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The unique shortest rank-state cycle in Beggar-My-Neighbour

Six tricks and 38 individual plays

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Abstract

Under chronological pickup and winner-led play, the minimum primitive rank-state cycle lengths for the standard 52-card inventory are six completed tricks and 38 individual plays. One cycle attains both minima and is unique under either criterion, up to cyclic phase and exchange of players. Its 30–22 witness has a short word certificate, and a 26–26 deal enters it after one trick. The trick-period proof excludes 1,201 face itineraries below six tricks and classifies all 21,407 six-trick itineraries. For the global individual-play result, an elementary bound reduces the problem to at most 18 tricks. Two independently implemented enumerators agree on all 43,808 candidate itineraries through 38 plays; exact integer gap constraints leave one itinerary and one rank-and-gap model. A short modular-arithmetic argument independently proves rigidity of that surviving model. With named players there are two oriented cycles, each with six boundary states and 181 backward-reachable rank states, including 19 balanced starts. We also classify period-four deck sizes and prove that fixed-period deck-size sets are semilinear.

1 Introduction and conventions

The classical card game Beggar-My-Neighbour (also called Beggar-Your-Neighbor) was studied as a problem of nontermination for many decades; see the problem statement of Paulhus [1] and the historical account of Casella et al. [2]. Casella and collaborators constructed a nonterminating standard-deck game with a 62-trick cycle, solving Conway's anti-Hilbert problem. More recent computational and dynamical investigations include Andorno et al. [3]. We minimize the primitive rank-state cycle length under the same deterministic rules and standard deck inventory, using both completed tricks and individual card plays as measures of time. The minima are six tricks and 38 plays. The same cycle uniquely attains either minimum, up to cyclic phase and player exchange. In particular, allowing more than six tricks yields neither a shorter individual-play cycle nor an additional cycle with 38 plays. We distinguish these primitive periods from the transient needed to enter a cycle and from periods that track the suits or physical identities of all cards.

The global exclusions and classifications are computer-assisted, using an exhaustive face-itinerary reduction and exact integer gap constraints. Independent enumeration implementations, reproducible source code, and two solver interfaces support the computations. The upper-bound witness is an exact hand-checkable word identity. Uniqueness within the surviving face itinerary also has an elementary modular-arithmetic proof; that positive-case argument is separate from the computer-assisted exclusion of every other itinerary.

Write C for any ordinary card and use J, Q, K, A for cards requiring payments of $1, 2, 3, 4$, respectively. A new face card interrupts the current payment and starts its own payment. A successful claimant appends

the entire table pile to the bottom of their hand in chronological play order and leads the next trick. Hands are queues, with the next card on the left.

A state here is (H_A, H_B, ℓ) at an empty-table boundary immediately after collection: both ordered words and the next leader are included. Equal face ranks are identified, as are all ordinary cards. Let T be the completed-trick map on states for which another trick can be completed. The primitive period in tricks is the least positive p with $T^p(s) = s$. Its individual-play period L is the number of cards played during those p tricks. Cyclic changes of the starting boundary leave both quantities unchanged.

Theorem 1 (Both minima and their unique attaining cycle)**.** *For inventory* $\mathrm{C}^{36}\mathrm{J}^4\mathrm{Q}^4\mathrm{K}^4\mathrm{A}^4$*,*

$$\boxed{p_{\min} = 6 \text{ tricks}, \qquad L_{\min} = 38 \text{ individual plays.}}$$

Both minima are taken over all nonempty hand splits. Up to cyclic phase and exchange of players, exactly one cycle has six tricks, and exactly one cycle has 38 individual plays; these are the same cycle. A 30–22 *state attains the minima, and a* 26–26 *deal reaches it after one trick. Thus both minima are also attained as eventual periods from balanced deals. With player names fixed, the attaining cycles comprise exactly two oriented orbits, each with six completed-trick boundary states.*

The existence proof is the explicit calculation in the next section. The trick-period exclusions and classification are given below; the global individual-play classification is in Section 11, with elementary rigidity in Section 9. The theorem concerns the minimum over all splits and the attainable eventual periods from balanced deals. It does not assign a minimum to every prescribed boundary split or minimize periods that distinguish all 52 physical cards.

2 A six-trick cycle with a word certificate

Player A leads from the following hands:

$$H_A = \mathrm{KCAC}^5\mathrm{QC}^3\mathrm{QCAC}^3\mathrm{QKCAC}^5\mathrm{QC}^2,$$
$$H_B = \mathrm{CJCKCJC}^2\mathrm{AKC}^5\mathrm{JC}^4\mathrm{JC}.$$

For copying into the verifier, the expanded words are

```
A: KCACCCCCQCCCQCACCCQKCACCCCCQCC
B: CJCKCJCCAKCCCCCJCCCCJC
```

Their lengths are 30 and 22. Together they contain 36 ordinary cards and four copies of each face rank.

2.1 The six tricks

In the table, a pile is written in chronological order. The two play counts record how many cards each player contributes to that pile.

Trick	Leader	A plays	B plays	Collected pile	Winner	Next split
1	A	2	2	$P_1 = \mathrm{KCJC}$	B	28–24
2	B	4	2	$P_2 = \mathrm{CAKC}^3$	B	24–28
3	B	2	2	$P_3 = \mathrm{C}^2\mathrm{JC}$	B	22–30
4	B	7	8	$P_4 = \mathrm{CQCAC}^3\mathrm{QKCAC}^4$	A	30–22
5	A	3	2	$P_5 = \mathrm{C}^3\mathrm{JC}$	B	27–25
6	B	1	3	$P_6 = \mathrm{CQC}^2$	A	30–22

For example, the fourth trick has the following sequence of face cards: player A's queen, B's ace, A's queen, B's king, A's ace. The gaps between them are $1, 3, 0, 1$, so every interruption is legal; B then pays four ordinary cards. The complete card-by-card trace is in the accompanying JSON and is also printed by the verifier.

2.2 Why both queues return

Set

$$U = \mathrm{KCAC}^5\mathrm{QC}^2, \qquad V = \mathrm{CJC}.$$

The starting words have the useful factorizations

$$\boxed{H_A = UP_4P_6, \qquad H_B = VP_1P_2P_3P_5.}$$

During the six tricks each player plays exactly 19 cards. No collected card has yet reached the front of either queue: A starts with 30 cards and B with 22. Their unplayed original suffixes are precisely U and V. Their captured piles are appended in the orders shown in the boxed identities. Consequently

$$T^6(H_A, H_B, A) = (UP_4P_6,\ VP_1P_2P_3P_5,\ A) = (H_A, H_B, A).$$

The pile lengths are $4, 6, 4, 15, 5, 4$, totaling 38 card plays. The winner word $BBBABA$ has least period six. A shorter state period dividing six would force the winner word to have that shorter period, so the displayed state has primitive period exactly six.

3 An equal deal enters the cycle after one trick

The following two words both have length 26; A leads:

```
A: ACCCQKCACCCCCQCCCQCACCCQKC
B: CCCCCJCCCCJCKCJCCAKCCCCCJC
```

A plays an ace, B pays four ordinary cards, and A collects AC^4. The resulting hands are

```
A: CCCQKCACCCCCQCCCQCACCCQKCACCCC
B: CJCCCCJCKCJCCAKCCCCCJC
```

with A again leading. This is exactly the boundary following trick 4 in the six-cycle. Thus the equal deal has a one-trick transient followed by primitive period six. It is not necessary for a 26–26 boundary itself to occur on the periodic orbit.

4 Word equations for a returning FIFO state

The central combinatorial simplification is independent of the card rules. Suppose one player begins with a queue S, plays a chronological word D during a proposed period, and collects a concatenated word R of captured piles during that period. If no collected cards have reached the front yet, then the return equation is $SR = DS$. More generally, the same equation holds as a concatenation of the chronological queue flow when D is taken to be the entire word removed from the queue during that interval.

Lemma 1 (Queue return equation)**.** *Let S, D, R be finite words with D nonempty and $|D| = |R|$. Then*

$$SR = DS \quad \Longleftrightarrow \quad S = \mathrm{pref}_{|S|}(D^\infty), \qquad R = \mathrm{rot}_{|S| \bmod |D|}(D).$$

Proof. The equality $SR = DS$ says that D is a prefix of SR and that deleting this prefix and adding R returns to S. Iterating the word equality yields that S is a prefix of the infinite periodic word D^∞. More explicitly, write $|S| = q|D| + r$ and $D = XY$ with $|X| = r$. Necessarily $S = D^q X$ and $R = YX$; conversely, then $SR = D^q XYX = D^{q+1}X = DS$. □

This lemma imposes only FIFO closure; legality of the intervening face-card challenge sequence must still be verified. In our six-trick witness, each player plays 19 cards, and their original queues are longer than 19. The packet identities in Section 2 instantiate the equation without replaying collected cards during the cycle.

5 Backward accessibility from a balanced deal

For the six-trick orbit, the complete inverse image under the completed-trick map can be enumerated directly, without a solver. Given a current state and its most recent winner, the most recent captured pile is a suffix of the winner's hand. For each possible nonempty suffix and either preceding leader, we recover the unique prospective ownership sequence from the face-payment rules; a candidate predecessor is accepted only if one forward trick reproduces the current state. Iterating these operations from all six cycle states gives the entire backward component.

Proposition 1 (Backward component of the displayed orbit)**.** *The six-trick orbit has exactly 181 backward-reachable rank states, including the six periodic states. Exactly 19 have a 26–26 hand split. The maximum distance from a backward-reachable state to the cycle is four completed tricks. Among states with player A leading, the A-hand sizes include every integer from 20 through 33.*

The proposition is certified by the standard-library script `backward_component.py`, which reconstructs all predecessors and forward-replays each one. Combining it with the global six-trick lower bound shows that an eventual period of six can be attained from all A-leading splits 20–32 through 33–19, and the explicit balanced deal above enters the cycle in one trick. The script also records the complete list of 19 balanced starts for independent inspection. These counts refer to the displayed named-player orientation. Section 10 accounts for its player-swapped copy and both components together. The global uniqueness theorem identifies these as all orbits attaining either minimum.

6 Why the face-itinerary search is exhaustive

Lemma 2 (Faces and winners)**.** *On a periodic orbit, both players win at least once, and each hand contains a face card at every completed-trick boundary. Within a trick the owners of successive face cards alternate. The final face card belongs to the winner.*

Proof. If a player never wins, their finite hand loses cards and cannot return. If a hand ever has no face card, it cannot win another pile and hence cannot acquire another face card. It therefore cannot lie on a periodic orbit. After a face is played, the other player must answer; any interrupting face is consequently owned by the other player. The last face makes the successful claim when the ensuing payment is completed. □

Suppose trick i has winner w_i and $m_i = 2k_i + e_i$ face cards, with $e_i \in \{0, 1\}$. Alternation implies that the loser contributes k_i faces, and the winner contributes $k_i + e_i$. Hence the winner gains exactly k_i faces. The loser must retain a face; no face in the pile can be used again before collection. Thus

$$1 \le m_i \le 15, \qquad 0 \le k_i \le 7.$$

Face balance over a return requires

$$\sum_{i:w_i=A} k_i = \sum_{i:w_i=B} k_i.$$

For fixed p, the exhaustive enumeration is therefore finite:

1. Enumerate nonconstant binary winner words of length p, up to cyclic rotation and exchange of the players. Choose a representative ending in A, so the initial leader is A.
2. Enumerate $k_i \in \{0, \dots, 7\}$ and $e_i \in \{0, 1\}$ satisfying the displayed bounds and face balance.
3. Try every initial A-face count $a = 1, \dots, 15$. Give the 16 faces distinct temporary labels and simulate their FIFO movement. Given w_i and m_i, the owner of face j of trick i is forced by alternation. Reject impossible queues and any boundary with no face in one hand.
4. At a possible return, flatten the two face queues into 16 slots. This gives a permutation of the initially labelled faces. Every permutation cycle must be monochromatic in rank. Its cycle sizes must therefore fit into four bins, each of total weight four.
5. For every surviving face itinerary, solve the exact ordinary-gap constraints described next, with total ordinary count 36.

This covers every initial card split. It enumerates the number of faces in each initial hand, but puts no restriction on the number of ordinary cards assigned to either hand beyond nonnegativity and their combined inventory.

7 Exact ordinary-gap constraints

With a faces in A's hand, write its word as

$$\mathrm{C}^{g_0} f_1 \mathrm{C}^{g_1} f_2 \cdots f_a \mathrm{C}^{g_a},$$

and similarly for B. The two hands together have exactly $16+2 = 18$ gaps. All initial gaps are nonnegative integers. For the standard deck their sum is 36. The gap model implements the following three operations exactly.

Before the first face. Let x be the current leading gap of the player who will expose the first face. Let $\delta = 0$ if this player is the leader and $\delta = 1$ otherwise. The other player's leading gap must be at least $x + \delta$; these ordinary cards are consumed. The pile begins with $2x + \delta$ ordinary cards.

Interruptions and final payment. After a face of penalty r, a prescribed interrupting face can occur after a leading gap g exactly when $0 \le g < r$. The gap is consumed and placed between the two faces in the pile. If no interruption is prescribed, the loser's leading gap must be at least r; exactly r cards are consumed and the pile ends in C^r.

Collection and closure. Append the pile to its winner, merging its first ordinary gap with the winner's existing trailing gap. Carry on with the winner as leader. At the end, require all 18 gaps to equal their original values. Rank equality is already imposed by giving each orbit of the face permutation one rank.

These are linear integer constraints, including when the rank attached to a permutation orbit is a variable in $\{1, 2, 3, 4\}$. For each rank r, the sum of the sizes of its assigned face orbits is constrained to equal four. The formulation is both necessary and sufficient for a legal return along the prescribed face itinerary. There is no floating-point tolerance or sampling in this calculation.

Counting individual plays. The same gap bookkeeping counts every play exactly. For trick i, let g_i be the leading ordinary gap of its first face supplier, let δ_i be one when that supplier is the nonleader and zero otherwise, let h_{ij} be the ordinary gap before face $j \geq 2$, and let r_i be the final face's penalty. The pile length is

$$\ell_i = m_i + 2g_i + \delta_i + \sum_{j=2}^{m_i} h_{ij} + r_i. \tag{1}$$

For a fixed itinerary this is an affine integer expression. Thus a bound on the total individual-play period is imposed exactly by adding $\sum_i \ell_i \leq B$ to the gap system.

7.1 The finite certificate for periods below six

Candidate period	Face itineraries	Rank assignments represented	Feasible at 36 Cs
2	2	48	0
3	10	288	0
4	163	20,328	0
5	1,026	134,592	0
Total	1,201	155,256	0

A one-trick return is impossible because its winner strictly gains cards. The table excludes every possible return of length two through five. Together with the explicit six-cycle, this proves the minimum trick-period assertion of Theorem 1. The table counts itineraries after symmetry and face-inventory filtering, not all physical deals and not distinct periodic orbits. The standalone checker regenerates the table from scratch; no precomputed candidate list is required.

8 Stronger short-period structure

The same calculation can leave the number of ordinary cards unrestricted. Periods two, three and five are impossible for *every* ordinary-card count when there are four cards of each face rank. Period four has an exact description.

8.1 Elementary obstructions for periods two and three

For period two, only two face itineraries survive the inventory test. Up to shifting the initial boundary and exchanging the players, the first has nine A-faces and trick face counts $(8, 9)$. If a_0, b_0 are initial leading gaps, B's leading gap returns as $b_0 - a_0 - r$ for a positive face penalty r. Closure would require $a_0 + r = 0$, which is impossible. The second itinerary is the same obstruction with eleven A-faces and counts $(9, 8)$.

For period three the winner representative is BBA. The ten admissible itineraries have the following immediate gap contradictions. Initially label the A-faces $0, \ldots, a-1$ and B-faces $a, \ldots, 15$; $r_j > 0$ is the penalty of label j. The symbol Δ denotes final minus initial gap.

A-faces	Face counts in the three tricks	Required gap difference
11	$(1,8,8), (3,6,8), (5,4,8), (7,2,8)$	$\Delta a_4 = -2b_0 - b_5 - r_{15} - 1 < 0$
9	$(2,6,9), (4,4,9), (6,2,9)$	$\Delta b_0 = -a_0 - r_8 < 0$
13	$(4,8,13), (6,6,13)$	$\Delta b_0 = -a_0 - r_{12} < 0$
11	$(7,1,7)$	$\Delta b_4 = b_4 + r_{14} > 0$

Each contradicts the required return of that gap.

8.2 An exact classification for period four

Proposition 2. *With four copies of each face rank and c ordinary cards, a four-trick cycle exists if and only if*

$$\boxed{c = 104 + 31n, \qquad n \in \mathbb{Z}_{\geq 0}.}$$

The complete face enumeration and unrestricted gap check leave only one four-trick itinerary, up to the symmetries already used: winners $BABA$, one face per trick, and initial face words $(rs)^4$ and $(tu)^4$. Here (r, s, t, u) is any permutation of the four penalties. This finite exclusion is computer-assisted. The remaining family can be solved algebraically.

Let q be B's initial leading ordinary gap. Define

$$\begin{aligned} E &= 2q - s + 2t + 1, & O &= 2q + r - 2s + 2u + 1, \\ D &= 2q + 2r - 2s + t + 1, & F &= 2q + u + 1. \end{aligned}$$

Return forces the two gap vectors to be

$$g_A = (E, O, E, O, E, O, E, O, s), \quad g_B = (q, D, F, D, F, D, F, D, u).$$

All legal-play inequalities reduce to

$$q \geq 0, \qquad q + r \geq s, \quad q + t \geq s, \quad q + u \geq s.$$

Since (r, s, t, u) permutes $1, 2, 3, 4$, these are equivalent to $q \geq s - 1$. Adding the gaps gives

$$c = 135 + 31(q - s) = 104 + 31n, \qquad n = q - s + 1 \geq 0.$$

Conversely the displayed gap words play as prescribed for every $n \geq 0$ and every rank permutation. Their period is four, since period two is impossible for this face inventory. In particular, the standard count $c = 36$ is excluded by a substantial margin.

9 The six-trick pattern is rigid

Suppress ordinary cards in the six-cycle and write the face words as

$$H_A : \ xyzzyzxyz, \qquad H_B : \ wxwyxww.$$

The trick face counts are $(2, 2, 1, 5, 1, 1)$, with winners $BBBABA$. The physical-face return permutation has four cycles of length four. Thus (x, y, z, w) must permute $(1, 2, 3, 4)$ if the inventory has four of each rank.

9.1 An elementary proof for the standard inventory

Write A's ordinary gaps as $(a_0, \ldots, a_9)$ and B's as $(b_0, \ldots, b_7)$. The linear gap-return equations have rank 17. With $u = b_6$, all their solutions are the affine formulas in Appendix A. Adding those 18 formulas gives the following identity for the total ordinary count C:

$$12C = 115u + 34x + 2y - 100z + 125w - 63. \tag{2}$$

Proposition 3 (Standard-inventory rigidity)**.** *For the prescribed face itinerary, the standard inventory admits exactly one rank-and-gap realization. It has* $(x, y, z, w) = (3, 4, 2, 1)$ *and* $u = 4$*, and is the witness of Section 2.*

Proof. Substitute $C = 36$ and $w = 10 - x - y - z$ in (2). This gives

$$115u = 91x + 123y + 225z - 755.$$

Because u is an integer,

$$-24x + 8y - 5z \equiv 65 \pmod{115}.$$

The bounds $1 \le x, y, z \le 4$ imply

$$-108 \le -24x + 8y - 5z \le 3.$$

The only integer in this interval congruent to 65 modulo 115 is -50. Hence $24x - 8y + 5z = 50$. Reduction modulo 8 forces $z = 2$, leaving $3x - y = 5$. The possibilities in $\{1, 2, 3, 4\}$ are $(x, y) = (2, 1)$ and $(3, 4)$; distinctness from $z = 2$ excludes the first. Consequently

$$\boxed{(x, y, z, w) = (3, 4, 2, 1) = (\mathrm{K}, \mathrm{A}, \mathrm{Q}, \mathrm{J}), \qquad u = 4.}$$

The affine formulas now fix all gaps:

$$g_A = (0, 1, 5, 3, 1, 3, 0, 1, 5, 2), \qquad g_B = (1, 1, 1, 2, 0, 5, 4, 1).$$

These are nonnegative integers totaling 36. The direct play in Section 2 proves legality and return, establishing existence as well as uniqueness. □

The standard-library checker `check_algebraic_rigidity.py` independently derives the 18 return equations by symbolic FIFO transport, verifies the physical face orbits, performs exact rational row reduction, checks the complete affine parameterization and (2), and verifies the modular deduction and a direct labelled-face replay. This positive-case rigidity proof does not depend on an SMT solver's second-model exclusion.

9.2 Rigidity even with unrestricted ordinary count

For this particular itinerary, *without specifying an ordinary-card count*, there is only one legal periodic realization: $(x, y, z, w) = (3, 4, 2, 1)$ and the 36 ordinary cards displayed above. The accompanying `check_rigidity.py` verifies this by exact rational arithmetic and ordinary forward play, without an SMT solver.

Here is a finite bound underlying that check. Let $u = b_6$ be B's gap before its last face. Linear return equations express every gap in u, x, y, z, w; in particular

$$a_0 = \frac{5u - 5w + 2x - 2y - 8z + 3}{12}.$$

The second trick requires $a_0 < y$, so

$$5u < 5w - 2x + 14y + 8z - 3 \le 85.$$

Consequently $u \in \{0, \ldots, 16\}$. All $24 \cdot 17$ possibilities can be checked with rational arithmetic. Only 21 produce nonnegative integral gap vectors; exactly one has legal play with the prescribed return.

For the successful ranks, all nonnegative integral solutions of the return *equalities* have the form

$$\begin{aligned} g_A &= (5n, 1+5n, 5+8n, 3+10n, 1+5n, \\ &\qquad 3+10n, 5n, 1+5n, 5+8n, 2), \\ g_B &= (1+8n, 1+10n, 1+3n, 2+10n, \\ &\qquad 5n, 5+6n, 4+12n, 1), \qquad n \geq 0. \end{aligned}$$

Their total ordinary count is $36 + 115n$. Already the first trick requires B's jack to interrupt A's king after $b_0 - a_0 = 1 + 3n$ ordinary cards. Thus $1 + 3n < 3$, forcing $n = 0$. This rational calculation certifies uniqueness for the one face-card itinerary considered here. The next section shows that no other six-trick itinerary admits any legal standard-deck realization.

10 Complete classification of the six-trick cycles

We now complete the period-six classification rather than merely exhibiting one cycle. The equivalence relation identifies cyclic shifts of the trick boundary and the exchange of players A and B. We continue to identify equal face ranks and ordinary cards. Distinguishing the physical card labels gives a different periodicity problem; it is not the convention used here.

Theorem 2 (Complete classification at six tricks)**.** *For the standard inventory* $C^{36}J^4Q^4K^4A^4$*, there is, up to cyclic time shift and exchange of the two players,* exactly one *primitive six-trick cycle. Its pile lengths are* $4, 6, 4, 15, 5, 4$*, totaling 38 individual plays. With player names fixed, there are exactly two distinct oriented periodic orbits (each containing six rank states); they are exchanged by interchanging A and B. Neither periodic orbit contains a* 26–26 *boundary, although both are accessible from balanced deals.*

Computer-assisted proof. By the face-only reduction established above, choose a winner word up to cyclic rotation and player exchange, with the last winner A (hence the initial leader A). There are exactly seven such length-six words. For each word enumerate the transfer values $0 \leq k_i \leq 7$, parity bits $e_i \in \{0, 1\}$ with $1 \leq 2k_i + e_i \leq 15$, balanced net face transfers, and initial A-face count $1 \leq a \leq 15$. Simulate every resulting face-only FIFO itinerary, including repeated appearance of a face card in separate tricks when allowed. Its return permutation on 16 distinct temporary face labels must be colorable by four ranks, each with multiplicity four.

An independent optimized C++ implementation and the original Python implementation agree *as sets* on all surviving itineraries for each winner-word type. The resulting counts are

Canonical winner word (B=1, A=0)	Face skeletons
111110	199
111100	2361
111010	2121
111000	5286
110110	2489
110100	4293
101010	4658
Total	21,407

These skeletons collectively represent 2,688,096 rank colorings, before imposing the ordinary gaps. For each skeleton, the 18 ordinary gaps are nonnegative integers summing to 36. The exact linear feasibility model of Section 7 imposes all interruptions, completed payments, queue availability, collection order,

and equality of the final gaps with the initial gaps. Native Z3 4.13.3 reports `unsat` for 21,406 skeletons and `sat` for exactly one. Every non-`sat`/`unsat` solver result is an error, not an exclusion.

The sole feasible skeleton has initial A-face count nine, winner word 111010 ($BBBABA$), and face counts $(2, 2, 1, 5, 1, 1)$. Its face-label return permutation has the four four-cycles

$$(0, 6, 13, 10),\ (1, 7, 4, 12),\ (2, 8, 5, 3),\ (9, 15, 14, 11).$$

Model enumeration yields exactly one solution of the entire system: after adding a blocking clause for the rank and gap assignment, Z3 reports `unsat`. The unique solution has orbit ranks $(3, 4, 2, 1)$ and ordinary-gap vector

$$(0, 1, 5, 3, 1, 3, 0, 1, 5, 2,\ 1, 1, 1, 2, 0, 5, 4, 1).$$

It reconstructs *exactly* the two hands in Section 2. Proposition 3 independently proves uniqueness for the standard inventory by modular arithmetic. The additional rational-arithmetic certificate of the preceding section establishes uniqueness even without a prior restriction on the number of ordinary cards.

The six displayed boundary states have distinct, primitive winner-word rotations. Their player-swapped states are not among those six, because the complement of $BBBABA$ is not a cyclic rotation of it. These are thus exactly two disjoint six-state orbits when A and B are fixed, or one symmetry class under exchange of players. The published standard-deck six-cycle contains no 26–26 boundary, as direct simulation shows; a 26–26 predecessor was exhibited above. □

Balanced starting deals. The backward component of each oriented orbit contains 181 states, of which 19 have a balanced hand split. These two components cannot intersect, since the completed-trick map is deterministic and their terminal cycles are disjoint. Consequently there are 38 distinct balanced rank states (including both possible leaders) whose eventual period is six: 19 in each component. Exactly 19 of these have A as the next leader, and 19 have B as the next leader. This strengthens the previously stated accessibility result but does not claim that every possible 26–26 *labelled* deal has period six.

11 Global minimum and uniqueness in individual plays

To compare the six-trick witness with cycles having more tricks, we classify every return with at most 38 plays, without restricting its number of tricks or its hand split.

11.1 An elementary finite range

Lemma 3 (Plays versus changes of winner)**.** *If a periodic return has p completed tricks, L individual plays, and c changes of winner in its cyclic winner word, then*

$$L \geq 2p + c \geq 2p + 2.$$

Proof. Every completed trick needs at least two plays. A two-play trick is the leader's jack followed by the other player's ordinary card. Its winner therefore equals its leader, who is the previous winner. Each change of winner requires at least three plays. Both players win on a periodic orbit, so the cyclic winner word has at least two changes. Summing the per-trick bounds proves the claim. □

Consequently $L \leq 38$ implies $p \leq 18$. A one-trick return is impossible because its winner gains cards. It is therefore sufficient to consider $p = 2, \ldots, 18$, including returns whose winner word has a smaller period than the full state. No primitivity assumption is used to discard cases.

11.2 Enumeration and exact play accounting

Use the exhaustive face-itinerary representation of Section 6. For a fixed winner word and face counts, let δ_i indicate that the first face of trick i belongs to its nonleader. Before solving the gap system, the necessary bound

$$L \geq \sum_i (m_i + \delta_i) + \min_\rho \sum_i \rho(f_i^{\text{last}})$$

can discard itineraries. Here f_i^{last} is the final labelled face, and ρ ranges over ranks constant on the returning face-permutation orbits, with inventory four for each rank. The minimum is computed exactly by a finite inventory dynamic program. It uses only the forced faces, initial ordinary offsets, and final payments; any remaining ordinary plays can only increase the total.

Partial itineraries also admit safe necessary bounds. Each remaining trick costs at least two plays. If r tricks remain and the current A-face count differs from its initial value by Δ, restoration needs at least $|\Delta|$ further face transfers. Those transfers require at least $2|\Delta|$ face events, so the remaining plays cost at least $2|\Delta| + r$. These bounds accelerate enumeration without eliminating a legal return.

The programs `bounded_play_skeletons.cpp` and `independent_play_skeletons.cpp` implement separate complete enumerations. Their surviving candidate sets agree exactly for every $p = 2, \ldots, 18$, after the same cyclic-phase and player-exchange reductions. For each surviving itinerary, the model imposes all ordinary-gap constraints of Section 7, the standard count 36, all rank inventories, and the exact bound $\sum_i \ell_i \leq 38$ from (1).

Tricks	Candidate itineraries	Feasible itineraries	Rank/gap models
2	2	0	0
3	10	0	0
4	145	0	0
5	565	0	0
6	4,234	1	1
7	7,119	0	0
8	13,140	0	0
9	8,530	0	0
10	5,178	0	0
11	3,340	0	0
12	1,340	0	0
13	166	0	0
14	35	0	0
15	4	0	0
16	0	0	0
17	0	0	0
18	0	0	0
Total	43,808	1	1

These counts include the necessary 38-play filters, so they are smaller than the unrestricted face-itinerary counts used for the earlier trick-period results. The global enumeration has 43,807 infeasible cases and one feasible case. Every solver response other than `sat` or `unsat` aborts the classification; a timeout is never an exclusion.

For every satisfiable case, `verify_unique38.py` records a model, blocks the complete assignment of all initial gaps and all orbit ranks, and continues until unsatisfiable. The sole feasible case has 22 essential

variables: 18 ordinary gaps and four orbit ranks. Blocking its full assignment yields `unsat`, so no second rank-and-gap realization is hidden in that case. The positive model and its blocked query were checked with Z3 5.4.1 and native Z3 4.13.3. Independently, Proposition 3 proves the same uniqueness by arithmetic. The sole realization is the six-trick witness and has exactly 38 plays.

Theorem 3 (Unique global minimum in individual plays)**.** *For the standard inventory, every rank-state cycle has at least 38 individual plays. Up to cyclic phase and exchange of players, exactly one cycle attains 38 plays; it is the six-trick cycle of Section 2.*

Computer-assisted exclusion and elementary positive case. Lemma 3 reduces every return with at most 38 plays to the finite range just classified. The exhaustive face enumeration and exact gap constraints leave only the displayed face itinerary. Proposition 3 gives its unique rank-and-gap realization, and direct play shows that this realization uses 38 plays. Thus there is no shorter cycle and no other equivalence class at equality. □

The equivalence relation uses only cyclic phase and exchange of the two players; different face ranks are not identified and queues are not reversed. Player exchange acts on both hands and the next leader. Since the complement of $BBBABA$ is not a cyclic rotation, the equality class has exactly two named-player orientations and twelve completed-trick boundary states. The balanced predecessors and backward components described above therefore account for all cycles attaining either of the two global minima. Together with Theorem 2, this completes Theorem 1.

12 General structural consequences

12.1 Fixed-period deck sizes form a semilinear set

Theorem 4. *Fix a finite face-card inventory and a positive integer p. The set of ordinary-card counts for which a completed-trick rank state satisfies $T^p(s) = s$ is effectively semilinear, and hence eventually periodic. The same holds for counts admitting a state of primitive period exactly p. More generally, the pairs (ordinary-card count, initial A-hand size) form an effectively semilinear subset of $\mathbb{N}^2$.*

Proof. Let f be the fixed number of face cards. Both players retain a face at each periodic boundary, so a trick contains at most $f - 1$ faces. For fixed p there are only finitely many possible face itineraries and rank assignments. Each is described by linear integer conditions on its $f + 2$ ordinary gaps. Their finite union is therefore definable in Presburger arithmetic. Adding the equations for total ordinary count and initial A-hand size, then projecting out the gaps, preserves Presburger definability. Presburger sets are effectively semilinear [4, Theorem 1.3].

For primitive period p, exclude every proper-divisor return on the *same initial state variables before projection*. Along a fixed itinerary, such a return is either impossible because of the leader or face word, or is expressed by linear gap equalities. Negating these equalities remains Presburger-definable. One must not subtract sets of ordinary counts after projection: one count can support cycles of more than one primitive period. □

The progression $104 + 31n$ for period four is a concrete example. The result also explains why an exact search based on gaps can classify families with arbitrarily many ordinary cards, rather than requiring a finite deck-size cutoff.

12.2 Lifting a rank cycle to individually labelled cards

If a rank state returns after p tricks, give each card a distinct label and follow those same p tricks. The resulting card-slot permutation preserves the game-relevant types. Repeating the rank itinerary applies the same permutation again. Hence the labelled period equals p times the order of this permutation when p is the primitive rank period.

For the displayed six-cycle the face-label cycles have lengths $4, 4, 4, 4$; the ordinary-label cycles have lengths $3, 3, 3, 3, 4, 9, 11$. Their least common multiple is 396, so the labelled-card period of this example is

$$6 \cdot 396 = 2376 \text{ tricks},$$

or $396 \cdot 38 = 15{,}048$ individual plays. This does not make 2376 a claimed minimum among labelled-card cycles.

13 Reproduction and scope of verification

13.1 Global classification through 38 plays

The repository contains both C++ enumerators, the compressed candidate set, the exact play-count model, the all-model classifier, the independent algebraic checker, and complete reports under `verification/play_period/`. With Python 3, `z3-solver`, and a C++20 compiler, the following commands regenerate both complete candidate sets, compare them exactly, classify every candidate, and check the elementary positive-case proof:

```
python3 verify_unique38.py --crosscheck --workers 2 \
  --json verification/play_period/fresh_crosscheck.json
python3 check_algebraic_rigidity.py \
  --json verification/play_period/algebraic_rigidity.json
```

The Make target `crosscheck-plays` runs the full crosscheck. The target `verify-algebraic` runs the arithmetic certificate. The target `verify-plays` checks the committed complete candidate archive and reruns all feasibility and model-exhaustion checks. The reports record all 43,808 candidates, the sole 22-variable assignment, its positive and blocked SMT queries, and the direct replay. Continuous integration runs the full enumeration crosscheck and publishes the verification report and those queries.

13.2 Trick periods and the structural results

The portable script `prove_minimum.py` requires Python 3 and `z3-solver`. It regenerates all face itineraries for periods two through five, checks their exact integer constraints, verifies the explicit six-cycle with two implementations of play, and checks the balanced entry. It raises an error on any solver result other than `unsat` in the lower-bound checks. No timeout or unresolved case is counted as an exclusion.

```
python -m pip install z3-solver
python prove_minimum.py --json minimum_verification.json
python prove_minimum.py --structural
python check_rigidity.py --json rigidity_verification.json
```

The optional `-smt-dir DIRECTORY` argument exports every ordinary-gap problem in SMT-LIB format for inspection or another solver. The optional structural checks remove the ordinary-count restriction for periods two, three and five and verify the period-four classification.

A second backend, `native_z3_check.py`, reuses the same exact gap models but serializes them directly as SMT-LIB and calls the native Z3 C library through Python's `ctypes`. It does not require the Python `z3-solver` package. A fresh run with native Z3 4.13.3 returned `unsat` for all 1,201 candidate face itineraries for periods two through five. This verifies the feasibility computations using a distinct solver interface, but it does not replace the need to audit the underlying model reduction.

To reproduce the exhaustive *period-six classification*, generate the face-only skeletons with the independent C++ enumerator and check them using the native solver interface:

```
c++ -O3 -std=c++17 skeleton6.cpp -o skeleton6
./skeleton6 skeleton6.tsv
python classify_six.py --json verification/six_full.json
python enumerate_models.py --json verification/six_models.json
python check_rigidity.py
```

The complete enumeration was also partitioned into disjoint consecutive ranges of skeleton indices, with all 21,407 cases covered. The `crosscheck_skeletons.py` script reproduces each of seven winner-word classes using the original Python face generator; the resulting sets of candidate skeletons agree exactly, not merely in size. These data and classification scripts are included in the repository.

The backward components and player-swap symmetry can be independently reproduced without third-party Python packages:

```
python backward_component.py --json verification/backward_component.json
python verify_orbit_symmetry.py --json verification/orbit_symmetry.json
```

The native solver path is reproduced separately with `python native_z3_check.py` and requires an installed Z3 shared library, but no Python Z3 bindings.

13.3 What is independently checked

The original trick-period proof was crosschecked using independently written gap bookkeeping, an independent winner-necklace enumerator, and two direct game simulators. The global individual-play calculation additionally compares two independently implemented complete face-itinerary enumerators as sets. The exact gap model is shared with the established trick-period framework; the algebraic rigidity checker independently derives its positive case's return equations. The positive model and blocked second-model query also have a separate native-Z3 check. These are reproducible computer-assisted proofs, not proof-assistant formalizations or collections of independently checked SMT proof derivations. The witness, balanced entry, and modular rigidity argument can be checked directly from the displayed words and equations.

14 Conclusion

For the standard inventory and the stated rank-state convention, the unique six-trick cycle is also the unique cycle with globally minimal individual-play period 38, up to phase and player exchange. The unrestricted six-trick classification and the global 38-play classification establish two distinct optimization

statements, attained by the same orbit. The two named-player orientations have twelve boundary states in total and exactly 38 balanced predecessors across their disjoint backward components. The proof of global exclusion is computer-assisted, while rigidity of its sole surviving itinerary follows from the short congruence argument above. We do not establish a minimum for fully distinguished physical-card periods or reachability from every prescribed initial hand split.

A Complete affine return formulas for the surviving itinerary

Let $u = b_6$. Applying the gap transport rules of Section 7 to the face itinerary with initial words $xyzzyzxyz$ and $wxwyxww$, winners $BBBABA$, and face counts $(2, 2, 1, 5, 1, 1)$ gives the complete parameterization

$$g_A = (a_0, a_1, a_2, a_3, a_1, a_3, a_0, a_1, a_2, z),$$
$$g_B = (b_0, b_1, b_2, b_3, a_0, b_5, u, w),$$

where

$$a_0 = \frac{5u - 5w + 2x - 2y - 8z + 3}{12}, \quad a_1 = a_0 + 2w - 1,$$
$$a_2 = \frac{2u + 4w + 2x + y - 2z - 3}{3}, \quad b_1 = 2a_0 + w,$$
$$a_3 = \frac{5u + 7w + 2x - 2y - 2z - 3}{6}, \quad b_3 = 2a_1 - w + 1,$$
$$b_0 = \frac{2u - 2w - x + y - 2z}{3},$$
$$b_2 = \frac{u - w - 2x + 2y - 1}{4},$$
$$b_5 = \frac{u - w + 2y - 1}{2}.$$

The coefficient matrix of the 18 gap-return equations has rank 17, with b_6 its only free coordinate. Substitution verifies the displayed family for every u, x, y, z, w, proving completeness of the parameterization. These claims are checked by exact rational row reduction and symbolic FIFO transport in `check_algebraic_rigidity.py`. Summing all 18 entries directly yields

$$12C = 115u + 34x + 2y - 100z + 125w - 63,$$

the identity used in Proposition 3.

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