)+i\pi$, and $\int_0^K\log(s+x)\,dx=(s+K)\log(s+$

$K)-(s+K)-s\log s+s$. Also $-2\pi\operatorname{Im}s=\operatorname{Re}(2\pi is)$. The real parts give (i), and by continuity (i) holds on $(0,B)$. Differentiating gives $\widetilde{\Psi}'=5\log s-3\log(B-s)-2\log(s+K)-\pi i$ and (ii). The bounds (iii) follow from $|d^j(s-p)^{-1}/ds^j|=j!/|s-p|^{j+1}$ and $|\arg|\le\pi$. □

The potential of a measure $g\,d\omega_0$ on $[-1,1]$, transported to a segment, has a closed form in the Joukowski variable. For $\zeta\notin[-1,1]$ let $\phi=\phi(\zeta)$ be the root of $\phi^2-2\zeta\phi+1=0$ with $|\phi|>1$.

Lemma 4.6 (segment potentials)**.** *Let ν be the image of $g\,d\omega_0$ under $x\mapsto M_c+h_cx$, with $g=g_0+\sum_{k\ge1}g_kT_k\ge0$ a polynomial. For $s\notin\operatorname{supp}\nu$ put $\zeta=(s-M_c)/h_c$ and*

$$G(s)=-g_0\log h_c-g_0\log(\phi/2)+\sum_k g_k\phi^{-k}/k.$$

(a) $U^\nu=\operatorname{Re}G$, and G is analytic off the segment. On the segment, at $\zeta=\cos\varphi$, $U^\nu=-g_0\log(|h_c|/2)+\sum_k(g_k/k)\cos k\varphi$.
(b) Put $c=(|\zeta-1|+|\zeta+1|)/2$ and $d=(|\zeta+1|-|\zeta-1|)/2$. Then $|\phi|=c+\sqrt{c^2-1}$ and $\operatorname{Re}\phi^{-k}=|\phi|^{-k}T_k(d)$.
(c) With $q=(\phi-\phi^{-1})/2$ one has $d\phi/d\zeta=\phi/q$, $dq/d\zeta=\zeta/q$ and $G'(s)=-(g_0+\sum_k g_k\phi^{-k})/(h_cq)$.
(d) $|G^{(m)}(s)|\le g_0(m-1)!/\operatorname{dist}(s,\operatorname{supp}\nu)^m$ for $m\ge1$, and $\operatorname{dist}(s,\operatorname{supp}\nu)\ge|h_c|(c-1)$.

Proof. (a) For $x\in[-1,1]$ the classical expansion $\log|\zeta-x|=\log(|\phi|/2)-\sum_k(2/k)T_k(x)\operatorname{Re}\phi^{-k}$ holds, and $\int T_kT_j\,d\omega_0=\delta_{jk}/2$ for $k\ge1$. Integrating against $g\,d\omega_0$ and adding $\log|h_c|$ gives $U^\nu=\operatorname{Re}G$. On the segment, $\phi=e^{i\varphi}$ by continuity. (b) Write $\zeta=\cosh(\alpha+i\beta)$ with $\alpha>0$. Then $|\zeta\pm1|=\cosh\alpha\pm\cos\beta$, so $c=\cosh\alpha$, $d=\cos\beta$ and $\phi=e^{\alpha+i\beta}$. (c) Differentiate $\phi=\zeta+q$ and $q^2=\zeta^2-1$. (d) Off the support, $G^{(m)}(s)=(-1)^m(m-1)!\int(s-u)^{-m}d\nu(u)$, and ν is positive of mass g_0. For the distance, $|\zeta-1|+|\zeta+1|\le2|\zeta-x|+2$ for $|x|\le1$. □

Lemma 4.6(b) evaluates the potential with square roots, one logarithm and the Chebyshev recurrence. Part (c) gives all derivatives of G at a point as rational functions of ϕ, q and h_c. Part (d) bounds the remainder of a Taylor expansion.

Proposition 4.7 (the cells)**.** *$W_\sigma\le W^*$ on the segment $[x_0,P]$ and on the ray segment $\{x_1+iy:11/5\le y\le33/10\}$.*

Proof. Parametrize the compact part of the upper half of Γ by four arcs: $s=M-h\cosh w$ with $0\le w\le146127/500000$, from A_0 towards the vertex; $s=M+h\cos\varphi$ with $\varphi=\pi\theta$, $0\le\theta\le1$, the support; $s=M+h\cosh w$ with $0\le w\le459681/1000000$, from A_1 to P; and $s=x_1+iy$ with $11/5\le y\le33/10$. These arcs cover $[x_0,P]$ and the ray segment. We cut them into 9, 26, 1 and 4 cells, with endpoints dyadic in θ, on a grid of step 2^{-12} in e^{-w}, and rational in y.

On each cell $t \mapsto W_\sigma(s(t))$ is real analytic, by Lemmas 4.5 and 4.6. We bound it by its Taylor polynomial of order 4 at the centre t_c plus a bound for the fifth derivative. The Taylor coefficients of $V \circ s$ come from Lemma 4.5(ii) and Faà di Bruno's formula. Those of $U^{s_\pm} \circ s$ come from Lemma 4.6(c), and on the support the own potential is the trigonometric polynomial of Lemma 4.6(a). The fifth derivative is bounded by Lemma 4.5(iii) and Lemma 4.6(d), since the Bell polynomials have nonnegative coefficients and $|s^{(k)}|$ is bounded on each arc by $|h|$, $|h|\cosh w$ or 1.

All values are taken at points where every quantity is rational. On the support the centre is $\varphi_c = 2\arctan\tau$ with τ rational, so $\cos\varphi_c$ and $\sin\varphi_c$ are rational. On the other two segment arcs the centre is $w_c = -\log\varrho$ with ϱ rational, so $\cosh w_c$ and $\sinh w_c$ are rational. The value $\phi(\zeta_c)$ is then rational in the given data, up to the square roots of Lemma 4.6(b), which are bounded on both sides by rationals. For the cross potential we choose a rational point $\tilde\phi$ near ϕ_c and use the exact point $\tilde s = M_c + h_c(\tilde\phi + \tilde\phi^{-1})/2$. The derivatives at s_c differ from those at $\tilde s$ by at most $|s_c - \tilde s|\, k!\, g_0/d^{k+1}$, by Lemma 4.6(d), where d is a lower bound for the distance to the support. On each of the 40 cells the resulting upper bound for W_σ is below W^*. The logarithms and π enter only through rational enclosures. □

Lemma 4.8 (the tail)**.** *Let $R_0 = 2091/1000$, $y_1 = 33/10$ and $s = x_1 + iy$ with $y \ge y_1$. Then $W_\sigma(s) \le g(y) := V(s) + 6\log(y + R_0)$ and $g'(y) \le 5x_1/y - 2\pi + 6/(y + R_0)$. Hence $W_\sigma \le g(y_1) < W^*$ for $y \ge y_1$.*

Proof. For u in the support, $|s - u| \le y + |x_1 - u| \le y + R_0$, so $-2U^\sigma(s) \le 6\log(y+R_0)$. By Lemma 4.5, $\partial_y V = 5\arctan(x_1/y) - 3\arctan((x_1 - B)/y) - 2\arctan((x_1 + K)/y) - 2\pi$. Since $x_1 > B$ the last two terms are negative, and $\arctan z \le z$. The bound for g' decreases in y and is $-1.988\ldots$ at y_1, so g decreases on $[y_1, \infty)$. Finally $g(y_1) = -13.2158\ldots < W^*$. □

Lemma 4.9 (the energy)**.** *$I(\sigma) = 2I(s_+) + 2I(s_+, s_-)$, with $I(s_+) = -c_0^2\log(|h|/2) + \sum_k c_k^2/(2k)$, and $I(\sigma) \le 208871/10^5$.*

Proof. The first identity is bilinearity and $I(s_-) = I(s_+)$. By Lemma 4.6(a), $I(s_+) = \int U^{s_+} ds_+$ and $\int T_k f\, d\omega_0 = c_k/2$. The cross term is $I(s_+, s_-) = \pi^{-1}\int_0^\pi g(\varphi) f(\cos\varphi)\, d\varphi$ with $g = U^{s_-}(M + h\cos\varphi)$. We integrate over the 26 support cells. On each, the Taylor polynomial of order 4 of gf is integrated exactly. The odd terms vanish, and the remainder is bounded by the fifth-derivative bounds of the proof of Proposition 4.7, with $|d^j \cos k\varphi/d\varphi^j| \le k^j$. □

Lemma 4.10 (the atoms)**.** *Let $\alpha(x) = \omega(x) - 2U^\sigma(-x)$ on $[0, 6]$. Then $\alpha'' \le -49/100$, and with $x^* = 2211/500$,*

$$\sup_{[0,6]} \alpha \le \alpha(x^*) + \frac{\alpha'(x^*)^2}{2 \cdot 49/100} \le W^* + 42.94856306.$$

Proof. One computes $\omega''(x) = -r/x - r/(K - x) + e/(B + x) - e/x \le -11/24 - 1/3$ on $(0, 6)$. Next $-2U^\sigma(-x) = \int \log((x + \operatorname{Re} u)^2 + (\operatorname{Im} u)^2) d\sigma(u)$, and the

second x-derivative of $\log(v^2+w^2)$ is at most $1/(4v^2)$. Here $v \geq \operatorname{Re} A_0 > 8/5$, so this term contributes at most $3/(4\cdot(8/5)^2) = 75/256$. Hence $\alpha'' < -0.4987$. A concave function with $\alpha'' \leq -m$ satisfies $\alpha(x) \leq \alpha(x^*) + \alpha'(x^*)(x-x^*) - m(x-x^*)^2/2 \leq \alpha(x^*) + \alpha'(x^*)^2/(2m)$. The values $\alpha(x^*) = 29.90146\ldots$ and $|\alpha'(x^*)| < 10^{-3}$ are computed from Lemma 4.6(a),(b) at the rational point $-x^*$. □

Proof of Theorem 4.3. The contour satisfies the definition. The measure σ is positive of mass 3, conjugation-invariant, and satisfies the mass bound. By symmetry, Proposition 4.7 and Lemma 4.8, $W_\sigma \leq W^*$ on Γ. By Lemma 4.10 the second and third hypotheses hold with $a^* = W^* + \rho$. Lemma 4.9 bounds $I(\sigma)$, and Theorem 4.1 gives the result. □

Corollary 4.11. *For fixed integers a, b with $b \neq 0$, the family $Q_n(X) = J_n(a,b,X)$ satisfies condition (b) of Theorem 3.1 at $\xi = a\zeta(3) + b\zeta(4)$ with $D = 33.4401$ and $\rho = 42.94856306$. The family $f_n(y) = J_n(1,-y,0)$ satisfies it at $\xi = y^*$ with the same constants.*

Proof. For Q_n, $c(a,b) \geq 1$ is fixed, and the factor c^{3n} is absorbed into εn^2. For f_n, put $y = y^* + t$ with $|t| \leq e^{-(\rho+\varepsilon)n}$. Then $f_n(y) = J_n(1, -y, \zeta(3) - y\zeta(4) + t\zeta(4))$, by $J_n(1,-y,0) = J_n(1,-y,X_y + t\zeta(4))$ with $X_y = \zeta(3) - y\zeta(4)$. Here $c(1,-y) = 3 + |y| \leq 6$, and $|t\zeta(4)| \leq c(1,-y)e^{-\rho n}$. Theorem 4.3 applies with $c \leq 6$. □

5. Local bounds for the denominators

The polynomial $J_n(a,b,X)$ has rational coefficients. We bound $v_p(J_n)$ from below for every prime p, as a Gauss valuation in (a,b,X), and put $m_n = \prod_{p \leq 6n} p^{-w(n,p)}$ for the resulting lower bounds $w(n,p)$. Then $m_n J_n(a,b,X)$ has integer coefficients. The bounds are of three kinds. For $p > 3n$ we use bases of $\mathbb{Z}_{(p)}[t]_{<3n}$ adapted to the residues of the nodes (Theorems 5.5 and 5.6). For $\sqrt{8n} < p < 3n$ we prove an exact law in the first and second order (Theorems 5.9 and 6.1). For the remaining small primes we use Lai's bound.

5.1. Bases and weighted determinants.

Write $B(F,G) = L_n(FG)$ for the bilinear form on $\mathbb{Q}[t]_{<3n}$. Its Gram matrix in the monomials has determinant $\prod_{i<3n} i!^2\, J_n$, and the monomials form a basis of $\mathbb{Z}_{(p)}[t]_{<3n}$ over $\mathbb{Z}_{(p)}$.

Lemma 5.1. *Let (f_x) be a basis of $\mathbb{Z}_{(p)}[t]_{<3n}$ over $\mathbb{Z}_{(p)}$. Then $v_p(J_n) = v_p \det[B(f_x,f_y)] - \nu_p$, where $\nu_p = 2\sum_{i<3n} v_p(i!)$. In particular this holds for the family*

$$e_{r,i} = (t-\rho_r)^i \prod_{r' \neq r} (t-\rho_{r'})^{\mu_{r'}} \quad (i < \mu_r), \qquad g_j = t^j \prod_r (t-\rho_r)^{\mu_r} \quad (j < d),$$

whenever the $\rho_r \in \mathbb{Z}$ are pairwise incongruent modulo p and $\sum_r \mu_r + d = 3n$.

Proof. A change of basis of determinant δ multiplies the Gram determinant by δ^2, and δ is a unit when both bases span $\mathbb{Z}_{(p)}[t]_{<3n}$. The family above spans $\mathbb{Z}_{(p)}[t]_{<3n}$ by the Chinese remainder theorem in $\mathbb{F}_p[t]$, since the factors $t-\rho_r$ are pairwise coprime modulo p. So its change of basis is unimodular. $\square$

We bound the determinant through the entries. Give each basis element a *potential* $\pi_x \in \frac{1}{2}\mathbb{Z}$.

Lemma 5.2. *Let $\mathcal{O}$ be a discrete valuation ring with $v(p)=1$ and M a symmetric matrix over its fraction field. Suppose the indices split into A, where $\pi_x \in \mathbb{Z}$, and B, where $\pi_x \in \mathbb{Z}+\frac{1}{2}$, with $v(M_{xy}) \geq \pi_x+\pi_y$ on $A\times A$ and $B\times B$ and $v(M_{xy}) \geq \pi_x+\pi_y+\frac{1}{2}$ on $A\times B$. Then*

$$v(\det M) \geq 2\sum_x \pi_x + \operatorname{corank} \overline{N}_{BB},$$

where $\overline{N}_{BB}$ is the reduction modulo p of $(p^{-\pi_x-\pi_y}M_{xy})_{x,y\in B}$. The same holds with the roles of A and B exchanged.

Proof. Over $\mathcal{O}[\varpi]$ with $\varpi^2=p$, put $N=D^{-1}MD^{-1}$ with $D=\operatorname{diag}(\varpi^{2\pi_x})$. The $A\times B$ entries of N lie in $\varpi\mathcal{O}[\varpi]$. Multiplying the B rows by ϖ and the B columns by ϖ^{-1} gives a matrix Z over $\mathcal{O}$ with $\det Z = \det N$, whose reduction is block triangular with diagonal blocks $\overline{N}_{AA}$ and $\overline{N}_{BB}$. So the reduction has rank at most $|A|+\operatorname{rank}\overline{N}_{BB}$, and the Smith form gives $v(\det Z) \geq \operatorname{corank}\overline{N}_{BB}$. $\square$

We also use the elementary fact that $\sum_{s\in\mathbb{F}_p} H(s)=0$ for every $H\in\mathbb{F}_p[s]$ of degree at most $p-2$.

5.2. **The nodes.** In this section and the next, p is the prime and ℓ denotes the second base-p digit of a node. For a node $k=s+\ell p$ with $0\leq s<p$, write $L_n(F)=\sum_k N_k(F)$ with $N_k(F)=c_k\big(U_kF'(-k)+V_kF(-k)\big)$. Write $6n=q_6p+r_6$ and $2n=q_2p+r_2$ with $0\leq r_6,r_2<p$. The residues $s\in[0,p)$ fall into four classes, each an interval:

class	residues	$\sigma(s)$	L_s
A	$s\leq r_6$, $s<p-r_2$	0	q_6
B	$s\leq r_6$, $s\geq p-r_2$	3	q_6
C	$s>r_6$, $s<p-r_2$	2	q_6-1
D	$s>r_6$, $s\geq p-r_2$	5	q_6-1

The nodes of residue s are $k=s+\ell p$ with $0\leq\ell\leq L_s$.

Lemma 5.3 (node valuations)**.** *Let $8n<p^2$ and $k=s+\ell p\leq 6n$.*

(i) $v_p(c_k)=\sigma(s)$.

(ii) If $\ell\geq 1$, then $U_k=p^{-3}u_\ell b+O(p^{-2})$ and $V_k=p^{-4}v_{s,\ell}b+O(p^{-3})$, with $u_\ell=-2H_\ell(3)$, $v_{s,\ell}=-2(H_\ell(3)\delta_{s,\ell}+3H_\ell(4))$ and $\delta_{s,\ell}=5H_\ell-3H_{\ell'}-2H_{\ell''}$, where $\ell'=\lfloor(2n+k)/p\rfloor$ and $\ell''=\lfloor(6n-k)/p\rfloor$. The

terms in a are smaller by one power of p, and the term in X is p-integral.

(iii) If $\ell = 0$, then $v(U_k) \geq 0$ and $v(V_k) \geq -1$.

Consequently, if $F = (t+s)^m R$ with $R \in \mathbb{Z}_{(p)}[a,b,X][t]$, then $v(N_k(F)) \geq \sigma(s) - 4 + m$ for $\ell \geq 1$, and $v(N_k(F)) \geq \sigma(s) - 1 + \min(m,1)$ for $\ell = 0$, with $N_k(F) = 0$ if $\ell = 0$ and $m \geq 2$.

Proof. Since $6n < p^2$, all numbers involved have at most two digits in base p. By Kummer's theorem $v_p\binom{6n}{k}$ is the number of carries in $k+(6n-k)$, which is $[s > r_6]$, and $v_p\binom{2n+k}{k} = [s \geq p - r_2]$. This gives (i). The multiples of p up to k, $2n+k$ and $6n-k$ are ip with $i \leq \ell$, ℓ', ℓ'', so $H_k(j) = p^{-j} H_\ell(j) + O(1)$ and $d_k = p^{-1}\delta_{s,\ell} + O(1)$. This gives (ii) and (iii). Finally $F(-k) = (-\ell p)^m R(-k)$ and $F'(-k) = O(p^{m-1})$. □

5.3. **Lai's bound and the large primes.** The following bounds are Lai's [19, Lemma 4.1], read for the Gauss valuation.

Proposition 5.4. *For every prime p, $v_p(J_n) \geq -12n\lfloor \log_p(8n) \rfloor$. If $p > 6n$, then $v_p(J_n) \geq 0$.*

Proof. Take the binomial basis $\binom{t}{i}$, $i < 3n$. Its change of basis from the monomials has determinant $\prod_{i<3n} i!^{-1}$, so $J_n = \det\big[B(\binom{t}{i}, \binom{t}{j})\big]$. The values of $\binom{t}{i}$ and its derivative at the integers $-k$ have $v_p \geq -\lfloor \log_p i \rfloor$. The numbers $H_k(j)$ with $k \leq 8n$ have $v_p \geq -j\lfloor \log_p(8n) \rfloor$. Each entry therefore has $v_p \geq -4\lfloor \log_p(8n) \rfloor$, and the determinant has $3n$ rows. For $p > 6n$ all quantities are p-integral. □

Theorem 5.5. *Let $3n < p \leq 7n/2$. Then $v_p(J_n) \geq -6(p-2n)$.*

Proof. Here $q_6 = 1$, $r_6 = 6n - p$, $q_2 = 0$ and $r_2 = 2n$. Put $\alpha = p - 2n$. The residues $s < \alpha$ form the class A, those in $[\alpha, 6n-p]$ the class B, and the rest the class D. Take $\rho_r = -r$ and $\mu_r = 2$ for $r < \alpha$, the other $\mu_r = 0$, and $d = 7n - 2p$. The potentials are $-2, -1$ for $e_{r,0}, e_{r,1}$ and 0 for the g_j. Then $2\sum \pi = -6\alpha$ and $\nu_p = 0$. By Lemma 5.3 every entry has $v \geq \pi_x + \pi_y$, except the entries $B(g_i, g_j)$ at the nodes $s+p$ with $s \in B$, which have $v \geq -1$. Their coefficient of p^{-1} is a constant times

$$\sum_{s \in B} \hat{c}_s \, \Pi(-s)^2 (-s)^{i+j}, \qquad \Pi = \prod_{r<\alpha} (t+r)^2,$$

where $\hat{c}_s$ is the unit part of c_{s+p}. By Anton's congruence for binomial coefficients modulo p [15, Theorem 1] and Wilson's reflection $m!(p-1-m)! \equiv (-1)^{m+1}$, $\hat{c}_s \Pi(-s)^2$ is a constant multiple of $\prod_{i<\alpha}(s-i)\prod_{j=1}^{2p-6n-1}(s+j)^2$ on B. This polynomial vanishes on A and D and its product with s^{i+j} has degree at most $p-4$. So the sum over B equals the sum over $\mathbb{F}_p$, which is 0. Lemmas 5.1 and 5.2 give the bound. □

Lai's bound in this range is $5n - 4p$, so Theorem 5.5 gains $7n - 2p$.

Theorem 5.6. *Let $n \geq 10$, $7n < 2p$ and $p \leq 6n$. Then $v_p(J_n) \geq w_7(n,p)$, where*

$$w_7(n,p) = \begin{cases} -2n-2p, & p \leq 4n, \\ -18n+2p-2, & 4n < p,\ 2p \leq 9n, \\ -54n+10p-10, & 2p > 9n. \end{cases}$$

These are Lai's bounds [19, Lemma 5.4]. We give a short proof by a basis with shifted roots, which works for the polynomial in (a, b, X).

Proof. Put $\alpha = p-2n$ and $A_6 = 6n-p$. Every node is $k = s+\ell p$ with $\ell \in \{0,1\}$, and $\ell = 1$ only for $s \leq A_6$. Choose integers $D_0, t_0, N_0 \geq 0$ with $D_0 = \min(\alpha, A_6+1)$, $t_0 \leq D_0$, $t_0 + D_0 + N_0 = 3n$, $t_0 = D_0$ if $N_0 > 0$, and $D_0 + N_0 \leq \alpha$. Put $\rho_r = -(r+p)$ for $r < D_0$ and $\rho_r = -r$ otherwise, and let $\mu_r = 2$ for $r < t_0$, $\mu_r = 1$ for $t_0 \leq r < D_0 + N_0$, and $\mu_r = 0$ otherwise. Give $e_{r,i}$ the potential $i-2$ for $r < D_0$, and potential 1 for $D_0 \leq r < D_0 + N_0$.

The root $-(r+p)$ sits on the node $k = r+p$ itself. If F has a zero of order at least 2 there, then $F(-k) = F'(-k) = 0$ and $N_k(F) = 0$. This replaces Lai's rank condition at these nodes. At the node $k = r$ the factor $(t+r+p)^m$ has value p^m, so $v(N_r(F)) \geq \sigma(r) + m - 1$. For $A_6 < s < \alpha$ all harmonic numbers in V_s have indices below p, so $v(V_s) \geq 0$ and $v(N_s(F)) \geq \sigma(s) = 2$. With Lemma 5.3 for the other nodes, every entry satisfies $v \geq \pi_x + \pi_y$. Lemmas 5.1 and 5.2 give $v_p(J_n) \geq 2\sum \pi = 2(N_0 - t_0 - 2D_0)$, since $\nu_p = 0$. The choices

range	D_0	t_0	N_0
$p \leq 4n$	α	$5n-p$	0
$4n < p$, $2p \leq 9n+1$	A_6+1	$p-3n-1$	0
$2p \geq 9n+2$	A_6+1	D_0	$2p-9n-2$

give $2(N_0 - t_0 - 2D_0) = w_7(n,p)$ in the three ranges. For $2p = 9n+1$ the bound obtained is larger than w_7. □

5.4. **The law in the range $\sqrt{8n} < p < 3n$.** Lai's bound in this range [19, Lemma 4.4] is a greedy count. The j-th index of the residue s costs the *increment* $2j - 6 + \sigma(s)$, plus 3 if $j > 2L_s$, and Lai's bound is the sum of the $3n$ smallest increments minus ν_p. We call this sum $\mathrm{E}(n,p)$. Let λ be the largest increment taken and d the number of increments taken that equal λ. Let μ_s be the number of indices of s with increment below λ, so that $d = 3n - \sum_s \mu_s$. The two classes with $\sigma \equiv \lambda \pmod 2$ are the *tie classes*: (A, C) if λ is even and (B, D) if λ is odd. Let Z be the smaller tie class, W the larger, and $\mathcal{N}$ the union of the other two classes. Put

$$x_1 = \max(0, \min(d - |Z|, |W| - d)), \qquad x_2 = \max(0, x_1 - |\mathcal{N}|).$$

Lemma 5.7 (shape of the greedy)**.** *Let $\sqrt{8n} < p < 3n$. Then $\lambda \geq 0$, every residue r satisfies $2\mu_r + \sigma(r) = \lambda + 4 + [r \in \mathcal{N}]$, and every tie residue satisfies*

$\mu_r + 1 \le 2L_r$. *Moreover*

$$2d + |\mathcal{N}| + 2 = p(2q_6 - 3q_2 - 2 - \lambda). \tag{5.1}$$

If $x_2 > 0$, *then* $\lambda = 2q_6 - 3q_2 - 3$, $2d + 2 = |Z| + |W|$, $x_1 = d - |Z|$ *and* $x_2 = x_1 - |\mathcal{N}|$.

Proof. The indices with negative increment are $j \le 2$ on A and $j = 1$ on B and C. There are $r_6 + 1 + p - r_2 < 3n$ of them, as one checks in the three ranges $p \le 3n/2$, $3n/2 < p \le 2n$ and $2n < p < 3n$. So $\lambda \ge 0$. The increment of the index $2L_c$ of a class c is $T_c = 4L_c - 6 + \sigma_c$, and the smallest is $T_C = 4q_6 - 8$. At least $12n - 2p - r_6 - r_2 + 1 \ge 3n$ indices have increment at most $4q_6 - 7$, so $\lambda \le T_c + 1$ for every class. On a tie class the parity then gives $\lambda \le T_c$. Hence the cap $j > 2L$ is never reached, and the classes are filled up to λ, which is the formula for μ_r. Summing it over r, with $\sum_r \sigma(r) = 2(p - 1 - r_6) + 3r_2$ and $3r_2 - r_6 = p(q_6 - 3q_2)$, gives (5.1). Write $k = 2q_6 - 3q_2 - 2 - \lambda \ge 1$. If $x_2 > 0$ and $k \ge 2$, then $2|Z| + 3|\mathcal{N}| < 2$, so all residues lie in one class. That class contains 0, hence is A, and then $r_6 = p - 1$ and $r_2 = 0$, which contradicts $r_6 \equiv 3r_2 \pmod p$. So $k = 1$ and the remaining identities follow. □

The unit parts of the weights are controlled by a product formula. By Anton's congruence [15, Theorem 1] the unit part of $c_{s+\ell p}$ is $\kappa_c(\ell)\hat{c}_s$ modulo p, where $\kappa_c(\ell)$ depends only on the class c and on ℓ, and $\hat{c}_s$ is a product of factorials of the low digits.

Lemma 5.8. *Put* $w_s = \hat{c}_s \prod_{r \ne s}(r - s)^{2\mu_r}$ *and* $Q(s) = \prod_{\beta \in \mathcal{N}}(s - \beta)$. *On each tie class* c *there is* $K_c \in \mathbb{F}_p^\times$ *with* $w_s \equiv K_c Q(s)$.

Proof. Inside a class, $\hat{c}_{s+1}/\hat{c}_s \equiv (s - r_6)^2(s - b_\varepsilon)^3(s - p + 1)^{-5}$ with $b_\varepsilon = p - r_2 - 1$. The ratio of the products is $\prod_{r \ne s}(r - s)^{2(\mu_{r+1} - \mu_r)}$, and the exponent is nonzero only at the class boundaries r_6, b_ε and $p - 1$. By Lemma 5.7, $2(\mu_{b+1} - \mu_b) = -(\sigma_{b+1} - \sigma_b) + ([b + 1 \in \mathcal{N}] - [b \in \mathcal{N}])$. The jumps of σ cancel the exponents from $\hat{c}$, and what remains is $\prod_b (s - b)^{[b+1 \in \mathcal{N}] - [b \in \mathcal{N}]} = Q(s + 1)/Q(s)$. □

Theorem 5.9. *For every prime* $\sqrt{8n} < p < 3n$, $v_p(J_n) \ge \mathrm{E}(n, p) + x_1$.

Proof. Take the family of Lemma 5.1 with $\rho_r = -r$, the multiplicities μ_r and d as above. Give $e_{r,i}$ the potential $i - 2 + \sigma(r)/2$ and each g_j the potential $\lambda/2$. Then $2\sum \pi$ is the sum of the $3n$ smallest increments, by Lemma 5.7.

Entries. By Lemmas 5.3 and 5.7, every entry has $v \ge \pi_x + \pi_y$, and equality can occur only for two elements of one residue, or for a pair involving the g_j at a tie residue. The elements of tie classes and the g_j have potentials of the same parity, and Lemma 5.2 applies with this group in the role of B.

The reduced form. By Lemma 5.3, the reduction of the tie part is $\overline{N}_T = \sum_{s \in Z \cup W} w_s E_s^\top H_{c(s)} E_s$. Here E_s maps a coefficient vector (x, h) to $(x_{s,0}, \dots, x_{s,\mu_s - 1}, h(s))$, with $h(s) = \sum_j h_j(-s)^j$, and H_c is a Hankel matrix $(\Phi_c(i + i'))_{i,i' \le \mu_c}$ of class constants. We replace the numbers $\Phi_c(m)$ of each

tie class by independent indeterminates. The actual matrix is a specialization of this generic matrix, and the rank does not increase under specialization. So it suffices to find x_1 independent kernel vectors of the generic matrix.

Kernel vectors. Let $h \in \mathbb{F}[s]$ vanish on Z, with $\deg h \le \min(d-1, |Z| + |W| - d - 1)$. For $s \in W$ put $x_s = -H_\mu^{-1}\varphi\, h(s)$, where H_μ is the leading $\mu \times \mu$ block of H_W and $\varphi = (\Phi_W(i+\mu))_{i<\mu}$, and put $x = 0$ elsewhere. Then $H_W E_s(x,h) = (0, \kappa_W h(s))$ with $\kappa_W = \Phi_W(2\mu) - \varphi^\top H_\mu^{-1}\varphi$. The pairing of (x,h) with any (x',h') is

$$\kappa_W \sum_{s\in W} w_s h(s)h'(s) = \kappa_W K_W \sum_{s\in\mathbb{F}_p} Q(s)h(s)h'(s),$$

by Lemma 5.8, because $h = 0$ on Z and $Q = 0$ on $\mathcal{N}$. The polynomial Qhh' has degree at most $|\mathcal{N}| + (|Z| + |W| - d - 1) + (d-1) = p - 2$, so the sum vanishes. The admissible h form a space of dimension x_1.

Lemmas 5.1 and 5.2 now give $v_p(J_n) \ge 2\sum \pi + x_1 - \nu_p$. □

6. The second level of the law

Theorem 6.1. *For every prime $\sqrt{8n} < p < 3n$, $v_p(J_n) \ge \mathrm{E}(n,p) + x_1 + x_2$.*

If $x_2 = 0$ this is Theorem 5.9, so we assume $x_2 > 0$. By Lemma 5.7,

$$\lambda = 2q_6 - 3q_2 - 3, \qquad 3|\mathcal{N}| + 2|Z| + 2x_2 - 2 = p - 4. \tag{6.1}$$

The idea is as follows. In Theorem 5.9 the gain came from a kernel of the reduced form. Here we need one more unit of valuation for x_2 elements. We raise their potentials, and we show that the entries still meet the required bounds. Most pairs meet them by the node valuations, if the basis elements at a residue of W are built from local orthogonal polynomials. The remaining pairs need a sum over W to vanish modulo p or modulo p^2. The second of these sums is a Faulhaber sum plus a sum of harmonic numbers. The two have opposite totals by (6.1), and they cancel.

Let W have the digit flags $\beta_W = [W \subseteq C \cup D]$ and $\varepsilon_W = [W \subseteq B \cup D]$, and write $\sigma = \sigma_W$, $L = L_W$, $\mu = \mu_W$ and $w = |W|$. Put $\mu'_r = \mu_r + 1$ for $r \notin W$ and $\mu'_r = \mu$ on W.

6.1. **Generic node constants.** By Lemma 5.3, the nodes $k = s + \ell p$ with $s \in W$ and $1 \le \ell \le L$ have leading constants $\tau_\ell = (\kappa_\ell u_\ell, \kappa_\ell v_\ell)$, independent of $s \in W$. We replace them by indeterminates. Let $T = (T^u_\ell, T^v_\ell)_{1\le\ell\le L}$ and define L_T by adding $b\,p^{\sigma-4}\mathrm{cl}_k\big[(T^u_\ell - \kappa_\ell u_\ell)\,pF'(-k) + (T^v_\ell - \kappa_\ell v_\ell)F(-k)\big]$ to $N_k(F)$ at these nodes, where cl_k is a fixed integer lift of the unit part of c_k/κ_ℓ. Let $J_n(T)$ be the determinant for L_T. At $T = \tau$ it is J_n, and τ is p-integral, so $v_p(J_n) \ge v_p(J_n(T))$ for the Gauss valuation in (a,b,X,T). From now on $\mathcal{O}$ is the valuation ring of this Gauss valuation on $\mathbb{Q}_p(a,b,X,T)$.

Lemma 6.2. *Let $1 \le L < p$ and $0 \le \mu \le 2L$, and let $\mathcal{L}_T(\Phi) = \sum_{\ell=1}^{L}(T^v_\ell\Phi(-\ell) + T^u_\ell\Phi'(-\ell))$. Then $\det(\mathcal{L}_T(y^{i+j}))_{i,j<\mu}$ is a nonzero element of $\mathbb{F}_p[T]$.*

Proof. Specialize. If $\mu \le L$, put $T^v_\ell = 1$ for $\ell \le \mu$ and all other $T = 0$. If $\mu > L$, put $T^u_\ell = 1$ for $\ell \le \mu - L$, $T^v_\ell = 1$ for $\mu - L < \ell \le L$, and all other $T = 0$. In both cases the form $(\phi, \psi) \mapsto \mathcal{L}(\phi\psi)$ on $\mathbb{F}_p[y]_{<\mu}$ is, by the Chinese remainder theorem, an orthogonal sum of the nondegenerate local forms $\phi\psi(-\ell)$ and $(\phi\psi)'(-\ell)$, since the nodes $-\ell$ are distinct modulo p. □

So there is a polynomial $A(y) = \sum_{j\le\mu} A_j y^j$ over $\mathcal{O}$, with leading coefficient $A_\mu = \Delta$ a unit of $\mathcal{O}$ (the Hankel determinant of Lemma 6.2), such that $\mathcal{L}_T(y^i A) = 0$ for $i < \mu$. For $s \in W$ put

$$P_s(t) = \sum_{j\le\mu} A_j p^{\mu-j}(t+s)^j.$$

Then $P_s(-s-\ell p) = p^\mu A(-\ell)$ and $pP_s'(-s-\ell p) = p^\mu A'(-\ell)$. The polynomial P_s is a root-shifted form of $\Delta(t+s)^\mu$: it is congruent to it modulo p.

6.2. **The basis.** Put $\Pi = \prod_{s\in W} P_s \prod_{r\notin W}(t+r)^{\mu'_r}$ and take

$$e_{r,i} = \frac{(t+r)^i\Pi}{(t+r)^{\mu'_r}}\ (r \notin W,\ i < \mu'_r), \qquad e_{s,i} = \frac{(t+s)^i\Pi}{P_s}\ (s \in W,\ i < \mu),$$

and $g_j = t^j\Pi$ for $j < x_2$. The count is $\sum_r \mu_r + |Z| + |\mathcal{N}| + x_2 = 3n$, by Lemma 5.7. Since $P_s \equiv \Delta(t+s)^\mu$ modulo p, this is a basis of $\mathcal{O}[t]_{<3n}$ with unimodular change of basis, as in Lemma 5.1. The potentials are $\pi(e_{r,i}) = i - 2 + \sigma(r)/2$ and $\pi(g_j) = \lambda/2 + 1$. By Lemma 5.7 and (6.1),

$$\begin{aligned} 2\sum \pi &= \sum_{3n \text{ smallest}} (\text{increments}) + |Z|\lambda + |\mathcal{N}|(\lambda+1) + x_2(\lambda+2) - d\lambda \\ &= \mathrm{E}(n,p) + \nu_p + x_1 + x_2, \end{aligned}$$

using $d = |Z| + |\mathcal{N}| + x_2$. So it suffices to show $v(L_T(f_x f_y)) \ge \pi_x + \pi_y$ for every pair, with the half-integral refinement of Lemma 5.2 between the two parity groups, and corank 0.

6.3. **Node jets.** For $F \in \mathcal{O}[t]$ and a node k put

$$\nu_k(F) = \min\big(vF(-k),\ v(pF'(-k))\big).$$

The map $F \mapsto (F(-k), pF'(-k))$ is multiplicative for the product

$$(f_0, f_1)(g_0, g_1) = (f_0 g_0,\ f_0 g_1 + f_1 g_0),$$

so $\nu_k(FG) \ge \nu_k(F) + \nu_k(G)$. Also $\nu_k(t+r) \ge [r \equiv k]$, and $\nu_k(P_s) \ge \mu[s \equiv k]$.

Lemma 6.3 (harmonic digits)**.** *Let $m = cp + e$ with $0 \le e < p$ and $c < p$. Then $H_m - p^{-1}H_c - H_e \in p\mathbb{Z}_{(p)}$. For $k = \ell p + s \le 6n$,*

$$d_k + p^{-1}\delta_{s,\ell} + \lambda_0(s) \in p\mathbb{Z}_{(p)}, \qquad \lambda_0(s) = 3H_{r_2+s-\varepsilon p} + 2H_{r_6-s+\beta p} - 5H_s,$$

with $(\beta, \varepsilon) = ([s > r_6], [s \ge p - r_2])$.

Proof. The terms $i = jp$ of H_m give $p^{-1}H_c$. Every other term satisfies $1/(jp+i) - 1/i \in p\mathbb{Z}_{(p)}$, which leaves $cH_{p-1} + H_e$, and $H_{p-1} \in p\mathbb{Z}_{(p)}$. Apply this to k, $2n + k$ and $6n - k$, whose digits are (ℓ, s), $(q_2 + \ell + \varepsilon, r_2 + s - \varepsilon p)$ and $(q_6 - \ell - \beta, r_6 - s + \beta p)$. □

By the same digit expansion of the factorials, $c_k = p^\sigma \kappa_\ell \mathrm{cl}_k$ modulo $p^{\sigma+2}$ with $\mathrm{cl}_k \equiv \bar{c}_s(1+p\ell\lambda_0(s))$ modulo p^2, where $\bar{c}_s$ is an integer lift of $\hat{c}_s(1+pe_0(s))$ and $e_0(s) = -2(q_6-\beta)H_{r_6-s+\beta p} + 3(q_2+\varepsilon)H_{r_2+s-\varepsilon p}$.

Lemma 6.4 (node jets)**.** *Let $k = \ell p + s$ with $\ell \geq 1$, $s \in W$, and $F \in \mathcal{O}[t]$ with $\nu_k(F) \geq e$. Put $f_0 = F(-k)$ and $f_1 = pF'(-k)$. Then the node term of L_T at k satisfies*

$$N_k^T(F) \equiv b\,p^{\sigma-4}\mathrm{cl}_k\big[T_\ell^u f_1 + T_\ell^v f_0\big] - b\,p^{\sigma-3}\kappa_\ell \mathrm{cl}_k \lambda_0(s) u_\ell f_0 + a\,p^{\sigma-3}\kappa_\ell \mathrm{cl}_k\big[u_\ell^a f_1 + v_\ell^a f_0\big]$$

modulo $p^{\sigma-2+e}$, with $u_\ell^a = -3H_\ell(2)$ and $v_\ell^a = -3(\delta_{s,\ell}H_\ell(2) + 2H_\ell(3))$. At the other nodes with $\ell \geq 1$, $v(N_k(F)) \geq \sigma(s) - 4 + \nu_k(F)$. At the nodes $k = s < p$, $v(N_s(F)) \geq \sigma(s) - 1 + \nu_s(F)$.

Proof. $N_k(F) = c_k(U_k p^{-1} f_1 + V_k f_0)$. By Lemma 6.3, $U_k = p^{-3}(bu_\ell + pau_\ell^a) + O(1)$ and $V_k = p^{-4}(bv_\ell + pav_\ell^a) - p^{-3}b\lambda_0 u_\ell + O(p^{-2})$. Add the modification of L_T. The b-terms then carry the factor $c_k/p^\sigma - \kappa_\ell \mathrm{cl}_k \equiv 0$ modulo p^2. For $\ell = 0$, $v(U_s) \geq 0$ and $v(V_s) \geq -1$. □

6.4. **Pairs at the residues of W.** For an element X of the basis and a residue s let $\mathrm{ord}_s(X)$ be i if $X = e_{s,i}$ and μ'_s otherwise, and put $\delta_X(s) = \pi_X - \mathrm{ord}_s(X) - (\sigma(s)-4)/2$. By Lemma 6.4 a pair (X, Y) meets its bound at all nodes of residue s if $\delta_X(s) + \delta_Y(s) \leq 0$ (same parity group) or $\leq -\frac{1}{2}$ (different groups). By Lemma 5.7, $\delta_X(s)$ is $\pi_X - \lambda/2 - 1$ at $s \in Z$, $\pi_X - \lambda/2 - \frac{3}{2}$ at $s \in \mathcal{N}$, and $\pi_X - \lambda/2$ at $s \in W$ when X is not an element of s. The largest potentials are $\lambda/2 + 1$ for the g_j, $\lambda/2 + \frac{1}{2}$ for the top elements of $\mathcal{N}$, $\lambda/2$ for the top elements of Z, and $\lambda/2 - \frac{1}{2}$ for the second elements of $\mathcal{N}$. All others are at most $\lambda/2 - 1$. A direct check gives:

- at the residues of Z and $\mathcal{N}$ every pair meets its bound;
- at the residues of W the only pairs that do not are (a) an element of s with g_j or with a top element of $\mathcal{N}$, with deficit 1; (b) g_j with a top element of Z or a second element of $\mathcal{N}$, and two top elements of Z and $\mathcal{N}$ or of $\mathcal{N}$, with deficit 1; (c) g_j with $g_{j'}$ or with a top element of $\mathcal{N}$, with deficit 2.

The nodes $k = s < p$ have room 3, so only the nodes with $\ell \geq 1$ matter.

Lemma 6.5 (own pairs)**.** *Let $s \in W$, $i < \mu$, and let Y be a basis element with the factor P_s. Then $\sum_\ell N^T_{s+\ell p}(e_{s,i}Y) \equiv 0$ modulo $p^{\sigma-3+\mu+i}$.*

Proof. Write $e_{s,i}Y = (t+s)^i P_s R$. The jet at the node ℓ is

$$p^{\mu+i}\big((y^iA)(-\ell),\ (y^iA)'(-\ell)\big)R(-s)$$

plus higher terms. By Lemma 6.4 the sum is $b\,p^{\sigma-4+\mu+i}\bar{c}_s R(-s)\mathcal{L}_T(y^iA)$ modulo $p^{\sigma-3+\mu+i}$, and $\mathcal{L}_T(y^iA) = 0$. □

For the pairs (b) and (c), write $XY = P_s^2 R_s$ for $s \in W$. Then $XY = \Pi^2 \phi$ for a rational function ϕ, and we put $\psi(x) = \phi(-x)$ and $G = \hat{Q}\psi$, with $\hat{Q}(x) = \prod_{r \notin W}(x-r)^{m_r}$, where $m_r = 3$ on $\mathcal{N}$ and 2 on Z. Then G is a polynomial, of degree at most $3|\mathcal{N}| + 2|Z| + 2x_2 - 2 = p - 4$ by (6.1). It vanishes at every residue outside W, to order at least 2 for the pairs (c).

Lemma 6.6 (unit parts on W)**.** *For $s \in W$ put $f(s) = \hat{c}_s \prod_{r \neq s}(r-s)^{2\mu'_r}$. There is a unit K with $f(s) \equiv K\hat{Q}(s)\big(1 + p\,\Xi(s)\big)$ modulo p^2, where $\Xi(s) = (\lambda - 1)H_s - 2\beta_W H_{r_6+p-s} + 3(1-\varepsilon_W)H_{r_2+s}$.*

Proof. Put $\beta = \beta_W$, $\varepsilon = \varepsilon_W$ and $P_c(s) = \prod_{c \le r < p,\, r \neq s}(s-r)$. By Lemma 5.7, $2\mu'_r + \sigma(r) = \lambda + 4 + m_r$ for every residue r, with $m_r = 0$ on W. Since $\sigma(r) = 2[r > r_6] + 3[r \ge p - r_2]$, this gives the exact identity

$$f(s)\,P_{r_6+1}(s)^2 P_{p-r_2}(s)^3 = \hat{c}_s\,\hat{Q}(s)\,P_0(s)^{\lambda+4} \qquad (s \in W).$$

Every factor is now a product of factorials of numbers in $[0, p)$. We use $m!(p-1-m)! \equiv (-1)^m (p-1)!(1 + pH_m)$ modulo p^2, which follows from $\prod_{i=1}^{p-1-m}(p-i) \equiv (-1)^{p-1-m}(p-1-m)!(1 - pH_{p-1-m})$ and $H_{p-1-m} \equiv H_m$. Write $u = (p-1)!$ and $\hat{c}_s = \big(r_6!/(s!\,(r_6+\beta p - s)!)\big)^2 \big((r_2 + s - \varepsilon p)!/(s!\,r_2!)\big)^3$.

First, $P_0(s) = (-1)^{p-1-s} s!(p-1-s)! \equiv u(1 + pH_s)$. Next, if $\beta = 0$ then $P_{r_6+1}(s) = \pm(p-1-s)!/(r_6 - s)!$ with a sign independent of s. If $\beta = 1$ then $P_{r_6+1}(s) = \pm(s - r_6 - 1)!\,(p-1-s)!$, and the reflection with $m = r_6 + p - s$ gives $(r_6 + p - s)!\,(s - r_6 - 1)! \equiv \pm u(1 + pH_{r_6+p-s})$. In both cases

$$\Big(\frac{r_6!}{s!\,(r_6+\beta p - s)!}\Big)^2 P_{r_6+1}(s)^{-2} \equiv \text{const} \cdot \big(1 - 2\beta p H_{r_6+p-s}\big)\,\big(s!(p-1-s)!\big)^{-2}.$$

In the same way, if $\varepsilon = 1$ then $P_{p-r_2}(s) = (s - p + r_2)!\,(-1)^s (p-1-s)!$, and if $\varepsilon = 0$ the reflection with $m = r_2 + s$ applies, so that

$$\Big(\frac{(r_2 + s - \varepsilon p)!}{s!\,r_2!}\Big)^3 P_{p-r_2}(s)^{-3} \equiv \text{const} \cdot (-1)^s \big(1 + 3(1-\varepsilon) p H_{r_2+s}\big)\,\big(s!(p-1-s)!\big)^{-3}.$$

Finally $(s!(p-1-s)!)^{-5} \equiv (-1)^s u^{-5}(1 - 5pH_s)$, and the two signs $(-1)^s$ cancel. Multiplying, $f(s)/\hat{Q}(s) \equiv K\big(1 + p\,\Xi(s)\big)$ with $\Xi = (\lambda + 4 - 5)H_s - 2\beta H_{r_6+p-s} + 3(1-\varepsilon)H_{r_2+s}$ and K a unit independent of s. □

We need two facts about sums over a complete residue system. First, for $G \in \mathbb{Z}_{(p)}[x]$ of degree at most $p-4$,

$$\sum_{s<p} G(s) \equiv p \int G \pmod{p^2}, \qquad \int G := \sum_e g_e B_e, \tag{6.2}$$

where $G = \sum_e g_e x^e$ and B_e are the Bernoulli numbers with $B_1 = -\frac{1}{2}$. This is Faulhaber's formula. Second, $\sum_{m<p} m^e H_m \equiv B_e \pmod{p}$ for $e \le p - 4$.

Hence, if G vanishes to order 2 at every residue outside W and $x(s)$ is $s+\alpha$ or $c-s$ with $x(W)\subseteq[0,p)$, then

$$\sum_{s\in W} G(s)H_{x(s)} \equiv \int G \pmod p. \tag{6.3}$$

For $x=c-s$ this follows from $H_{p-1-m}\equiv H_m$. For $x=s+\alpha$, the residues $s+\alpha$ run over a complete system as s runs over W and its complement, the terms off W vanish, and the shift $\int G(x+1)=\int G+G'(0)$ changes $\int G$ only by values of G' at residues outside W, which vanish.

Lemma 6.7 (window sums)**.** *Let (X,Y) be a pair of type (b) or (c) at the residues of W. Then $v\big(\sum_{s\in W}\sum_\ell N^T_{s+\ell p}(XY)\big)\ge \sigma+2\mu-4+\theta$, where $\theta=1$ for type (b) and $\theta=2$ for type (c).*

Proof. For $\ell\ge 1$ the jets of $F=P_s^2R_s$ are $f_0=p^{2\mu}A(-\ell)^2R_s(-k)$ and $f_1=p^{2\mu}\big(2AA'(-\ell)R_s(-k)+pA(-\ell)^2R_s'(-k)\big)$. Insert these into Lemma 6.4, together with $R_s(-k)\equiv R_s(-s)-\ell pR_s'(-s)$ and $\mathrm{cl}_k\equiv\bar c_s(1+p\ell\lambda_0)$. Summing over ℓ, with $(yA^2)'=A^2+2yAA'$, gives modulo $p^{\sigma+2\mu-2}$

$$p^{\sigma+2\mu-4}\bar c_s\Big\{b\big[R_s(-s)\mathcal{L}_T(A^2)+p(R_s'(-s)-\lambda_0R_s(-s))\mathcal{L}_T(yA^2) \\ +p\lambda_0R_s(-s)\mathcal{L}^u_{T-\tau}(A^2)\big]+paR_s(-s)\mathcal{L}^a(A^2)\Big\},$$

with three local functionals independent of s. By Lemma 6.6 and $P_r(-s)\equiv\Delta(r-s)^\mu$,

$$\bar c_sR_s(-s)\equiv KG(s)\{\Delta^{2w-2}(1+p(\Xi+e_0)(s)) \\ +2p\Delta^{2w-3}A_{\mu-1}\sigma_W(s)\}\pmod{p^2},$$

with $\sigma_W(s)=\sum_{r\in W,r\ne s}1/(r-s)=H_{w_1-s}-H_{s-w_0}$ for $W=[w_0,w_1]$, and $\bar c_sR_s'(-s)$ is $K\Delta^{2w-2}$ times a polynomial that vanishes off W, modulo p.

For type (b) it suffices to work modulo $p^{\sigma+2\mu-3}$, where the sum is a unit times $\sum_{s\in W}G(s)$. Since G vanishes at the residues outside W, this is $\sum_{s\in\mathbb{F}_p}G(s)=0$ in $\mathbb{F}_p$.

For type (c) the sums $\sum_W G$, $\sum_W G\sigma_W$, $\sum_W G\lambda_0$ and the sum of the derivative terms must be divisible by p, and $\sum_W G(1+p(\Xi+e_0))$ by p^2. The first and the derivative sum vanish as before. By (6.3), $\sum_W G\sigma_W\equiv(1-1)\int G$ and $\sum_W G\lambda_0\equiv(3+2-5)\int G$, both 0. By (6.2) and (6.3),

$$\sum_{s\in W}G(s)(1+p(\Xi+e_0)(s))\equiv p(1+\omega)\int G\pmod{p^2},$$

where ω is the total weight of the harmonic numbers in $\Xi+e_0$: $\omega=(\lambda-1)-2\beta_W+3(1-\varepsilon_W)-2(q_6-\beta_W)+3(q_2+\varepsilon_W)=\lambda+2-2q_6+3q_2$. By (6.1), $1+\omega=0$. □

Proof of Theorem 6.1. Every entry $L_T(XY)$ is a sum of node terms. By Lemma 6.4 and the check above, every node term meets $\pi_X+\pi_Y$, except the nodes of W for the pairs (a), (b) and (c), whose sums over W meet it by

Lemmas 6.5 and 6.7. Lemma 5.2, applied with corank 0, gives $v(\det) \geq 2\sum \pi$, and Lemma 5.1 gives $v_p(J_n(T)) \geq \mathrm{E}(n,p) + x_1 + x_2$. □

7. The prime sum

We now sum the local bounds. For $n \geq 10$ and a prime p put

$$w(n,p) = \begin{cases} 0, & p > 6n, \\ w_7(n,p), & 7n < 2p \leq 12n, \\ -6(p-2n), & 3n < p,\ 2p \leq 7n, \\ \max(\mathrm{E}(n,p) + x_1 + x_2,\ -12n\lfloor \log_p 8n \rfloor), & \sqrt{8n} < p < 3n, \\ -12n\lfloor \log_p 8n \rfloor, & \text{otherwise}, \end{cases}$$

and $m_n = \prod_{p \leq 6n} p^{-w(n,p)}$. By Proposition 5.4 and Theorems 5.5, 5.6 and 6.1, $v_p(m_n J_n) \geq 0$ for every prime p. So $m_n J_n(a,b,X)$ has integer coefficients.

7.1. **The law on 22 pieces.** For $\sqrt{8n} < p < 3n$ write $q = q_6 = \lfloor 6n/p \rfloor \geq 2$ and $t = \{6n/p\}$. The class sizes are affine in p and n with bounded error, and so are d, x_1 and x_2. The next proposition is the result of this bookkeeping.

Proposition 7.1. *Let $n \geq 10$ and $\sqrt{8n} < p < 3n$. Let $(r, [t_0, t_1])$ be the row of Table 2 with $r = q \bmod 6$ and $t \in [t_0, t_1)$. Then*

$$\mathrm{E}(n,p) + x_1 + x_2 \geq p\, c_0(q) + n\, c_1(q) - 1000(q^2+1).$$

Proof. Write $r_3 = q \bmod 3$, so that $\{2n/p\} = (r_3 + t)/3$, and $\theta = \{3n/p\} = ((q \bmod 2) + t)/2$. Up to an error of at most 2 each, the class sizes are p times the masses

$$m_A = t,\ m_B = 0,\ m_C = 1 - \tfrac{r_3+t}{3} - t,\ m_D = \tfrac{r_3+t}{3} \quad \text{if } t < \tfrac{3-r_3}{4},$$

$$m_A = 1 - \tfrac{r_3+t}{3},\ m_B = t - m_A,\ m_C = 0,\ m_D = 1 - t \quad \text{otherwise.}$$

Put $\lambda_c = q + r_3 - 3$. Since $q_2 = (q - r_3)/3$, (5.1) reads $2d + |\mathcal{N}| + 2 = kp$ with $k = \lambda_c + 1 - \lambda \geq 1$, and $k \leq 2$ because $d \leq |Z| + |W|$. So $\lambda = \lambda_c$, or $\lambda = \lambda_c - 1$ and then $|\mathcal{N}| \leq 2$. Let $h(\lambda', \sigma) = j^2 + (\sigma - 5)j$ with $j = \lceil (\lambda' + 4 - \sigma)/2 \rceil$. This is the sum of the increments below λ' at a residue of class σ. Let N_c be the number of residues in the two classes with $\sigma \not\equiv \lambda_c \pmod 2$. If $\lambda = \lambda_c$, the sum of the $3n$ smallest increments is $\sum_c |c|\, h(\lambda_c, \sigma_c) + d\lambda_c$ with $2d = p - N_c - 2$. If $\lambda = \lambda_c - 1$, then $h(\lambda_c - 1, \sigma) = h(\lambda_c, \sigma) - (\lambda_c - 1)[\sigma \not\equiv \lambda_c]$ and $N_c = p - |\mathcal{N}|$, and the sum is $\sum_c |c|\, h(\lambda_c, \sigma_c) + \frac{1}{2}(p - N_c - 2)(\lambda_c - 1)$. In both cases it equals $\sum_c |c|\, h(\lambda_c, \sigma_c) + \frac{1}{2}(p - N_c - 2)\lambda_c + O(q)$, and $|h| \leq 12q^2$. The tie classes for λ_c are (A, C) if $\lfloor q/3 \rfloor$ is odd and (B, D) if it is even. Write $m_Z \leq m_W$ for their masses and $m_{\mathcal{N}}$ for the mass of the other two. If $\lambda = \lambda_c$, then $2d + 2 = |Z| + |W|$, so $x_1 = (|W| - |Z|)/2 - 1$ and $x_2 = (x_1 - |\mathcal{N}|)_+$. If $\lambda = \lambda_c - 1$, the classes of the parity of λ_c have at most two residues, so m_Z and m_W are $O(1/p)$. In both cases

$$x_1 + x_2 \geq p\Big(\frac{m_W - m_Z}{2} + \Big(\frac{m_W - m_Z}{2} - m_{\mathcal{N}}\Big)_+\Big) - 5.$$

r	t	$c_0(q)$	$c_1(q)$
0	$[0,\frac{3}{4}]$	$2q+\frac{q^2}{2}$	$-18-3q$
0	$[\frac{3}{4},\frac{6}{7}]$	$-2-\frac{2}{3}q+\frac{q^2}{2}$	$-2-3q$
0	$[\frac{6}{7},1]$	$-4-3q+\frac{q^2}{2}$	$12-3q$
1	$[0,\frac{1}{2}]$	$-\frac{1}{2}+2q+\frac{q^2}{2}$	$-15-3q$
1	$[\frac{1}{2},\frac{5}{7}]$	$-\frac{11}{6}-\frac{2}{3}q+\frac{q^2}{2}$	$1-3q$
1	$[\frac{5}{7},1]$	$-\frac{7}{2}-3q+\frac{q^2}{2}$	$15-3q$
2	$[0,\frac{1}{4}]$	$\frac{5}{2}q+\frac{q^2}{2}$	$-15-3q$
2	$[\frac{1}{4},\frac{2}{5}]$	$-\frac{1}{3}+\frac{7}{6}q+\frac{q^2}{2}$	$-7-3q$
2	$[\frac{2}{5},\frac{4}{7}]$	$-\frac{2}{3}+\frac{1}{3}q+\frac{q^2}{2}$	$-2-3q$
2	$[\frac{4}{7},\frac{2}{3}]$	$-2-2q+\frac{q^2}{2}$	$12-3q$
2	$[\frac{2}{3},1]$	$-3-\frac{7}{2}q+\frac{q^2}{2}$	$21-3q$
3	$[0,\frac{1}{3}]$	$1+\frac{9}{2}q+\frac{q^2}{2}$	$-33-3q$
3	$[\frac{1}{3},\frac{3}{7}]$	$\frac{1}{2}+3q+\frac{q^2}{2}$	$-24-3q$
3	$[\frac{3}{7},\frac{3}{5}]$	$-\frac{1}{2}+\frac{2}{3}q+\frac{q^2}{2}$	$-10-3q$
3	$[\frac{3}{5},\frac{3}{4}]$	$-1-\frac{1}{6}q+\frac{q^2}{2}$	$-5-3q$
3	$[\frac{3}{4},1]$	$-2-\frac{3}{2}q+\frac{q^2}{2}$	$3-3q$
4	$[0,\frac{2}{7}]$	$4q+\frac{q^2}{2}$	$-27-3q$
4	$[\frac{2}{7},\frac{1}{2}]$	$-\frac{2}{3}+\frac{5}{3}q+\frac{q^2}{2}$	$-13-3q$
4	$[\frac{1}{2},1]$	$-2-q+\frac{q^2}{2}$	$3-3q$
5	$[0,\frac{1}{7}]$	$-\frac{1}{2}+4q+\frac{q^2}{2}$	$-24-3q$
5	$[\frac{1}{7},\frac{1}{4}]$	$-\frac{5}{6}+\frac{5}{3}q+\frac{q^2}{2}$	$-10-3q$
5	$[\frac{1}{4},1]$	$-\frac{3}{2}-q+\frac{q^2}{2}$	$6-3q$

TABLE 2. The law on 22 pieces: $r=q \bmod 6$, the range of $t=\{6n/p\}$, and the coefficients of $p\,c_0(q)+n\,c_1(q)$.

Also $\nu_p = 2\sum_{i<3n}\lfloor i/p\rfloor$ since $p^2>3n$, and this is exactly $p(9x^2-3x+\theta(1-\theta))$ with $x=n/p$. On each piece the result is $p\,c_0(q)+n\,c_1(q)$: the terms in t^2 cancel between ν_p and the increments, and with $t=6x-q$ the rest is affine in x. The errors are $O(q^2)$, with constant at most 1000. □

Remark 7.2. The formal proof carries out the case analysis of Proposition 7.1 piece by piece. The constant 1000 is generous and plays no role, because $\sum_{p>n/M} q^2\log p = O(M^2 n)$.

7.2. **The integrals.** Put $y=p/n$. On a piece, $p\,c_0(q)+n\,c_1(q) = n\big(c_0(q)y+c_1(q)\big)$, and the piece is the interval $6/(q+t_1) < y \le 6/(q+t_0)$. The weighted prime number theorem $\sum_{\alpha n<p\le\beta n}\log p = (\beta-\alpha+o(1))n$ gives, for each

piece κ with $q \geq 2$ fixed,

$$\sum_{p\in\kappa}(-w(n,p))\log p \leq (I_\kappa + o(1))n^2, \qquad I_\kappa = -\int_\kappa (c_0(q)y + c_1(q))\,dy.$$

Lemma 7.3. *(i) The pieces with $2 \leq q \leq 5$, which cover $n < p < 3n$, have $\sum I_\kappa = 2263/280$.*

(ii) For $3n < p \leq 7n/2$ the bound of Theorem 5.5 contributes $15/4$ per n^2, and for $7n/2 < p \leq 6n$ the bound of Theorem 5.6 contributes $47/4$.

(iii) For $m \geq 1$, the pieces with $6m \leq q < 6m+6$, which cover $n/(m+1) < p \leq n/m$, have $\sum I_\kappa = f(m) := -\sum_e d_e/(m+e)$, with the 23 nodes e and weights d_e of Table 3.

e	0	$\frac{1}{8}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{4}{9}$	$\frac{1}{2}$
d_e	-12	8	7	-12	8	7	$-\frac{27}{2}$	4	$\frac{5}{2}$	7	$\frac{9}{2}$	-30
e	$\frac{5}{9}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	1	
d_e	$\frac{9}{2}$	7	$\frac{5}{2}$	4	$-\frac{27}{2}$	7	8	-12	7	8	-3	

TABLE 3. The nodes e and weights d_e. One has $\sum_e d_e = 0$ and $\sum_e d_e e = 9/2$.

Proof. (i) and (iii) are exact integrations of linear functions over the pieces. In (iii), on the piece with endpoints $x_e = m + e$ in the variable $x = 1/y$, the substitution $q = 6x_e - t_e$ turns the endpoint terms of $\int (c_0/x^3 + c_1/x^2)\,dx$ into a constant, which cancels inside the piece, plus $d/x_e + f/x_e^2$. Summing over the six residues r, the coefficients of $1/x_e^2$ cancel and those of $1/x_e$ are the d_e. (ii) is the integral of $6(y-2)$ over $(3, 7/2]$ and of $2+2y$, $18-2y$ and $54-10y$ over $(7/2, 4]$, $(4, 9/2]$ and $(9/2, 6]$. □

Thus the exact part of the content is $2263/280 + 15/4 + 47/4 = 6603/280$. It remains to bound the partial sums of f.

Lemma 7.4. *Put $S_M = \sum_{1\leq m<M} f(m)$ and $R_C = 3573565641/700000000$. Then $S_M \leq R_C$ for every $M \geq 1$.*

Proof. Positivity. From $e/(m+e) = e/m - e^2/(m(m+e))$ and $\sum d_e = 0$, $\sum d_e e = 9/2$,

$$f(m) = \frac{9/2}{m^2} - \frac{1}{m^2}\sum_e \frac{d_e e^2}{m+e} \geq \frac{9}{2m^2} - \frac{\Sigma_2}{m^3}, \qquad \Sigma_2 = \sum_e |d_e| e^2 = \frac{20383}{360}.$$

So $f(m) > 0$ for $m \geq 13$, and an exact computation gives $f(m) > 0$ for $m \leq 12$. Hence S_M increases with M.

The tail. For $K = 6$, $1/(m+e) = \sum_{j<K}(-e)^j/m^{j+1} + (-e)^K/(m^K(m+e))$, so $f(m) = \sum_{k=2}^{6} a_k m^{-k} + r(m)$ with $a_k = (-1)^k \sum_e d_e e^{k-1}$ and $|r(m)| \leq$

$\Sigma_6 m^{-7}$, $\Sigma_6=\sum_e |d_e|e^6$. Here $a_2,a_4,a_6>0$ and $a_3,a_5<0$. For $k\ge 2$ and $M\ge 1$, convexity gives

$$\frac{1}{(k-1)M^{k-1}}+\frac{1}{2M^k}\le\sum_{m\ge M} m^{-k}\le\frac{1}{(k-1)(M-\frac12)^{k-1}}.$$

Using the upper bound for positive coefficients and for Σ_6, and the lower bound for negative coefficients, we get $\sum_{m\ge1000} f(m)\le T$ for an explicit rational $T=0.0045006\ldots$.

The partial sum. S_{1000} is a sum of 22977 rationals. Rounding each term $1/(m+e)$ outward to a multiple of 2^{-80} gives an upper bound $\hat S$ with $\hat S-S_{1000}<2\cdot10^{-19}$, and $\hat S+T\le R_C$ with a margin of $5.76\cdot10^{-9}$. Since $f>0$, $S_M\le S_{1000}+T\le R_C$ for every M. □

Theorem 7.5. *For every $\varepsilon>0$ and all large n, m_n is a positive rational number, $m_nJ_n(a,b,X)\in\mathbb{Z}[a,b,X]$, and*

$$\log m_n\le(\Lambda+\varepsilon)n^2,\qquad \Lambda=\frac{2868723663}{10^8}=28.68723663.$$

Proof. Fix $M\ge 48/\varepsilon$. Split the primes $p\le 6n$ into $p\le n/M$, $n/M<p<3n$ and $3n<p\le 6n$. For $p\le n/M$, Proposition 5.4 gives $\sum_{p\le n/M}12n\lfloor\log_p 8n\rfloor\log p\le(12/M+o(1))n^2$. For $n/M<p<3n$ we use Proposition 7.1 on each piece. The pieces with $q<6M$ are finite in number, and the error terms $1000(q^2+1)$ contribute $O(M^2n\log n)$. On $\sqrt{8n}<p\le n/M$ the crude bound is used, so the pieces with $q\ge6M$ are not needed. Lemma 7.3 gives

$$\log m_n=\sum_{p\le6n}(-w(n,p))\log p\le\Big(\frac{6603}{280}+S_M+\frac{12}{M}+\frac{\varepsilon}{2}\Big)n^2$$

for large n. By Lemma 7.4, $6603/280+S_M\le 6603/280+R_C=2868723663/10^8$. □

The rational bound of Theorem 7.5 rounds up an exact constant.

Theorem 7.6. *The law on $(0,1]$ has the integral $\sum_{m\ge1}f(m)=\sum_e d_e\psi(1+e)=5.10509376672\ldots$, and*

$$\frac{6603}{280}+\sum_e d_e\psi(1+e)=95-36\log2+\tfrac{153}{2}\log3-\tfrac{25}{4}\log5-49\log7$$
$$-8\sqrt2\log(1+\sqrt2)+\tfrac{5\sqrt5}{2}\log\phi+\tfrac92\Xi,$$

where $\phi=(1+\sqrt5)/2$ and $\Xi=\psi(4/9)+\psi(5/9)+2\gamma$. The value is $28.6872366238631\ldots$.

Proof. Since $\sum_e d_e=0$, $\sum_{m\ge1}f(m)=\sum_e d_e\sum_{m\ge1}(1/m-1/(m+e))=\sum_e d_e(\psi(1+e)+\gamma)=\sum_e d_e\psi(1+e)$. With $\psi(1+e)=\psi(e)+1/e$ the rational part is $\sum_{e>0}d_e/e=19997/280$. Gauss's digamma theorem evaluates the symmetric pairs: $\psi(\frac12)=-\gamma-2\log2$, $\psi(\frac13)+\psi(\frac23)=-2\gamma-3\log3$, $\psi(\frac14)+\psi(\frac34)=-2\gamma-6\log2$, $\psi(\frac16)+\psi(\frac56)=-2\gamma-4\log2-3\log3$, $\sum_{k=1}^6\psi(\frac k7)=$

$-6\gamma - 7\log 7$, $\psi(\frac{1}{8}) + \psi(\frac{7}{8}) = -2\gamma - 8\log 2 - 2\sqrt{2}\log(1+\sqrt{2})$, $\psi(\frac{3}{8}) + \psi(\frac{5}{8}) = -2\gamma - 8\log 2 + 2\sqrt{2}\log(1+\sqrt{2})$ and $\psi(\frac{2}{5}) + \psi(\frac{3}{5}) = -2\gamma - \frac{5}{2}\log 5 + \sqrt{5}\log\phi$. The constant γ cancels because $\sum d_e = 0$. □

Remark 7.7. Theorem 7.5 is the form proved in Lean. Its constant 28.68723663 is the closed form of Theorem 7.6 rounded up by less than 10^{-8}. The formal proof checks only the rational partial sums of Lemma 7.4, and it does not use Theorem 7.6. The paper states the closed form because it shows where the constant comes from. Theorem 7.6 is not part of the formal proof.

8. Nonvanishing at prime-tied sizes

Let ℓ be a prime. We show that $J_n(a, b, X) \neq 0$ for three kinds of sizes n tied to ℓ, at all points with ℓ-adically bounded coordinates. In all cases the ℓ-adic valuation of a Gram determinant is computed exactly by Lemma 5.2 and its equality case. The first two kinds give one size for each prime. The third gives every size in an interval. We write v for v_ℓ and $\mathbb{Z}_{(\ell)}$ for the rationals with $v \geq 0$.

8.1. **Lai's primes.**

Theorem 8.1. *Let $m \geq 1$, $n = 2m$, and let $\ell = 9m + 1$ be prime. Let $a', b, X' \in \mathbb{Z}_{(\ell)}$ with $3a' + 2b \not\equiv 0 \pmod{\ell}$. Then $J_n(a'/\ell, b, X'/\ell^3) \neq 0$.*

For $a' = \ell a$ and $X' = \ell^3 c$ with integers a, b and c this is Lai's Proposition 3.1 [19], where b is an ℓ-adic unit. The new points are the coefficient a'/ℓ and the tolerance $v(X) \geq -3$.

Proof. Write $a = a'/\ell$, $X = X'/\ell^3$ and $\epsilon = 3a' + 2b$. Put $s = 3m = 6n - \ell + 1$, so that the nodes $k \in [\ell, 6n]$ are $k = \ell + j$ with $0 \leq j < s$. Every index in the harmonic sums is at most $8n < 2\ell$, and $3n < \ell$.

Step 1: local sizes. For $k < \ell$, $H_k(i) \in \mathbb{Z}_{(\ell)}$, and for $\ell \leq k < 2\ell$, $H_k(i) - \ell^{-i} \in \mathbb{Z}_{(\ell)}$. So $d_k \in \ell^{-1}\mathbb{Z}_{(\ell)}$ for every k. For $k < \ell$, $U_k \in \ell^{-1}\mathbb{Z}_{(\ell)}$ and $V_k \in \ell^{-3}\mathbb{Z}_{(\ell)}$. For $k = \ell + j$, $U_k \equiv -\ell^{-3}\epsilon \pmod{\ell^{-1}}$ and $V_k \in \ell^{-4}\mathbb{Z}_{(\ell)}$. By Kummer's theorem c_k is a unit for $\ell \leq k \leq 6n$, $\ell^2 \mid c_k$ for $s \leq k < \ell$, and $c_k \in \mathbb{Z}_{(\ell)}$ for $k < s$.

Step 2: Lai's lattice. For $i < s$ let $F_i(t) = \prod_{j<s, j\neq i}(t + j)/(j - i)$ and $\gamma_i = F_i'(-i)$, and put

$$P_i = \ell^2(1 - 2\gamma_i(t + i))F_i^2, \qquad Q_i = \ell(t + i)F_i^2.$$

They have degree at most $2s - 1 = 3n - 1$, and at $-j$ $(j < s)$ one has $P_i(-j) = \ell^2\delta_{ij}$, $P_i'(-j) = 0$, $Q_i(-j) = 0$ and $Q_i'(-j) = \ell\delta_{ij}$. The Hermite map $f \mapsto (f(-j), f'(-j))_{j<s}$ is an isomorphism of $\mathbb{Z}_{(\ell)}[t]_{<2s}$ onto $\mathbb{Z}_{(\ell)}^{2s}$, since its determinant is a power of a Vandermonde determinant in $0, \ldots, s - 1$. So the P_i, Q_i span a lattice of index $\ell^{3s} = \ell^{9n/2}$, and $v\det[B(t^i, t^j)] = v\det G - 9n$, where G is the Gram matrix of the P_i, Q_i.

Step 3: the Gram matrix modulo ℓ. At the nodes $k<s$, $Q_j(-k)=0$, so every contribution lies in $\ell\mathbb{Z}_{(\ell)}$. This exact vanishing is what allows $v(X)\geq -3$. At the nodes $s\leq k<\ell$, $\ell^2\mid c_k$ and every contribution lies in $\ell\mathbb{Z}_{(\ell)}$. At $k=\ell+j$, Taylor expansion from $-j$ gives $P_i(-k)\equiv \ell^2\delta_{ij}$, $Q_i(-k)\equiv -\ell^2\delta_{ij}$ and $Q_i'(-k)\equiv \ell\delta_{ij}$ to the relevant order. With $A_j=c_k\ell^4V_k$ and $\bar{c}_j$ the residue of c_k, the node contributes, modulo ℓ,

$$\begin{pmatrix} A_j & -\bar{c}_j\epsilon - A_j \\ -\bar{c}_j\epsilon - A_j & 2\bar{c}_j\epsilon + A_j \end{pmatrix}$$

to the block of (P_j,Q_j), and nothing else. Its determinant is $-\bar{c}_j^2\epsilon^2$, whatever A_j is. So $\det G\equiv \prod_{j<s}(-\bar{c}_j^2\epsilon^2)\not\equiv 0$. Hence $v(\prod i!^2J_n)=-9n$, and $\prod_{i<3n} i!^2$ is a unit. □

The proof gives $v_\ell(J_n(a'/\ell,b,X'/\ell^3))=-9n$.

Corollary 8.2. *Let $m\geq 1$, $n=2m$, and let $\ell=9m+1$ be prime.*

(a) Let $a,b\in\mathbb{Z}$ with $\ell\nmid b$. Then $J_n(a,b,p/q)\neq 0$ for all integers p and $q>0$ with $\ell^4\nmid q$.

(b) Let p,q be integers with $q\neq 0$ and $\ell\nmid p$. Then $f_n(p/q)\neq 0$.

Proof. (a) Take $a'=\ell a$ and $X'=\ell^3p/q$. Then $v(X')\geq 0$ and $3a'+2b\equiv 2b\not\equiv 0$. (b) By (2.4), $q^{3n}f_n(p/q)=J_n(q,-p,0)$. Take $a'=\ell q$, $b=-p$ and $X'=0$, so that $3a'+2b\equiv -2p\not\equiv 0$. □

8.2. **The lines $3\ell=7n+1$ and $3\ell=7n+2$.** Lai's primes have density $1/6$ among the primes, in the sense of Theorem 3.1, and the sizes tied to them grow with slope $9/2$. A second family of exact nonvanishing comes from primes ℓ with $2n<\ell<8n/3$.

Theorem 8.3. *Let $n\geq 7$ and let ℓ be a prime with $3\ell\in\{7n+1,7n+2\}$ and $\ell\notin\{73,1039\}$. Let $a\in\mathbb{Z}_{(\ell)}$, $b\in\mathbb{Z}_{(\ell)}^\times$ and $X'\in\mathbb{Z}_{(\ell)}$. Then $J_n(a,b,X'/\ell^3)\neq 0$.*

Proof. Put $\alpha_0=\ell-2n$ and $A_2=6n-2\ell$, and write $k=s+j\ell$ with $0\leq s<\ell$. Since $6n<3\ell$, $j\leq 2$. In the notation of Section 5 the residues fall into the classes $T_1=[0,\alpha_0)$ with $\sigma=0$ and $j\leq 2$, $T_2=[\alpha_0,A_2]$ with $\sigma=3$ and $j\leq 2$, and $T_4=(A_2,\ell)$ with $\sigma=5$ and $j\leq 1$. Put $\varkappa=3\ell-7n$. Then $|T_2|=n+1-\varkappa$ and $|T_4|=n-1+\varkappa$.

Step 1: node sizes. For $j\geq 1$, Lemma 5.3 gives $U_k=\ell^{-3}u_jb+O(\ell^{-2})$ and $V_k=\ell^{-4}E_jb+O(\ell^{-3})$, with $u_1=-2$, $u_2=-9/4$, and $E_j=-6,-105/8$ on T_1, $E_j=-3,-87/8$ on T_2 and $E_1=-7$ on T_4. The terms in a and X are one order smaller. For $j=0$, $v(U_k)\geq 0$ and $v(V_k)\geq -3$. The unit parts of $c_{s+j\ell}$ are $(1,4,1)\hat{c}_s$ on T_1 and $(1,32,27)\hat{c}_s$ on T_2.

Step 2: the basis. Put $\Pi=\prod_{r\in T_1}(t+r)^3\prod_{r\in T_2}(t+r)$, of degree $2n+1$, and $\Psi_r=\Pi/(t+r)^3$. The basis consists of $e_{r,i}=(t+r)^i\Psi_r$ for $r\in T_1$, $i<3$, with potential $i-2$; of $e_r=\Pi/(t+r)$ for $r\in T_2$, with potential $-\frac{1}{2}$; and of $g_m=t^m\Pi$ for $m<n-1$, with potential $\frac{1}{2}$. It has $3n$ elements and

$2\sum\pi = 5n - 3\ell - 2$. By Lemma 5.3 and Step 1 every entry has $v \geq \pi_x + \pi_y$, with $+\frac{1}{2}$ between the two groups. At the nodes with $j = 0$ every term lies strictly above its threshold: on T_1 only $e_{s,0}$ and $e_{s,1}$ do not vanish to order 3, on T_2 only e_s does not vanish, and on T_4 the weight has $v = 5$. So X never enters the reductions. In the proof of Lemma 5.2 the reduction is block triangular, so

$$v(\det G) = 2\sum\pi \iff \det \overline{N}_{AA} \cdot \det \overline{N}_{BB} \neq 0 \text{ in } \mathbb{F}_\ell.$$

The reductions use only the leading terms, so they are $\bar{b}$ times their values at $(a, b, X) = (0, 1, 0)$.

Step 3: $\overline{N}_{AA}$. The elements of different residues of T_1 meet only above the threshold. At the nodes $-r-j\ell$, $e_{r,i} = (-j\ell)^i \Psi_r(-r) + O(\ell^{i+1})$. Hence $\overline{N}_{AA}$ is block diagonal with blocks $\hat{c}_r \Psi_r(-r)^2 M_0$, where $M_0 = [h_{i+i'}]_{i,i'<3}$ and $h_m = \sum_{j=1,2} \binom{2}{j}^2 (-j)^{m-1}(mu_j - jE_j)$. One finds $h = (-\frac{297}{8}, 40, -\frac{103}{2}, 78, -130)$ and $\det M_0 = 1095/2 = 3 \cdot 5 \cdot 73/2$.

Step 4: $\overline{N}_{BB}$. The element e_r meets $e_{r'}$ only for $r = r'$, at the nodes of T_2, with value $-\frac{3117}{8}\hat{c}_r \Pi'(-r)^2$. The pairing of e_r with g_m is $\frac{1117}{2}\hat{c}_r \Pi'(-r)^2(-r)^m$. The pairing of g_m with $g_{m'}$ is $-\frac{1799}{2}\hat{c}_r \Pi'(-r)^2(-r)^{m+m'}$ at each residue of T_2 plus $-7\hat{c}_{s,1}\Pi(-s)^2(-s)^{m+m'}$ at each residue of T_4. By Anton's congruence [15, Theorem 1] and Wilson's theorem, $\hat{c}_s \Pi'(-s)^2 = -KD(s)$ on T_2 and $\hat{c}_{s,1}\Pi(-s)^2 = -32KD(s)$ on T_4, with $D(s) = \prod_{i<\alpha_0}(s-i)$ and a unit K. The diagonal block is invertible since $3117 = 3 \cdot 1039$, and its Schur complement is

$$S_{mm'} = -K\Big[c_2 \sum_{T_2} D(r)(-r)^{m+m'} - 224 \sum_{T_4} D(s)(-s)^{m+m'}\Big],$$

where

$$c_2 = -\frac{1799}{2} + \Big(\frac{1117}{2}\Big)^2 \frac{8}{3117} = -\frac{616727}{6234}.$$

The polynomial $D(s)s^{m+m'}$ vanishes on T_1 and has degree at most $\ell - 4$, so its sums over T_2 and T_4 are opposite. Hence $S = K\frac{779689}{6234}[\sum_{s\in T_4} D(s)(-s)^{m+m'}]$, with $779689 = 883^2$, and it remains to show

$$\mathcal{H} = \det\Big[\sum_{s\in T_4} D(s)s^{m+m'}\Big]_{m,m'<n-1} \neq 0.$$

Step 5: duality. For points x_s with weights ω_s $(s = 1, \ldots, N)$ and $\nu < N$, write $H_\nu(\omega) = \det[\sum_s \omega_s x_s^{i+j}]_{i,j<\nu}$. The Cauchy–Binet formula expresses it as a sum over subsets of size ν of $\prod\omega$ times a squared Vandermonde determinant. Passing to the complements gives

$$H_\nu(\omega) = \prod_s \omega_s \cdot V^2 \cdot H_{N-\nu}(\omega^*), \qquad \omega_s^* = \frac{1}{\omega_s \prod_{t\neq s}(x_s - x_t)^2},$$

with V the Vandermonde determinant of all the points. We apply this to T_4 with $\omega = D$, $\nu = n - 1$ and $N - \nu = \varkappa$. By Wilson's theorem,

$s!(\ell-1-s)!\equiv(-1)^{s+1}$, so $\prod_{t\in T_4,t\neq s}(s-t)^2\equiv\big((s-A_2-1)!/s!\big)^2$. Since $\alpha_0\le A_2+1$, the dual weight is the polynomial

$$\omega_s^*\equiv\Phi(s)=\prod_{i\le A_2}(s-i)\cdot\prod_{i=\alpha_0}^{A_2}(s-i),$$

which is monic of degree $\ell+2-2\varkappa$ and vanishes off T_4. So the dual entries are full power sums $\tau_m=\sum_{s\in\mathbb{F}_\ell}\Phi(s)s^m$, and $\sum_{s\in\mathbb{F}_\ell}s^j=-1$ if $(\ell-1)\mid j>0$ and 0 otherwise. For $\varkappa=1$, $H_1=\tau_0=-[s^{\ell-1}]\Phi$ is the sum of the roots of Φ, which is $(31n^2-7n-2)/18\equiv-1/49$, since $n\equiv-1/7$. For $\varkappa=2$, $\tau_0=0$, $\tau_1=-1$, and $H_2=\tau_0\tau_2-\tau_1^2=-1$.

Step 6. On both lines $\det\overline{N}_{AA}\det\overline{N}_{BB}\neq0$ unless ℓ divides $2\cdot3\cdot5\cdot7\cdot73\cdot883\cdot1039$. Since $n\ge7$ gives $\ell\ge17$, and $3\cdot883$ lies on neither line, this excludes only $\ell=73$ (with $n=31$) and $\ell=1039$ (with $n=445$). □

The proof gives $v_\ell(J_n(a,b,X'/\ell^3))=5n-3\ell-2-2(3n-\ell)=-n-\ell-2$.

Corollary 8.4. *Let n and ℓ be as in Theorem 8.3. If $a,b\in\mathbb{Z}$ with $\ell\nmid b$, then $J_n(a,b,p/q)\neq0$ whenever $\ell^4\nmid q$. If $\ell\nmid p$ and $q\neq0$, then $f_n(p/q)\neq0$.*

Proof. As for Corollary 8.2, with $(a,b,X')=(a,b,\ell^3p/q)$ and $(q,-p,0)$. □

8.3. **The band $8n/3<\ell<3n-1$.** The two families above give nonvanishing at one size n for each prime. In this subsection a prime ℓ is good for every size n with $8n/3<\ell<3n-1$, and for every direction. The proof shows that a reduced Gram matrix is nonsingular modulo ℓ.

Let n be a positive integer and let ℓ be a prime with $8n<3\ell$ and $\ell+1<3n$. Then $8n<3\ell<9n-3$, so $n\ge4$. No integer ℓ satisfies both inequalities for $n=4$, 5 or 6, so $n\ge7$ and $\ell>8n/3$ gives $\ell\ge19$. Put

$$\alpha_0=\ell-2n,\qquad A_2=6n-2\ell,\qquad \gamma=\frac{\alpha_0-1}{2},\qquad g=3n-1-\ell.$$

Then α_0 is odd, $6n=2\ell+A_2$, $0\le A_2<\alpha_0$, $A_2=2g+2$ and

$$\ell=3\alpha_0+A_2=6\gamma+2g+5. \tag{8.1}$$

In the notation of Section 5 we have $q_6=2$, $r_6=A_2$, $q_2=0$ and $r_2=2n$, so the class B is empty. The residues $s\in[0,\ell)$ form three intervals:

$$\begin{aligned}A&=[0,A_2] && (\sigma=0,\ k=s+j\ell,\ j\le2),\\ C&=(A_2,\alpha_0) && (\sigma=2,\ j\le1),\\ D&=[\alpha_0,\ell) && (\sigma=5,\ j\le1).\end{aligned}$$

We put $\mathrm{w}_{n,\ell}=-21n+7\ell-7$.

Theorem 8.5. *Let n be a positive integer and let ℓ be a prime with $8n/3<\ell<3n-1$ and $\ell\notin\{2,3,5,13,24359\}$. Let $a\in\mathbb{Z}_{(\ell)}$, $b\in\mathbb{Z}_{(\ell)}^\times$ and $X\in\mathbb{Q}$ with $\ell^3X\in\mathbb{Z}_{(\ell)}$. Then $J_n(a,b,X)\neq0$.*

Since the band forces $\ell \geq 19$, the exclusion of 2, 3, 5 and 13 is automatic. The prime 24359 lies in the band exactly for $8120 < n \leq 9134$. The proof gives more than nonvanishing. It shows that $v_\ell(J_n(a,b,X)) = \mathrm{w}_{n,\ell}$, and this value is the bound of Theorem 6.1 in this range. So the law is sharp at every such point. This exact valuation is not part of the formal proof, which uses only the nonvanishing.

Corollary 8.6. *Let n and ℓ be as in Theorem 8.5.*

(a) Let $a, b \in \mathbb{Z}$ with $\ell \nmid b$. Then $J_n(a,b,p/q) \neq 0$ for all integers p and $q > 0$ with $\ell^4 \nmid q$.

(b) Let p, q be integers with $q \neq 0$ and $\ell \nmid p$. Then $f_n(p/q) \neq 0$.

Proof. (a) Apply Theorem 8.5 with $X = p/q$. Since $\ell^4 \nmid q$, we have $\ell^3 X \in \mathbb{Z}_{(\ell)}$. (b) By (2.4), $q^{3n} f_n(p/q) = J_n(q, -p, 0)$, and Theorem 8.5 applies with $(a,b,X) = (q,-p,0)$. □

The proof has four parts: a basis and a reduction to one point, the integral blocks, the half-integral block and its Hankel determinant, and the assembly.

8.3.1. *The basis and the reduction to one point.* Let $\mu_r = 4$ for $r \in A$, $\mu_r = 3$ for $r \in C$, and $\Pi = \prod_{r<\alpha_0} (t+r)^{\mu_r}$. Put

$$e_{r,i} = (t+r)^i \, \Pi/(t+r)^{\mu_r} \quad (r < \alpha_0,\ i < \mu_r), \qquad z_j = t^j \Pi \quad (j < g).$$

The roots $-r$ are pairwise incongruent modulo ℓ, and $\sum_r \mu_r + g = 3\alpha_0 + A_2 + 1 + g = \ell + 1 + g = 3n$. So this is a basis in the sense of Lemma 5.1. Give $e_{r,i}$ the potential $i - 2$ if $r \in A$ and $i - 1$ if $r \in C$, and give z_j the potential $\frac{3}{2}$. The elements $e_{r,i}$ form the *integral group* and the z_j the *half group*. Put $K_0 = 32(\ell-1)!^3 A_2!^2/(2n)!^3$, a unit. Then $2\sum \pi = -4(A_2+1) + 3g = 5\ell - 15n - 7$. Also $\nu_\ell = 2(3n - \ell)$, because $\ell^2 > 3n$. So

$$2\sum \pi - \nu_\ell = \mathrm{w}_{n,\ell}. \tag{8.2}$$

The entries of the Gram matrix are linear forms in (a,b,X). Write $G_{xy} = aG^a_{xy} + bG^b_{xy} + XG^X_{xy}$, and let $\mathrm{thr}(x,y) = \pi_x + \pi_y$ if x, y lie in the same group and $\pi_x + \pi_y + \frac{1}{2}$ otherwise.

Lemma 8.7 (entries)**.** *For every pair of basis elements, $v(G^b_{xy}) \geq \mathrm{thr}$, $v(G^a_{xy}) \geq \mathrm{thr} + 1$ and $v(G^X_{xy}) \geq \mathrm{thr} + 4$.*

Proof. We use the elementary fact that $\sum_{s \in \mathbb{F}_\ell} H(s) = 0$ for $H \in \mathbb{F}_\ell[s]$ of degree at most $\ell - 2$ (M1), and (6.2) and (6.3) (M2).

The b-part. In the band $\lambda = 1$ is odd, so the tie classes are (B, D) with $Z = B = \emptyset$ and $W = D$, and $\mu_W = 0$. The basis above is the basis of the proof of Theorem 6.1, which gives $v(G^b) \geq \mathrm{thr}$.

Own nodes, $j \geq 1$. By Lemma 5.3(ii) the terms in a are smaller than those in b by one power of ℓ, and the term in X is ℓ-integral, so by four powers. Hence the bounds with $+1$ and $+4$ hold at every node of A and C with $j \geq 1$.

Nodes with $j=0$ on A and C. At these nodes $v(U_k)\ge 0$ and $v(V_k)\ge -1$ in each of the three variables, so the parts in a and b have the same lower bound. As in the proof of Theorem 6.1, this bound exceeds the threshold by 3, which covers the a-part. The term in X of the node $k=r$ is $6c_rF(-r)$. Here $F(-r)$ vanishes unless both elements are $e_{r,0}$. In that case the coefficient of X has $v\ge\sigma(r)$, and $\sigma(r)=\mathrm{thr}+4$, since $\mathrm{thr}=-4$ on A and -2 on C.

The D-nodes, a-part. At $k=\ell+s$ with $s\in D$ the leading order of the a-part is ℓ^2, one above the b-part. The thresholds 1 and 2 hold outright, or after (M1) when the polynomial $G=\Omega\psi$ of Lemma 8.12 vanishes at the residues outside D. For $z_j\times z_{j'}$ and (top element of $A)\times z_j$ the threshold is 3. The node data of Lemma 8.12 below give $-\frac{15}{2}-3\ell\lambda_0$ and -3 for the coefficients of F and $\ell F'$ in the a-part. So $\sum_D(\cdots)/\ell^2\equiv -\frac{15}{2}\ell\int G+\ell\frac{15}{2}\int G=0 \pmod{\ell^2}$ by (M2), because the total weight of the harmonic numbers in λ_0 is 0 and in the expansion of $c_{\ell+s}$ it is -1.

The D-nodes, X-part. Here $c_k\sim\ell^5$ for $j=0$ and $j=1$, and $c_{\ell+s}/c_s\equiv 8(1+\ell(2H_{s_2}-5H_s+3H_{s_1}))$ with $s_1=s-\alpha_0$ and $s_2=s-A_2-1$. For the pairs of threshold 3 the sum $\sum_D[c_sF(-s)+c_{\ell+s}F(-\ell-s)]/(\ell^5K_0/8)$ is $\equiv 9\ell\int G+\ell(-1+8(-1))\int G=0 \pmod{\ell^2}$ by (M2). So $v\ge 7=\mathrm{thr}+4$. The other pairs need at most 6 and follow from (M1). □

The next lemma moves the computation to one point.

Lemma 8.8 (transfer)**.** *Let $F\in\mathbb{Z}_\ell[\alpha,Y]$ with $F(0,0)\in\mathbb{Z}_\ell^\times$. Then $F(\alpha,Y)\in\mathbb{Z}_\ell^\times$ for all $\alpha,Y\in\ell\mathbb{Z}_\ell$.*

Proof. $F(\alpha,Y)\equiv F(0,0) \pmod{\ell}$. □

We apply this to $F(\alpha,Y)=\ell^{-\mathrm{w}_{n,\ell}}J_n(\alpha/\ell,1,Y/\ell^4)$. By Lemma 8.7 the matrix $N(\alpha,Y)=D^{-1}GD^{-1}$ of Lemma 5.2, with $G=G^b+(\alpha/\ell)G^a+(Y/\ell^4)G^X$, has entries in $\mathbb{Z}_\ell[\varpi][\alpha,Y]$, where $\varpi^2=\ell$. So F has integral coefficients. For $a\in\mathbb{Z}_{(\ell)}$ and $\ell^3X\in\mathbb{Z}_{(\ell)}$ the point $\alpha=\ell a$, $Y=\ell^4X$ lies in $\ell\mathbb{Z}_\ell$. By the homogeneity (2.3), $J_n(a,b,X)=b^{3n}J_n(a/b,1,X/b)$ for a unit b. Hence Theorem 8.5 holds as soon as the reduction of $N(0,0)$ has nonzero determinant.

By the proof of Lemma 5.2 this reduction is block triangular. Its diagonal blocks are $\overline{N}_{\mathrm{int}}$ on the integral group and $\overline{N}_{\mathrm{half}}$ on the half group. Recall that $\ell\ge 19$. We show that both blocks have nonzero determinant when $\ell\ne 24359$. At $(a,b,X)=(0,1,0)$ we have $U_k=-2H_k(3)$ and $V_k=U_kd_k-6H_k(4)$, and $L(F)=\sum_k c_k(U_kF'(-k)+V_kF(-k))$.

8.3.2. *The integral blocks.* Put $\Psi_r=\Pi/(t+r)^{\mu_r}$, and let $\hat{c}_r$ be the unit part of c_r, so that $c_{r+j\ell}\equiv\kappa_j\hat{c}_r$ by Lucas's theorem, with $\kappa=(1,4,1)$ on A and $(1,1)$ on C. Let $P_r(s)=D_{\alpha_0}(s)/(s-r)$ with $D_m(s)=\prod_{i<m}(s-i)$.

Lemma 8.9 (unit identities)**.** *For $r\in A$, $K_0P_r(r)^3\equiv 32\,\hat{c}_r\Psi_r(-r)^2 \pmod{\ell}$. For $r\in C$, $K_0P_r(r)^3\equiv 8\,\hat{c}_r\Psi_r(-r)^2$.*

Proof. By Anton's congruence, $c_{r+\ell} \equiv 4\hat{c}_r$ on A, and $c_{r+\ell}/\ell^2 \equiv c_r/\ell^2$ on C. Write $P_r(r) = r!(-1)^{\alpha_0-1-r}(\alpha_0-1-r)!$ and $\Psi_r(-r)^2$ as factorials. The ratio $K_0 P_r(r)^3/(\hat{c}_r\Psi_r(-r)^2)$ reduces to $-32(-1)^{\alpha_0-1-r}/[(2n+r)!(\alpha_0-1-r)!]^3$ on A, and to $-8(-1)^{\alpha_0-1-r}/[\cdots]^3$ on C. Since $(2n+r)+(\alpha_0-1-r) = \ell-1$, Wilson's theorem gives $(2n+r)!(\alpha_0-1-r)! \equiv (-1)^{r+1}$. With α_0 odd, this gives 32 and 8. □

Lemma 8.10 (the blocks of A)**.** *For $r \in A$ the reduced block of $e_{r,0},\dots,e_{r,3}$ is $\hat{c}_r\Psi_r(-r)^2 M_A$, where $M_A = [\chi_{i+i'}]_{i,i'<4}$ and $\chi = (-\frac{297}{8}, 40, -\frac{103}{2}, 78, -130, 0, 0)$. Its determinant is $-16466684 = -2^2 \cdot 13^2 \cdot 24359$.*

Proof. Own nodes. At $k = r + j\ell$ ($j = 1, 2$), Lemma 5.3 gives $U_k = \ell^{-3}u_j + \cdots$ and $V_k = \ell^{-4}E_j + \cdots$ with $u = (-2, -\frac{9}{4})$, $E = (-6, -\frac{105}{8})$. For the pair (i, i') with $m = i + i'$ the own nodes contribute $\Phi_A(m)\hat{c}_r\Psi_r(-r)^2\ell^{m-4}$ with $\Phi_A(m) = \sum_{j=1,2}\kappa_j(-j)^{m-1}(mu_j - jE_j)$. This gives $\Phi_A = (-\frac{297}{8}, 40, -\frac{103}{2}, 78, -130, 224, -384)$ for $m = 0,\dots,6$.

The D-nodes. At $k = \ell + s$, $s \in D$, the contribution of the pair is $-7\ell K_0 \sum_D G + \ell^2 K_0 S_1 + \cdots$ with $G = D^3_{\alpha_0}\psi$ and $\psi(s) = (r-s)^{m-8}$. For $m \le 4$ it lies above the threshold. For $m = 5$, G is regular at r with $G(r) = -P_r(r)^3$, so $\sum_D G \equiv -G(r)$ by (M1), and the entry is $\ell[224 - 7\cdot 32]\hat{c}_r\Psi_r(-r)^2 = 0$ by Lemma 8.9. For $m = 6$, $G = (s-r)P_r^3$ has a simple zero at r. At level ℓ^2 the D-part is $K_0[5(\Omega\psi')(r) + 10(\Omega\Lambda_1\psi)(r) - 18G'(r)] = 12K_0P_r(r)^3 = 384\,\hat{c}_r\Psi_r(-r)^2$. The terms in $\int G$ cancel, and the shift corrections are $G'(r)$ for the shifts $s-\alpha_0$ and $s-A_2-1$. So the entry is $-384+384 = 0$.

The determinant of M_A is an exact computation with rational numbers. □

Lemma 8.11 (the blocks of C)**.** *For $r \in C$ the reduced block of $e_{r,0}, e_{r,1}, e_{r,2}$ is $\hat{c}_r\Psi_r(-r)^2\left[\begin{smallmatrix} -10 & 8 & -6 \\ 8 & -6 & -52 \\ -6 & -52 & -130 \end{smallmatrix}\right]$, and its determinant is 2^{15}.*

Proof. The own node $j = 1$ gives $\Phi_C(m) = (-1)^{m-1}(10-2m)$. For $m = 3$ the D-part is $-7\ell K_0 P_r(r)^3 = -56\,\hat{c}_r\Psi_r(-r)^2\ell$, so the entry is $4 - 56 = -52$. For $m = 4$ the D-part is $K_0P_r^3[5(-2) + 10\cdot 3 - 36] = -16K_0P_r^3 = -128\,\hat{c}_r\Psi_r(-r)^2\ell^2$, since $\mu_r = 3$ and only the shift $s-\alpha_0$ crosses r. The entry is $-2-128 = -130$. The determinant is $2t(245t+88)$ at $t = 8$, which is 2^{15}. □

Entries between elements of different residues $r \neq r'$ vanish in the reduction. At the own nodes they lie strictly above the threshold, since an element that is not own at a node vanishes there to order μ_r. At the D-nodes the pairs of threshold 2 have G vanishing to order at least 2 at both residues. Then (M1) and (M2) give $S_1 \equiv 7\int G$, which cancels $-7\int G$. The pairs of threshold at most 1 follow from (M1). Hence

$$\det \overline{N}_{\mathrm{int}} = \prod_{r\in A}(\hat{c}_r\Psi_r(-r)^2)^4 \det M_A \cdot \prod_{r\in C}(\hat{c}_r\Psi_r(-r)^2)^3\, 2^{15},$$

which is nonzero for $\ell \notin \{2, 13, 24359\}$.

8.3.3. *The half-integral block.* Let $\mathcal{V}$ be the linear functional on $\mathbb{Q}[x]$ with $\mathcal{V}(x^e) = eB_{e-1}$, where $B_1 = -\frac{1}{2}$, and put $\int g = \sum_e g_e B_e$, so that $\mathcal{V}(g) = \int g'$. Let $\Omega = D_{\alpha_0}^3$.

Lemma 8.12. *Let $\varphi \in \mathbb{Z}_{(\ell)}[t]$ with $\deg\varphi \le 2g-2$, and put $\psi(x) = \varphi(-x)$ and $G = \Omega\psi$. Then*

$$L(\Pi^2\varphi) \equiv 15K_0\,\ell^3\,\mathcal{V}(G) \pmod{\ell^4}.$$

The degree of G is at most $3\alpha_0 + 2g - 2 = \ell - 4$ by (8.1). The proof has four steps.

Step 1: summation lemmas. Let $g_0 \in \mathbb{Z}_{(\ell)}[x]$ have degree at most $\ell-3$ and put $\mathcal{G}(x) = \sum_{u<x} g_0(u)$. This is a polynomial divisible by x, by Faulhaber's formula. Put $q = \mathcal{G}/x$ and $\mathcal{T}(g_0) = \int q$. By the theorem of von Staudt, $\int g_0$, $\mathcal{V}(g_0)$ and $\mathcal{T}(g_0)$ lie in $\mathbb{Z}_{(\ell)}$. Wolstenholme's theorem gives $H_{\ell-1} \equiv 0 \pmod{\ell^2}$ and $H_{\ell-1}(2) \equiv 0 \pmod{\ell}$. With Faulhaber's formula this gives

$$\begin{aligned}
\sum_{u<\ell} g_0(u) &\equiv \ell\int g_0 + \tfrac{\ell^2}{2}\mathcal{V}(g_0)\ (\ell^3),\\
\sum_{u<\ell} g_0(u)H_u &\equiv \int g_0 - \ell\,\mathcal{T}(g_0)\ (\ell^2),\\
\sum_{u<\ell} g_0(u)H_u(2) &\equiv \tfrac{1}{2}\mathcal{V}(g_0)\ (\ell),\\
\sum_{u<\ell} g_0(u)H_u^2 &\equiv -2\mathcal{T}(g_0) - \tfrac{1}{2}\mathcal{V}(g_0)\ (\ell).
\end{aligned}$$

For the second, $\sum_u g_0(u)H_u = \mathcal{G}(\ell)H_{\ell-1} - \sum_{i=1}^{\ell-1} q(i)$ by summation by parts, and the first formula applied to q gives the claim, since $q(0) = \int g_0$. The third is the same computation with $\mathcal{G}(i)/i^2$. For the fourth, $H_u^2 = H_u(2) + 2\sum_{j\le u} H_{j-1}/j$, and the second formula for q finishes. By Lemma 8.13 below these apply to every weight in the proof, and sums over D equal sums over $[0,\ell)$.

Step 2: the nodes. Write $k = j\ell + s$. By Lemma 5.3, $v(c_k) = \sigma(s)$. The polynomial $F = \Pi^2\varphi$ vanishes to order 8 at $-r$ for $r \in A$ and to order 6 for $r \in C$, and $H_k(3)$, $H_k(4)$, d_k have $v \ge -3, -4, -1$. So the nodes with $s \in A\cup C$ and $j = 0$ vanish. The nodes with $s \in A$ and $j \ge 1$ have $v \ge 0{-}4{+}7$, those with $s \in C$ and $j = 1$ have $v \ge 2{-}4{+}5$, and those with $s \in D$ and $j = 0$ have $v \ge 5{-}1$. Hence every node other than $k = \ell{+}s$ with $s \in D$ contributes 0 modulo ℓ^4. So $L(\Pi^2\varphi) \equiv T = \sum_{s\in D}[c_{\ell+s}V_{\ell+s}F(-\ell-s) + c_{\ell+s}U_{\ell+s}F'(-\ell-s)] \pmod{\ell^4}$.

Step 3: the node term. Fix $s \in D$, $k = \ell + s$, $s_1 = s - \alpha_0$, $s_2 = s - A_2 - 1$. Write $\langle x, y\rangle = 1 + \ell x + \ell^2(y + x^2/2)$, so that $\langle x,y\rangle\langle x',y'\rangle \equiv \langle x+x', y+y'\rangle \pmod{\ell^3}$ and $\prod_{y=1}^m (1 + j\ell/y) \equiv \langle jH_m, -\frac{j^2}{2}H_m(2)\rangle$. Digit expansions of the factorials, the reflection $(\ell-1)! = (\ell-1-m)!(-1)^m m!\prod_{y\le m}(1-\ell/y)$ and

Wolstenholme's theorem give

$$c_{\ell+s} \equiv \ell^5 K_0' \frac{s_1!^3 s_2!^2}{s!^5} \langle \kappa_1 + 6H_{s_1} - 5H_s - 2H_{s_2},\ \kappa_2 - 6H_{s_1}(2) + \tfrac{5}{2}H_s(2) - H_{s_2}(2) \rangle$$

modulo a relative error ℓ^3, with $K_0' = 32A_2!^2(\alpha_0 - 1)!^3$ and constants κ_1, κ_2 independent of s. Also $\Pi(-s)^2 = s!^8 s_2!^{-2} s_1!^{-6}$ exactly, so $s_1!^3 s_2!^2\, \Pi(-s)^2/s!^5 = \Omega(s)$, and $\Pi(-k)^2/\Pi(-s)^2 \equiv \langle 2\Lambda_1(s), -\Lambda_2(s) \rangle$ with $\Lambda_j(s) = \sum_r \mu_r (s-r)^{-j}$. Next, $H_k(3) = \ell^{-3} + (\text{integral})$ and $H_k(4) = \ell^{-4} + (\text{integral})$, and

$$\ell d_k \equiv \tfrac{1}{2} + \ell\lambda_0 + \ell^2\delta_1, \qquad \ell^4 V_k \equiv -7 - 2\ell\lambda_0 - 2\ell^2\delta_1 \pmod{\ell^3}, \qquad \ell^3 U_k \equiv -2,$$

with $\lambda_0 = 5H_s - 3H_{s_1} - 2H_{s_2}$ and $\delta_1 = -5H_s(2) + 6H_{s_1}(2) - 2H_{s_2}(2)$. With $\psi(x) = \varphi(-x)$ we have $F(-k) = \Pi(-k)^2\psi(k)$ and $F'(-k) = \Pi(-k)^2(-2\Lambda_1(k)\psi(k) - \psi'(k))$. Taylor expansion gives, modulo ℓ^3,

$$\begin{aligned} \ell^4 \big(V_k F(-k) + U_k F'(-k)\big)/\Pi(-k)^2 &\equiv 2\ell(\psi' + \ell\psi'') \\ &\quad + (-7 - 2\ell\lambda_0 - 2\ell^2\delta_1 + 4\ell\Lambda_1 - 4\ell^2\Lambda_2)(\psi + \ell\psi' + \tfrac{\ell^2}{2}\psi''). \end{aligned}$$

Step 4: collection. By the exact identities $H_{s_1} = H_s - \delta_N(s)$, $H_{s_2} = H_s - \delta_A(s)$, and the same for the second order sums, with $\delta_N(x) = \sum_{i<\alpha_0}(x-i)^{-1}$ and $\delta_A(x) = \sum_{i\le A_2}(x-i)^{-1}$, all harmonic numbers collapse to H_s, $H_s(2)$ and rational functions with poles at the zeros of D_{α_0}. Since $\Lambda_1 = 3\delta_N + \delta_A$, we get $e_1 = -H_s + 4\delta_A$, $\varepsilon_2 = -\frac{9}{2}H_s(2) + 3\delta_N^{(2)}$, $-2\lambda_0 + 4\Lambda_1 = 6\delta_N$ and $-2\delta_1 - 4\Lambda_2 = 2H_s(2) - 8\delta_A^{(2)}$, where e_1, ε_2 are the arguments of $\langle\cdot,\cdot\rangle$ in the expansion of $c_{\ell+s}\Pi(-k)^2$ beyond the constants. With $W_1 = \Omega\psi'$ and $W_2 = \Omega\psi''$, the expansion of $\Omega\langle e_1, \varepsilon_2\rangle \times (\text{bracket})$ is

$$\begin{aligned} &-7G + \ell\big[-5W_1 + 7GH_s - 28G\delta_A + 6G\delta_N\big] \\ &+ \ell^2\big[-\tfrac{3}{2}W_2 - \tfrac{7}{2}GH_s^2 + \tfrac{67}{2}GH_s(2) + 5H_sW_1 + 28GH_s\delta_A - 6GH_s\delta_N + R\big], \end{aligned}$$

where R is a combination, with integral coefficients, of terms without harmonic numbers. The functions G, W_1, W_2, $G\delta_N$, $G\delta_A$, $G\delta_N^{(2)}$, $G\delta_A^{(2)}$, $G\delta_A^2$, $G\delta_A\delta_N$, $W_1\delta_A$ and $W_1\delta_N$ are polynomials of degree at most $\ell - 4$ that vanish on $[0, \alpha_0)$ (Lemma 8.13). Summing the three brackets over D with Step 1, with the accuracies ℓ^3, ℓ^2 and ℓ:

- bracket 0: $-7\ell \int G - \frac{7}{2}\ell^2\mathcal{V}(G)$;
- bracket 1: $7\int G - 7\ell\mathcal{T}(G) - 5\ell\int W_1 - 28\ell\int G\delta_A + 6\ell\int G\delta_N$;
- bracket 2: $7\mathcal{T}(G) + \frac{7}{4}\mathcal{V}(G) + \frac{67}{4}\mathcal{V}(G) + 5\int W_1 + 28\int G\delta_A - 6\int G\delta_N$, the terms without harmonic numbers being 0 by the first formula of Step 1.

In the total of bracket 0, ℓ times bracket 1 and ℓ^2 times bracket 2, the coefficient of ℓ is $-7\int G + 7\int G = 0$. The terms $\mathcal{T}$, $\int W_1$, $\int G\delta_A$ and $\int G\delta_N$ cancel in the coefficient of ℓ^2, which is $(-\frac{7}{2} + \frac{7}{4} + \frac{67}{4})\mathcal{V}(G) = 15\mathcal{V}(G)$. So $\sum_{s\in D}\cdots \equiv 15\ell^2\mathcal{V}(G) \pmod{\ell^3}$. The constant factor $\langle\kappa_1, \kappa_2\rangle \equiv 1 \pmod{\ell}$ does not matter, because the sum is 0 modulo ℓ^2. Hence $T \equiv \ell K_0' \cdot 15\ell^2\mathcal{V}(G) \pmod{\ell^4}$.

Finally $K_0 \equiv K_0'$ (mod ℓ), because Wilson's theorem gives $(2n)!(\alpha_0-1)! \equiv -1$. □

Lemma 8.13. *Let $\delta_N^{(j)}(x)=\sum_{i<\alpha_0}(x-i)^{-j}$ and $\delta_A^{(j)}(x)=\sum_{i\le A_2}(x-i)^{-j}$. The functions listed in Step 4 are polynomials in $\mathbb{Z}_{(\ell)}[x]$ of degree at most $\ell-4$ that vanish at every $x\in[0,\alpha_0)$.*

Proof. Ω has a zero of order 3 at each $i<\alpha_0$ and each factor has a pole of order at most 2 there. The degree of G is at most $\ell-4$ and the others are smaller. □

Lemma 8.12 makes $\overline{N}_{\mathrm{half}}$ explicit. For $\varphi=t^{j+j'}$ we have $\psi=(-x)^{j+j'}$, so up to the unit b and the unit scaling of the basis

$$\det \overline{N}_{\mathrm{half}} = (15K_0)^g \det\bigl[\mathcal{V}(\Omega\, x^{j+j'})\bigr]_{j,j'<g},$$

because $\det[(-1)^{j+j'}h_{j+j'}]=\det[h_{j+j'}]$. The following closed form shows that this is a unit.

Lemma 8.14 (the Hankel determinant)**.** *Let $\gamma\ge 0$, $\alpha_0=2\gamma+1$ and $g\ge 1$. Then*

$$\det\bigl[\mathcal{V}(D_{\alpha_0}(x)^3x^{i+j})\bigr]_{0\le i,j<g} = (-1)^{\gamma g+\lfloor g/2\rfloor}\prod_{k<\lceil g/2\rceil}h_k^+\prod_{k<\lfloor g/2\rfloor}h_k^-,$$

where

$$h_k^+=\frac{k!\,(\gamma+k)!^3\,(2\gamma+1+k)!^3\,(k+3\gamma+1)!}{(2k+3\gamma+1)!\,(2k+3\gamma+2)!},$$
$$h_k^-=\frac{k!\,(\gamma+1+k)!^3\,(2\gamma+1+k)!^3\,(k+3\gamma+2)!}{(2k+3\gamma+2)!\,(2k+3\gamma+3)!}.$$

All factorial arguments are at most $g+3\gamma+1$. So the determinant is nonzero, and no prime $\ell\ge g+3\gamma+2$ divides its numerator or its denominator.

By (8.1), $\ell=6\gamma+2g+5>g+3\gamma+1$. Since $\ell\ge 19$, the factor $15K_0$ is a unit as well. So $\ell\nmid\det\overline{N}_{\mathrm{half}}$. The numbers $h_k^\pm$ are the squared norms of Wilson polynomials with parameters $(0,\gamma+1,\gamma+1,\gamma+1)$ and $(1,\gamma+1,\gamma+1,\gamma+1)$ [37, 18]. The identification of the weight is a remark and not a step of the proof. We prove the lemma by finite algebra. Put $\varsigma=\gamma+1$ and $(y)_k=y(y+1)\cdots(y+k-1)$.

Two facts on $\mathcal{V}$. (L0) $\mathcal{V}(f(x+1)-f(x))=f''(0)$. By linearity take $f=x^m$. Then $\mathcal{V}((x+1)^m-x^m)=\sum_{e<m}\binom{m}{e}eB_{e-1}=m\sum_{e'<m-1}\binom{m-1}{e'}B_{e'}$, which is $m[m=2]=(x^m)''(0)$ by $\sum_{k<N}\binom{N}{k}B_k=[N=1]$. (R0) $\mathcal{V}(g(-x))=-\mathcal{V}(g)-g''(0)$, since $B_{e-1}=0$ for even $e\ge 4$. By induction, $\mathcal{V}(g(x+m))=\mathcal{V}(g)+\sum_{j<m}g''(j)$ and $\mathcal{V}(g(m-x))=-\mathcal{V}(g)-\sum_{j\le m}g''(j)$.

The centred functional. Put $\mathcal{Z}[g]=(-1)^\gamma\mathcal{V}(D_{\alpha_0}(x)^3g(x-\gamma))$ on $\mathbb{Q}[z]$, so that $\mathcal{V}(D_{\alpha_0}^3x^e)=(-1)^\gamma\mathcal{Z}[(z+\gamma)^e]$.

- *Evenness:* $\mathcal{Z}[g(-z)]=\mathcal{Z}[g(z)]$. Indeed $D_{\alpha_0}(2\gamma-x)=-D_{\alpha_0}(x)$, and the reflection with $N=2\gamma$ applies, because the second derivatives of $D_{\alpha_0}^3g(x-\gamma)$ vanish at $0,\dots,2\gamma$.

- *Pearson equation:* $\mathcal{Z}[(z+\varsigma)^3 g(z+1)] = \mathcal{Z}[(z-\varsigma)^3 g(z)]$. Put $f = D^3_{\alpha_0+1} g(x-\gamma)$. Then $f(x+1) - f(x) = D^3_{\alpha_0}[(x+1)^3 g(x+1-\gamma) - (x-\alpha_0)^3 g(x-\gamma)]$, and $f''(0) = 0$, so (L0) applies.

Put $F_j = (z)_j(-z)_j$, $E_k = (\varsigma+z)_k(\varsigma-z)_k$ and $F^1_j = (1+z)_j(1-z)_j$. These are even, of degree j, k, j in z^2, and $z^2 F^1_j = -F_{j+1}$.

Beta evaluation. For $j, k \geq 0$,

$$\mathcal{Z}[F_j E_k] = \mathcal{Z}[1]\,\frac{(\varsigma)_{j+k}(\varsigma)^2_j(2\varsigma)^2_k}{(3\varsigma)_{j+k}}.$$

Let $g(z) = (z)_j(1-z)_j(\varsigma+z)_k(\varsigma+1-z)_k$ and $T(z) = (z+\varsigma)^3 g(z+1)$. The Pearson equation gives $\mathcal{Z}[T] = \mathcal{Z}[(z-\varsigma)^3 g] = \mathcal{Z}[-T(-z)] = -\mathcal{Z}[T]$ by evenness, so $\mathcal{Z}[T] = 0$. Evenness also gives $\mathcal{Z}[T] = \frac{1}{2}\mathcal{Z}[T(z) + T(-z)]$. With $zT(z) = P(z)F_jE_k$, $P = (\varsigma+z)^2(\varsigma+k+z)(z+j)$, we get $(P(z)-P(-z))/z = 2[(3\varsigma+j+k)z^2 + 2\varsigma j(\varsigma+k) + \varsigma^2(\varsigma+j+k)]$, hence $\mathcal{Z}[F_jE_k((3\varsigma+j+k)z^2 + 2\varsigma j(\varsigma+k) + \varsigma^2(\varsigma+j+k))] = 0$. Using $z^2E_k = (\varsigma+k)^2E_k - E_{k+1}$ and $z^2F_j = j^2F_j - F_{j+1}$ gives the two recursions

$$(3\varsigma+j+k)\mathcal{Z}[F_jE_{k+1}] = (\varsigma+j+k)(2\varsigma+k)^2\mathcal{Z}[F_jE_k],$$
$$(3\varsigma+j)\mathcal{Z}[F_{j+1}] = (\varsigma+j)^3\mathcal{Z}[F_j],$$

which the right side satisfies. Induction on j and then on k gives the formula.

The mass. $\mathcal{Z}[1] = (2\gamma+1)!^3\gamma!^3/(3\gamma+2)!$. Write $\mathcal{Z}_\gamma$ for $\mathcal{Z}$ to show the dependence on γ. Then $D_{2\gamma+1}(x) = \Phi_\gamma(x-\gamma)$ with $\Phi_\gamma(z) = z\prod_{i=1}^{\gamma}(z^2-i^2)$, and (L0) gives $\mathcal{Z}_\gamma[g] = -\mathcal{Z}_{\gamma-1}[(z^2-\gamma^2)^3 g]$. So $\mathcal{Z}_\gamma[1] = \mathcal{Z}_{\gamma-1}[u^3]$ with $u = \gamma^2 - z^2 = E_1$ at parameter $\gamma-1$, where $\varsigma = \gamma$. Write $u^3 = E_3 - (6\gamma+5)E_2 + (2\gamma+1)^2E_1$ and use the Beta evaluation at $j=0$. A short computation gives $\mathcal{Z}_\gamma[1]/\mathcal{Z}_{\gamma-1}[1] = ((2\gamma+1)2\gamma)^3\gamma^3/(3\gamma(3\gamma+1)(3\gamma+2))$, and $\mathcal{Z}_0[1] = \mathcal{V}(x^3) = 3B_2 = \frac{1}{2}$.

Splitting. The matrix $[\mathcal{Z}[(z+\gamma)^{i+j}]]$ equals $U[\mathcal{Z}[z^{i+j}]]U^\top$ with U lower unitriangular, so the determinant in the lemma is $(-1)^{\gamma g}\det[\mathcal{Z}[z^{i+j}]]$. By evenness $\mathcal{Z}[z^{\text{odd}}] = 0$, and ordering the indices as (even, odd) makes the matrix block diagonal. The blocks are the Hankel matrices of $\mathcal{Z}_+(P) = \mathcal{Z}[P(z^2)]$ of size $\lceil g/2\rceil$ and $\mathcal{Z}_-(P) = \mathcal{Z}[z^2P(z^2)]$ of size $\lfloor g/2\rfloor$. Let $\mathcal{K}$ be a linear functional on $\mathbb{Q}[w]$. If $\mathcal{P}_m$ are monic of degree m with $\mathcal{K}(\mathcal{P}_m q) = 0$ for $\deg q < m$, then $\det[\mathcal{K}(w^{i+j})]_{i,j<r} = \prod_{m<r}\mathcal{K}(\mathcal{P}_m w^m)$. Indeed $[\mathcal{K}(\mathcal{P}_i\mathcal{P}_j)] = V[\mathcal{K}(w^{i+j})]V^\top$ with V unitriangular.

The orthogonal polynomials. For $m \geq 0$ put

$$\mathcal{Q}_m = \sum_{k\leq m}\frac{(-m)_k(m+3\varsigma-1)_k}{k!(\varsigma)_k(2\varsigma)^2_k}E_k, \qquad \mathcal{R}_m = \sum_{k\leq m}\frac{(-m)_k(m+3\varsigma)_k}{k!(\varsigma+1)_k(2\varsigma)^2_k}E_k.$$

For $j \leq m$ the Beta evaluation gives

$$\mathcal{Z}[\mathcal{Q}_mF_j] = \mathcal{Z}[1]\frac{(\varsigma)^3_j}{(3\varsigma)_j}\sum_k\frac{(-m)_k(m+3\varsigma-1)_k(\varsigma+j)_k}{k!(\varsigma)_k(3\varsigma+j)_k}.$$

This sum is balanced. Pfaff–Saalschütz [1, Theorem 2.2.6] with $\alpha = m+3\varsigma-1$, $\beta = \varsigma + j$ and $\gamma' = \varsigma$ gives a product containing $(-j)_m$, which is zero for $j < m$. So $\mathcal{Q}_m$ is orthogonal for $\mathcal{Z}_+$ to all polynomials in z^2 of degree less than m. Let $\hat{\mathcal{Q}}_m$ be the monic multiple of $\mathcal{Q}_m$ in the variable $w = z^2$. At $j = m$ we divide by the leading coefficient $(m+3\varsigma-1)_m/((\varsigma)_m(2\varsigma)_m^2)$ of $\mathcal{Q}_m$ and by the leading coefficient $(-1)^m$ of F_m. The product then evaluates to $\mathcal{Z}_+(\hat{\mathcal{Q}}_m w^m) = h_m^+$. For the odd block, $\mathcal{Z}_-(\mathcal{R}_m F_j^1) = -\mathcal{Z}[\mathcal{R}_m F_{j+1}]$. The same argument with $\gamma' = \varsigma + 1$ and $\beta = \varsigma + j + 1$ gives orthogonality for $j < m$ and $\mathcal{Z}_-(\hat{\mathcal{R}}_m w^m) = -h_m^-$. The factorial arithmetic uses $(3\varsigma)_m = (m+3\gamma+2)!/(3\gamma+2)!$ and $(m+3\varsigma-1)_m = (2m+3\gamma+1)!/(m+3\gamma+1)!$. The product of the norms, with the sign $(-1)^{\gamma g}$ from the shift and $(-1)^{\lfloor g/2\rfloor}$ from the odd block, is the formula. The largest factorial arguments are $2m+3\gamma+2 \le g+3\gamma+1$ in the even block and $2m+3\gamma+3 \le g+3\gamma+1$ in the odd block. □

Remark. Put $W(y) = |\Gamma(\gamma+1+iy)|^6 \cosh(\pi y)/\pi$. A Barnes-integral evaluation gives $\mathcal{Z}[g] = \int_{\mathbb{R}} g(iy)W(y)\,dy$, and W is the Wilson weight with parameters $(0, \gamma+1, \gamma+1, \gamma+1)$. Its odd part y^2W is the Wilson weight with $(1, \gamma+1, \gamma+1, \gamma+1)$. This explains the two families h^+, h^-. We do not use it.

8.3.4. *Proof of Theorem 8.5.* Lemma 8.10, Lemma 8.11 and the cross entries give $\det \overline{N}_{\text{int}} \neq 0$ for $\ell \notin \{2, 13, 24359\}$. Lemmas 8.12 and 8.14 give $\det \overline{N}_{\text{half}} \neq 0$ for $\ell \ge 19$. So $\det \overline{N}(0,0) \neq 0$. By (8.2) and Lemma 8.8, the valuation of $J_n(a,b,X)$ is $\mathrm{w}_{n,\ell}$ at every point of the theorem. In particular $J_n(a,b,X) \neq 0$. □

9. Proof of Theorem 1.1

Put $D = 33.4401$, $\Lambda = 28.68723663$, $\eta = D - \Lambda = 4.75286337$, $\rho = 42.94856306$ and $\kappa = 3$. The modulus is $M = 126$, and there are four tied families:

- the band: all unit classes, $c = \frac{8}{3}$, $c' = 3$, $R(\ell) = \{n : 8n < 3\ell,\ \ell+1 < 3n\}$;
- Lai's primes: $S = \{s : s \equiv 1 \pmod{18}\}$, $c = c' = \frac{9}{2}$, $\ell = \frac{9}{2}n + 1$;
- the first line: $S = \{s : s \equiv 5 \pmod 7\}$, $c = c' = \frac{7}{3}$, $\ell = \frac{7}{3}n + \frac{1}{3}$;
- the second line: $S = \{s : s \equiv 3 \pmod 7\}$, $c = c' = \frac{7}{3}$, $\ell = \frac{7}{3}n + \frac{2}{3}$.

The band has the required property with $C = 2$ for $\ell \ge 100$. Indeed the interval $(\frac{\ell+1}{3}, \frac{3\ell}{8})$ has length at least 3, so it contains an integer within 2 of every $y \in [\frac{\ell}{3}, \frac{3\ell}{8}]$. Each of the three other sets S has 6 of the 36 unit classes. Lai's set meets each of the two lines in one class, and the two lines are disjoint. The finite exceptional set E consists of the primes below 100 and the primes 1039 and 24359.

Part (a). Since $a\zeta(3)+b\pi^4 = a\zeta(3)+90b\zeta(4)$, it suffices to prove the bound of part (a) for $a\zeta(3)+b\zeta(4)$ with $a, b \in \mathbb{Z}$ and $b \neq 0$. Let $\xi = a\zeta(3)+b\zeta(4)$ and $P_n = Q_n$. By (2.4) and Theorem 7.5, $q^{3n} m_n Q_n(p/q) = m_n J_n(qa, qb, p) \in \mathbb{Z}$,

so (a) of Theorem 3.1 holds with $d_n = m_n$. Corollary 4.11 gives (b). Take $W(p,q) = q$ and $K = 1$. For each family add the prime divisors of b to the exceptional set, and let $\mathrm{Bad}(\ell, p, q)$ be $\ell^4 \mid q$, with $e = 4$. Corollaries 8.6, 8.2 and 8.4 give (c). Put $u_0 = 3/\eta$ and $\beta = 7/800$. The windows $(\frac{8}{3}u, 3w]$, $(\frac{9}{2}u, \frac{9}{2}w]$ and $(\frac{7}{3}u, \frac{7}{3}w]$ with $w = u_0 + \beta$ are pairwise disjoint, because $\frac{7}{3}w < \frac{8}{3}u_0$ and $3w < \frac{9}{2}u_0$. So

$$\mathcal{M}(u_0, u_0+\beta) = 4\Big[\Big(\frac{u_0}{3} + 3\beta\Big) + \frac{1}{6}\cdot\frac{9}{2}\beta + 2\cdot\frac{1}{6}\cdot\frac{7}{3}\beta\Big] = \frac{4}{\eta} + \frac{163}{9}\beta = 1.00007\ldots > 1.$$

Theorem 3.1 applies, and $\rho(3/\eta + \beta) = 27.48487\ldots < 27.49$.

Part (b). Let $\xi = y^* = \zeta(3)/\zeta(4)$ and $P_n = f_n$. As above, $q^{3n} m_n f_n(p/q) = m_n J_n(q, -p, 0) \in \mathbb{Z}$, and Corollary 4.11 gives (b). Since $1 < y^* < 2$, every p/q with $|y^* - p/q| < 1$ has $0 < p < 3q$, and we take $W(p,q) = |p|$ and $K = 3$. Let $\mathrm{Bad}(\ell, p, q)$ be $\ell \mid p$, with $e = 1$. Corollaries 8.6, 8.2 and 8.4 give (c). With $\beta = 9593/50000$ the windows of the band and of the two lines overlap, and the other windows are disjoint from them. The band meets each line in a window of length $\frac{7}{3}w - \frac{8}{3}u_0 = \frac{1}{3}(7\beta - u_0)$ and density $\frac{1}{6}$, so

$$\mathcal{M}(u_0, u_0+\beta) = \Big(\frac{u_0}{3} + 3\beta\Big) + \frac{1}{6}\cdot\frac{9}{2}\beta + 2\cdot\frac{1}{6}\cdot\frac{7}{3}\beta - 2\cdot\frac{1}{6}\cdot\frac{1}{3}(7\beta - u_0) = \frac{4u_0}{9} + \frac{15}{4}\beta = 1.0000076\ldots > 1.$$

Since $\rho(3/\eta + \beta) = 35.34918\ldots$, Theorem 3.1 with $\mu = 35.3495$ gives $|y^* - p/q| > q^{-35.3495}$ for all integers p and all large q. Finally $\zeta(3)/\pi^4 = y^*/90$, because $\zeta(4) = \pi^4/90$. Lemma 3.2 with $r = 1/90$, $\mu' = 35.3495$ and $\mu = 35.35$ gives part (b) of Theorem 1.1. □

10. Remarks

The numbers in this section are numerical evidence. They are not part of the formal proof.

The decay. The cell computation of Theorem 4.3 gives $D \geq 33.44015$. Optimizing the measure on wedge contours suggests that the best value of this method for Lai's determinants is close to 33.46. On a vertical line, weak duality caps the method at about 33.25. Bending the contour is therefore necessary for the constant of Theorem 4.3. Lai's exponent 31.61 holds at the constant only, and Theorem 4.3 improves it on a neighbourhood.

The denominators. At more than 3000 pairs (n, p) with $n \leq 200$ we computed $v_p(J_n)$ exactly. It never lies below the bounds of Section 5, and in the range $\sqrt{8n} < p < 3n$ it equals $\mathrm{E}(n,p) + x_1 + x_2$ except at a few primes where a class constant of the proofs vanishes modulo p. This suggests that $\Lambda = 28.6872366\ldots$ is the true exponential rate of the denominators of $J_n(a, b, X)$ as a polynomial in three variables.

Nonvanishing. The tolerance $\ell^4 \nmid q$ is best possible for all three families: at the tested pairs, the roots of $J_n(1,1,X)$ have $v_\ell(X) = -4$. For primes $5n/2 < \ell < 8n/3$ the reduced half block is the Hankel matrix of a weight on two intervals of residues. We know no closed form for its determinants, which have large prime factors. At 124 pairs (n,ℓ) with $n < 90$ the prime ℓ does not divide it. For one size n the primes of the band carry the mass $\sum_\ell e \log \ell \sim \frac{4}{3}n$ for the plane and $\frac{1}{3}n$ for the ratio. A mass of $1.585n$ would give the exponent 27.11. The exponent of part (b) is larger than that of part (a) because the coefficient $b = -p$ of the ratio family grows with q. So only one power of each tied prime is available.

The axes $a = 0$ and $b = 0$. For $b = 0$ the number is $a\zeta(3)$. Theorem 1.1 does not apply, and the known measures for $\zeta(3)$ are much smaller. For $a = 0$ the number is a rational multiple of π^4, and the bound 12.51 of Marcovecchio and Zudilin is better than ours.

11. Formalization

Theorem 1.1 is formalized in Lean 4 [6] with Mathlib [35], together with every theorem, proposition and lemma its proof uses. The remarks, Theorem 7.6, the exact valuations stated after Theorems 8.1, 8.3 and 8.5, and the remark on Wilson weights after Lemma 8.14 are not part of the formalization. The development is in the directories `CatalanCore/Lai` and `CatalanCore/Measure` of the repository

https://github.com/slitsey/catalan-irrationality-measure

Theorem 1.1 consists of the theorems

```
LaiIrrationalityMeasure.lai_plane_measure_27_49
LaiIrrationalityMeasure.lai_ratio_measure_35_35
```

in `SolutionLai.lean`. Their statements use only Mathlib. The numbers $\zeta(3)$ and $\zeta(4)$ are written as the series $\sum_{k\ge0}(k+1)^{-3}$ and $\sum_{k\ge0}(k+1)^{-4}$, and π is Mathlib's `Real.pi`. The first theorem states, for integers a and $b \neq 0$, that $q^{-27.49} < |a\zeta(3)+b\zeta(4)-p/q|$ for all integers p and all sufficiently large q. Applied to $(a, 90b)$ it gives part (a), since $\pi^4 = 90\zeta(4)$. This is the only place where the formal statement differs in form from Theorem 1.1. The second states $q^{-35.35} < |\zeta(3)/\pi^4 - p/q|$ in the same form. The modules contain no `sorry` and no `native_decide`. The command `#printaxioms` reports only `propext`, `Classical.choice` and `Quot.sound`. The proof is kernel-checked. It has not been run through the independent checker `lakecomparator` [20].

The prime number theorem in arithmetic progressions is derived from the Wiener–Ikehara theorem of the PNT+ project [36] and Mathlib's results on primes in progressions. The proofs were checked with Lean `v4.35.0-rc2` and the matching Mathlib.

The formal proof differs from the text in one point of organization. It applies the criterion to the families m_nQ_n and m_nf_n, which have integer values at p/q after multiplication by q^{3n}, with content exponent 0 and decay

exponent $D - \Lambda$. This is equivalent to the form of Theorem 3.1 used in Section 9.

The results of the paper correspond to the following declarations. Their files are under `CatalanCore`. We give the names with a short prefix only. The full namespaces are `CatalanCore.Lai`, `CatalanCore.Measure`, `CatalanCore.Content` and `CatalanCore.NT`.

- Proposition 2.1 and (2.3): `laiR_partial_fractions`, `laiL_eval_moment` and `isHomogeneous_laiJ`. The identity $\zeta(4) = \pi^4/90$ is `zeta4_eq_pi`.
- Theorem 3.1: `measure_of_family_tiedNV_index`, with the structure `TiedNV`, whose families have a slope interval $[c, c']$, the mass condition `NT.UnionWindowOK` and the prime number theorem in progressions `NT.pnt_class` and `NT.theta_interval_asymp`. Lemma 3.2: `plain_measure_smul`.
- Theorem 4.1: `lai_decay_dual`. Theorem 4.3 and Corollary 4.11: `lai_wedge_D`, `lai_plane_decay` and `lai_ratio_decay`, applied to the certificate `laiDecayData`.
- Proposition 5.4: `lai_vp_lower`, `lai_vp_nonneg` and `lai_small_primes`. Theorem 5.5: `thm0`. Theorem 5.6: `band7_proof`.
- Lemma 5.7: `lemmaS` and `lemmaT`. Theorem 5.9: `lai_L1`. Theorem 6.1: `lai_L2`.
- Proposition 7.1: `bridge_proof`. Lemma 7.4: `c01_partial_le`. Lemma 7.3 and Theorem 7.5: `sum_signed_le` and `lai_content`.
- Theorem 8.1 and Corollary 8.2: `lai_nv_core`, `lai_nv_plane` and `lai_nv_ratio`. Theorem 8.3 and Corollary 8.4: `nv7_core`, `nv7_plane` and `nv7_ratio`.
- Theorem 8.5: `B83Main.b83_nonvanishing`, with the corollaries `nv_plane_B83` and `nv_ratio_B83` for Corollary 8.6. Lemma 8.9: `u83_A` and `u83_C`. Lemmas 8.10 and 8.11: `det_MA` and `det_MC`, with `det_intBlock_band_ne`. Lemma 8.7: `lb06_point` and the weighted bounds in the file `CatalanCore/Lai/B83Weighted.lean`. Lemma 8.12: `lemma_G3_monomial`, with Wolstenholme's theorem `wolstenholme_one` and `wolstenholme_two`. Lemma 8.14: `wilson_hankel`, `wilson_hankel_padic_unit` and the Pfaff–Saalschütz summation `saalschutz`. The unit of the half block: `half_block_unit`. The assembly of the Gram determinant: `det_gramF_ne_zero`.
- Section 9: `lai_roots_of`.

The certificate data are the coefficients of H, the 40 cells, the trigonometric values at the cell centres, and the 22977 terms of the partial sum in Lemma 7.4. They were computed with floating-point optimization and the interval arithmetic of Arb [17], and Lean re-checks them with rational arithmetic. They play no role in the proof beyond satisfying the checks.

Disclosure of AI use

The mathematics, the Lean formalization, the certificate generators and the text of this paper were produced by AI systems at the request of the author. The author posed the problems, directed the work and is responsible for the paper. The systems were Claude Opus 5.5, Claude Sonnet 5.5 and Claude Haiku 5.5 (Anthropic), running in Claude Code, and GPT-6.1 Sol (OpenAI), running in Codex. The construction of the determinants and the linear independence of 1, $\zeta(3)$ and π^4 are due to Lai [19].

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Email address: sean.litsey.math@gmail.com