Hecke calculations for the conditional vanishing of $K_{4k}(\mathbb{Z})$ through degree 104

Leo Pintar École Polytechnique

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Abstract. In degree 104, two rank-324 Hecke operators over $\mathbb{Q}(\sqrt{-3})$, together with classical tests, leave four candidate primes. An integer evaluation matrix of rank 2,925 constructed using $\mathbb{Q}(\sqrt{-7})$ has determinant coprime to their product. Together with the certificates for earlier degrees, these computations imply $K_{4k}(\mathbb{Z}) = 0$ for $1 \le k \le 26$, assuming the strict Selmer comparison, coverage of the admissible representations, and a simultaneous residual trace condition. The supplement contains the matrices or their generators, exact residues, and replay code.

1. The degree-104 certificate

Put $e = 52$, $E_d = \mathbb{Q}(\sqrt{-d})$ for $d = 3, 7$, and $b_j = |\operatorname{num}(B_{j,\eta}/j)|$, where η is the quadratic character of conductor 3 and the numerator is taken in lowest terms. Define p_1, p_2, p_3, p_4 by

$$p_1 = 43\,279\,339\,111, \qquad p_2 = 490\,440\,141\,803,$$
$$p_3 = 63\,421\,406\,191\,963\,401\,438\,831\,238\,997, \qquad p_4 = b_{105}/2162.$$

Primality tests and exact arithmetic give $b_{53} = 2p_1p_2p_3$ and $b_{105} = 2 \cdot 23 \cdot 47\, p_4$. Write $M = p_1p_2p_3p_4$.

First field. On the rank-324 negative subspace for E_3, let X be the shifted inert-2 operator and Z the sum of the two shifted split-7 operators. In the integral normalization of the supplement the shift added at 2 is $3 \cdot 4^{51} - (4^{104} + 4^{52} + 1)$. The shift in each direction at 7 is $-(7^{104} + 7^{52} + 1)$. Reconstruction from the neighbor sums gives

$$\gcd(\det X, \det Z) = 3^{1578}5^{33}11^{31}19^{18}23^6 \cdot 29 \cdot 79\, p_1p_2p_3.$$

The two matrices have a common kernel line modulo each of p_1, p_2, p_3. With the comparisons in Section 2, the classical branch requires $\ell \mid b_{105}$ and the self-dual cuspidal branch requires $\ell \mid b_{53}$. Thus every possible prime $\ell \ge 2^{31}$ appears in M.

Second field. At the two places above 2 in E_7, the scaled target eigenvalue in each direction is $u = 2^{104} + 2^{52} + 1$. The weight-53 and weight-105 level-seven classical tests exclude the self-dual and classical branches at all four primes. For the remaining system, conjugation J exchanges distinct eigensystems, so $v - Jv \neq 0$ for a corresponding eigenvector v. This difference is an eigenvector for the sum of the directed operators. Under the simultaneous trace condition its eigenvalue reduces to $t = 2u$.

The two lattice classes give directed operators with blocks $\left(\begin{smallmatrix} A & B \\ D & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} A' & B' \\ D' & 0 \end{smallmatrix}\right)$. As $\gcd(t, M) = 1$, elimination of the second class gives the operator on the standard negative subspace

$$L = t^2 I - t(A + A') - (B + B')(D + D').$$

Its dimension is $\binom{27}{3} = 2925$. Let $F_j = P_j - JP_j$ be the signed orbit basis of bidegree $(51, 51)$ modulo $x \cdot y$. At the integral incidence pairs in the supplement (seed 128711) evaluate $V_{ij} = F_j(x_i, y_i)$ and $W_{ij} = (LF_j)(x_i, y_i)$. Then $W = V[L]$ over $\mathbb{Q}$. The scaled correspondences preserve a lattice at each p_i, so the proposed eigenvalue forces positive p_i-adic valuation of $\det L$ and hence $p_i \mid \det W$.

The table gives the least nonnegative residues $r_i = \det W \bmod p_i$. The three lines for r_4 form one decimal integer when read consecutively.

i	Bit length of p_i	r_i in decimal
1	36	32189186543
2	39	445745447561
3	96	224502128465254697645526729 27
4	432	385227761542015351632264879402615235505177920
		099210521658726589115415728765387796061514558
		743454746609756795606967109316896574903 1

Thus $\gcd(\det W, M) = 1$. The computation generates one matrix modulo the 602-bit product M, reduces it into each prime field, and computes the four determinants there. Direct evaluation of all 40 oriented contributions gives a separate check on three complete rows.

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2. Arithmetic hypotheses and the range through degree 104

Fix $e = 2k$, $1 \le k \le 26$, and a prime $\ell \ge 2^{31}$. Write ω for the residual cyclotomic character and S_e for the subgroup of $H^1(\mathbb{Q}, (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(-e))$ consisting of classes unramified everywhere and trivial at 2 and ℓ. The deduction uses the following three inputs in the fields and at the primes tested.

Strict Selmer comparison. A nonzero $K_{2e}(\mathbb{Z})/\ell$ gives a nonzero class in S_e. Assume that the length in Alpöge's `OneCharU_s`, for $N = 2$, trivial character and coefficients $\mathbb{Z}_\ell$, measures this strict group. The required implication, at a fixed coefficient embedding, is

$$S_e \neq 0 \implies \forall \sigma \in G_{E_d}\ \exists\, \rho_\sigma \text{ admissible}: \quad \operatorname{tr} \bar{\rho}_\sigma(\sigma) = (1 + \omega^e + \omega^{-e})(\sigma).$$

If all differences at one σ were units, the valuation sum in the bound would be zero. The source defines the bound in `NumberU3US.lean` and constructs it inside `corGK_of`.

Coverage. Assume the systems admitted by `Adm3U_s` and `Adm2A` occur with the tested weights, levels, twists and Frobenius normalizations. This includes the transfer to the compact unitary space of weight $(e-1, 0, 1-e)$, the self-dual reduction and the classical spaces. For the two-class model see Loeffler, *Explicit calculations of automorphic forms for definite unitary groups*.

Simultaneous residual traces. For each field separately, assume that $S_e \neq 0$ supplies one admitted system at one coefficient prime realizing the target traces at all places compared together. The pointwise implication above permits ρ_σ to vary. A proposed way to fix a single system interpolates the product of the candidate trace differences on ordinary inertia. For the proposed bound of $s = 2e^3 + 6(4e + 3)$ systems, even $2es = 29\,378\,128$ at $e = 52$ lies below $\ell - 1$. The simultaneous assertion is retained as a hypothesis.

Conditional conclusion. Under these inputs $K_{4k}(\mathbb{Z}) = 0$ for $1 \le k \le 26$. For odd ℓ, the cyclotomic comparison detects the quotient by ℓ in the even ω^{-e}-component of the cyclotomic class group. Vandiver's conjecture has been checked throughout the required range $\ell < 2^{31}$. The known 2-primary calculation treats $\ell = 2$. See Jordan et al., Theorem 5.1, Hart–Harvey–Ong, *Irregular primes to two billion*, and Weibel, *The K-book*, VI.10.

For $e = 2, 4, \ldots, 50$, let C_q, U_q be the classical and compact comparison determinants over E_3 at $q = 2, 7$. Their archived integer values satisfy, for every prime $\ell > 337$,

$$v_\ell(\gcd(C_2U_2, C_7U_7)) = v_\ell(b_{e+1}) + 3v_\ell(b_{2e+1}).$$

Thus all large primes requiring a second-field test divide $b_{e+1}b_{2e+1}$. In degrees $52, 56, 60, 64, 68$ the first field leaves respectively $4, 2, 4, 4, 3$ primes. The second-field determinants exclude them. In degree 72 the test works modulo a product containing all survivors. The supplement includes these cases and the earlier degrees.

For degrees $76, 80, \ldots, 100$, removing certified small factors from $b_{e+1}b_{2e+1}$ gives a test modulus M_e. The block elimination of Section 1 gives a signed evaluation matrix W_e. The recorded results are

Degree	76	80	84	88	92	96	100
Matrix dimension	1140	1330	1540	1771	2024	2300	2600
Bit length of M_e	368	419	426	475	489	515	527
$\gcd(\det W_e, M_e)$	1	1	1	1	1	1	1

Section 1 supplies the four-prime exclusion for degree 104. These groups are finite, so exclusion at every prime gives the conditional conclusion.

Prior work and reproducibility. Alpöge suggested this application in Step 17, footnote 6. A 21 November report in the Grothendieck–Montpellier archive gives a conditional calculation through degree 48. The v1.0 supplement includes the additional fourteen degrees $52, 56, \ldots, 104$. After installing the dependencies in `requirements.txt`, run `python reproduce.py` for the degree-104 calculation. The program rebuilds both rank-324 matrices and the rank-2,925 matrix from their defining sums.

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