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AN IRRATIONALITY MEASURE FOR $\zeta(5)$

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Abstract. Let μ denote the irrationality exponent. We prove that $|\zeta(5) - p/q| > q^{-87}$ for all integers p and all sufficiently large integers q, so $\mu(\zeta(5)) \le 87$. Fauzan proved that $\zeta(5)$ is irrational with rationally normalized Hankel determinants, and his argument gives $\mu(\zeta(5)) \le 260$. Giraudin lowered this to 171 by a sharper bound for a relative matrix norm. We keep Fauzan's determinants and improve the two other constants of the method. On the arithmetic side a reflection symmetry modulo p of the reduced Gram form gives extra divisibility of the determinants by the primes p with $K/200 \le p \le K/3$, a second local bound with positive row weights is used for the primes $K/3 < p \le K$, and the prime sum is evaluated exactly over a longer range with a cutoff that grows with K. The normalizing factor then grows at most like $e^{1.288003K^2}$. On the analytic side we bound the logarithmic energy by a comparison measure on one interval, a Chebyshev series of degree 60 against the arcsine density whose positivity is an exact sum of squares. Its potential and energy have closed forms, and the pointwise inequality is checked on 110 cells in exact arithmetic. This gives decay at the rate $e^{-1.38500086K^2}$. With Giraudin's form of the relative norm estimate the exponent is below 86.54. The proof is formalized in Lean 4.

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1. Introduction

1.1. **The result.** The *irrationality exponent* $\mu(\xi)$ of a real number ξ is the infimum of the real numbers μ such that $|\xi - p/q| > q^{-\mu}$ for all integers p and all sufficiently large integers q. It does not change when ξ is multiplied by a nonzero rational number or translated by a rational number. A finite bound for $\mu(\xi)$ is an irrationality measure for ξ, and it shows that ξ is irrational. We write

$$\zeta(5) = \sum_{n \geq 1} \frac{1}{n^5} = 1.03692\,77551\ldots$$

Theorem 1.1. *There is an integer q_0 such that*

$$\left|\zeta(5) - \frac{p}{q}\right| > q^{-87}$$

for all integers p and all integers $q \geq q_0$. In particular $\mu(\zeta(5)) \leq 87$.

The proof gives the exponent $\lambda\rho/(D - \Lambda) < 86.535$, with the constants of Table 1. Theorem 1.1 has been verified in the Lean 4 proof assistant, together with every theorem, proposition and lemma used in its proof. The formal proof has no unproved steps and uses only the standard axioms. Section 10 describes it. The remarks are not part of the formal proof. The formalization is available at `https://github.com/slitsey/catalan-irrationality-measure`. The use of AI in this work is disclosed at the end of the paper.

1.2. **Background.** *Odd zeta values.* Apéry proved that $\zeta(3)$ is irrational [1], and Beukers gave his integral proof [3]. For the higher odd values the classical results concern families. Rivoal [29] and Ball and Rivoal [2] proved that infinitely many of the numbers $\zeta(2k+1)$ are irrational. Their bound for the dimension of the $\mathbb{Q}$-span of $1, \zeta(3), \ldots, \zeta(s)$ grows like $\log s$. Zudilin proved that at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational [34, 35]. Fischler, Sprang and Zudilin showed that the number of irrational values among $\zeta(3), \ldots, \zeta(s)$ grows faster than any power of $\log s$ [11]. Fischler improved the dimension bound to a multiple of $\sqrt{s/\log s}$ with Siegel's lemma [9]. Fischler, Lai, Sprang and Zudilin raised the count of irrational values to a multiple of $\sqrt{s \log\log s/\log s}$ with a Hankel form of the construction [10].

The number $\zeta(5)$. Brown and Zudilin constructed simultaneous rational approximations to $\zeta(3)$ and $\zeta(5)$ from cellular integrals on the moduli space $\mathcal{M}_{0,8}$. They obtained an effective sequence of rationals with $0 < |\zeta(5) - p/q| < q^{-0.86}$ [4]. This exponent does not imply irrationality. Fauzan proved that $\zeta(5)$ is irrational [7]. His approximations are integer polynomials of growing degree, rationally normalized Hankel determinants of a functional whose value at $\zeta(5)$ is integration against a positive weight. The same paper proves $\mu(\zeta(5)) \leq 260$ [7, Corollary 1.2]. Giraudin replaced two worst-case estimates in Fauzan's bound for a relative matrix norm by closed-form functions of the node, and obtained $\mu(\zeta(5)) \leq 171$ [15]. His theorem is stated under

the hypotheses of Fauzan's paper. Fasten [6], Firsching [8] and Romik [30] verified Fauzan's irrationality theorem in Lean. Romik's formalization takes the prime number theorem as an axiom. As far as we know, 260 and 171 are the only published irrationality measures for $\zeta(5)$. Theorem 1.1 improves them to 87, and its proof is unconditional and formally verified.

Recent results. Since September 2026 several new results have appeared in this area, some obtained with AI systems. OpenAI proved that Catalan's constant G is irrational [25] and that $\mu(\pi) = 2$ [26], both with Lean formalizations. Liu, Jiang and Zhang proved $\mu(\log r) = 2$ for rational $r > 0$, $r \neq 1$ [23]. Giraudin improved the bounds for $\mu(\zeta(2))$ and $\mu(\zeta(3))$ [13, 14]. Lai combined Fauzan's determinants with simultaneous approximations to show that 1, $\zeta(3)$ and $\zeta(4)$ are linearly independent [18]. Leibtag, Barkan, Kalisch, Weinbaum, Shalyt and Kaminer used conservative matrix fields to improve the measures for $\zeta(2)$ and $\zeta(3)$, and they also proved this linear independence [20].

The method. Irrationality proofs with determinants of growing size go back to Zudilin's determinantal approach [36]. The measure method we use, a family of integer polynomials whose values near the constant are small and nonzero, is a one-variable form of the criteria of Nesterenko [24] used by Ball and Rivoal [2]. The arithmetic of the coefficients is handled prime by prime, as in the work of Hata [16], Rhin and Viola [27, 28] and Zudilin [35]. The size of the determinants is controlled by Andréief's identity [12] and a logarithmic energy problem with an external field [31].

1.3. **The companion papers.** This paper applies one method to a third constant. The method consists of Hankel or Löwner determinants, denominators bounded prime by prime, decay proved through an energy problem, and a formal proof of every step. The first companion paper [21] refines OpenAI's determinant for G and proves $\mu(G) \leq 270$. The second [22] uses Löwner determinants of digamma combinations and proves measures on three planes of numbers, spanned modulo $\mathbb{Q}$ by $L(2, \chi_{-3})$ and π^2, by $\zeta(3)$ and π^2, and by $\zeta(3)$ and π^4. We use the definitions of μ, the measure criterion and the potential-theoretic language of these papers. The construction here is Fauzan's, with his notation.

1.4. **What a measure requires.** Fauzan's irrationality proof needs three facts about his polynomials Q_n of degree $h = 37n$: integral coefficients, $Q_n(\zeta(5)) > 0$, and $Q_n(\zeta(5)) < e^{-cn^2}$. If $\zeta(5) = a/b$, then $b^h Q_n(a/b)$ is a positive integer below 1 for large n. A measure needs more. Let p/q be a rational close to $\zeta(5)$. Three estimates are needed.

(i) *Denominators.* A normalizing factor m_K with $m_K F_K \in \mathbb{Z}[X]$ and $\log m_K \leq (\Lambda + o(1))K^2$.
(ii) *Decay on a neighbourhood.* $|F_K(x)| \leq e^{-(D-o(1))K^2}$ for every real x with $|x - \zeta(5)| \leq e^{-(\rho+o(1))K}$, not only at $x = \zeta(5)$.

(iii) *Nonvanishing near the point.* $F_K(p/q) \neq 0$ for every rational p/q in the same neighbourhood.

Then $\mu(\zeta(5)) \leq \lambda\rho/(D-\Lambda)$, where $\lambda = h/K$ is the degree per unit of scale. For Fauzan's determinants the matrix is affine in the variable, and it is positive definite at $\zeta(5)$. A relative norm bound for the coefficient matrix of the variable therefore gives (ii) and (iii) together, with the neighbourhood exponent ρ of the norm. The constant ρ was bounded by Fauzan [7, Appendix C] and sharpened by Giraudin [15]. The exponent is very sensitive to the gap $D-\Lambda$. Fauzan's constants give $D-\Lambda \approx 0.0494$, and ours give $D-\Lambda > 0.0969$. This halves the exponent.

1.5. **New ideas.** Our proof keeps Fauzan's determinants unchanged and rests on the following ideas.

(1) *The parity gain.* For a prime p in the inner range, Fauzan bounds the p-adic valuation of the determinant by row weights attached to the classes of poles modulo p. On the rows of integral weight we compute the leading term of the Gram form modulo p. It is a sum over residue classes of local bilinear forms, and each local form is odd under the reflection $z \mapsto -1-z$ of its pole set. The classes whose space of local polynomials has more even than odd elements under this reflection give radical vectors of the reduced form, and a dimension count bounds the loss from the coupling between classes. Each radical vector raises the valuation of the determinant by one. For $K/200 \leq p \leq K/3$ this gives $p\, g_{\mathrm{par}}(K/p) + O(M)$ extra units, with an explicit density $0 \leq g_{\mathrm{par}} \leq \frac{1}{2}$ (Theorem 6.1).

(2) *A second bound in the outer range.* For $K/3 < p \leq K$ we also apply Fauzan's class construction of the inner range, with weights that may be positive. The polynomial part of the functional is kept, so the bound holds for the whole matrix. Its cost is the inner profile R continued to $1 < K/p < 3$, and the minimum with Fauzan's outer bound saves $3203/163200$ in Λ (Theorem 5.1).

(3) *An exact prime sum.* Fauzan's inner profile R has a closed form $R(x) = xF(x) + Q(x)$ with an explicit sawtooth F and $-\frac{1}{2} \leq Q \leq \frac{13}{8}$. We integrate $R(x)x^{-3}$ exactly over $[3, 200]$ and bound the tail by one integration by parts. We let the cutoff between the small and the inner primes grow like $K^{1/4}$. The error terms of the local bounds are uniform in the cutoff, so the terms of order $1/M$ of a fixed cutoff disappear (Theorem 7.1). Together with items 1 and 2 this gives $\Lambda = 1.288003$, against 1.31758.

(4) *A comparison measure for the energy problem.* Fauzan bounds the energy with a sum of sixteen arcsine measures. We use one measure $\sigma = f\,\omega_{[0,b]}$, where $\omega_{[0,b]}$ is the arcsine probability measure of $[0, b]$ and f is a Chebyshev series of degree 60. Its positivity is an exact sum of squares. Its potential and energy have closed forms, inside and outside the support. The pointwise inequality of the energy method is checked on 110 cells by Taylor polynomials of order four with explicit remainders.

A layer-cake estimate controls the hard edge of σ in the regularization. This gives $D = 1.38500086$ against 1.36699 (Theorem 8.5).

The relative norm exponent $\rho = 9.07427$ follows Giraudin's argument [15] with another choice of parameters (Section 4). The measure criterion is Fauzan's argument of [7, Appendix C], stated for a family with a rational normalizing factor (Section 3).

	Λ	D	ρ	$\mu(\zeta(5)) \leq$
Fauzan [7]	1.31758	1.36699	13.866	260
Giraudin [15]	1.31758	1.36699	9.09	171
This paper	1.288003	1.38500086	9.07427	87

TABLE 1. The three constants of the measure method for Fauzan's determinants, with $\lambda = 37/40$. The bound is $\lambda\rho/(D-\Lambda)$, rounded up. Fauzan's Λ is his A_{100000}, and his D is $-U$ in [7, (6.4)]. His paper and Giraudin's use the margin $D - \Lambda > 79/1600$.

1.6. **Organization.** Section 2 recalls Fauzan's construction. Section 3 states the measure criterion and reduces nonvanishing and decay near $\zeta(5)$ to decay at $\zeta(5)$ and a relative norm bound. Section 4 proves the norm bound. Sections 5, 6 and 7 prove the bound for the normalizing factor: the local bounds, the parity gain and the prime sum. Section 8 proves the decay. Section 9 completes the proof. Section 10 describes the formalization. The disclosure of AI use follows it.

1.7. **Notation.** We write v_p for the p-adic valuation. For a polynomial A with rational coefficients, $v_p^{\mathrm{G}}(A)$ is the minimum of the valuations of its coefficients, the Gauss valuation, with $v_p^{\mathrm{G}}(0) = +\infty$. We write $\{y\} = y - \lfloor y \rfloor$ for the fractional part, and $y_+ = \max(y, 0)$. We write $\xi = \zeta(5)$ throughout.

2. Fauzan's construction

We recall the construction of [7, §2]. Throughout,

(2.1)
$$K = 40n, \quad N = 3n, \quad h = 37n, \quad \alpha = \tfrac{3}{40}, \quad \lambda = \tfrac{37}{40}, \quad H = 1 + 2\alpha = \tfrac{23}{20}.$$

Put $D_m(t) = \prod_{j=1}^{m}(t + j^2)$ and $H_j^{(5)} = \sum_{v=1}^{j} v^{-5}$. With $B_1 = -\frac{1}{2}$, the $\mathbb{Q}$-linear functional $\mathcal{F}_X$ with values in $\mathbb{Q}[X]$ is defined by (Fauzan writes μ_X; we keep μ for the irrationality exponent)

$$\mathcal{F}_X(t^e) = \frac{(-1)^e B_{2e+2}(2e+3)(2e+4)(2e+5)}{24} \qquad (e \geq 0), \tag{2.2}$$

$$\mathcal{F}_X\Big(\frac{1}{t+j^2}\Big) = j^4(X - H_j^{(5)}) - \frac{1}{4} + \frac{1}{2j} \qquad (j \geq 1),$$

and extended to rational functions with simple poles at negative integer squares by polynomial division and partial fractions. Then

$$G_K(X) = \Big[\mathcal{F}_X\Big(\frac{D_N(t)^6 t^{i+j}}{D_K(t)}\Big)\Big]_{0\le i,j<h}, \qquad \Delta_K = \det G_K, \tag{2.3}$$

$$S_K = \frac{(K!)^{2h}4^{h-1}}{(N!)^{12h}\prod_{i=1}^{h-1}((2i)!)^2}, \qquad F_K = S_K\Delta_K \in \mathbb{Q}[X].$$

The entries of $G_K(X)$ are affine in X. We write $M_K = [X]G_K(X)$ for the matrix of coefficients of X and $G = G_K(\xi)$.

Proposition 2.1 (Fauzan)**.** *(a) For every rational function R in the domain of $\mathcal{F}_X$,*

$$\mathcal{F}_\xi(R) = \int_0^\infty R(y^2)\,w(y)\,dy, \qquad w(y) = \frac{(2\pi)^4 y^5}{12}\sum_{\ell\ge1}\ell^4 e^{-2\pi\ell y}.$$

(b) G is positive definite. In particular $F_K(\xi) > 0$.
(c) $\deg\Delta_K = h$.

Part (a) comes from Euler's formula for $\zeta(2e+2)$ and Hermite's integral for the Hurwitz zeta function [7, (2.10)]. Part (b) follows because $w > 0$, and part (c) from the Vandermonde factorization $M_K = V\,\mathrm{diag}(\dots)V^T$ [7, (2.9)]. The integral representation also gives Andréief's formula

$$\Delta_K(\xi) = \frac{1}{h!}\int_{(0,\infty)^h}\prod_{i<j}(y_i^2-y_j^2)^2\prod_{i=1}^h\frac{D_N(y_i^2)^6}{D_K(y_i^2)}\,w(y_i)\,dy. \tag{2.4}$$

3. The measure criterion

3.1. **The deduction.** The criterion is that of the companion papers [21, 22], in the form with nonvanishing near the point. Let (P_n) be polynomials with rational coefficients, indexed by $n \ge n_0$, at the scale $K = 40n$.

Theorem 3.1 (measure criterion)**.** *Let ξ be real and let $\Lambda < D$, $\rho > 0$ and $\gamma > 0$ be real numbers. Suppose that $\deg P_n \le \gamma K$, and that for every $\varepsilon > 0$ and all large n the following hold.*

(a) There is a positive integer d_n with $\log d_n \le (\Lambda+\varepsilon)K^2$ and $q^{\deg P_n} d_n P_n(p/q) \in \mathbb{Z}$ for all integers p and $q > 0$.
(b) $|P_n(\xi+t)| \le e^{-(D-\varepsilon)K^2}$ for every real t with $|t| \le e^{-(\rho+\varepsilon)K}$.
(c) $P_n(x) \ne 0$ for every rational x with $|\xi - x| \le e^{-(\rho+\varepsilon)K}$.

Then ξ is irrational. Moreover, for every $\mu_0 > \gamma\rho/(D-\Lambda)$ there is an integer q_0 such that $|\xi - p/q| > q^{-\mu_0}$ for all coprime integers p and q with $q \ge q_0$.

Proof. Let $\eta = D - \Lambda$. If $\xi = a/b$ were rational, with $b > 0$, then ξ lies in every window of (c). So $b^{\deg P_n} d_n P_n(\xi)$ is a nonzero integer. By (a) and (b) with $t = 0$ its absolute value is at most $b^{\gamma K} e^{-(\eta-2\varepsilon)K^2}$, which is below 1 for small ε and large n. This is impossible, so ξ is irrational.

Now let $\mu_0 > \mu_1 > \gamma\rho/\eta$, and suppose $|\xi - p/q| \le q^{-\mu_0}$ with q large. Take the least n with $\rho K \ge \mu_1 \log q$. Then $\rho K \le \mu_1 \log q + 40\rho$, so $|\xi - p/q| \le q^{-\mu_0} \le e^{-(\rho+\varepsilon)K}$ for a small fixed ε and large q. By (c), $z = q^{\deg P_n} d_n P_n(p/q)$ is a nonzero integer, so $|z| \ge 1$. By (a) and (b), $|z| \le q^{\gamma K} e^{(\Lambda+\varepsilon)K^2 - (D-\varepsilon)K^2}$. Hence $(\eta - 2\varepsilon)K \le \gamma \log q$. But $\eta K \ge (\mu_1\eta/\rho)\log q$ and $\mu_1\eta/(\gamma\rho) > 1$. This is a contradiction for small ε and large q. □

The second statement concerns fractions in lowest terms. The following step, used also in the companion papers, removes this restriction once ξ is known to be irrational.

Lemma 3.2. *Let ξ be irrational and $E \ge 0$. Suppose that for every $e > 0$ there is q_0 with $|\xi - p/q| > q^{-(E+e)}$ for all coprime p and q with $q \ge q_0$. Then for every $\mu_0 > E$ there is q_1 with $|\xi - p/q| > q^{-\mu_0}$ for all integers p and all integers $q \ge q_1$.*

Proof. Take $e = \mu_0 - E$ and the corresponding q_0. Write $p/q = p'/q'$ in lowest terms, so $q' \le q$. If $q' \ge q_0$, then $|\xi - p/q| > q'^{-\mu_0} \ge q^{-\mu_0}$. The fractions with $q' < q_0$ have finitely many denominators. Since ξ is irrational, $c = \min_{q'<q_0} \min_{p'} |\xi - p'/q'|$ is positive. For $q > c^{-1/\mu_0}$ we get $|\xi - p/q| \ge c > q^{-\mu_0}$. □

Fauzan's normalizing factor m_K is a positive rational number with exponents of both signs. A positive exponent divides F_K by a power of p, and a negative one clears a denominator. So m_K is not an integer multiple of a denominator. We apply Theorem 3.1 to the family $P_n = m_K F_K$ itself. Its coefficients are integers, so (a) holds with $d_n = 1$ and $\Lambda = 0$. If $\log m_K \le (\Lambda + o(1))K^2$ and F_K satisfies (b) with the rate D, then P_n satisfies (b) with the rate $D - \Lambda$. The exponent is the same.

3.2. **Nearby nonvanishing from a relative norm bound.** The following lemma is the version of [7, Appendix C] that we use.

Lemma 3.3 (relative norm)**.** *Let G be a real positive definite $h \times h$ matrix and M a real symmetric $h \times h$ matrix with $|v^T M v| \le \beta\, v^T G v$ for all $v \in \mathbb{R}^h$. If $|\delta|\beta h \le \frac{1}{2}$, then $0 < \det(G + \delta M) \le 2 \det G$.*

Proof. Write $G + \delta M = G^{1/2}(I + \delta B)G^{1/2}$ with $B = G^{-1/2} M G^{-1/2}$. The eigenvalues β_i of B satisfy $|\beta_i| \le \beta$. So $1 + \delta\beta_i \ge \frac{1}{2} > 0$, and $\prod_i (1 + \delta\beta_i) \le e^{|\delta|\beta h} \le e^{1/2} \le 2$. □

Definition 3.4. We say that ρ is a *norm exponent* if $|v^T M_K v| \le e^{\rho K}\, v^T G v$ for all $v \in \mathbb{R}^h$ and all large n.

Since $G_K(x) = G + (x - \xi)M_K$, Lemma 3.3 gives the following.

Corollary 3.5. *Let ρ be a norm exponent. If $|x - \xi| e^{\rho K} h \le \frac{1}{2}$, then $0 < F_K(x) \le 2F_K(\xi)$. Hence for every $\varepsilon > 0$ and all large n, $F_K(x) \ne 0$ and $|F_K(x)| \le 2F_K(\xi)$ whenever $|x - \xi| \le e^{-(\rho+\varepsilon)K}$.*

The second statement uses $2he^{-\varepsilon K} \le 1$ for large K. So the decay at ξ alone gives (b), and positivity gives (c).

4. The norm exponent

We follow Giraudin [15], who refines [7, Proposition C.1]. The field of the energy problem is

$$V(t) = 2\pi\sqrt{t} + \int_0^1 \log(t+u^2)\,du - 6\int_0^\alpha \log(t+u^2)\,du \qquad (t \ge 0). \tag{4.1}$$

Let $g(x) = x(\log|x| - 1)$ with $g(0) = 0$, and for $\alpha < u \le 1$ put

$$L(u) = 6\big[g(\alpha - u) + g(\alpha + u)\big] - g(1-u) - g(1+u).$$

For $0 < a < b$ put $Y(u) = (2u^2 + a + b)/(b-a)$ and $\Phi_{a,b}(u) = L(u) + 2\lambda \operatorname{arcosh} Y(u)$.

Proposition 4.1. *The number $\rho = 907427/10^5$ is a norm exponent.*

Proof. We list the steps. Write $u_j = j/K$, let $q \in \mathbb{R}[t]$ have degree less than h, and put $p(t) = q(K^2 t)$.

(1) *Residues.* $q^T M_K q = A_K \sum_{N<j\le K} r_j p(-u_j^2)^2$ with $A_K = K^{12N-2K+6}$ and $r_j = u_j^4 \prod_{l\le N}(u_l^2 - u_j^2)^5 / \prod_{N<k\le K, k\ne j}(u_k^2 - u_j^2)$. This is the Vandermonde factorization of M_K.

(2) *Nodes.* $|r_j| \le \exp(KL(u_j))$. The logarithm of $|r_j|$ is a Riemann sum of the monotone functions $\log|u^2 - v^2|$, compared with the antiderivatives $g(u+v) \mp g(u-v)$ by the mean value theorem.

(3) *Gram matrix.* For $W \ge \max_{[a,b]} V$, $q^T G q \ge A_K c_{a,b} e^{-KW} \int_a^b p(t)^2\,dt$ with $c_{a,b} = (2\pi)^4 a^3/(24(1+b))$. Keep the term $\ell = 1$ of w, restrict to $y = K\sqrt{t}$ with $a \le t \le b$, and compare the products D_N^6/D_K with e^{-KV} by Riemann sums.

(4) *Legendre polynomials.* If $\deg P < h$ and $u > 0$, then $P(-u^2)^2 \le h^2 e^{2h \operatorname{arcosh} Y(u)} (b-a)^{-1} \int_a^b P^2$. Expand P in the Legendre polynomials of $[a,b]$, use Cauchy–Schwarz, and bound the Legendre polynomials outside the interval through Rodrigues' formula.

(5) *Assembly.* Steps (1)–(4) give $|q^T M_K q| \le h^3((b-a)c_{a,b})^{-1} \exp\big(K(W + \sup_{\alpha<u\le1} \Phi_{a,b})\big)\, q^T G q$.

(6) *Shape of V.* With $y = \sqrt{t}$, the derivative of $V(y^2)$ is $2(\pi + \arctan(1/y) - 6\arctan(\alpha/y))$. This is increasing on $(0, \frac{3}{20}]$ and positive on $[\frac{3}{20}, \infty)$. So $\max_{[a,b]} V = \max(V(a), V(b))$.

(7) *Concavity.* If $ab \le \alpha^4$, then $\Phi'_{a,b}$ is decreasing on $(\alpha, 1)$. So $\Phi_{a,b}(u) \le \Phi_{a,b}(u_0) + \Phi'_{a,b}(u_0)(u - u_0)$ for every $u_0 \in (\alpha, 1)$.

(8) *Numbers.* Take $a = 10^{-8}$, $b = 242976/10^6$ and $u_0 = 973660012/10^9$. Then $V(a)$ and $V(b)$ are at most $W = 30449413/10^7$, $\Phi_{a,b}(u_0) \le 60293244/10^7$ and $|\Phi'_{a,b}(u_0)| \le 10^{-7}$. By (7), $\sup_{\alpha<u\le1} \Phi_{a,b} \le 60293245/10^7$. Hence $W + \sup \Phi_{a,b} \le 9.0742658 < \rho$, and the margin absorbs the factor $h^3((b-a)c_{a,b})^{-1}$ for large K.

The numbers in (8) are rational enclosures of finitely many logarithms and arctangents. □

Steps (1)–(6) are Giraudin's Lemmas 2, 3 and 5 and his Proposition 7 [15]. He takes $a = 10^{-3}$, $b = 49/200$ and obtains 9.09. He locates the maximum of $\Phi_{a,b}$ by its unique critical point. We use the tangent bound of step (7) instead, which needs one evaluation of $\Phi_{a,b}$ and of its derivative.

5. Local bounds for the denominators

5.1. **Fauzan's bounds.** Let p be a prime. A lower bound $v_p^{\mathrm{G}}(F_K) \geq L_p$ allows the factor p^{-L_p} in the normalization. If $L_p < 0$ this clears a denominator, and if $L_p > 0$ it removes a power of p common to all coefficients. Fauzan proves such bounds in three ranges [7, §§3–5]. Fix a cutoff $M \geq 40$ with $K \geq 200M^2$.

Small primes, $pM \leq K$. Here $v_p^{\mathrm{G}}(F_K) \geq -6h\lfloor \log_p 5K \rfloor - hv_p(24)$ [7, (3.12)].

Inner primes, $K/M < p \leq K/3$. Let $m = (p-1)/2$. For $0 \leq a \leq m$ and $A \geq 1$ let $\ell_A(a) = \#\{1 \leq j \leq A : j \equiv \pm a \pmod p\}$, and put $b_a = 3\ell_N(a)$. With the reserve $L_0 = 4M + 10$ define integers T and E by $mT + E = h - L_0 + 3(N - \lfloor N/p \rfloor)$ and $0 \leq E < m$. The value $\epsilon_a = 1$ is given to E classes $a \neq 0$ of largest $\ell_K(a)$, and $\epsilon_a = 0$ to the others. Put $L_a = T - b_a + \epsilon_a$. The rows of the basis are

$$E_{a,i}(t) = \prod_{c \neq a} (t + c^2)^{L_c} \, (t + a^2)^i \qquad (0 \leq a \leq m,\ 0 \leq i < L_a), \tag{5.1}$$

and for $a \neq 0$ the row (a, i) has the weight

$$w_{a,i} = i + b_a - \frac{\ell_K(a) + 4}{2}. \tag{5.2}$$

The zero class has the weights of [7, (4.7)]. The basis is unimodular over $\mathbb{Z}_{(p)}$, the entry in rows s and t has Gauss valuation at least $w_s + w_t$, and so $v_p^{\mathrm{G}}(\Delta_K) \geq \gamma_p^{\mathrm{in}} = 2\sum_s w_s$ [7, Proposition 4.1].

Outer primes, $K/3 < p \leq K$. Fauzan separates the residue terms from a polynomial correction of bounded rank. With nonpositive half-integer weights and a complementary-minor lemma he obtains an explicit integer γ_p^{out} with $v_p^{\mathrm{G}}(\Delta_K) \geq \gamma_p^{\mathrm{out}}$ [7, (4.14) and Proposition 4.3]. For $p > K$ the matrix is integral, and for $p > 2h$ the factor S_K is a p-adic unit.

5.2. **The outer extension.** Fauzan uses the class basis (5.1) only in the inner range. It can be used in the outer range too. There the weights (5.2) are not clipped at zero, and the polynomial part of the functional must be kept in the local bounds.

Theorem 5.1 (outer extension)**.** *Let $p \geq 101$ be a prime with $K/3 < p \leq K$. Reserve $L_0 = 8$ rows at the zero class, and define T, E by $mT + E = h - 8 + 3N$ with $0 \leq E < m$. Give the E extra rows to the classes of largest*

$2\ell_K(a) + [a \le N]$, *and put* $L_a = T - 3[a \le N] + \epsilon_a$. *With the weights* (5.2) *on the classes* $a \neq 0$,

$$v_p^{\mathrm{G}}(\Delta_K) \ge \gamma_p^{\mathrm{ext}} = 2\sum_s w_s,$$

and γ_p^{ext} *is an integer. Moreover* $|\gamma_p^{\mathrm{ext}} - p\,\Gamma(K/p)| \le 83$, *where* Γ *is Fauzan's inner function* [7, (5.4)].

Proof sketch. Since $p \ge 101$ and $p \le K$ we have $n \ge 3$. Since $p > 2N$, $\ell_N(a) = [a \le N]$ for $a \neq 0$, so $b_a = 3[a \le N]$. Counting rows gives $2 \le T \le 6$ and $L_0 + \sum_a L_a = h$. If $T = 2$, every class with $a \le N$ receives an extra row, because the key $2\ell_K(a) + [a \le N]$ puts these classes first among those of equal $\ell_K(a)$. Hence $L_a \ge 0$ for every a. We then apply the class bound of the inner range [7, Proposition 4.1] to the whole local rational function, polynomial part included. It needs three conditions. The first is separation and degree, $2K < p^2$ and $104n + 6 < p^2$, which hold because $p > K/3$. The second is that the top weights of two classes $a, c \neq 0$ differ by at most 1, which holds because ℓ_K takes two consecutive values and the extra rows go to the larger ones. The third compares the top weight of each class $a \neq 0$, minus one, with the zero class. It holds because $T \le 6$, $\lfloor K/p \rfloor \le 2$ and $L_0 = 8$. The approximation by $p\Gamma(K/p)$ is the count of classes by their pole numbers, which are floor functions of K/p and a/p. □

5.3. **The normalizing factor.** Fauzan's inner profile is $R(x) = -\Gamma(x) - \mathcal{N}(x)$ [7, (5.6)], where $\mathcal{N}(x) = 2\lambda x\lfloor x \rfloor - 12\lambda x\lfloor \alpha x \rfloor - 2J(\lambda x)$ and $J(u) = \lfloor 2u \rfloor u - \lfloor 2u \rfloor(\lfloor 2u \rfloor + 1)/4$. It has a closed form. Put $\tau = \{2Hx\}$, $s = \{2x\}$, $\vartheta = \{2\lambda x\}$, $t_e = \min(\{x\}, 1 - \{x\})$, $t_n = \min(\{\alpha x\}, 1 - \{\alpha x\})$, and let ϵ_f and ϵ_g be $+1$ or -1 according as $\{x\} < \frac{1}{2}$ and $\{\alpha x\} < \frac{1}{2}$.

Lemma 5.2 (closed profile)**.** $R(x) = xF(x) + Q(x)$, *where* $F(x) = 4\lambda + 2\lambda\{x\} - 12\lambda\{\alpha x\}$ *and*

$$Q(x) = \tfrac{1}{2}\big[\tau(\tau - s) - (\tau - s)_+ + \vartheta(1 - \vartheta)\big] + 9t_n(1 - 2t_n) - 3\epsilon_f\epsilon_g\big(\min(t_e, t_n) - 2t_e t_n\big).$$

Moreover $-\frac{1}{2} \le Q \le \frac{13}{8}$.

The proof is a direct evaluation of the integral in Γ, cell by cell in the fractional parts. The scalar factor satisfies $0 \le v_p(S_K) - p\mathcal{N}(K/p) \le \lfloor 2h/p \rfloor$ whenever $p^2 > 2K$ (Legendre's formula). Fauzan's inner estimate becomes uniform in the cutoff:

$$|-v_p(S_K) - \gamma_p^{\mathrm{in}} - p\,R(K/p)| \le 7501M^2 \qquad (K/M < p \le K/3). \tag{5.3}$$

In the outer range $|(-v_p(S_K) - \gamma_p^{\mathrm{ext}}) - p\,R(K/p)| \le 86$. For $K/3 < p \le 2h$, Fauzan's outer cost is $-v_p(S_K) - [p \le K]\,\gamma_p^{\mathrm{out}} = K\,T_0(p/K) + O(1)$, with

error at most 10, where

$$T_0(y) = R_0(y) - d(y) - 2\lambda\lfloor 1/y\rfloor + \sum_{j=1}^{5}(2\lambda - jy)_+$$

is the integrand of [7, (5.10)], with R_0 and d as in [7, (5.8), (5.9)].

Definition 5.3 (the local exponents)**.** Let $M \geq 40$ and $K \geq 200M^2$. For a prime $p \leq 2h$ put

$$L_p = \begin{cases} -6h\lfloor \log_p 5K\rfloor - hv_p(24), & pM \leq K, \\ v_p(S_K) + \gamma_p^{\mathrm{in}} + [K \leq 200p]\, k_p, & K/M < p \leq K/3, \\ v_p(S_K) + \max(\gamma_p^{\mathrm{out}}, \gamma_p^{\mathrm{ext}}), & K/3 < p \leq K, \\ v_p(S_K), & K < p \leq 2h, \end{cases}$$

where k_p is the parity gain of Section 6. Put $m_K = \prod_{p\leq 2h} p^{-L_p}$. Since $K \geq 200M^2 \geq 320000$, every prime in the third case satisfies $p > K/3 > 101$, so Theorem 5.1 applies.

Proposition 5.4. *m_K is a positive rational number and $m_K F_K \in \mathbb{Z}[X]$. Also $\log m_K = -\sum_{p\leq 2h} L_p \log p$.*

Proof. At each prime $p \leq 2h$ the bound $v_p^{\mathrm{G}}(F_K) \geq L_p$ holds. For the small primes it is Fauzan's [7, (3.12)]. In the inner range it is [7, Proposition 4.1], and Theorem 6.1 below when $K \leq 200p$. In the outer range it is Fauzan's [7, Proposition 4.3] or Theorem 5.1, whichever is larger. For $p > K$ it is [7, Proposition 4.3]. At $p > 2h$ both S_K and Δ_K are p-integral. □

6. The parity gain

6.1. **Statement.** Let p be an inner prime, $K/M < p \leq K/3$, with the allocation of Section 5 and $L_0 = 4M + 10$. The gain is used only when $K \leq 200p$. By (5.2), $w_{a,i}$ is an integer exactly when $\ell_K(a)$ is even. The counts $\ell_K(a)$ for $a \neq 0$ take two consecutive values. Let $\mathcal{S} = \{a \neq 0 : \ell_K(a) \text{ even}\}$ and let $\mathcal{E} = \{a : \epsilon_a = 1\}$. Let $\mathcal{S}_0$ and $\mathcal{S}_1$ be the sets of $a \in \mathcal{S}$ with $\ell_N(a)$ even and odd, respectively. Define

$$k_p = \begin{cases} \big[\#(\mathcal{S}_0 \setminus \mathcal{E}) - \#\{b \in \mathcal{S}_1 \cap \mathcal{E} : L_b > 0\} - L_0\big]_+, & T \text{ odd}, \\ \big[\#(\mathcal{S}_0 \cap \mathcal{E}) - \#(\mathcal{S}_1 \setminus \mathcal{E}) - L_0\big]_+, & T \text{ even}. \end{cases}$$

Theorem 6.1 (parity gain)**.** *For every inner prime p with $K \leq 200p$, $v_p^{\mathrm{G}}(\Delta_K) \geq \gamma_p^{\mathrm{in}} + k_p$.*

The count k_p has a density. Put t_e, t_n as in Lemma 5.2, $s' = \frac{1}{2} - t_e$, $r_n = (t_n - t_e)_+$, $T(x) = \lfloor \frac{23}{10}x\rfloor$ and $u = \frac{23}{20}x - \frac{1}{2}T(x)$. Let $e = (u - t_e)_+$ if $\{x\} < \frac{1}{2}$ and $e = \min(u, s')$ otherwise. Define

$$g_{\mathrm{par}}(x) = \begin{cases} (e - r_n)_+, & T(x) \text{ even}, \\ (s' - e - r_n)_+, & T(x) \text{ odd}. \end{cases}$$

Then $0 \le g_{\mathrm{par}} \le \frac{1}{2}$, and g_{par} is periodic with period 40.

Proposition 6.2. *For every inner prime p, $k_p \ge p\, g_{\mathrm{par}}(K/p) - 30M$.*

The proof counts the classes $a \in [1, m]$ by the values of $\ell_K(a)$, $\ell_N(a)$ and ϵ_a. These depend on a/p through floor functions of $K/p \pm a/p$ and $N/p \pm a/p$, and each type of class fills an interval of a/p up to $O(1)$ points. The reserve L_0 and the boundary effects cost at most $30M$.

6.2. **The leading term modulo p.** We now describe the proof of Theorem 6.1. Following Fauzan [7, §3], the substitution $t = -x^2$ turns $\mathcal{F}_X$ into a functional τ_X on rational functions of x with simple integer poles, $\mathcal{F}_X(R) = \tau_X(x^5 R(-x^2))$. On polynomials $\tau(P) = \mathcal{L}(P''')/24$, where $\mathcal{L}(x^k) = B_k$, and $\tau_X(1/(x-r)) = H^{(5)}_{d(r)} - X$ with $d(r) = r$ for $r \ge 0$ and $d(r) = -r-1$ for $r < 0$.

We first replace the factors $t + c^2$ of (5.1) by the centred factors $t + c^2 - cp$. Under $x = c + pz$ the centred factor becomes $p(-2c(z + \frac{1}{2}) - pz^2)$, with leading part proportional to $z + \frac{1}{2}$. The centred basis is again unimodular, and its entries satisfy the same bounds $w_s + w_t$. So Fauzan's bound $v_p^{\mathrm{G}}(\Delta_K) \ge \gamma_p^{\mathrm{in}}$ holds for it, with the same weights.

For a finite set $I \subset \mathbb{Z}$ put $B_I(z) = \prod_{k \in I}(z - k)$. For a residue c modulo p let I_c be the set of k with $c + pk$ a pole, the local pole set. For a polynomial W over $\mathbb{F}_p$ define the local symbol

$$C_I(W) = \tau\Big(\Big\lfloor \frac{W}{B_I} \Big\rfloor\Big) + \sum_{k \in I} \mathrm{res}_{z=k} \frac{W(z)}{B_I(z)}\, H^{(5)}_{d(k)} \pmod p, \tag{6.1}$$

where $\lfloor \cdot \rfloor$ is the polynomial part, and the bilinear form $C_I(U, V) = C_I(UV)$. Both parts are needed. The polynomial part has degree up to about p, and it contributes at the same level as the residues.

Lemma 6.3 (leading term of τ_X)**.** *Let $A \in \mathbb{Q}[x]$ and let Π be a product of distinct linear factors $x - r$, with the poles distinct modulo p inside each residue class and the degrees below p^2. Suppose that $A(c + pz)$ has Gauss valuation at least $e(c)$ for every c, and put $\beta = \min_c(e(c) - \#\{r \equiv c\})$. Then $p^{4-\beta}\tau_X(A/\Pi)$ has integral constant term, and its reduction modulo p is the sum over the residues c with $e(c) - \#\{r \equiv c\} = \beta$ of $u_c^{-1} C_{I_c}(\bar{a}_c)$. Here $\bar{a}_c$ is the reduction of $p^{-e(c)} A(c + pz)$ and u_c is the product of $c - r$ over the poles $r \not\equiv c$. The coefficient of X in $p^{4-\beta}\tau_X(A/\Pi)$ has valuation at least 5.*

The proof has three ingredients. The distribution identity $\tau(P) = p^{-4} \sum_{c<p} \tau(P(c + pz))$ follows from the difference equation $\tau(P(x+1)) - \tau(P) = P''''(0)/24$ and Newton's expansion. The Taylor coefficients of the binomial polynomials at $c + pz$ of degree above p are divisible by p^2, and the Tate-algebra bound of [7, Lemma 3.1] disposes of them. Finally $p^5 H^{(5)}_{d(c+pk)} \equiv H^{(5)}_{d(k)} \pmod{p^5}$, since the multiples of p up to $d(c + pk)$ are the $p\,j$ with $j \le d(k)$.

6.3. **Reflection and radical vectors.** Write J for the reflection $W(z) \mapsto W(-1-z)$. If I is symmetric under $k \mapsto -1-k$ and $\#I$ is even, then B_I is J-invariant and

$$C_I(JU, JV) = -C_I(U, V). \tag{6.2}$$

This comes from the reflection identity $\tau_X(g(-1-x)) = -\tau_X(g(x))$ [7, (3.2)] and from $H^{(5)}_{d(-1-k)} = H^{(5)}_{d(k)}$. In particular $C_I(U, V) = 0$ when U and V are both J-invariant.

Normalize the Gram matrix by $\tilde{G}_{st} = p^{-\lfloor w_s \rfloor - \lfloor w_t \rfloor} G_{st}$. For rows s and t of integral weight, Lemma 6.3 identifies $\tilde{G}_{st} \bmod p$. The residues at the minimal level lie in the classes of $\mathcal{S}$, and for such a class the local pole set has $\ell_K(a)$ elements, an even number, and is symmetric. The residues c and $p-c$ give equal contributions. The result is

$$\tilde{G}_{st} \equiv \kappa_s \kappa_t \sum_{a \in \mathcal{S}} \delta_a \, C_{I_a}(\Psi_a e_s, \Psi_a e_t) \pmod{p}. \tag{6.3}$$

Here $\kappa_{a,i} = (-2a)^i$, the δ_a are units, and e_s is the unit vector of the row s. The variable z is the local variable shifted by $\frac{1}{2}$, so that J acts by $z \mapsto -z$. The polynomial B_a is the cube of the local pole polynomial of D_N, and it is J-invariant when $\ell_N(a)$ is even. Call a class $b \in \mathcal{E}$ with $L_b > 0$ active, and write $t_b(v) = v_{b,L_b-1}$ for the coordinate of its top row. For a vector v on the rows of the classes in $\mathcal{S}$, the local image at a is

$$\Psi_a v = B_a\Big(\sum_{i < L_a} v_{a,i} z^i + [a \notin \mathcal{E}] \, s_a(v) \, z^{L_a} \Big), \qquad s_a(v) = \sum_{b \text{ active}} c_{ab} \, t_b(v),$$

with constants $c_{ab} \in \mathbb{F}_p$. So a row of another class b reaches the class a only through the top row of an active class b, and only when a is not extra.

We now count radical vectors. For $a \in \mathcal{S}_0$ we have $L_a \equiv T + \epsilon_a \pmod 2$. Let V be the space of vectors supported on the rows (a, i) with $a \in \mathcal{S}_0$ and i even. Its dimension is the number of rows (a, i) with $a \in \mathcal{S}_0$ and i odd, plus the number of $a \in \mathcal{S}_0$ with L_a odd. By (6.2), the form (6.3) vanishes between two vectors whose local images are J-invariant at every class of $\mathcal{S}_0$ and zero at every class of $\mathcal{S}_1$.

Let T be odd. Then L_a is odd for $a \in \mathcal{S}_0 \setminus \mathcal{E}$ and even for $a \in \mathcal{S}_0 \cap \mathcal{E}$. The top row of a class in $\mathcal{S}_0 \cap \mathcal{E}$ has odd index, so t_b vanishes on V for every active b in $\mathcal{S}_0$, and also for every $b \in \mathcal{S}_1$. Hence $\Psi_a v$ is J-invariant for $a \in \mathcal{S}_0$ and zero for $a \in \mathcal{S}_1$, for every $v \in V$. The form between $v \in V$ and e_r can then be nonzero only when r is a row (a, i) with $a \in \mathcal{S}_0$ and i odd, or the top row of an active class in $\mathcal{S}_1$. These are the only linear conditions. So the radical contains at least $\#(\mathcal{S}_0 \setminus \mathcal{E}) - \#\{b \in \mathcal{S}_1 \cap \mathcal{E} : L_b > 0\}$ independent vectors of V.

Let T be even. Then L_a is odd exactly for $a \in \mathcal{S}_0 \cap \mathcal{E}$, and for $a \in \mathcal{S}_0 \setminus \mathcal{E}$ the power z^{L_a} is J-invariant. So $\Psi_a v$ is J-invariant for $a \in \mathcal{S}_0$. For $a \in \mathcal{S}_1 \cap \mathcal{E}$ we have $\Psi_a v = 0$. For $a \in \mathcal{S}_1 \setminus \mathcal{E}$ we impose the condition $s_a(v) = 0$, which makes $\Psi_a v = 0$. After these conditions the form between v and e_r can be

nonzero only when r is a row (a, i) with $a \in \mathcal{S}_0$ and i odd. So the radical contains at least $\#(\mathcal{S}_0 \cap \mathcal{E}) - \#(\mathcal{S}_1 \setminus \mathcal{E})$ independent vectors of V.

In both cases the vectors are supported on rows of integral weight. The L_0 rows of the zero class impose at most L_0 further linear conditions. So at least k_p independent vectors $u_1, \dots, u_{k_p}$ over $\mathbb{F}_p$ remain, with

$$\sum_s u_{j,s} \tilde{G}_{st} \equiv 0 \pmod{p} \quad \text{for every row } t \text{ of integral weight.}$$

Lemma 6.4 (weight raising)**.** *Let E be a $\mathbb{Z}_{(p)}$-unimodular basis with half-integral weights w for which the Hankel entries satisfy $v_p^{\mathrm{G}}(G_{st}) \geq w_s + w_t$. Suppose that k independent vectors over $\mathbb{F}_p$, supported on the rows of integral weight, lie in the left radical of $\tilde{G} \bmod p$ against every row of integral weight. Then $v_p^{\mathrm{G}}(\Delta_K) \geq 2\sum_s w_s + k$.*

Proof. Complete the lifts of the k vectors to a unimodular change of basis that mixes only rows of integral weight. In the new basis the k retained rows have entries of valuation at least $w_s + w_t + 1$ against the rows of integral weight. Against a row of half-integral weight the bound $w_s + w_t$ is a half-integer, so the integral valuation is at least $w_s + w_t + \frac{1}{2}$. Hence each retained row has its weight raised by $\frac{1}{2}$, and the determinant gains k. □

Theorem 6.1 follows from Lemma 6.4 with the vectors above.

Remark 6.5. In the inner range $K/p \leq 200$ the gain is about $p\, g_{\mathrm{par}}(K/p)$ units, against the cost $pR(K/p)$ of Fauzan's bound. We use the gain only for $K/p \leq 200$, where its integral against x^{-3} is computed exactly.

7. The prime sum

The prime number theorem converts the sums over p into integrals. For a bounded piecewise continuous function f on $[c_0, c_1] \subset (0, \infty)$,

$$\frac{1}{K} \sum_{c_0 K < p \leq c_1 K} f(p/K) \log p \longrightarrow \int_{c_0}^{c_1} f(y)\, dy \qquad (K \to \infty). \tag{7.1}$$

For crude bounds we use Chebyshev's inequality $\theta(y) = \sum_{p \leq y} \log p \leq y \log 4$.

Theorem 7.1 (denominator exponent)**.** *Let $M(K) = \max(40, \lfloor K^{1/4} \rfloor)$ and let m_K be the normalizing factor of Definition 5.3 with $M = M(K)$. Then for all large n, $m_K F_K \in \mathbb{Z}[X]$ and*

$$\log m_K \leq \Lambda K^2, \qquad \Lambda = \frac{1288003}{10^6}.$$

The theorem rests on the following integrals.

Proposition 7.2. *(a)* $\displaystyle\int_{1/3}^{2\lambda} T_0(y)\, dy = \frac{127751}{96000}.$

(b) $\displaystyle S_{\mathrm{out}} = \int_1^3 \left[xT_0(1/x) - R(x)\right]_+ \frac{dx}{x^3} = \frac{3203}{163200}.$

(c) $\displaystyle\int_3^{200} R(x)\frac{dx}{x^3} = I_{200}$, *an explicit rational number with* $I_{200} = -0.01297481\ldots$

(d) $\displaystyle\int_{200}^{\infty} R(x)\frac{dx}{x^3} \le -\frac{103109}{24000000}$.

(e) $\displaystyle\int_3^{200} g_{\mathrm{par}}(x)\frac{dx}{x^3} > \frac{5847}{10^6}$.

Proof. Part (a) is Fauzan's [7, (5.10)]. In (a), (b), (c) and (e) the integrands are polynomials in x and x^{-1} on finitely many intervals, on which the floor functions are constant and the minima and positive parts do not change branch. The breakpoints are rational, so the integrals are rational numbers, computed exactly.

For (d) write $R = xF + Q$ by Lemma 5.2, and put $f = \{x\}$, $g = \{\alpha x\}$, $G_0(v) = v^2/2 - v^3/3 - v/6$ and $B_4(v) = v^2(1-v)^2 - 1/30$. Let

$$P = 74g(1-g) - \lambda f(1-f), \qquad C = \tfrac{74}{\alpha}G_0(g) - \lambda G_0(f),$$
$$D_3 = -\frac{74B_4(g)}{12\alpha^2} + \frac{\lambda B_4(f)}{12}.$$

These functions are continuous and periodic in f and g. Off $\mathbb{Z}\cup\alpha^{-1}\mathbb{Z}$ they satisfy $P' = F + \lambda$, $C' = P - \bar{P}$ and $D_3' = C$, where $\bar{P} = 2923/240$. Put $\Psi = \lambda/x + (P-\bar{P})/x^2 + 2C/x^3 + 6D_3/x^4$. Then $\Psi' = F/x^2 - 24D_3/x^5$. Since $D_3 \le D_+ = 74/(360\alpha^2) + 7\lambda/2880$ and $Q \le \frac{13}{8}$, the function $\Omega = \Psi - 6D_+/x^4 - \frac{13}{16}x^{-2}$ satisfies $R(x)x^{-3} \le \Omega'(x)$ off a countable set. It is continuous, and $\Omega(x) \to 0$ as $x \to \infty$. So $\int_{200}^{\infty} R(x)x^{-3}dx \le -\Omega(200)$. At $x = 200$ both f and g vanish, so $P = C = 0$ and $D_3 = 74/(360\alpha^2) - \lambda/360$. Hence

$$\int_{200}^{\infty}\frac{R(x)}{x^3}\,dx \le -\frac{\lambda}{200} + \frac{\bar{P} + 13/16}{200^2} + \frac{\lambda}{32\cdot 200^4} \le -\frac{103109}{24000000}.$$

□

Proof of Theorem 7.1. By Proposition 5.4, $\log m_K = \sum_p (-L_p)\log p$. We bound each band.

Small primes, $p \le K/M$. Here $-L_p \le 6h\lfloor \log_p 5K\rfloor + hv_p(24)$. The primes $p \le \sqrt{5K}$ contribute $O(K^{3/2}\log K)$. For larger p the floor is 1, and Chebyshev's inequality bounds the sum by $7K\log 4\cdot K/M$. Since $M(K)\to\infty$, the band contributes $o(K^2)$.

Inner primes, $K/M < p \le K/3$. By (5.3) and Proposition 6.2, $-L_p \le pR(K/p) - [K \le 200p]\,p\,g_{\mathrm{par}}(K/p) + 190M^3 + 30M$. The error sums to $O(M^3K) = O(K^{7/4})$. Put $x_1 = 10^8$. For $K/x_1 < p \le K/3$, (7.1) in the variable $x = K/p$ turns the main terms into $K^2(\int_3^{x_1} R(x)x^{-3}dx - \int_3^{200} g_{\mathrm{par}}(x)x^{-3}dx) + o(K^2)$. By Lemma 5.2, $|F| \le 8\lambda$ and $|Q| \le \frac{13}{8}$, so $|R(x)| \le 22x$ for $x \ge 1$. So $p|R(K/p)| \le 22(K+p) \le 44K$ for $K/M < p \le K/x_1$, and Chebyshev's inequality bounds this band by $(44\log 4/x_1)K^2$.

Outer primes, $K/3 < p \le 2h$. Here $-L_p \le \min(KT_0(p/K),\, pR(K/p)) + O(1)$ for $p \le K$, and $-L_p = KT_0(p/K) + O(1)$ for $K < p \le 2h$. By (7.1) the sum is at most $K^2(\int_{1/3}^{2\lambda} T_0 - S_{\mathrm{out}}) + o(K^2)$.

Adding the bands and using Proposition 7.2, $\limsup K^{-2} \log m_K$ is at most

$$\frac{127751}{96000} - \frac{3203}{163200} + I_{200} - \frac{103109}{24000000} - \frac{5847}{10^6} + \frac{44\log 4}{10^8} + \int_{10^8}^{\infty} \frac{|R(x)|}{x^3}\, dx.$$

The first five terms add up to a number in $(1.287995331,\, 1.287995332)$. Since $|R(x)| \le 22x$, the last two terms are at most $44\log 4/10^8 + 22/10^8 < 10^{-6}$, and the sum is below Λ. □

Remark 7.3. Fauzan's bound at the cutoff $M = 100000$ is $A_{100000} = 1.31757\ldots$ [7, (5.20)]. Of the gain of 0.0296, the outer extension gives $3203/163200 \approx 0.0196$ and the parity gain at least 0.0058. The remaining 0.0041 comes from the exact integral to 200, the sharper tail and the growing cutoff, which removes the terms of order $1/M$. Computations of the exact content of Δ_K for $K \le 800$ suggest that the true exponent is near 1.2735. They are not part of the proof.

Remark 7.4 (the exact constant). The cutoffs at 200 in Proposition 7.2 are not needed. The same argument gives $\limsup K^{-2} \log m_K \le \Lambda^*$, with

$$\Lambda^* = \frac{127751}{96000} - \frac{3203}{163200} + \int_3^{\infty} R(x)\frac{dx}{x^3} - \int_3^{\infty} g_{\mathrm{par}}(x)\frac{dx}{x^3} = 1.2879797838\ldots$$

Both integrands are periodic of period 40 up to terms that are polynomial in the period index. Summing over periods gives a finite linear combination of digamma values $\psi(u/40)$ at rational u. The coefficients sum to zero, and the trigamma terms cancel identically. By Gauss's digamma theorem, Λ^* is therefore an explicit rational combination of 1, of $\pi\cot(\pi r)$ and of $\log\sin(\pi r)$ at rationals r, with denominators dividing one of 80, 92, 74, 43, 49, 9 and 12. The formal proof uses the rational bound Λ of Theorem 7.1 instead. With Λ^* and the same decay certificate the method gives $\mu(\zeta(5)) \le 86.52$. This value is not formalized.

8. Decay

8.1. **The energy bound.** Fauzan bounds $\Delta_K(\xi)$ through (2.4). After the substitution $y_i = K\sqrt{t_i}$, the logarithm of the integrand is, up to lower order terms, K^2 times a discrete logarithmic energy of the points t_i in the field V of (4.1). We use the logarithmic potential and energy in Fauzan's sign convention,

$$U^{\sigma}(t) = \int \log|t-u|\, d\sigma(u), \qquad I(\sigma) = \iint \log|t-u|\, d\sigma(t)\, d\sigma(u).$$

Let $\omega_{[0,b]}$ be the arcsine probability measure of $[0,b]$. We say that a positive measure σ has an *arcsine majorant* if $\sigma \le \Theta\,\omega_{[0,b]}$ for a constant Θ.

Theorem 8.1 (energy bound)**.** *Let $0<b<4$ and let σ be a positive measure of mass λ on $[0,b]$ with an arcsine majorant. Let $M_0 \ge -8$ satisfy*

$$2U^\sigma(t) - V(t) \le M_0 \qquad (0 \le t \le 4). \tag{8.1}$$

Put $C^* = -2\lambda + 12\alpha\lambda(1-\log\alpha) + 3\lambda^2 - 2\lambda^2\log(2\lambda)$. Then for every $n \ge 1$,

$$\log F_K(\xi) \le \bigl(C^* + \lambda M_0 - I(\sigma)\bigr)K^2 + 24K\log K + O(K).$$

The constant C^* comes from Stirling's formula for S_K and from the node sums, as in [7, (6.3)]. The proof follows [7, §6], with three changes.

(a) *The tail.* For $t \ge 4$ the inequality $2U^\sigma - V + \frac12\sqrt{t} \le M_0$ holds for every such σ and every $M_0 \ge -8$. Indeed $U^\sigma(t) \le \lambda\log t$, $\int_0^1 \log(t+u^2)du \ge \log t$ and $\int_0^\alpha \log(t+u^2)du \le \alpha\log t + \alpha^3/(3t)$. With $s=\sqrt{t}\ge 2$ the resulting bound $1.3\log t + 2\alpha^3/t - (2\pi - \frac12)s$ is at most -8.96.

(b) *The zero-mass lemma.* Let $\sigma_1,\dots,\sigma_r$ be finite measures of bounded support on $\mathbb{C}$ and $s_1,\dots,s_r$ real numbers with $\sum_k s_k\sigma_k(\mathbb{C}) = 0$. If the logarithm is integrable, then $\sum_{k,l} s_k s_l \iint \log|z-w|\,d\sigma_k\,d\sigma_l \le 0$. The proof truncates the kernel, writes it through Frullani's integral as a superposition of Gaussian kernels, and uses their positive definiteness.

(c) *The regularization.* The points t_i are replaced by circles of radius ϵ and compared with $K\sigma$ through (b). The cross term of a circle and σ is bounded by $U^\sigma(t_i)$ plus $\int \log^+(\epsilon/|t-u|)\,d\sigma(u) \le 2\sqrt{2}\,\Theta\sqrt{\epsilon/b}$. This follows from the layer-cake formula and $\omega_{[0,b]}([x-r,x+r]) \le \sqrt{2r/b}$. It also shows that σ has no atoms and that $\log|t-u|$ is σ-integrable.

With these, the configuration inequality gives, for distinct $t_i \ge 0$,

$$2\sum_{i<j}\log|t_i-t_j| - K\sum_i V(t_i) + \sum_i \sqrt{t_i} \le (\lambda M_0 - I(\sigma))K^2 + O(K\log K).$$

The weight bound $w(y) \le 8192(1+y)^5 e^{-2\pi y}$, the comparison of the products D_N^6/D_K with e^{-KV} and Stirling's formula for S_K then give Theorem 8.1.

8.2. **The comparison measure.** Let T_k be the Chebyshev polynomials and let

$$f(x) = \sum_{k=0}^{60} c_k T_k(x), \qquad \sigma = f\bigl(\tfrac{2t}{b}-1\bigr)\,\omega_{[0,b]}(dt), \qquad b = \Bigl(\frac{43}{50}\Bigr)^2.$$

Lemma 8.2 (closed forms)**.** *Suppose $f \ge 0$ on $[-1,1]$. Then σ has mass c_0 and $\sigma \le \bigl(\sum_k |c_k|\bigr)\,\omega_{[0,b]}$. With $x = 2t/b - 1$,*

$$U^\sigma(t) = c_0\log\frac{b}{4} - \sum_{k\ge1}\frac{c_k T_k(x)}{k} \qquad (0 \le t \le b),$$

$$U^\sigma(t) = c_0\Bigl(\log\frac{b}{4} + w\Bigr) - \sum_{k\ge1}\frac{c_k e^{-kw}}{k} \qquad \bigl(t = \tfrac{b}{2}(1+\cosh w),\ w \ge 0\bigr),$$

and

$$I(\sigma) = c_0^2 \log \frac{b}{4} - \sum_{k \geq 1} \frac{c_k^2}{2k}.$$

Proof. For $y \in [-1,1]$ and $x = \cos\theta \in [-1,1]$, $\log|y-x| = -\log 2 - \sum_{k\geq 1} \frac{2}{k} T_k(y) \cos k\theta$. For y outside, $y = \cosh w$, the same expansion holds with $\cos k\theta$ replaced by $e^{-kw}\cos k\theta$ and $-\log 2$ by $w - \log 2$. Integrating against $f(\cos\theta)\, d\theta/\pi$ and using orthogonality gives the potentials, after the change of scale from $[-1,1]$ to $[0,b]$. Integrating the inner formula against σ and using $\int T_k \, d\sigma = c_k/2$ for $k \geq 1$ gives the energy. □

Lemma 8.3 (positivity)**.** *Let $h_0, \dots, h_{60}$ be real numbers, not all zero. Put $a_0 = \sum_j h_j^2$, $a_k = 2\sum_j h_j h_{j+k}$ for $k \geq 1$, and $c_k = (\lambda/a_0)a_k$. Then $f(\cos\theta) = (\lambda/a_0)\bigl|\sum_j h_j e^{ij\theta}\bigr|^2 \geq 0$, and $c_0 = \lambda$.*

The proof is the expansion of the square. We take for $h_0, \dots, h_{60}$ the integers of the formal proof. They begin $-2573, -2919, -6471, -5888, \dots$. These data were found by numerical optimization and play no other role in the proof. Put $M_0 = -3330031/500000 = -6.660062$.

8.3. **The cells.** Condition (8.1) is checked on two regions. On $[0,b]$ put $t = b\cos^2\varphi$ with $0 \leq \varphi \leq \pi/2$. By Lemma 8.2, $2U^\sigma(t) - V(t)$ becomes

$$g_A(\varphi) = 2\lambda \log \frac{b}{4} - 2\sum_{k=1}^{60} \frac{c_k \cos 2k\varphi}{k} - V(b\cos^2\varphi).$$

On $[b,4]$ put $t = \frac{b}{2}(1+\cosh x)$ with $0 \leq x \leq 3$. This covers $[b,4]$, since $\frac{b}{2}(1+\cosh 3) > 4$. Then

$$g_B(x) = 2\lambda\Bigl(\log\frac{b}{4} + x\Bigr) - 2\sum_{k=1}^{60} \frac{c_k e^{-kx}}{k} - V\bigl(\tfrac{b}{2}(1+\cosh x)\bigr).$$

In both regions the field is written in $y = \sqrt{t}$, with $\int_0^A \log(y^2+u^2)\, du = A\log(y^2+A^2) - 2A + 2y\arctan(A/y)$. Its derivatives in y have poles only at $y = \pm i$ and $y = \pm i\alpha$.

Proposition 8.4 (cell certificate)**.** *$g_A \leq M_0$ on $[0,\pi/2]$ and $g_B \leq M_0$ on $[0,3]$. Hence (8.1) holds.*

Proof. Region A is covered by 100 cells $[\varphi_c - r, \varphi_c + r]$ with centres $\varphi_c = 2\arctan m$, m rational. Then $\cos\varphi_c$ and $\sin\varphi_c$ are rational, and so are all $\cos 2k\varphi_c$ and $\sin 2k\varphi_c$, by the rotation recurrence. On each cell the function is bounded by its Taylor polynomial of order four at the centre and a Lagrange remainder of order five. The quadratic part is maximized exactly. The remainder of the trigonometric sum is bounded by $2\sum_k |c_k|(2k)^5/k$. The remainder of the field is bounded by Faà di Bruno's formula, with the derivatives of $y = \sqrt{b}\cos\varphi$ and the distances to the poles of the field. The values of V and its first derivative at the centre need rational enclosures

of log and arctan at rational points. The overlaps of consecutive cells are checked with $\arctan m' - \arctan m \le (m'-m)/(1+mm')$. Region B is covered in the same way by 10 cells with centres $x_c = 2\log r$, $r \ge 1$ rational, so that e^{-kx_c} and $\cosh(x_c/2)$ are rational. Every cell inequality is a finite computation in rational arithmetic, carried out by the Lean kernel. □

Theorem 8.5 (decay). *$F_K(\xi) \le e^{-(D-o(1))K^2}$ with $D = 138500086/10^8$.*

Proof. By Lemma 8.3 and Lemma 8.2, σ is a positive measure of mass λ on $[0,b]$ with an arcsine majorant. By Proposition 8.4 it satisfies (8.1). Theorem 8.1 gives $\limsup K^{-2}\log F_K(\xi) \le -(I(\sigma) - \lambda M_0 - C^*)$. By Lemma 8.2, $I(\sigma) - \lambda M_0 - C^* = \lambda^2\log(b/4) - \sum_k c_k^2/(2k) - \lambda M_0 - C^* = 1.38500086079\ldots$. The inequality with D needs three rational enclosures of logarithms and exact arithmetic. □

Remark 8.6. Fauzan's measure, a sum of sixteen arcsine measures on nested intervals, gives $D = 1.36699$ [7, Lemma 6.1]. The value D of the method is the maximum of $I(\sigma) - \lambda M_0(\sigma) - C^*$ over measures of mass λ, where $M_0(\sigma)$ is the supremum of $2U^\sigma - V$. It is attained at the equilibrium measure of mass λ in the field $V/2$ [31]. Numerical solution of this problem gives $D \approx 1.38551$. So the remaining room on the analytic side is about $5\cdot 10^{-4}$. This value is a numerical observation and is not part of the proof.

9. Proof of Theorem 1.1

Let m_K be the normalizing factor of Theorem 7.1 and put $P_n = m_K F_K$, of degree $h = \lambda K$. Its coefficients are integers, so condition (a) of Theorem 3.1 holds with $d_n = 1$ and the exponent 0. By Proposition 4.1, $\rho = 907427/10^5$ is a norm exponent. Let $\varepsilon > 0$ and $|t| \le e^{-(\rho+\varepsilon)K}$. Corollary 3.5 gives $0 < F_K(\xi+t) \le 2F_K(\xi)$. With Theorems 7.1 and 8.5,

$$|P_n(\xi+t)| \le 2e^{\Lambda K^2} e^{-(D-\varepsilon)K^2},$$

so (b) holds with the rate $D - \Lambda$. Corollary 3.5 and $m_K > 0$ give (c). Theorem 3.1 applies with $\gamma = \lambda$, and

$$\frac{\lambda\rho}{D-\Lambda} = \frac{37}{40}\cdot\frac{9.07427}{1.38500086 - 1.288003} = 86.5349\ldots < 87.$$

Put $E = \lambda\rho/(D-\Lambda) = 839369975/9699786$. Theorem 3.1 shows that $\zeta(5)$ is irrational and that the hypothesis of Lemma 3.2 holds with this E. Since $E < 87$, Lemma 3.2 with $\mu_0 = 87$ gives an integer q_0 such that $|\zeta(5) - p/q| > q^{-87}$ for all integers p and all integers $q \ge q_0$. This is Theorem 1.1.

10. Formalization

Theorem 1.1 is formalized in Lean 4 [5] with Mathlib [32], together with every theorem, proposition and lemma its proof uses. The remarks are not part of the formalization. The development is in the directory `CatalanCore/Zeta5` of the repository

https://github.com/slitsey/catalan-irrationality-measure

Theorem 1.1 is the theorem

`Zeta5IrrationalityMeasure.zeta5_irrationalityMeasure_87`

in `SolutionZeta5.lean`. Its statement uses only Mathlib. The number $\zeta(5)$ is written as the series $\sum_{n\geq 0}(n+1)^{-5}$, and the conclusion is $q^{-87} < |\zeta(5)-p/q|$ for all integers p and all sufficiently large q. In the development $\zeta(5)$ is `Apery.zeta5`, and `riemannZeta_five` identifies it with Mathlib's $\zeta(5)$. The root is proved in the module `CatalanCore.Zeta5.Data.Public`. There `exponentE` is the exponent $E = 839369975/9699786 < 87$, `zeta5_exact` is the second statement of Theorem 3.1 at this exponent, `irrational_of_family_nnv` is its first statement, and `TrigammaDifferences.Aux.eventually_of_point` is Lemma 3.2. The modules contain no `sorry` and no `native_decide`. The command `#printaxioms` reports only `propext`, `Classical.choice` and `Quot.sound`. The proof is kernel-checked. It has not been run through the independent checker `lakecomparator` [19].

Fauzan's construction, the integral representation, positivity, the degree, Andréief's formula and parts of the energy argument are ported from Firsching's formalization [8], under the Apache License 2.0, at commit `7fe7367`. The ported files are in `CatalanCore/Zeta5/Port` and are listed in the repository's `NOTICE`. The prime number theorem is taken from the PNT+ project [33], through OpenAI's workspace [25]. The weighted prime sums of Section 7 use the prime-sum files of the same workspace (`OAI/NumberTheory/ZetaFive`). The proofs were checked with Lean `v4.35.0-rc2` and the matching Mathlib.

The results of the paper correspond to the following declarations. Their files are under `CatalanCore/Zeta5`. We give the names with a short prefix only. The full namespaces are `CatalanCore.Zeta5` and its subnamespaces, `CatalanCore.Uniform` and `Apery`.

- Theorem 3.1: `measure_of_family_nnv_period` (the period 40 of the scale) and `Data.Public.irrational_of_family_nnv`, applied in `Data.Public.zeta5_exact` to the family $m_K F_K$. Lemma 3.2: `TrigammaDifferences.Aux.eventually_of_point`.
- Lemma 3.3: `Apery.det_pencil_bounds`. Corollary 3.5: `F_pencil_bounds`, `zeta5_nearbyNV` and `zeta5_hasScaledDecay`.
- Proposition 2.1: `Apery.moment_rep`, `Apery.G_posDef`, `Apery.natDegree_Δ`, and (2.4) is `Apery.Δ_eq_integral`.
- Proposition 4.1: `Rho.zeta5_rho_bound`, from `Rho.residue_identity`, `Rho.rj_abs_le`, `Rho.gram_lower`, `Rho.node_bound`, `Rho.Vf_le_max`, `Rho.Phi_tangent` and `Rho.rho_numerics`.
- Theorem 5.1: `outer_extension` and `outer_ext_gamma`. The cost estimates are `inner_cost_gamma`, `outer_ext_cost` and `outer_cost`. Lemma 5.2 is `neg_Gam_sub_N0` with `Profile.Qc_ge` and `Profile.Qc_le`.

- Definition 5.3 and Proposition 5.4: `Content.LpC`, `Content.mC`, `Content.log_mC`, `Content.GV_F_LpC` and `Content.normaliser_bound`.
- Theorem 6.1: `Apery.parity_gain`, a proof of the statement `Apery.ParityGain`. Its parts are `tauX_lead` (Lemma 6.3), `fullSym_odd` and `locSymb_pc` (the reflection), `lead_entry` (the identification (6.3)), `radical_T_odd` and `radical_T_even` (the radical vectors), `cclass_unimodular` and `centred_entry` (the centred basis), and `parity_gain_delta` (Lemma 6.4). Proposition 6.2 is `Apery.kpar_ge`, with $0 \le g_{\mathrm{par}} \le \frac{1}{2}$ in `Profile.gref_nonneg` and `Profile.gref_le_half`.
- Proposition 7.2: `Sum.integral_T0`, `Sum.integral_head`, `Sum.integral_R_mid`, `Sum.integral_R_tail` and `Sum.integral_gref`. Their sum is `Sum.lambdaQ_value`.
- Theorem 7.1: `Content.content_bound_signed`, through `Content.log_mC_le`, `Content.pt_bound` and the prime sums `Sum.weighted_prime_sum_tendsto` and `Sum.theta_le_linear`. The bound $|R(x)| \le 22x$ is `Profile.Rg_bound`, and the band bound $(44 \log 4/x_1)K^2$ is the second term in the statement of `Content.log_mC_le`. The tail in Proposition 7.2(d) is `Sum.TL200`.
- Theorem 8.1: `Energy.real_bound`, with the zero-mass lemma `Energy.energy_log_nonpos`, the regularization `Energy.integral_regTerm_le`, the configuration inequality `Energy.config_ineq` and the tail `Decay.cellHyp_tail`.
- Lemmas 8.2 and 8.3: `Potential.logPot_nu_inside`, `Potential.logPot_nu_cosh`, `Potential.energy_nu`, `Potential.chebF_sos` and `Potential.sup_le_sumAbs`.
- Proposition 8.4: `Cells.cellA_sound`, `Cells.cellB_sound` and `Cells.hcell_of_cells`, applied to the 100 and 10 cells of `Data/CheckA1.lean`–`CheckA4.lean` and `Data/CheckB.lean`.
- Theorem 8.5: `Decay.zeta5_decay`, with the constant inequality `Cells.energy_const_sound`.

The certificate data are the 61 integers h_j, the cells, and the rational breakpoints of the integrals of Proposition 7.2. They were computed with floating-point optimization and the interval arithmetic of Arb [17], and Lean re-checks them with rational arithmetic. They play no role in the proof beyond satisfying the checks.

Disclosure of AI use

The mathematics, the Lean formalization, the certificate generators and the text of this paper were produced by AI systems at the request of the author. The author posed the problems, directed the work and is responsible for the paper. The systems were Claude Opus 5.5, Claude Sonnet 5.5 and Claude Haiku 5.5 (Anthropic), running in Claude Code, and GPT-6.1 Sol

(OpenAI), running in Codex. The construction and the irrationality proof of $\zeta(5)$ are due to Fauzan [7].

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