Hecke calculations for the conditional vanishing of $K_{4k}(\mathbb{Z})$ through degree 104

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Abstract. Following a suggestion of Alpöge [1], we combine the Grothendieck–Montpellier archive's joint residual representation theorem and Bernoulli unit criterion [2] with explicit Hecke determinants. Under the strict Selmer comparison in Alpöge's Step 17 construction, we obtain $K_{4k}(\mathbb{Z}) = 0$ for $1 \le k \le 26$. At degree 104 two matrices of rank 324 exclude all large primes outside a set of four, and a matrix of rank 2,925 excludes those four. The cited Montpellier checkpoint reports vanishing through degree 80 under its source convention.

1. From a strict Selmer class to a compact Hecke eigensystem

Put $e = 2k$, $E_d = \mathbb{Q}(\sqrt{-d})$ for $d = 3, 7$, and let $\ell \ge 2^{31}$ be prime. Let S_e consist of the classes in $H^1(\mathbb{Q}, (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(-e))$ trivial on all finite inertia groups and on the decomposition groups at $2, \ell$. The cyclotomic comparison [7] gives $K_{2e}(\mathbb{Z})/\ell \ne 0 \Rightarrow S_e \ne 0$. Indeed the unramified cyclotomic class descends by the prime-to-ℓ extension degree. The prime above ℓ is principal, and multiplication by $2^{-e} - 1$ on $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ is surjective, giving the strict conditions.

Hypothesis and theorem. Use the closed inputs of Alpöge's pinned Step 17 construction [1]. Assume its strict length `Vlen`, at level 2 with trivial character and coefficients $\mathbb{Z}_\ell$, is positive whenever $S_e \ne 0$, in the fields and at the primes used below. Then

$$\boxed{K_{4k}(\mathbb{Z}) = 0 \quad (1 \le k \le 26).}$$

One coefficient prime. The hypotheses `GD` hold in both fields. Their class numbers are one, d is prime to 2ℓ, and ramification at d gives $E_d \not\subset \mathbb{Q}(\zeta_\ell)$. The character is trivial, e is even, $0 < 2e < \ell - 1$, and the coefficient ramification index is one. In [2, K17-S17-JOINT], positive length forces an actual deformation point. Otherwise the model-length bound at the identity has an empty covering set, forcing `Vlen` to vanish. The integral representation of this point reduces at one coefficient prime on all of G_{E_d} to

$$\bar{\rho}_f|_{G_{E_d}} = \omega^e \oplus 1 \oplus \omega^{-e},$$

where ω is the residual cyclotomic character. All the trace tests in a given field therefore apply to one eigenvector.

Choice of field. Write $\eta_d = \chi_{-d}$, $b_{n,d} = |\operatorname{num}(B_{n,\eta_d}/n)|$ in lowest terms and $A_d = b_{e+1,d} b_{2e+1,d}$. By [2, FV.3], the units $B_{1,\eta_d\omega^a}$ needed for $a = e, 2e$ are tested by $B_{a+1,\eta_d}/(a+1)$, using the Teichmüller lift of ω. The third quantity B_{1,η_d} is $-1/3$ or -1, respectively. At $e = 52$ the two denominators are $3, 9$ and $1, 49$, respectively. Exact arithmetic gives $\gcd(A_3, A_7) = 4$. Use E_3 when $\ell \nmid A_3$, and E_7 otherwise. All three units now hold in the selected field. By [2, ES.5] every actual point there is cuspidal of rank three and is not essentially self-dual.

Compact level. The level characters at $N = 2$ are trivial since $\mu_2 \subset \mathbb{Q}$. The level theorem in [1] applied to $\rho\chi_\ell^{-1}$ gives an everywhere finite-spherical conjugate-self-dual Π on GL_3/E_d. Its central character is trivial by its infinity type and class number one. The finite vector for the full stabilizer of the standard Hermitian lattice occurs on the definite unitary group in weight $(e-1, 0, 1-e)$. Indeed the global parameter is simple generic with trivial reduced component group, so [3] permits the compact member at infinity and the spherical members at good places. At ramified d, the bounded roots [4] are $z, 1, z^{-1}$. They are the stable lift of the unramified torus character with values z on a uniformizer and 1 on $U(1)$. In normalized induction from the Borel B, the function supported on BJ_d with value 1 at the identity is fixed by the full stabilizer J_d. Compact invariants are exact, giving a fixed vector in a constituent of the required packet [3, §4.7].

2. Normalization of the Hecke operators

Set $u_Q = Q^{2e} + Q^e + 1$. At geometric Frobenius of norm Q, the twist by χ_ℓ^{-1} and the source's algebraic Satake normalization both multiply by Q. Thus $\operatorname{tr}\rho$ is the unitary Satake trace. At split p the neighbor operator H_w has eigenvalue $p\sum z_i$, so $p^{e-1}H_w$ has residual target u_p in each direction. At inert 2 over E_3, the 18 distance-two neighbors have horospherical counts $1, 1, 16$, giving $H = 4z + 1 + 4z^{-1}$ on the eigenline. The base-change roots $z, 1, z^{-1}$ [5] give trace $(H+3)/4$.

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3. The degree-104 certificate and the finite range

Conjugation J exchanges split directions and fixes H. The characteristic-zero eigensystems of v, Jv are distinct since $\Pi^c = \Pi^\vee \neq \Pi$ by the unit criterion and Frobenius compatibility. Thus $v - Jv \neq 0$ has the same eigenvalues as v for the invariant operators. For any such operator, if its integral characteristic polynomial has a root $s \equiv 0 \bmod \lambda$, its constant term vanishes modulo λ. This applies even if $v \equiv Jv \bmod \lambda$.

Take $e = 52$ and set $p_1 = 43279339111$, $p_2 = 490440141803$, $p_3 = 63421406191963401438831238997$ and $p_4 = b_{105,3}/2162$. The certificates give $b_{53,3} = 2p_1p_2p_3$ and $b_{105,3} = 2 \cdot 23 \cdot 47p_4$, with all four p_i prime. Write $M = \prod_i p_i$. Over E_3 at $\ell \nmid M$, a witness would force both shifted determinants on the rank-324 negative space to vanish modulo ℓ. In terms of the unweighted neighbor operators the two shifts and the recorded gcd are

$$X = 4^{51}(H + 3I) - u_4 I, \qquad Z = 7^{51}(H_w + H_{\bar{w}}) - 2u_7 I.$$

$$\gcd(\det X, \det Z) = 3^{1578}5^{33}11^{31}19^{18}23^6 \cdot 29 \cdot 79\, p_1p_2p_3.$$

Over E_7 at each $\ell \mid M$, J preserves both lattice classes [6]. Their scaled split-2 sum has blocks $\left(\begin{smallmatrix} A & B \\ D & 0 \end{smallmatrix}\right)$. Its eigenvalue s on $v - Jv = (x, y)$ satisfies $s \equiv t = 2u_2 \bmod \lambda$. Since $\gcd(t, M) = 1$, $s \neq 0$ and $x \neq 0$. Eliminate y to get $F(s) = 0$, where $F(T) = \det((T^2 I - TA - BD)|_{H_0^-})$ on the standard-class negative space of rank $\binom{27}{3} = 2925$. The scaled correspondences preserve lattices away from 2, so $F(t) \equiv 0 \bmod \lambda$. Integral evaluation of the signed polynomial basis gives $W = V[t^2 I - tA - BD]$ on this space, hence $\det W = \det V\, F(t)$. The four certified residues give $\gcd(\det W, M) = 1$, a contradiction.

A cross-check from the adjoint space. For weight 53, level 3 and character η_3, let a_q be the T_q-eigenvalue and Δ_q the adjoint trace at the tested place minus $1 + Q^{52} + Q^{-52}$, with $Q = 4, 7$ respectively. The identity $\operatorname{tr}\mathrm{Ad}^0(g) = (\operatorname{tr} g)^2/\det g - 1$ gives

$$2^{104}\Delta_2 = (a_2^2 + 2^{53})^2 - (1 + 2^{104})^2, \qquad 7^{52}\Delta_7 = a_7^2 - (1 + 7^{52})^2.$$

Their determinants on the 16×16 Hecke matrices have gcd $3^{76}5^{12}11^2 19^2 (p_1p_2p_3)^2$.

Lower degrees. For even $2 \leq e \leq 50$, the odd part of $\gcd(A_3, A_7)$ is 1 except at $e = 12, 18, 36, 42$, where it is $47, 37, 37^2, 43^2$, respectively. The same field choice works. Over E_3, the gcd of the two products C_2U_2, C_7U_7 of classical and compact determinants has large prime support exactly that of A_3. A stable source forces both compact factors to vanish. Through degree 72 the remaining primes are excluded by directed compact determinants or cyclotomic tests. In degrees $76, 80, \ldots, 100$ the seven signed matrices have determinants coprime to the moduli M_e containing all large survivors. For $\ell < 2^{31}$, [7, 8] exclude odd torsion and [9] treats the prime 2. Finiteness gives the theorem.

Degree	76	80	84	88	92	96	100	104
Signed rank over E_7	1140	1330	1540	1771	2024	2300	2600	2925
Modulus size (bits)	368	419	426	475	489	515	527	602
$\gcd(\det W_e, M_e)$	1	1	1	1	1	1	1	1

The detailed proof and certificates and verifiers give the degree-by-degree checks, exact residues and matrix generators.

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