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AN ARAKELOV-THEORETIC PERSPECTIVE ON THE ALPÖGE–CLAUDE ARGUMENT ON RH

Abstract. We describe an Arakelov-theoretic perspective on the argument on the Riemann hypothesis for Dirichlet L-functions contained in the Lean documents of Alpöge. We show that the argument is an instance of the slope method of Bost, in the form used by Calegari, Dimitrov and Tang. The evaluation map sends cusp forms on a unitary Shimura variety to forms on a finite set of CM abelian varieties, and we interpret it as an evaluation of sections on special cycles. To define it, we write cusp forms as theta lifts and evaluate them through a chain of correspondences given by the jets at the origin of theta functions. The two global inputs are a zero estimate, which we view as a quantitative equidistribution statement for the special cycles, and a quantitative arithmetic Hilbert–Samuel theorem. Every step that we do not prove is stated as a hypothesis, and for each hypothesis we say which mathematics it uses. We do not claim a proof of the Riemann hypothesis. For a more complete account of the argument and of its history we refer to the note of Robin Zhang, which the present note supplements.

1. Introduction

1.1. **The goal.** The Lean documents of Alpöge [Alp26] contain an argument for the generalized Riemann hypothesis for Dirichlet L-functions. A more complete account of the argument and of its history is given in the note of Robin Zhang [Zha26], and the present note only supplements it. We concentrate on one perspective, namely a way to organize the Arakelov-geometric part of the argument, and we make three points.

(1) The argument is an instance of the slope method, with one evaluation map, one slope inequality, and the usual list of inputs. Our reference point is the use of the slope method by Calegari, Dimitrov and Tang [CDT25].
(2) The evaluation map has a geometric description. Morally, a section on a unitary Shimura variety is evaluated on a special cycle of CM points. Concretely, the map is computed on finite sets of CM abelian varieties, by the jets at the origin of theta functions.
(3) The two global inputs are quantitative forms of classical statements of Arakelov geometry. The first is a zero estimate, which we view as a quantitative equidistribution statement for the special cycles. The second is a quantitative form of the arithmetic Hilbert–Samuel theorem.

We now say what this note does not do. We do not prove the Riemann hypothesis, and we do not verify the argument of [Alp26]. The statements on Hermitian lattices in §2 are extracted from [Alp26]. We state them in the form of a slope inequality and indicate their proofs, and we refer to [Zha26, §4.4] for the details. Every other step is stated as a hypothesis, and for each hypothesis we say which mathematics it uses. Some of these hypotheses are schematic: they involve explicit numbers and small errors which we do not specify. We come back to them in §6.

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1.2. **The slope method.** We first recall the method. Let $\mathrm{ev}\colon \overline{N} \to \overline{V}$ be a map between Hermitian lattices over a number field, where we think of N as a lattice of sections and of ev as an evaluation map. If ev is injective, then, schematically,

$$\mu(\overline{N}) \leqslant \mu(\overline{V}) + h_{\mathrm{fin}}(\mathrm{ev}) + h_{\infty}(\mathrm{ev}). \tag{1.2.1}$$

Here μ is the slope, and h_{fin} and h_{∞} are the heights of ev at the finite and at the infinite places, see §2. The height at the finite places is non-positive when ev is integral, and it is negative when the values of ev are divisible. To use this inequality we need five things. These are a zero estimate, which gives the injectivity, the slopes of the domain and of the codomain, and the heights of ev at the finite and at the infinite places.

1.3. **The framework.** We now describe the framework, starting with the unknown, then the evaluation map, and finally the way in which the slope method bounds the unknown.

1.3.1. *The unknown.* Let Λ be the von Mangoldt function and γ Euler's constant. For $y \geqslant 1$ put

$$G(y) = \sum_{n \leqslant y} \Lambda(n)\Big(\frac{y}{n} - 1\Big) - y\big(\log y - 1 - \gamma\big). \tag{1.3.1}$$

The Riemann hypothesis is equivalent to $G(y) = O(y^{1/2+\varepsilon})$. Since $\Lambda \geqslant 0$, a theorem of Landau shows that the upper bound $G(y) \leqslant y^{1/2+\varepsilon}$ suffices. For a Dirichlet character χ one uses a combination G_{Ξ} of G and of its twists by the powers of χ, with positive coefficients, and it suffices to bound $G_{\Xi}(x)$ by $x^{1/2+C\varepsilon}$ for x in arbitrarily long intervals. We refer to [**Zha26**, Proposition 1.2 and §2] for this reduction. Hence the unknown is a sum over primes, and we only need an upper bound at a point x, which we fix from now on. This is what a slope inequality can give, provided that the primes contribute to the slopes. The natural question is therefore how to make the primes contribute to a slope, and this leads to the evaluation map below.

1.3.2. *The evaluation map.* The source is a space $\mathcal{S}$ of cusp forms on a unitary Shimura variety Sh. The target is the space of forms on a finite set of CM abelian varieties, which are isogenous to powers of an elliptic curve E_0 with complex multiplication by an auxiliary imaginary quadratic field K. We call such a finite set a Shimura set. The intuition is that we evaluate a section on Sh, with multiplicity about a, on a special cycle of CM points. We define the evaluation map

$$\mathrm{ev}_u\colon R \longrightarrow V_u, \qquad u \in \mathcal{U},$$

as follows. We write a cusp form as a theta lift from a large definite unitary group, and we evaluate the theta lift through m correspondences between Shimura sets, each given by the jet of order about a of a theta function at the origin of an abelian variety. The domain $\overline{R}$ consists of the theta lifts of integral forms, with the Petersson norm, and the codomain $\overline{V}_u$ is a lattice of values at CM abelian varieties. The construction is summarized in Figure 1.

1.3.3. *The three steps.* The argument then has three steps.

The local bound. At every prime p inert in K the values are divisible by a power of p. Hence the height of ev_u at the finite places is negative. More precisely, $-h_{\mathrm{fin}}(\mathrm{ev}_u)$ is bounded from below by a sum over primes $\mathcal{B}_u$, and we write $\mathcal{B}$ for the mean of $\mathcal{B}_u$ over u.

The global bound. A slope inequality of the form (1.2.1), with the mean over u on the right-hand side, bounds the mean of $-h_{\mathrm{fin}}(\mathrm{ev}_u)$ from above by the mean of

$\mu(\overline{V}_u) - \big(\mu(\overline{R}) - h_\infty(\mathrm{ev}_u)\big)$. It rests on a zero estimate, which we view as a quantitative equidistribution statement: the special cycles, taken together, sample the sections evenly.

The computation. The slope of $\overline{V}_u$ is a height of CM abelian varieties, and it is at most an explicit number μ_u^{val}. The difference $\mu(\overline{R}) - h_\infty(\mathrm{ev}_u)$ is bounded from below by an explicit number μ_u^{dom}, by a quantitative arithmetic Hilbert–Samuel bound. The two slopes are not known with the same precision. The heights of the CM abelian varieties are computed exactly, while the arithmetic volumes of the Shimura varieties are only bounded from below, by a lower bound for the height of every point and the inequality of Zhang, see §4.4.8.

Up to errors, the three steps give

$$\mathcal{B} \leqslant \mathbb{E}_u\big[-h_{\mathrm{fin}}(\mathrm{ev}_u)\big] \leqslant \mathbb{E}_u\big[\mu(\overline{V}_u) - \big(\mu(\overline{R}) - h_\infty(\mathrm{ev}_u)\big)\big] \leqslant \mathbb{E}_u\big[\mu_u^{\mathrm{val}} - \mu_u^{\mathrm{dom}}\big], \quad (1.3.2)$$

where $\mathbb{E}_u$ is the mean over u. After we identify the explicit terms, the difference between the two ends of (1.3.2) is a positive multiple of $G(x)$. Hence (1.3.2) is the upper bound for G that we want.

1.3.4. *The origin of the divisibility.* The divisibility comes from an elementary fact: the n-th derivative of a function is $n!$ times its Taylor coefficient. At a prime p of supersingular reduction the Taylor coefficients of a theta function have denominators of size about $p^{n/(p^2-1)}$ only, while $n!$ is divisible by $p^{n/(p-1)}$. Hence the jets of order n are divisible by about $p^{n\gamma_p}$, where

$$\gamma_p = \frac{1}{p-1} - \frac{1}{p^2-1} = \sum_{e \text{ odd}} p^{-e}.$$

At a prime of ordinary reduction there is no such divisibility. Now the supersingular primes are the inert primes, and $\sum_{p \text{ inert}} \log p \sum_{e \text{ odd}} n/p^e$ is the part of $\log n!$ that belongs to the non-trivial character of K, which is how G enters.

1.4. **Comparison with Calegari–Dimitrov–Tang.** The following table compares the ingredients with those of [CDT25], and names the mathematics behind each of them.

	in [CDT25]	here	mathematics used
domain	polynomials	theta lifts of integral forms $\overline{R}$, Petersson norm, §3.7	basis problem, seesaw pairs
evaluation map	jets at a point	m correspondences given by jets of theta functions, §3	theta functions on CM abelian varieties
height at finite places	denominators	negative at the inert primes, by divisibility, §4.1	formal groups of height 2, theorem of the cube
slope of codomain	trivial	height of CM points, §4.2	Chowla–Selberg and Colmez formulas
zero estimate	orders of vanishing	equidistribution of the special cycles, in the second moment, §4.3	Bessel periods, relative trace formula
slope inequality	successive evaluations	in the mean, Thm. 2.3.3	Harder–Narasimhan filtrations
slope of domain	Hilbert–Samuel, asymptotic, with the exact volume of Bost–Charles	m Shimura varieties, quantitative Hilbert–Samuel, a lower bound for the volumes, §4.4	inner product formula, Chow forms, integral theta lifts, matrix lemma, inequality of Zhang
heights at ∞ unknown	Jensen's formula a rank	only $\mu(\overline{R}) - h_\infty$ is bounded a sum over primes	the same norm identity Landau's theorem

There are three differences. First, the roles of the known and the unknown are exchanged. In [CDT25] the prime number theorem is an input and the output is a bound for a rank. Here all ranks and periods are known and the output is a bound for a sum over primes. Secondly, there are many codomains, indexed by u, and we need the inequality with the mean over u. A zero estimate is a quantitative algebraic equidistribution statement, since it says that the points do not lie on a hypersurface of small degree. For the mean this is not enough. We need its analytic counterpart, namely that the points sample the sections evenly. Thirdly, the evaluation map is a composition of m correspondences. The main terms are proportional to m, some errors do not grow with m, and others grow like m^2, so that the choice $m \approx \sqrt{x}$ gives the exponent $\frac{1}{2}$. Every term must therefore be controlled with relative precision $x^{-1/2}$, in a family in which the degree of the field and all ranks grow with x.

2. The slope inequality

In this section we state the two slope inequalities and indicate their proofs. The first is the usual one, while the second has the mean over a family of codomains on the right-hand side and rests on a stronger zero estimate. Both are statements on Hermitian lattices, and they do not involve automorphic forms. The second inequality and the zero estimate behind it are extracted from [Alp26]. In the transposed form, with the period maps going from the lattices $\overline{V}_i^\vee$ to $\overline{N}^\vee$, they are Proposition 4.3 and Lemma 4.2 of [Zha26]. What we add is the formulation as a slope inequality for an evaluation map, with a zero estimate as its input. We use the second inequality in §4.3 to obtain the global bound.

2.1. **Conventions.** Let $\Bbbk$ be a number field. A **Hermitian lattice** $\overline{N}$ over $\Bbbk$ is a finitely generated projective $\mathcal{O}_\Bbbk$-module N with Hermitian norms on $N \otimes_\sigma \mathbb{C}$ for all embeddings $\sigma\colon \Bbbk \to \mathbb{C}$, compatible with complex conjugation. For a rational section $\varsigma \neq 0$ of $\det N$, its **normalized degree** and **slope** are

$$\widehat{\deg}_\Bbbk \overline{N} = \frac{1}{[\Bbbk : \mathbb{Q}]}\Big(\sum_{\mathfrak{p}} \operatorname{ord}_\mathfrak{p}(\varsigma) \log \mathrm{N}\,\mathfrak{p} - \sum_\sigma \log \|\varsigma\|_\sigma\Big), \qquad \hat{\mu}(\overline{N}) = \frac{\widehat{\deg}_\Bbbk \overline{N}}{\operatorname{rk} N}. \tag{2.1.1}$$

We work with squared norms, so we use the doubled slope $\mu = 2\hat{\mu}$ throughout. For a fractional ideal $\mathfrak{a}$ we have

$$\mu(\overline{\mathfrak{a} \cdot N}) = \mu(\overline{N}) - \frac{2 \log \mathrm{N}\,\mathfrak{a}}{[\Bbbk : \mathbb{Q}]}. \tag{2.1.2}$$

We write $\mu_{\max}$ and $\mu_{\min}$ for the largest and the smallest slope of the Harder–Narasimhan filtration, and $N^{>\xi}$ for the step of the filtration whose graded pieces have slope larger than ξ.

Lemma 2.1.1 (An index and a determinant). *Let $\phi\colon N \to N'$ be an injective map between Hermitian lattices of the same rank, and let $I(\phi) = [\Bbbk : \mathbb{Q}]^{-1} \sum_\mathfrak{p} \operatorname{length}(N'/\phi N)_\mathfrak{p} \log \mathrm{N}\,\mathfrak{p}$. Then*

$$\widehat{\deg}_\Bbbk \overline{N}' - \widehat{\deg}_\Bbbk \overline{N} = I(\phi) - \frac{1}{[\Bbbk : \mathbb{Q}]} \sum_\sigma \log \|\det \phi_\sigma\|\,.$$

The lemma follows by evaluating (2.1.1) at ς and at $(\det \phi)(\varsigma)$. It puts the logarithm of an index and the logarithm of a norm into one identity. A divisibility at a prime is a lower bound for the index, and an estimate for a complex period is an upper bound for the determinant. We use it in this way in §4.4.

2.2. **The simple slope inequality.** Let $\mathrm{ev}\colon N \otimes \Bbbk \to V \otimes \Bbbk$ be a nonzero $\Bbbk$-linear map between Hermitian lattices. Its **height at the infinite places** is $h_\infty(\mathrm{ev}) = [\Bbbk : \mathbb{Q}]^{-1} \sum_\sigma \log \|\mathrm{ev}_\sigma\|^2$, with the operator norms. To define its height at the finite places, let $J(\mathrm{ev})$ be the smallest fractional ideal J of $\Bbbk$ such that $\mathrm{ev}(N) \subset JV$. We put

$$h_{\mathrm{fin}}(\mathrm{ev}) = -\frac{2 \log \mathrm{N}\, J(\mathrm{ev})}{[\Bbbk : \mathbb{Q}]}, \qquad h(\mathrm{ev}) = h_{\mathrm{fin}}(\mathrm{ev}) + h_\infty(\mathrm{ev}). \tag{2.2.1}$$

We say that ev is **integral** if $\mathrm{ev}(N) \subset V$. In this case $J(\mathrm{ev})$ is an integral ideal and $h_{\mathrm{fin}}(\mathrm{ev}) \leqslant 0$. If moreover the values of ev are divisible by an ideal J, then $h_{\mathrm{fin}}(\mathrm{ev}) \leqslant -2 \log \mathrm{N}\, J / [\Bbbk : \mathbb{Q}]$. The following is the simple slope inequality of Bost [**Bos01**, Proposition 4.5].

PROPOSITION 2.2.1. *If* ev *is injective, then* $\mu(\overline{N}) \leqslant \mu_{\max}(\overline{V}) + h(\mathrm{ev})$.

The proof has two parts, one for each kind of place. Let $N_0 = \mathrm{ev}(N)$, with the norms induced from $\overline{V}$. By definition N_0 is a sublattice of $J(\mathrm{ev})V$, so that $\mu(\overline{N}_0) \leqslant \mu_{\max}(\overline{V}) + h_{\mathrm{fin}}(\mathrm{ev})$ by (2.1.2). Since ev is injective, Lemma 2.1.1 and Hadamard's inequality give $\mu(\overline{N}) \leqslant \mu(\overline{N}_0) + h_\infty(\mathrm{ev})$. The same argument, applied to the quotients of N, shows that ev sends $N^{>\xi} \otimes \Bbbk$ into $V^{>\xi - h(\mathrm{ev})} \otimes \Bbbk$. We use the inequality in this form below.

2.2.2. *What an application requires.* To deduce something from Proposition 2.2.1, we need the following.

(1) A *zero estimate*: the evaluation map is injective. This is a quantitative algebraic equidistribution statement, since it says that the points of evaluation do not lie on a hypersurface of small degree.
(2) The slope of the domain, from below.
(3) The height of the evaluation map at the finite places, from above. It is non-positive when the map is integral, and any divisibility of the values at a prime makes it negative.
(4) The height of the evaluation map at the infinite places, from above.
(5) The slope of the codomain, from above.

Here (3) produces the sum over primes. We treat the five points in §4, in the order (3), (5), (1), (2) and (4).

2.3. **The slope inequality in the mean.** Given the simple slope inequality, the next natural question is whether it applies when there are many codomains. In our situation the codomain is a sum of many lattices $\overline{V}_i$, with different slopes and different heights. We need the inequality with the mean over i on the right-hand side. Injectivity does not give this, as the following example shows. Let the domain be a line of slope $H > 0$, and let the codomain consist of n lines. The first has slope H and receives an isometry from the domain, while the others have slope 0 and receive the zero map. Then the evaluation map is integral, injective and of height 0, while the mean slope of the codomain is H/n. To obtain the mean, we need to know that all the conditions are used evenly. In other words, we need an equidistribution statement in the analytic sense, and not only its algebraic form. This leads to the notion of a frame, which we formulate at one embedding σ.

Let $\overline{N}$ and $\overline{V}_i$, $i \in I$, be Hermitian lattices over $\Bbbk$, of ranks D_0 and d_i, and let $\mathrm{ev}_i\colon N \otimes \Bbbk \to V_i \otimes \Bbbk$ be $\Bbbk$-linear. Put $\rho = \sum_i d_i$. We write N_σ and $V_{i,\sigma}$ for the complex vector spaces attached to σ.

DEFINITION 2.3.1. A **frame with parameters** (δ, α, η) for the family (ev_i) consists of the following data.

(a) Hilbert space structures on the spaces $V_{i,\sigma}$.
(b) A Hilbert space $\mathcal{H}$ containing N_σ as a subspace of dimension at least $(1-\delta)\dim \mathcal{H}$.
(c) Linear maps $B_i \colon \mathcal{H} \to V_{i,\sigma}$ extending $\mathrm{ev}_{i,\sigma}$.
(d) Numbers $c_i > 0$.

We put $Q = \sum_i c_i B_i^* B_i$, and we require that

$$\sum_i \|c_i B_i B_i^* - 1\|_{\mathrm{HS}}^2 \leqslant \alpha\, \rho, \qquad \operatorname{tr} Q^2 \leqslant (1+\eta)\, \frac{(\operatorname{tr} Q)^2}{\dim \mathcal{H}}. \tag{2.3.1}$$

This definition may look strange, but both conditions have a simple meaning. The operator Q measures a section by its values, since $\langle Qf, f\rangle = \sum_i c_i \|B_i f\|^2$. The second condition says that Q is almost scalar. To see this, recall that $\operatorname{tr} Q^2 \geqslant (\operatorname{tr} Q)^2/\dim \mathcal{H}$, with equality exactly for scalar Q. Hence the weighted sum of the squared norms of the values of f is almost proportional to the squared norm of f. The first condition says that the conditions imposed by one evaluation map are almost orthonormal.

We view the second condition as an equidistribution condition. Suppose that the maps B_i are evaluations at points. Then $\langle Qf, f\rangle$ is a weighted sum of the values of $|f|^2$ at these points, and Q is scalar exactly when this sum is a constant multiple of the integral $\|f\|^2$ for every f. Hence the condition says that the points are equidistributed, when we test them against the functions $|f|^2$ with $f \in \mathcal{H}$. The condition on $\operatorname{tr} Q^2$ bounds the difference between Q and a scalar in the Hilbert–Schmidt norm. So the equidistribution is required in the second moment over f, and not for each f.

LEMMA 2.3.2 (Zero estimate). *Assume* (2.3.1) *with* $\alpha \leqslant \frac{1}{4}$. *Let* $0 \leqslant \beta \leqslant 1$, *and let* $V_i' \subset V_{i,\sigma}$ *be subspaces with* $\sum_i \dim V_i' \leqslant \beta\rho$. *Then the space* $\mathcal{H}'$ *of all* $f \in \mathcal{H}$ *with* $B_i f \in V_i'$ *for all* i *has dimension at most* $(\eta + 4\beta + 4\sqrt{\alpha})\dim \mathcal{H}$.

For $V_i' = 0$ the lemma bounds the dimension of the space of sections on which all the evaluation maps vanish. It says more: the sections that satisfy all but a small proportion of the conditions form a space of small dimension, and this stronger form is what we need below. So the lemma is a zero estimate in the usual sense, and we keep this name. The difference is that its hypothesis is an analytic equidistribution condition, while a usual zero estimate is proved by algebraic means.

A proof of the transposed statement is given in [**Zha26**, Lemma 4.2].

Let $h_i \in \mathbb{R}$ and put

$$\bar\mu = \frac{1}{\rho}\sum_i d_i\big(\mu(\overline{V}_i) + h_i\big), \qquad H_- = \bar\mu - \min_i\big(\mu_{\min}(\overline{V}_i) + h_i\big). \tag{2.3.2}$$

The number $\bar\mu$ is the mean of the slopes of the codomains, corrected by the heights and weighted by the ranks. The number H_- measures how far the smallest slope lies below the mean.

THEOREM 2.3.3 (Slope inequality in the mean). *Suppose that* $h(\mathrm{ev}_i) \leqslant h_i$ *for all* i, *and that the family* (ev_i) *admits a frame with parameters* (δ, α, η), $\delta < 1$ *and* $\alpha \leqslant \frac{1}{4}$. *Let* $H_+ \geqslant \max(0, \mu_{\max}(\overline{N}) - \bar\mu)$. *Then*

$$\mu(\overline{N}) \leqslant \bar\mu + \frac{1}{1-\delta}\Big(4H_- \log\Big(1 + \frac{H_+}{H_-}\Big) + \big(\eta + 4\sqrt{\alpha}\big)H_+\Big). \tag{2.3.3}$$

We now explain the proof, and we refer to [Zha26, Proposition 4.3] for the details, in the transposed form. The difficulty is that the simple slope inequality compares $\overline{N}$ with one codomain at a time, and that it only sees the largest slope. The strategy is to compare ranks instead of slopes, at every level $\xi \geqslant 0$ of the Harder–Narasimhan filtrations. There are three steps. In the comparison step we apply the simple slope inequality to each map ev_i, which sends $N^{>\bar{\mu}+\xi}$ into $V_i^{>\bar{\mu}+\xi-h_i}$. In the rank step we apply the zero estimate. The parts of large slope of the codomains are small, so the sections of large slope satisfy most of the conditions, and by Lemma 2.3.2 there are few of them. In the integration step we integrate the resulting bound on ranks over $\xi \in [0, H_+]$, which gives (2.3.3).

3. The evaluation map

In the usual slope method one evaluates sections of a line bundle at points, with multiplicities. We want the same here, namely a map from a space of cusp forms on a Shimura variety Sh to the forms on a Shimura set of CM abelian varieties. The difficulty is that the map cannot be defined directly on Sh. For $m > 2$ the cusp forms in question are not holomorphic sections, see Remark 3.3.4. Moreover neither the integrality nor the divisibility of the values is visible there. Our strategy has four steps.

(1) We write a cusp form as a theta lift from a large definite unitary group. This is the basis problem, see §3.4.
(2) We evaluate the theta lift on Shimura sets, by a finite sum and by m correspondences given by the jets at the origin of theta functions, see §3.5.
(3) We show that the result does not depend on the lift chosen in the first step. This is the seesaw identity, see §3.6.
(4) We take as integral structure on the cusp forms the one given by the theta lifts of integral forms, see §3.7.

The construction is summarized in Figure 1 in §3.8. The output of this section is the family of maps $\mathrm{ev}_u \colon R \to V_u$, to which we apply the slope method in §4.

3.1. **The data.** Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field, with d a prime. The **host field** E is an abelian CM field containing K, whose Galois characters include the powers of χ. Let Φ be the CM type of E induced from K, and let A_0 be an abelian variety with complex multiplication by $\mathcal{O}_E$ of type Φ. It is isogenous to a power of an elliptic curve E_0 with complex multiplication by K. We call a prime **inert** if it is inert in K. A subset of Φ, the **top places**, carries the large multiplicities.

There are even integers k and m and an odd integer t, and we put $r = k + m - 1$. We also put $b = \exp(x^\varepsilon)$ and $a = xb$. The number a is the size of the multiplicities, and $m \approx \sqrt{x}$ is the number of correspondences. The degree of E and the ranks k and t are powers of x.

3.2. **Shimura sets.** We now introduce the finite sets on which all our computations take place. Let X be a definite Hermitian space over E. Let $\mathcal{Z}(X)$ be the finite set of classes of self-dual $\mathcal{O}_E$-lattices $L \subset X$ in a fixed genus, and let Γ_L be the finite stabilizer of L. To a lattice L we attach the abelian variety

$$A_L = L \otimes_{\mathcal{O}_E} A_0.$$

The Hermitian form of L gives it a principal polarization. Hence $\mathcal{Z}(X)$ is a finite set of CM abelian varieties. It is the Shimura variety of the definite group $\mathrm{U}(X)$, which has dimension zero, and we call it a **Shimura set**.

For $\tau \in \Phi$ let L_τ be the τ-component of L. For each τ let λ_τ be a highest weight of $\mathrm{GL}(L_\tau)$. We put $\lambda = (\lambda_\tau)_{\tau\in\Phi}$, and we let $\mathcal{V}_\lambda(L)$ be the corresponding irreducible representation of $\prod_\tau \mathrm{GL}(L_\tau)$. A **form of weight** λ on $\mathcal{Z}(X)$ is a family $\phi(L) \in \mathcal{V}_\lambda(L)$, invariant under the groups Γ_L. These are the algebraic automorphic forms on $\mathrm{U}(X)$. Throughout, we fix a number field $\Bbbk$ containing E over which all the forms that we consider are defined. The integral forms then form a Hermitian lattice over $\Bbbk$, with the norm $\sum_L |\Gamma_L|^{-1} \|\phi(L)\|^2$.

3.3. **The source and the target.** We first say what the map should be, by describing its source, its target and the map itself. We then explain why this description cannot serve as a definition.

3.3.1. *The source.* Let Y_0 be a Hermitian space over E of dimension r and of signature $(k + \frac{m}{2} - 1, \frac{m}{2})$ at every place. Let Sh be the Shimura variety of $\mathrm{U}(Y_0)$, and let $\mathcal{S}$ be a space of cusp forms on $\mathrm{U}(Y_0)$, of a fixed type at the infinite places, with the Petersson norm. We put $D_0 = \dim \mathcal{S}$.

3.3.2. *The target.* Let $X_0 \subset Y_0$ be a definite subspace of dimension k. Its orthogonal complement is the sum of a line and of $\frac{m}{2} - 1$ hyperbolic planes. The codomains are indexed by a finite set $\mathcal{U}$. Each $u \in \mathcal{U}$ determines a weight λ_u for X_0, whose first entry is about a at every top place, together with a level structure at an auxiliary prime, which we suppress from the notation. Let $\overline{V}_u$ be the dual of the lattice of integral forms of weight λ_u on the Shimura set $\mathcal{Z}(X_0)$, and let d_u be its rank. We call it the **lattice of values**, and we write $V_u^\vee$ for the lattice of forms itself. It is defined as a dual because a value will be given as a functional on the forms $\phi \in V_u^\vee$.

3.3.3. *The map we want.* For $f \in \mathcal{S}$ and a form ϕ on $\mathcal{Z}(X_0)$, let $B_u(\phi, f)$ be the period of f against ϕ along $\mathrm{U}(X_0) \subset \mathrm{U}(Y_0)$. It is a period of Bessel type [GGP12]. The map we want is

$$\mathcal{S} \longrightarrow V_u \otimes \mathbb{C}, \qquad f \longmapsto B_u(\,\cdot\,, f).$$

Remark 3.3.4. For $m = 2$ the Shimura variety Sh is a ball quotient, and the image of $\mathcal{Z}(X_0)$ is a finite set of CM points. The map above sends a holomorphic cusp form to a component of its jet at these points. Hence it is an evaluation of sections at points with multiplicities, as in [CDT25]. For $m > 2$ the period B_u involves an integration over a unipotent group, and it vanishes on holomorphic forms. Hence the forms in $\mathcal{S}$ are not holomorphic sections, and the period is not an evaluation at points. For this reason B_u is *not* at our disposal as a definition.

3.4. **Step 1: cusp forms as theta lifts.** The first step is to write a cusp form as a theta lift. There are Hermitian spaces $Y_1, \dots, Y_m$ over E with $\dim Y_j = r + jt$, ending with a definite space Y_m. There are theta lifts from $\mathrm{U}(Y_{j+1})$ down to $\mathrm{U}(Y_j)$. We call their composition

$$\iota\colon \{\text{forms on } \mathcal{Z}(Y_m)\} \longrightarrow \mathcal{S}$$

the **descent**. Here the forms on the Shimura set $\mathcal{Z}(Y_m)$ are of the relevant weights.

Hypothesis 3.4.1 (Basis problem). The descent is surjective: every $f \in \mathcal{S}$ is of the form $f = \iota(F)$ for a form F on $\mathcal{Z}(Y_m)$.

In classical terms, the hypothesis says that the cusp forms in $\mathcal{S}$ are spanned by theta series attached to the definite space Y_m. Equivalently, the adjoint of ι, which is the theta

lift in the upward direction, is injective on $\mathcal{S}$. We stress that the form F is not unique, because the descent has a kernel, and we deal with this in Step 3.

3.5. **Step 2: the evaluation of a theta lift.** We now evaluate a theta lift $\iota(F)$. The key point is that the construction only involves F and Shimura sets. We first attach a theta function to a pair of points of two Shimura sets, then use its jets to define a correspondence, and finally compose m correspondences with a period.

3.5.1. *The theta function.* Let X and X' be definite Hermitian spaces, and let $L \in \mathcal{Z}(X)$ and $L' \in \mathcal{Z}(X')$. The lattice $\mathrm{Hom}_{\mathcal{O}_E}(L, L')$ is again a self-dual Hermitian lattice. We put

$$A_{L,L'} = \mathrm{Hom}_{\mathcal{O}_E}(L, L') \otimes_{\mathcal{O}_E} A_0 = \mathrm{Hom}_{\mathcal{O}_E}(L, A_{L'}).$$

It is a principally polarized abelian variety. Let $\theta_{L,L'}$ be a theta function, that is, a nonzero section of a symmetric line bundle defining the polarization, with a suitable integral normalization.

3.5.2. *Jets at the origin.* The tangent space of $A_{L,L'}$ at the origin is the sum over $\tau \in \Phi$ of the spaces $\mathrm{Hom}(L_\tau, L'_\tau) \otimes \mathrm{Lie}_\tau(A_0)$. We differentiate $\theta_{L,L'}$ at the origin along invariant vector fields. The jets of all orders form an element of the completed symmetric algebra of the cotangent space. By the Cauchy decomposition

$$\mathrm{Sym}\big(\mathrm{Hom}(L_\tau, L'_\tau)^\vee\big) = \bigoplus_{\lambda_\tau} \mathcal{V}_{\lambda_\tau}(L_\tau) \otimes \mathcal{V}_{\lambda_\tau}(L'_\tau)^\vee,$$

the jet of $\theta_{L,L'}$ defines, for every weight λ, a linear map $\vartheta_\lambda(L, L') \colon \mathcal{V}_\lambda(L) \to \mathcal{V}_\lambda(L')$. The component of weight λ involves the jets of order $|\lambda|$.

3.5.3. *The correspondence.* For a form ϕ of weight λ on $\mathcal{Z}(X)$ we put

$$(\Theta\phi)(L') = \sum_{L \in \mathcal{Z}(X)} \frac{1}{|\Gamma_L|}\, \zeta(L, L')\, \vartheta_\lambda(L, L')\, \phi(L), \tag{3.5.1}$$

where $\zeta(L, L')$ is a number of absolute value 1. Up to a twist by a power of the determinant, the result is a form on $\mathcal{Z}(X')$. We stress that the definition only uses the two Shimura sets, the abelian varieties $A_{L,L'}$ over their product, and one section of a line bundle on each of them.

3.5.4. *The chain and the period.* There are definite Hermitian spaces $X_0 \subset X_1 \subset \cdots \subset X_m$ with $\dim X_j = k + j(t+1)$, starting with the space X_0 of the target. Hence we have correspondences

$$\mathcal{Z}(X_0) \xrightarrow{\Theta_0} \mathcal{Z}(X_1) \xrightarrow{\Theta_1} \cdots \xrightarrow{\Theta_{m-1}} \mathcal{Z}(X_m).$$

The space Y_m is the orthogonal complement of an anisotropic vector $z \in X_m$. A lattice L'' in Y_m gives the lattice $L'' \perp \mathcal{O}_E z$ in X_m, and this defines a map $\mathcal{Z}(Y_m) \to \mathcal{Z}(X_m)$. For a form ϕ' on $\mathcal{Z}(X_m)$ and a form F on $\mathcal{Z}(Y_m)$, the **period** is the finite sum

$$P(\phi', F) = \sum_{L'' \in \mathcal{Z}(Y_m)} \frac{1}{|\Gamma_{L''}|} \big\langle \phi'(L'' \perp \mathcal{O}_E z), F(L'') \big\rangle.$$

3.5.5. *The value of a theta lift.* For a form F on $\mathcal{Z}(Y_m)$ we define $\widetilde{\mathrm{ev}}_u(F) \in V_u \otimes \mathbb{C}$ by

$$\big\langle \phi, \widetilde{\mathrm{ev}}_u(F) \big\rangle = P\big(\Theta_{m-1} \cdots \Theta_0\, \phi,\ F\big) \qquad (\phi \in V_u^\vee). \tag{3.5.2}$$

In words, we push the form F forward to $\mathcal{Z}(X_m)$, and we then transport it down the chain by the transposed correspondences. Every step is a finite sum, and every kernel is a jet of order about a of a theta function at the origin of an abelian variety.

3.6. **Step 3: independence of the lift.** The value $\widetilde{\mathrm{ev}}_u(F)$ is defined in terms of F, while we want a function of the cusp form $\iota(F)$. Hence we have to check that it does not depend on the choice of F.

HYPOTHESIS 3.6.1 (Seesaw identity). For every form ϕ on $\mathcal{Z}(X_0)$ of weight λ_u and every form F on $\mathcal{Z}(Y_m)$ we have $\big\langle \phi, \widetilde{\mathrm{ev}}_u(F)\big\rangle = B_u\big(\phi, \iota(F)\big)$.

It follows that $\widetilde{\mathrm{ev}}_u$ vanishes on the kernel of the descent. Together with Hypothesis 3.4.1, this allows us to make the following definition.

DEFINITION 3.6.2. The **evaluation map** $\mathrm{ev}_u \colon \mathcal{S} \to V_u \otimes \mathbb{C}$ is defined by $\mathrm{ev}_u(f) = \widetilde{\mathrm{ev}}_u(F)$ for any form F on $\mathcal{Z}(Y_m)$ with $\iota(F) = f$.

By construction we have $\big\langle \phi, \mathrm{ev}_u(f)\big\rangle = B_u(\phi, f)$. Hence ev_u is the map of §3.3, and for $m = 2$ it is an evaluation at CM points. What we have gained from the detour is a formula for this map in which every step is a finite sum on Shimura sets.

3.7. **Step 4: the integral structure.** It remains to say which cusp forms are integral. The difficulty is that for $m > 2$ the forms in $\mathcal{S}$ are not sections of a line bundle on Sh, so that Sh gives no integral structure. Our solution is to use the theta lifts. Let M be the lattice of integral forms on $\mathcal{Z}(Y_m)$, of the relevant weights. The **domain** of the slope method is

$$\overline{R} = \iota(M) \subset \mathcal{S},$$

with the Petersson norm. In other words, we call a cusp form integral if it is the theta lift of an integral form. By Hypothesis 3.4.1 the lattice R spans $\mathcal{S}$, and its rank is D_0.

This definition may look strange, but it is essential: with this choice the values are computed on integral data. For $f = \iota(F)$ with $F \in M$ we have $\mathrm{ev}_u(f) = \widetilde{\mathrm{ev}}_u(F)$, which is a finite sum of jets of theta functions applied to an integral form. Hence the integrality of $\mathrm{ev}_u \colon R \to V_u$, and every divisibility of its values, is a property of these jets, which we study in §4.1. The price is paid on the side of the norm. The integral structure comes from the Shimura set $\mathcal{Z}(Y_m)$ and the norm from Sh, and we have to compare the two, which we do in §4.4.

Finally we note that the total number of conditions $\rho = \sum_u d_u$ exceeds D_0 by a power of x.

3.8. **Summary: the evaluation map as a path.** Figure 1 summarizes the construction. The value $\mathrm{ev}_u(f)$ is obtained by following the three solid arrows: we lift f to a form F on $\mathcal{Z}(Y_m)$, we go across to $\mathcal{Z}(X_m)$, and we come down to $\mathcal{Z}(X_0)$. The dashed arrow at the bottom is the map we want, namely the period along $X_0 \subset Y_0$. The seesaw identity says that the path does not depend on F and that it agrees with the dashed arrow.

We now explain how the four corners are related. The space X_m is not related to Y_0 directly. At the bottom X_0 lies in Y_0, with codimension $m-1$. At the top Y_m lies in X_m, with codimension 1. To see this, note that $\dim Y_j - \dim X_j = m-1-j$, which drops by one at each step. According to [**Zha26**, §4.1] there are isometric embeddings $X_j \subset Y_j$ for all $j < m$, and the direction reverses only at the last stage. Hence the detour replaces a period of codimension $m-1$ on a Shimura variety by a period of codimension 1 on a Shimura set.

Each tower has its own role. The tower of the points is where we treat the finite places. The tower of the sections carries the integral structure at the top and the Petersson norm

at the bottom. Its two ends carry the same vector space with two different norms, which we compare in §4.4.

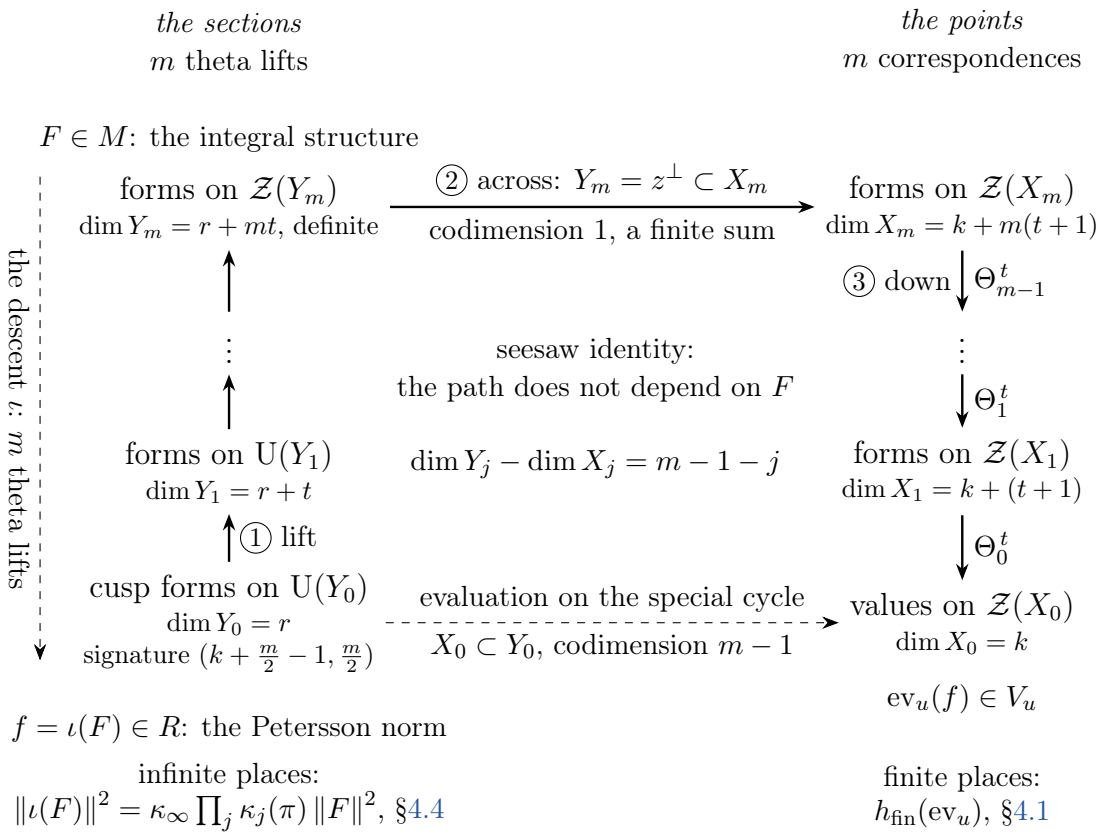

FIGURE 1. The evaluation map as a path. A cusp form $f = \iota(F)$ is lifted to the Shimura set $\mathcal{Z}(Y_m)$, sent across to $\mathcal{Z}(X_m)$ by a finite sum, and brought down to $\mathcal{Z}(X_0)$. The dashed arrow is the period along $X_0 \subset Y_0$, which for $m = 2$ is an evaluation of holomorphic sections at CM points.

3.8.1. *Mathematics used.* The abelian varieties A_L and $A_{L,L'}$ are given by Serre's tensor construction, and Θ is the theta lift for the pair of compact groups $(\mathrm{U}(X), \mathrm{U}(X'))$. In the Fock model its kernel is a theta series with harmonic polynomial coefficients, and these are the jets of $\theta_{L,L'}$. Hypothesis 3.4.1 is the basis problem of Eichler over $\mathbb{C}$. It amounts to the non-vanishing of the theta lifts in the upward direction, as given by the inner product formula of Rallis when the target group is large. The seesaw identity is the seesaw principle for dual pairs, applied m times along the two chains. The principle is classical, but here the constants must be exact.

4. THE INPUTS OF THE SLOPE METHOD

We now go through the five points of §2.2.2 for the maps $\mathrm{ev}_u \colon R \to V_u$. For each point we give the statement that we need and the mathematics that it uses, and we combine these statements in §5. Throughout, the letters e denote small non-negative errors.

4.1. **The height at the finite places.** This is point (3) of the list. In the usual slope method the height of the evaluation map at the finite places is only known to be non-positive, because the map is integral. Here it is the source of the main term. We state the bound that we need. We only indicate the mechanism behind it, and we do not go into the local estimates.

4.1.1. *Divisibility and the height.* Suppose that $\mathrm{ev}_u(R) \subset J_u V_u$ for an ideal J_u supported at the inert primes. By (2.2.1) we then have

$$h_{\mathrm{fin}}(\mathrm{ev}_u) \leqslant -\frac{2\log \mathrm{N}\, J_u}{[\Bbbk : \mathbb{Q}]}. \tag{4.1.1}$$

Hence a divisibility of the values at the inert primes is an upper bound for the height at the finite places. By §3.5, to obtain it we have to bound from below the p-adic valuation of jets of high order of theta functions at the origin of a power of E_0.

4.1.2. *The shape of the gain.* Let p be an inert prime of good reduction. The abelian varieties $A_{L,L'}$ have supersingular reduction at p, and their formal group has height 2. By the mechanism of §1.3.4, a jet of order n of a function is then divisible by about $p^{n\gamma_p}$, where $\gamma_p = \sum_{e \text{ odd}} p^{-e}$. The theta function is a section and not a function, and this causes a loss. Consider a jet of order a at a place of Φ and of order w at the conjugate place. The statement that we need is that the loss is at most w, so that the valuation of this jet at p is at least

$$\sum_{e \text{ odd}} \max\Big(\frac{a}{p^e} - w,\, 0\Big) \tag{4.1.2}$$

up to an error which is logarithmic in the orders. We call this lower bound the **gain** at p. In words, it is the order at the top place divided by p^e, truncated by the order at the conjugate place. We do not prove this statement here, and it is part of Hypothesis 4.1.4 below. The logarithmic error is harmless, because the orders are of size $b = \exp(x^{\varepsilon})$ and fewer than x primes have a positive gain.

4.1.3. *The local bound.* We now pass from one theta function to the evaluation map. Let $\mathcal{B}_u$ be the sum over the inert primes of the gains (4.1.2) of all the correspondences, with the weights $2\log p$ of (4.1.1). We write it out in the simplest case in §5.2.2.

Hypothesis 4.1.4 (Local bound). There is a small error $\mathrm{err}_{\mathrm{fin}} \geqslant 0$ such that $h_{\mathrm{fin}}(\mathrm{ev}_u) \leqslant -\mathcal{B}_u + \mathrm{err}_{\mathrm{fin}}$.

This is the only place where the primes enter.

4.1.5. *Mathematics used.* The formal group of a CM abelian variety at an inert prime is a Lubin–Tate group of height 2, and the gain for functions is a p-adic Cauchy estimate on a polydisc of radius $p^{-1/(p^2-1)}$. The passage from functions to sections uses the theorem of the cube. For an elliptic curve and one variable, the convergence of the sigma function at a supersingular prime is studied in [**BKY17**]. We do not use p-adic Hodge theory at the inert primes.

4.2. **The slope of the lattice of values.** This is point (5) of the list. In the simplest applications of the slope method the codomain is a sum of fibres of a line bundle at points, and its slope is a height of these points. Here the points are the CM abelian varieties of $\mathcal{Z}(X_0)$ and the fibres are dual to the representations $\mathcal{V}_{\lambda_u}(L)$. We state the bound that we need, and we then explain where its explicit terms come from.

HYPOTHESIS 4.2.1 (Slope of the lattice of values). There are an explicit number μ_u^{val} and a small error $\mathrm{err}_{\mathrm{val}} \geqslant 0$ such that $\mu(\overline{V}_u) \leqslant \mu_u^{\mathrm{val}} + \mathrm{err}_{\mathrm{val}}$.

4.2.2. *Mathematics used.* The slope of the representation lattices is explicit, by the Borel–Weil theorem. The other ingredient is the height of the CM abelian varieties. Let ω be the line of invariant differentials of E_0 over a field of good reduction, with the metric $\|\varpi\|^2 = \left|\frac{i}{2\pi}\int \varpi\wedge\overline{\varpi}\right|$ for a differential ϖ, and let $-D$ be the discriminant of K. The Chowla–Selberg formula reads [**KRY04**]

$$\log\Omega := \gamma + \log\frac{2\pi}{\sqrt{D}} - \frac{L'}{L}(1,\chi_K) = -2\,\widehat{\deg}\,\overline{\omega}. \tag{4.2.1}$$

A jet of order a of a theta function is a section of $\omega^{\otimes a}$ twisted by a line of degree 0, hence of a line of slope $-a\log\Omega$, and its normalization by $a!$ contributes $a\log a - a$. For the abelian CM field E the height is given by the formula of Colmez [**Col93**].

4.3. **The zero estimate.** This is point (1), in the form required by Theorem 2.3.3: we need the family $(\mathrm{ev}_u)_{u\in\mathcal{U}}$ to admit a frame. By §2.3 this is a quantitative equidistribution statement. We first describe the frame, then explain in which sense the special cycles are equidistributed, and finally state the hypothesis and deduce the global bound.

4.3.1. *The frame.* Let $\mathcal{H}$ be the full space of cusp forms on $\mathrm{U}(Y_0)$ of the relevant type. It contains $R\otimes\mathbb{C}$. As frame map we take the Bessel period $f \mapsto B_u(\,\cdot\,, f)$, defined on all of $\mathcal{H}$, which we still denote by B_u. By the seesaw identity it extends ev_u. Let $c_u > 0$ be the weights of the frame, and put $Q_{u,u'} = \sqrt{c_u c_{u'}}\, B_u B_{u'}^*$, so that $\operatorname{tr} Q^2 = \sum_{u,u'} \|Q_{u,u'}\|_{\mathrm{HS}}^2$. Since ρ is much larger than $\dim\mathcal{H}$, the conditions (2.3.1) follow from the pairwise bounds

$$\|Q_{u,u'}\|_{\mathrm{HS}}^2 \leqslant (1+\eta_0)\frac{d_u d_{u'}}{\dim\mathcal{H}} \quad (u\neq u'), \qquad \|Q_{u,u} - 1\|_{\mathrm{HS}}^2 \leqslant \alpha_0\, d_u. \tag{4.3.1}$$

The right-hand side of the first bound is what we would find for two independent random families of conditions.

4.3.2. *The zero estimate as equidistribution.* We now explain what the existence of a frame means here. That Q is almost scalar means that the evaluation maps, taken together, sample the functions $|f|^2$ evenly. For $m = 2$ the maps B_u take jets at the finite sets of CM points of Remark 3.3.4, and the statement is that these sets are equidistributed on Sh. The equidistribution holds in a specific sense, which we now make precise. The test functions are the functions $|f|^2$ with $f \in \mathcal{H}$, so they oscillate at the scale of the weight. The statement is in the mean over the family $u \in \mathcal{U}$, and in the second moment over f. The pairwise bounds (4.3.1) express it on pairs: two families of conditions are as independent as two random families. For $m > 2$ the periods are not evaluations at points, and the precise statement is that Q is almost scalar.

This statement should not be confused with the equidistribution of a single Galois orbit of CM points against a fixed test function. The latter is tied to subconvexity, and we do not need it. Here we average over the family and over f, but the test functions vary with the weight. On the other hand our statement is stronger than a zero estimate in the usual sense, which is its algebraic counterpart.

HYPOTHESIS 4.3.3 (Zero estimate, as quantitative equidistribution). The family $(\mathrm{ev}_u)_{u\in\mathcal{U}}$ admits a frame whose parameters δ, α and η are bounded by a large negative power of x.

Let $\mathbb{E}_u[\,\cdot\,]$ be the mean over $u \in \mathcal{U}$ weighted by the ranks d_u, and let $h_u \geqslant h_\infty(\mathrm{ev}_u)$. Applying Theorem 2.3.3 to the maps $\mathrm{ev}_u \colon \overline{R} \to \overline{V}_u$, with the bounds $h_{\mathrm{fin}}(\mathrm{ev}_u) + h_u$ for their heights, we obtain the global bound.

COROLLARY 4.3.4 (Global bound). *Assume Hypothesis 4.3.3. Then* $\mu(\overline{R}) \leqslant \mathbb{E}_u\big[\mu(\overline{V}_u) + h_{\mathrm{fin}}(\mathrm{ev}_u) + h_u\big] + \mathrm{err}_{\mathrm{slope}}$, *where* $\mathrm{err}_{\mathrm{slope}}$ *is the loss in* (2.3.3).

4.3.5. *Mathematics used.* The adjoint B_u^* is a Poincaré series of Bessel type, and the bounds (4.3.1) are to be read off from the geometric side of a relative trace formula for Bessel periods. The local theory of these periods is available [**GGP12**]. What is not in the literature is the uniformity. The rank, the weight and the level grow together, and the error must be a negative power of x.

4.4. **The domain and the heights at infinity.** These are points (2) and (4). In the usual slope method the slope of the domain is given by the arithmetic Hilbert–Samuel theorem on the variety whose sections form the domain. Here the domain $\overline{R}$ is not a lattice of sections on Sh. The natural question is therefore where a Hilbert–Samuel estimate can be applied. The answer is that the slope of $\overline{R}$ is expressed through lattices of sections on m other Shimura varieties, one for each correspondence. We first introduce these Shimura varieties, then state the norm identity and deduce a formula for the slope of the domain, and finally say which mathematics the resulting bounds use.

4.4.1. *The Shimura varieties* Sh_j. For $1 \leqslant j \leqslant m$ let W_j be a Hermitian space over E of rank r and of signature $(k-1+j, m-j)$ at every place, which is isometric to Y_0 at all finite places. Hence the groups $\mathrm{U}(W_j)$ are inner forms of $\mathrm{U}(Y_0)$. Let Sh_j be the Shimura variety of $\mathrm{U}(W_j)$. Let $\mathcal{S}_j$ be the space of holomorphic forms on it of a weight determined by the j-th correspondence, with the Petersson norm, and let $\overline{\mathcal{M}}_j$ be the Hermitian lattice of integral forms. These forms are the sections of a line bundle on a flag bundle Fl_j over Sh_j. The **type** π of a form is the finite part of the automorphic representation it generates. The spaces $\mathcal{S}$ and $\mathcal{S}_j$ decompose according to the type, and the same types occur in all of them with the same multiplicities.

Let X_j' be a definite Hermitian space of rank $r+t$, and let $\overline{T}_j$ be the lattice of integral forms on $\mathcal{Z}(X_j')$. There is a theta lift θ_j from these forms to $\mathcal{S}_j$. It is the construction of §3.5.1 in a family. The abelian scheme $\mathrm{Hom}_{\mathcal{O}_E}(L, \mathcal{A}_j)$ over Sh_j carries a theta function, and its jets along the zero section are sections of automorphic bundles. On each type π the operator $\theta_j\theta_j^*$ is a scalar $\kappa_j(\pi) > 0$.

HYPOTHESIS 4.4.2 (Norm identity). There is a function $\kappa_\infty > 0$ of the infinite place such that for every $F \in M \otimes \mathbb{C}$ of type π and orthogonal to the kernel of the descent,

$$\|\iota(F)\|^2 = \kappa_\infty \prod_{j=1}^{m} \kappa_j(\pi)\ \|F\|^2 . \tag{4.4.1}$$

In other words, the Petersson norm of the descent is the norm on the Shimura set $\mathcal{Z}(Y_m)$ multiplied by the constants of the m theta lifts θ_j.

4.4.3. *The slope of the domain and the heights.* Our goal is a lower bound for $\mu(\overline{R}) - h_\infty(\mathrm{ev}_u)$. The difficulty is that neither term is accessible directly. The lattice $\overline{R}$ is not a lattice of sections, and the height involves the constant κ_∞. The key point is that Lemma 2.1.1, applied to ι and to each θ_j, expresses both terms through the same constants. By (3.5.2) the map $\widetilde{\mathrm{ev}}_u$ is defined on M by finite sums. Let $h_u^{\mathcal{Z}}$ be its height at

the infinite places for the norm of M on the Shimura set $\mathcal{Z}(Y_m)$, and let I_j be the index of $\theta_j(T_j)$ in $\mathcal{M}_j$. We write $\mathbb{E}[\log \kappa_\infty]$ for the mean of $\log \kappa_\infty$ over the infinite places, and $\mathbb{E}[\log \kappa_j]$ for the mean of $\log \kappa_j(\pi)$ over the infinite places and over the types, weighted by their multiplicities.

PROPOSITION 4.4.4 (Domain and heights). *Assume Hypothesis 4.4.2. Put* $h_u = h_u^{\mathcal{Z}} - \mathbb{E}[\log \kappa_\infty] - \sum_j \min_\pi \log \kappa_j(\pi)$. *Then* $h_\infty(\mathrm{ev}_u) \leqslant h_u$ *and*

$$\mu(\overline{R}) - h_u \;\geqslant\; \sum_{j=1}^{m} \Big(\mu(\overline{\mathcal{M}}_j) + \min_\pi \log \kappa_j(\pi)\Big) - h_u^{\mathcal{Z}} - \mathrm{err}_{\mathrm{ind}},$$

where $\mathrm{err}_{\mathrm{ind}}$ *is a sum of terms at the finite places, which contains the indices* $\frac{2}{D_0} I_j$.

Proof. Let $M_\perp$ be the projection of M to the orthogonal complement of the kernel of ι. Lemma 2.1.1 for $\iota\colon M_\perp \to R$, whose determinant is given by (4.4.1), reads $\mu(\overline{R}) = \mu(\overline{M}_\perp) - \mathbb{E}[\log \kappa_\infty] - \sum_j \mathbb{E}[\log \kappa_j]$, and for θ_j it reads $\mu(\overline{\mathcal{M}}_j) = \mu(\overline{T}_j) - \mathbb{E}[\log \kappa_j] + \frac{2}{D_0} I_j$. Hence

$$\mu(\overline{R}) = \sum_j \mu(\overline{\mathcal{M}}_j) - \mathbb{E}[\log \kappa_\infty] - \mathrm{err}_{\mathrm{ind}}, \qquad \mathrm{err}_{\mathrm{ind}} = \sum_j \mu(\overline{T}_j) + \frac{2}{D_0} \sum_j I_j - \mu(\overline{M}_\perp).$$

The slopes $\mu(\overline{M}_\perp)$ and $\mu(\overline{T}_j)$ are slopes of lattices of forms on Shimura sets, whose norms are defined over a number field. By the product formula they are sums of terms at the finite places. Finally, passing from the norm of M to the Petersson norm divides the norm of ev_u by the square root of the factor in (4.4.1), which gives $h_\infty(\mathrm{ev}_u) \leqslant h_u$. Subtracting the two relations, we obtain the proposition. □

We now briefly explain what the proposition says. The constant κ_∞ cancels in the difference, which is why we need not compute the heights at the infinite places. Each of the m Shimura varieties Sh_j contributes once, which matches the m correspondences on the side of the codomain. To make the right-hand side explicit, we need a lower bound for the slopes $\mu(\overline{\mathcal{M}}_j)$, a lower bound for the constants $\kappa_j(\pi)$, and an upper bound for $\mathrm{err}_{\mathrm{ind}}$. We record the outcome as a hypothesis.

HYPOTHESIS 4.4.5 (Bound for the domain). There are explicit numbers μ_u^{dom} and a small error $\mathrm{err}_{\mathrm{dom}} \geqslant 0$ such that $\mu(\overline{R}) - h_u \geqslant \mu_u^{\mathrm{dom}} - \mathrm{err}_{\mathrm{dom}}$.

4.4.6. *Mathematics used.* The identity (4.4.1) has the form of the inner product formula of Rallis, applied at each step of the descent. The constants $\kappa_j(\pi)$ are values of L-functions times local zeta integrals, which are explicit at the infinite places for holomorphic discrete series. This is the reason for the inner forms W_j. By Remark 3.3.4 the forms on $\mathrm{U}(Y_0)$ are not holomorphic for $m > 2$, and the holomorphic members of their packets live on the Sh_j. To our knowledge, the factorization (4.4.1) over the inner forms is not in the literature.

The lower bound for the slopes $\mu(\overline{\mathcal{M}}_j)$ is a quantitative form of the arithmetic Hilbert–Samuel theorem. The lattice $\overline{\mathcal{M}}_j$ is a lattice of sections of a line bundle $\mathcal{L}^{\otimes \nu}$ on Fl_j, a variety of dimension Δ. The asymptotic theorem of Gillet and Soulé [**GS92**], in the forms given by Zhang [**Zha95**] and Yuan [**Yua08**], does not suffice, because the variety, its dimension and the field vary with x. We need a bound at a given degree ν, with explicit errors.

HYPOTHESIS 4.4.7 (Quantitative arithmetic Hilbert–Samuel). For each connected component Z of Fl_j, of degree $\deg_Z$ and normalized height $\mathrm{ht}_Z = \hat{c}_1(\overline{\mathcal{L}})^{\Delta+1}/(\Delta+1)\deg_Z$,

$$\binom{\nu-1}{\Delta}\deg_Z\cdot\Big(\nu\,\mathrm{ht}_Z - 3(\Delta+1)\,|\mathrm{ht}_Z| - C_1\log\big(2+|\mathrm{ht}_Z|\big) - C_2\Big) \leqslant \widehat{\deg} H^0\big(Z,\mathcal{L}^{\otimes\nu}\big),$$

with constants C_1 and C_2 within the error budget.

The main term is $\binom{\nu}{\Delta+1}\hat{c}_1(\overline{\mathcal{L}})^{\Delta+1}$, which differs from the main term of the arithmetic Hilbert–Samuel theorem by a factor $1-O(\Delta^2/\nu)$. The method is that of Chow forms, and the known bounds of this kind lose a factor exponential in Δ, see [**DP99**, Proposition 4.2] and [**MY25**, Proposition 3.6]. The results of algebraic and arithmetic geometry that enter the argument of [**Alp26**] at this point are listed in [**Zha26**, §4.3].

4.4.8. *The heights of the Shimura varieties.* Hypothesis 4.4.7 involves the number ht_Z, which is the normalized arithmetic volume of $\overline{\mathcal{L}}$ on a component of Fl_j. The natural question is how this number is known. In [**CDT25**] the corresponding number is computed exactly. There the arithmetic Hilbert–Samuel theorem and a theorem of Bost and Charles express the degree of the domain through the double integral

$$\iint_{\mathbb{T}^2}\log|\varphi(z)-\varphi(w)|\;d\mu(z)\,d\mu(w),$$

where φ is the analytic map of the holonomy bound and μ is the Haar measure of the unit circle $\mathbb{T}$, see [**BC22**, Theorem 5.4.1 and Proposition 5.4.2].

In principle an exact formula is also available here. The arithmetic volume of a Shimura variety is expected to be a combination of logarithmic derivatives of L-functions, and this is known in many cases. We mention the modular curves [**KÖ1**], the Hilbert modular surfaces [**BBGK07**], the Shimura varieties of orthogonal type [**HÏ4**], and the unitary Shimura varieties of signature $(n-1,1)$ [**BH21**]. These results do not apply to our setting, in which the spaces W_j have general signature, and their rank and the field E vary with x.

We therefore do not compute ht_Z, and we only bound it from below, in two steps. The first step is the inequality of Zhang on successive minima [**Zha95**, Theorem 5.2]: if every algebraic point of Z has height at least e, then $\mathrm{ht}_Z \geqslant e$. The second step is a lower bound for the height of every point. A point of Fl_j is an abelian variety with an action of $\mathcal{O}_E$ and a flag in its Lie algebra, and $\mathcal{L}$ is a product of determinants of quotients of its Hodge bundle. Hence the height of the point is bounded from below as soon as the slopes of the Lie algebra are bounded from above. The statement that we need is the following. Let A be a principally polarized abelian variety of dimension g over a number field, with an action of $\mathcal{O}_E$ and with a semi-abelian model $\mathcal{A}$ over the ring of integers. Let $\overline{\mathrm{Lie}}\,\mathcal{A}$ be the Lie algebra of $\mathcal{A}$ with the Hermitian metrics given by the polarization, and let $h(A)$ be the Faltings height. Then

$$\mu_{\max}\big(\overline{\mathrm{Lie}}\,\mathcal{A}\big) \;\leqslant\; C\log\big(2+g+\max(h(A),0)\big) \tag{4.4.2}$$

with an absolute constant C. A bound of this kind is known. For the usual slope $\hat{\mu}$, Gaudron proves that

$$\hat{\mu}_{\max}\big(\overline{\mathrm{Lie}}\,\mathcal{A}\big) \;\leqslant\; h(A) + \frac{g}{2}\log\max\big(1,h(A)\big) + \frac{3g}{2}\log(15.4\,g),$$

see [**Gau19**, Proposition 4.2]. The proof follows a method of Bost [**Bos96**, §5.3.4]. It evaluates the first jets of the sections of $L^{\otimes 2}$ at a torsion point, and it rests on the matrix lemma of Autissier [**Aut13**] and on the formula of Bost for the degree of $H^0(A,L)$. This

bound is linear in the Faltings height, while (4.4.2) is logarithmic, and hence stronger. In [Alp26] it is deduced from a theta function of small height on A. For $g = 1$ both bounds follow from the lower bound of Bost for the Faltings height [GR14, Corollaire 8.4]. Finally the Hodge metrics are singular along the boundary of Sh_j, and they are modified there before the inequality of Zhang is applied, see [Zha26, §4.3].

We now briefly explain what this says about the comparison of slopes. The two slopes are known with different precision. The slope of the codomain is computed exactly, by the Chowla–Selberg formula (4.2.1), and it carries the constants of the main term. The heights ht_Z are only bounded from below, with a constant that is not explicit. In particular they contribute to the error $\mathrm{err}_{\mathrm{dom}}$ and not to the main term.

4.4.9. *The comparison of integral structures.* Finally the term $\mathrm{err}_{\mathrm{ind}}$ compares two integral structures on $\mathcal{S}_j$, namely the forms that are integral on Sh_j and the theta lifts of integral forms. The upper bound for the index I_j is an integral form of the basis problem of Eichler. Results of this kind are known for GL_2 and quaternion algebras, see Emerton [Eme02], Hida [Hid05] and Prasanna [Pra06]. For unitary groups of growing rank this is the least established input. The argument of [Alp26] for this step, through Galois cohomology, is outlined in [Zha26, §5].

5. The comparison of slopes

In this section we combine the estimates of §4. We first deduce the slope comparison, which is the chain (1.3.2) of the introduction. We then show on the simplest example how its two sides produce the function G, and we explain the choice of the number m of correspondences.

5.1. **The slope comparison.** We first recall the four estimates, and we write $\mathcal{B} = \mathbb{E}_u[\mathcal{B}_u]$.

$$\mu(\overline{V}_u) \leqslant \mu_u^{\mathrm{val}} + \mathrm{err}_{\mathrm{val}} \qquad \text{(Hypothesis 4.2.1)}, \tag{5.1.1}$$

$$h_{\mathrm{fin}}(\mathrm{ev}_u) \leqslant -\mathcal{B}_u + \mathrm{err}_{\mathrm{fin}} \qquad \text{(Hypothesis 4.1.4)}, \tag{5.1.2}$$

$$\mu(\overline{R}) \leqslant \mathbb{E}_u\big[\mu(\overline{V}_u) + h_{\mathrm{fin}}(\mathrm{ev}_u) + h_u\big] + \mathrm{err}_{\mathrm{slope}} \qquad \text{(Corollary 4.3.4)}, \tag{5.1.3}$$

$$\mu(\overline{R}) - h_u \geqslant \mu_u^{\mathrm{dom}} - \mathrm{err}_{\mathrm{dom}} \qquad \text{(Hypothesis 4.4.5)}. \tag{5.1.4}$$

Proposition 5.1.1 (Slope comparison). *If* (5.1.1)–(5.1.4) *hold, then*

$$\mathcal{B} \leqslant \mathbb{E}_u\big[\mu_u^{\mathrm{val}} - \mu_u^{\mathrm{dom}}\big] + \mathrm{err}_{\mathrm{val}} + \mathrm{err}_{\mathrm{fin}} + \mathrm{err}_{\mathrm{dom}} + \mathrm{err}_{\mathrm{slope}}.$$

Proof. The estimate (5.1.3) reads $\mathbb{E}_u[-h_{\mathrm{fin}}(\mathrm{ev}_u)] \leqslant \mathbb{E}_u[\mu(\overline{V}_u) - (\mu(\overline{R}) - h_u)] + \mathrm{err}_{\mathrm{slope}}$. On the right we bound the first term by (5.1.1) and the second by (5.1.4). By (5.1.2) the left-hand side is at least $\mathcal{B} - \mathrm{err}_{\mathrm{fin}}$. □

This is the chain (1.3.2). It says that the sum over primes $\mathcal{B}$ is bounded by explicit quantities, in which neither the slope of the domain nor the heights at the infinite places appear separately. In the notation of [Zha26], the four estimates correspond to the inequalities (4.6) to (4.9) there, and the proposition to its inequality (4.10).

5.2. **The two sides.** We now identify the two sides of Proposition 5.1.1. We first describe the main term, and then compute the sum over primes in the simplest case.

5.2.1. *The main term.* Let a_τ be the multiplicity at a top place τ, and let $w_j(\tau') > 0$ be certain smaller weights of the j-th correspondence at the places $\tau' \in \Phi$. We call them the **nodes**. Up to errors, we have

$$\mathbb{E}_u\big[\mu_u^{\mathrm{val}} - \mu_u^{\mathrm{dom}}\big] = \sum_{j=1}^{m} \sum_{\tau \text{ top}} a_\tau \Big(\log a_\tau - 1 - \gamma + \frac{L'}{L}(1, \chi_K) - \frac{1}{|\Phi|} \sum_{\tau' \in \Phi} \log w_j(\tau') \Big). \quad (5.2.1)$$

The term $a_\tau(\log a_\tau - 1)$ is the logarithm of a factorial and comes from the normalization of jets of order a_τ. The constants γ and $\frac{L'}{L}(1, \chi_K)$ come from the CM period (4.2.1). Each correspondence contributes once.

5.2.2. *The sum over primes in one row.* The simplest case shows the mechanism. Take $E = K$, one correspondence, one multiplicity a and one node $w \leqslant a$, and put $x = a/w$. By (4.1.2) the gain at an inert prime p is $\sum_{e \text{ odd}} (a/p^e - w)_+$. With the weights $2 \log p$, and a similar term at the ramified prime d, the sum over primes is

$$\mathcal{B} = 2 \sum_{\chi_K(p) = -1} \log p \sum_{e \text{ odd}} \Big(\frac{a}{p^e} - w \Big)_+ + \log d \sum_{e \geqslant 1} \Big(\frac{a}{d^e} - w \Big)_+ = w \sum_{n \leqslant x} \Lambda(n) \big(1 - \chi_K(n)\big) \Big(\frac{x}{n} - 1 \Big).$$

Here we use that $1 - \chi_K(p^e)$ equals 2 for odd powers of inert primes, 1 for powers of d and 0 otherwise. The main term (5.2.1) is $a(\log x - 1 - \gamma + \frac{L'}{L}(1, \chi_K))$. Let $g_{\chi_K}(x) = \sum_{n \leqslant x} \Lambda(n) \chi_K(n)(x/n - 1) + x \, \frac{L'}{L}(1, \chi_K)$ be the twist of G by χ_K. Then

$$\mathcal{B} - \text{main term} = w\big(G(x) - g_{\chi_K}(x)\big), \quad (5.2.2)$$

and Proposition 5.1.1 is an upper bound for $G - g_{\chi_K}$ at the point $x = a/w$. This shows the mechanism: the slope inequality compares complete logarithmic derivatives of L-functions, which come from heights, with sums over primes truncated at x, which come from the gain.

In general (5.2.2) becomes an expansion over the characters of the Galois group of E, and an average isolates the combination $G_\Xi(x)$. We refer to [**Zha26**, §3 and §4.6] for this step, and for the choice of the top places.

5.3. **The number of correspondences.** It remains to explain the choice of m. After the average, the left-hand side of the comparison is about $n_T \, m \, b \, G_\Xi(x)$, where n_T is the number of top places. The errors are of two kinds. The first is of size $a = xb$ per top place and does not grow with m. The second is of size b per correspondence and per place, and there are m of them in each of the m terms of the main term. After division by $n_T m b$ the two contribute $x^{1+O(\varepsilon)}/m$ and $m \, x^{O(\varepsilon)}$. The choice $m \approx \sqrt{x}$ balances them and gives $G_\Xi(x) \leqslant x^{1/2 + C\varepsilon}$. The simultaneous choice of all the parameters is described in [**Zha26**, §6].

We end with a remark on the role of m. The argument of [**Alp26**] is presented there as a descendant of the method of Stepanov, with m in the part of the order of vanishing, see [**Zha26**, §4.1]. In our setting this reads as follows. The order a of the jets is a multiplicity in the usual sense, but it does not amplify the unknown: by (5.2.2) it determines the point $x = a/w$ at which G is evaluated. What amplifies $G(x)$ is the repetition of the same evaluation. If we think of the primes as the points and of the p-adic valuation as the order of vanishing, then each correspondence adds once the same order of vanishing at every inert prime up to x. So the chain of m correspondences is what produces a value that vanishes to order m at all these points.

6. Remarks on the inputs

We end with three remarks on the inputs.

First, the inputs are of unequal weight. The evaluation map, the divisibility and the main term are statements on finite sets of CM abelian varieties and on one section of one line bundle. The zero estimate, the norm identity and the indices are of automorphic nature, and they are the inputs for which the literature gives the least. Two inputs are quantitative forms of classical statements of Arakelov geometry. The zero estimate is a quantitative equidistribution statement for the special cycles, and the bound of Hypothesis 4.4.7 is a quantitative form of the arithmetic Hilbert–Samuel theorem. In both cases what we need is a version with explicit errors, uniform in the rank.

Secondly, the classical theorems quoted above are stated for a fixed group or a fixed variety. Here the degree of E and the ranks are powers of x, and each term is needed with relative precision $x^{-1/2}$. Hence the uniform versions have to be established even where the statement is classical.

Thirdly, several of our hypotheses are schematic, and some of our descriptions are our own reading of [**Alp26**]. This is the case for the description of the evaluation map as an evaluation at CM points when $m = 2$, for the statement that the period vanishes on holomorphic forms when $m > 2$, and for the factorization of the norm over the inner forms in Hypothesis 4.4.2. We have not verified them against the Lean documents beyond the definitions.

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