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THE EGAMI–MATSUMOTO CONJECTURE UNDER THE RIEMANN HYPOTHESIS ALONE: A NATURAL BOUNDARY OF THE GOLDBACH GENERATING SERIES

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Abstract. The Goldbach generating series $\Phi_2(s) = \sum_{m,n\geq 1} \Lambda(m)\Lambda(n)(m+n)^{-s}$ continues meromorphically to $\operatorname{Re} s > 1$ under the Riemann Hypothesis (RH), by a theorem of Egami and Matsumoto, who conjectured that the line $\operatorname{Re} s = 1$ is its natural boundary. The complete proofs so far have assumed, besides RH, a hypothesis on additive relations among the ordinates of the zeros of ζ. We prove the conjecture under RH alone, and the corresponding statement for the series $\Phi_r(s) = \sum \Lambda(k_1)\cdots\Lambda(k_r)(k_1+\cdots+k_r)^{-s}$ at $\operatorname{Re} s = r-1$, for every $r \geq 2$. RH can be weakened: the continuation and the boundary hold under the hypothesis H_∞ that for every $\delta > 0$ only finitely many zeros have real part greater than $\frac{1}{2}+\delta$. The new step is a phase estimate for products of two Gamma values at zeros of large ordinates of opposite signs. It shows that the pairs of zeros whose sums land on one point do not cancel when every such pair consists of zeros whose ordinates exceed in modulus a threshold depending only on a bound for the imaginary part of the point; points of this kind accumulate at every point of the line, and under RH they are dense on it. Under H_∞, if RH fails then Φ_2 has a simple pole with positive residue at the real point $2B \in (1,2)$, where B is the supremum of the real parts of the zeros; so, under H_∞, RH holds if and only if Φ_2 has no pole in the real interval $(1,2)$. A criterion of this kind, through the poles at the points $1+\rho$, follows from work of Bhowmik, Halupczok, Matsumoto and Suzuki; what is new is the pole at $2B$, whose residue is a sum of positive terms. The theorems are checked in the Lean 4 proof assistant, as implications from H_∞ (or RH) and classical theorems on ζ and Γ stated as hypotheses.

1. Introduction

1.1. The series and the conjecture.

Let Λ be the von Mangoldt function and

$$\Phi_2(s) = \sum_{m\geq 1}\sum_{n\geq 1} \frac{\Lambda(m)\Lambda(n)}{(m+n)^s} = \sum_{N\geq 2} \frac{G_2(N)}{N^s}, \qquad G_2(N) = \sum_{m+n=N} \Lambda(m)\Lambda(n). \tag{1}$$

Since $G_2(N) \leq N\log^2 N$, the series converges absolutely for $\operatorname{Re} s > 2$; $G_2(N)$ is the weighted number of representations of N as a sum of two prime powers. The series appears in [17, (7), p. 174]. Egami and Matsumoto proved [18, Theorem 2.1] that under the Riemann Hypothesis (RH) Φ_2 continues meromorphically to $\operatorname{Re} s > 1$, with simple poles at $s = 2$ and at $s = 1+\rho$ for each non-trivial zero ρ of ζ, and no other poles there, and they proposed [18, Conjecture 2.1]:

The line $\operatorname{Re} s = 1$ *is the natural boundary of* $\Phi_2(s)$.

Here, for a function f meromorphic on a half-plane $\operatorname{Re} s > a$, the line $\operatorname{Re} s = a$ is a *natural boundary* of f if f has no meromorphic continuation to a neighbourhood of any point of the line; precisely, if for no point s_1 with $\operatorname{Re} s_1 = a$, no $\varepsilon > 0$ and no function g meromorphic on the

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disc $|s-s_1|<\varepsilon$ does g coincide with f at every point of the half-disc $\{|s-s_1|<\varepsilon,\ \operatorname{Re} s>a\}$ at which f is analytic.

The continuation of Φ_2 contains a double sum over pairs of zeros in which the term of the pair (ρ,ρ') has a simple pole at the *pair sum* $\rho+\rho'$, and under RH the pair sums are dense on the line $\operatorname{Re} s=1$ [18, Lemma 4.1]. Under RH they lie on the line itself, not in the half-plane, and a point $s_1\neq 1$ of the line is tested by a radial limit of the whole continuation: as $\eta\downarrow 0$, $\eta\,\Phi_2(s_1+\eta)$ tends to a nonzero multiple of the sum of the coefficients of the pairs with sum s_1 (Corollary B(iv) below, with $r=2$; part (iii) gives the regularized limit at $s_1=1$). A natural boundary follows once these sums are shown to be nonzero, so that the pairs landing on one point do not cancel, at a dense set of points of the line. The complete proofs so far get this from a hypothesis on additive relations among the ordinates of the zeros: hypothesis (A) of [18], that two pairs of ordinates with the same nonzero sum are the same pair up to order, or a hypothesis that implies it. Under RH and (A), at most one pair, up to order, lands on each point of the line other than 1. The proof of Egami and Matsumoto assumes their hypothesis (B), which implies (A) and bounds from below, by a quantity that decays exponentially in the ordinates, the distance between a nonzero sum of two ordinates and the sum of any pair that is not the same pair up to order. Section 9 states both.

This paper uses no hypothesis on the ordinates, and it weakens RH. Let $\mathcal{Z}$ be the set of non-trivial zeros of ζ, write $\rho=\beta+i\gamma$ for $\rho\in\mathcal{Z}$, and let $m(\rho)$ be the multiplicity of ρ.

(H_∞) *For every $\delta>0$, only finitely many $\rho\in\mathcal{Z}$ have $\beta>\frac12+\delta$.*

In words: zeros may leave the critical line, but for each fixed amount only finitely many leave it by more than that amount. RH implies H_∞,[1] and so does the existence of only finitely many zeros off the line; H_∞ is not known to imply RH, and it remains unproved. In the notation of [10, Theorem 2(4)] it is the case $\sigma_0=\frac12$ (Section 9). Put $B=\sup_{\rho\in\mathcal{Z}}\beta$. The proofs use two consequences of H_∞ (Lemma 2.1): $B<1$, and the approach of high zeros to the critical line, $\beta\to\frac12$ as $|\gamma|\to\infty$, which a fixed zero-free half-plane does not give. The bound $B<1$ gives the margin of the phase estimate of Section 5 and the cutoff of Section 7; the approach to the line gives the continuation (Section 4) and the interior poles of Section 6.

1.2. **The results.**

Theorem A. *Assume* H_∞.

(i) Φ_2 *has a meromorphic continuation to the half-plane* $\operatorname{Re} s>1$.
(ii) *The line* $\operatorname{Re} s=1$ *is a natural boundary of* Φ_2.
(iii) *For every integer* $r\geq 2$, *the series* $\Phi_r(s)=\sum_{k_1,\dots,k_r\geq 1}\Lambda(k_1)\cdots\Lambda(k_r)(k_1+\cdots+k_r)^{-s}$ *has a meromorphic continuation to* $\operatorname{Re} s>r-1$, *and the line* $\operatorname{Re} s=r-1$ *is a natural boundary of* Φ_r.

By the identity theorem the continuations are unique, and Φ_2 and Φ_r denote them. Part (i) says "meromorphic" and no more. Under H_∞ the continuation may have poles that RH excludes, among the pair sums in $\operatorname{Re} s>1$, which we call *candidate poles*: a zero with $\beta>\frac12$ and its conjugate have the real sum $2\beta\in(1,2)$, and any two zeros to the right of the critical line have a sum with real part greater than 1. Away from 2 and the points $1+\rho$, a candidate is a pole exactly when the pairs that land on it do not cancel, and Theorem A does not list them; Theorem C below settles one. Under RH no pair sum lies in $\operatorname{Re} s>1$, and the poles are those

[1] Alpöge [1] has announced a proof of RH and of the Generalized Riemann Hypothesis for Dirichlet L-functions, released as a Lean formalization without an accompanying prose proof; we have not built or examined it. If the proof of RH stands, the conclusions of Theorem A and Corollary B hold unconditionally; in particular $\operatorname{Re} s=1$ is a natural boundary of Φ_2.

found by Egami and Matsumoto. That is the first part of Corollary B below, which collects what RH adds; parts (iii) and (iv) need one definition.

Let $a_\rho = m(\rho)\Gamma(\rho)$ for $\rho \in \mathcal{Z}$ and, for real τ,

$$\mu\{\tau\} = \sum_{\substack{(\rho,\rho')\in\mathcal{Z}\times\mathcal{Z}\\ \gamma+\gamma'=\tau}} a_\rho a_{\rho'}, \tag{2}$$

summed over ordered pairs, so that (ρ,ρ') and (ρ',ρ) both appear when $\rho \neq \rho'$; the sum converges absolutely (Lemma 3.4). We call it the *atom* at τ.

Corollary B. *Assume RH.*

(i) *The continuation of Theorem A(i) is analytic on* $\operatorname{Re} s > 1$ *except for simple poles at* $s = 2$*, with residue* 1*, and at* $s = 1+\rho$ *for each* $\rho \in \mathcal{Z}$*, with residue* $-2m(\rho)/\rho$*.*

(ii) *The line* $\operatorname{Re} s = 1$ *is a natural boundary of* Φ_2*.*

(iii) *Let* $M(s) = 2\psi(s-1)/(s-1) - 1/(\Gamma(s)(s-2))$*, with* $\psi = -\zeta'/\zeta$*, and* $H = \Phi_2 - M$*. Then* H *is holomorphic on* $\operatorname{Re} s > 1$*, and for every real* τ

$$\lim_{\eta\downarrow 0} \eta\, H(1+i\tau+\eta) = \frac{\mu\{\tau\}}{\Gamma(1+i\tau)}.$$

(iv) *For every integer* $r \geq 2$ *the continuation of* Φ_r *is analytic on* $\operatorname{Re} s > r-1$ *except for simple poles at* $s = r$*, with residue* $1/(r-1)!$*, and at* $s = r-1+\rho$ *for each* $\rho \in \mathcal{Z}$*, with residue* $-r\, m(\rho)\Gamma(\rho)/\Gamma(r-1+\rho)$*; and for real* $\tau \neq 0$

$$\lim_{\eta\downarrow 0} \eta\, \Phi_r(r-1+i\tau+\eta) = \binom{r}{2} \frac{\mu\{\tau\}}{\Gamma(r-1+i\tau)}.$$

Part (ii) is the conjecture of Egami and Matsumoto under RH alone, and the boundary in Theorem A(iii) is, under RH, Conjecture 6.1 of [18]. Multiple zeros and coincident sums $\gamma_1+\gamma_2 = \gamma_3+\gamma_4$ are allowed. We know of no earlier complete proof of the natural boundary of Φ_2 under RH alone, and of none that does not assume RH (Section 9). Part (iii) is stated for $H = \Phi_2 - M$, and not for Φ_2, because of the point $\tau = 0$. For $\tau \neq 0$ the main term M is analytic at $1+i\tau$, so $\eta\,\Phi_2(1+i\tau+\eta)$ and $\eta\, H(1+i\tau+\eta)$ have the same limit; at $\tau = 0$ the term $2\psi(s-1)/(s-1)$ of M has a pole at $s = 1$, which adds its residue $2\psi(0)$ to the limit of $\eta\,\Phi_2(1+\eta)$ (Remark 8.3.1). In (iv) the point $\tau = 0$ is left out: for $r = 2$ it is covered by (iii), and for $r \geq 3$ the limit at $\tau = 0$ is not treated in this paper.

We return to H_∞ alone. If some zero lies off the line, there is one candidate pole at which nothing can cancel: the pairs of zeros with sum $2B$ are exactly the pairs $(\rho,\bar\rho)$ with $\beta = B$, and each carries the positive coefficient $a_\rho a_{\bar\rho} = m(\rho)^2|\Gamma(\rho)|^2$.

Theorem C. *Assume* H_∞*.*

(i) *If some* $\rho \in \mathcal{Z}$ *has* $\beta \neq \frac12$*, then* $\frac12 < B < 1$*, the set of* $\rho \in \mathcal{Z}$ *with* $\beta = B$ *is finite and not empty, and the real point* $2B \in (1,2)$ *is a simple pole of* Φ_2 *with residue*

$$\frac{1}{\Gamma(2B)} \sum_{\rho\in\mathcal{Z},\ \beta=B} m(\rho)^2|\Gamma(\rho)|^2 > 0.$$

The sum runs over both ρ *and* $\bar\rho$*, so each conjugate pair contributes* $2m(\rho)^2|\Gamma(\rho)|^2$*.*

(ii) *RH holds if and only if* Φ_2 *has no pole in the real interval* $(1,2)$*.*

The interval in (ii) is open: it excludes $s = 2$, a pole under RH. Only $2B$ is shown to be a pole; other real points of $(1,2)$ can be pair sums, for instance 2β for a zero with $\frac12 < \beta < B$, and Theorem C says nothing about them. Part (ii) is new only in form: when $B < 1$, which

H_∞ implies, a criterion for RH through the poles of Φ_2 at the points $1+\rho$ follows from [4, Proposition 3] (Section 9). What Theorem C adds is the pole at $2B$, with its residue.

1.3. **The idea.** Write $F(x)=\sum \Lambda(n)e^{-nx}$. Integration term by term gives

$$\Gamma(s)\Phi_2(s)=\int_0^\infty F(x)^2x^{s-1}\,dx \qquad (\operatorname{Re} s>2),$$

and, since F decays exponentially as $x\to\infty$, the continuation of Φ_2 is decided by the behaviour of F as $x\downarrow 0$. There $1/x$ is the main term of F; it comes from the pole of $-\zeta'/\zeta$ at 1 (Lemma 3.5). Put $h(x)=F(x)-1/x$. Then $F^2=x^{-2}+2x^{-1}h+h^2$, and the first two terms give a function meromorphic on the whole plane. The explicit formula writes $h=-Z+c_0+V$, with

$$Z(x)=\sum_{\rho\in\mathcal{Z}} a_\rho x^{-\rho},$$

a constant c_0 and $V(x)=O(x^{1/2})$, and the Mellin transform of Z^2 over $(0,1]$ is the *pair series*

$$\mathcal{C}(s)=\sum_{(\rho,\rho')\in\mathcal{Z}\times\mathcal{Z}} \frac{a_\rho a_{\rho'}}{s-\rho-\rho'},$$

a sum of simple fractions whose denominators vanish at the pair sums. Under RH these lie on the line $\operatorname{Re} s=1$, and $\mathcal{C}(s)=\int_{\mathbb{R}} d\mu(\tau)/(s-1-i\tau)$ is the Cauchy transform of the finite complex measure $\mu=\sum_{(\rho,\rho')} a_\rho a_{\rho'}\delta_{\gamma+\gamma'}$ on $\mathbb{R}$, where δ_t is the unit mass at t; the mass of μ at τ is the atom $\mu\{\tau\}$ of (2). Along the normal $\eta\,\mathcal{C}(1+i\tau+\eta)$ tends to the atom $\mu\{\tau\}$, because each term $\eta/(\eta+i(\tau-\gamma-\gamma'))$ has modulus at most 1 and tends to 0 unless $\gamma+\gamma'=\tau$ (the limit is that of [6, p. 297]; Section 1.4). The factor η thus removes every pair sum other than τ, however close to τ. So no hypothesis that keeps distinct pair sums apart, such as (B), is needed, and what is left to prove is that the pairs with sum exactly τ do not cancel, which is where the earlier proofs use (A). Nonzero atoms at points accumulating at $1+iy$ obstruct a meromorphic continuation there, for the following reason. The other terms of the continuation extend meromorphically across the line (step (1) below), so a meromorphic extension of Φ_2 to a neighbourhood of a point $1+iy$ would give one of $\mathcal{C}$. Such an extension is analytic at every point $1+i\tau_0$ close enough to $1+iy$ and different from it, so η times its value at $1+i\tau_0+\eta$ tends to 0. This contradicts a nonzero atom $\mu\{\tau_0\}$, and so nonzero atoms at points τ_0 arbitrarily close to y exclude the extension (Lemma 6.2).

Off the critical line both of these facts fail: the pair sums need not lie on the line $\operatorname{Re} s=1$, and the bound on the terms, which needs $\beta+\beta'\le 1$, can fail. A zero with $\beta>\frac12$ pairs with its conjugate to give the real pair sum 2β inside $\operatorname{Re} s>1$, and zeros off the line can give pair sums with real part $\beta+\beta'\neq 1$. For such a pair the term becomes $\eta/(\eta+1-\beta-\beta'+i(\tau-\gamma-\gamma'))$, whose denominator vanishes for $\eta=\beta+\beta'-1$ and $\tau=\gamma+\gamma'$ when $\beta+\beta'>1$. The proof under H_∞ has five steps.

(1) An exact decomposition with no hypothesis (Section 3). In $h^2=Z^2-2c_0Z+\big((c_0+V)^2-2ZV\big)$ the term Z^2 gives $\mathcal{C}$, and the cross term $-2c_0Z$ has Mellin transform $-2c_0L(s)$ over $(0,1]$, where $L(s)=\sum_\rho a_\rho/(s-\rho)$ is meromorphic on $\mathbb{C}$. Moving it into the meromorphic part gives $\Phi_2=M_0+(\mathcal{C}+\mathcal{A})/\Gamma$, with M_0 meromorphic on $\mathbb{C}$ and $\mathcal{A}$ holomorphic on $\operatorname{Re} s>\frac12$, because what is left under the integral over $(0,1]$ is $O(x^{-1/2})$. The other terms extend meromorphically across the line, so whatever obstructs continuation of Φ_2 across it is in $\mathcal{C}$.

(2) Continuation (Section 4). Under H_∞ high zeros, those of large $|\gamma|$, have real parts near $\frac12$, so pairs of high zeros have sums near the line $\operatorname{Re} s=1$, never deep inside $\operatorname{Re} s>1$. A pair of one low and one much higher zero has a sum of large imaginary part, and the pairs of two low zeros are finitely many. So a compact subset of $\operatorname{Re} s>1$ contains only finitely many pair sums, and

the remaining terms converge normally on it: $\mathcal{C}$, and with it Φ_2, is meromorphic on $\operatorname{Re} s > 1$. The residue of $\mathcal{C}$ at a point s_0 is $\nu(s_0)$, the sum of $a_\rho a_{\rho'}$ over all representations $s_0 = \rho + \rho'$.

(3) Non-cancellation (Section 5). A candidate s_0 that is neither 2 nor one of the points $1+\rho$, for instance any candidate with $1 < \operatorname{Re} s_0 < 2-B$, is a pole when $\nu(s_0) \neq 0$ (Proposition 4.2(c)), and $\nu(s_0)$ may have many terms. Several pairs may land on the same point, and their coefficients might cancel; the earlier complete proofs of the boundary exclude this by a hypothesis on the ordinates (Section 9). The new step uses a freedom in the criterion: a natural boundary needs non-cancellation only at points that accumulate at every point of the line, and those points may be chosen. Earlier work also chose points, $2\rho_1$ in [10, Theorem 2(3)] and $1+2i\gamma_1$ in [6], but not for a boundary; what is new here is the choice of points all of whose representations are high, together with the phase cone, on the critical line and for unequal real parts. In each bounded range of imaginary parts only finitely many points have a representation in which one of the two zeros lies below a given height (Lemma 6.1). We discard these points. At every point that remains any number of representations may coincide; no coincidence has to be ruled out, and the following estimate applies to all the representations together. If every representation of a point with imaginary part τ uses two high zeros, that is, zeros with $|\gamma|$ beyond a threshold at least as large as $|\tau|$, each is a pair of opposite signs, because two ordinates of the same sign, both larger in modulus than $|\tau|$, cannot add up to τ. Up to order such a pair is $\rho = b + i(u+\tau)$, $\rho' = c - iu$ with u large, and Stirling's formula gives

$$\Gamma\big(b + i(u+\tau)\big)\Gamma(c-iu) = 2\pi e^{-\pi u - \pi\tau/2} u^{b+c-1} e^{i(\tau \log u + \pi(b-c)/2)}\big(1 + O(1/u)\big).$$

The modulus decays like $e^{-\pi u}$, so the lowest representation, at height u_0, dominates every representation more than $\sqrt{u_0}$ above it, and all of them together, because the total multiplicity of the representations in each unit interval of heights is at most a constant times the square of the logarithm of the height (Lemma 5.5). Those within $\sqrt{u_0}$ of it have phases that spread, in the limit, by at most $\pi(B - \frac{1}{2})$ about the common direction $\tau \log u_0$; the proof works with a wider cone, of half-angle $\pi B/2 < \pi/2$, which absorbs the drift and the error terms (Lemma 5.2). All of them lie in the open half-plane centred on the common direction, so none can cancel the lowest. The cone is centred on the direction $\tau \log u_0$, not on the argument of the lowest weight, which may lie at the edge of the sector. On the critical line, where $b = c = \frac{1}{2}$, the phase spread shrinks to the slow drift $\tau \log(u/u_0)$; this is the case that RH needs.

(4) Two cases (Section 6). If infinitely many zeros lie off the line, there are zeros ρ_n with $\beta_n > \frac{1}{2}$ and $|\gamma_n| \to \infty$. For every real y, Littlewood's gap theorem supplies zeros ρ'_n with $\gamma'_n - \gamma_n \to y$, and the symmetry $\rho \mapsto 1 - \bar\rho$ lets us take $\beta'_n \geq \frac{1}{2}$, and H_∞ makes both β_n and β'_n tend to $\frac{1}{2}$. The pair sums $\bar\rho_n + \rho'_n$, with real part $\beta_n + \beta'_n > 1$, then tend to $1 + iy$ from inside $\operatorname{Re} s > 1$, and eventually all their representations are high; by step (3) they are poles, and poles accumulating at $1 + iy$ forbid a meromorphic extension there. If only finitely many zeros lie off the line, the pairs that involve one of them form a function meromorphic across the line, and what is left is the pair series $\mathcal{C}_{\mathrm{c}}$ over the zeros on the line, for which the argument under RH runs: radial limits, nonzero atoms dense by Littlewood's theorem and step (3) with $b = c = \frac{1}{2}$, and blow-up along the normal. Each case needs its own argument. In the first the termwise bound behind the radial limit can fail, for pairs with $\beta + \beta' > 1$, whose denominators vanish inside $\operatorname{Re} s > 1$, and the proof uses the accumulating interior poles instead; in the second interior poles cannot be used, because the pair sums inside $\operatorname{Re} s > 1$ then keep a positive distance from the line: their real parts take only finitely many values.

(5) Sums of r prime powers (Section 7). Expanding $F^r = (1/x+h)^r$, keeping exact the finitely many zeros with $\beta > \frac{1}{2} + (1-B)/6$ and bounding the rest, gives $\Phi_r(s) = c_r(s)\Phi_2(s-r+2)$ plus a function meromorphic across the line $\operatorname{Re} s = r-1$, with $c_r(s) = \binom{r}{2}\prod_{j=1}^{r-2}(s-j)^{-1}$. Since

c_r has no zeros, an extension of Φ_r across a point of $\operatorname{Re} s = r-1$ would extend Φ_2 across the corresponding point of $\operatorname{Re} s = 1$.

Under RH, $\mathcal{C}$ has no poles in $\operatorname{Re} s > 1$, the case of infinitely many zeros off the line does not occur, and the poles and radial limits come out explicitly (Section 8.1). Theorem C(i) is steps (1) and (2) at the single point $s_0 = 2B$: every term of $\nu(2B)$ is positive, so no phase estimate is needed, and M_0 is analytic there: if RH fails then $2B \in (1,2)$, so $2B \neq 2$, and $2B$ is not a point $1+\rho$ because ζ has no real zero in $(0,1)$ (Section 8.2).

Section 2 collects the classical inputs. Sections 3–6 prove Theorem A(i) and (ii), Section 7 proves Theorem A(iii), and Section 8 proves Corollary B and Theorem C and adds remarks not used in the proofs. Section 9 reviews the earlier work, and Section 10 describes the formal verification.

1.4. **What is earlier and what is new.** Earlier:

- the continuation of Φ_2 and of Φ_r under RH, with residues [18, Theorems 2.1 and 6.1]; the continuation of Φ_2 alone, with no residues, had been announced in [17, Theorem 3]; with no hypothesis, Φ_2 continues to $\operatorname{Re} s > 2B$ [4, Proposition 3];
- the hypothesis H_∞ (their $\sigma_0 = \frac{1}{2}$), and the continuation to $\operatorname{Re} s > 1$ and the boundary under it together with their Conjecture 2, stated in [10, Theorem 2(4)] (in our reading, Section 9); the proof of the continuation is left there as parallel to [18, Theorem 2.1] [10, p. 7];
- Φ_2 as a sum of simple poles at the pair sums plus a remainder: [18, (3.9) and (4.2)], [10, Lemma 4] and, with no hypothesis, [13, §5, (28)], which with the argument of Section 4 also gives Theorem A(i) (compared in Section 9);
- the smoothing F, its square and its explicit formula with the constant $-\log 2\pi$, used at $x = 1/N$ and without the Mellin transform in [11, §4, (42)–(43)];
- the termwise radial limit by dominated convergence [6, p. 297], and the radial coefficient for a single representation [18, (4.11)];
- the comparison of the lowest pair with all much higher pairs [10, pp. 8–9], [6, pp. 297–298];
- the density of pair sums from Littlewood's gap theorem [38], [47, Theorem 9.12, p. 224], as in [10, Lemma 3], [46, p. 488] and, for single ordinates, [48, Remark 2]; the blow-up criterion [18, Lemma 4.2], [10, proof of Theorem 2(2)];
- the division by whether infinitely or finitely many zeros lie off the line, with zeros off the line making zeros or poles accumulate at a boundary [8, Theorem 1 and proof of Theorem 2], which Section 6 follows;
- a criterion for RH through the poles of Φ_2 at the points $1+\rho$ [4, Proposition 3, case $q=1$]; the point $2B$ among the candidate poles $\rho+\rho'$ of [41, §5].

New here:

- the natural boundary of Φ_2 and of Φ_r under RH with no hypothesis on the ordinates; under H_∞, a complete proof of the continuation, and the boundary with no hypothesis on the ordinates;
- the phase cone (Lemma 5.2), on the critical line and for unequal real parts, and the choice of points all of whose representations are high (Section 5);
- the decomposition (13) in our normalization, proved directly, with a remainder holomorphic on $\operatorname{Re} s > \frac{1}{2}$ under no hypothesis (Section 3);
- the subtraction of the main term M in the radial limit, and with it the value at $\tau = 0$ (Corollary B(iii), Remark 8.3.1);
- the reduction of Φ_r to Φ_2 under H_∞, with neither RH nor the circle-method kernel bound of [10] (Section 7, Remark 8.3.2);

- in Theorem C, that $2B$ is a pole, with its residue: the pairs $(\rho, \bar{\rho})$ landing on it do not cancel.

2. Notation and classical inputs

The proof rests on three classical theorems on the zeros of ζ, stated below as (Z1)–(Z3), on the Euler product, the non-vanishing of ζ on $\operatorname{Re} s = 1$ and the functional equation [47, (2.1.1), p. 13], which place the non-trivial zeros in $0 < \operatorname{Re} s < 1$, and on four standard facts about Γ, (G1)–(G4). Two further facts about ζ, (Z4) and (Z5), are proved here. Of the three theorems, the local count (Z1) makes the sums over the zeros converge (Lemma 3.4) and bounds the number of representations in each unit interval of heights (Lemma 5.5); the local expansion (Z3), with (Z1), gives the explicit formula (Lemma 3.5); and Littlewood's gap theorem (Z2) gives the pair sums near every point of the line in Section 6. No explicit constant is tracked: C denotes a positive constant whose value may change from one occurrence to the next, and a constant referred to later carries a subscript.

Throughout, $s = \sigma + it$, w and z denote complex variables, $\mathbb{H} = \{s : \operatorname{Re} s > 1\}$ (a right half-plane), and $D(z_0, r) = \{|s - z_0| < r\}$. Let

$$\psi(z) = -\frac{\zeta'(z)}{\zeta(z)}, \qquad N(t) = \sum_{\rho \in \mathcal{Z},\ 0<\gamma\le t} m(\rho) \quad (t \in \mathbb{R}). \tag{3}$$

For $\operatorname{Re} z > 1$, $\psi(z) = \sum_{n\ge1} \Lambda(n) n^{-z}$, so $|\psi(z)| \le \psi(\operatorname{Re} z)$ there; ψ is meromorphic on $\mathbb{C}$, with simple poles exactly at $z = 1$, with residue 1, and at the zeros of ζ, with residue $-m(\rho)$ at ρ. (This ψ is neither Chebyshev's function nor the digamma function, which carry the same letter in several of the papers cited.) For $c \in \mathbb{R}$ and f integrable on the line $\operatorname{Re} w = c$,

$$\int_{(c)} f(w)\, dw = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(w)\, dw = \frac{1}{2\pi} \int_{\mathbb{R}} f(c + iy)\, dy. \tag{4}$$

The function ζ has no zeros on $\operatorname{Re} s \ge 1$ (none for $\sigma > 1$, by the Euler product; none on $\sigma = 1$, by Hadamard [26] and de la Vallée Poussin [47, §3.1, p. 45]), and none with $\operatorname{Re} s \le 0$ except $-2, -4, \dots$ [47, §2.12, p. 30] (on $\operatorname{Re} s = 0$ because $\zeta(0) = -\frac{1}{2}$ [42, 25.6.1] and the functional equation would carry a zero there to one on $\operatorname{Re} s = 1$). So $0 < \beta < 1$ for every $\rho \in \mathcal{Z}$. Only finitely many zeros have $|\gamma| \le T$, for any T, so N is finite. Since $\zeta(\bar{z}) = \overline{\zeta(z)}$, the map $\rho \mapsto \bar{\rho}$ preserves $\mathcal{Z}$ and multiplicities. The functional equation $\zeta(s) = 2^s \pi^{s-1} \sin(\frac{1}{2}\pi s) \Gamma(1-s) \zeta(1-s)$ [47, (2.1.1)], whose factors other than $\zeta(1-s)$ have neither zeros nor poles in $0 < \operatorname{Re} s < 1$, shows the same for $\rho \mapsto 1 - \rho$, hence for $\rho \mapsto 1 - \bar{\rho}$, which keeps the ordinate and replaces β by $1 - \beta$. We call these the symmetries of the zeros.

2.1. Classical inputs on the zeros.

The citations are to Titchmarsh [47]. A sum over zeros runs over the distinct zeros $\rho \in \mathcal{Z}$; a count "with multiplicity" weights each ρ by $m(\rho)$, as $N(t)$ does.

- (Z1) There is a constant C_1 such that $N(t+1) - N(t) \le C_1 \log(t+3)$ for all $t \ge 0$. This is [47, Theorem 9.2, p. 211], $N(T+1) - N(T) = O(\log T)$ for large T; the form for all $t \ge 0$ follows by enlarging the constant, since N is bounded on bounded sets.
- (Z2) (Littlewood's gap theorem.) There is a constant A such that for all sufficiently large t some $\rho \in \mathcal{Z}$ has $|\gamma - t| < A/\log\log\log t$. This is [38], [47, Theorem 9.12, p. 224]; it is unconditional.

(Z3) (The local expansion of ψ.) There is a constant C such that for every z with $-1 \le \operatorname{Re} z \le 2$, $|\operatorname{Im} z| \ge 2$ and $\zeta(z) \ne 0$,

$$\Big|\psi(z) + \sum_{\rho\in\mathcal{Z},\ |\gamma-\operatorname{Im} z|\le 1} \frac{m(\rho)}{z-\rho}\Big| \le C \log|\operatorname{Im} z|.$$

This is [47, Theorem 9.6(A), p. 217], stated for ζ'/ζ: $\zeta'(s)/\zeta(s) = \sum_{|t-\gamma|\le 1} 1/(s-\rho) + O(\log t)$ uniformly for $-1 \le \sigma \le 2$ and large t. Since $\psi = -\zeta'/\zeta$, the sign of the sum changes, and we write it with the multiplicities $m(\rho)$, as ζ'/ζ has residue $m(\rho)$ at ρ. Titchmarsh states it for large t; the extension to all $|\operatorname{Im} z| \ge 2$ is elementary: on the compact set $-1 \le \sigma \le 2$, $2 \le t \le t_0$ the left side is bounded, being continuous away from the zeros and the cutoffs $|t-\gamma| = 1$ and bounded near each of them (near a zero ρ_0, $\psi(z) + m(\rho_0)/(z-\rho_0)$ is analytic), and negative t follows from positive t by conjugation.

(Z4) The ordinates are unbounded above: by (Z2) at a $t > T + A$ large enough that $\log\log\log t > 1$, some ordinate satisfies $\gamma > t - A > T$.

(Z5) $\zeta(\sigma) \ne 0$ for real $0 < \sigma < 1$. The series $\sum_{n\ge1}(-1)^{n-1}n^{-s}$ converges for $\operatorname{Re} s > 0$, by Abel summation, to an analytic function. On $\operatorname{Re} s > 1$ its sum is $(1-2^{1-s})\zeta(s)$, and by the identity theorem this identity extends to all $s \ne 1$ with $\operatorname{Re} s > 0$. For real $0 < \sigma < 1$ the series is alternating with decreasing terms, so its sum is at least $1 - 2^{-\sigma} > 0$, while $1 - 2^{1-\sigma} < 0$; hence $\zeta(\sigma) \ne 0$.

2.2. **The hypothesis.** Recall that $B = \sup_{\rho\in\mathcal{Z}} \beta$. The lemma records the two consequences of H_∞ that the proofs use: the zeros lie in a strip strictly inside the critical strip, and zeros of large ordinate lie near the critical line. Part (a) supplies the bound $B < 1$, on which the strip of Proposition 4.2(c), the margin of the phase cone (Lemma 5.2) and the cutoff of Section 7 depend. Part (b) is used twice, in Lemma 4.1 and in Proposition 6.3. Besides the lemma, the hypothesis is used directly only where a set of zeros has to be finite: the set $\mathcal{E}_*$ of Section 7, and the zeros with $\beta = B$ in the proof of Theorem C.

Lemma 2.1. *Assume* H_∞.

(a) $\frac12 \le B < 1$, *and* $1 - B \le \beta \le B$ *for every* $\rho \in \mathcal{Z}$.

(b) *For every* $\delta > 0$ *there is* T *such that* $|\beta - \frac12| \le \delta$ *for every* $\rho \in \mathcal{Z}$ *with* $|\gamma| > T$.

Proof. (a) By H_∞ with $\delta = \frac14$, only finitely many zeros have $\beta > \frac34$, and each has $\beta < 1$; so B is at most the largest of $\frac34$ and these finitely many real parts, and $B < 1$. $\mathcal{Z}$ is not empty by (Z4), and of ρ and $1-\bar\rho$ one has real part at least $\frac12$, so $B \ge \frac12$. For every ρ, $1-\beta$ is the real part of the zero $1-\bar\rho$, so $1 - \beta \le B$.

(b) The zeros with $\beta > \frac12 + \delta$ are finitely many; take T larger than the modulus of their ordinates. If $|\gamma| > T$ then $\beta \le \frac12 + \delta$, and the same applied to $1-\bar\rho$, which has the same ordinate, gives $1 - \beta \le \frac12 + \delta$. □

2.3. **The Gamma function.**

(G1) Γ has no zeros and $1/\Gamma$ is entire [42, §5.2(i)]; $\Gamma(z+1) = z\Gamma(z)$ [42, 5.5.1]; $\Gamma(1) = \Gamma(2) = 1$ [42, 5.4.1]; $\Gamma(\bar z) = \overline{\Gamma(z)}$.

(G2) For real x, $|\Gamma(\frac12 + ix)|^2 = \pi/\cosh\pi x$, by the reflection formula [42, 5.5.3] at $z = \frac12 + ix$, where $1 - z = \bar z$ and $\sin\pi z = \cosh\pi x$.

(G3) For $0 < a \le a'$ and every integer $n \ge 0$ there is C such that $|\Gamma(z)|(1 + |\operatorname{Im} z|)^n \le C$ for $a \le \operatorname{Re} z \le a'$, by Stirling's formula [42, 5.11.9].

(G4) (Stirling's formula on vertical strips.) Let Log be the principal logarithm and $\mathrm{St}(z) = (z-\frac{1}{2})\operatorname{Log} z - z + \frac{1}{2}\log 2\pi$. For real $x_1 \le x_2$ there are C and T_0 such that

$$\left|\Gamma(z) - e^{\mathrm{St}(z)}\right| \le \frac{C}{|\operatorname{Im} z|}\left|e^{\mathrm{St}(z)}\right| \qquad (x_1 \le \operatorname{Re} z \le x_2,\ |\operatorname{Im} z| \ge T_0).$$

It follows from the multiplicative form [42, 5.11.3]: $\Gamma(z) = e^{-z}z^z(2\pi/z)^{1/2}\,\Gamma^*(z)$ with $\Gamma^*(z) = 1 + O(1/z)$ uniformly in $|\arg z| \le \pi - \delta$ for each $\delta > 0$. With principal powers [42, §4.2(iv)] the prefactor $e^{-z}z^z(2\pi/z)^{1/2}$ equals $e^{\mathrm{St}(z)}$, the only logarithm being the principal $\operatorname{Log} z$ inside St. The part of the strip with $|\operatorname{Im} z| \ge T_0$ lies in such a sector, and there $|z| \ge |\operatorname{Im} z|$, so $\Gamma(z) = e^{\mathrm{St}(z)}(1+r)$ with $|r| \le C/|\operatorname{Im} z|$, which is the multiplicative inequality (G4). Unlike (G3), (G4) gives the argument of Γ and not only its size; Section 5 needs the argument.

3. The decomposition

This section proves an identity, valid with no hypothesis on the location of the zeros, in which one function, the pair series $\mathcal{C}$, carries the possible obstruction to meromorphic continuation of Φ_2 across the line $\operatorname{Re} s = 1$; the other terms extend meromorphically across it (Proposition 3.7). This is step (1) of Section 1.3. The ingredients, one to a subsection, are the Mellin transform of F^2, a count of the zeros that makes $\sum |a_\rho|$ converge, and the explicit formula for h; the last subsection combines them. Nothing in this section uses H_∞ or RH.

3.1. The Mellin transform.

For $x > 0$ put

$$F(x) = \sum_{n\ge 1}\Lambda(n)e^{-nx}, \qquad h(x) = F(x) - \frac{1}{x}, \qquad k_x(w) = x^{-w}\Gamma(w)\psi(w). \tag{5}$$

F is non-negative and continuous on $(0,\infty)$, and $F(x) \le Ce^{-x}$ for $x \ge 1$, since $\Lambda(n) \le n$. Integrating the series termwise, with $\int_0^\infty e^{-nx}x^{z-1}\,dx = \Gamma(z)n^{-z}$ (Tonelli, then Fubini),

$$\begin{aligned} \int_0^\infty F(x)x^{z-1}\,dx &= \Gamma(z)\psi(z) \quad (\operatorname{Re} z > 1),\\ \int_0^\infty F(x)^2x^{s-1}\,dx &= \Gamma(s)\Phi_2(s) \quad (\operatorname{Re} s > 2), \end{aligned} \tag{6}$$

both absolutely convergent; for the second, $F^2 = \sum_{m,n}\Lambda(m)\Lambda(n)e^{-(m+n)x}$, and the series (1) converges absolutely for $\operatorname{Re} s > 2$. Mellin inversion recovers F from the first: for $c > 1$ and $x > 0$, k_x is integrable on the line $\operatorname{Re} w = c$, by (G3) and $|\psi(c+iy)| \le \psi(c)$, and

$$F(x) = \int_{(c)} k_x(w)\,dw. \tag{7}$$

(Apply the inversion theorem for an L^1 function with L^1 Fourier transform [2, 11.76] to $\varphi(v) = F(e^{-v})e^{-cv}$, whose transform $\int_{\mathbb{R}}\varphi(v)e^{-iyv}\,dv = \Gamma(c+iy)\psi(c+iy)$ is the first integral in (6) after the substitution $x = e^{-v}$; the identity holds almost everywhere, hence everywhere, both sides being continuous in v.) Moving the line of integration in (7) to the left across $w = 1$, where ψ has a simple pole with residue 1, produces the term $1/x$; $h = F - 1/x$ is what remains, and Lemma 3.5 will express it through the zeros. Now split the two Mellin integrals of (6) at $x = 1$. Their tails over $[1,\infty)$, the functions e_1 and E_2 below, are entire in the exponent, because $F \le Ce^{-x}$ there, and the integrals over $(0,1]$ decide the behaviour at the line. On $(0,1]$ expand

$F^2 = x^{-2} + 2x^{-1}h + h^2$: the first term integrates to $1/(s-2)$, the cross term is expressed through the transform of F, and the integral R of h^2 is left for the explicit formula (Lemma 3.1). Write

$$e_1(z) = \int_1^\infty F(x)x^{z-1}\,dx, \qquad E_2(s) = \int_1^\infty F(x)^2 x^{s-1}\,dx, \qquad R(s) = \int_0^1 h(x)^2 x^{s-1}\,dx, \tag{8}$$

$$M(s) = \frac{2\psi(s-1)}{s-1} - \frac{1}{\Gamma(s)(s-2)}. \tag{9}$$

Lemma 3.1. *For* $\operatorname{Re} s > 2$ *the integral* $R(s)$ *converges absolutely and*

$$\Gamma(s)\Phi_2(s) = \Gamma(s)M(s) + E_2(s) - 2e_1(s-1) + R(s).$$

Proof. For $\operatorname{Re} z > 1$, $F(x)x^{z-1}$ and x^{z-2} are integrable on $(0,1]$, hence so is $h(x)x^{z-1}$, and (6) gives

$$\Gamma(z)\psi(z) = \frac{1}{z-1} + \int_0^1 h(x)x^{z-1}\,dx + e_1(z). \tag{10}$$

For $\operatorname{Re} s > 2$, on $(0,1]$, $F^2 = x^{-2} + 2x^{-1}h + h^2$; the functions $F(x)^2x^{s-1}$, x^{s-3} and $h(x)x^{s-2}$ are integrable there (the last by the case $z = s-1$ above), hence so is $h(x)^2x^{s-1}$, and (6) gives

$$\Gamma(s)\Phi_2(s) = \frac{1}{s-2} + 2\int_0^1 h(x)x^{s-2}\,dx + R(s) + E_2(s).$$

Replacing the middle term by (10) at $z = s-1$, $\Gamma(s)\Phi_2(s) = 2\Gamma(s-1)\psi(s-1) - \frac{1}{s-2} + E_2(s) - 2e_1(s-1) + R(s)$, and $2\Gamma(s-1)\psi(s-1) - \frac{1}{s-2} = \Gamma(s)M(s)$ since $\Gamma(s-1)/\Gamma(s) = 1/(s-1)$. □

In this identity ΓM is meromorphic on $\mathbb{C}$, and E_2 and e_1 are entire. So the continuation of Φ_2 beyond $\operatorname{Re} s > 2$ is that of R, the Mellin transform of h^2 over $(0,1]$, and the rest of the section computes R from an explicit formula for h.

3.2. **The zeros.** The explicit formula expresses h through the sum $Z(x) = \sum_\rho a_\rho x^{-\rho}$ of Section 1.3, and the pair series comes from its square, so the series $\sum |a_\rho|$ has to converge, and with it the double series $\sum |a_\rho a_{\rho'}|$. The three lemmas of this subsection give that: a count of the zeros near a given height, from (Z1); the decay of Γ on a vertical strip, from (G3), stated on the strip $-\frac{1}{2} \le \operatorname{Re} w \le 2$ because the contour of Lemma 3.5 reaches the line $\operatorname{Re} w = -\frac{1}{2}$; and their combination, Lemma 3.4. The count is used again in the proof of the explicit formula (Lemma 3.5) and for the number of representations in each unit interval of heights (Lemma 5.5).

Lemma 3.2 (Counting)**.**

(a) *There is a constant* C_2 *such that for every real* t *the set* $\mathcal{W}_t = \{\rho \in \mathcal{Z} : |\gamma - t| \le 1\}$ *is finite and* $\sum_{\rho\in\mathcal{W}_t} m(\rho) \le C_2 \log(|t|+3)$.
(b) *For every* $\rho \in \mathcal{Z}$, $m(\rho) \le C_2 \log(|\gamma|+3)$.
(c) $\sum_{\rho\in\mathcal{Z}} (1+|\gamma|)^{-2} < \infty$.

Proof. (a) $\mathcal{W}_t$ is finite since the zeros up to any height are. By conjugate symmetry, $\sum_{\mathcal{W}_t} m(\rho) \le 2\sum' m(\rho) + n_0$, where $\sum'$ runs over the zeros with $\gamma > 0$ and $|\gamma - |t|| \le 1$, and n_0 is the total multiplicity of the zeros with $\gamma = 0$, a finite constant ($n_0 = 0$ by (Z5); the argument does not need this). The zeros in $\sum'$ lie in at most three intervals $(v, v+1]$ with integer $0 \le v \le |t|+1$, so by (Z1) $\sum' \le 3C_1 \log(|t|+4) \le 6C_1 \log(|t|+3)$; and $n_0 \le \frac{n_0}{\log 3}\log(|t|+3)$.

(b) $\rho \in \mathcal{W}_\gamma$; apply (a).

(c) The zeros with $\lfloor\gamma\rfloor = n \in \mathbb{Z}$ lie in $\mathcal{W}_{n+1/2}$, so they number at most $C_2\log(|n|+4)$, and for them $1+|\gamma| \ge (1+|n|)/2$. Hence $\sum_\rho (1+|\gamma|)^{-2} \le \sum_{n\in\mathbb{Z}} 4C_2 \log(|n|+4)(1+|n|)^{-2} < \infty$. □

Lemma 3.3 (Gamma on the strip)**.** *For every integer $n \geq 0$ there is C such that, for $|\operatorname{Im} w| \geq 1$ and $-\frac{1}{2} \leq \operatorname{Re} w \leq 2$, $|\Gamma(w)| \leq C(1 + |\operatorname{Im} w|)^{-n}$.*

Proof. $\Gamma(w) = \Gamma(w+1)/w$ by (G1), with $\frac{1}{2} \leq \operatorname{Re}(w+1) \leq 3$ and $|1/w| \leq 1/|\operatorname{Im} w| \leq 1$; apply (G3). □

Lemma 3.4.

(a) *$S := \sum_{\rho \in \mathcal{Z}} |a_\rho| < \infty$, and the double series $\sum_{(\rho,\rho') \in \mathcal{Z}\times\mathcal{Z}} |a_\rho a_{\rho'}|$ converges, with sum S^2.*

(b) *For every $x > 0$ the series $Z(x) = \sum_\rho a_\rho x^{-\rho}$ converges absolutely, and $|Z(x)| \leq S x^{-1}$ for $0 < x \leq 1$.*

Proof. (a) Only finitely many zeros have $|\gamma| < 1$. For the others $|\operatorname{Im} \rho| = |\gamma| \geq 1$ and $0 < \beta < 1$, so Lemma 3.3 with $n = 3$ and Lemma 3.2(b) give $|a_\rho| \leq C_2 \log(|\gamma| + 3) \cdot C(1 + |\gamma|)^{-3} \leq 2C_2C(1 + |\gamma|)^{-2}$; this is summable by Lemma 3.2(c). The double series is the product of two copies of $\sum |a_\rho|$.

(b) $|x^{-\rho}| = x^{-\beta}$ lies between $\min(1, x^{-1})$ and $\max(1, x^{-1})$, as $0 < \beta < 1$; for $0 < x \leq 1$ it is at most x^{-1}. □

3.3. **The explicit formula.** Hardy and Littlewood [27, §2.2, (2.214)–(2.2141), p. 136] prove, for $\operatorname{Re} y > 0$ and with no hypothesis, $F(y) = 1/y - Z(y) + \Omega_1(y) + y^2 \log(1/y)\,\Omega_2(y)$ with Ω_1 and Ω_2 entire (their Φ, Φ_1, Φ_2, renamed here). Languasco and Zaccagnini [35, Lemma 4.1 and its (4.4), p. 1951], following Linnik [37, §4, p. 4], give the version with the contour stopped at $\operatorname{Re} w = -\frac{1}{2}$. The bound printed for the remainder in their Lemma 4.1 omits the constant term $-(\zeta'/\zeta)(0)$ that their (4.4) contains; Languasco [33, Lemma 5.1 and §6] corrects it, and with it the error term of [35, Theorem 1], from $O_k(N^{1/2})$ to $O_k(N)$ [33, Theorem 2.2]. In Lemma 3.5 that constant is c_0. Neither they nor Hardy and Littlewood estimate the horizontal sides of the contour, so we prove the form we need.

That form is $h = -Z + c_0 + V$ with $V(x) = O(x^{1/2})$, and the exponent matters. With no hypothesis on the zeros, the bound $Z(x) = O(x^{-1})$ on $(0, 1]$ is enough: the product ZV, which appears in the remainder of Proposition 3.7, is then $O(x^{-1/2})$: small enough for a Mellin transform over $(0, 1]$ that is holomorphic on $\operatorname{Re} s > \frac{1}{2}$. The proof moves the line of integration in (7) from $\operatorname{Re} w = 2$ to $\operatorname{Re} w = -\frac{1}{2}$. The pole of ψ at 1 gives $1/x$, the zeros give $-Z(x)$, the pole of Γ at 0 gives c_0, and V is the integral on the new line. The horizontal sides are taken at heights T_n that keep a distance from the ordinates, so that (Z3) bounds ψ on them.

Lemma 3.5 (The explicit formula)**.** *For every $x > 0$, k_x is integrable on the line $\operatorname{Re} w = -\frac{1}{2}$, and*

$$h(x) = -Z(x) + c_0 + V(x), \qquad c_0 = \psi(0), \qquad V(x) = \int_{(-1/2)} k_x(w)\,dw, \tag{11}$$

with $|V(x)| \leq K_0 x^{1/2}$ for a constant K_0.

Proof. The line $\operatorname{Re} w = -\frac{1}{2}$. No pole of k_x lies on it: the poles of ψ are 1 and the zeros of ζ, none of which has real part $-\frac{1}{2}$, and the poles of Γ are $0, -1, \ldots$. For $|t| \geq 2$, (Z3) at $z = -\frac{1}{2} + it$ gives $\psi(z) = -\sum_{|\gamma - t| \leq 1} m(\rho)/(z - \rho) + O(\log|t|)$; each term $m(\rho)/(z-\rho)$ has modulus at most $2m(\rho)$, because $|z - \rho| \geq \beta + \frac{1}{2} > \frac{1}{2}$, and $\sum_{|\gamma - t| \leq 1} m(\rho) \leq C_2 \log(|t| + 3)$ (Lemma 3.2(a)). On $|t| \leq 2$, ψ is continuous. Hence $|\psi(-\frac{1}{2} + it)| \leq C \log(|t| + 3)$ for all real t. By Lemma 3.3 for $|t| \geq 1$ and continuity for $|t| \leq 1$, $|\Gamma(-\frac{1}{2} + it)| \leq C(1 + |t|)^{-2}$ for all real t. With $|x^{-w}| = x^{1/2}$, k_x is integrable on the line and $|V(x)| \leq K_0 x^{1/2}$ with $K_0 = \frac{1}{2\pi} \int_{\mathbb{R}} |\Gamma(-\frac{1}{2} + it)\psi(-\frac{1}{2} + it)|\,dt < \infty$.

Good heights. Let $n \geq 2$ be an integer. The ordinates in $[n-1, n+2]$ lie in $\mathcal{W}_{n-1/2} \cup \mathcal{W}_{n+1/2} \cup \mathcal{W}_{n+3/2}$, so their total multiplicity ω_n is at most $3C_2 \log(n + 5) \leq C \log n$ (Lemma 3.2(a)). Put

$\delta_n = 1/(4\omega_n+4)$. The set of $t \in [n, n+1]$ within δ_n of one of these ordinates has measure at most $2\delta_n\omega_n < \frac{1}{2}$; choose $T_n \in [n, n+1]$ outside it. Let $z = \sigma + iT_n$ with $-1 \le \sigma \le 2$. Every zero in the sum of (Z3) at z has $|\gamma - T_n| \le 1$, hence $\gamma \in [n-1, n+2]$ and $|z-\rho| \ge |T_n - \gamma| \ge \delta_n$; in particular $\zeta(z) \ne 0$, and the sum has modulus at most $\omega_n/\delta_n \le C\log^2 n$. Hence $|\psi(\sigma + iT_n)| \le C\log^2 n$ for $-1 \le \sigma \le 2$, and the same at $\sigma - iT_n$, since the zeros with $|\gamma + T_n| \le 1$ are the conjugates of those with $|\gamma - T_n| \le 1$ and (Z3) applies at $\operatorname{Im} z = -T_n \le -2$.

The rectangle. Let Q_n have vertices $2 \pm iT_n$ and $-\frac{1}{2} \pm iT_n$. On ∂Q_n, k_x has no pole: T_n is not an ordinate, $T_n \ge 2$, the poles of Γ are real, the left side was dealt with above, and on the right side $\operatorname{Re} w = 2$ neither ψ nor Γ has a pole. Inside Q_n its poles are $w = 1$, where ψ has residue 1, giving the residue $x^{-1}\Gamma(1) = 1/x$; $w = 0$, where Γ has a simple pole with residue 1 and ψ is analytic since $\zeta(0) = -\frac{1}{2} \ne 0$ [42, 25.6.1], giving $\psi(0)$; and $w = \rho \in \mathcal{Z}$ with $|\gamma| < T_n$, finitely many, where ψ has a simple pole with residue $-m(\rho)$, giving $-a_\rho x^{-\rho}$. No other zero of ζ and no other pole of Γ lies inside the rectangle Q_n. The residue theorem gives

$$\frac{1}{2\pi i}\Big(\int_{2-iT_n}^{2+iT_n} - \int_{-1/2-iT_n}^{-1/2+iT_n}\Big) k_x(w)\,dw + \mathcal{H}_n = \frac{1}{x} + \psi(0) - \sum_{|\gamma|<T_n} a_\rho x^{-\rho},$$

where $\mathcal{H}_n$ is the contribution of the two horizontal sides, each of length $\frac{5}{2}$. On them $|x^{-w}| \le x^{-2} + x^{1/2}$, $|\Gamma(w)| \le C(1+T_n)^{-2}$ by Lemma 3.3 and $|\psi(w)| \le C\log^2 n$, so $\mathcal{H}_n \to 0$ as $n \to \infty$.

The limit. As $n \to \infty$, the first integral tends to $F(x)$ (by (7) with $c = 2$), the second to $V(x)$ by integrability, and the finite sums, over the increasing finite sets $\{|\gamma| < T_n\}$ that exhaust $\mathcal{Z}$, to $Z(x)$ by absolute convergence (Lemma 3.4(b)). This proves (11). □

The contour is moved across the whole critical strip, so the proof crosses every zero, wherever it lies; that is why no hypothesis on the zeros enters. The value $c_0 = \psi(0) = -\zeta'(0)/\zeta(0) = -\log 2\pi$ [42, 25.6.1, 25.6.11] is not needed in the proofs.

3.4. **The regularized identity.** The pair series $\mathcal{C}$ and the single series L defined below are sums of simple fractions with summable coefficients. One lemma handles both: such a sum is meromorphic on a disc as soon as all but finitely many of the points where its denominators vanish keep a fixed distance from the disc, and its residue at the centre of the disc is the sum of the coefficients of the terms with a pole there. It is applied to L in Proposition 3.7(a) and to $\mathcal{C}$ in Proposition 4.2(a).

Lemma 3.6 (Cauchy sums)**.** *Let $(p_i)_{i\in\iota}$ be complex numbers and $(a_i)_{i\in\iota}$ complex coefficients, ι countable, with $\sum_i |a_i| < \infty$. Let $z_0 \in \mathbb{C}$, $r > 0$, $d > 0$, and let $\mathcal{E} \subseteq \iota$ be finite with $|w - p_i| \ge d$ for all $w \in D(z_0, r)$ and $i \notin \mathcal{E}$. Then $f(w) = \sum_i a_i/(w - p_i)$ converges absolutely at every $w \in D(z_0, r)$ other than the p_i with $i \in \mathcal{E}$, f is meromorphic on $D(z_0, r)$, and $(w - z_0)f(w) \to \sum_{p_i = z_0} a_i$ as $w \to z_0$.*

Proof. Split f into the finite sum over $\mathcal{E}$, a rational function with at most simple poles at the p_i, $i \in \mathcal{E}$, and the sum over $i \notin \mathcal{E}$, whose terms are holomorphic on $D(z_0, r)$ and bounded there by $|a_i|/d$; by the Weierstrass M-test the second sum converges uniformly to a holomorphic function. Every i with $p_i = z_0$ lies in $\mathcal{E}$, since $|z_0 - p_i| \ge d$ otherwise. As $w \to z_0$, $(w - z_0)a_i/(w - p_i)$ tends to a_i if $p_i = z_0$ and to 0 otherwise, and $(w - z_0)$ times the holomorphic part tends to 0. □

By the explicit formula, on $(0, 1]$ the square h^2 is the sum of three parts, Z^2, $-2c_0 Z$ and $(c_0 + V)^2 - 2ZV$. The definitions that follow name the Mellin transform over $(0, 1]$ of each: $\mathcal{C}$ for Z^2, L for Z, and the integral in $\mathcal{A}$ for the third part, to which the entire tails of Lemma 3.1

are added. Put

$$L(s)=\sum_{\rho\in\mathcal{Z}}\frac{a_\rho}{s-\rho},\qquad \mathcal{C}(s)=\sum_{(\rho,\rho')\in\mathcal{Z}\times\mathcal{Z}}\frac{a_\rho a_{\rho'}}{s-\rho-\rho'},\qquad M_0(s)=M(s)-\frac{2c_0L(s)}{\Gamma(s)},$$
$$\mathcal{A}(s)=E_2(s)-2e_1(s-1)+\int_0^1\Big(\big(c_0+V(x)\big)^2-2Z(x)V(x)\Big)x^{s-1}\,dx, \tag{12}$$

sums over ordered pairs. For $\operatorname{Re}s>2$ the series $\mathcal{C}(s)$ converges absolutely, since $|s-\rho-\rho'|\ge\operatorname{Re}s-2$, and uniformly on $\operatorname{Re}s\ge2+\varepsilon$; so $\mathcal{C}$ is holomorphic on $\operatorname{Re}s>2$. Section 4 continues it.

Proposition 3.7 (The regularized identity)**.**

(a) *L and M_0 are meromorphic on $\mathbb{C}$.*
(b) *$\mathcal{A}$ is holomorphic on $\operatorname{Re}s>\frac12$.*
(c) *For $\operatorname{Re}s>2$,*

$$\Phi_2(s)=M_0(s)+\frac{\mathcal{C}(s)+\mathcal{A}(s)}{\Gamma(s)}. \tag{13}$$

Proof. (a) Given z_0 and $r>0$, let $\mathcal{E}$ be the finite set of zeros with $|\gamma-\operatorname{Im}z_0|\le r+1$. For $w\in D(z_0,r)$ and $\rho\notin\mathcal{E}$, $|w-\rho|\ge|\operatorname{Im}w-\gamma|\ge1$. Lemma 3.6 with $d=1$ and Lemma 3.4(a) make L meromorphic on $D(z_0,r)$. M is meromorphic on $\mathbb{C}$ since ψ is and $1/\Gamma$ is entire, and so is M_0.

(b) E_2 and e_1 are entire. On $(0,1]$, $|c_0+V(x)|\le|c_0|+K_0$ and $|2Z(x)V(x)|\le2SK_0x^{-1/2}$ (Lemmas 3.4(b) and 3.5), so the integrand of $\mathcal{A}$ is bounded by $Cx^{\operatorname{Re}s-3/2}$ with $C=(|c_0|+K_0)^2+2SK_0$. This is integrable on $(0,1]$ locally uniformly on $\operatorname{Re}s>\frac12$, and $V=h+Z-c_0$ is continuous on $(0,\infty)$; so the integral is holomorphic there.

(c) Let $\sigma=\operatorname{Re}s>2$. On $(0,1]$, (11) gives

$$h^2=(-Z+c_0+V)^2=Z^2-2c_0Z+\big((c_0+V)^2-2ZV\big).$$

By Lemma 3.4 and the Cauchy product, $Z(x)^2=\sum_{(\rho,\rho')}a_\rho a_{\rho'}x^{-\rho-\rho'}$, and $|a_\rho a_{\rho'}x^{-\rho-\rho'}x^{s-1}|\le|a_\rho a_{\rho'}|x^{\sigma-3}$; since $\sum|a_\rho a_{\rho'}|\int_0^1x^{\sigma-3}\,dx=S^2/(\sigma-2)$, Fubini's theorem gives $\int_0^1Z^2x^{s-1}\,dx=\sum a_\rho a_{\rho'}/(s-\rho-\rho')=\mathcal{C}(s)$. In the same way $|a_\rho x^{-\rho}x^{s-1}|\le|a_\rho|x^{\sigma-2}$ gives $\int_0^1Zx^{s-1}\,dx=L(s)$. Hence $R(s)=\mathcal{C}(s)-2c_0L(s)+\mathcal{A}(s)-E_2(s)+2e_1(s-1)$, and Lemma 3.1 becomes $\Gamma(s)\Phi_2(s)=\Gamma(s)M(s)-2c_0L(s)+\mathcal{C}(s)+\mathcal{A}(s)$. Divide by $\Gamma(s)\ne0$. □

The subtraction of $2c_0L$ is what lets the remainder $\mathcal{A}$ be holomorphic across the line with no hypothesis on the zeros; the identity itself, on $\operatorname{Re}s>2$, needs none either way. Left in the remainder, the term $-2c_0Z(x)x^{s-1}$ has on $(0,1]$ the bound $2|c_0|S\,x^{\operatorname{Re}s-1-B}$, from $|x^{-\rho}|=x^{-\beta}\le x^{-B}$, and this bound gives convergence for $\operatorname{Re}s>B$, a half-plane that contains the line $\operatorname{Re}s=1$ only if $B<1$. Taken out as $-2c_0L$, which is meromorphic on $\mathbb{C}$, the term needs no bound. Identities of the shape of (13) are in [18], [10] and [13]; Section 9 compares them with ours.

The identity (13) is proved for $\operatorname{Re}s>2$, but every term on its right side except $\mathcal{C}$ is already defined across the line $\operatorname{Re}s=1$: M_0 on all of $\mathbb{C}$, $\mathcal{A}$ on $\operatorname{Re}s>\frac12$, and $1/\Gamma$ everywhere. The rest of the proof of Theorem A(i) and (ii) is about $\mathcal{C}$.

4. Continuation under H_∞

This section is step (2): it continues $\mathcal{C}$, and with it Φ_2, to $\mathbb{H}$. The series $\mathcal{C}$, a sum of simple fractions whose denominators vanish at the pair sums, is meromorphic wherever the pair sums do not accumulate. Under H_∞ the distinct pair sums have no accumulation point off the line $\operatorname{Re}s=1$: in a bounded set of pair sums, if one ordinate tends to infinity then so does the other,

and both real parts tend to $\frac{1}{2}$. The continuation needs only that they do not accumulate in $\operatorname{Re} s > 1$ (Lemma 4.1), and this gives Theorem A(i) and the residues (Proposition 4.2). With no hypothesis on the zeros, Φ_2 continues to $\operatorname{Re} s > 2B$ [4, Proposition 3], a half-plane that reaches the line only if $B = \frac{1}{2}$; what H_∞ adds is the strip $1 < \operatorname{Re} s \le 2B$, in which pair sums can lie. The continuation under H_∞ together with their Conjecture 2 is stated in [10, Theorem 2(4)], with its proof left there as parallel to an earlier one (Section 1.4). For $s_0 \in \mathbb{C}$ let

$$\mathcal{R}(s_0) = \{(\rho, \rho') \in \mathcal{Z} \times \mathcal{Z} : \rho + \rho' = s_0\}, \qquad \nu(s_0) = \sum_{(\rho,\rho') \in \mathcal{R}(s_0)} a_\rho a_{\rho'}, \tag{14}$$

the representations of s_0 as a pair sum and the coefficient they carry, an absolutely convergent sum by Lemma 3.4(a). Under RH, $\nu(1 + i\tau) = \mu\{\tau\}$ of (2).

For a given compact set $K \subset \mathbb{H}$, the lemma below sorts the pairs of zeros into three kinds, by a height T. If both ordinates exceed T in modulus, the real parts are close to $\frac{1}{2}$ by Lemma 2.1(b), so the pair sum has real part close to 1 and stays to the left of K. If one ordinate is at most T in modulus and the other is much larger, the pair sum has imaginary part beyond that of every point of K. The pairs that remain have both ordinates bounded, and they are finitely many.

Lemma 4.1 (Local finiteness)**.** *Assume* H_∞*, and let* $K \subset \mathbb{H}$ *be compact. There are a finite set* $\mathcal{E}$ *of ordered pairs of zeros and* $d > 0$ *such that* $|s - \rho - \rho'| \ge d$ *for all* $s \in K$ *and* $(\rho, \rho') \notin \mathcal{E}$*. In particular only finitely many ordered pairs have* $\rho + \rho' \in K$*.*

Proof. Let $e = \min_{s \in K} \operatorname{Re} s - 1 > 0$ and $Y = \max_{s \in K} |\operatorname{Im} s|$. Lemma 2.1(b) with $\delta = e/4$ gives T such that $\beta \le \frac{1}{2} + e/4$ whenever $|\gamma| > T$. If $|\gamma| > T$ and $|\gamma'| > T$, then for $s \in K$, $\operatorname{Re}(s - \rho - \rho') \ge 1 + e - 1 - e/2 = e/2$. If $|\gamma| \le T$ and $|\gamma'| > T + Y + 1$, then $|\operatorname{Im}(s - \rho - \rho')| \ge |\gamma'| - |\gamma| - |\operatorname{Im} s| > 1$; likewise with the roles exchanged. Every other pair has $|\gamma|, |\gamma'| \le T + Y + 1$, and these form a finite set $\mathcal{E}$. Take $d = \min(e/2, 1)$. A pair with $\rho + \rho' \in K$ has $|s - \rho - \rho'| = 0$ at $s = \rho + \rho'$, so it lies in $\mathcal{E}$. □

This is where H_∞ enters the continuation: if infinitely many zeros had real part at least $\frac{1}{2} + \delta$, their pair sums with their conjugates, the real points $2\beta \ge 1 + 2\delta$, could accumulate inside $\operatorname{Re} s > 1$, and the argument would stop there.

With local finiteness, Lemma 3.6 applies to $\mathcal{C}$ on every disc whose closure lies in $\mathbb{H}$. The proposition states what follows, in three parts: (a) is the continuation of $\mathcal{C}$ with its residues, (b) is Theorem A(i), and (c) is the criterion for a pole on which Section 6 and the proof of Theorem C rely. The points 2 and $1 + \rho$ are set aside in (c) because there M_0 may have a pole of its own; the strip $1 < \operatorname{Re} s < 2 - B$ contains none of them, and it is where Proposition 6.3 finds its poles.

Proposition 4.2 (Continuation and residues)**.** *Assume* H_∞*.*

(a) $\mathcal{C}$ *is meromorphic on* $\mathbb{H}$*. For every* $s_0 \in \mathbb{H}$ *the set* $\mathcal{R}(s_0)$ *is finite and* $(s - s_0)\mathcal{C}(s) \to \nu(s_0)$ *as* $s \to s_0$*.*

(b) *The right side of* (13) *is meromorphic on* $\mathbb{H}$ *and equals* Φ_2 *on* $\operatorname{Re} s > 2$*: it is the continuation of Theorem A(i).*

(c) *Let* $s_0 \in \mathbb{H}$ *be neither* 2 *nor one of the points* $1 + \rho$*,* $\rho \in \mathcal{Z}$*. Then* M_0 *is analytic at* s_0*, and* $(s - s_0)\Phi_2(s) \to \nu(s_0)/\Gamma(s_0)$ *as* $s \to s_0$*. In particular, if* $\nu(s_0) \ne 0$*, then* s_0 *is a simple pole of* Φ_2*. All of this holds at every point of the strip* $1 < \operatorname{Re} s < 2 - B$*, which contains neither* 2 *nor any of the points* $1 + \rho$*.*

Proof. (a) Given $z_0 \in \mathbb{H}$, choose $r > 0$ with the closed disc $K = \overline{D(z_0, r)}$ inside $\mathbb{H}$. Lemma 4.1 for K and Lemma 3.6, with the pairs as index set, the points $\rho + \rho'$ and the coefficients $a_\rho a_{\rho'}$, whose moduli have sum S^2, make $\mathcal{C}$ meromorphic on $D(z_0, r)$; on $\operatorname{Re} s > 2$ it is the series of Section 3. Lemma 4.1 for $K = \{s_0\}$ shows that $\mathcal{R}(s_0)$ is finite, and Lemma 3.6 gives the limit.

(b) M_0 is meromorphic on $\mathbb{C}$, $\mathcal{A}$ is holomorphic on $\operatorname{Re} s > \frac{1}{2}$ and $1/\Gamma$ is entire (Proposition 3.7); combine with (a) and (13).

(c) The poles of $\psi(s-1)$ are at $s-1=1$, at $s-1 \in \mathcal{Z}$ and at $s-1 \in \{-2,-4,\dots\}$; in $\mathbb{H}$ the first two give $s=2$ and $s=1+\rho$, and the third none. The other terms of M have poles only at $s=1 \notin \mathbb{H}$ and $s=2$, and the poles of L are among the zeros, with real part less than 1. So M_0 is analytic at every point of $\mathbb{H}$ other than 2 and the points $1+\rho$. At such a point s_0, $(s-s_0)M_0(s) \to 0$, $(s-s_0)\mathcal{A}(s)/\Gamma(s) \to 0$, and $(s-s_0)\mathcal{C}(s)/\Gamma(s) \to \nu(s_0)/\Gamma(s_0)$ by (a). A nonzero limit of $(s-s_0)f(s)$ at s_0 means a pole of order exactly one. For the strip: $1-B>0$ by Lemma 2.1(a), so the strip $1 < \operatorname{Re} s < 2-B$ is not empty. It lies in $\operatorname{Re} s < \frac{3}{2}$, so it does not contain 2; and $\operatorname{Re}(1+\rho) = 1+\beta \geq 2-B$ by Lemma 2.1(a), so it contains none of the points $1+\rho$. □

Which pair sums lie in $\mathbb{H}$? A pair sum $\rho+\rho'$ has real part $\beta+\beta'$, which exceeds 1 only if one of the two zeros has $\beta > \frac{1}{2}$. Under RH there is none: $\mathcal{C}$ is holomorphic on $\mathbb{H}$, and the poles of Φ_2 there are those of M_0, the points 2 and $1+\rho$ (Section 8.1). Under H_∞ there may be zeros with $\beta > \frac{1}{2}$, and then pair sums do lie in $\mathbb{H}$; these are the candidate poles of Section 1.2. By part (c), a pair sum $s_0 \in \mathbb{H}$ other than 2 and the points $1+\rho$ is a pole when $\nu(s_0) \neq 0$. The proof of Theorem C applies this at the real point $2B$, where every term of ν is positive (Section 8.2). At a general point the terms of $\nu(s_0)$ can have different arguments, and whether they can cancel is the subject of the next section.

5. Non-cancellation: the phase cone

This section is step (3), and it contains the new estimate of the paper. By Proposition 4.2(c) a pair sum s_0 in the strip $1 < \operatorname{Re} s_0 < 2-B$ is a simple pole of Φ_2 when $\nu(s_0) \neq 0$, but s_0 may have many representations, and their weights might cancel. This section shows that they cannot when every representation of s_0 uses two high zeros (Theorem 5.6). The proof has three ingredients, one lemma each. Stirling's formula gives the modulus and the phase of a weight (Lemma 5.1). The weights at heights within $\sqrt{u_0}$ of the lowest height u_0 lie, after one common rotation, in a cone of half-angle less than a right angle (Lemma 5.2). The weights above that window are exponentially smaller than the lowest (Lemma 5.3); this comparison with the lowest pair goes back to [10, pp. 8–9] and [6, pp. 297–298] (Section 1.4); the phase cone, on the critical line and for unequal real parts, is new. Theorem 5.4 combines the three for an abstract family of weights, so that the analysis is separated from the zeros, which enter only through a lowest element and a $\log^2$ count of the representations at each height; the last subsection then matches the zeros to the family. H_∞ is used only through the bound $B<1$, and the results are stated for any $B \in [\frac{1}{2}, 1)$. They are applied twice in Section 6: at points inside $\mathbb{H}$, where the two zeros of a pair may have different real parts (Proposition 6.3), and at points of the line, to pairs of zeros on the critical line, with $B=\frac{1}{2}$ (Theorem 6.8).

We call a zero *high* when $|\gamma|$ exceeds the threshold X of Theorem 5.6, and *low* otherwise; a representation is high when both of its zeros are high, and low when at least one of them is low. To show that the whole coefficient $\nu(s_0)$ is not zero, every term of it has to be controlled. So the theorem is applied at points all of whose representations are high, and a point with even one low representation is discarded. (The theorem itself is stated for any non-empty family of high representations of s_0 whose zeros have real parts in $[1-B, B]$; Section 6.2 needs that form.) Why must every one be high? There are two reasons. The estimates of this section hold only for heights above a threshold U, and an opposite-sign pair at a lower height could be the lowest term of the sum without being covered by them. And same-sign pairs have to be excluded. A same-sign pair with $\gamma+\gamma'=\tau$ has both ordinates in $(0,\tau)$ if they are positive and in $(\tau,0)$ if they are negative, so such pairs are low; and on the critical line (G2) gives the product of the

two Gamma values the modulus $2\pi e^{-\pi|\tau|/2}$, up to a relative error that is exponentially small in the smaller of $|\gamma|$ and $|\gamma'|$, so the Gamma moduli give no such pair an exponential advantage over the others, and no lowest one dominates. That is why the threshold is at least the bound Y on $|\tau|$: two high zeros then have ordinates of opposite signs. Little is lost by the assumption: in a bounded range of imaginary parts only finitely many points have a low representation (Lemma 6.1), and Section 6 avoids them.

5.1. **The weights.** Two high zeros whose sum has bounded imaginary part have ordinates of opposite signs, so a representation of $s_0 = \alpha + i\tau$ can be written $\rho = b + i(u+\tau)$, $\rho' = c - iu$ with u large, and its weight is $m(\rho)m(\rho')$ times

$$w_{b,c}(\tau,u) = \Gamma\big(b + i(u+\tau)\big)\,\Gamma(c-iu). \tag{15}$$

For $u > 0$ put

$$\Omega_{b,c}(\tau,u) = 2\pi e^{-\pi u - \pi\tau/2} u^{b+c-1} \exp\Big(i\big(\tau\log u + \tfrac{\pi}{2}(b-c)\big)\Big). \tag{16}$$

This is the main term of $w_{b,c}(\tau,u)$ for large u, as the next lemma shows, and the argument rests on three features of it. The factor $e^{-\pi u}$ makes the weights decay with the height. The exponent of the power u^{b+c-1} is the same for all representations of one point, since $b+c = \operatorname{Re} s_0$. The argument $\tau\log u + \frac{\pi}{2}(b-c)$ turns slowly with u, at the rate τ/u, and is offset by a term that depends only on the real parts.

Lemma 5.1 (Stirling for the weights)**.** *Let $\frac12 \le B < 1$ and $Y > 0$. There are K and $U \ge Y+1$ such that for all $b, c \in [1-B, B]$, $|\tau| \le Y$ and $u \ge U$,*

$$\big|w_{b,c}(\tau,u) - \Omega_{b,c}(\tau,u)\big| \le \frac{K}{u}\,\big|\Omega_{b,c}(\tau,u)\big|.$$

Proof. Let T_0 be the constant of (G4) for the strip $0 \le \operatorname{Re} z \le 1$, and take $U \ge \max(2Y, Y+2, Y+T_0)$, so that $U \ge Y+1$. For $u \ge U$ and $|\tau| \le Y$, both $t = u$ and $t = u+\tau$ then satisfy $t \ge u - Y \ge \max(2, T_0)$, and $u + \tau \ge u/2$; these are the conditions used below.

Let $b \in [0,1]$ and $t \ge 2$. Then $\operatorname{Log}(b+it) = \log t + i\pi/2 + \operatorname{Log}(1-ib/t)$ and $|\operatorname{Log}(1-ib/t) + ib/t| \le (b/t)^2 \le t^{-2}$. Substituting into $\mathrm{St}(z)$ of (G4) at $z = b+it$ and collecting terms,

$$\mathrm{St}(b+it) = (b-\tfrac12)\log t - \tfrac{\pi}{2}t + \tfrac12\log 2\pi + i\big(t\log t - t + \tfrac{\pi}{2}(b-\tfrac12)\big) + \varepsilon, \qquad |\varepsilon| \le \frac{4}{t}.$$

With (G4) on the strip $0 \le \operatorname{Re} z \le 1$, this gives, uniformly in $b \in [0,1]$,

$$\Gamma(b+it) = \sqrt{2\pi}\,t^{b-1/2}e^{-\pi t/2}\exp\Big(i\big(t\log t - t + \tfrac{\pi}{2}(b-\tfrac12)\big)\Big)\big(1 + O(t^{-1})\big) \qquad (t\to\infty). \tag{17}$$

Apply (17) with $t = u+\tau \ge u/2$ (for $u \ge 2Y$), and with $t = u$ to $\Gamma(c-iu) = \overline{\Gamma(c+iu)}$ (G1). The product has modulus $2\pi(u+\tau)^{b-1/2}u^{c-1/2}e^{-\pi u - \pi\tau/2}(1+O(1/u))$, and $(u+\tau)^{b-1/2} = u^{b-1/2}(1+\tau/u)^{b-1/2} = u^{b-1/2}(1+O_Y(1/u))$. The argument of the leading Stirling product is

$$(u+\tau)\log(u+\tau) - (u+\tau) - u\log u + u + \tfrac{\pi}{2}(b-c) = \tau\log u + \tfrac{\pi}{2}(b-c) + \Big((u+\tau)\log\big(1+\tfrac{\tau}{u}\big) - \tau\Big),$$

and the last bracket is $O_Y(1/u)$ by Taylor's theorem. Collecting, $w_{b,c}(\tau,u) = \Omega_{b,c}(\tau,u)(1 + O_Y(1/u))$, uniformly in $b, c \in [1-B, B] \subset [0,1]$ and $|\tau| \le Y$. □

The phase $\frac{\pi}{2}(b-c)$ is what zeros off the line add: it vanishes when $b = c = \frac12$, and off the line it offsets the phase of each weight by up to $\pm\frac{\pi}{2}(2B-1)$. For the zeros of ζ under H_∞ the offset tends to 0 as both heights grow, because then $b - c = o(1)$ by Lemma 2.1(b); but that lemma gives no quantitative rate (under RH the offset is 0), and so no bound that would put the offset inside the error term of Lemma 5.1. That is why the phase is kept in (16). The proofs do not use this decay of the offset: Lemma 5.2 allows every offset up to $\pi(B - \frac12)$, which $B < 1$ keeps

short of a right angle, so its threshold depends on B and Y alone, and not on how fast the zeros approach the critical line. On the critical line the argument of the weight is $\tau \log u + o(1)$, which changes slowly with u; this slow change is all that the proof under RH has to control.

The next lemma is the phase estimate. It concerns the weights at heights u in a window $[u_0, u_0 + \sqrt{u_0}]$ above a lowest height u_0. In such a window the weights need not be small compared with the weight at u_0, and there may be many of them, so they cannot be treated as an error; they are compared by phase instead. Across a window of width ℓ the phase $\tau \log u$ turns by $\tau \log(1 + \ell/u_0)$, at most $Y\ell/u_0$ in modulus, so the phase tolerates any width up to a small multiple, depending on B, of u_0/Y. The tail estimate of Theorem 5.4, on the other side, needs a width large enough for $e^{-\pi\ell}$ to outweigh the count of the representations above the window. That count, with multiplicity, is of order at most $\log^2 u$ in each unit interval of heights near u (condition (3) of Theorem 5.4), and by Lemma 5.3 a weight at distance between n and $n+1$ above u_0 is at most a constant times $(n+2)e^{-\pi n}$ times the lowest. So for large u_0 a window whose width is a large enough constant times $\log\log u_0$ would already do, since $e^{-\pi\ell}\log^2 u_0$ is then small. The proof takes $\sqrt{u_0}$, which for large u_0 lies between the two limits and for which the crude bound $\log^2 y \le 4y$ for $y \ge 1$ is enough, so that no constant has to be tracked.

Lemma 5.2 (The phase cone)**.** *Let $\frac12 \le B < 1$, $Y > 0$, and $\lambda_B = \sin\big(\frac{\pi}{2}(1-B)\big) > 0$. There is U such that for all $b, c \in [1-B, B]$, $|\tau| \le Y$ and $U \le u_0 \le u \le u_0 + \sqrt{u_0}$,*

$$\operatorname{Re}\Big(e^{-i\tau \log u_0}\, w_{b,c}(\tau, u)\Big) \ge \lambda_B \,\big|w_{b,c}(\tau, u)\big|.$$

Since $\lambda_B = \cos(\frac{\pi}{2}B)$, the conclusion says that the rotated weight $e^{-i\tau\log u_0}w$ lies in the sector $|\arg| \le \frac{\pi}{2}B < \frac{\pi}{2}$, the cone of Section 1.3. The proof puts $\theta = \pi(1-B)$: the offsets $\frac{\pi}{2}(b-c)$ fill the sector $|\arg| \le \frac{\pi}{2} - \theta = \pi(B - \frac12)$, the limiting spread of Section 1.3, and θ is the margin to a right angle, half of which is spent on the drift across the window and on the error of Lemma 5.1.

Proof. Put $\theta = \pi(1-B) \in (0, \frac{\pi}{2}]$. The rotated main term $e^{-i\tau\log u_0}\Omega_{b,c}(\tau, u)$ has argument $\varphi = \tau\log(u/u_0) + \frac{\pi}{2}(b-c)$. Here $|\tau\log(u/u_0)| \le Y\log(1+u_0^{-1/2}) \le Y u_0^{-1/2} \le \theta/4$ once $u_0 \ge (4Y/\theta)^2$, and $|b-c| \le 2B-1$, so $|\frac{\pi}{2}(b-c)| \le \frac{\pi}{2} - \theta$. Hence $|\varphi| \le \frac{\pi}{2} - \frac34\theta$ and $\operatorname{Re}(e^{-i\tau\log u_0}\Omega) \ge \sin(\frac34\theta)|\Omega|$. With $\varepsilon = K/u_0$ from Lemma 5.1, $|w - \Omega| \le \varepsilon|\Omega|$ and $|w| \le (1+\varepsilon)|\Omega|$, so

$$\operatorname{Re}\big(e^{-i\tau\log u_0}w\big) \ge \big(\sin(\tfrac34\theta) - \varepsilon\big)|\Omega| \ge \sin(\tfrac12\theta)\,|w|$$

as soon as $(1 + \sin\frac12\theta)\varepsilon \le \sin\frac34\theta - \sin\frac12\theta$, which is positive since $0 < \frac12\theta < \frac34\theta < \frac{\pi}{2}$. Take U at least the U of Lemma 5.1, at least $(4Y/\theta)^2$, and so large that K/U satisfies this inequality. Finally $\sin(\frac12\theta) = \lambda_B$. □

The weights are rotated by $-\tau\log u_0$, that is, multiplied by $e^{-i\tau\log u_0}$, which turns the direction $\tau\log u_0$ into the positive real axis. That direction depends only on τ and the height; it is not the argument of any particular weight. The weight of the lowest representation may sit near one edge of the sector of the offsets, $|\arg| \le \frac{\pi}{2} - \theta$, and a second weight near the opposite edge; their arguments then differ by about $\pi - 2\theta = \pi(2B-1)$, more than a right angle when $B > \frac34$, yet both have positive real part after the common rotation. That is all the argument needs: each weight of the window contributes a non-negative amount in one fixed direction, and the lowest contributes a definite positive amount.

An example on the line. Let $\gamma_2 < \gamma_3$ be the second and third positive ordinates and $b = c = \frac12$. The point $1 + i\tau$ with $\tau = \gamma_3 - \gamma_2 = 3.9888\ldots$ is represented by the zeros with ordinates γ_3 and $-\gamma_2$, at the height $u_0 = \gamma_2 = 21.022\ldots$, and $|w_{1/2,1/2}(\tau, u_0)| = 2.48\ldots\cdot 10^{-31}$. Across the window $[u_0, u_0 + \sqrt{u_0}]$, of width $4.58\ldots$, the modulus falls by the factor $e^{-\pi\sqrt{u_0}} = 5.55\ldots\cdot 10^{-7}$

(by (G2), up to a factor $1+O(e^{-2\pi u_0})$), while the argument turns by only $0.726\ldots$ radians, against $\tau\log(1+\sqrt{u_0}/u_0)\approx 0.787$ from (16). The threshold U of Lemma 5.2 is not explicit; the example illustrates the phase drift, not the lemma. These values were computed in floating point with the mpmath library at 40 significant digits, using its values of the ordinates; they are numerical illustrations, not certified enclosures.

Above the window the phases are not controlled, and the weights are compared with the lowest one by modulus alone.

Lemma 5.3 (Decay)**.** *Let $\frac12\le B<1$ and $Y>0$. There are K_1 and U such that for all $b,c,b',c'\in[1-B,B]$ with $b+c=b'+c'$, all $|\tau|\le Y$ and $U\le u_0\le u$,*

$$\bigl|w_{b,c}(\tau,u)\bigr|\le K_1(1+u-u_0)e^{-\pi(u-u_0)}\bigl|w_{b',c'}(\tau,u_0)\bigr|.$$

Proof. Take $U\ge 1$ from Lemma 5.1 so large that $K/U\le\frac12$; then $\frac12|\Omega|\le|w|\le\frac32|\Omega|$ for both weights. With $\alpha=b+c=b'+c'\in(0,2)$,

$$\frac{|\Omega_{b,c}(\tau,u)|}{|\Omega_{b',c'}(\tau,u_0)|}=\Bigl(\frac{u}{u_0}\Bigr)^{\alpha-1}e^{-\pi(u-u_0)}.$$

If $\alpha\le1$ the power is at most 1; if $1<\alpha<2$ it is at most $u/u_0\le1+(u-u_0)/u_0\le1+u-u_0$. So the lemma holds with $K_1=3$. □

The common value of $b+c$ is what keeps the powers u^{b+c-1} under control in the ratio, which is $(u/u_0)^{b+c-1}$; with two different exponents the ratio of the powers would depend on u_0 and not only on u/u_0. All representations of one point s_0 share this value: $b+c=\operatorname{Re}s_0$.

5.2. **Non-cancellation for an abstract family.** The theorem is stated for an arbitrary family of heights u_i, with real parts b_i, c_i and non-negative integer multiplicities q_i, which may vanish, finitely many in each unit interval of heights, so that the analytic argument is separated from the zeros. Its hypotheses are what the three lemmas need: (1) is the common value of $b+c$ of Lemma 5.3; (2) provides a lowest element, above the thresholds of the lemmas and with positive multiplicity; (3) bounds the total multiplicity in each unit interval of heights, and is the only place where the count of the zeros enters (Lemma 5.5). The proof splits the family at the height $u_0+\sqrt{u_0}$. After rotation by $-\tau\log u_0$ the head, up to that height, has real part at least $\lambda_B|w_0|$, by the phase cone; the tail has modulus at most half of that, by the decay set against the count. So the sum cannot vanish.

Theorem 5.4 (Non-cancellation for an abstract family)**.** *Let $\frac12\le B<1$, $Y>0$ and $\kappa\ge0$. There is U with the following property. Let $|\tau|\le Y$, let ι be a set, $b,c:\iota\to[1-B,B]$, $u:\iota\to\mathbb{R}$, $q:\iota\to\mathbb{N}=\{0,1,2,\ldots\}$, and $i_0\in\iota$, such that*

1. *$b_i+c_i=b_{i_0}+c_{i_0}$ for all $i\in\iota$;*
2. *$U\le u_{i_0}\le u_i$ for all $i\in\iota$, and $q_{i_0}\ge1$;*
3. *for every real v the set $\{i:u_i\in[v,v+1)\}$ is finite, and for $v\ge0$ its total weight satisfies $\sum_{u_i\in[v,v+1)}q_i\le\kappa\log^2(v+Y+4)$.*

Then the series $\sum_{i\in\iota}q_iw_{b_i,c_i}(\tau,u_i)$ converges absolutely and its sum is nonzero.

Proof. Let K_1,U_1 be given by Lemma 5.3, U_2 by Lemma 5.2, and put $d_n=4\kappa K_1(Y+4)(n+2)^3e^{-\pi n}$ for $n\ge0$; $d_n|w_0|$ will bound the contribution of the nth bin of the tail below. The series $\sum d_n$ converges; choose n_0 with $\sum_{n\ge n_0}d_n\le\lambda_B/2$. Put $U=\max\bigl(U_1,U_2,1,(n_0+1)^2\bigr)$.

Let data as in the statement be given, write $u_0=u_{i_0}$, $w_i=w_{b_i,c_i}(\tau,u_i)$ and $w_0=w_{i_0}$, so $u_0\ge U$ and $w_0\ne0$ by (G1). For $n\ge0$ let $I_n=\{i:u_i\in[u_0+n,u_0+n+1)\}$; these sets are finite and partition ι. By Lemma 5.3 and (1), for $i\in I_n$,

$$|w_i|\le K_1(1+u_i-u_0)e^{-\pi(u_i-u_0)}|w_0|\le K_1(n+2)e^{-\pi n}|w_0|,$$

and by (3) and $\log^2 y \le 4y$ ($y \ge 1$), $\sum_{i\in I_n} q_i \le 4\kappa(u_0+n+Y+4)$. Hence $\sum_i q_i|w_i| \le |w_0|\sum_n 4\kappa K_1(u_0+n+Y+4)(n+2)e^{-\pi n} < \infty$. It remains to show that $\operatorname{Re}\bigl(e^{-i\tau\log u_0}\sum_i q_i w_i\bigr) > 0$.

Head. If $u_i \le u_0+\sqrt{u_0}$, Lemma 5.2 gives $\operatorname{Re}(e^{-i\tau\log u_0} q_i w_i) \ge \lambda_B q_i|w_i| \ge 0$; for $i = i_0$ this is at least $\lambda_B|w_0|$, as $q_{i_0} \ge 1$.

Tail. Let $n \ge 0$ be a bin index with $n+1 > \sqrt{u_0}$. Then $n+1 > \sqrt{u_0} \ge n_0+1$, so $n \ge n_0$; and $u_0 < (n+1)^2$, whence $u_0+n+Y+4 \le (Y+4)(n+2)^2$ and $\sum_{i\in I_n} q_i|w_i| \le d_n|w_0|$. If $u_i > u_0+\sqrt{u_0}$ then the bin index n of i satisfies $n+1 > u_i-u_0 > \sqrt{u_0}$, so it is one of these. Hence

$$\sum_{u_i > u_0+\sqrt{u_0}} q_i|w_i| \le \sum_{n+1>\sqrt{u_0}} d_n|w_0| \le \sum_{n\ge n_0} d_n|w_0| \le \tfrac12\lambda_B|w_0|.$$

Conclusion. $\operatorname{Re}\bigl(e^{-i\tau\log u_0}\sum_i q_i w_i\bigr) \ge \lambda_B|w_0| - \tfrac12\lambda_B|w_0| > 0$. □

5.3. **The zero side.** It remains to match the representations of a point s_0 to the abstract family. Lemma 5.5 supplies condition (3), from the local count of the zeros. The proof of Theorem 5.6 supplies the rest. Since both zeros of a representation are high and the imaginary part of s_0 is bounded, their ordinates have opposite signs, and the pair can be written as a weight $w_{b,c}(\tau,u)$ with $u > X$; this is where the assumption that every pair in $\mathcal{J}$ is high is used, for the opposite signs and for the threshold U of Theorem 5.4. The theorem is stated for a subset $\mathcal{J}$ of the representations because Section 6.2 applies it to the pairs of zeros on the critical line alone, with $B = \frac12$; in Section 6.1, $\mathcal{J}$ is the set of all representations.

Lemma 5.5. *Let $Y > 0$. There is κ_0 such that for every $s_0 \in \mathbb{C}$ with $|\operatorname{Im} s_0| \le Y$ and every $v \ge 0$, the pairs $(\rho,\rho') \in \mathcal{R}(s_0)$ with $\gamma' \in (-v-1,-v]$ are finitely many, and*

$$\sum_{(\rho,\rho')\in\mathcal{R}(s_0),\ \gamma'\in(-v-1,-v]} m(\rho)m(\rho') \le \kappa_0\log^2(v+Y+4).$$

Proof. The zero ρ' lies in $\mathcal{W}_{-v-1/2}$, a finite set of total multiplicity at most $C_2\log(v+\frac72) \le C_2\log(v+Y+4)$ (Lemma 3.2(a)), and $\rho = s_0-\rho'$ is determined by ρ'. Its ordinate satisfies $|\gamma| \le |\operatorname{Im} s_0|+|\gamma'| \le Y+v+1$, so $m(\rho) \le C_2\log(v+Y+4)$ (Lemma 3.2(b)). Take $\kappa_0 = C_2^2$. □

Theorem 5.6 (Non-cancellation)**.** *Let $\frac12 \le B < 1$ and $Y > 0$. There is $X \ge Y$ with the following property. Let $s_0 \in \mathbb{C}$ with $|\operatorname{Im} s_0| \le Y$, and let $\mathcal{J} \subseteq \mathcal{R}(s_0)$ be non-empty, with $1-B \le \beta,\beta' \le B$ and $|\gamma|,|\gamma'| > X$ for every $(\rho,\rho') \in \mathcal{J}$. Then $\sum_{(\rho,\rho')\in\mathcal{J}} a_\rho a_{\rho'} \ne 0$.*

In particular, assume H_∞, and let $B = \sup\beta$. For every $Y > 0$ there is $X \ge Y$ such that $\nu(s_0) \ne 0$ whenever $|\operatorname{Im} s_0| \le Y$, $\mathcal{R}(s_0)$ is non-empty, and every $(\rho,\rho') \in \mathcal{R}(s_0)$ has $|\gamma| > X$ and $|\gamma'| > X$.

Proof. Let κ_0 be given by Lemma 5.5, U by Theorem 5.4 for B, Y and $\kappa = 2\kappa_0$, and put $X = \max(Y,U)$. Write $\tau = \operatorname{Im} s_0$.

Orientation. Let $(\rho,\rho') \in \mathcal{J}$. Then $\gamma+\gamma' = \tau$, $|\tau| \le Y \le X$ and $|\gamma|,|\gamma'| > X$, so the two ordinates have opposite signs: if both were positive their sum would exceed $2X \ge Y$, and if both were negative it would be below $-2X \le -Y$. If $\gamma' < 0$ put $(b,c,u) = (\beta,\beta',-\gamma')$; then $\rho = b+i(u+\tau)$ and $\rho' = c-iu$. If $\gamma < 0$ put $(b,c,u) = (\beta',\beta,-\gamma)$; then $\rho' = b+i(u+\tau)$ and $\rho = c-iu$. In both cases $u > X$, $b+c = \operatorname{Re} s_0$, $b,c \in [1-B,B]$, and

$$a_\rho a_{\rho'} = q\, w_{b,c}(\tau,u), \qquad q = m(\rho)m(\rho') \ge 1.$$

The family. Apply Theorem 5.4 with $\iota = \mathcal{J}$ and these b,c,u,q. Condition (1) holds. For $v \in \mathbb{R}$, the $i \in \iota$ with $u_i \in [v,v+1)$ are the pairs whose negative ordinate lies in $(-v-1,-v]$; by Lemma 5.5, applied to the pairs themselves and to the reversed pairs (ρ',ρ), which also lie in $\mathcal{R}(s_0)$, they are finitely many, with total weight at most $2\kappa_0\log^2(v+Y+4)$ for $v \ge 0$; this is

(3). Since $\mathcal{J}$ is non-empty and the values u_i below any bound form a finite set, some i_0 has the least u_{i_0}; $u_{i_0} > X \geq U$ and $q_{i_0} \geq 1$, which is (2). So $\sum_{\mathcal{J}} a_\rho a_{\rho'} = \sum_i q_i w_{b_i,c_i}(\tau, u_i) \neq 0$.

Under H_∞. By Lemma 2.1(a), $\frac{1}{2} \leq B < 1$ and every zero has $1 - B \leq \beta \leq B$; take $\mathcal{J} = \mathcal{R}(s_0)$. □

Theorem 5.6 takes the place of the uniqueness that hypothesis (A) supplies in the earlier proofs (Section 9); what replaces (B) is the factor η in the limit along the normal (Lemma 6.6). Where, under RH, hypothesis (A) leaves at each pair sum $1 + i\tau$ with $\tau \neq 0$ a single representation up to order, the theorem allows any number of representations, with multiple zeros and coincident sums, as long as all of them are high. The pairs of height close to the lowest, where a coincidence would matter most, form the head of Theorem 5.4, and none of them can cancel the lowest.

6. The natural boundary

This section proves Theorem A(ii); it is step (4). The sections before it leave one thing to do. By Proposition 4.2 a pair sum $s_0 \in \mathbb{H}$, other than 2 and the points $1 + \rho$, is a pole when its coefficient $\nu(s_0)$ is not zero, and by Theorem 5.6 such a coefficient is nonzero when every representation of s_0 is high; on the line the same theorem is applied to the nonempty family of representations by two zeros of the critical line, each of them high. The condition that every representation be high is a condition on the point, not on the zeros. In each bounded range of imaginary parts only finitely many points have a low representation (Lemma 6.1), and these are discarded; Littlewood's gap theorem puts infinitely many pair sums of two high zeros in every neighbourhood of every point of the line, so some of them are among the points kept. So points with a nonzero coefficient accumulate at every point of the line, and it remains to turn this accumulation into the absence of a meromorphic extension. How that is done depends on where the points lie.

Let $\mathcal{Z}_{\mathrm{off}} = \{\rho \in \mathcal{Z} : \beta \neq \frac{1}{2}\}$ be the zeros off the line and $\mathcal{Z}_{\mathrm{c}} = \mathcal{Z} \setminus \mathcal{Z}_{\mathrm{off}}$ the zeros on it. By the symmetry $\rho \mapsto 1 - \bar{\rho}$, $\mathcal{Z}_{\mathrm{off}}$ is infinite exactly when infinitely many zeros have $\beta > \frac{1}{2}$. If $\mathcal{Z}_{\mathrm{off}}$ is infinite, pair sums involving an off-line zero give poles inside $\mathbb{H}$ that accumulate at every point of the line (Section 6.1), and accumulating poles exclude an extension; no radial limit is taken here, since the termwise bound behind it fails for pairs with $\beta + \beta' > 1$. If $\mathcal{Z}_{\mathrm{off}}$ is finite, the off-line zeros contribute a function meromorphic across the line, and the argument under RH runs on the rest (Section 6.2): the points found are then on the line itself. The limits along the normal are nonzero at points of the line that accumulate at every point of it, while a meromorphic extension near a point of the line would make these limits 0 at every other point of the line close to it (Lemma 6.2). The two cases need different arguments for this reason. Two preliminary lemmas come first. Lemma 6.1 serves both cases and Lemma 6.2 the second; neither uses a hypothesis. The division into the two cases follows [8] (Section 1.4).

The first lemma is the reason why so little is lost by the restriction to high representations. A representation of a point of bounded imaginary part that uses one low zero uses two zeros of bounded ordinate, and there are only finitely many of those.

Lemma 6.1 (Low representations)**.** *For all real X and Y the set*

$$\mathcal{E}(X,Y) = \big\{ s \in \mathbb{C} : \ |\operatorname{Im} s| \leq Y,\ \exists (\rho,\rho') \in \mathcal{R}(s) \text{ with } |\gamma| \leq X \text{ or } |\gamma'| \leq X \big\}$$

is finite.

Proof. If $(\rho,\rho') \in \mathcal{R}(s)$, $|\operatorname{Im} s| \leq Y$ and $|\gamma| \leq X$, then $|\gamma'| \leq |\operatorname{Im} s| + |\gamma| \leq |X| + |Y|$, and symmetrically. So every $s \in \mathcal{E}(X,Y)$ is a sum of two zeros whose ordinates have modulus at most $|X| + |Y|$, and there are finitely many such zeros (Section 2). □

The second lemma is the criterion of Section 1.3, the blow-up criterion of [18, Lemma 4.2] and [10, proof of Theorem 2(2)]: nonzero limits along the normal, at points of the line that accumulate at $1+iy$, exclude a meromorphic extension at $1+iy$. It is stated for any G holomorphic on $\mathbb{H}$, and Theorem 6.9 applies it to $G=\mathcal{C}_{\mathrm{c}}$. The extension is required to agree with G only outside an exceptional set with no accumulation point, because that is the form in which the lemma is applied: the function that Theorem 6.9 builds from a candidate extension of Φ_2 is known to agree with $\mathcal{C}_{\mathrm{c}}$ only off the exceptional set of the extension and the poles of the other terms, a set with no accumulation point in the open half-disc.

Lemma 6.2 (Line blow-up)**.** *Let G be holomorphic on $\mathbb{H}$ and $y\in\mathbb{R}$. Suppose that for every $\varepsilon>0$ there are $\tau_0\neq y$ with $|\tau_0-y|<\varepsilon$ and $c\neq 0$ such that $\eta\,G(1+i\tau_0+\eta)\to c$ as $\eta\downarrow 0$. Then for no $r>0$ is there a function g meromorphic on $D(1+iy,r)$ with $g=G$ at every point of $D(1+iy,r)\cap\mathbb{H}$ outside a set with no accumulation point in $D(1+iy,r)\cap\mathbb{H}$.*

Proof. Let $s_1=1+iy$ and suppose g is meromorphic on $D=D(s_1,r)$ and agrees with G on $D\cap\mathbb{H}$ outside such a set. The poles of g are isolated, so there is $\delta\in(0,r]$ with g analytic on $D(s_1,\delta)\setminus\{s_1\}$. Choose τ_0 and c as in the hypothesis for $\varepsilon=\delta$ and put $s_0=1+i\tau_0$; then $s_0\in D(s_1,\delta)\setminus\{s_1\}$, so g is analytic on a disc N about s_0. On $N\cap\mathbb{H}$ the continuous functions g and G agree outside a set with no accumulation point there, hence everywhere on $N\cap\mathbb{H}$. For all small $\eta>0$, $s_0+\eta\in N\cap\mathbb{H}$, so $\eta\,G(s_0+\eta)=\eta\,g(s_0+\eta)\to 0\cdot g(s_0)=0$, contradicting $c\neq 0$. □

6.1. **Infinitely many zeros off the line.** In this case the singular points are found inside $\mathbb{H}$. A zero ρ_n to the right of the critical line, at a large height, is conjugated and paired with a zero ρ_n' whose ordinate is close to γ_n+y; Littlewood's theorem (Z2) provides such a zero, and the symmetry $\rho\mapsto 1-\bar\rho$ lets us take $\beta_n'\geq\frac12$. The sum $s_n=\bar\rho_n+\rho_n'$ has real part greater than 1, because $\beta_n>\frac12$ and $\beta_n'\geq\frac12$, and it tends to $1+iy$, because both real parts tend to $\frac12$. This is the second use of the approach of zeros of large ordinate to the critical line; the first was Lemma 4.1. That s_n is a pole needs $\nu(s_n)\neq 0$, and here Theorem 5.6 is applied to zeros whose real parts may differ.

Proposition 6.3 (Interior poles)**.** *Assume H_∞ and that $\mathcal{Z}_{\mathrm{off}}$ is infinite. For every real y there are pairwise distinct points $s_n\in\mathbb{H}$ with $s_n\to 1+iy$, each a simple pole of Φ_2.*

Proof. Construction. Infinitely many zeros have $\beta>\frac12$. Only finitely many lie below any height, so there are such zeros $\rho_n=\beta_n+i\gamma_n$ with $|\gamma_n|\to\infty$; replacing ρ_n by $\bar\rho_n$ where needed, $\gamma_n\to+\infty$. For large n, (Z2) at $t=\gamma_n+y$ gives a zero ρ_n' with $|\gamma_n'-\gamma_n-y|<A/\log\log\log(\gamma_n+y)$; replacing ρ_n' by $1-\bar\rho_n'$ where needed, which keeps its ordinate, $\beta_n'\geq\frac12$. Both ordinates tend to $+\infty$, so $\beta_n,\beta_n'\to\frac12$ by Lemma 2.1(b), and

$$s_n=\bar\rho_n+\rho_n'=(\beta_n+\beta_n')+i(\gamma_n'-\gamma_n)\longrightarrow 1+iy,\qquad \operatorname{Re}s_n>1.$$

The real parts exceed 1 and tend to 1, so infinitely many of the s_n are distinct; pass to a subsequence of distinct points.

Every representation is high. Let $B=\sup\beta<1$, $Y=|y|+1$, and X be given by Theorem 5.6. The set $\mathcal{E}(X,Y)$ is finite (Lemma 6.1) and the s_n are pairwise distinct, so $s_n\notin\mathcal{E}(X,Y)$ for large n; and $|\operatorname{Im}s_n|\leq Y$ for large n, since $s_n\to 1+iy$. Then $\mathcal{R}(s_n)$ contains $(\bar\rho_n,\rho_n')$, and every pair in it has both ordinates of modulus greater than X; Theorem 5.6 gives $\nu(s_n)\neq 0$.

Poles. For large n, $1<\operatorname{Re}s_n<2-B$, and Proposition 4.2(c) makes s_n a simple pole. □

Poles accumulating at $1+iy$ exclude a meromorphic extension there. As in Lemma 6.2, the corollary allows the extension to disagree with Φ_2 on a set with no accumulation point in the

half-disc. The proof of Theorem A(iii) needs that form: the function it builds from an extension of Φ_r is known to agree with Φ_2 only off such a set.

Corollary 6.4. *Assume* H_∞ *and that* $\mathcal{Z}_{\mathrm{off}}$ *is infinite. Then for no* $y \in \mathbb{R}$, $r > 0$ *is there a function* g *meromorphic on* $D(1+iy, r)$ *that agrees with* Φ_2 *at every point of* $D(1+iy, r) \cap \mathbb{H}$ *outside a set with no accumulation point in* $D(1+iy, r) \cap \mathbb{H}$.

Proof. Suppose there is such a g, and take s_n from Proposition 6.3. For large n, $s_n \in D(1+iy, r) \cap \mathbb{H}$; near s_n both g and Φ_2 are meromorphic and agree on a punctured neighbourhood outside a set with no accumulation point, hence on a punctured neighbourhood, so g has a pole at s_n. Then the poles of g accumulate at $1+iy$, an interior point of its disc of meromorphy, which is impossible. □

6.2. **Finitely many zeros off the line.** With finitely many zeros off the line, we split them off and run the argument under RH on the rest. (In this case H_∞ holds by its definition; Lemma 6.5 and Theorem 6.9 keep it as a hypothesis because they use the results of Section 4.) The steps are these. The pairs that involve an off-line zero form a function meromorphic on $\mathbb{C}$ (Lemma 6.5). Each off-line zero ρ_0 still pairs with infinitely many zeros, but the pairs with ρ_0 in a given place sum to $a_{\rho_0} L(s-\rho_0)$, a shift of the single series L, which is meromorphic on $\mathbb{C}$; and there are finitely many ρ_0. Some of these pairs have their sums on the line itself, for instance ρ_0 with $1-\bar{\rho}_0$, whose sum has real part 1. They do no harm, since they lie in a function meromorphic across the line. What is left, $\mathcal{C}_{\mathrm{c}}$, has a limit along the normal at every point of the line (Lemma 6.6); a τ represented by two high zeros of the critical line lies in every interval (Lemma 6.7); the nonzero atoms are dense (Theorem 6.8); and Lemma 6.2 concludes (Theorem 6.9).

Split the pair series by whether a pair involves a zero off the line:

$$\mathcal{C}_{\mathrm{c}}(s) = \sum_{(\rho,\rho')\in\mathcal{Z}_{\mathrm{c}}\times\mathcal{Z}_{\mathrm{c}}} \frac{a_\rho a_{\rho'}}{s-\rho-\rho'}, \qquad \mathcal{C}_{\mathrm{ex}} = \mathcal{C} - \mathcal{C}_{\mathrm{c}}, \qquad W = M_0 + \frac{\mathcal{C}_{\mathrm{ex}} + \mathcal{A}}{\Gamma}, \tag{18}$$

and for real τ let

$$\mu_{\mathrm{c}}\{\tau\} = \sum_{\substack{(\rho,\rho')\in\mathcal{Z}_{\mathrm{c}}\times\mathcal{Z}_{\mathrm{c}} \\ \gamma+\gamma'=\tau}} a_\rho a_{\rho'},$$

absolutely convergent by Lemma 3.4(a). As in Section 1.3, these are the masses of a finite complex measure μ_{c} on $\mathbb{R}$, of which $\mathcal{C}_{\mathrm{c}}(s) = \int_{\mathbb{R}} d\mu_{\mathrm{c}}(\tau)/(s-1-i\tau)$ is the Cauchy transform on $\mathbb{H}$, and Lemma 6.6 recovers the mass at one point from the values along the normal. Under RH, $\mu_{\mathrm{c}} = \mu$.

Lemma 6.5 (The split)**.** *Assume* H_∞ *and that* $\mathcal{Z}_{\mathrm{off}}$ *is finite. Then* $\mathcal{C}_{\mathrm{c}}$ *is holomorphic on* $\mathbb{H}$, $\mathcal{C}_{\mathrm{ex}}$ *is meromorphic on* $\mathbb{C}$, W *is meromorphic on* $\operatorname{Re} s > \frac{1}{2}$, *and* $\Phi_2 = W + \mathcal{C}_{\mathrm{c}}/\Gamma$ *on* $\mathbb{H}$.

Proof. For $\rho, \rho' \in \mathcal{Z}_{\mathrm{c}}$ and $s \in \mathbb{H}$, $|s-\rho-\rho'| \geq \operatorname{Re} s - 1$, so $\mathcal{C}_{\mathrm{c}}$ converges uniformly on $\operatorname{Re} s \geq 1+\varepsilon$ and is holomorphic on $\mathbb{H}$. Let $\operatorname{Re} s > 2$ first. There the series for $\mathcal{C}$ and for $\mathcal{C}_{\mathrm{c}}$ converge absolutely, so $\mathcal{C}_{\mathrm{ex}}(s)$ is the sum of $a_\rho a_{\rho'}/(s-\rho-\rho')$ over the ordered pairs with at least one entry in $\mathcal{Z}_{\mathrm{off}}$, and by inclusion and exclusion over the finite set $\mathcal{Z}_{\mathrm{off}}$

$$\mathcal{C}_{\mathrm{ex}}(s) = 2\sum_{\rho_0\in\mathcal{Z}_{\mathrm{off}}} a_{\rho_0} L(s-\rho_0) - \sum_{(\rho_0,\rho_1)\in\mathcal{Z}_{\mathrm{off}}\times\mathcal{Z}_{\mathrm{off}}} \frac{a_{\rho_0}a_{\rho_1}}{s-\rho_0-\rho_1},$$

since $\sum_{\rho'} a_{\rho'}/(s-\rho_0-\rho') = L(s-\rho_0)$ and each ordered pair with exactly one, or two, entries in $\mathcal{Z}_{\mathrm{off}}$ is counted once on the right. The right side is meromorphic on $\mathbb{C}$, because L is (Proposition 3.7). It equals $\mathcal{C}-\mathcal{C}_{\mathrm{c}}$ on $\operatorname{Re} s > 2$, hence on $\mathbb{H}$ by the identity theorem, since $\mathcal{C}$ is meromorphic

on $\mathbb{H}$ (Proposition 4.2(a)); so it continues $\mathcal{C}_{\mathrm{ex}}$ to $\mathbb{C}$. Then W is meromorphic on $\operatorname{Re} s > \frac{1}{2}$ by Proposition 3.7, and $\Phi_2 = W + \mathcal{C}_{\mathrm{c}}/\Gamma$ on $\mathbb{H}$ follows from (13) and Proposition 4.2(b). □

Lemma 6.6 (Radial limit)**.** *For every real τ, $\lim_{\eta\downarrow 0} \eta\, \mathcal{C}_{\mathrm{c}}(1+i\tau+\eta) = \mu_{\mathrm{c}}\{\tau\}$.*

Proof. For $\eta > 0$ and $s = 1 + i\tau + \eta$, $s - \rho - \rho' = \eta + i(\tau - \gamma - \gamma')$ for $\rho, \rho' \in \mathcal{Z}_{\mathrm{c}}$, so

$$\eta\, \mathcal{C}_{\mathrm{c}}(s) = \sum_{(\rho,\rho')\in \mathcal{Z}_{\mathrm{c}}\times\mathcal{Z}_{\mathrm{c}}} a_\rho a_{\rho'} \frac{\eta}{\eta + i(\tau - \gamma - \gamma')}.$$

Each term has modulus at most $|a_\rho a_{\rho'}|$, and tends, as $\eta \downarrow 0$, to $a_\rho a_{\rho'}$ if $\gamma + \gamma' = \tau$ and to 0 otherwise. By dominated convergence for series, with the majorant of Lemma 3.4(a), the sum tends to $\mu_{\mathrm{c}}\{\tau\}$. □

The factor η removes every pair sum other than τ, however close; this is what replaces hypothesis (B) of [18], which is stated in Section 9. The termwise limit is that of [6, p. 297] (Section 1.4). The bound of each term by $|a_\rho a_{\rho'}|$ uses $\beta + \beta' \le 1$, which holds when both zeros are on the critical line. For a pair with $\beta+\beta' > 1$ the term is $a_\rho a_{\rho'}\, \eta/(\eta+1-\beta-\beta'+i(\tau-\gamma-\gamma'))$, and its denominator vanishes at $\eta = \beta + \beta' - 1$ when $\tau = \gamma + \gamma'$. This is why the pairs that involve an off-line zero were split off first, and why Section 6.1 takes no radial limit.

It remains to find, near every point of the line, a τ with $\mu_{\mathrm{c}}\{\tau\} \neq 0$. The next lemma puts a τ represented by two high zeros of the critical line in every interval, so that the finitely many τ of a bounded range that have a low representation can be avoided. It rests on Littlewood's gap theorem (Z2), which is unconditional; no zero-density estimate is used.

Lemma 6.7 (High pairs in every interval)**.** *Assume that $\mathcal{Z}_{\mathrm{off}}$ is finite. For every real X and all real $a < b$ there are $\tau \in (a, b)$ and $\rho, \rho' \in \mathcal{Z}_{\mathrm{c}}$ with $\gamma + \gamma' = \tau$, $|\gamma| > X$ and $|\gamma'| > X$.*

Proof. Let $\ell = b - a$, and let $T_{\mathrm{off}} \ge 0$ exceed the modulus of every ordinate of a zero in $\mathcal{Z}_{\mathrm{off}}$. Since $\log\log\log t \to \infty$, (Z2) gives T_1 such that for every $t \ge T_1$ some $\rho \in \mathcal{Z}$ has $t < \gamma < t + \ell$: apply (Z2) at $t+\ell/2$, with T_1 so large that $t+\ell/2$ is beyond the threshold of (Z2) and $A/\log\log\log(t+\ell/2) < \ell/2$ for $t \ge T_1$. By (Z4) pick $\rho_1 \in \mathcal{Z}$ with $\gamma_1 > \max(T_1 - a,\ X,\ X - a,\ T_{\mathrm{off}},\ T_{\mathrm{off}} - a)$. Then $\gamma_1 + a \ge T_1$, so there is $\rho_2 \in \mathcal{Z}$ with $\gamma_1 + a < \gamma_2 < \gamma_1 + b$. Both ordinates exceed T_{off}, so $\rho_1, \rho_2 \in \mathcal{Z}_{\mathrm{c}}$, and $\bar\rho_1 \in \mathcal{Z}_{\mathrm{c}}$. Put $\rho = \rho_2$, $\rho' = \bar\rho_1$, $\tau = \gamma_2 - \gamma_1 \in (a, b)$; the ordinates are $\gamma_2 > \gamma_1 + a > X$ and $-\gamma_1$, with $\gamma_1 > X$. □

The theorem that follows assembles the pieces. Remove from an interval the finitely many τ with a low representation (Lemma 6.1); find in what is left a τ represented by two high zeros of the critical line (Lemma 6.7); and apply Theorem 5.6, with $B = \frac{1}{2}$, to the pairs of zeros of the critical line that represent it.

Theorem 6.8 (Density of nonzero atoms)**.** *Assume that $\mathcal{Z}_{\mathrm{off}}$ is finite. For every real y and every $\varepsilon > 0$ there is τ_0 with $\tau_0 \neq y$, $|\tau_0 - y| < \varepsilon$ and $\mu_{\mathrm{c}}\{\tau_0\} \neq 0$.*

Proof. Put $Y = |y| + \varepsilon$ and let $X \ge Y$ be given by Theorem 5.6 with $B = \frac{1}{2}$. The set $E = \{\tau \in \mathbb{R} : 1 + i\tau \in \mathcal{E}(X, Y)\}$ is finite (Lemma 6.1), so the interval $(y - \varepsilon, y + \varepsilon)$ minus $E \cup \{0, y\}$ contains an open interval (a, b). Lemma 6.7 gives $\tau \in (a, b)$ and $\rho, \rho' \in \mathcal{Z}_{\mathrm{c}}$ with $\rho + \rho' = 1 + i\tau$ and both ordinates of modulus greater than X. Let $\mathcal{J}$ be the set of pairs in $\mathcal{R}(1 + i\tau)$ with both entries in $\mathcal{Z}_{\mathrm{c}}$; it is non-empty, its entries have real part $\frac{1}{2} \in [1 - B, B]$, and since $\tau \notin E$ all its ordinates have modulus greater than X. Also $|\tau| \le Y$. Theorem 5.6 gives $\mu_{\mathrm{c}}\{\tau\} = \sum_{\mathcal{J}} a_\rho a_{\rho'} \neq 0$, and $\tau \neq y$, $|\tau - y| < \varepsilon$; take $\tau_0 = \tau$. □

It remains to pass from $\mathcal{C}_{\mathrm{c}}$ to Φ_2. A candidate extension of Φ_2 near a point of the line, less the function W, which is meromorphic across the line, and multiplied by Γ, would be an extension of $\mathcal{C}_{\mathrm{c}}$, and Lemma 6.2 excludes one.

Theorem 6.9. *Assume* H_∞ *and that* $\mathcal{Z}_{\mathrm{off}}$ *is finite. Then for no* $y \in \mathbb{R}$, $r > 0$ *is there a function* g *meromorphic on* $D(1+iy, r)$ *that agrees with* Φ_2 *at every point of* $D(1+iy, r) \cap \mathbb{H}$ *outside a set with no accumulation point in* $D(1+iy, r) \cap \mathbb{H}$.

Proof. Suppose there is such a g; shrinking the disc, take $r \le \frac{1}{2}$, so that $D = D(1+iy, r)$ lies in $\operatorname{Re} s > \frac{1}{2}$. There W is meromorphic (Lemma 6.5) and Γ is analytic without zeros, so $g_1 = \Gamma \cdot (g - W)$ is meromorphic on D. On $D \cap \mathbb{H}$, outside the exceptional set of g and the poles of W and of Φ_2, which together have no accumulation point in $D \cap \mathbb{H}$, Lemma 6.5 gives $g_1 = \Gamma(\Phi_2 - W) = \mathcal{C}_{\mathrm{c}}$. By Lemma 6.6 and Theorem 6.8, $G = \mathcal{C}_{\mathrm{c}}$, holomorphic on $\mathbb{H}$, satisfies the hypothesis of Lemma 6.2 at y, with $c = \mu_{\mathrm{c}}\{\tau_0\} \neq 0$. Lemma 6.2 rules out g_1. □

Proof of Theorem A(i) and (ii). Part (i) is Proposition 4.2(b). For (ii), in the sense of Section 1.1, let g be meromorphic on $D(s_1, \varepsilon)$, $\operatorname{Re} s_1 = 1$, and coincide with Φ_2 at every point of $D(s_1, \varepsilon) \cap \mathbb{H}$ where Φ_2 is analytic. The poles of Φ_2 in $\mathbb{H}$ have no accumulation point in $\mathbb{H}$, so g agrees with Φ_2 outside a set with no accumulation point in $D(s_1, \varepsilon) \cap \mathbb{H}$. Under H_∞, either $\mathcal{Z}_{\mathrm{off}}$ is infinite, and Corollary 6.4 applies, or it is finite, and Theorem 6.9 applies. □

In this proof H_∞ entered at three points: the local finiteness of the pair sums (Lemma 4.1); the bound $B < 1$, which gives the phase cone its margin and makes the strip of Proposition 4.2(c) non-empty; and the approach of the zeros ρ_n and ρ_n' to the critical line in Proposition 6.3. The decomposition of Section 3, the local count of the zeros and Littlewood's theorem need no hypothesis, and RH was used nowhere. No formula for the radial limit of Φ_2 under H_∞ is proved, and none is used; the limits of Corollary B(iii) and (iv) are statements under RH.

7. Sums of more than two prime powers

For $r \ge 2$ let

$$\Phi_r(s) = \sum_{k_1, \dots, k_r \ge 1} \frac{\Lambda(k_1) \cdots \Lambda(k_r)}{(k_1 + \cdots + k_r)^s},$$

which converges absolutely for $\operatorname{Re} s > r$, with Φ_2 as before. This section proves Theorem A(iii) by reducing it to Theorem A(ii); it is step (5). The reduction is an identity between Mellin transforms (Proposition 7.4), and nothing from Sections 5 and 6 is used until the last proof, which quotes the boundary of Φ_2. In $F^r = (1/x + h)^r$ the terms with at most two factors h are rewritten through F^2 and F, and they give Φ_2 shifted by $r-2$ times a factor with no zeros. The terms with three or more factors h need separate treatment. Under RH, $h = O(x^{-b})$ for every $b > \frac{1}{2}$, and they are too small to affect the continuation to the line $\operatorname{Re} s = r - 1$. Under H_∞ alone that bound need not hold: if a zero has $\beta > b$, its term $-a_\rho x^{-\rho}$ gives the Mellin transform of h over $(0, 1]$ a pole at ρ, which $h = O(x^{-b})$ would forbid; we keep the finitely many zeros with real part above a cutoff exactly, as powers $x^{-\rho}$ whose Mellin transforms are explicit, and bound only the rest. In the reduction itself H_∞ is used in two ways: it makes the set of zeros above the cutoff finite, and through $B < 1$ it leaves room for the cutoff. The last proof uses it once more, through Theorem A(i) and the boundary of Φ_2. For the earlier treatments of Φ_r see Section 9 and Remark 8.3.2.

7.1. **The cutoff.** Assume H_∞ for the rest of this section, let $B = \sup \beta$, and put

$$\delta_* = \frac{1 - B}{2} \in (0, \tfrac{1}{4}], \qquad \sigma_* = \tfrac{1}{2} + \frac{\delta_*}{3}, \qquad \mathcal{E}_* = \{\rho \in \mathcal{Z} : \beta > \sigma_*\}. \tag{19}$$

By H_∞, $\mathcal{E}_*$ is finite. Split $h = h_{\mathrm{ex}} + g$ with

$$h_{\mathrm{ex}}(x) = -\sum_{\rho \in \mathcal{E}_*} a_\rho x^{-\rho}, \qquad g(x) = h(x) - h_{\mathrm{ex}}(x) = -\sum_{\rho \notin \mathcal{E}_*} a_\rho x^{-\rho} + c_0 + V(x), \tag{20}$$

the second form by Lemma 3.5. A zero with real part exactly σ_* stays in g. The part h_{ex} is a finite sum of powers of x, which can be integrated exactly. The part g contains all but finitely many of the zeros, and a product of two or more factors g is not integrated exactly but bounded, using $g(x) = O(x^{-\sigma_*})$ with σ_* close to $\frac{1}{2}$. The cubic term $\binom{r}{3}x^{3-r}h^3$ of the expansion of $F^r = (1/x + h)^r$ shows why the split is needed. Bounding all of h by x^{-B} makes $x^{3-r}h^3$ of size x^{3-r-3B}, which gives integrability against x^{s-1} only for $\operatorname{Re} s > r - 3 + 3B$, and this lies left of $r - 1$ only when $B < \frac{2}{3}$. The split removes this restriction (Lemma 7.3 and the paragraph after it). The margins δ_* and $\delta_*/3$ are chosen so that the exponents counted in the proof of Lemma 7.3 leave the margin δ_* beyond the line $\operatorname{Re} s = r - 1$.

Lemma 7.1. *There are constants L_* and K_* with $|h_{\mathrm{ex}}(x)| \le L_* x^{-B}$ and $|g(x)| \le K_* x^{-\sigma_*}$ for $0 < x \le 1$.*

Proof. For $0 < x \le 1$, $|x^{-\rho}| = x^{-\beta} \le x^{-B}$, which gives the first bound with $L_* = \sum_{\mathcal{E}_*} |a_\rho|$. Every zero left in g has $\beta \le \sigma_*$, so $|x^{-\rho}| \le x^{-\sigma_*}$; also $1 \le x^{-\sigma_*}$ and $|V(x)| \le K_0 x^{1/2} \le K_0 x^{-\sigma_*}$. Take $K_* = \sum_{\rho \notin \mathcal{E}_*} |a_\rho| + |c_0| + K_0$. □

For $j \ge 1$ let $E_j(z) = \int_1^\infty F(x)^j x^{z-1}\,dx$, entire since $F(x) \le Ce^{-x}$ for $x \ge 1$ (so E_2 is the function of (8), and $E_1 = e_1$), and let

$$G_*(z) = \Gamma(z)\psi(z) - E_1(z) - \frac{1}{z-1} + \sum_{\rho \in \mathcal{E}_*} \frac{a_\rho}{z - \rho}. \tag{21}$$

The function G_* is the Mellin transform of g over $(0, 1]$, continued to $\mathbb{C}$, as the next lemma shows. It is needed for the terms of the expansion of F^r below that have exactly one factor g beside two or more factors h_{ex}, for which the bounds of Lemma 7.1 are not always enough and which are computed exactly instead. The sign in (21) cancels the residue $-a_\rho$ of $\Gamma\psi$ at an extracted zero: the zeros of $\mathcal{E}_*$ have been removed from g, and G_* has no pole at them.

Lemma 7.2. *G_* is meromorphic on $\mathbb{C}$, and $G_*(z) = \int_0^1 g(x)x^{z-1}\,dx$ for $\operatorname{Re} z > 1$, the integral converging absolutely.*

Proof. Meromorphy is clear. For $\operatorname{Re} z > 1$, (6) gives $\int_0^1 F(x)x^{z-1}\,dx = \Gamma(z)\psi(z) - E_1(z)$, and $\int_0^1 x^{-1}x^{z-1}\,dx = 1/(z-1)$ and $\int_0^1 x^{-\rho}x^{z-1}\,dx = 1/(z-\rho)$, since $\operatorname{Re}(z - \rho) > 0$. Subtract $\int_0^1 (x^{-1} + h_{\mathrm{ex}})x^{z-1}\,dx$ from the first. □

7.2. **The reduction.** Let $C_r = \binom{r}{2}$ and $A_r = \frac{1}{2}(r-1)(r-2)$. With $F = 1/x + h$ the binomial theorem gives, for $x > 0$,

$$F^r = C_r x^{2-r}F^2 - r(r-2)x^{1-r}F + A_r x^{-r} + Q_r, \qquad Q_r = \sum_{k=3}^{r} \binom{r}{k} x^{k-r}h^k. \tag{22}$$

Both sides agree in the powers h^0, h^1 and h^2: on the right their coefficients are $C_r - r(r-2) + A_r = 1$, $2C_r - r(r-2) = r$ and C_r, the binomial coefficients on the left; the higher powers are Q_r. With $h = h_{\mathrm{ex}} + g$, split $Q_r = P_r + R_r$ by the number of factors g:

$$P_r = \sum_{k=3}^{r} \binom{r}{k} x^{k-r}\bigl(h_{\mathrm{ex}}^k + k h_{\mathrm{ex}}^{k-1} g\bigr), \qquad R_r = \sum_{k=3}^{r} \binom{r}{k} x^{k-r} \sum_{j=2}^{k} \binom{k}{j} h_{\mathrm{ex}}^{k-j} g^j. \tag{23}$$

P_r collects the terms with at most one factor g. Their Mellin transforms over $(0, 1]$ are computed exactly, from powers of x and from G_*, and give the function $\mathcal{P}_r$ below, which is meromorphic on $\mathbb{C}$. R_r collects the terms with two or more factors g, which are small enough for their transform $\mathcal{H}_r$ to be holomorphic beyond the line $\operatorname{Re} s = r - 1$ (Lemma 7.3). For an ordered tuple

$\boldsymbol{\rho} = (\rho_1, \dots, \rho_p) \in \mathcal{E}_*^p$, repetitions allowed, put $\xi(\boldsymbol{\rho}) = \rho_1 + \cdots + \rho_p$ and $a(\boldsymbol{\rho}) = (-a_{\rho_1}) \cdots (-a_{\rho_p})$, so that $h_{\mathrm{ex}}^p = \sum_{\boldsymbol{\rho} \in \mathcal{E}_*^p} a(\boldsymbol{\rho}) x^{-\xi(\boldsymbol{\rho})}$, and let

$$\mathcal{P}_r(s) = \sum_{k=3}^{r} \binom{r}{k} \Big(\sum_{\boldsymbol{\rho} \in \mathcal{E}_*^k} \frac{a(\boldsymbol{\rho})}{s - r + k - \xi(\boldsymbol{\rho})} + k \sum_{\boldsymbol{\rho} \in \mathcal{E}_*^{k-1}} a(\boldsymbol{\rho})\, G_*\big(s - r + k - \xi(\boldsymbol{\rho})\big) \Big), \tag{24}$$

$$\mathcal{H}_r(s) = \int_0^1 R_r(x) x^{s-1}\, dx.$$

Lemma 7.3. *$\mathcal{P}_r$ is meromorphic on $\mathbb{C}$, and $\mathcal{P}_r(s) = \int_0^1 P_r(x) x^{s-1}\, dx$ for $\operatorname{Re} s > r$. The integral $\mathcal{H}_r$ converges absolutely and is holomorphic on $\operatorname{Re} s > r - 1 - \delta_*$.*

Proof. $\mathcal{P}_r$ is a finite sum of functions meromorphic on $\mathbb{C}$ (Lemma 7.2). Let $\operatorname{Re} s > r$. For $\boldsymbol{\rho} \in \mathcal{E}_*^k$, $\operatorname{Re}(s - r + k - \xi(\boldsymbol{\rho})) > k - kB > 0$, so $\int_0^1 x^{k-r} x^{-\xi(\boldsymbol{\rho})} x^{s-1}\, dx = 1/(s - r + k - \xi(\boldsymbol{\rho}))$. For $\boldsymbol{\rho} \in \mathcal{E}_*^{k-1}$, $\operatorname{Re}(s - r + k - \xi(\boldsymbol{\rho})) > k - (k-1)B = 1 + 2(k-1)\delta_* > 1$, so Lemma 7.2 gives $\int_0^1 x^{k-r} x^{-\xi(\boldsymbol{\rho})} g(x) x^{s-1}\, dx = G_*(s - r + k - \xi(\boldsymbol{\rho}))$. Summing proves the formula.

For $\mathcal{H}_r$, by Lemma 7.1 the term of R_r with $p = k - j$ factors h_{ex} and $j \geq 2$ factors g is at most $\binom{r}{k}\binom{k}{j} L_*^p K_*^j x^{-r + 2p\delta_* + j(1-\sigma_*)}$ on $(0,1]$, where $p + j = k \geq 3$ and $1 - B = 2\delta_*$. If $p \geq 1$, then $2p\delta_* + j(1-\sigma_*) \geq 2\delta_* + 2(\frac{1}{2} - \frac{\delta_*}{3}) = 1 + \frac{4}{3}\delta_* \geq 1 + \delta_*$. If $p = 0$, then $j \geq 3$ and $j(1-\sigma_*) \geq 3(\frac{1}{2} - \frac{\delta_*}{3}) = \frac{3}{2} - \delta_* \geq 1 + \delta_*$, the last step because $\delta_* \leq \frac{1}{4}$. Since $x \leq 1$, $|R_r(x)| \leq U_r x^{-(r-1-\delta_*)}$ with $U_r = \sum_{k,j} \binom{r}{k}\binom{k}{j} L_*^{k-j} K_*^j$, and R_r is continuous on $(0,1]$. So $|R_r(x) x^{s-1}| \leq U_r x^{\operatorname{Re} s - r + \delta_*}$, integrable on $(0,1]$ locally uniformly on $\operatorname{Re} s > r - 1 - \delta_*$, and $\mathcal{H}_r$ is holomorphic there. □

For the cubic term, which the crude bound after (20) carries across the line only when $B < \frac{2}{3}$, the cutoff works as follows: h_{ex}^3 and $3h_{\mathrm{ex}}^2 g$ go into the explicit $\mathcal{P}_r$, whose poles may lie to the right of the line but do not prevent continuation across it, and the bounds make $x^{3-r} \cdot 3h_{\mathrm{ex}} g^2$ and $x^{3-r} g^3$ absolutely integrable against x^{s-1} for $\operatorname{Re} s > r - 3 + B + 2\sigma_*$ and $\operatorname{Re} s > r - 3 + 3\sigma_*$, both thresholds less than $r - 1$.

The proposition assembles the pieces. Recall that P_r and R_r are functions of x, and that $\mathcal{P}_r$ and $\mathcal{H}_r$ are their Mellin transforms over $(0,1]$. In (26) the first term carries Φ_2, shifted by $r-2$, and the other two extend across the line $\operatorname{Re} s = r - 1$: M_r is meromorphic on $\mathbb{C}$, and $\mathcal{H}_r/\Gamma$ is holomorphic on $\operatorname{Re} s > r - 1 - \delta_*$.

Proposition 7.4 (The reduction)**.** *Let $r \geq 3$, and put*

$$c_r(s) = C_r \frac{\Gamma(s - r + 2)}{\Gamma(s)} = C_r \prod_{j=1}^{r-2} (s - j)^{-1},$$

$$\widetilde{E}_r(s) = E_r(s) - C_r E_2(s - r + 2) + r(r-2) E_1(s - r + 1), \tag{25}$$

$$M_r(s) = \frac{-r(r-2)\Gamma(s - r + 1)\psi(s - r + 1) + \frac{A_r}{s-r} + \widetilde{E}_r(s) + \mathcal{P}_r(s)}{\Gamma(s)}.$$

Then M_r is meromorphic on $\mathbb{C}$, and for $\operatorname{Re} s > r$

$$\Phi_r(s) = c_r(s)\Phi_2(s - r + 2) + M_r(s) + \frac{\mathcal{H}_r(s)}{\Gamma(s)}. \tag{26}$$

Proof. M_r is meromorphic on $\mathbb{C}$ because Γ, ψ, the E_j and $\mathcal{P}_r$ are and $1/\Gamma$ is entire. As in (6), $\int_0^\infty F(x)^r x^{s-1}\, dx = \Gamma(s)\Phi_r(s)$ for $\operatorname{Re} s > r$, the series converging absolutely there. Split the integral at $x = 1$ and expand the integrand on $(0,1]$ by (22). The term $C_r x^{2-r} F^2$ gives

$C_r\big(\Gamma(w)\Phi_2(w) - E_2(w)\big)$ with $w = s - r + 2$, $\operatorname{Re} w > 2$; the term $-r(r-2)x^{1-r}F$ gives $-r(r-2)\big(\Gamma(z)\psi(z) - E_1(z)\big)$ with $z = s - r + 1$, $\operatorname{Re} z > 1$; the term $A_r x^{-r}$ gives $A_r/(s-r)$; and $Q_r = P_r + R_r$ gives $\mathcal{P}_r(s) + \mathcal{H}_r(s)$ by Lemma 7.3. With the integral over $[1,\infty)$, which is $E_r(s)$,

$$\Gamma(s)\Phi_r(s) = C_r\Gamma(s-r+2)\Phi_2(s-r+2) - r(r-2)\Gamma(s-r+1)\psi(s-r+1) + \frac{A_r}{s-r} + \widetilde{E}_r(s) + \mathcal{P}_r(s) + \mathcal{H}_r(s).$$

Divide by $\Gamma(s)$, using $\Gamma(s-r+2)/\Gamma(s) = \prod_{j=1}^{r-2}(s-j)^{-1}$ (G1). □

The coefficient $C_r = \binom{r}{2}$ counts the ways to choose the two factors of $F^r = (1/x + h)^r$ that contribute h to the term with h^2; the other $r-2$ factors contribute the main term $1/x$, and their product x^{2-r} is the shift by $r-2$.

Proof of Theorem A(iii). For $r = 2$ this is Theorem A(i) and (ii). Let $r \geq 3$. On $\operatorname{Re} s > r - 1$ the right side of (26) is meromorphic: $\Phi_2(s-r+2)$ is meromorphic there by Theorem A(i), as $\operatorname{Re}(s-r+2) > 1$; c_r is a rational function; M_r is meromorphic on $\mathbb{C}$; and $\mathcal{H}_r/\Gamma$ is holomorphic on $\operatorname{Re} s > r - 1 - \delta_*$. It equals Φ_r on $\operatorname{Re} s > r$, so it is the continuation.

For the boundary, let $s_1 = r - 1 + i\tau$ and suppose f is meromorphic on a disc $D(s_1, \varepsilon)$ and agrees with Φ_r at every point of the disc with $\operatorname{Re} s > r - 1$ outside a set with no accumulation point there. Shrink ε below $\min(\delta_*/2, \frac{1}{4})$. On the smaller disc M_r and $\mathcal{H}_r/\Gamma$ are meromorphic, and c_r is holomorphic without zeros, its poles being at $1, \dots, r-2$, at distance at least 1 from s_1. So

$$\frac{f(s) - M_r(s) - \mathcal{H}_r(s)/\Gamma(s)}{c_r(s)}$$

is meromorphic on the disc and agrees with $\Phi_2(s-r+2)$ at the points of the disc with $\operatorname{Re} s > r-1$, outside a set with no accumulation point there. Translating by $r-2$ gives a function meromorphic on $D(1 + i\tau, \varepsilon)$ that agrees with Φ_2 in this sense on its part in $\mathbb{H}$, contrary to Corollary 6.4 or Theorem 6.9. This includes $\tau = 0$; poles of M_r near s_1 are allowed. Finally, the poles of Φ_r in $\operatorname{Re} s > r - 1$ have no accumulation point there, so a function that coincides with Φ_r wherever Φ_r is analytic agrees with it outside such a set; this gives the natural boundary in the sense of Section 1.1, as in the proof for Φ_2. □

All of the difficulty of the boundary in Theorem A(iii) lies in the case $r = 2$: the proof above uses Theorem A(i) for the continuation, and the non-cancellation of Section 5 and the two cases of Section 6 only through Corollary 6.4 and Theorem 6.9. The identity (26) by itself gives both directions. By it $\Phi_r(s) - c_r(s)\Phi_2(s-r+2)$ continues to a function meromorphic on $\operatorname{Re} s > r - 1 - \delta_*$, and c_r has neither zeros nor poles at distance less than 1 from the line $\operatorname{Re} s = r - 1$; so a meromorphic extension of either of Φ_r and Φ_2 across a point of its line would give one of the other across the corresponding point. Only the direction from Φ_r to Φ_2 is used here. As Bhowmik and Schlage-Puchta put it under RH, "the analytic behavior of Φ_r is completely determined by the behavior of Φ_2" [10, published version, p. 45].

8. Proofs of Corollary B and Theorem C, and remarks

This section derives Corollary B (Section 8.1) and Theorem C (Section 8.2) from Sections 3–7. The remarks of Section 8.3 comment on the results and are not used in any proof.

8.1. **Proof of Corollary B.** Assume RH. Every zero has $\beta = \frac{1}{2}$, so H_∞ holds with $B = \frac{1}{2}$, $\mathcal{Z}_{\mathrm{off}}$ is empty, $\mathcal{Z}_{\mathrm{c}} = \mathcal{Z}$, $\mathcal{C} = \mathcal{C}_{\mathrm{c}}$ and $\mu = \mu_{\mathrm{c}}$.

Continuation and (ii). RH implies H_∞, so Theorem A(i) gives the meromorphic continuation to $\mathbb{H}$ and Theorem A(ii) the natural boundary. What remains is to place the poles and residues of (i), and to prove (iii) and (iv). Under RH every pair sum $\rho+\rho'$ has real part 1, so $|s-\rho-\rho'| \geq \operatorname{Re} s - 1$ for $s \in \mathbb{H}$; hence $\mathcal{C}$ converges uniformly on $\operatorname{Re} s \geq 1+\varepsilon$ and is holomorphic on $\mathbb{H}$.

(i). The poles of L are the zeros, on $\operatorname{Re} s = \frac{1}{2}$, so L/Γ is holomorphic on $\operatorname{Re} s > \frac{1}{2}$, and so is $\mathcal{A}/\Gamma$ (Proposition 3.7). Since $M_0 = M - 2c_0 L/\Gamma$, (13) and Proposition 4.2(b) give, on $\mathbb{H}$,

$$\Phi_2 = M + H, \qquad H = \frac{\mathcal{C} + \mathcal{A} - 2c_0 L}{\Gamma},$$

with H holomorphic on $\mathbb{H}$. The poles of M in $\mathbb{H}$ are among those of $\psi(s-1)$, at $s-1=1$ and $s-1 \in \mathcal{Z}$, that is, at $s=2$ and at the points $1+\rho$, and of $1/(s-2)$; the factor $1/(s-1)$ has its pole at $s=1 \notin \mathbb{H}$. At every other $s \in \mathbb{H}$, $\zeta(s-1) \neq 0$ (for $\operatorname{Re}(s-1) \geq 1$ as ζ has no zeros there, and for $0 < \operatorname{Re}(s-1) < 1$ as a zero would make s one of the points $1+\rho$) and $s-1 \neq 1$, so M and Φ_2 are analytic at s. At $s=2$ the terms of $(s-2)M(s)$ tend to 2 and -1, and $(s-2)H(s) \to 0$, so the residue is 1. At $s = 1+\rho$, which is not 2 since $\beta < 1$, $(s-1-\rho)\cdot 2\psi(s-1)/(s-1) \to -2m(\rho)/\rho \neq 0$ and the other terms tend to 0. A nonzero limit of $(s-\alpha)f(s)$ at α means a pole of order exactly one.

(iii). Let $\tau \in \mathbb{R}$ and $s = 1 + i\tau + \eta$ with $0 < \eta \leq 1$. As $\eta \downarrow 0$: $\eta\mathcal{A}(s) \to 0$ and $\eta L(s) \to 0$, since $\mathcal{A}$ and L are analytic at $1+i\tau$; $\eta\mathcal{C}(s) \to \mu\{\tau\}$ by Lemma 6.6; and $1/\Gamma(s) \to 1/\Gamma(1+i\tau)$, with $\Gamma(1+i\tau) \neq 0$ by (G1). So $\eta H(s) \to \mu\{\tau\}/\Gamma(1+i\tau)$.

(iv). For $r=2$ the residues are (i). Let $r \geq 3$. Under RH, $\delta_* = \frac{1}{4}$ and $\sigma_* = \frac{7}{12}$, so $\mathcal{E}_*$ is empty, $h_{\mathrm{ex}} = 0$ and $\mathcal{P}_r = 0$, since every term of (24) has at least two factors from $\mathcal{E}_*$. So in (26)

$$M_r(s) = \frac{-r(r-2)\Gamma(s-r+1)\psi(s-r+1) + \frac{A_r}{s-r} + \widetilde{E}_r(s)}{\Gamma(s)},$$

and $\mathcal{H}_r/\Gamma$ is holomorphic on $\operatorname{Re} s > r - \frac{5}{4}$. On $\operatorname{Re} s > r-1$, $\Gamma(s-r+1)$ is analytic, and the poles of $\psi(s-r+1)$ there are at $s=r$ and $s = r-1+\rho$; the term $A_r/(s-r)$ has its pole at $s=r$; the poles of $c_r(s)\Phi_2(s-r+2)$ are at the same points by (i), and c_r is analytic there. Each of these terms has at most a simple pole at each of these points, so the continuation of Φ_r has at most a simple pole there, and elsewhere in $\operatorname{Re} s > r-1$ it is analytic. At $s=r$ the residue is

$$\frac{C_r\Gamma(2)\cdot 1 - r(r-2)\Gamma(1)\cdot 1 + A_r}{\Gamma(r)} = \frac{1}{(r-1)!},$$

by the coefficient identity after (22). At $s = r-1+\rho$, which is not r, with $\Gamma(1+\rho) = \rho\Gamma(\rho)$, it is

$$\frac{C_r\,\rho\Gamma(\rho)\cdot\big(-2m(\rho)/\rho\big) + r(r-2)\Gamma(\rho)m(\rho)}{\Gamma(r-1+\rho)} = \frac{-r\,m(\rho)\Gamma(\rho)}{\Gamma(r-1+\rho)} \neq 0.$$

Both residues are nonzero, so these poles are simple.

For the limit, let $\tau \neq 0$ and $s = r-1+i\tau+\eta$. M_r is analytic at $r-1+i\tau$, since $\Gamma(i\tau)$ is finite, $\zeta(i\tau)$ is finite and nonzero (Section 2) and $i\tau \neq 1$; so $\eta M_r(s) \to 0$, and $\eta\mathcal{H}_r(s)/\Gamma(s) \to 0$. Likewise M is analytic at $1+i\tau$, as $\psi(i\tau)$ is finite and $i\tau \neq 0$, so by (iii) $\eta\Phi_2(1+i\tau+\eta) \to \mu\{\tau\}/\Gamma(1+i\tau)$. Since $c_r(r-1+i\tau) = C_r\Gamma(1+i\tau)/\Gamma(r-1+i\tau)$, (26) gives

$$\eta\,\Phi_r(r-1+i\tau+\eta) \to C_r\frac{\Gamma(1+i\tau)}{\Gamma(r-1+i\tau)}\cdot\frac{\mu\{\tau\}}{\Gamma(1+i\tau)} = \binom{r}{2}\frac{\mu\{\tau\}}{\Gamma(r-1+i\tau)}.$$

For $r=2$ the same argument, with $c_2 = 1$ and no M_2, is the line just above. This proves Corollary B. □

8.2. **Proof of Theorem C.** Assume H_∞. Part (i) takes three steps: B is attained; every representation of $2B$ is a zero paired with its conjugate, so that $\nu(2B) > 0$; and $2B$ is none of the points excluded in Proposition 4.2(c), which then gives the pole. Part (ii) follows from (i) and Corollary B(i). The third step and part (ii) also use (Z5): ζ has no real zero in $(0,1)$.

(i), B is attained. Let some $\rho_0 \in \mathcal{Z}$ have $\beta_0 \neq \frac{1}{2}$. Of the zeros ρ_0 and $1-\bar{\rho}_0$ one has real part greater than $\frac{1}{2}$, so $B > \frac{1}{2}$, and $B < 1$ by Lemma 2.1(a). Put $\delta = \frac{1}{2}(B-\frac{1}{2})$, so that $\frac{1}{2}+\delta < B$. By H_∞ only finitely many zeros have $\beta > \frac{1}{2}+\delta$, and there is at least one, since B is the supremum of the real parts; every other zero has $\beta \leq \frac{1}{2}+\delta < B$. So B is the largest real part among finitely many zeros: it is attained, and only finitely many zeros have $\beta = B$.

(i), the coefficient $\nu(2B)$. The representations of $2B$ are the pairs $(\rho, \bar{\rho})$ with $\beta = B$. Indeed $\rho + \rho' = 2B$ means $\beta + \beta' = 2B$ and $\gamma + \gamma' = 0$; since $\beta, \beta' \leq B$, this forces $\beta = \beta' = B$ and $\rho' = \bar{\rho}$. Conversely, every $\rho \in \mathcal{Z}$ with $\beta = B$ gives the representation $(\rho, \bar{\rho})$, since $\bar{\rho} \in \mathcal{Z}$, and $m(\bar{\rho}) = m(\rho)$. By (G1), $a_\rho a_{\bar{\rho}} = m(\rho)^2 \Gamma(\rho)\overline{\Gamma(\rho)} = m(\rho)^2 |\Gamma(\rho)|^2$, so

$$\nu(2B) = \sum_{\rho \in \mathcal{Z},\ \beta = B} m(\rho)^2 |\Gamma(\rho)|^2,$$

a finite sum, not empty, of positive terms, since Γ has no zeros.

(i), the pole. The point $2B$ lies in $\mathbb{H}$, since $B > \frac{1}{2}$. It is not 2, since $B < 1$, and it is not $1+\rho$ for any $\rho \in \mathcal{Z}$: that would make $\rho = 2B - 1$ a real zero in $(0,1)$. Proposition 4.2(c) then gives $(s-2B)\Phi_2(s) \to \nu(2B)/\Gamma(2B)$, which is positive because $\Gamma(2B) > 0$. So $2B$ is a simple pole with this residue.

(ii). If RH fails, some $\rho \in \mathcal{Z}$ has $\beta \neq \frac{1}{2}$, and by (i) the point $2B \in (1,2)$ is a pole. If RH holds, Corollary B(i) leaves as possible poles in $\mathbb{H}$ only 2 and the points $1+\rho$. The point 2 is not in $(1,2)$, and a point $1+\rho$ in $(1,2)$ would make ρ a real zero in $(0,1)$. So Φ_2 has no pole in $(1,2)$. □

8.3. **Remarks.**

Remark 8.3.1 (The radial coefficient in [18] and in [6]). Assume RH. For real $\tau \neq 0$, M is analytic at $1+i\tau$, and Corollary B(iii) gives $\lim_{\eta \downarrow 0} \eta\, \Phi_2(1+i\tau+\eta) = \mu\{\tau\}/\Gamma(1+i\tau)$. If τ has a unique unordered representation $\tau = \gamma + \gamma'$ by the ordinates of zeros ρ, ρ', then $\mu\{\tau\} = 2m(\rho)m(\rho')\Gamma(\rho)\Gamma(\rho')$ if $\gamma \neq \gamma'$, from the ordered pairs (ρ, ρ') and (ρ', ρ), and $\mu\{\tau\} = m(\rho)^2\Gamma(\rho)^2$ if $\gamma = \gamma'$, from the single pair (ρ, ρ). In the first case this agrees with [18, (4.11)], obtained there under RH and (B) and printed with $s \to 1+i\tau$; the approach along the normal is the one the proof there uses [18, p. 12]. That formula is printed for any unique representation, and hypothesis (A) does not exclude the diagonal (Section 9 states (A) and (B)); in the second case its factor 2 is absent from ours. This does not affect [18, Theorem 2.2], whose proof uses (4.11) only through the nonvanishing of the residue. At $\tau = 2\gamma_1$, with γ_1 the first positive ordinate, the diagonal term is the main term of [6, (3.3)], with which ours agrees.

For $\tau = 0$ the term $2\psi(s-1)/(s-1)$ of M has a pole at $s = 1$ with residue $2\psi(0) = -2\log 2\pi$, so

$$\lim_{\sigma \downarrow 1} (\sigma - 1)\Phi_2(\sigma) = \mu\{0\} - 2\log 2\pi, \qquad \mu\{0\} = \sum_{\rho \in \mathcal{Z}} m(\rho)^2 |\Gamma(\rho)|^2,$$

the sum running over the ordered pairs $(\rho, \bar{\rho})$ of all zeros. This limit is negative. For real t we have $\cosh \pi t \geq (\frac{1}{4}+t^2)^2$: the first three terms of the power series give $\cosh \pi t \geq 1 + \frac{1}{2}\pi^2 t^2 + \frac{1}{24}\pi^4 t^4$, which is at least $\frac{1}{16} + \frac{1}{2}t^2 + t^4$ term by term. With (G2), and since a sum of squares of non-negative numbers is at most the square of their sum,

$$\mu\{0\} = \sum_{\rho \in \mathcal{Z}} \frac{\pi\, m(\rho)^2}{\cosh \pi\gamma} \leq \pi \sum_{\rho \in \mathcal{Z}} \frac{m(\rho)^2}{(\frac{1}{4}+\gamma^2)^2} \leq \pi D^2, \qquad D = \sum_{\rho \in \mathcal{Z}} \frac{m(\rho)}{\frac{1}{4}+\gamma^2}.$$

Davenport [16, Chapter 12] gives $\sum_\rho 1/\rho = 1 + \frac{1}{2}\gamma_{\mathrm{E}} - \frac{1}{2}\log 4\pi$, where γ_{E} is Euler's constant, the zeros are counted with multiplicity, and the terms of ρ and $\bar\rho$ are taken together. Under RH, $1/\rho + 1/\bar\rho = 1/(\frac{1}{4} + \gamma^2)$, so D is twice that sum: $D = 2 + \gamma_{\mathrm{E}} - \log 4\pi < 0.05$. Hence $\mu\{0\} < 0.008$ and $\mu\{0\} - 2\log 2\pi < -3.66$.

The value printed at $t = 0$ in [6, p. 297], $\sum_\rho |\Gamma(\rho)|^2 = 3.259\ldots \cdot 10^{-19}$, agrees, to the digits printed, with half of $\mu\{0\} \approx 6.518 \cdot 10^{-19}$, that is, with one contribution from each conjugate pair of zeros; no term $-2\log 2\pi$ from M appears in it. The constant called c_2 in [6, Proposition 1.3], which we write c_2^{BRS} to keep it apart from the factor $c_r(s)$ of Section 7, is $\lim_{\sigma\downarrow 2}(\sigma - 2)\int_1^\infty \Delta_2(x) x^{-\sigma}\,dx$, where $\Delta_2(x) = \sum_{n\le x} G_2(n) - \frac{1}{2}x^2 - H_2(x)$ and $H_2(x) = -2\sum_\rho x^{1+\rho}/(\rho(1+\rho))$ [6, p. 293]. It is the limit above. By partial summation $\int_1^\infty \big(\sum_{n\le x} G_2(n)\big) x^{-s-1}\,dx = \Phi_2(s)/s$ for $\operatorname{Re} s > 2$, and the same transforms of $\frac{1}{2}x^2$ and of H_2, which are $1/(2(s-2))$ and a sum over the zeros with poles only at the points $1+\rho$, are analytic at $s = 1$; so $\int_1^\infty \Delta_2(x) x^{-s-1}\,dx$ equals $\Phi_2(s)/s$ plus a function analytic at $s = 1$, first for $\operatorname{Re} s > 2$ and then for $\operatorname{Re} s > 1$, where the integral converges because $\Delta_2(x) = O(x\log^3 x)$ [34, Theorem 1]. Hence $c_2^{\mathrm{BRS}} = \mu\{0\} - 2\log 2\pi < -3.66$, where [6, p. 298] use $3.259\ldots \cdot 10^{-19}$. We write $f(x) = \Omega_+(x)$ if $\limsup_{x\to\infty} f(x)/x > 0$, and $f(x) = \Omega_-(x)$ if $\liminf_{x\to\infty} f(x)/x < 0$. From the limit at $2\gamma_1$ they find that the hypothesis (1.2) of their Theorem 1.2 holds with a constant at least 0.4713, and conclude that $\Delta_2(x)$, less their value of the constant, is "infinitely often" larger than 0.4713 and smaller than -0.4713 [6, p. 298]; with the factor x of their Theorem 1.2, in their reading this is $\Delta_2(x) - c_2^{\mathrm{BRS}} x > 0.4713\,x$ and $\Delta_2(x) - c_2^{\mathrm{BRS}} x < -0.4713\,x$, each for arbitrarily large x (with the factor $1/s$ of the partial summation, the diagonal contribution at $2\gamma_1$ has modulus approximately 0.0167; the conclusion below does not depend on it). Since $c_2^{\mathrm{BRS}} < -3.66$, the first gives only $\Delta_2(x) > (c_2^{\mathrm{BRS}} + 0.4713)\,x$, a negative multiple of x, and so not the conclusion $\Delta_2(x) = \Omega_+(x)$ of their Corollary 1.5. The second gives $\Delta_2(x) = \Omega_-(x)$, which also follows from the real limit alone: that limit forces $\liminf_{x\to\infty} \Delta_2(x)/x \le c_2^{\mathrm{BRS}} < 0$. Whether Ω_+ holds by another argument we do not examine.

The value $\mu\{0\} \approx 6.518 \cdot 10^{-19}$, with which the limit is about -3.6758, is a numerical illustration, not a certified enclosure, and the bound above does not use it: it is the sum over the first ten positive ordinates and their conjugates, with $m(\rho) = 1$ for these zeros and $|\Gamma(\frac{1}{2} + i\gamma)|^2 = \pi/\cosh \pi\gamma$ by (G2), computed in floating point with the mpmath library at 40 significant digits, using its values of the ordinates.

Remark 8.3.2 (The kernel bound in [10]). Bhowmik and Schlage-Puchta state the expansion of Φ_r through Φ_2 under RH, with an error holomorphic on $\operatorname{Re} s > r - \frac{11}{10}$ [10, arXiv v1, Theorem 4; published version, Theorem 3], and prove it by the circle method with a bound $\min(x^3, \alpha^{-3})$ [10, arXiv v1, pp. 3–4; published version, pp. 45, 47] for the kernel $T_3(\alpha) = \sum_{|n|\le x} (x - |n|)^2 e^{2\pi i n\alpha}$, x a large integer, which does not hold as printed: $T_3(\frac{1}{2}) = x^2 + 2\sum_{n=1}^{x-1}(x-n)^2(-1)^n = x$, while $\min(x^3, 2^3) = 8$, so no bound $\ll \min(x^3, \alpha^{-3})$ holds uniformly in x. We have not examined whether their theorem survives with a corrected bound; Proposition 7.4 gives the reduction independently, through the Mellin transform instead of the circle method and under H_∞ instead of RH, and under RH its error term $\mathcal{H}_r/\Gamma$ is holomorphic on $\operatorname{Re} s > r - \frac{5}{4}$ (Section 8.1).

Remark 8.3.3 (Thresholds). The proofs use only the existence of the threshold X, which depends on B: the margin $\lambda_B = \cos(\frac{\pi}{2}B)$ of Lemma 5.2 tends to 0 as $B \to 1$, and no uniformity as $B \to 1$ is claimed. The tail condition of Theorem 5.4 depends on the constant κ_0 of Lemma 5.5, hence on the constant C_1 of (Z1), which we have not made explicit and which a finite table of zeros cannot replace, since the tail needs the local count at all heights.

9. Earlier work

This section states the hypotheses (A) and (B) to which the earlier sections refer, and compares the earlier results and their proofs with ours; nothing in it is used in the proofs. In brief, Theorem 5.6 takes the place of the uniqueness that (A) supplies, at the points all of whose representations are high, and the limit along the normal (Lemma 6.6) separates distinct pair sums without the quantitative separation of (B). Bhowmik and Schlage-Puchta had already replaced (B) by (A), and the termwise limit along the normal, by dominated convergence, is that of [6, p. 297]; the details follow.

Under RH write $\rho = \frac{1}{2} + i\gamma$ and let I be the set of ordinates. Egami and Matsumoto state two hypotheses on I [18, p. 7]:

(A) If $\gamma_j \in I$ $(1 \le j \le 4)$ and $\gamma_1+\gamma_2 = \gamma_3+\gamma_4$ $(\neq 0)$, then (γ_3,γ_4) equals (γ_1,γ_2) or (γ_2,γ_1).
(B) There exists a constant α, with $0 < \alpha < \pi/2$, such that if $\gamma_j \in I$ $(1 \le j \le 4)$, $\gamma_1+\gamma_2 \neq 0$, and (γ_3,γ_4) is neither equal to (γ_1,γ_2) nor to (γ_2,γ_1), then $|(\gamma_1+\gamma_2)-(\gamma_3+\gamma_4)| \ge \exp(-\alpha(|\gamma_1|+|\gamma_2|+|\gamma_3|+|\gamma_4|))$.

Hypothesis (A) is a special case of the conjectured linear independence of the positive ordinates over $\mathbb{Q}$, the conjecture used by Ingham [28] (see also [23, p. 50]), and (B) implies (A). Theorem 2.2 of [18] states that under RH and (B) the conjecture holds: (A) makes the residue at $1+i\tau$ a single nonzero product of Gamma values, and (B) bounds the near-coincident terms of the remaining sum [18, (4.9)–(4.10)]. For Φ_r they prove the continuation with its residues [18, Theorem 6.1] and deduce the boundary, their Conjecture 6.1, under RH, (B) and a further hypothesis $(\mathrm{C})_k$ $(k \le r)$ [18, Theorem 6.2]; under those hypotheses the radial specialization of [18, (6.6)] agrees with the limit in Corollary B(iv) at a nonzero sum having a unique unordered representation by two distinct zeros.

Bhowmik and Schlage-Puchta [10] replaced (B) by (A), their Conjecture 1, under RH. Their Theorem 2 has four parts. (1) Under RH, Φ_r has a natural boundary at $\operatorname{Re} s = r-1$ for all $r \ge 2$ if and only if Φ_2 has one at $\operatorname{Re} s = 1$. (2) Under RH and Conjecture 1, Φ_2 has a natural boundary at $\operatorname{Re} s = 1$. (3) Under RH, Φ_2 has a singularity at $2\rho_1$, where $\rho_1 = \frac{1}{2} + 14.1347\ldots i$ is the first zero of ζ. (4) Let σ_0 be the infimum of the real numbers σ for which ζ has only finitely many zeros with real part greater than σ, and let $\mathcal{D}_2$ be the domain of meromorphic continuation of Φ_2; if their Conjecture 2 holds, a form of (B) for the ordinates of all non-trivial zeros, then $\{\operatorname{Re} s > 2\sigma_0\} \subseteq \mathcal{D}_2 \subseteq \{\operatorname{Re} s > 1\}$. Their proof of (2) and (3) compares a chosen pair with all pairs of much greater height, which are exponentially smaller; among the finitely many pairs of comparable height that remain, (2) rules out an exact coincidence by Conjecture 1 and (3) by a finite computation at $2\rho_1$; they remark that the finite-range form of Conjecture 1 that this needs "seems as unreachable as Conjecture 1" in general [10, p. 9]. Section 5 deals with that finite set, and Theorem C(i) has its precedent in (3): one named point is shown to be singular because the pairs of zeros whose sums land on it cannot cancel. H_∞ is the case $\sigma_0 = \frac{1}{2}$ of (4), which then reads $\mathcal{D}_2 = \{\operatorname{Re} s > 1\}$. We read Conjecture 2, as (B), with the restriction $\gamma_1+\gamma_2 \neq 0$, since as printed it is violated by the pairs $(\gamma,-\gamma)$ and $(\gamma',-\gamma')$, which have the same sum 0; so read, it does not cover the real points 2β, and their remark that under it every sum $\rho+\rho'$ of two zeros is a singularity of Φ_2 [10, p. 7] does not give the pole at $2B$. Apart from the density of the pair sums, which they prove in full without RH [10, Lemma 3], the proof of (4) is by reference to [18, Theorem 2.1 and Lemma 4.2], both proved under RH [10, p. 7]; for that reason we do not count (4) as a complete proof. Remark 8.3.2 concerns their expansion of Φ_r through Φ_2.

Bhowmik, Ramaré and Schlage-Puchta [6] take the radial limit of the pair contribution termwise, by dominated convergence [6, p. 297], as Lemma 6.6 does, and work at the ordinate sum $2\gamma_1$, where the only same-sign representation is the diagonal pair and a finite check

rules out the other pairs of comparable height [6, pp. 297–298]. Remark 8.3.1 compares their coefficients with ours.

Tanigawa and Zhai [46, Corollary 1.4] state, under RH alone, that the series of a class containing Φ_2 and Φ_r "cannot be continued meromorphically beyond some half-plane" [46, p. 469]. Egami and Matsumoto write of the proof of the natural-boundary claim in [46, Theorem 1.3], which the corollary applies, that it "seems incomplete; some condition similar to our (B) below seems to be necessary to verify their argument" [18, p. 5], and we read it the same way. The proofs for Φ_2 and Φ_r end by concluding that the series is meromorphic on the half-plane [46, p. 484; p. 487, Case 2], and the corollary reads the boundary off the density, on the line, of the sums of poles of the factors [46, p. 488]; the missing step is that accumulating poles of the individual terms do not by themselves make the limit points singular for the sum, because the residues of the terms that land on one point may cancel. The objection is to the proof, not to the statement.

Table 1 lists the hypotheses under which the natural boundary of Φ_2 has been stated. Every earlier complete proof of the full natural boundary listed there assumes RH.

Source	Hypotheses	Remark
[18, Theorem 2.2]	RH and (B)	(B) implies (A)
[10, Theorem 2(2)]	RH and their Conjecture 1, which is (A)	
[41, Theorem 5.4]	RH and linear independence of the positive ordinates	survey statement
[10, Theorem 2(4)]	H_∞ and their Conjecture 2, a form of (B) for all zeros, in our reading with $\gamma_1 + \gamma_2 \neq 0$	proof partly by reference
[46, Corollary 1.4]	RH	half-plane form; proof read as incomplete (see text)
Theorem A	H_∞	

TABLE 1. Hypotheses under which $\operatorname{Re} s = 1$ has been stated to be a natural boundary of Φ_2. For [10, Theorem 2(4)], H_∞ is the case $\sigma_0 = \frac{1}{2}$ of their statement.

Egami and Matsumoto [19, Conjecture 1.1] restate the conjecture as open, and for a function-field analogue of Φ_2 they prove, with no hypothesis, continuation to the whole plane when the characteristics of the two function fields are equal and the natural boundary $\operatorname{Re} s = 2$ when they differ [19, Theorems 2.2 and 2.3]. Other analogues are stated under hypotheses that limit coincidences among the ordinates: the series over $m \equiv a$, $n \equiv b \pmod q$ under an analogue of (B) and the Generalized Riemann Hypothesis up to a possible exceptional zero [45, Theorems 2 and 3]; the Liouville analogue $\sum \lambda(m)\lambda(n)(m+n)^{-s}$ under RH, simple zeros and linear independence [14, Corollary 2]; and, under RH and a hypothesis that limits coincidences to finitely many, a family in which the term for (m, n) carries the weight $H(\alpha \log(m+n))$, for suitable periodic H and $\alpha > 0$, with natural boundary $\operatorname{Re} s = \frac{3}{2}$, whose singularities come from single zeros [48, Theorem 1]. For a random version of Φ_r, with independent signs ± 1 on the terms, Bhowmik and Matsumoto prove that the abscissa of convergence is almost surely a natural boundary and lies at or to the left of $\operatorname{Re} s = r/2$ [5, Propositions 1 and 2]. Peter's continuation theorem for Dirichlet series attached to polynomials [44, Theorem 1.3, pp. 246–247] requires auxiliary series with only finitely many poles in each vertical strip of finite width, and does not apply to Φ_r, whose auxiliary series is $-\zeta'/\zeta$. Natural boundaries of Dirichlet series over primes go back to the prime zeta function $P(s) = \sum_p p^{-s}$, with natural boundary $\operatorname{Re} s = 0$ (Kluyver [29] and Landau [31, pp. 157–158] for $e^{P(s)}$ under RH, Landau and Walfisz [32] for

P without RH), and to the Euler products $\prod_p h(p^{-s})$ of Estermann [20] and Dahlquist [15, Theorem I, p. 533].

An identity of the shape of (13) is not new. Egami and Matsumoto [18, (3.9) and (4.2)], whose identity is restated as [10, Lemma 4], and Cantarini, Gambini and Zaccagnini, with no hypothesis [13, §5, (28) and the display after it], write Φ_2 as a sum of simple poles at the pair sums plus a remainder. The identity of [13] is obtained from the explicit formula of [12, Theorem] for the Cesàro mean of order 1, which extends [35, Theorem 1], as corrected in [33, Theorem 2.2 and §6]. In [13], apart from explicit terms with poles at $s = 2$ and $s = 1 + \rho$, the error term is holomorphic on $\operatorname{Re} s > 1$, and with the remainder of [13, Remark 8] it is holomorphic to the left of $\operatorname{Re} s = 1$, though they do not put it in that form. So what (13) adds is not a remainder that reaches further left, but the normalization: the pair series $\mathcal{C}$ carries the poles, and $\mathcal{A}/\Gamma$ is holomorphic on $\operatorname{Re} s > \frac{1}{2}$ under no hypothesis on the zeros.

RH implies H_∞, and H_∞ implies the Lindelöf Hypothesis by Backlund's criterion [47, Theorem 13.5, p. 331]; zero-density estimates, whose bounds grow with the height, do not give H_∞. H_∞ implies $B < 1$, a zero-free half-plane.[2] Such a half-plane is related to the mean value of the Goldbach coefficients. Under RH, $\sum_{N \le X} G_2(N) = \frac{1}{2}X^2 - 2\sum_\rho X^{\rho+1}/(\rho(\rho+1)) + O(X \log^3 X)$ [34, Theorem 1]; this improves [9, Theorem 1.1] and comes after Fujii's error term $O(X^{3/2})$ [21, Theorem, p. 174] and his explicit term of order $X^{3/2}$, the sum over the poles at $1 + \rho$, which he found with the main term $\sum_{n \le X} nS_2(n)$, $S_2(n)$ the singular series [22, Theorem]. The estimate $\sum_{N \le X} G_2(N) = \frac{1}{2}X^2 + O(X^{2-\delta})$ for some $\delta > 0$ implies $B < 1$ [7, Theorem 2.1]. RH is equivalent to $\sum_{2N \le x}(G(2N) - J(2N)) \ll x^{3/2+o(1)}$, where $G(2N) = \sum_{p+q=2N} \log p \log q$ is taken over primes and $J(2N)$ is the main term of Hardy and Littlewood [24, Theorem 1A]. The corrigendum [25] keeps Theorem 1A.

A criterion for RH through the poles of Φ_2 is known. When $B < 1$, $1 + B$ is the infimum of the $\sigma \ge \frac{3}{2}$ for which $\Phi_2(s) - 1/(s-2)$ extends analytically to the half-plane to the right of σ; this is the case $q = 1$ of [4, Proposition 3], written out for Φ_2 in [3, Section 4, (13)]. So, when $B < 1$, which H_∞ implies, RH holds if and only if $\Phi_2(s) - 1/(s-2)$ extends analytically to $\operatorname{Re} s > \frac{3}{2}$, and Theorem C(ii) is new only in form. That criterion detects a zero off the line through the pole at $1 + \rho$; Theorem C detects it through the pairs $(\rho, \bar{\rho})$, whose weights $m(\rho)^2|\Gamma(\rho)|^2$ are positive and cannot cancel.

For individual even integers, Li and Liu [36], in a preprint, derive the binary Goldbach conjecture for all sufficiently large even integers from the Generalized Riemann Hypothesis for Dirichlet L-functions together with either of two further spectral hypotheses, their Hypotheses S and R [36, (2.28), p. 12; (2.30), p. 13; Theorem 6.1, p. 35]; our results concern the generating series.

10. Formal verification

Theorems A and C and Corollary B have been checked in the Lean 4 proof assistant (Lean 4.34.0, Mathlib [40] at tag v4.34.0, PrimeNumberTheoremAnd [30] at commit `c39a751`), in two developments: Theorems A and C in a development under H_∞, and Corollary B in a development under RH. Each formal theorem is an implication from named hypotheses, each written in Lean as a proposition about Mathlib's `riemannZeta`, its zeros and Γ and taken as a hypothesis, not as an axiom; what is machine-checked is that the hypotheses imply the theorem. Everything else, from (Z4), (Z5) and (G1)–(G3) to Mellin inversion and the residue theorem, is proved in the developments or taken from the two libraries. Table 2 matches the formal names of the classical inputs, which differ between the two developments, with the paper's labels.

[2] A zero-free half-plane with $B \le 7/8$ is asserted in an unrefereed 2026 preprint of OpenAI [43], with a Lean formalization in its repository, and a sharper bound in an unrefereed preprint of Liu [39]. We use neither: a fixed zero-free half-plane, unlike H_∞, does not make high zeros approach the critical line.

Paper	H_∞ development	RH development
H_∞	`Hinf`	(none)
RH	`RiemannHypothesis` (Mathlib's; in Theorem C(ii) only)	`RiemannHypothesis`
symmetries of the zeros, with multiplicities (derived in Section 2)	`Z1`	(none)
(Z1), the local count	second clause of `Z2`	`RiemannVonMangoldt`
Riemann–von Mangoldt formula for $N(T)$ [47, Theorem 9.4, p. 214] (not used)	first clause of `Z2`	(none)
(Z2), Littlewood's gap theorem	`ZLittlewood`	`LittlewoodGap`
(Z3), the local expansion of ψ	`Z3`	`LogDerivZetaLocal`
(Z5), $\zeta(\sigma) \neq 0$ on $(0,1)$	`ZetaNoZeroIoo`, a premise proved in the development	(none)
(G4), Stirling on strips	`StirlingStrip`	(none)

TABLE 2. Inputs of Section 2 and the formal hypotheses that state them; "(none)" means that the development does not assume the input. Note that `Z1` is not (Z1), and that `RiemannVonMangoldt` is the local count (Z1), not the Riemann–von Mangoldt formula.

Theorem A is `Phi2HInf.phi2_hinf_natural_boundary`, for parts (i) and (ii), and `Phi2HInf.phiR_hinf_natural_boundary`, for part (iii) and every $r \geq 2$, both in the H_∞ development, with hypotheses `Hinf`, `Z1`, `Z2`, `Z3`, `ZLittlewood` and `StirlingStrip`. These are slightly stronger than the paper's inputs: `Z1` assumes the symmetries of the zeros, which Section 2 derives from the functional equation, and `Z2` adds to the local count (Z1) the Riemann–von Mangoldt formula, which is not used. Their boundary clause excludes agreement outside a set without accumulation point; the natural boundary in the sense of Section 1.1 is the case where the agreement is at every point at which Φ_2 is analytic, since the poles of Φ_2 in $\mathbb{H}$ have no accumulation point in $\mathbb{H}$, and that form is stated separately, with the same hypotheses, as `Phi2HInf.phi2_hinf_natural_boundary_classical` and `Phi2HInf.phiR_hinf_natural_boundary_classical`.

The intermediate results of Sections 3–7 and of Section 8.2 have formal counterparts in the same development, which the repository documentation lists by section.

Corollary B is, in the RH development, `phi2_natural_boundary`, whose first clause is part (i) with the residues and whose second, codiscrete clause, applied to that continuation, is part (ii); `Phi2Proof.tendsto_eta_Hphi2`, which is part (iii); and `phiR_natural_boundary` and `phiR_radial_limit`, which are part (iv). Their hypotheses are `RiemannHypothesis`, `RiemannVonMangoldt`, `LittlewoodGap` and `LogDerivZetaLocal`; the radial limits need only the first two and the last, that is, RH and the paper's (Z1) and (Z3). That development proves Corollary B by its own route, a contour shift under RH that gives $h = O(x^{-b})$ for every $b > \frac{1}{2}$, and defines H as $(E_2(s) - 2e_1(s-1) + R(s))/\Gamma(s)$, which equals $\Phi_2 - M$ on $\mathbb{H}$ by Lemma 3.1 and the identity theorem; the derivation of Section 8.1 from the machinery of Theorem A is not formalized in that form. It takes no Stirling hypothesis: on the critical line it has the modulus of the weights from the reflection formula, as in (G2), and it proves its own phase cone there from Euler's limit formula for Γ. The hypothesis (G4) belongs to the H_∞ development alone, where the real parts vary.

Theorem C is, in the H_∞ development, `Phi2HInf.phi2_hinf_pole_twoBsup`, which asserts every clause of part (i) for the continuation given by the right side of (13); `Phi2HInf.phi2_hinf_`

`pole_twoBsup_of_continuation`, which gives the residue and the order -1 for every function meromorphic on $\operatorname{Re} s > 1$ that agrees with the series on $\operatorname{Re} s > 2$ outside a set without accumulation point there; and `Phi2HInf.phi2_hinf_rh_iff_no_real_pole_of_continuation`, which is part (ii) for every such function, with "no pole at x" written as a non-negative order at x. Their hypotheses are `Hinf`, `Z1`, `Z2`, `Z3` and `StirlingStrip`, without `ZLittlewood`; the first two also assume a zero with $\beta \neq \frac{1}{2}$. `StirlingStrip` is carried in the signatures of the continuation lemmas; the formal proof of Theorem C uses it only by passing it to lemmas that do not use it. Each also takes (Z5) as a premise, `ZetaNoZeroIoo`, which the development proves as `Phi2HInf.zetaNoZeroIoo`; applying that proof removes the premise. The development proves the implication of part (ii) from RH directly, without Corollary B(i).

Not formalized: the expository prose of Sections 1.1, 1.3 and 1.4 (for instance the bound $G_2(N) \leq N \log^2 N$, the density of the pair sums under RH, and the outline of the proof), the comments after the statements in Section 1.2, the section openers and unnumbered remarks of Sections 3–7, with the numerical example of Section 5, the remarks of Section 8.3 with their numerical values, among them the bound for $\mu\{0\}$ and the comparison with [6] in Remark 8.3.1, and all of Section 9; the proofs use none of these. Of Section 2, the remarks on notation are not formalized. The formal set of non-trivial zeros is by definition the set of zeros of ζ in the strip $0 < \operatorname{Re} s < 1$ (Appendix A). The facts quoted in Section 2 to place the non-trivial zeros in that strip are used by the formal proof of Theorem C(ii), to pass from RH for the zeros in the strip to Mathlib's statement of RH; they are supplied by the development and by Mathlib, so they add no hypothesis. The proof of (Z5) printed there is not the one formalized. Three passages of Section 2 are checked by hand: the derivation of the symmetries of the zeros from the functional equation, which the H_∞ development assumes as `Z1`, and the passages from Titchmarsh's statements to the forms (Z1) and (Z3), and from [42] to (G4), which the formal theorems take as hypotheses in those forms. The kernel checks the stated implications; whether the formal definitions say what the prose says is a matter of reading them, and for that purpose Appendix A prints the hypotheses, the series, the natural-boundary predicate and the main statements, and describes the definitions it does not print, which the release gives in full.

Both developments are available at `github.com/lpkralv/phi2-natural-boundary`. Release `v1.2` contains both, about 15,700 lines in 170 files, of which the H_∞ development with the files it needs beyond release `v1.1` is about 5,700 lines in 69 files; release `v1.1` contains the RH development alone, about 10,000 lines in 101 files. Both build against the same pinned versions of Mathlib and PrimeNumberTheoremAnd. Neither contains `sorry`, and `#print axioms` on the main theorems of both lists only the three standard axioms of Lean's logic (`propext`, `Classical.choice`, `Quot.sound`).

Acknowledgement and statement on the use of AI

The author is a computer scientist, not a mathematician. He directed an AI research agent through a sequence of questions on natural boundaries of Goldbach series over primes and prime powers, chose at each step the direction to pursue, and decided to attack the Egami–Matsumoto conjecture under RH alone. He designed the verification methodology: independent adversarial reviews, re-derivation and small-case checks of every result before it was recorded as proved, and a formalization in Lean (Section 10). The mathematics is largely the work of AI models (Anthropic Claude, OpenAI GPT) under his direction, including the idea of Section 5, the phase alignment of opposite-sign pairs, the reduction of Section 7. Multiple models reviewed the argument. An Anthropic Claude model drafted the manuscript, which the author edited. AI agents wrote the Lean development. The author did not write or check the Lean development line by line; he relies on the Lean kernel and the checks of Section 10. No human mathematician

has checked the proof line by line. There is no co-reader: queries about the proof will be answered by the author, with the same AI assistance. No AI system is an author. The author takes full responsibility for the content.

The author thanks Anna DeConti, formerly of Osbourn Park High School, who taught him calculus and encouraged his explorations in mathematics.

Appendix A. The Lean statements

This appendix prints the hypotheses, the series, the natural-boundary predicate and the main formal statements of Theorem A, Corollary B and Theorem C, with the basic definitions under them, copied verbatim from the developments (docstrings and proofs omitted; line breaks adjusted). The other definitions that the statements use, `phi2Poles`, `HasPhi2Residues`, `phiRPoles`, `HasPhiRResidues`, `Hphi2`, `pairAtom`, `PhiRc` and `Phi2cHInf`, are described in the glossary below and given in full in the release. The forms of the boundary statements with pointwise agreement, in place of agreement off a set without accumulation point, `Phi2HInf.phi2_hinf_natural_boundary_classical`, `Phi2HInf.phiR_hinf_natural_boundary_classical` and, in the RH development, `phi2_natural_boundary_classical`, are not printed; they, and every other declaration named in Section 10, are in the README of the release, listed by section.

All declarations live in the namespace `GoldbachBoundaries`: the hypotheses of the H_∞ development and `Bsup` in its sub-namespace `Paper2`, except `StirlingStrip`, which with the premise `ZetaNoZeroIoo`, `collisionIdx`, `collisionAtom` and the H_∞ theorems is in `Phi2HInf`; the three hypotheses of the RH development in `Paper2` as well; and `zeroCoef` and `tendsto_eta_Hphi2` in `Phi2Proof`; of the definitions not printed, `pairAtom` is in `Phi2Proof` and `PhiRc` in `PhiRProof`. The formal names of the classical inputs are not the paper's labels: `Z1` states the symmetries of the zeros of Section 2, `Z2` the Riemann–von Mangoldt formula together with the paper's (Z1), `Z3` is the paper's (Z3), and `ZLittlewood`, which is word for word `LittlewoodGap`, is the paper's (Z2) (Table 2).

`Phi2` is the series (1) as a function on $\mathbb{C}$, and `PhiR r` is Φ_r; where the series does not converge absolutely the Lean sum takes Lean's default value 0, which plays no part, since the statements use `Phi2` only on $\operatorname{Re} s > 2$ and `PhiR r` only on $\operatorname{Re} s > r$. `HH` is $\mathbb{H}$, `halfPlane a` is $\{\operatorname{Re} s > a\}$, `lineOne` is $\{\operatorname{Re} s = 1\}$ and `lineAt a` is $\{\operatorname{Re} s = a\}$. `NontrivialZeros`, `zetaMult` and `Nzero` are $\mathcal{Z}$, $m(\rho)$ and $N(t)$; in `zetaMult`, `untopD 0` and `toNat` turn the order of ζ at ρ, an integer or $\top$, into a natural number. `I` is the imaginary unit, `conj` complex conjugation, `z.re` and `z.im` the real and imaginary parts of z, `‖z‖` the modulus, and `Complex.log` the principal logarithm. `IsNaturalBoundary F U L` says that no function meromorphic on a disc centred on L agrees with F on the part of the disc in U off a set without accumulation point there; `NoMeromorphicExtensionAt F U` z_0 says this for the discs centred at the one point z_0, and `IsNaturalBoundary` asserts it at every z_0 in L. `HasResidue F` α `c` means $(s-\alpha)F(s) \to c$ as $s \to \alpha$ with $s \neq \alpha$ (this is the filter $\mathcal{N}$`[≠]` α in its definition), and `meromorphicOrderAt F` α `= -1` means that F has a simple pole at α; `((-1 : ℤ) : WithTop ℤ)` is the order -1.

In the RH statements, `phi2Poles` is the set $\{2\} \cup \{1+\rho : \rho \in \mathcal{Z}\}$, `HasPhi2Residues res` says that `res 2 = 1` and `res (1 + ρ)` $= -2m(\rho)/\rho$ for every non-trivial zero ρ, `phiRPoles r` is the pole set of Corollary B(iv) and `HasPhiRResidues r res` fixes its residues; `Hphi2` is H of Corollary B(iii) in the form $(E_2(s) - 2e_1(s-1) + R(s))/\Gamma(s)$, `Phi2Proof.pairAtom` τ is $\mu\{\tau\}$ of (2), `PhiRProof.PhiRc r` is the continuation of Φ_r that the RH development constructs, `r.choose 2` is $\binom{r}{2}$, and `Tendsto f (𝓝[>] 0) (𝓝 L)` means $f(\eta) \to L$ as $\eta \downarrow 0$. In the statements for Theorem C, `Bsup` is B, `zeroCoef` ρ is a_ρ, `collisionIdx s` is the set $\mathcal{R}(s)$ of (14), the ordered pairs of zeros with sum s, `collisionAtom s` is $\nu(s)$ of (14), the sum over that

set, `Phi2cHInf` is the continuation given by the right side of (13), `Gamma` is Γ, here and in the RH statements, and the sum $\sum'$ over the zeros ρ with `(ρ : ℂ).re = Bsup` runs over the zeros with $\beta = B$. In the definition of `Bsup`, `re '' NontrivialZeros` is the set of real parts of the zeros and `sSup` its supremum. Mathlib gives `sSup` the value 0 on a set of real numbers that is empty or not bounded above; this plays no part, since the set is bounded above by 1 and `Bsup` occurs only in statements with the hypothesis `hoff`, which makes it non-empty. $\sum'$ is the sum of a family, which Mathlib sets to 0 when the family is not summable; `{x : S // p x}` is the type of the x in S with $p(x)$, and `Fin r → ℕ` the type of r-tuples of natural numbers.

The names `RiemannHypothesis`, `riemannZeta`, `deriv`, `Complex.exp`, `Complex.Gamma`, `Real.pi`, `Real.log`, `AnalyticAt`, `AnalyticOnNhd`, `MeromorphicOn`, `meromorphicOrderAt`, `codiscreteWithin`, `EqOn`, `Tendsto`, `ball`, `atTop`, `sSup` and `ArithmeticFunction.vonMangoldt` are Mathlib's: `deriv` is the derivative, `Complex.exp` and `Complex.Gamma` are the complex exponential and Gamma functions, `Real.pi` is π, `Real.log` is the real logarithm, `ball z r` is the open disc $D(z,r)$, and `ArithmeticFunction.vonMangoldt` is Λ; $\forall^{\mathrm{f}}$ `t in atTop` means "for all sufficiently large t", and $\sum^{\mathrm{f}}$ is a finite sum; $F =^{\mathrm{f}}$ [`codiscreteWithin` S] G means that F and G agree on S outside a set with no accumulation point in S.

Basic definitions.

```
def halfPlane (a : ℝ) : Set ℂ := {s | a < s.re}

abbrev HH : Set ℂ := halfPlane 1

def lineOne : Set ℂ := {s | s.re = 1}

def lineAt (a : ℝ) : Set ℂ := {s | s.re = a}

def NontrivialZeros : Set ℂ := {ρ | riemannZeta ρ = 0 ∧ 0 < ρ.re ∧ ρ.re < 1}

def zetaMult (ρ : ℂ) : ℕ := ((meromorphicOrderAt riemannZeta ρ).untopD 0).toNat

def HasResidue (f : ℂ → ℂ) (a c : ℂ) : Prop :=
  Tendsto (fun s => (s - a) * f s) (𝓝[≠] a) (𝓝 c)
```

The H_∞ *development (Theorem A).*

```
def Hinf : Prop := ∀ δ : ℝ, 0 < δ → {ρ | ρ ∈ NontrivialZeros ∧ 1 / 2 + δ < ρ.re}.Finite

def Z1 : Prop :=
  ∀ ρ ∈ NontrivialZeros,
    (conj ρ ∈ NontrivialZeros ∧ zetaMult (conj ρ) = zetaMult ρ) ∧
      (1 - conj ρ ∈ NontrivialZeros ∧ zetaMult (1 - conj ρ) = zetaMult ρ)

def Nzero (t : ℝ) : ℝ :=
  ∑ᶠ ρ ∈ {ρ : ℂ | ρ ∈ NontrivialZeros ∧ 0 < ρ.im ∧ ρ.im ≤ t}, (zetaMult ρ : ℝ)

def Z2 : Prop :=
  (∃ C : ℝ, ∀ᶠ t in atTop,
      |Nzero t - (t / (2 * Real.pi) * Real.log (t / (2 * Real.pi)) - t / (2 * Real.pi))| ≤
        C * Real.log t) ∧
    ∃ C₁ : ℝ, ∀ t : ℝ, 0 ≤ t → Nzero (t + 1) - Nzero t ≤ C₁ * Real.log (t + 3)

def Z3 : Prop :=
  ∃ C : ℝ, ∀ z : ℂ, -1 ≤ z.re → z.re ≤ 2 → 2 ≤ |z.im| → riemannZeta z ≠ 0 →
    ‖deriv riemannZeta z / riemannZeta z -
        ∑ᶠ ρ ∈ {ρ : ℂ | ρ ∈ NontrivialZeros ∧ |ρ.im - z.im| ≤ 1},
          (zetaMult ρ : ℂ) / (z - ρ)‖ ≤
```

```
      C * Real.log |z.im|

def ZLittlewood : Prop :=
  ∃ a : ℝ, ∀ᶠ t in atTop, ∃ ρ ∈ NontrivialZeros,
    |ρ.im - t| < a / Real.log (Real.log (Real.log t))

def StirlingStrip : Prop :=
  ∀ x₁ x₂ : ℝ, x₁ ≤ x₂ → ∃ C T₀ : ℝ, 0 ≤ C ∧
    ∀ z : ℂ, x₁ ≤ z.re → z.re ≤ x₂ → T₀ ≤ |z.im| →
      ‖Complex.Gamma z -
          Complex.exp ((z - 1 / 2) * Complex.log z - z +
            (1 / 2) * Complex.log (2 * (Real.pi : ℂ)))‖ ≤
        C / |z.im| *
          ‖Complex.exp ((z - 1 / 2) * Complex.log z - z +
            (1 / 2) * Complex.log (2 * (Real.pi : ℂ)))‖

def NoMeromorphicExtensionAt (F : ℂ → ℂ) (U : Set ℂ) (z₀ : ℂ) : Prop :=
  ∀ r : ℝ, 0 < r → ¬ ∃ f : ℂ → ℂ, MeromorphicOn f (ball z₀ r) ∧
    f =ᶠ[codiscreteWithin (ball z₀ r ∩ U)] F

def IsNaturalBoundary (F : ℂ → ℂ) (U L : Set ℂ) : Prop :=
  ∀ z₀ ∈ L, NoMeromorphicExtensionAt F U z₀

def Phi2 (s : ℂ) : ℂ :=
  ∑' mn : ℕ × ℕ, ((ArithmeticFunction.vonMangoldt mn.1 : ℝ) : ℂ) *
    ((ArithmeticFunction.vonMangoldt mn.2 : ℝ) : ℂ) * (((mn.1 + mn.2 : ℕ) : ℂ)) ^ (-s)

def PhiR (r : ℕ) (s : ℂ) : ℂ :=
  ∑' k : Fin r → ℕ, (∏ i, ((ArithmeticFunction.vonMangoldt (k i) : ℝ) : ℂ)) *
    (((∑ i, k i : ℕ) : ℂ)) ^ (-s)

theorem phi2_hinf_natural_boundary (hH : Hinf) (hZ1 : Z1) (hZ2 : Z2) (hZ3 : Z3)
    (hL : ZLittlewood) (hStir : StirlingStrip) :
    (∃ F : ℂ → ℂ, MeromorphicOn F HH ∧ EqOn F Phi2 (halfPlane 2)) ∧
    (∀ F : ℂ → ℂ, MeromorphicOn F HH → F =ᶠ[codiscreteWithin (halfPlane 2)] Phi2 →
      IsNaturalBoundary F HH lineOne) :=

theorem phiR_hinf_natural_boundary (hH : Hinf) (hZ1 : Z1) (hZ2 : Z2) (hZ3 : Z3)
    (hL : ZLittlewood) (hStir : StirlingStrip) {r : ℕ} (hr : 2 ≤ r) :
    (∃ F : ℂ → ℂ, MeromorphicOn F (halfPlane ((r : ℝ) - 1)) ∧
      EqOn F (PhiR r) (halfPlane r)) ∧
    (∀ F : ℂ → ℂ, MeromorphicOn F (halfPlane ((r : ℝ) - 1)) →
      F =ᶠ[codiscreteWithin (halfPlane r)] PhiR r →
      IsNaturalBoundary F (halfPlane ((r : ℝ) - 1)) (lineAt ((r : ℝ) - 1))) :=
```

The H_∞ *development (Theorem C).* The hypotheses are those above.

```
def ZetaNoZeroIoo : Prop := ∀ σ : ℝ, 0 < σ → σ < 1 → riemannZeta σ ≠ 0

def Bsup : ℝ := sSup (re '' NontrivialZeros)

def zeroCoef (ρ : ℂ) : ℂ := (zetaMult ρ : ℂ) * Gamma ρ

def collisionIdx (s : ℂ) := {p : NontrivialZeros × NontrivialZeros //
  (p.1 : ℂ) + p.2 = s}

def collisionAtom (s : ℂ) : ℂ :=
  ∑' p : collisionIdx s, zeroCoef p.1.1 * zeroCoef p.1.2
```

```
theorem phi2_hinf_pole_twoBsup (hH : Hinf) (hZ1 : Z1) (hZ2 : Z2) (hZ3 : Z3)
    (hStir : StirlingStrip) (hζ : ZetaNoZeroIoo) (hoff : ∃ ρ ∈ NontrivialZeros, ρ.re ≠ 1 / 2) :
    (1 / 2 < Bsup ∧ Bsup < 1) ∧
    ({ρ | ρ ∈ NontrivialZeros ∧ ρ.re = Bsup}.Finite ∧
      {ρ | ρ ∈ NontrivialZeros ∧ ρ.re = Bsup}.Nonempty) ∧
    (collisionAtom (2 * (Bsup : ℂ)) =
        ((∑' ρ : {ρ : NontrivialZeros // (ρ : ℂ).re = Bsup},
          ((zetaMult (ρ : ℂ) : ℝ) ^ 2 * ‖Gamma (ρ : ℂ)‖ ^ 2) : ℝ) : ℂ) ∧
      0 < (collisionAtom (2 * (Bsup : ℂ))).re ∧ (collisionAtom (2 * (Bsup : ℂ))).im = 0) ∧
    HasResidue Phi2cHInf (2 * (Bsup : ℂ))
      (collisionAtom (2 * (Bsup : ℂ)) / Gamma (2 * (Bsup : ℂ))) ∧
    (0 < (collisionAtom (2 * (Bsup : ℂ)) / Gamma (2 * (Bsup : ℂ))).re ∧
      (collisionAtom (2 * (Bsup : ℂ)) / Gamma (2 * (Bsup : ℂ))).im = 0) ∧
    meromorphicOrderAt Phi2cHInf (2 * (Bsup : ℂ)) = ((-1 : ℤ) : WithTop ℤ) ∧
    ¬ AnalyticAt ℂ Phi2cHInf (2 * (Bsup : ℂ)) :=

theorem phi2_hinf_pole_twoBsup_of_continuation (hH : Hinf) (hZ1 : Z1) (hZ2 : Z2) (hZ3 : Z3)
    (hStir : StirlingStrip) (hζ : ZetaNoZeroIoo) (hoff : ∃ ρ ∈ NontrivialZeros, ρ.re ≠ 1 / 2)
    (F : ℂ → ℂ) (hF : MeromorphicOn F HH) (hFΦ : F =ᶠ[codiscreteWithin (halfPlane 2)] Phi2) :
    HasResidue F (2 * (Bsup : ℂ)) (collisionAtom (2 * (Bsup : ℂ)) / Gamma (2 * (Bsup : ℂ))) ∧
    meromorphicOrderAt F (2 * (Bsup : ℂ)) = ((-1 : ℤ) : WithTop ℤ) :=

theorem phi2_hinf_rh_iff_no_real_pole_of_continuation (hH : Hinf) (hZ1 : Z1) (hZ2 : Z2)
    (hZ3 : Z3) (hStir : StirlingStrip) (hζ : ZetaNoZeroIoo)
    (F : ℂ → ℂ) (hF : MeromorphicOn F HH) (hFΦ : F =ᶠ[codiscreteWithin (halfPlane 2)] Phi2) :
    RiemannHypothesis ↔ ∀ x : ℝ, 1 < x → x < 2 → 0 ≤ meromorphicOrderAt F x :=
```

The RH development (Corollary B). The definitions of `NoMeromorphicExtensionAt`, `IsNaturalBoundary`, `Phi2` and `PhiR` are those above.

```
def RiemannVonMangoldt : Prop :=
  ∃ C₁ : ℝ, ∀ t : ℝ, 0 ≤ t → Nzero (t + 1) - Nzero t ≤ C₁ * Real.log (t + 3)

def LogDerivZetaLocal : Prop :=
  ∃ C : ℝ, ∀ z : ℂ, -1 ≤ z.re → z.re ≤ 2 → 2 ≤ |z.im| → riemannZeta z ≠ 0 →
    ‖deriv riemannZeta z / riemannZeta z -
        ∑ᶠ ρ ∈ {ρ : ℂ | ρ ∈ NontrivialZeros ∧ |ρ.im - z.im| ≤ 1},
          (zetaMult ρ : ℂ) / (z - ρ)‖ ≤
      C * Real.log |z.im|

def LittlewoodGap : Prop :=
  ∃ a : ℝ, ∀ᶠ t in atTop, ∃ ρ ∈ NontrivialZeros,
    |ρ.im - t| < a / Real.log (Real.log (Real.log t))

theorem phi2_natural_boundary (hRH : RiemannHypothesis) (hRvM : RiemannVonMangoldt)
    (hLD : LogDerivZetaLocal) (hL : LittlewoodGap) :
    (∃ F : ℂ → ℂ, MeromorphicOn F HH ∧ EqOn F Phi2 (halfPlane 2) ∧
      AnalyticOnNhd ℂ F (HH \ phi2Poles) ∧
      (∀ α ∈ phi2Poles, meromorphicOrderAt F α = -1) ∧
      ∃ res : ℂ → ℂ, (∀ α ∈ phi2Poles, HasResidue F α (res α)) ∧ HasPhi2Residues res ∧
        ∀ α ∈ phi2Poles, res α ≠ 0) ∧
    (∀ F : ℂ → ℂ, MeromorphicOn F HH → F =ᶠ[codiscreteWithin (halfPlane 2)] Phi2 →
      IsNaturalBoundary F HH lineOne) :=

theorem tendsto_eta_Hphi2 (hRH : RiemannHypothesis) (hRvM : RiemannVonMangoldt)
    (hLD : LogDerivZetaLocal) (τ : ℝ) :
    Tendsto (fun η : ℝ => (η : ℂ) * Hphi2 (1 + τ * I + η)) (𝒩[>] 0)
      (𝒩 (pairAtom τ / Gamma (1 + τ * I))) :=
```

```
theorem phiR_natural_boundary (hRH : RiemannHypothesis) (hRvM : RiemannVonMangoldt)
    (hLD : LogDerivZetaLocal) (hL : LittlewoodGap) {r : ℕ} (hr : 2 ≤ r) :
    (∃ F : ℂ → ℂ, MeromorphicOn F (halfPlane ((r : ℝ) - 1)) ∧ EqOn F (PhiR r) (halfPlane r) ∧
      AnalyticOnNhd ℂ F (halfPlane ((r : ℝ) - 1) \ phiRPoles r) ∧
      (∀ α ∈ phiRPoles r, meromorphicOrderAt F α = -1) ∧
      ∃ res : ℂ → ℂ, (∀ α ∈ phiRPoles r, HasResidue F α (res α)) ∧ HasPhiRResidues r res ∧
        ∀ α ∈ phiRPoles r, res α ≠ 0) ∧
    (∀ F : ℂ → ℂ, MeromorphicOn F (halfPlane ((r : ℝ) - 1)) →
      F =ᶠ[codiscreteWithin (halfPlane r)] PhiR r →
      IsNaturalBoundary F (halfPlane ((r : ℝ) - 1)) (lineAt ((r : ℝ) - 1))) :=

theorem phiR_radial_limit (hRH : RiemannHypothesis) (hRvM : RiemannVonMangoldt)
    (hLD : LogDerivZetaLocal) {r : ℕ} (hr : 2 ≤ r) {τ : ℝ} (hτ : τ ≠ 0) :
    Tendsto (fun η : ℝ => (η : ℂ) * PhiRProof.PhiRc r ((r : ℂ) - 1 + τ * I + η)) (𝓝[>] 0)
      (𝓝 (((r.choose 2 : ℕ) : ℂ) * Phi2Proof.pairAtom τ / Gamma ((r : ℂ) - 1 + τ * I))) :=
```

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