No post-Lie algebra structures on pairs of perfect and nilpotent Lie algebras

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Abstract

Let $\mathfrak{g}$ and $\mathfrak{n}$ be finite-dimensional Lie algebras on the same vector space over a field of characteristic zero, with $\mathfrak{n}$ nilpotent. We prove that if the pair $(\mathfrak{g}, \mathfrak{n})$ carries a post-Lie algebra structure all of whose right multiplications have trace zero, then $\mathfrak{g}$ is solvable. Hence no nonzero perfect Lie algebra $\mathfrak{g}$ admits a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n}$ nilpotent. This answers a question of Burde and Dekimpe in the negative and proves the nilpotent case of a conjecture of Burde, Dekimpe and Monadjem. Nonexistence was known for abelian $\mathfrak{n}$, by a theorem of Helmstetter on left-symmetric algebras; for two-step nilpotent $\mathfrak{n}$, by Burde and Dekimpe; and for the complex perfect non-semisimple $\mathfrak{g}$ of dimensions five and six, by Burde and Dekimpe and by Burde, Dekimpe and Monadjem. The proof extends Helmstetter's argument. A post-Lie algebra structure embeds $\mathfrak{g}$ into the holomorph of $\mathfrak{n}$. Conjugations chosen by Whitehead's first lemma along the lower central series of $\mathfrak{n}$ make the translation part of this embedding vanish on a Levi factor of $\mathfrak{g}$, while its determinant stays equal to one, as in Kriebel's treatment of semisimple $\mathfrak{n}$. The proof is formalized in Lean with Mathlib, assuming Levi's theorem and Whitehead's first lemma as stated in Weibel's textbook.

This work has not yet been reviewed by experts. The proofs, the Lean formalization and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

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1 Introduction

1.1 Main results

Throughout, k is a field of characteristic zero and all Lie algebras are finite-dimensional over k. Following Burde and Dekimpe [4, Definition 2.1], let $\mathfrak{g} = (V, [\,,\,])$ and $\mathfrak{n} = (V, \{\,,\,\})$ be two Lie brackets on one vector space V. A *post-Lie algebra structure* on the pair $(\mathfrak{g}, \mathfrak{n})$ is a bilinear product $x \cdot y$ on V such that, for all $x, y, z \in V$,

$$x \cdot y - y \cdot x = [x, y] - \{x, y\}, \tag{1}$$

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z), \tag{2}$$

$$x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\}. \tag{3}$$

We write $L(x)y = x \cdot y$ and $R(x)y = y \cdot x$. By (3) each $L(x)$ is a derivation of $\mathfrak{n}$, and by (2) the map $L\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{n})$ is a representation of $\mathfrak{g}$ [4, §2]. When $\mathfrak{n}$ is abelian, (1) and (2) say that $(V, \cdot)$ is a left-symmetric algebra (an algebra whose associator $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$ is symmetric in x and y) whose commutator is the bracket of $\mathfrak{g}$.

Theorem 1.1. *Let $x \cdot y$ be a post-Lie algebra structure on a pair $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n}$ nilpotent. If $\operatorname{tr} R(x) = 0$ for every $x \in V$, then $\mathfrak{g}$ is solvable.*

Corollary 1.2. *Let $\mathfrak{g} \neq 0$ be perfect, that is, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and let $\mathfrak{n}$ be nilpotent. Then there is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.*

For $\mathfrak{g} = 0$ the zero product is a post-Lie algebra structure, so the hypothesis $\mathfrak{g} \neq 0$ is needed. The trace hypothesis in Theorem 1.1 cannot be dropped: the product of matrices $x \cdot y = xy$ on $V = \mathfrak{gl}_m(k)$ is a post-Lie algebra structure on $(\mathfrak{gl}_m(k), \mathfrak{n})$ with $\mathfrak{n}$ abelian, since associativity gives (2), and $\mathfrak{gl}_m(k)$ is not solvable for $m \geq 2$; here $\operatorname{tr} R(x) = m \operatorname{tr} x$.

1.2 The question and prior work

Post-Lie algebras were introduced by Vallette [20]. As Burde and Dekimpe recall [4, §1], post-Lie algebra structures on pairs of Lie algebras were introduced independently by Burde, Dekimpe and Vercammen [7], in the study of NIL-affine actions of Lie groups, that is, affine actions on a nilpotent Lie group N by elements of $N \rtimes \mathrm{Aut}(N)$.

Burde and Dekimpe [4] studied their existence in terms of the structure of $\mathfrak{g}$ and $\mathfrak{n}$. For perfect $\mathfrak{g}$ they proved: "Proposition 3.3. Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras over a field of characteristic zero, where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is abelian or 2-step nilpotent. Then there is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$." In its proof the abelian case is taken from Segal [18, Theorem 1 and Proposition 1] and, over $\mathbb{C}$, from Helmstetter [13, Proposition 20], and the two-step case is reduced to it by [7, Proposition 4.2]. They also recall that $\mathfrak{n}$ cannot be solvable and not nilpotent [7, Proposition 4.4] (their Proposition 3.4), and that $\mathfrak{g}$ cannot be semisimple when $\mathfrak{n}$ is solvable [5, Theorem 4.2] (their Theorem 3.2). This led them to ask:

Question 1.3 ([4, Question 3.5])**.** *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras, where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is nilpotent. Does there exist a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$?*

They showed that no structure exists for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes V(2)$, the semidirect product with the two-dimensional irreducible module [4, Proposition 3.6]. In a report of joint work with Dekimpe, Burde then stated two conjectures [17, p. 2689]: "Conjecture. Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras, where $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \ltimes V(n)$ and $\mathfrak{n}$ is nilpotent. Then there is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$," where n is an integer and $V(n)$ the natural representation of $\mathfrak{sl}_n(\mathbb{C})$, and, as "a more general conjecture": "Conjecture. Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras, where $\mathfrak{g}$ is perfect non-semisimple, and $\mathfrak{n}$ is nilpotent. Then there is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$."

Burde, Dekimpe and Monadjem [6] studied the pairs with $\mathfrak{g}$ perfect and not semisimple, in seven cases for $\mathfrak{n}$, of which case (b) is "$\mathfrak{n}$ is nilpotent non-abelian" [6, §3]. In their introduction they write that for the cases "where $\mathfrak{g}$ is perfect non-semisimple, and $\mathfrak{n}$ is either nilpotent, simple, or semisimple, we are not able [to] solve the existence question in general. We conjecture that there do not exist post-Lie algebra structures in these cases." In case (b) they proved nonexistence for the two six-dimensional complex perfect Lie algebras that are not semisimple, $\mathfrak{sl}_2(\mathbb{C}) \ltimes V(3)$ and $\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{n}_3(\mathbb{C})$, where $\mathfrak{n}_3(\mathbb{C})$ is the three-dimensional Heisenberg algebra [6, Proposition 3.2].

Corollary 1.2 answers Question 3.5 of [4] in the negative, and proves case (b) of the conjecture of [6] and both conjectures of [17, p. 2689], over every field of characteristic zero.

The case $\mathfrak{n}$ abelian of Theorem 1.1 is due to Helmstetter. A left-symmetric algebra is *complete* if every $I+R(x)$ is bijective [3, Definition 3.30], and in characteristic zero this holds if and only if every $R(x)$ is nilpotent, and if and only if $\operatorname{tr} R(x) = 0$ for all x [3, Theorem 3.31], a theorem of Segal [18]. Helmstetter calls a left-symmetric algebra nilpotent when every right multiplication is nilpotent [13, Definition 17], and proved: "Si A est une algèbre symétrique à gauche nilpotente, l'algèbre de Lie A est résoluble" (if A is a nilpotent left-symmetric algebra, its Lie algebra is solvable) [13, Proposition 20]. The proof of Helmstetter's Corollary 21, that for a finite-dimensional left-symmetric algebra "il est impossible que l'algèbre de Lie A soit égale à l'algèbre de Lie dérivée $[A, A]$" (the Lie algebra A cannot equal its derived algebra), deduces nilpotency from $\operatorname{tr} R_a = 0$ for every a. Fox restates these results, as due to Helmstetter, in [12, Lemma 5.3]: "(1) The underlying Lie algebra of a complete finite-dimensional LSA over k is solvable." Burde recalls the geometric origin: if a Lie group G acts simply transitively by affine transformations of $\mathbb{R}^n$, then "Auslander proved that in this case G is solvable" [1], [3, Remark 2.28].

Several analogues concern groups. A Lie skew brace is a Lie group with a second Lie group structure on the same manifold, linked by the skew-brace identity, and its tangent object is a post-Lie algebra [10, §1]. Damele and Loi proved, in their Theorem 1.1(S2): "Let $(G, \cdot, \circ)$ be a connected LSB [Lie skew brace]. ... If $(G, \cdot)$ is solvable, then $(G, \circ)$ is solvable" [10]. Of the existence question for nilpotent $\mathfrak{n}$ and perfect $\mathfrak{g}$, which they attribute to [6], they write: "While we are currently not able to answer this question definitively, we can nevertheless state that any such PLA [post-Lie algebra], if it exists, would not be integrable into an LSB due to condition (S2) in Theorem 1.1" [10, §1]. Theorem 1.1 needs no Lie group and no integration. For finite groups, Byott proved "if G is nilpotent then Γ is soluble" for a Hopf–Galois structure of type G on a Galois extension with group Γ [8, Theorem 1]; for skew braces this reads "Let A be a finite skew brace whose additive group $(A, \cdot)$ is nilpotent. Then $(A, \circ)$ is solvable" [19, Corollary 1.23], and Damele has extended it to finite skew braces whose additive derived subgroup is nilpotent [9]. On simply connected nilpotent Lie groups, Deré and Origlia recall that "if a group acts simply transitively on a 1-connected nilpotent Lie group, it must be solvable" [11, §1].

For semisimple $\mathfrak{n}$, Kriebel's preprint gives a proof of nonexistence for perfect non-semisimple complex $\mathfrak{g}$ [14, Theorem 1 and Corollary 3]. There "the integrated left-right action has a holomorphic infinitesimal determinant. Unimodularity makes this determinant invariant" [14, abstract], and Kriebel credits Baues's determinant relative invariant of an étale affine representation as its predecessor [2, Theorem 5.19 and Corollary 5.20].

1.3 Proof overview and what is new

Helmstetter's proof, in the form given by Fox [12, proof of Lemma 5.3], runs as follows. If $\mathfrak{g}$ is not solvable it contains a nonzero semisimple subalgebra $\mathfrak{s}$; the identity map $\mathfrak{s} \to V$ is a 1-cocycle for the representation L, so by Whitehead's first lemma there is $a \in V$ with $x = L(x)a = R(a)x$ for $x \in \mathfrak{s}$. Hence $I - R(a)$ is singular, which contradicts completeness (take $x = -a$).

For nilpotent $\mathfrak{n}$ both steps need a replacement. By a theorem of Burde, Dekimpe and Vercammen [7, Proposition 2.13], a post-Lie algebra structure is the same as an embedding $x \mapsto (x, L(x))$ of $\mathfrak{g}$ into the holomorph $\mathfrak{n} \rtimes \operatorname{Der}(\mathfrak{n})$. Its first component, the translation part, is the identity map $P = \mathrm{id}_V$, which is a cocycle for L with respect to the bracket of $\mathfrak{n}$ (Section 2). Conjugating the embedding by the exponential of $-\operatorname{ad}(z, 0)$ for $z \in \mathfrak{n}$ changes P to a new cocycle P^z. We show two things.

- When $\operatorname{tr} L = \operatorname{tr} \operatorname{ad}_{\mathfrak{g}}$, which is equivalent to $\operatorname{tr} R = 0$, every conjugation preserves $\det P$ (Lemma 3.1). This step follows the determinant method Kriebel used for semisimple $\mathfrak{n}$ [14],

after Baues [2] and Helmstetter's characteristic polynomial: there the unimodularity of $\mathfrak{g}$ makes a determinant invariant, and here the trace hypothesis does. What is new is that for nilpotent $\mathfrak{n}$ the invariance is an identity of polynomials, with no integration. The proof puts tz in place of z, so that the translation part becomes a matrix P_t with entries in $k[t]$, and shows that $\det P_t$ has derivative zero by Jacobi's formula and a trace identity extracted from the cocycle equation. For $\mathfrak{n}$ abelian, $P_t = I + tR(z)$, and the lemma says that Helmstetter's characteristic polynomial $\det(I + R(x))$ [12, §3] equals one.

- Let $\mathfrak{n} = \mathfrak{n}^1 \supseteq \mathfrak{n}^2 \supseteq \cdots \supseteq \mathfrak{n}^{r+1} = 0$ be the lower central series and $\mathfrak{s}$ a Levi factor of $\mathfrak{g}$. Whitehead's first lemma, applied to the successive quotients $\mathfrak{n}^i/\mathfrak{n}^{i+1}$, gives elements $z_1, \dots, z_r$ whose conjugations move $P(\mathfrak{s})$ into $\mathfrak{n}^2, \dots, \mathfrak{n}^{r+1} = 0$ (Lemma 4.1). In the abelian case one application suffices. When $\mathfrak{n}$ is not abelian the cocycle equation is nonlinear and becomes linear only modulo the next term of the series, so the lemma is applied r times; this step has no abelian analogue.

After the r conjugations the determinant is still one, but the cocycle vanishes on $\mathfrak{s} \neq 0$, a contradiction.

The proof uses Levi's theorem and Whitehead's first lemma, quoted in Section 6. Everything else is proved in a Lean formalization [15, 16], described there; the two theorems enter it as named axioms.

2 Cocycles and conjugation

Let $\mathfrak{n} = (V, \{\,,\,\})$ be a Lie algebra and $\mathfrak{h} = \mathfrak{n} \rtimes \mathrm{Der}(\mathfrak{n})$ its holomorph, the vector space $V \oplus \mathrm{Der}(\mathfrak{n})$ with bracket

$$[(a, D), (b, E)] = (\{a, b\} + Db - Ea,\ DE - ED).$$

Let $\mathfrak{g} = (V, [\,,\,])$ be a Lie algebra on the same space and $L\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{n})$ a representation. Call a linear map $P\colon V \to V$ a *cocycle* (for L) if

$$P[a, b] = \{Pa, Pb\} + L(a)Pb - L(b)Pa \qquad (a, b \in V), \tag{4}$$

that is, if $x \mapsto (Px, L(x))$ is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$. For a post-Lie algebra structure, (1) says exactly that $P = \mathrm{id}_V$ is a cocycle for L; together with (2) and (3) this is [7, Proposition 2.13].

Two trace identities will be used. First, (1) gives $R(x) = L(x) - \mathrm{ad}_{\mathfrak{g}}(x) + \mathrm{ad}_{\mathfrak{n}}(x)$. If $\mathfrak{n}$ is nilpotent, each $\mathrm{ad}_{\mathfrak{n}}(x)$ is nilpotent and has trace zero, so

$$\operatorname{tr} R(x) = 0 \text{ for all } x \quad \Longleftrightarrow \quad \operatorname{tr} L(x) = \operatorname{tr} \mathrm{ad}_{\mathfrak{g}}(x) \text{ for all } x. \tag{5}$$

Second, if $\mathfrak{g}$ is perfect, every element of $\mathfrak{g}$ is a sum of brackets, and L and $\mathrm{ad}_{\mathfrak{g}}$ are representations, so $\operatorname{tr} L(x) = \operatorname{tr} \mathrm{ad}_{\mathfrak{g}}(x) = 0$ for all x.

Now let $\mathfrak{n}$ be nilpotent and $z \in V$. The derivation $A = \mathrm{ad}_{\mathfrak{h}}(z, 0)$ of $\mathfrak{h}$ is $A(a, D) = (\{z, a\} - Dz, 0)$. It maps $\mathfrak{h}$ into $V \oplus 0$ and acts there as $\mathrm{ad}_{\mathfrak{n}}(z)$, so it is nilpotent, and $\exp(-A)$ is an automorphism of $\mathfrak{h}$ (a finite sum since A is nilpotent, with coefficients $1/j!$ since k has characteristic zero). Because A has no $\mathrm{Der}(\mathfrak{n})$-component, for a cocycle P the homomorphism $\exp(-A) \circ (P, L)$ again has the form $x \mapsto (P^z x, L(x))$, and P^z is a cocycle. We call P^z the *conjugate* of P by z. For $\mathfrak{n}$ abelian, $A^2 = 0$ and $P^z = P + \rho(z)$, where $\rho(z)x = L(x)z$; in particular $\mathrm{id}^z = I + R(z)$.

3 The determinant is invariant

Lemma 3.1. *Let $\mathfrak{n}$ be nilpotent, let $\operatorname{tr} L(x) = \operatorname{tr}\operatorname{ad}_{\mathfrak{g}}(x)$ for all $x \in V$, and let P be a cocycle with $\det P \neq 0$. Then $\det P^z = \det P$ for every $z \in V$.*

Proof. Extend all maps $k[t]$-linearly to $V[t] = V \otimes k[t]$. The derivation tA of $\mathfrak{h}[t]$ is nilpotent, and $\psi_t = \exp(-tA) \circ (P, L)$ is a homomorphism $\mathfrak{g}[t] \to \mathfrak{h}[t]$ of the form $\psi_t(x) = (P_t x, L(x))$, with P_t a matrix over $k[t]$, $P_0 = P$ and $P_1 = P^z$. Differentiating in t gives $\psi_t' = -A\psi_t$, whose first component is

$$P_t' = -\operatorname{ad}_{\mathfrak{n}}(z)\, P_t + \rho(z), \qquad \rho(w)\colon x \mapsto L(x)w. \tag{6}$$

Let $F = \det P_t \in k[t]$ and $Q = \operatorname{adj}(P_t)$, so $QP_t = P_tQ = F\,I$. Since ψ_t is a homomorphism, P_t satisfies (4) over $k[t]$. Read as an identity of maps in b, this says

$$P_t \operatorname{ad}_{\mathfrak{g}}(a) = (\operatorname{ad}_{\mathfrak{n}}(P_t a) + L(a))P_t - \rho(P_t a) \qquad (a \in V[t]).$$

Multiply on the right by Q and take traces. Since $\operatorname{tr}(P_t \operatorname{ad}_{\mathfrak{g}}(a)Q) = \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(a)QP_t) = F \operatorname{tr}\operatorname{ad}_{\mathfrak{g}}(a)$ and $\operatorname{tr}\operatorname{ad}_{\mathfrak{n}}(P_t a) = 0$,

$$\operatorname{tr}\big(\rho(P_t a)Q\big) = F\big(\operatorname{tr} L(a) - \operatorname{tr}\operatorname{ad}_{\mathfrak{g}}(a)\big) = 0 \qquad (a \in V[t]).$$

For $w \in V[t]$ take $a = Qw$; then $P_t a = Fw$, so $F \operatorname{tr}(\rho(w)Q) = 0$. As $F(0) = \det P \neq 0$, F is a nonzero element of the domain $k[t]$, hence $\operatorname{tr}(\rho(w)Q) = 0$ for all w. Jacobi's formula $F' = \operatorname{tr}(QP_t')$ and (6) now give

$$F' = -\operatorname{tr}(Q \operatorname{ad}_{\mathfrak{n}}(z)P_t) + \operatorname{tr}(Q\rho(z)) = -F \operatorname{tr}\operatorname{ad}_{\mathfrak{n}}(z) + 0 = 0.$$

In characteristic zero F is constant, so $\det P^z = F(1) = F(0) = \det P$. □

4 Pushing a Levi factor down the lower central series

Let $\mathfrak{n} = \mathfrak{n}^1 \supseteq \mathfrak{n}^2 \supseteq \cdots \supseteq \mathfrak{n}^{r+1} = 0$ be the lower central series, $\mathfrak{n}^{i+1} = \{\mathfrak{n}, \mathfrak{n}^i\}$, and recall that $\{\mathfrak{n}^i, \mathfrak{n}^j\} \subseteq \mathfrak{n}^{i+j}$ and that derivations preserve each $\mathfrak{n}^i$.

Lemma 4.1. *Let $\mathfrak{n}$ be nilpotent, $\mathfrak{s} \subseteq \mathfrak{g}$ a semisimple subalgebra, and P a cocycle with $P(\mathfrak{s}) \subseteq \mathfrak{n}^i$ for some $i \geq 1$. Then there is $z \in \mathfrak{n}^i$ with $P^z(\mathfrak{s}) \subseteq \mathfrak{n}^{i+1}$.*

Proof. The space $M = \mathfrak{n}^i/\mathfrak{n}^{i+1}$ is an $\mathfrak{s}$-module through L. For $x, y \in \mathfrak{s}$ we have $\{Px, Py\} \in \mathfrak{n}^{2i} \subseteq \mathfrak{n}^{i+1}$, so (4) shows that $c\colon \mathfrak{s} \to M$, $c(x) = Px + \mathfrak{n}^{i+1}$, satisfies $c([x,y]) = L(x)c(y) - L(y)c(x)$. By Whitehead's first lemma there is $w \in \mathfrak{n}^i$ with $Px \equiv L(x)w \pmod{\mathfrak{n}^{i+1}}$ for all $x \in \mathfrak{s}$. Put $z = -w$ and $A = \operatorname{ad}_{\mathfrak{h}}(z, 0)$. For $x \in \mathfrak{s}$, $A(Px, L(x)) = (\{z, Px\} - L(x)z, 0)$ has first component in $\mathfrak{n}^i$, and each further application of A brackets with $z \in \mathfrak{n}^i$, so $A^j(Px, L(x))$ has first component in $\mathfrak{n}^{2i} \subseteq \mathfrak{n}^{i+1}$ for $j \geq 2$. Since $\{z, Px\} \in \mathfrak{n}^{i+1}$ too,

$$P^z x \equiv Px + L(x)z = Px - L(x)w \equiv 0 \pmod{\mathfrak{n}^{i+1}}.$$ □

5 Proofs of the main results

Proof of Theorem 1.1. Suppose $\mathfrak{g}$ is not solvable. By Levi's theorem $\mathfrak{g}$ has a nonzero semisimple subalgebra $\mathfrak{s}$. By (5), $\operatorname{tr} L(x) = \operatorname{tr}\operatorname{ad}_{\mathfrak{g}}(x)$ for all x. Start from the cocycle $P^{(0)} = \mathrm{id}_V$, with $P^{(0)}(\mathfrak{s}) \subseteq \mathfrak{n}^1$. Applying Lemma 4.1 r times gives $z_1, \ldots, z_r$ and cocycles $P^{(i)} = (P^{(i-1)})^{z_i}$ with $P^{(i)}(\mathfrak{s}) \subseteq \mathfrak{n}^{i+1}$. By Lemma 3.1 and induction, $\det P^{(i)} = \det P^{(0)} = 1$ for every i. But $P^{(r)}(\mathfrak{s}) \subseteq \mathfrak{n}^{r+1} = 0$ and $\mathfrak{s} \neq 0$, so $P^{(r)}$ is singular, a contradiction. □

Proof of Corollary 1.2. Suppose $(\mathfrak{g}, \mathfrak{n})$ has a post-Lie algebra structure. As $\mathfrak{g}$ is perfect, $\operatorname{tr} L = \operatorname{tr} \operatorname{ad}_{\mathfrak{g}} = 0$ (Section 2), so $\operatorname{tr} R = 0$ by (5), and $\mathfrak{g}$ is solvable by Theorem 1.1. A solvable perfect Lie algebra is zero, since its derived series is constant. □

6 Formal verification

The theorem and corollary are proved in Lean 4 [15] with Mathlib [16]. The definition of a post-Lie algebra structure, in `PostLie/Definitions.lean`, takes two Lie algebras `g` and `n` over a field `k` with `[CharZero k]`, both finite-dimensional, and a linear equivalence `e : g ≃ₗ[k] n`, which plays the role of the common space V:

```
structure Product (e : g ≃ₗ[k] n) where
  mul : g →ₗ[k] g →ₗ[k] g
  skew : ∀ x y, mul x y - mul y x = ⁅x, y⁆ - e.symm ⁅e x, e y⁆
  left_lie : ∀ x y z, mul ⁅x, y⁆ z = mul x (mul y z) - mul y (mul x z)
  derivation : ∀ x y z, mul x (e.symm ⁅e y, e z⁆) =
    e.symm ⁅e (mul x y), e z⁆ + e.symm ⁅e y, e (mul x z)⁆
```

Here `mul x y` is $x \cdot y$ and `e.symm ⁅e y, e z⁆` is $\{y, z\}$, so the three fields are (1)–(3). With `right p x`, the map $y \mapsto y \cdot x$, the two results in `PostLie/Statement.lean` are

```
theorem solvable_of_trace_right_eq_zero (e : g ≃ₗ[k] n) (p : Product e)
    [LieRing.IsNilpotent n]
    (htrace : ∀ x, LinearMap.trace k g (right p x) = 0) :
    LieAlgebra.IsSolvable g

theorem no_product_of_perfect (e : g ≃ₗ[k] n)
    [LieRing.IsNilpotent n] [Nontrivial g]
    (hperfect : ⁅(⊤ : LieIdeal k g), (⊤ : LieIdeal k g)⁆ = ⊤) :
    ¬ Nonempty (Product e)
```

They are Theorem 1.1 and Corollary 1.2: `Nontrivial g` says $\mathfrak{g} \neq 0$, and `hperfect` says $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. A build check prints the axioms each theorem depends on and fails unless they are exactly Lean's `propext`, `Classical.choice` and `Quot.sound` and the following two.

- `PostLie.levi_nonzero_semisimple`: if `g` is not solvable, it has a nontrivial subalgebra `s` with `LieAlgebra.IsSemisimple k s`. This follows from Levi's theorem [21, Theorem 7.8.13]: "If $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field of characteristic zero, then there is a semisimple Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ (called a Levi factor of $\mathfrak{g}$) such that $\mathfrak{g}$ is isomorphic to the semidirect product $\mathfrak{g} = (\operatorname{rad} \mathfrak{g}) \rtimes \mathfrak{s}$." If $\mathfrak{g}$ is not solvable, then $\operatorname{rad} \mathfrak{g} \neq \mathfrak{g}$ and $\mathfrak{s} \neq 0$.
- `PostLie.whitehead_first`: for a finite-dimensional semisimple `s`, a representation ρ of `s` on a finite-dimensional `M` and a linear `c : s →ₗ[k] M` with `c ⁅x, y⁆ = ρ x (c y) - ρ y (c x)`, there is `m` with `c x = ρ x m` for all `x`. This is Whitehead's first lemma [21, Corollary 7.8.10]: "Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0. If M is any finite-dimensional $\mathfrak{g}$-module, then $H^1(\mathfrak{g}, M) = 0$. That is, every derivation from $\mathfrak{g}$ into M is an inner derivation."

Weibel calls a Lie algebra semisimple when its radical is zero, and these are the finite products of simple Lie algebras [21, Theorem 7.8.5]. Such a product satisfies Mathlib's definition, which

asks that the ideal lattice be generated by independent non-abelian atoms: the atoms are the simple factors. Conversely, Mathlib proves that its semisimple Lie algebras have zero radical (`LieAlgebra.IsSemisimple.instHasTrivialRadical`). So each axiom is an instance of the statement it quotes. Everything else is proved, including Jacobi's formula over $k[t]$, the exponential of the nilpotent derivation A and the two lemmas. In the formal proof a cocycle is a linear map from `g` to `n`, the conjugation by tz is the exponential series of tA truncated to a polynomial in t, whose value at $t = 1$ is Mathlib's exponential of A, and Lemma 3.1 is the theorem `det_translated`. To reproduce the check, run `lake exe cache get` and `lake build` in the `lean` folder of the accompanying files; with the Mathlib cache the build takes about two minutes and under 3 GB of memory.

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