Counterexamples to the conjectured characterization of Hamiltonian degree sequences in digraphs

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First published November 4, 2026; this version November 5, 2026 (v2)
Zenodo archive (Lean development in `lean`, `checks`, `data`): https://doi.org/10.5281/zenodo.23969956

hexagon:2611.00844v1

Abstract

We show that the conjecture of Kühn, Osthus and Treglown (2008), proposed as a digraph analogue of Chvátal's theorem, fails for every order $n \geq 7$ and holds for $3 \leq n \leq 6$. It states that every digraph whose separately sorted out- and in-degree sequences satisfy two Chvátal-type conditions, and a third condition when the order is even, has a Hamilton cycle. It strengthens Nash-Williams' conjecture of 1975, which assumes strong connectivity instead of the third condition. Guninski found 6-vertex counterexamples to Nash-Williams' conjecture on MathOverflow in 2013 and 2020; the 2013 example was confirmed there by Royle and Thomas and the 2020 ones by DeBiasio. All of them violate the third condition. Our counterexamples are an explicit 7-vertex digraph and an explicit family, and the small orders are settled by exhaustive search. Hence Nash-Williams' conjecture fails for every $n \geq 7$, in addition to Guninski's case $n = 6$, and the same search shows that it holds for $3 \leq n \leq 5$. The family also shows that for every $n \geq 8$ the conditions of either conjecture do not even force a cycle through every pair of vertices, answering a question recorded by Kühn and Osthus after Bermond and Thomassen. The counterexamples are checked in Lean.

This work has not yet been reviewed by experts. The proofs, the Lean formalization and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

2020 Mathematics Subject Classification. 05C20, 05C45, 05C07.
Keywords. Hamilton cycle, digraph, degree sequence, Chvátal's theorem, Nash-Williams' conjecture.

1 Introduction

1.1 The conjectures

Chvátal's theorem [4] says that a graph on $n \geq 3$ vertices with degrees $d_1 \leq \cdots \leq d_n$ has a Hamilton cycle if, for every $i < n/2$, $d_i \geq i + 1$ or $d_{n-i} \geq n - i$. In the words of Kühn, Osthus and Treglown [12], "This condition on the degree sequence is best possible in the sense that for any degree sequence violating this condition there is a corresponding graph with no Hamilton cycle." More precisely, as they add in version 2 of [12], for every graphical sequence violating the condition there is a non-Hamiltonian graph whose degree sequence $d'_1 \leq \cdots \leq d'_n$ satisfies $d'_i \geq d_i$ for all i. So the condition is sufficient, and best possible among conditions preserved when degrees increase, but it is not necessary: $(2, 2, 2, 2, 2)$ violates it at $i = 2$, since $d_2 \leq 2$ and $d_3 < 3$, yet its only realization,

the 5-cycle, is Hamiltonian. We say that a condition on degree sequences forces Hamiltonicity if every (di)graph whose degree sequences satisfy it has a Hamilton cycle.

For digraphs we follow the conventions of [12]: "The digraphs we consider do not have loops and we allow at most one edge in each direction between any pair of vertices." We call the edges arcs and write $u \to v$ for an arc from u to v. For a digraph G on n vertices, $d_1^+ \le \cdots \le d_n^+$ is its out-degree sequence and $d_1^- \le \cdots \le d_n^-$ its in-degree sequence, each sorted separately; as [12] notes, "the terms d_i^+ and d_i^- do not necessarily correspond to the degree of the same vertex of G." Consider the following conditions, where (i) and (ii) are required for every integer i with $1 \le i < n/2$:

(i) $d_i^+ \ge i+1$ or $d_{n-i}^- \ge n-i$;

(ii) $d_i^- \ge i+1$ or $d_{n-i}^+ \ge n-i$;

(iii) (only when n is even) $d_{n/2}^+ \ge n/2$ or $d_{n/2}^- \ge n/2$.

In words, (i) says: if at least i vertices have out-degree at most i, then at least $i+1$ vertices have in-degree at least $n-i$. Condition (ii) is the same with out- and in-degrees exchanged.

Conjecture A (Nash-Williams [15]; [12, Conjecture 1], [11, Conjecture 6])**.** *Every strongly connected digraph on $n \ge 3$ vertices satisfying (i) and (ii) has a Hamilton cycle.*

Nash-Williams observed that doubling every edge of a graph shows that Conjecture A would imply Chvátal's theorem (see [12, Section 2]). Kühn, Osthus and Treglown observed that with the connectivity assumption Conjecture A "does not seem to be a precise digraph analogue of Chvátal's theorem: in such an analogue, we would ask for a complete characterization of all digraph degree sequences which force Hamiltonicity" [12, Section 2]. They replaced strong connectivity by condition (iii) and showed that (i)–(iii) imply strong connectivity, so that Conjecture B below would follow from Conjecture A. They also showed that each condition is needed: for each $i < n/2$ they give a non-Hamiltonian digraph satisfying all the conditions except (i) at index i, and for even n one satisfying (i) and (ii) but not (iii) [12, Section 2]. They wrote: "If true, the following conjecture would provide the desired characterization." As for Chvátal's theorem, the characterization meant is a sufficient condition that is best possible in this sense, not an exact test: the degree sequences of a directed triangle violate (i) for $i = 1$, yet every digraph with these degree sequences is a directed triangle and so is Hamiltonian.

Conjecture B (Kühn, Osthus and Treglown [12, Conjecture 6 of version 1; Conjecture 8 of version 2])**.** *Every digraph on $n \ge 3$ vertices satisfying (i), (ii) and, if n is even, (iii) has a Hamilton cycle.*

1.2 Main results

Theorem 1. *(a) For every $n \ge 7$ there is a strongly connected digraph on n vertices that satisfies (i), (ii) and, if n is even, (iii), and has no Hamilton cycle. Hence Conjecture B is false for every $n \ge 7$.*

(b) For $3 \le n \le 6$, every digraph on n vertices satisfying (i), (ii) and, if n is even, (iii) has a Hamilton cycle. For $3 \le n \le 5$, every strongly connected digraph on n vertices satisfying (i) and (ii) has a Hamilton cycle.

Corollary 2. *Conjecture A is false for every $n \ge 6$ and true for $3 \le n \le 5$. The case $n = 6$ is due to Guninski [8, 9].*

Kühn and Osthus add after Conjecture A: "It is even an open problem whether the conditions imply the existence of a cycle through any pair of given vertices" [11, Section 2.3]; the same sentence follows Conjecture 1 in [12], both citing Bermond and Thomassen [3]. A cycle through two given vertices is a weaker requirement than a Hamilton cycle.

Theorem 3. *For $n = 6$ and every $n \geq 8$ there is a strongly connected digraph on n vertices satisfying (i) and (ii), and for even $n \geq 8$ also (iii), in which some two vertices lie on no common cycle.*

For $n = 6$ the example is isomorphic to Guninski's digraph g_2 of [9]. The case $n = 7$ is not settled here: the 7-vertex counterexample below has a cycle through every pair of vertices.

1.3 Prior work

Nash-Williams posed Conjecture A in 1975 [15], and in 2008 Kühn, Osthus and Treglown wrote: "No progress has been made on this conjecture so far" [12]. They proved an approximate version for large digraphs [12, Theorem 2], and Christofides, Keevash, Kühn and Osthus proved a stronger approximate version [5, Theorem 5]. The conditions of these theorems carry slack proportional to n, and the examples here do not contradict them.

Georgi Guninski posted the first counterexample to Conjecture A on MathOverflow in March 2013 [8], a 6-vertex digraph whose out- and in-degree sequences are both $(2, 2, 2, 4, 4, 4)$. There Gordon Royle confirmed it ("yes, it is a counterexample"), Hugh Thomas explained how to check by hand that it has no Hamilton cycle, and Gabriel C. Drummond-Cole found the statement "identical up to changes in notation" to Nash-Williams' (Conjecture 2, p. 327). In August 2020 Louis DeBiasio explained the example through a theorem of Amar and Manoussakis on bipartite digraphs [2] (see [1, Theorem 1.7]), showed that the analogous construction for $n = 4k$ "just barely fails the Nash-Williams condition", and wrote "my guess is that the conjecture is safe for all $n \neq 6$" [8]. Days later Guninski posted two further 6-vertex counterexamples, g_2 and g_3 (his g_1 is isomorphic to the 2013 example, and g_3 is g_2 minus one arc), which DeBiasio verified there, and asked: "Can we get infinitely many counterexamples from the small ones?" [9]. Guninski also proposed an infinite family of complete multipartite digraphs [10] and withdrew it there as the result of a program bug; DeBiasio pointed out that its digraphs are symmetric, and for symmetric digraphs the conjecture reduces to Chvátal's theorem.

These counterexamples to Conjecture A are Guninski's, and the case $n = 6$ of Corollary 2 rests entirely on them. All of them violate (iii), so they leave Conjecture B open; Theorem 1(b) shows that every 6-vertex counterexample to Conjecture A violates (iii). What is new here is the refutation of Conjecture B, the counterexamples to both conjectures for every $n \geq 7$, the exhaustive verification of the small cases, and the answer to the pair question for $n \geq 8$.

Two recent papers are nearby. Darbinyan [6] constructs "infinitely many k-strong, where $k \geq 2$, digraphs which show that two famous conjectures of Nash-Williams and a conjecture of Kühn et al. for Hamiltonicity of digraphs are best possible in the sense that they become false if exactly one of the degree conditions is not valid"; such digraphs violate a condition and are not counterexamples. Its abstract adds: "This answers the two questions posed by Kühn et al." Section 2 of [12] poses two questions of this kind: "We do not know whether a similar construction as above also exists for this case" (for Conjecture A with n odd and $i = \lfloor n/2 \rfloor$), and "It would also be interesting to find an analogous construction as above for Conjecture 3" (Nash-Williams' weaker Pósa-type conjecture). DeBiasio [7] recently gave a regularity-free proof of a theorem of Taylor on oriented Hamilton cycles in robust outexpanders.

1.4 The idea

Guninski's 2013 example rests on an independent set of three vertices joined to the other three by a non-Hamiltonian bipartite digraph, and DeBiasio's analysis shows that this structure does not extend to larger orders. Our examples use instead that conditions (i) and (ii) pair the sorted out-degrees with the sorted in-degrees, and so cannot see which vertex has which degree. In the family F_u below, the four vertices of smallest out-degree have large in-degree, so condition (i) holds through its second alternative, while the vertices of smallest in-degree have out-degree $n-1$, so condition (ii) holds through its second alternative. The only arc leaving the set $L = \{t, r_1, r_2\}$ is $t \to w$, so on a Hamilton cycle t precedes w. The vertex s has out-neighbors only t and w, so it must precede t, and then the cycle enters and leaves L at t and misses r_1 and r_2. In G_7, the two vertices 2 and 6 of in-degree two force the successor of vertex 3, and then the path $4 \to 0 \to 1 \to 3$, until the cycle would need the missing arc $5 \to 4$.

2 The counterexamples

A Hamilton cycle $v_1 \to v_2 \to \cdots \to v_n \to v_1$ defines a successor map f with $f(v_j) = v_{j+1}$, a cyclic permutation of the vertex set.

2.1 Seven vertices

Let G_7 have vertices $0, \ldots, 6$ and out-neighborhoods

$$N^+(0) = \{1,4\},\ N^+(1) = \{3,4\},\ N^+(2) = \{1,3,4,5,6\},\ N^+(3) = \{0,1,2,4,5,6\},$$

$$N^+(4) = \{0,1,3,5\},\ N^+(5) = \{1,3\},\ N^+(6) = \{1,2,3,4,5\}.$$

Figure 1 shows it.

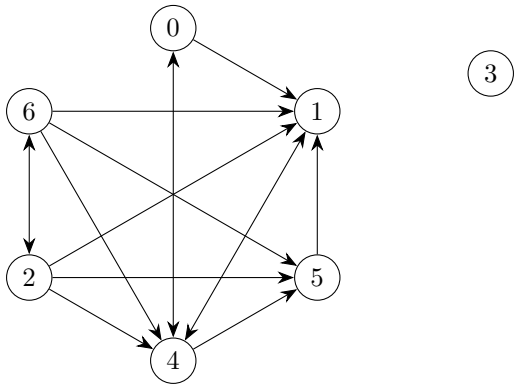

Figure 1: The digraph G_7. Vertex 3 sends an arc to every other vertex and receives arcs from 1, 2, 4, 5 and 6; those arcs are not drawn. A double-headed line is a pair of opposite arcs.

Lemma 4. *G_7 is strongly connected, satisfies (i) and (ii), and has no Hamilton cycle.*

Proof. The in-neighborhoods are $N^-(0) = \{3,4\}$, $N^-(1) = \{0,2,3,4,5,6\}$, $N^-(2) = \{3,6\}$, $N^-(3) = \{1,2,4,5,6\}$, $N^-(4) = \{0,1,2,3,6\}$, $N^-(5) = \{2,3,4,6\}$ and $N^-(6) = \{2,3\}$. Both degree sequences are $(2,2,2,4,5,5,6)$. For $i = 1$ the first alternatives of (i) and (ii) hold, since $d_1^+ = d_1^- = 2$. For $i = 2,3$ the second alternatives hold, since $d_{7-i}^+ = d_{7-i}^- = 7-i$. Vertex 3 sends

an arc to every other vertex, and every other vertex reaches 3 ($0 \to 1 \to 3$, and the others directly), so G_7 is strongly connected.

Suppose f is the successor map of a Hamilton cycle. Since $N^-(2) = \{3, 6\}$ and $N^-(6) = \{2, 3\}$, and $2 \to 6 \to 2$ would close the cycle too early, either $f(3) = 2$ and $f(2) = 6$, or $f(3) = 6$ and $f(6) = 2$; in both cases $f(3) \in \{2, 6\}$. Hence the predecessor of 0, which lies in $N^-(0) = \{3, 4\}$, is 4, and $f(0) \in \{1, 4\} \setminus \{4\}$ gives $f(0) = 1$. Then $f(1) \in \{3, 4\}$, and $f(1) = 4$ would close the cycle $4 \to 0 \to 1 \to 4$ too early, so $f(1) = 3$. The cycle is therefore $4 \to 0 \to 1 \to 3 \to x \to y \to 5 \to 4$ with $\{x, y\} = \{2, 6\}$, which needs the arc $5 \to 4$. There is no such arc. □

2.2 Six vertices

Let G_6 have vertices $0, \ldots, 5$ and out-neighborhoods $N^+(0) = \{1, 2, 3, 4, 5\}$, $N^+(1) = \{4, 5\}$, $N^+(2) = \{1, 5\}$, $N^+(3) = \{1, 4\}$, $N^+(4) = \{0, 1, 3, 5\}$ and $N^+(5) = \{0, 1, 2, 4\}$. It is isomorphic to Guninski's g_2 [9].

Lemma 5. *G_6 is strongly connected, satisfies (i) and (ii), and has no cycle through both 2 and 3; in particular it has no Hamilton cycle.*

Proof. Both degree sequences are $(2, 2, 2, 4, 4, 5)$. For $i = 1$ the first alternatives of (i) and (ii) hold, and for $i = 2$ the second, since $d_4^+ = d_4^- = 4$. (Condition (iii) fails, since $d_3^+ = d_3^- = 2$.) Vertex 0 sends an arc to every other vertex, and every vertex reaches 0 ($4 \to 0$, $5 \to 0$, $1 \to 4$, $2 \to 5$, $3 \to 4$), so G_6 is strongly connected.

Suppose a cycle passes through 2 and 3, say $3 \to P \to 2 \to Q \to 3$ with paths P and Q that share no vertex. The predecessor of 2 is 0 or 5, that of 3 is 0 or 4, the successor of 2 is 1 or 5, and that of 3 is 1 or 4. If P ends at 0, then Q ends at 4, so P starts at 1; since $N^+(1) = \{4, 5\}$, P is $1 \to 5 \to 0$, and 2 has no successor left. If P ends at 5, then Q starts at 1 and continues to 4, since $N^+(1) = \{4, 5\}$; so Q is $1 \to 4$ or $1 \to 4 \to 0$, and 3 has no successor left. Both cases are impossible. □

2.3 A family for every $n \geq 8$

For an integer $u \geq 4$ let F_u have $n = u + 4$ vertices: a set U of u vertices with a distinguished vertex $w \in U$, and four further vertices s, t, r_1, r_2. Its arcs are

- $x \to y$ for every $x \in U$ and every vertex $y \neq x$;
- $a \to b$ for all distinct $a, b \in L = \{t, r_1, r_2\}$;
- $s \to t$, $s \to w$ and $t \to w$.

Figure 2 shows the arcs that do not start in U.

Lemma 6. *For every $u \geq 4$, F_u is strongly connected, satisfies (i), (ii) and, if $n = u + 4$ is even, (iii), and has no Hamilton cycle. Moreover s lies on no cycle with r_1, and none with r_2.*

Proof. Out-degrees: s, r_1 and r_2 have out-degree 2, t has 3, and every vertex of U has $n - 1$. In-degrees: each vertex of $U \setminus \{w\}$ receives arcs only from the other $u - 1$ vertices of U; s receives arcs from U alone; w receives arcs from $U \setminus \{w\}$, s and t; r_1 and r_2 from U and the other two vertices of L; and t from every other vertex. So

$$(d_1^+, \ldots, d_n^+) = (2, 2, 2, 3, n-1, \ldots, n-1),$$
$$(d_1^-, \ldots, d_n^-) = (n-5, \ldots, n-5,\ n-4,\ n-3,\ n-2,\ n-2,\ n-1),$$

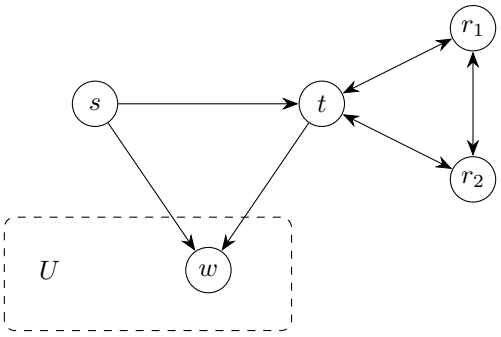

Figure 2: The arcs of F_u that do not start in U. Every vertex of U also sends an arc to every other vertex.

with $u-1$ entries $n-5$, so that $d^-_{n-j} = n-j$ for $j = 2,3,4$. Condition (i) holds for $i = 1$ since $d^+_1 = 2$, for $i = 2,3,4$ since $d^-_{n-i} = n-i$, and for $i \geq 5$ since $d^+_i = n-1$. For condition (ii), every $i < n/2$ has $n-i > n/2 \geq 4$, so $d^+_{n-i} = n-1 \geq n-i$. If n is even, then $d^+_{n/2} = n-1$ when $n \geq 10$, and $d^-_4 = n-4 = 4$ when $n = 8$; so (iii) holds. The vertex w sends an arc to every other vertex, and every vertex reaches w (vertices of U, s and t directly, r_1 and r_2 through t), so F_u is strongly connected.

The only arc leaving L is $t \to w$. Let C be a cycle through s that meets L. It must leave L, so it uses $t \to w$; it uses that arc only once, so it makes a single visit to L, which ends at t. Since t precedes w on C, the vertex s, whose out-neighbors are t and w, precedes t. So the visit to L also begins at t and consists of t alone. Hence no cycle passes through s and r_1, or s and r_2, and in particular no cycle passes through every vertex. □

Proof of Theorems 1(a) and 3 and of Corollary 2. For $n = 7$ take G_7 (Lemma 4; condition (iii) is void for odd n), and for $n \geq 8$ take F_{n-4} (Lemma 6). These digraphs are strongly connected, so they are counterexamples to Conjecture A too; for $n = 6$, Guninski's counterexamples, among them G_6 (Lemma 5), refute it. The case $3 \leq n \leq 5$ of Corollary 2 is Theorem 1(b). Theorem 3 is Lemma 5 for $n = 6$ and Lemma 6 for $n \geq 8$. □

3 Small orders

Theorem 1(b) is proved by exhaustive search over all labeled digraphs on $3 \leq n \leq 6$ vertices: each of the $\binom{n}{2}$ pairs of vertices carries no arc, one of two arcs, or both, giving $4^{\binom{n}{2}}$ digraphs, or $2^{30} \approx 1.07 \times 10^9$ for $n = 6$. For each digraph the search tests (i), (ii) and (iii), strong connectivity and Hamiltonicity. Table 1 lists the counts. A digraph is counted in the last column when it satisfies (i), (ii) and, for even n, (iii), and has no Hamilton cycle; it is counted in the fourth when it is strongly connected, satisfies (i) and (ii), and has no Hamilton cycle.

Two independent programs, which use different enumerations and different tests for the conditions, strong connectivity and Hamiltonicity (Appendix A), produce identical output. The run at $n = 6$ takes about 2.5 minutes on one core for the first and 1.5 minutes for the second.

4 Verification and reproducibility

Formal verification. The counterexamples are checked in Lean 4 with the Mathlib library [13, 14]. A digraph on n vertices is a Boolean relation on `Fin n` with no loops, so a pair of vertices may

n	digraphs	satisfying (i), (ii)	counterexamples to Conj. A	counterexamples to Conj. B
3	64	7	0	0
4	4096	186	0	0
5	1048576	40140	0	0
6	1073741824	44387496	1080	0

Table 1: Labeled digraphs on n vertices. The 1080 labeled counterexamples to Conjecture A on six vertices all violate (iii).

carry arcs in both directions:

```
structure Digraph (n : ℕ) where
  adj : Fin n → Fin n → Bool
  loopless : ∀ v, adj v v = false
```

`G.dplus i` and `G.dminus i` are the i-th entries of the sorted out- and in-degree lists. The hypotheses read

```
def NWHyp (G : Digraph n) : Prop :=
  ∀ i, 1 ≤ i → 2 * i < n →
    (G.dplus i ≥ i + 1 ∨ G.dminus (n - i) ≥ n - i) ∧
    (G.dminus i ≥ i + 1 ∨ G.dplus (n - i) ≥ n - i)

def KOTHyp (G : Digraph n) : Prop :=
  NWHyp G ∧ (n % 2 = 0 → G.dplus (n / 2) ≥ n / 2 ∨ G.dminus (n / 2) ≥ n / 2)
```

A Hamilton cycle (`HasHamCycle`) is a bijection c of `Fin n` with an arc from $c(k)$ to $c(k+1 \bmod n)$ for every k; `StronglyConnected` is reflexive-transitive reachability for every ordered pair; and `CycleThrough G a b` asks for an injective map c: `Fin m` $\to$ `Fin n` with arcs from $c(k)$ to $c(k+1 \bmod m)$ whose image contains a and b. The digraph F_u is `F u` on the vertices $0, \dots, u+3$, with $U = \{0, \dots, u-1\}$, $w = 0$, $s = u$, $t = u+1$, $r_1 = u+2$ and $r_2 = u+3$. The main formal theorems are

```
theorem kot_counterexample_all (n : ℕ) (hn : 7 ≤ n) :
    ∃ G : Digraph n, KOTHyp G ∧ ¬ HasHamCycle G
theorem counterexample_all (n : ℕ) (hn : 6 ≤ n) :
    ∃ G : Digraph n, StronglyConnected G ∧ NWHyp G ∧ ¬ HasHamCycle G
theorem F_counterexample (u : ℕ) (hu : 4 ≤ u) :
    StronglyConnected (F u) ∧ NWHyp (F u) ∧ ¬ HasHamCycle (F u)
theorem F_no_cycle_through_s_r1 (u : ℕ) :
    ¬ CycleThrough (F u) ⟨u, _⟩ ⟨u + 2, _⟩
theorem G6_no_cycle_through_2_3 : ¬ CycleThrough G6 2 3
```

together with `G6_counterexample` and `G7_counterexample` (strong connectivity, `NWHyp` and no Hamilton cycle), `F_no_cycle_through_s_r2`, the lemma `F_kot3` (condition (iii) for `F u` when $u \geq 4$ and $u+4$ is even), and `not_kotConjecture6` and `not_nashWilliamsConjecture`, the negations of Conjectures B and A stated for all $n \geq 3$. Theorem 1(a) is `kot_counterexample_all`, whose witnesses are G_7 and `F (n-4)`, together with the strong connectivity in `G7_counterexample` and `F_counterexample`; no single formal statement combines the three properties. Corollary 2 for $n \geq 6$ is `counterexample_all`. Theorem 3 combines

`G6_counterexample` and `G6_no_cycle_through_2_3` for $n = 6$, and `F_counterexample`, `F_kot3` and `F_no_cycle_through_s_r1` for $n \geq 8$; the pair statements hold for every u. The degree conditions for F_u are proved for all $u \geq 4$ through the counting form of Appendix A, and the statements about G_6 and G_7 by decision procedures evaluated in Lean's kernel. The build checks the statements and that each theorem depends only on the axioms `propext`, `Classical.choice` and `Quot.sound`. Theorem 1(b) is not in Lean.

Computational checks. Two further programs, written independently of each other and of the Lean development, read G_6 and G_7 from files and build F_{n-4} for $8 \leq n \leq 18$ and $8 \leq n \leq 14$ respectively. They confirm strong connectivity, conditions (i)–(iii) and non-Hamiltonicity, and both reject four altered inputs: G_6 with the arc $2 \to 3$ added and G_7 with $0 \to 2$ added, which are Hamiltonian; G_6 without $4 \to 3$, which violates (ii); and G_6 with a loop. The two enumerators of Section 3 complete the checks. The files, with the commands that run them, are published with this paper, in the folder `results/hamiltonian-degree-sequences` of its repository, together with the Lean development.

Appendix A The programs

Both the checkers and the enumerators come in two independent versions. The first enumerator assigns each pair of vertices one of its four states; it tests the conditions in their sorted-sequence form, strong connectivity by a reachability closure, and Hamiltonicity by the Held–Karp dynamic program over vertex subsets. The second runs through the $2^{n(n-1)}$ bit words over the ordered pairs of distinct vertices. It tests the conditions in their counting form, $d_i^+ \geq i+1$ if and only if at most $i-1$ vertices have out-degree at most i, and $d_{n-i}^- \geq n-i$ if and only if at least $i+1$ vertices have in-degree at least $n-i$ (and likewise with out- and in-degrees exchanged); it tests strong connectivity by Warshall's transitive closure and Hamiltonicity by depth-first search over paths from a fixed vertex. The two checkers of Section 4 use the same pairs of methods: the first in Python with graph search in both directions for strong connectivity, the second in C.

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