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A BOUND FOR THE IRRATIONALITY EXPONENT OF REAL ELLIPTIC PERIODS

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Abstract. We prove that every nonzero real period of an elliptic curve over $\mathbb{Q}$, with or without complex multiplication, has irrationality exponent at most 3. Equivalently, if $L \subset \mathbb{C}$ is a lattice whose Weierstrass invariants $g_2(L)$ and $g_3(L)$ are rational, every nonzero real $\omega \in L$ satisfies $\mu(\omega) \leq 3$. G. and D. Chudnovsky proved $\mu(\omega) = 2$ when the curve has complex multiplication. Their proof uses an interpolation determinant of jets at centres on a two-dimensional grid of periods. Without complex multiplication the only algebraic multiples of a real period inside the lattice are its integer multiples, so the centres lie on a line. We show that the method still works on this one-dimensional grid. The arithmetic is carried out over $\mathbb{Z}$ in the chart at the origin. A multiplier made periodic along the real period removes the Gaussian factor in the transport of the determinant. The analytic estimate then costs a factor K^2 in place of K, where K is the number of centres. The two estimates still contradict each other for every exponent larger than 3. More generally, within the method an analytic cost K^e gives exactly the exponents larger than $e+1$. The bound is ineffective. The proof is formalized in Lean 4.

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1. Introduction

1.1. **The result.** The *irrationality exponent* $\mu(\xi)$ of a real number ξ is the infimum of the real numbers ν such that $|\xi - p/q| > q^{-\nu}$ for all integers p and all sufficiently large integers q. It does not change when ξ is multiplied by a nonzero rational number. For irrational ξ Dirichlet's theorem gives $\mu(\xi) \geq 2$.

Theorem 1.1. *Let E be an elliptic curve over $\mathbb{Q}$, let η be a nonzero invariant differential on E defined over $\mathbb{Q}$, and let ω be a nonzero real period of η. For every $\varepsilon > 0$ there is an integer $q_0 = q_0(\varepsilon, \omega)$ such that*

$$\left|\omega - \frac{p}{q}\right| > q^{-3-\varepsilon}$$

for all integers p and all integers $q \geq q_0$. In particular ω is irrational and $2 \leq \mu(\omega) \leq 3$.

The curve may or may not have complex multiplication. For $E : y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Q}$ and $\eta = dx/(2y)$, the periods are the points of a lattice in $\mathbb{C}$, which we now describe.

For a lattice $L \subset \mathbb{C}$ we write $g_2(L) = 60 \sum' \lambda^{-4}$ and $g_3(L) = 140 \sum' \lambda^{-6}$, the sums running over $\lambda \in L \setminus \{0\}$. These are the invariants of the Weierstrass function $\wp_L$, which satisfies $\wp_L'^2 = 4\wp_L^3 - g_2(L)\wp_L - g_3(L)$. The curve $y^2 = x^3 + Ax + B$ with $4A^3 + 27B^2 \neq 0$ is uniformized by the lattice Λ with $g_2(\Lambda) = -4A$ and $g_3(\Lambda) = -4B$, through $x = \wp_\Lambda(z)$ and $y = \wp_\Lambda'(z)/2$ [18, Section VI.5]. The differential $dx/(2y)$ pulls back to dz, so its periods form Λ. Conversely, every lattice L with $g_2(L), g_3(L) \in \mathbb{Q}$ arises in this way from the curve $y^2 = x^3 - \frac{1}{4}g_2(L)x - \frac{1}{4}g_3(L)$. Theorem 1.1 is therefore equivalent to the following statement about lattices, which is the form we prove (Section 7).

Theorem 1.2. *Let $L \subset \mathbb{C}$ be a lattice with $g_2(L), g_3(L) \in \mathbb{Q}$, and let $\omega \in L$ be real and nonzero. For every $\varepsilon > 0$ there is an integer $q_0 = q_0(\varepsilon, \omega)$ such that $|\omega - p/q| > q^{-3-\varepsilon}$ for all integers p and all integers $q \geq q_0$.*

Theorem 1.1 applies to classical constants. The complete elliptic integral of the first kind is

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} = \frac{\pi}{2\,M(1, \sqrt{1-k^2})} \qquad (0 < k < 1),$$

where $M(a, b)$ is the arithmetic-geometric mean of Gauss.

Corollary 1.3. *Let $0 < k < 1$ with $k^2 \in \mathbb{Q}$. Then $\mu(K(k)) \leq 3$, and equivalently $\mu(\pi/M(1, \sqrt{1-k^2})) \leq 3$.*

Proof. Put $\lambda = k^2$. The curve $y^2 = (1 - x^2)(1 - \lambda x^2)$ is smooth of genus one because $\lambda \neq 0, 1$, and it has the rational point $(1, 0)$. So it is an elliptic curve over $\mathbb{Q}$, and dx/y is an invariant differential defined over $\mathbb{Q}$. Its integral over the closed loop around the segment $[-1, 1]$ equals $2\int_{-1}^{1} dx/\sqrt{(1-x^2)(1-\lambda x^2)} = 4K(k)$, a nonzero real period. Theorem 1.1 gives $\mu(4K(k)) \leq 3$, and μ does not change under multiplication by 4. □

When the Legendre curve has complex multiplication, for instance $k^2 = \frac{1}{2}$ (the lemniscate case), the exponent is 2 by the work of G. and D. Chudnovsky [2]. Otherwise the corollary gives, to our knowledge, the first explicit bound. For example, $k = \frac{1}{2}$ gives the curve with $j = 2^8(\lambda^2 - \lambda + 1)^3/(\lambda^2(1-\lambda)^2) = 35152/9$. The j-invariant of a curve with complex multiplication is an algebraic integer, and $35152/9$ is not an integer. So this curve has no complex multiplication, and $\mu(K(\frac{1}{2})) \leq 3$. In the same way, Theorem 1.1 bounds by 3 the exponent of the real period Ω_E of every elliptic curve E over $\mathbb{Q}$ in the tables of Cremona and the LMFDB, for instance the curve `11a1` of smallest conductor.

The proof does not give $q_0(\varepsilon, \omega)$: the bound is ineffective. Theorem 1.2 is formalized in Lean 4 with Mathlib as `EllipticPeriodMeasure.real_period_measure_three`. The proof uses no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. The source is at https://github.com/slitsey/elliptic-periods-irrationality. The use of AI in this work is disclosed at the end of the paper.

1.2. **Background.** *Rational approximation.* Roth proved that $\mu(\alpha) = 2$ for every irrational algebraic number α [15]. For transcendental numbers the exact value of μ is known in few cases.

Elliptic periods. Siegel proved that at least one of two basis periods of $\wp$ is transcendental when g_2 and g_3 are algebraic [17]. Under complex multiplication this makes every nonzero period transcendental. Schneider proved that every nonzero period of such a $\wp$ is transcendental, with or without complex multiplication [16]. He also proved that if τ and the modular invariant $j(\tau)$ are both algebraic, then τ is imaginary quadratic [16]. We use this second theorem in Remark 1.4. Both results, and their later history, are described in Waldschmidt's survey [19, Section 3.1].

Measures. Fel'dman gave measures of transcendence for the numbers in the theorems of Siegel and Schneider [4, 5]. Masser determined the linear relations among 1, the periods, the quasi-periods and $2\pi i$ of one Weierstrass function, and he gave measures of linear independence for them [9]. Philippon and Waldschmidt proved effective lower bounds for linear forms in logarithms on commutative algebraic groups. They include the first such measures for elliptic logarithms that hold without complex multiplication [12, 13]. Hirata-Kohno and David refined these bounds [6, 7, 3]. As recorded in [19, Section 4.3], it follows from work of David, Hirata-Kohno and Gaudron that these nonvanishing linear forms are not Liouville numbers. So $\mu(\omega)$ is finite for every nonzero real period ω, with an exponent that depends on the curve. We have not found a numerical value of such an exponent in the literature. For curves without complex multiplication, Theorem 1.1 gives, to our knowledge, the first explicit bound, and it is uniform over all curves over $\mathbb{Q}$.

Recent results. Since September 2026 several results of exponent 2 have appeared, some obtained with AI systems. In October 2026 OpenAI proved

that $\mu(\pi) = 2$, with a Lean formalization [11]. Liu, Jiang and Zhang proved $\mu(\log r) = 2$ for rational $r > 0$, $r \neq 1$ [8]. G. V. Chudnovsky proved $\mu(\varpi) = 2$ for the lemniscate period $\varpi = 2\int_0^1 (1-t^4)^{-1/2}\,dt$ [1]. G. V. and D. V. Chudnovsky extended this to every nonzero real period of an elliptic curve over $\mathbb{Q}$ with complex multiplication, for an invariant differential defined over $\mathbb{Q}$. They also proved $\mu(\varpi^n) = O(n \log 2n)$ [2]. The paper [2] is a preprint. Its results that we use are stated exactly where they are used, in Theorem 3.1, Lemma 4.1 and Lemma 5.2.

1.3. **What the method needs without complex multiplication.** The proof of [2] works on $E \times \mathbb{G}_a^m$. It takes m rational approximations p_i/q_i to a real period ω, and a set of K periods $\gamma_j = \lambda_j\omega \in \Lambda$. The centres are the points $(\lambda_j p_1/q_1, \dots, \lambda_j p_m/q_m)$ of the additive factor. The determinant Δ is formed from the weighted jets of a space of polynomials at these centres. Three estimates are compared. An interpolation theorem with separated weights shows that $\Delta \neq 0$. Clearing denominators gives a lower bound for $|\Delta|$, since λ_j lies in an order of an imaginary quadratic field. An analytic upper bound comes from the periodicity of the elliptic coordinates and a collision estimate. The centres λ_j run over a rectangle $\{a + bi\sqrt{d}\}$ in the order, so they lie in a disc of radius of order $\sqrt{K}$. The multiplier that clears the poles grows like $\exp(c|z|^2)$, and the analytic estimate costs a term of order K/w_0, where w_0 is the source weight of the elliptic coordinates. Apart from the irrationality of ω, complex multiplication enters [2] only through this two-dimensional grid.

Remark 1.4. Let Λ have algebraic invariants and no complex multiplication, and let Ω generate $\Lambda \cap \mathbb{R}$. If α is algebraic and $\alpha\Omega \in \Lambda$, then $\alpha \in \mathbb{Z}$. Indeed, write $\Lambda = \mathbb{Z}\Omega + \mathbb{Z}\Omega'$ and $\alpha\Omega = a\Omega + b\Omega'$. If $b \neq 0$, then $\tau = \Omega'/\Omega$ is algebraic. Since $j(\tau)$ is algebraic, Schneider's theorem makes τ imaginary quadratic, and Λ has complex multiplication. So $b = 0$ and $\alpha = a$.

So without complex multiplication the multipliers λ_j must be integers. We take the centres jp_i/q_i with $0 \leq j < K$. They lie on a line, at distance up to $K\Omega$ from the origin. The multiplier then costs a term of order K^2/w_0. The question is whether the parameters still close, and at which exponent.

1.4. **New ideas.** We keep the construction of [2] and its nonvanishing theorem. The proof rests on the following ideas.

(1) *A one-dimensional real grid.* The periods $j\Omega$, $0 \leq j < K$, replace the two-dimensional grid of complex multiplication, and the centres are $c_{ji} = jp_i/q_i$. These centres are distinct in each coordinate, which is all that the nonvanishing theorem of [2] asks (Theorem 3.1). The entries lie in $\mathbb{Q}$, and the lower bound needs only $|n| \geq 1$ for a nonzero integer n.
(2) *The origin chart for every curve.* For $y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Z}$ we use the chart of [2] at the origin, $u = 1/(x-a)$, $t = y/(x-a)^2$, with any integer a such that $a^3 + Aa + B \neq 0$. Its elliptic logarithm

has the denominators n only. The whole arithmetic part of [2] uses nothing else. We give it over $\mathbb{Z}$ (Proposition 4.2). No torsion point and no endomorphism enters.

(3) *The periodised multiplier.* Let $\psi(z) = \sigma(z - z_Q)\sigma(z + z_Q)$ be the sigma-type multiplier of [2] and let η_Ω be the quasi-period of Ω. The function $\widetilde{\psi}(z) = \psi(z)\exp(-\eta_\Omega z^2/\Omega)$ is periodic with period Ω. In the transport of the determinant it removes the factor $\prod_j |\psi(j\Omega)|$, which is a Gaussian in j (Lemma 5.1). Legendre's relation gives its growth across the strip, $\log|\widetilde{\psi}(z)| \le c + \chi(|\operatorname{Im} z| + Y)^2$ for a constant c, with $\chi = 2\pi/(\Omega Y)$ and Y the height of the period parallelogram (Lemma 2.6).

(4) *The analytic cost K^2/w_0 and the exponent 3.* The collision estimate needs a disc around all centres, of radius of order K. On it the multiplier costs a term $C_\Lambda K^2/(2w_0)$, with an explicit constant C_Λ of the lattice (Theorem 5.3). With explicit rational choices of all parameters the comparison closes for every exponent $\nu > 3$ (Section 6).

(5) *The threshold $e + 1$.* If the analytic cost is K^e/w_0, the parameters close exactly when $\nu > e + 1$ (Proposition 8.1). The case $e = 1$ is the exponent 2 of complex multiplication and of π. The case $e = 2$ is the exponent 3 of this paper.

1.5. **Constants, organization and notation.** The exponent 3 is the only constant in Theorem 1.2. The constants of the proof, such as C_Λ, depend on the lattice and do not enter the result.

Section 2 reduces Theorem 1.2 to an integral model and constructs the chart and the multiplier. Section 3 defines the determinant and states the nonvanishing theorem. Sections 4 and 5 prove the lower and upper bounds. Section 6 chooses the parameters and Section 7 proves the theorem. Section 8 explains why the exponent is 3 and what would give 2.

Logarithms are natural, $\mathbb{N}$ contains 0, and $\log^+ x = \max(0, \log x)$. For a series f in T and $V = (V_1, \dots, V_m)$, $[T^s V^\beta]f$ is the coefficient of $T^s \prod_i V_i^{\beta_i}$. For $\alpha, \beta \in \mathbb{N}^m$ we write $\beta \le \alpha$ coordinatewise, $|\alpha| = \sum_i \alpha_i$ and $\binom{\alpha}{\beta} = \prod_i \binom{\alpha_i}{\beta_i}$, which is 0 unless $\beta \le \alpha$.

2. The lattice, the origin chart and the multiplier

2.1. **Reduction to an integral model.** The first lemma replaces L by a lattice whose curve has integer coefficients.

Lemma 2.1. *Let L be a lattice with $g_2(L), g_3(L) \in \mathbb{Q}$. There is a positive integer d such that $\Lambda = d^{-1}L$ satisfies $g_2(\Lambda) = -4A$ and $g_3(\Lambda) = -4B$ with $A, B \in \mathbb{Z}$ and $4A^3 + 27B^2 \neq 0$. The functions $x = \wp_\Lambda$ and $y = \wp'_\Lambda/2$ satisfy $y^2 = x^3 + Ax + B$ and $dx/(2y) = dz$.*

Proof. For $c \in \mathbb{C}^\times$ one has $g_2(cL) = c^{-4}g_2(L)$ and $g_3(cL) = c^{-6}g_3(L)$. Let s be a common denominator of $g_2(L)$ and $g_3(L)$ and put $d = 2s$. Then $d^4 g_2(L)/4$ and $d^6 g_3(L)/4$ are integers, and we define $A = -d^4 g_2(L)/4$ and $B = -d^6 g_3(L)/4$. The discriminant $g_2^3 - 27g_3^2$ of a lattice is nonzero, and

for Λ it equals $-16(4A^3+27B^2)$. The differential equation of $\wp_\Lambda$ gives $y^2=\wp_\Lambda'^2/4=x^3+Ax+B$, and $dx/(2y)=\wp_\Lambda' dz/\wp_\Lambda'=dz$. □

A nonzero real $\omega\in L$ gives the nonzero real period $\omega/d\in\Lambda$. The group $\Lambda\cap\mathbb{R}$ is discrete and nonzero, so $\Lambda\cap\mathbb{R}=\mathbb{Z}\Omega$ for a unique $\Omega>0$. Thus $\omega/d=n\Omega$ with a nonzero integer n, and $\omega=N\Omega$ with $N=dn$. Integer multiples do not change the exponent, in the following form.

Lemma 2.2. *Let $N\neq 0$ be an integer and $\nu'<\nu$. If infinitely many pairs (p,q) with $q\geq 1$ satisfy $|N\xi-p/q|\leq q^{-\nu}$, then infinitely many pairs (p',q') with $q'\geq 1$ satisfy $|\xi-p'/q'|\leq q'^{-\nu'}$.*

Proof. For bounded q only finitely many p satisfy the inequality, so the denominators q are unbounded. Put $p'=p\operatorname{sgn}N$ and $q'=|N|q$. Then $|\xi-p'/q'|\leq q^{-\nu}/|N|$. This is at most $(|N|q)^{-\nu'}$ as soon as $q^{\nu-\nu'}\geq|N|^{\nu'-1}$, which holds for all large q. □

From now on we fix $A,B\in\mathbb{Z}$ with $4A^3+27B^2\neq 0$, the cubic $f(X)=X^3+AX+B$, the lattice Λ of Lemma 2.1 and the generator $\Omega>0$ of $\Lambda\cap\mathbb{R}$. We choose $\Omega'\in\Lambda$ with $\Lambda=\mathbb{Z}\Omega+\mathbb{Z}\Omega'$ and $Y=\operatorname{Im}\Omega'>0$.

2.2. **The origin chart.** This chart is the one of [2]. Fix an integer a with $f(a)\neq 0$. One of $0,1,2,3$ will do, since a cubic has at most three roots. Put

$$u=\frac{1}{x-a},\qquad t=\frac{y}{(x-a)^2}.$$

Multiplying $y^2=f(x)$ by u^4 gives

$$t^2=F(u),\qquad F(U)=U+3aU^2+(3a^2+A)U^3+f(a)U^4\in\mathbb{Z}[U]. \tag{2.1}$$

The decisive facts are $F(0)=0$ and $F'(0)=1$.

Proposition 2.3 ([2, Proposition 3.1])**.** *There is a unique $U(T)\in\mathbb{Z}[[T]]$ with $U(0)=0$ and $F(U(T))=T^2$. The series $-1/F'(U(T))$ lies in $\mathbb{Z}[[T]]$. The series $L(T)\in\mathbb{Q}[[T]]$ with $L(0)=0$ and $L'(T)=-1/F'(U(T))$ satisfies $n[T^n]L\in\mathbb{Z}$ for $n\geq 1$.*

Proof. The linear coefficient of F is 1, so F has a compositional inverse $G\in\mathbb{Z}[[S]]$ with $G(0)=0$. Put $U(T)=G(T^2)$. Any solution W with $W(0)=0$ satisfies $W=G(F(W))=G(T^2)$, which gives uniqueness. The constant term of $F'(U(T))$ is 1, so its inverse lies in $\mathbb{Z}[[T]]$. If $-1/F'(U(T))=\sum_k v_kT^k$, then $[T^n]L=v_{n-1}/n$. □

Next we relate the chart to the uniformization. The function $u=1/(\wp_\Lambda-a)$ vanishes at the points of Λ. Let $Q=(a,y_Q)$ be a point of the curve with $x(Q)=a$, and let z_Q be an elliptic logarithm of Q. Since $y_Q^2=f(a)\neq 0$, the point Q is not of order 2, so the cosets $z_Q+\Lambda$ and $-z_Q+\Lambda$ are distinct and $\wp_\Lambda'(z_Q)\neq 0$. Hence u has simple poles exactly at $\pm z_Q+\Lambda$. The function $t=\frac{1}{2}\wp_\Lambda' u^2$ has double poles there. At $z=0$ the triple pole of $\wp_\Lambda'$ meets the zero of order four of u^2, so t vanishes at the points of Λ and has no other poles.

Proposition 2.4 ([2, Proposition 3.3])**.** *Write $\tau(z) = t(z)$ near $z = 0$. Then $\tau'(0) = -1$, and as convergent germs at $z = 0$*

$$u(z) = U(\tau(z)), \qquad L(\tau(z)) = z.$$

Proof. Near 0 one has $\wp_\Lambda = z^{-2} + O(z^2)$ and $\wp'_\Lambda = -2z^{-3} + O(z)$. So $u = z^2 + O(z^4)$ and $t = -z + O(z^3)$. Thus τ is a local coordinate, and $u = W(\tau)$ for a convergent series W with $W(0) = 0$. The identity $t^2 = F(u)$ gives $F(W(T)) = T^2$, and Proposition 2.3 gives $W = U$. From $x = a + 1/u$ and $y = t/u^2$ we get $dz = dx/(2y) = -du/(2t)$. Differentiating $t^2 = F(u)$ gives $2t\,dt = F'(u)\,du$, hence $dz = -dt/F'(u) = L'(\tau)\,d\tau$. Both z and $L(\tau(z))$ vanish at 0, so they are equal. □

2.3. **The periodised multiplier.** Let σ be the Weierstrass sigma function of Λ. We use the classical facts in the form of [10, Section 23.2]. The function σ is entire with simple zeros exactly on Λ. For $\lambda \in \Lambda$ there are $\varepsilon_\lambda \in \{\pm 1\}$ and $\eta_\lambda \in \mathbb{C}$ with

$$\sigma(z + \lambda) = \varepsilon_\lambda e^{\eta_\lambda (z + \lambda/2)} \sigma(z), \tag{2.2}$$

and $\lambda \mapsto \eta_\lambda$ is additive. Legendre's relation reads

$$\eta_\Omega \Omega' - \eta_{\Omega'} \Omega = 2\pi i. \tag{2.3}$$

In the notation of [10], $\Omega = 2\omega_1$, $\Omega' = 2\omega_3$, $\eta_\Omega = 2\eta_1$ and $\eta_{\Omega'} = 2\eta_3$. Then (2.2) is (23.2.17) and (2.3) is four times (23.2.14).

Put

$$\psi(z) = \sigma(z - z_Q)\,\sigma(z + z_Q), \qquad \widetilde{\psi}(z) = \psi(z) \exp\Big(-\frac{\eta_\Omega}{\Omega} z^2\Big).$$

The function ψ is the multiplier of [2]. The factor $\exp(-\eta_\Omega z^2/\Omega)$ is new.

Lemma 2.5. *(a) $\psi(z + \lambda) = e^{\eta_\lambda (2z + \lambda)} \psi(z)$ for $\lambda \in \Lambda$, and $\widetilde{\psi}(z + \Omega) = \widetilde{\psi}(z)$.*

(b) $\widetilde{\psi}(z + n\Omega') = \widetilde{\psi}(z) \exp(-\frac{2\pi i}{\Omega}(2nz + n^2\Omega'))$ for $n \in \mathbb{Z}$.

(c) The functions $\widetilde{\psi}$, $\widetilde{\psi} u$ and $\widetilde{\psi}^2 t$ are entire, and $\widetilde{\psi}(0) = \psi(0) = -\sigma(z_Q)^2 \neq 0$.

Proof. (a) Apply (2.2) at $z - z_Q$ and $z + z_Q$. The signs give $\varepsilon_\lambda^2 = 1$ and the exponents add to $\eta_\lambda(2z + \lambda)$. For $\lambda = \Omega$ the exponential factor of $\widetilde{\psi}$ changes by $\exp(-\eta_\Omega(2z + \Omega))$, which cancels.

(b) With $\lambda = n\Omega'$ the exponent becomes $(n\eta_{\Omega'} - n\eta_\Omega \Omega'/\Omega)(2z + n\Omega')$. By (2.3) the first factor is $-2\pi i n/\Omega$.

(c) The zeros of ψ are simple and lie exactly on $\pm z_Q + \Lambda$. They cancel the simple poles of u and the double poles of t. Finally $z_Q \notin \Lambda$ and σ is odd. □

For integers $D \geq 1$ and $k, b \geq 0$ with $k + 2b \leq D$ put

$$\widetilde{g}_{D,k,b} = \widetilde{\psi}^D u^k t^b = \widetilde{\psi}^{D-k-2b} (\widetilde{\psi} u)^k (\widetilde{\psi}^2 t)^b.$$

It is entire and Ω-periodic. Let $S_* \geq 1$ be the maximum of 1 and the suprema of $|\widetilde{\psi}|$, $|\widetilde{\psi}u|$ and $|\widetilde{\psi}^2 t|$ on the rectangle $\{x+iy : 0 \leq x \leq \Omega,\ 0 \leq y \leq Y\}$. Put

$$\chi = \frac{2\pi}{\Omega Y}.$$

Lemma 2.6. *For all $z \in \mathbb{C}$ and all admissible (D,k,b),*

$$|\widetilde{g}_{D,k,b}(z)| \leq S_*^D \exp(D\chi(|\operatorname{Im} z| + Y)^2).$$

Proof. Write $z = z_1 + n\Omega' + l\Omega$ with $n, l \in \mathbb{Z}$, $0 \leq \operatorname{Re} z_1 < \Omega$ and $0 \leq y_1 = \operatorname{Im} z_1 < Y$. The functions u, t are Λ-periodic and $\widetilde{\psi}$ is Ω-periodic. By Lemma 2.5(b), each of $\widetilde{\psi}$ and $\widetilde{\psi}u$ is multiplied by a factor of modulus $\exp(\frac{2\pi}{\Omega}(2ny_1 + n^2 Y))$ in passing from z_1 to z, and $\widetilde{\psi}^2 t$ by its square. The product defining $\widetilde{g}_{D,k,b}$ therefore acquires this factor to the power $(D-k-2b)+k+2b = D$. At z_1 the product has $D - b \leq D$ factors, each bounded by S_*.

It remains to bound $2ny_1 + n^2 Y$ by $Y^{-1}(|y|+Y)^2$, where $y = \operatorname{Im} z = y_1 + nY$. If $n \geq 0$, then $2ny_1 + n^2 Y \leq (n+1)^2 Y$ and $(n+1)Y \leq y + Y$. If $n < 0$, then $2ny_1 + n^2 Y \leq n^2 Y$, and $|n|Y = y_1 - y < Y + |y|$. In both cases the factor is at most $\exp(\chi(|y|+Y)^2)$. □

3. The interpolation determinant

This section sets up the determinant of [2] for the one-dimensional grid, and states its nonvanishing.

3.1. **Weights, rows and columns.** Fix integers $m, K \geq 1$, positive rationals w_0, v_0 and $\theta \in (0,1)$, and a cutoff $H > 0$. Fix integers $p_i \neq 0$ and $q_i \geq 2$ for $1 \leq i \leq m$, and put $w_i = \lceil \log q_i \rceil$ and $W(\beta) = \sum_i w_i \beta_i$ for $\beta \in \mathbb{N}^m$. The target monomials and their number are

$$\mathcal{J}_H = \{(s,\beta) \in \mathbb{N}^{1+m} : v_0 s + W(\beta)/\theta < H\}, \qquad J_H = |\mathcal{J}_H|, \qquad M = K J_H.$$

The source space is

$$\mathcal{F}_H = \operatorname{span}_{\mathbb{C}}\{u^k t^b Z^\alpha :\ k \in \mathbb{N},\ b \in \{0,1\},\ \alpha \in \mathbb{N}^m,\ 2w_0(k+2b) + W(\alpha) \leq H\}.$$

The centres are

$$c_{ji} = j\,\frac{p_i}{q_i} \qquad (0 \leq j < K,\ 1 \leq i \leq m).$$

For each i they are distinct as j varies, since $p_i \neq 0$. For $P \in \mathcal{F}_H$, centre j and $(s,\beta) \in \mathcal{J}_H$, the coefficient

$$[T^s V^\beta]\, P(U(T), T, c_{j1} + V_1 + L(T), \dots, c_{jm} + V_m + L(T)) \tag{3.1}$$

is the (s,β)-jet of P at the centre j, read in the chart of Proposition 2.3. Choose M monomials $P_c = u^{k_c} t^{b_c} Z^{\alpha_c}$ of $\mathcal{F}_H$, so that

$$b_c \in \{0,1\}, \qquad 2w_0(k_c + 2b_c) + W(\alpha_c) \leq H. \tag{S}$$

The matrix A has rows $r = (j, s, \beta)$ with $0 \le j < K$ and $(s, \beta) \in \mathcal{J}_H$, and columns c. Its entries are the coefficients (3.1) for $P = P_c$. Taylor expansion in V gives

$$A_{(j,s,\beta),c} = \binom{\alpha_c}{\beta}[T^s]\Big(U(T)^{k_c}T^{b_c}\prod_i (c_{ji} + L(T))^{\alpha_{ci}-\beta_i}\Big). \tag{3.2}$$

We write $\Delta = \det A$ and

$$\bar{b} = \frac{1}{MH}\sum_{(j,s,\beta)} W(\beta).$$

The target cutoff gives $W(\beta) < \theta H$ for every row, so $0 \le \bar{b} < \theta$.

3.2. **Nonvanishing.** The nonvanishing theorem of [2] is a separated-weight interpolation theorem in the chart at the origin. We state it as it is proved there. For $n \in \mathbb{N}$ and $s > 0$ put

$$A_n(s) = 2(n+2)^{n+2}\Big(1 + \frac{n+2}{s}\Big)^{n+2}.$$

Theorem 3.1 ([2, Corollary 13.5])**.** *Let A, B, a be integers with $4A^3+27B^2 \neq 0$ and $f(a) \neq 0$. Let $m, K \ge 1$ and let w_0, v_0, θ be positive rationals with*

$$\theta < 1, \qquad K\frac{w_0}{v_0}\theta^m < 1, \qquad K\theta^m < 1. \tag{3.3}$$

Choose a rational $\sigma_0 > 0$ with

$$(1+3\sigma_0)^{m+1}K\frac{w_0}{v_0}\theta^m < 1, \qquad (1+3\sigma_0)^m K\theta^m < 1, \qquad (1+\sigma_0)\theta < 1.$$

Put $\varepsilon_0 = \min\{\sigma_0, \sigma_0 v_0/(4(1+\sigma_0)w_0)\}$, $C_4 = A_m(\sigma_0) + 4\cdot 2^{m+3}A_{m+1}(\varepsilon_0)$ and $q_* = \theta^m \min(1, w_0)/\max(1, v_0)$. Let $w_1, \dots, w_m$ be positive rationals with

$$w_i > \max\Big\{1, \frac{C_4}{q_*}\prod_{1\le l<i} w_l\Big\} \qquad (1 \le i \le m). \tag{3.4}$$

There is a positive integer D_{EPA}, depending on these data but not on any centres, with the following property. For every array $(c_{ji}) \in \mathbb{C}^{K\times m}$ such that $j \mapsto c_{ji}$ is injective for each i, there is H_0 such that for every integer $H \ge H_0$ divisible by D_{EPA} the map from $\mathcal{F}_H$ to $\prod_j \mathbb{C}^{\mathcal{J}_H}$ given by the coefficients (3.1) is surjective.

The theorem is stated in [2] for every curve $y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Z}$ and $4A^3 + 27B^2 \neq 0$, and for every array whose coordinates are injective in j. Its proof, in [2, Sections 4–13], uses the elliptic curve through four facts only. The quartic F is squarefree, which follows from $4A^3 + 27B^2 \neq 0$. Its leading coefficient $f(a)$ is nonzero. The differential $dx/(2y)$ has no algebraic primitive on any finite cover. Finally $F'(0) = 1$. None of these uses an endomorphism. The centres enter through the injectivity of each coordinate, and in one place through their distinctness as points of $\mathbb{C}^m$. Complex multiplication is mentioned in those sections only to fix the curve. So Theorem 3.1 applies to every curve and to the collinear centres jp_i/q_i.

Remark 3.2. The collinear centres do impose a condition, which the thresholds already contain. They lie on the line $u = t = 0$, $Z_i = \lambda p_i/q_i$. Its pole divisor has degree $1/w_1$ in the weighted sense of [2], and it meets each of the K centres with contact θ/w_m. The curve inequality behind Theorem 3.1 therefore contains $(1+\sigma_0)K\theta \le w_m/w_1$. For $m \ge 2$ this follows from (3.4), since $w_m/w_1 > C_4/q_* \ge \theta^{-m} > K > (1+\sigma_0)K\theta$. For $m = 1$ it follows from the condition $(1+3\sigma_0)K\theta < 1$ on σ_0.

If the coefficient map is surjective, the matrix with all M rows and with the columns indexed by all monomials of $\mathcal{F}_H$ has rank M. Some choice of M columns then gives $\Delta \neq 0$. These columns satisfy (S).

4. The arithmetic lower bound

The entries (3.2) contain the full logarithm L, whose coefficients of high order have large denominators. The first lemma shows that a fixed truncation of L leaves Δ unchanged.

Weight the monomial T^sV^β by $v_0 s + W(\beta)/\theta$. Let $\mathcal{I}_H$ be the closed ideal of $\mathbb{C}[[T,V]]$ spanned by the monomials of weight at least H. The quotient $\mathbb{C}[[T,V]]/\mathcal{I}_H$ has the basis $\mathcal{J}_H$.

Lemma 4.1 ([2, Lemma 15.1])**.** *Let $T_1,\dots,T_m$ be positive integers with $T_i v_0 \ge w_i/\theta$, and let L_i^{tr} be the sum of the terms of L of degree less than T_i. Let A^{tr} be the matrix (3.2) with L replaced by L_i^{tr} in the factor with index i. Then $\det A^{\mathrm{tr}} = \det A$ for every choice of the columns.*

Proof. Put $R_i = L - L_i^{\mathrm{tr}}$ and let Φ be the substitution $V_i \mapsto V_i + R_i(T)$. Every term of R_i has degree at least T_i, so its weight is at least $T_i v_0 \ge w_i/\theta$, the weight of V_i. Hence Φ and its inverse $V_i \mapsto V_i - R_i(T)$ do not decrease weight. They preserve $\mathcal{I}_H$ and induce inverse linear maps of $\mathbb{C}[[T,V]]/\mathcal{I}_H$. The map Φ carries the truncated evaluation at each centre to the full one. Order the basis $\mathcal{J}_H$ by $|\beta|$. The image of T^sV^β is T^sV^β plus terms of smaller $|\beta|$, so the matrix of Φ is unitriangular. Thus each centre block of A is a matrix of determinant 1 times the corresponding block of A^{tr}. □

Proposition 4.2. *Let $F_0 \ge 1/\theta$ and suppose $\Delta \neq 0$. Then*

$$\frac{\log|\Delta|}{MH} \ge -(1-\bar{b}) - E_{\mathrm{ar}}, \qquad E_{\mathrm{ar}} = \frac{c_0 F_0}{v_0}, \qquad c_0 = 1.03883.$$

This is [2, Lemma 15.2 and Corollary 15.3] with $\mathbb{Z}$ in place of the order $\mathbb{Z}[i\sqrt{d}]$, and with the constant of Rosser and Schoenfeld in place of $\log 4 + 4$. We give the proof.

Proof. Put $T_i = \lceil F_0 w_i/v_0 \rceil$ and $\ell_i = \mathrm{lcm}(1,\dots,T_i-1)$, with $\ell_i = 1$ if $T_i \le 2$. Then $T_i v_0 \ge F_0 w_i \ge w_i/\theta$, so Lemma 4.1 gives $\Delta = \det A^{\mathrm{tr}}$. By Proposition 2.3, $U \in \mathbb{Z}[[T]]$ and $\ell_i L_i^{\mathrm{tr}} \in \mathbb{Z}[T]$. Hence

$$q_i\ell_i\Big(j\frac{p_i}{q_i} + L_i^{\mathrm{tr}}\Big) = \ell_i j p_i + q_i\ell_i L_i^{\mathrm{tr}} \in \mathbb{Z}[T].$$

By (3.2), $\prod_i (q_i \ell_i)^{\alpha_{ci}-\beta_{ri}} A^{\mathrm{tr}}_{rc}$ is an integer whenever $\beta_r \le \alpha_c$, and $A^{\mathrm{tr}}_{rc} = 0$ otherwise.

Put $d_i = \sum_c \alpha_{ci} - \sum_r \beta_{ri}$. Since $\Delta \neq 0$, some permutation φ of the Leibniz expansion gives a nonzero product. Then $\beta_r \le \alpha_{\varphi(r)}$ for all r, and $d_i = \sum_r (\alpha_{\varphi(r)i} - \beta_{ri}) \ge 0$. The same identity holds for every permutation. So $\Delta \prod_i (q_i \ell_i)^{d_i}$ is a sum of integers. It is a nonzero integer, and

$$\log |\Delta| \ge -\sum_i d_i (\log q_i + \log \ell_i).$$

Now $\log q_i \le w_i$. The number $\log \operatorname{lcm}(1, \dots, n)$ is Chebyshev's function at n, and Rosser and Schoenfeld proved that it is less than $c_0 n$ for $n \ge 1$ [14, Theorem 12]. Hence $\log \ell_i \le c_0 (T_i - 1) < c_0 F_0 w_i / v_0$. By (S), $\sum_i w_i d_i = \sum_c W(\alpha_c) - \sum_r W(\beta_r) \le MH(1 - \bar{b})$. Therefore

$$\log |\Delta| \ge -MH(1-\bar{b})\Big(1 + \frac{c_0 F_0}{v_0}\Big) \ge -MH\big((1-\bar{b}) + E_{\mathrm{ar}}\big). \qquad \square$$

The error E_{ar} depends on F_0 and v_0 only. It does not involve K or the grid.

5. The analytic upper bound

The upper bound follows [2, Section 16]. The grid enters at two places: the transport of the determinant uses $j\Omega \in \Lambda$, and the collision estimate uses a disc containing all centres. Put

$$D = \Big\lceil \frac{H}{2w_0} \Big\rceil, \qquad \delta_{ji} = c_{ji} - j\Omega = j\Big(\frac{p_i}{q_i} - \Omega\Big).$$

By (S), $k_c + 2b_c \le D$ for every column.

5.1. **Transport.**

Lemma 5.1. *Let $\widetilde{A}$ be the matrix with entries*

$$\widetilde{A}_{(j,s,\beta),c} = [z^s]\, \widetilde{g}_{D,k_c,b_c}(j\Omega + z) \binom{\alpha_c}{\beta} \prod_i (j\Omega + z + \delta_{ji})^{\alpha_{ci}-\beta_i}.$$

Then $|\det \widetilde{A}| = |\Delta|\, |\psi(0)|^{DJ_H K}$.

Proof. Fix a centre j and fix β. The set of s with $(s, \beta) \in \mathcal{J}_H$ is an initial segment of $\mathbb{N}$. Substitute $T = \tau(z)$ in the series (3.1). The coefficient of $z^s V^\beta$ is a combination of the coefficients of $T^{s'} V^\beta$ with $s' \le s$, with diagonal coefficient $\tau'(0)^s = (-1)^s$. This is a row operation of determinant ± 1 on the block of j.

By Proposition 2.4 the substituted series is $u(z)^k t(z)^b \prod_i (c_{ji} + V_i + z)^{\alpha_i}$. Since $j\Omega \in \Lambda$, the germs of u and t at 0 and at $j\Omega$ agree. Also $c_{ji} + z = j\Omega + z + \delta_{ji}$. Taking the coefficient of V^β gives $u(j\Omega+z)^k t(j\Omega+z)^b \binom{\alpha}{\beta} \prod_i (j\Omega + z + \delta_{ji})^{\alpha_i - \beta_i}$.

Multiply this by $\widetilde{\psi}(j\Omega+z)^D = \widetilde{\psi}(z)^D$, which is a series with constant term $\psi(0)^D$. On the coefficients of z^s this is again lower triangular, with diagonal

$\psi(0)^D$. Over the block of j it multiplies the determinant by $\psi(0)^{DJ_H}$. The product is $\widetilde{g}_{D,k,b}(j\Omega+z)\binom{\alpha}{\beta}\prod_i(j\Omega+z+\delta_{ji})^{\alpha_i-\beta_i}$. The K blocks give the identity. $\square$

The corresponding identity of [2, Lemma 16.1] has the factor $\prod_j|\psi(\gamma_j)|^{DJ_H}$. For $\gamma_j=j\Omega$, Lemma 2.5(a) gives $\psi(j\Omega)=e^{j^2\eta_\Omega\Omega}\psi(0)$. The periodised multiplier removes this Gaussian in j.

5.2. **Expansion in the displacements.** Let $\mathcal{T}_H=\{\xi\in\mathbb{N}^m : W(\xi)\le H\}$. For a column c and $\xi\in\mathcal{T}_H$ define the entire function

$$G_{c,\xi}(\zeta)=\binom{\alpha_c}{\xi}\,\widetilde{g}_{D,k_c,b_c}(\zeta)\,\zeta^{|\alpha_c|-|\xi|},$$

which is 0 unless $\xi\le\alpha_c$. For a row with centre j and transverse index β put $s(\beta,\xi,j)=\binom{\xi}{\beta}\prod_i\delta_{ji}^{\xi_i-\beta_i}$. The identity

$$\binom{\alpha_i}{\beta_i}(\zeta+\delta)^{\alpha_i-\beta_i}=\sum_{\beta_i\le\xi_i\le\alpha_i}\binom{\xi_i}{\beta_i}\delta^{\xi_i-\beta_i}\binom{\alpha_i}{\xi_i}\zeta^{\alpha_i-\xi_i}$$

with $\zeta=j\Omega+z$ expresses each row of $\widetilde{A}$ as a combination of rows of entire functions. Since $W(\alpha_c)\le H$, only $\xi\in\mathcal{T}_H$ occur. Multilinearity in the rows gives the finite identity

$$\det\widetilde{A}=\sum_{g}\Big(\prod_{r}s(\beta_r,g(r),j(r))\Big)\det B_g, \tag{5.1}$$

over all maps g from the rows to $\mathcal{T}_H$, where $(B_g)_{r,c}=[z^{s_r}]\,G_{c,g(r)}(j(r)\Omega+z)$ and $r=(j(r),s_r,\beta_r)$.

5.3. **Collision.** The next lemma is the collision estimate of [2]. It uses only that the same entire functions are expanded at all the centres, and that the centres lie in a disc of half the radius.

Lemma 5.2 ([2, Lemma 16.2])**.** *Let $R>0$ with $(K-1)\Omega+1\le R/2$, and suppose $|G_{c,\xi}(\zeta)|\le e^{H\mathcal{G}}$ for $|\zeta|\le R$ and all c,ξ. For a map g put $n_\xi=|g^{-1}(\xi)|$. Let $\kappa=(\log 2)/4$, $s_0=2^{-1/2}$ and $\rho_H=(\log M+\kappa-\log(1-s_0))/H$. Then*

$$|\det B_g|\le\exp\Big(-\kappa\sum_{\xi}n_\xi^2+MH(\mathcal{G}+\rho_H)\Big).$$

Proof. Put $a_{c,\xi,l}=R^l[\zeta^l]G_{c,\xi}$, so $|a_{c,\xi,l}|\le e^{H\mathcal{G}}$ by Cauchy's inequality. Put $b_{r,l}=[z^{s_r}]((j(r)\Omega+z)/R)^l$. Cauchy's inequality on $|z|=1$ gives $|b_{r,l}|\le((j(r)\Omega+1)/R)^l\le 2^{-l}$. So $(B_g)_{r,c}=\sum_{l\ge0}b_{r,l}a_{c,g(r),l}$, with absolute convergence. Expanding the determinant by multilinearity in the rows,

$$\det B_g=\sum_{(l_r)}\Big(\prod_r b_{r,l_r}\Big)\det\big[a_{c,g(r),l_r}\big]_{r,c}.$$

Two rows with equal pairs $(g(r),l_r)$ are equal, so the determinant vanishes. In a surviving term the l_r are distinct within each class $g^{-1}(\xi)$, and $\sum_r l_r\ge$

$\sum_\xi \binom{n_\xi}{2}$. Each coefficient determinant is at most $M!\,e^{MH\mathcal{G}}$. Write $2^{-\sum l_r} = s_0^{\sum l_r} s_0^{\sum l_r}$. Bound the first factor by $s_0^{\sum_\xi \binom{n_\xi}{2}}$ and sum the second over all (l_r). This gives

$$|\det B_g| \le M!\,e^{MH\mathcal{G}} s_0^{\sum_\xi \binom{n_\xi}{2}} (1-s_0)^{-M}.$$

Finally $M! \le M^M$, $\sum_\xi n_\xi = M$ and $\log s_0 = -2\kappa$. □

5.4. **The estimate.** Put

$$r_* = 2(\Omega+1), \qquad C_\Lambda = \log S_* + \log^+ \frac{1}{|\psi(0)|} + \chi(r_*+Y)^2.$$

These constants depend only on Λ, a and the choice of z_Q.

Theorem 5.3. *Assume $\Delta \neq 0$. Let $\nu \ge 1$ with $|\Omega - p_i/q_i| \le q_i^{-\nu}$ for $1 \le i \le m$, and let $0 < w_{\min} \le \min_i w_i$. Fix $a_0, \gamma \in (0,1)$ and put*

$$N_H = |\{\xi \in \mathbb{N}^m : W(\xi) \le a_0 H\}|, \qquad c_H = \frac{\kappa\gamma^2 M}{H N_H}.$$

Then

$$\frac{\log|\Delta|}{MH} \le E_{\mathrm{an}} + e_H + \max\{-c_H,\ -\nu(a_0(1-\gamma) - \bar b)\},$$

where

$$E_{\mathrm{an}} = \frac{C_\Lambda K^2}{2w_0} + \frac{\nu + \log(4r_*K^2)}{w_{\min}}, \qquad e_H = \frac{C_\Lambda K^2}{H} + \rho_H + \frac{\log|\mathcal{T}_H|}{H}.$$

Proof. Displacements. Since $\log q_i > w_i - 1$, we have $|\delta_{ji}| \le K q_i^{-\nu} \le \exp(C_\delta - \nu w_i)$ with $C_\delta = \nu + \log K > 0$. A nonzero $s(\beta,\xi,j)$ has $\beta \le \xi$, and $|\xi| \le H/w_{\min}$ for $\xi \in \mathcal{T}_H$. Since $\binom{\xi}{\beta} \le 2^{|\xi|}$,

$$|s(\beta,\xi,j)| \le \exp\Big(-\nu(W(\xi) - W(\beta)) + \frac{H(\log 2 + C_\delta)}{w_{\min}}\Big). \tag{5.2}$$

The disc. Take $R = r_*K$. Then $(K-1)\Omega + 1 \le (\Omega+1)K = R/2$. On $|\zeta| \le R$ we have $|\operatorname{Im}\zeta| \le R$ and $R + Y \le (r_* + Y)K$. Lemma 2.6, $\binom{\alpha_c}{\xi} \le 2^{|\alpha_c|}$ and $|\alpha_c| \le H/w_{\min}$ give $|G_{c,\xi}| \le e^{H\mathcal{G}}$ with

$$\mathcal{G} = \frac{D}{H}(\log S_* + \chi(r_*+Y)^2K^2) + \frac{\log(2R)}{w_{\min}}.$$

Two cases. Consider a map g whose scalar product in (5.1) is nonzero. Then $W(g(r)) \ge W(\beta_r)$ for every row, so $\sum_r W(g(r)) \ge MH\bar b$. Suppose first that $\sum_r W(g(r)) \ge a_0(1-\gamma)MH$. The product of the bounds (5.2) contains the factor $\exp(-\nu MH(a_0(1-\gamma) - \bar b))$, and we use Lemma 5.2 without its first term. Otherwise more than γM rows have $W(g(r)) \le a_0 H$. Indeed, if at least $(1-\gamma)M$ rows had $W(g(r)) > a_0 H$, the sum would be at least $a_0(1-\gamma)MH$. These rows use at most N_H values of g, so the Cauchy–Schwarz inequality gives $\sum_\xi n_\xi^2 \ge \gamma^2 M^2/N_H$. Lemma 5.2 then gives

the factor $\exp(-c_H MH)$, and the factor $\exp(-\nu\sum_r(W(g(r))-W(\beta_r)))$ is at most 1. There are $|\mathcal{T}_H|^M$ maps g. Hence

$$\frac{\log|\det\widetilde{A}|}{MH} \le \frac{\log 2+C_\delta}{w_{\min}}+\mathcal{G}+\rho_H+\frac{\log|\mathcal{T}_H|}{H}+\max\{-c_H,-\nu(a_0(1-\gamma)-\bar{b})\}.$$

The multiplier. By Lemma 5.1 and $MH = KJ_HH$,

$$\frac{\log|\Delta|}{MH} \le \frac{\log|\det\widetilde{A}|}{MH}+\frac{D}{H}\log^+\frac{1}{|\psi(0)|}.$$

The terms with the factor D/H add up to at most $(D/H)C_\Lambda K^2$, since $K\ge 1$. Moreover $D/H \le 1/(2w_0)+1/H$. Finally $\log 2+C_\delta+\log(2R) = \nu+\log(4r_*K^2)$. □

Corollary 5.4 ([2, Corollary 16.4])**.** *With all data other than H fixed, $e_H\to 0$ and*

$$c_H \to c_\infty = \frac{\kappa\gamma^2K\theta^m}{(m+1)v_0a_0^m} \qquad (H\to\infty).$$

Proof. The number of points of $\mathbb{N}^n$ in the simplex $\{\sum_i a_ix_i < X\}$ is asymptotic to $X^n/(n!\prod_i a_i)$. Hence $J_H \sim \theta^mH^{m+1}/((m+1)!\,v_0\prod_i w_i)$ and $N_H \sim a_0^mH^m/(m!\prod_i w_i)$. This gives the limit of $c_H = \kappa\gamma^2KJ_H/(HN_H)$. The numbers M and $|\mathcal{T}_H|$ grow polynomially in H, so $e_H\to 0$. □

6. The parameters

In this section $\nu>3$ is rational. We choose the remaining parameters explicitly. Put

$$b=\frac{5}{6}-\frac{1}{2\nu}, \qquad h=\frac{\nu-3}{4\nu}, \qquad \theta=1-h,$$

$$a_0=1-bh, \qquad \gamma=\frac{h(\nu-3)}{12\nu}, \qquad F_0=\frac{1}{\theta},$$

and

$$g=\nu\big(a_0(1-\gamma)-\theta\big)-(1-\theta).$$

Lemma 6.1. *(a)* $2/3<b<1-1/\nu$ *and* $0<h<1/4$. *Hence* $0<\theta<a_0<1$.
(b) $a_0^3<\theta^2$.
(c) $g\ge h(\nu-3)/12>0$.

Proof. (a) We have $3b-2=(\nu-3)/(2\nu)=2h>0$ and $1-1/\nu-b=(\nu-3)/(6\nu)>0$. Since $b<1$, $a_0>\theta$.

(b) Expanding, $a_0^3-\theta^2 = h\big((2-3b)+(3b^2-1)h-b^3h^2\big)$. With $2-3b=-2h$ the bracket equals $3(b^2-1)h-b^3h^2<0$.

(c) We have $a_0-\theta=h(1-b)$ and $\nu(1-b)-1=(\nu-3)/6$. Hence $g=h(\nu-3)/6-\nu\gamma a_0\ge h(\nu-3)/6-\nu\gamma=h(\nu-3)/12$. □

By (b), $a_0^{3/2} < \theta$, that is, $a_0/\theta < a_0^{-1/2}$. Choose rationals c_1, c_2 with

$$\frac{a_0}{\theta} < c_1 < a_0^{-1/2}, \qquad a_0 < c_2 < \min\{c_1^{-2},\ c_1\theta\}.$$

The second interval is not empty, because $c_1 < a_0^{-1/2}$ means $a_0 < c_1^{-2}$ and $c_1 > a_0/\theta$ means $a_0 < c_1\theta$. Also $c_1 > 1$ and $c_1\theta < a_0^{-1/2}\theta < a_0^{1/2} < 1$. For integers $m \geq 1$ put

$$K = \lfloor c_1^m \rfloor, \qquad w_0 = c_2^{-m}, \qquad v_0 = 4K\theta^m w_0. \tag{6.1}$$

These are rational. Let E_{ar} be as in Proposition 4.2 and c_∞ as in Corollary 5.4.

Proposition 6.2. *As $m \to \infty$:*

(P1) $K\theta^m \to 0$;
(P2) $K(w_0/v_0)\theta^m = 1/4$ *for every* m;
(P3) $K^2/w_0 \to 0$;
(P4) $v_0 \to \infty$, *so* $E_{\mathrm{ar}} \to 0$;
(P5) $c_\infty \to \infty$.

Proof. For $c_1^m \geq 2$ we have $c_1^m/2 \leq K \leq c_1^m$. (P1) $K\theta^m \leq (c_1\theta)^m$. (P2) is the definition of v_0. (P3) $K^2/w_0 \leq (c_1^2 c_2)^m$ and $c_1^2 c_2 < 1$. (P4) $v_0 \geq 2(c_1\theta/c_2)^m$ and $c_1\theta > c_2$. (P5) Substituting v_0, the factor K cancels and $c_\infty = \kappa\gamma^2(c_2/a_0)^m/(4(m+1))$, with $c_2 > a_0$. □

The factor K cancels from c_∞ because v_0 is proportional to K. This is why the analytic cost K^2/w_0 and the collision rate c_∞ meet in the single condition $a_0^3 < \theta^2$.

7. Proof of Theorems 1.1 and 1.2

Theorem 7.1. *For every $\nu > 3$ there are only finitely many pairs (p, q) of integers with $q \geq 1$ and $|\Omega - p/q| \leq q^{-\nu}$.*

Proof. Decreasing ν only enlarges the set of pairs, so we may assume ν rational. Suppose that infinitely many pairs satisfy the inequality. Their denominators are unbounded. Since $\Omega \neq 0$, all but finitely many have $p \neq 0$ and $q \geq 2$. Take the parameters of Section 6 and the constant C_Λ of Theorem 5.3.

Step 1. By Proposition 6.2 we may fix m so large that (3.3) holds, $c_\infty > 2$, and

$$E_{\mathrm{ar}} + \frac{C_\Lambda K^2}{2w_0} < \min\Big(\frac{g}{4}, \frac{1}{4}\Big).$$

This fixes K, w_0, v_0 and θ. Choose σ_0, C_4 and q_* as in Theorem 3.1.

Step 2. Choose pairs $(p_1, q_1), \dots, (p_m, q_m)$ from the given infinite set in turn. Each w_i is to satisfy (3.4), which involves only $w_1, \dots, w_{i-1}$. Each is also to satisfy $w_i \geq w_*$, where

$$\frac{\nu + \log(4r_* K^2)}{w_*} < \min\Big(\frac{g}{4}, \frac{1}{4}\Big).$$

Unbounded denominators allow every choice. Put $w_{\min} = w_*$. Then $E = E_{\mathrm{ar}} + E_{\mathrm{an}} < \min(g/2, 1/2)$.

Step 3. Theorem 3.1 gives D_{EPA}. The centres $c_{ji} = jp_i/q_i$ are injective in j for each i, so there is H_0 such that for every $H \geq H_0$ divisible by D_{EPA} some choice of columns gives $\Delta_H \neq 0$. Let $\bar{b}_H \in [0, \theta)$ be the corresponding mean.

Step 4. Proposition 4.2 and Theorem 5.3 apply to the same Δ_H:

$$\frac{\log|\Delta_H|}{MH} \geq -(1 - \bar{b}_H) - E_{\mathrm{ar}},$$
$$\frac{\log|\Delta_H|}{MH} \leq E_{\mathrm{an}} + e_H + \max\{-c_H, -\nu(a_0(1-\gamma) - \bar{b}_H)\}.$$

If the maximum is $-c_H$, subtracting the lower bound from the upper bound gives $0 \leq E + e_H - c_H + 1 - \bar{b}_H \leq 1 + E + e_H - c_H$. By Corollary 5.4, $e_H \to 0$ and $c_H \to c_\infty > 2$, while $E < 1/2$. So the right side is negative for large H. In the other case the difference is

$$E + e_H - (\nu a_0(1-\gamma) - 1 - (\nu - 1)\bar{b}_H) \leq E + e_H - g,$$

because $\nu > 1$ and $\bar{b}_H < \theta$. Since $E < g/2$, this is negative for large H. Both cases are impossible for large H divisible by D_{EPA}. □

Proof of Theorem 1.2. By Lemma 2.1 and the remarks after it, $\omega = N\Omega$ with N a nonzero integer, for the lattice $\Lambda = d^{-1}L$. Let $\varepsilon > 0$. If infinitely many pairs satisfied $|\omega - p/q| \leq q^{-3-\varepsilon}$, Lemma 2.2 with $\nu' = 3 + \varepsilon/2$ would contradict Theorem 7.1. So the finitely many exceptions have bounded q, and q_0 exists. If ω were a rational a/b, the pairs (na, nb) would give infinitely many exceptions. So ω is irrational, and Dirichlet's theorem gives $\mu(\omega) \geq 2$. □

Proof of Theorem 1.1. Take a Weierstrass equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ of E over $\mathbb{Q}$. With $y_1 = y + (a_1x + a_3)/2$ and $x_1 = x + (a_2 + a_1^2/4)/3$ it becomes $y_1^2 = x_1^3 + A_1x_1 + B_1$ with $A_1, B_1 \in \mathbb{Q}$, and $dx/(2y + a_1x + a_3) = dx_1/(2y_1)$. The invariant differentials defined over $\mathbb{Q}$ are the rational multiples of this differential, so $\eta = r\,dx_1/(2y_1)$ with $r \in \mathbb{Q}^\times$. By [18, Section VI.5] the periods of $dx_1/(2y_1)$ form the lattice Λ_1 with $g_2(\Lambda_1) = -4A_1$ and $g_3(\Lambda_1) = -4B_1$. The periods of η form $r\Lambda_1$, whose invariants $r^{-4}g_2(\Lambda_1)$ and $r^{-6}g_3(\Lambda_1)$ are rational. Theorem 1.2 applies to $L = r\Lambda_1$. □

8. Why the exponent is three

The exponent 3 comes from the analytic term $C_\Lambda K^2/(2w_0)$. This section explains how that term determines the exponent, why it arises, and what would remove it.

8.1. **The threshold for a cost of order K^e.** Suppose that the analytic estimate of Theorem 5.3 held with CK^e/w_0 in place of $C_\Lambda K^2/(2w_0)$, for some $e \geq 1$. Take $K = \lfloor c_1^m \rfloor$, $w_0 = c_2^{-m}$ and $v_0 = CK\theta^m w_0$ with fixed c_1, c_2, C, as in (6.1). Up to the constant this choice of v_0 is the best one. The interpolation theorem needs $v_0 > K\theta^m w_0$, and c_∞ decreases as v_0 grows. The proof of Section 7 needs four rates. The analytic term must tend to 0, which needs $c_1^e c_2 \leq 1$. The collision rate c_∞, proportional to $(c_2/a_0)^m/(m+1)$, must tend to infinity, which needs $c_2 > a_0$. The arithmetic error $c_0 F_0/v_0$ must stay small, which needs v_0 bounded below and so $c_1\theta \geq c_2$. Finally $g > 0$ needs $\nu(a_0 - \theta) > 1 - \theta$. The first three give $a_0 < c_2 \leq \min(c_1^{-e}, c_1\theta)$, hence $a_0/\theta < c_1 < a_0^{-1/e}$ and

$$a_0^{e+1} < \theta^e. \tag{8.1}$$

Proposition 8.1. *Let $e \geq 1$ be real and $\nu > 1$. There are $0 < \theta < a_0 < 1$ with $a_0^{e+1} < \theta^e$ and $\nu(a_0 - \theta) > 1 - \theta$ if and only if $\nu > e + 1$.*

Proof. Write $\theta = x^{e+1}$ with $0 < x < 1$. Condition (8.1) says $a_0 < x^e$. Since $1 - \theta > 0$, the second condition requires $a_0 > \theta$ and $\nu > (1-\theta)/(a_0 - \theta)$. For $\theta < a_0 < x^e$,

$$\frac{1-\theta}{a_0 - \theta} > r_e(x), \qquad r_e(x) = \frac{1 - x^{e+1}}{x^e - x^{e+1}}.$$

The function $k_e(x) = 1 - (e+1)x^e + ex^{e+1}$ vanishes at $x = 1$, and $k_e'(x) = e(e+1)x^{e-1}(x-1) < 0$ on $(0,1)$. So $k_e > 0$ on $(0,1)$, which says $r_e(x) > e+1$. Hence both conditions force $\nu > e+1$. Conversely, let $\nu > e+1$. By l'Hôpital's rule $r_e(x) \to e+1$ as $x \to 1$, so we can fix x with $r_e(x) < \nu$. By continuity some a_0 slightly below x^e satisfies both conditions. □

When (8.1) holds, choose $c_1 \in (a_0/\theta, a_0^{-1/e})$, $c_2 \in (a_0, \min(c_1^{-e}, c_1\theta))$ and γ small. Then all four rates hold strictly, as in Proposition 6.2. So within this family of parameters the exponent obtained is exactly $e + 1$. For $e = 1$ the window is $a_0^2 < \theta$. This is the condition of [2, Theorem 17.1] and of the proof for π [11], and it gives the exponent 2. For $e = 2$ it is Lemma 6.1(b), and it gives 3.

8.2. **The growth of the multiplier along the grid.** The cost K^2 is a property of the disc. The collision estimate expands the same entire functions at all the centres, so it needs a disc of radius R of order $K\Omega$ around the origin. On that disc the multiplier attains the growth $\exp(D\chi R^2)$ near $\pm iR$, by Lemma 2.5(b). No other multiplier does better by more than a constant factor in the exponent.

Remark 8.2. Let Φ be entire with $\Phi(0) \neq 0$, vanishing to order at least D at every point of $\pm z_Q + \Lambda$. Such a function is needed to clear the poles of u^D. Let $n(r)$ count these points in $|z| < r$. Jensen's formula gives

$$\log \max_{|z|=R} |\Phi| \geq \log |\Phi(0)| + D \int_0^R \frac{n(r)}{r}\, dr.$$

Comparing areas, $n(r) \geq 2\pi r^2/(\Omega Y) - O(r+1)$. Hence $\log \max_{|z|=R} |\Phi| \geq \log|\Phi(0)| + D\pi R^2/(\Omega Y) - O(D(R+1))$. With R of order K and D/H close to $1/(2w_0)$, any Cauchy estimate on the disc costs a term of order K^2/w_0.

The entries of $\widetilde{A}$ themselves are not large. By periodicity $\widetilde{g}_{D,k,b}(j\Omega + z) = \widetilde{g}_{D,k,b}(z)$, so the entries are bounded on the strip $|\operatorname{Im} z| \leq 1$ independently of j, up to the polynomial factors. The loss comes from bounding the Taylor coefficients at 0 through the maximum on $|\zeta| = R$.

8.3. **A collision estimate without that growth.** Suppose the analytic estimate held with a cost CK/w_0, for a constant C depending only on Λ. By Proposition 8.1 with $e = 1$, the proof of Section 7 would then give $\mu(\omega) = 2$ for every nonzero real ω in every lattice with rational invariants. Such an estimate would have to separate the rows of different centres without expanding at a single point. The monomials ζ^l achieve this separation in Lemma 5.2: rows of the same tag at different centres share a coefficient vector only when they share the exponent l. A basis adapted to the strip, on which translation by Ω acts diagonally, would keep the entries bounded. It would need a new counting argument for coincident rows. We do not know such an estimate.

8.4. **Algebraic centres and norms.** A two-dimensional grid of periods with algebraic multipliers is available only under complex multiplication, by Remark 1.4. Other algebraic centres bring a different loss. Suppose the cleared determinant is a nonzero algebraic integer of a number field k of degree n. The lower bound then comes from $|N_{k/\mathbb{Q}}(\Delta)| \geq 1$, and it needs upper bounds for all conjugates of Δ. The conjugates are determinants at the conjugate centres, which are not close to periods. They obey only the trivial bound, without the collision or displacement gains. The arithmetic side then loses a factor of order n. Under complex multiplication with the rectangular grid, the only other embedding of $\mathbb{Q}(i\sqrt{d})$ is complex conjugation, and $|a + bi\sqrt{d}|^2$ is a positive integer, so nothing is lost. For the powers ϖ^n of the lemniscate period the natural centres involve $(p/q)^{1/n}$, of degree n. There G. and D. Chudnovsky use the norm from $\mathbb{Q}(i, (p/q)^{1/n})$ to $\mathbb{Q}(i)$, and they obtain $\mu(\varpi^n) = O(n \log 2n)$ [2, Theorem 18.1].

Disclosure of AI use

The mathematics and the text of this paper were produced by AI systems at the request of the author. The author posed the problem, directed the work and is responsible for the paper. The systems were Claude Opus 5.5, Claude Sonnet 5.5 and Claude Fable 5.2 (Anthropic), running in Claude Code. The construction, the nonvanishing theorem and the method of the proof are due to G. V. Chudnovsky and D. V. Chudnovsky [1, 2], building on OpenAI's proof for π [11]. The Lean files reuse code from OpenAI's formalization for Catalan's constant and from the Tau Ceti project, both under the Apache License 2.0.

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